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Liftable pairs of functors and initial objects

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Abstract
Let $\mathcal{A}$ and $\mathcal{B}$ be monoidal categories and let $R : \mathcal{A} \to \mathcal{B}$ be a lax monoidal functor. If R has a left adjoint L, it is well-known that the two adjoints induce functors $\overline{R} = \mathsf{Alg}(R) : \mathsf{Alg}(\mathcal{A}) \to \mathsf{Alg}(\mathcal{B})$ and $\underline{L} = \mathsf{Coalg}(L) : \mathsf{Coalg}(\mathcal{B}) \to \mathsf{Coalg}(\mathcal{A})$ respectively. The pair (L, R) is called *liftable* if the functor $\overline{R}$ has a left adjoint and if the functor $\underline{L}$ has a right adjoint. A pleasing fact is that, when $\mathcal{A}$, $\mathcal{B}$ and R are moreover braided, a liftable pair of functors as above gives rise to an adjunction at the level of bialgebras. In this note, sufficient conditions on the category $\mathcal{A}$ for $\overline{R}$ to possess a left adjoint, are given. Natively these conditions involve the existence of suitable colimits that we interpret as objects which are simultaneously initial in four distinguished categories (among which the category of epi-induced objects), allowing for an explicit construction of $\overline{L}$, under the appropriate hypotheses. This is achieved by introducing a relative version of the notion of weakly coreflective subcategory, which turns out to be a useful tool to compare the initial objects in the involved categories. We apply our results to obtain an analogue of Sweedler's finite dual for the category of vector spaces graded by an abelian group G endowed with a bicharacter. When the bicharacter on G is skew-symmetric, a lifted adjunction as mentioned above is explicitly described, inducing an auto-adjunction on the category of bialgebras "colored" by G.

Keywords Monoidal categories · Liftable pairs · Initial objects · Weakly coreflective subcategories · Group graded vector spaces

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1 Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be monoidal categories and let $L \dashv R : \mathcal{A} \to \mathcal{B}$ be adjoint functors. It is well-known that, if L can be endowed with the structure of a colax monoidal functor, then R becomes a lax monoidal functor, and the other way around. Letting $(R, \phi_2, \phi_0) : \mathcal{A} \to \mathcal{B}$ now be a lax monoidal functor, R induces a functor $\overline{R} = \mathsf{Alg}(R) : \mathsf{Alg}(\mathcal{A}) \to \mathsf{Alg}(\mathcal{B})$ between the respective categories of algebra objects. Dually, a colax monoidal functor $(L, \psi_2, \psi_0) : \mathcal{B} \to \mathcal{A}$ colifts to a functor $\underline{L} = \mathsf{Coalg}(L) : \mathsf{Coalg}(\mathcal{B}) \to \mathsf{Coalg}(\mathcal{A})$ between the respective categories of coalgebra objects. In the article [18], an adjoint pair of functors (L, R) between monoidal categories $\mathcal{A}$ and $\mathcal{B}$ such that R is a lax monoidal functor (or, equivalently, L is colax monoidal) is called *liftable* if the functor $\overline{R}$ has a left adjoint, say $\overline{L}$, and if the functor $\underline{L}$ has a right adjoint, say $\underline{R}$. If $\mathcal{A}$ and $\mathcal{B}$ come both endowed with a braiding, it is shown in loc. cit. that such a liftable pair of functors (L, R) gives rise to an adjunction between the respective categories of bialgebra objects

$$\mathsf{Bialg}(\mathcal{A}) \underset{\underline{\overline{L}}=\mathsf{Coalg}(\overline{L})}{\overset{\overline{\underline{R}}=\mathsf{Alg}(\underline{R})}{\rightleftarrows}} \mathsf{Bialg}(\mathcal{B})$$

provided the functor R enjoys the property of being braided with respect to the braidings of $\mathcal{A}$ and $\mathcal{B}$ (cf. [18, Theorem 2.7]).

A prototypical example of a liftable pair of functors is obtained by taking $\mathcal{B}$ to be the symmetric monoidal category Vec of k-vector spaces (k a field) where the symmetry is just the twist. Putting $\mathcal{A}$ to be the opposite category $\mathsf{Vec}^{\mathrm{op}}$ of Vec and taking the vector space dual $X^* = \mathrm{Hom}_k(X, k)$, one obtains a (covariant) adjunction $L \dashv R : \mathcal{A} \to \mathcal{B}$ with $L = (-)^*$ and $R = (-)^*$. The functor R satisfies the necessary conditions to induce a functor $\overline{R} = \mathsf{Alg}(R) : \mathsf{Alg}(\mathcal{A}) \to \mathsf{Alg}(\mathcal{B})$. Explicitly, one obtains that $\overline{R}$ is the well-known functor that computes the dual algebra of a k-coalgebra (remark that the functor $\underline{L} = \mathsf{Coalg}(L) : \mathsf{Coalg}(\mathcal{B}) \to \mathsf{Coalg}(\mathcal{A})$ is exactly the same functor). A left adjoint $\overline{L} = (-)^\circ$ for $\overline{R}$ is given by the functor that assigns the so-called finite dual coalgebra A° to a k -algebra A; this construction is originally due to Sweedler, see [33]. Noticing that the very same construction provides a right adjoint for the functor $\underline{L}$, one obtains that the pair (L, R) is indeed liftable and, applying the above-cited theorem, one recovers the result that the finite dual induces an auto-adjunction on the category of k -bialgebras (cf. [1, page 87], for instance). Generalizations of this construction have been studied by different authors, see e.g. [8, 15, 29] and [31].

Let us go back to our general setting of a functor $R : \mathcal{A} \to \mathcal{B}$ as at the beginning of this introductory section and notice that being liftable really *is* a condition: there exist examples of lax monoidal functors R between monoidal categories that have a left adjoint L, but for which $\overline{R}$ does not have a left adjoint (cf. [7, Example 4.2]). One aim of this paper is to give sufficient conditions on the category $\mathcal{A}$ for the functor $\overline{R} = \mathsf{Alg}(R) : \mathsf{Alg}(\mathcal{A}) \to \mathsf{Alg}(\mathcal{B})$ to possess a left adjoint $\overline{L}$. As we will see, these conditions involve the existence of suitable colimits that we manage to interpret as objects which are simultaneously initial in four distinguished categories, among them the category of epi-induced objects, which allows for an explicit construction of $\overline{L}$, under the appropriate hypotheses. This is performed by introducing a relative version of the notion of weakly coreflective subcategory that allows, among other things, to identify the initial objects in these categories and that we find is of independent interest.

In the article [7], a context, appeared in [18], where the liftability assumption can be proved to hold is studied: a so-called *pre-rigid* braided monoidal category $\mathcal{C}$ always allows for a liftable pair of adjoint functors $L \dashv R : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$, with $L = (-)^*$ and $R = (-)^*$, provided $\overline{R}$ has a left adjoint. In the present paper, we consider $\mathcal{C}$ to be the category of vector spaces graded by an abelian group G. When being given a skew-symmetric bicharacter on G, the lifted adjunction between bialgebras in $\mathcal{C}$ can be explicitly computed and provides a G-graded version of Sweedler's classical finite dual construction. To the best of our knowledge, this application does not appear elsewhere in literature. Let us sketch in more detail how we go about this computation.

In Section 2, we start by recalling some of the notions we use in the paper, among them the one of liftable pair of adjoint functors and its behaviour on braided categories. Then, in Subsection 1.3, we present sufficient conditions for $\overline{R}$ to possess a left adjoint, provided some extra conditions on the category $\mathcal{A}$ hold (cf. Theorem 2.5). This is obtained by slightly improving results by Dubuc [16, 17] and by Tambara [34]. An advantage of our treatment in this section is the fact that the construction of the adjoint $\overline{L}$ can be given explicitly by means of a specific colimit in $\mathsf{Alg}(\mathcal{A})$. In Section 3, we consider the notion of weakly coreflective subcategory and we introduce a relative version of it in order to reinterpret this colimit as a suitable initial object and obtain an explicit description for $\overline{L}\,\overline{B}$, for every algebra $\overline{B}$ in $\mathcal{B}$. This turns out to be a useful tool to compare the initial objects in the involved categories. An instance of this fact is Proposition 3.5, where we prove that, if $\mathcal{C}$ is a replete posetal weakly coreflective full subcategory of a category $\mathcal{D}$, then $\mathcal{C}$ and $\mathcal{D}$ have the same initial objects, if any. Then we study pullbacks of a relative weakly coreflective subcategory along relative fibrations. More precisely, in Proposition 3.7, we prove that, if $\mathcal{C}$ is a weakly $\mathcal{E}$-coreflective full subcategory of a category $\mathcal{D}$, then the pullback $\mathcal{C}'$ of $\mathcal{C}$ along an $\mathcal{E}$-fibration $V : \mathcal{D}' \to \mathcal{D}$ is a weakly coreflective full subcategory of $\mathcal{D}'$.

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{U} & \mathcal{C} \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{D}' & \xrightarrow{V} & \mathcal{D} \end{array} \qquad\qquad \begin{array}{ccc} \mathsf{IndAlg}^{\mathsf{e}}(\overline{B}) & \xrightarrow{U} & \mathsf{IndObj}^{\mathsf{e}}(\overline{B}) \\ \downarrow & \lrcorner & \downarrow \\ \mathsf{IndAlg}(\overline{B}) & \xrightarrow{V} & \mathsf{IndObj}(\overline{B}) \end{array}$$

These results will be applied in Section 2 to the particular pullback represented in the diagram above, which involves the categories $\mathsf{IndObj}(\overline{B})$, $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$, $\mathsf{IndAlg}(\overline{B})$ and $\mathsf{IndAlg}^{\mathsf{e}}(\overline{B})$. We have mentioned that the functor $\overline{L}$ can be given explicitly in terms of a specific colimit. First, in Proposition 4.4, we reinterpret this colimit as an initial object in the category $\mathsf{IndAlg}(\overline{B})$ of induced algebras of $\overline{B}$ leading us to Theorem 4.5. Then, under suitable assumptions, we will see in Theorem 4.21 that $\mathsf{IndAlg}(\overline{B})$ can be replaced by other three categories, precisely $\mathsf{IndObj}(\overline{B})$, $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$ and $\mathsf{IndAlg}^{\mathsf{e}}(\overline{B})$, that can be more easy to handle in practice, among them the category $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$ of epi-induced objects. Then, in Proposition 4.23, we provide a construction of an initial object in $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$. Putting together these results we obtain Proposition 4.24 giving an explicit description for $\overline{L}\,\overline{B}$. By taking $\mathcal{C}^{\mathrm{op}}$ instead of $\mathcal{A}$, we get Proposition 4.26 that will be applied to the main example we are concerned in Section 5, namely the category Vec_G of G-graded vector spaces. This category is a particular instance of a pre-rigid category as it is monoidal closed. This led us to look for an analogue of Sweedler's finite dual in the general context of pre-rigid braided monoidal categories. In [18], a braided monoidal category $\mathcal{C}$ is called pre-rigid if for every object X there exists an object X^* and a morphism $\mathrm{ev}_X : X^* \otimes X \to \mathbb{I}$ such that the map

$$\mathrm{Hom}_{\mathcal{C}}(T, X^*) \to \mathrm{Hom}_{\mathcal{C}}(T \otimes X, \mathbb{I}) : u \mapsto \mathrm{ev}_X \circ (u \otimes X)$$

is bijective for every object T in $\mathcal{C}$. In this framework, consider the functor $R := (-)^* : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$. It turns out, see Proposition 5.2 and Lemma 5.5, that the adjunction (L, R) is liftable, whenever the functor $\overline{R}$ has a left adjoint. In Proposition 5.7 we present conditions guaranteeing that this happens. Moreover, in Corollary 5.8 we find a pre-rigid analogue of [31, Proposition 8].

In Subsection 4.2 we deal with the case when $\mathcal{C}$ is taken to be the braided monoidal category Vec_G^α of vector spaces graded by an abelian group G, where the braiding depends on a bicharacter $\alpha : G \times G \to k \setminus \{0\}$ on G. In case α is skew-symmetric, our theory gives rise to auto-adjunctions on the categories of bialgebras "colored" by G. As a consequence of arguments settled in the slightly more general setting in Remark 4.6, the lifted functors in this example can be described explicitly. The paper concludes with hinting at why one could expect that explicit descriptions as in case of Vec_G could be carried out, more generally, for the category of comodules over a coquasi-bialgebra.

2 Preliminaries and first results

We begin our exposition by recalling some notions we need in the paper, among them the one of liftable pair of adjoint functors and its behaviour on braided categories, from [18]. Then we will present sufficient conditions for the functor induced by a lax monoidal right adjoint at the level of algebras to possess a left adjoint, see Theorem 2.5. This is obtained by slightly improving results by Dubuc and by Tambara.

2.1 Some notational conventions

When X is an object in a category $\mathcal{C}$, we will denote the identity morphism on X by 1_X or X for short. For categories $\mathcal{C}$ and $\mathcal{D}$, a functor $F : \mathcal{C} \to \mathcal{D}$ will be the name for a covariant functor; it will only be a contravariant one if it is explicitly mentioned. By $\mathsf{id}_{\mathcal{C}}$ we denote the identity functor on $\mathcal{C}$. For any functor $F : \mathcal{C} \to \mathcal{D}$, we denote Id_F (or sometimes -in order to lighten notation in some computations- just F, if the context does not allow for confusion) the natural transformation defined by $\mathrm{Id}_{FX} = 1_{FX}$.

Let $\mathcal{C}$ be a category. Denote by $\mathcal{C}^{\mathrm{op}}$ the opposite category of $\mathcal{C}$. Using the notation of [25, page 12], an object X and a morphism $f : X \to Y$ in $\mathcal{C}$ will be denoted by X^{op} and $f^{\mathrm{op}} : Y^{\mathrm{op}} \to X^{\mathrm{op}}$ when regarded as object and morphism in $\mathcal{C}^{\mathrm{op}}$. Given a functor $F : \mathcal{C} \to \mathcal{D}$, one defines its opposite functor $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ by setting $F^{\mathrm{op}}X^{\mathrm{op}} = (FX)^{\mathrm{op}}$ and $F^{\mathrm{op}}f^{\mathrm{op}} = (Ff)^{\mathrm{op}}$. If $\alpha : F \to G$ is a natural transformation, its opposite α^{op} is given by $(\alpha^{\mathrm{op}})_{X^{\mathrm{op}}} := (\alpha_X)^{\mathrm{op}}$ for every object X.

Throughout the paper, we will work in the setting of monoidal categories. With respect to the material presented below, it is useful to recall the following notation. Let $(\mathcal{M}, \otimes, \mathbb{I}, a, l, r)$ be a monoidal category. Following [32, 0.1.4, 1.4], we have that $\mathcal{M}^{\mathrm{op}}$ is also monoidal, the monoidal structure being given by

$$X^{\mathrm{op}} \otimes Y^{\mathrm{op}} := (X \otimes Y)^{\mathrm{op}}, \qquad \text{the unit object is } \mathbb{I}^{\mathrm{op}}$$
$$a_{X^{\mathrm{op}},Y^{\mathrm{op}},Z^{\mathrm{op}}} := \left(a^{-1}_{X,Y,Z}\right)^{\mathrm{op}}, \qquad l_{X^{\mathrm{op}}} := \left(l^{-1}_X\right)^{\mathrm{op}}, \qquad r_{X^{\mathrm{op}}} := \left(r^{-1}_X\right)^{\mathrm{op}}.$$

If $\mathcal{M}$ is moreover braided (with braiding c), then so is $\mathcal{M}^{\mathrm{op}}$, the braiding being given by

$$c_{X^{\mathrm{op}},Y^{\mathrm{op}}} := \left(c_{X,Y}^{-1}\right)^{\mathrm{op}}.$$

Unless explicitly stated, we will assume monoidal categories to be strict from now on. By Mac Lane's Coherence Theorem, this does not impose restrictions on the obtained results. We will moreover consider braided monoidal categories. A basic reference for these notions is [21], for instance.

Recall (see e.g. [4, Definition 4.1]) that a functor $F : \mathcal{A} \to \mathcal{B}$ between monoidal categories $(\mathcal{A}, \otimes, \mathbb{I}_{\mathcal{A}})$ and $(\mathcal{B}, \otimes', \mathbb{I}_{\mathcal{B}})$ is said to be a lax monoidal functor if it comes equipped with a family of natural morphisms $\phi_2(X, Y) : F(X) \otimes' F(Y) \to F(X \otimes Y)$, for $X, Y \in \mathcal{A}$, and a $\mathcal{B}$-morphism $\phi_0 : \mathbb{I}_{\mathcal{B}} \to F(\mathbb{I}_{\mathcal{A}})$, satisfying the known suitable compatibility conditions with respect to the associativity and unit constraints of $\mathcal{A}$ and $\mathcal{B}$. Dually, colax monoidal functors are defined.

Also recall that given a lax monoidal functor (F, ϕ_2, ϕ_0), then $(F^{\mathrm{op}}, \phi_2^{\mathrm{op}}, \phi_0^{\mathrm{op}})$ is a colax monoidal functor, where we set $\phi_2^{\mathrm{op}}(X^{\mathrm{op}}, Y^{\mathrm{op}}) := \phi_2(X, Y)^{\mathrm{op}}$, see e.g. [4, Proposition 3.7].

2.2 Liftability of adjoint pairs

Let $(L : \mathcal{B} \to \mathcal{A}, R : \mathcal{A} \to \mathcal{B})$ be an adjunction with unit η and counit ϵ. It is known, see e.g. [4, Proposition 3.84], that if (L, ψ_2, ψ_0) is a colax monoidal functor, then (R, ϕ_2, ϕ_0) is a lax monoidal functor where, for every $X, Y \in \mathcal{A}$,

$$\phi_2(X,Y) := \left(RX \otimes RY \xrightarrow{\eta(RX \otimes RY)} RL(RX \otimes RY) \xrightarrow{R\psi_2(RX,RY)} R(LRX \otimes LRY) \xrightarrow{R(\epsilon_X \otimes \epsilon_Y)} R(X \otimes Y) \right),$$

$$\phi_0 := \left(\mathbb{I}_{\mathcal{B}} \xrightarrow{\eta_{\mathbb{I}_{\mathcal{B}}}} RL(\mathbb{I}_{\mathcal{B}}) \xrightarrow{R\psi_0} R(\mathbb{I}_{\mathcal{A}}) \right).$$

Conversely, if (R, ϕ_2, ϕ_0) is a lax monoidal functor, then (L, ψ_2, ψ_0) is a colax monoidal functor where, for every $X, Y \in \mathcal{B}$

$$\psi_2(X,Y) := \left(L(X \otimes Y) \xrightarrow{L(\eta_X \otimes \eta_Y)} L(RLX \otimes RLY) \xrightarrow{L\phi_2(LX,LY)} LR(LX \otimes LY) \xrightarrow{\epsilon(LX \otimes LY)} LX \otimes LY \right), \tag{1}$$

$$\psi_0 := \left(L(\mathbb{I}_{\mathcal{B}}) \xrightarrow{L\phi_0} LR(\mathbb{I}_{\mathcal{A}}) \xrightarrow{\epsilon_{\mathbb{I}_{\mathcal{A}}}} \mathbb{I}_{\mathcal{A}} \right). \tag{2}$$

Let $(R, \phi_2, \phi_0) : \mathcal{A} \to \mathcal{B}$ be a lax monoidal functor. It is well-known that R induces a functor $\overline{R} := \mathsf{Alg}(R) : \mathsf{Alg}(\mathcal{A}) \to \mathsf{Alg}(\mathcal{B})$ such that the diagram on the right-hand side in (3) commutes (cf. [13, Proposition 6.1, page 52]; see also [4, Proposition 4.29]). Explicitly,

$$\overline{R}(A,m,u) = \left(RA,\ RA \otimes RA \xrightarrow{\phi_2(A,A)} R(A \otimes A) \xrightarrow{Rm} RA\ ,\ \mathbb{I}_{\mathcal{B}} \xrightarrow{\phi_0} R\mathbb{I}_{\mathcal{A}} \xrightarrow{Ru} RA \right).$$

Dually, a colax monoidal functor $(L, \psi_2, \psi_0) : \mathcal{B} \to \mathcal{A}$ colifts to a functor $\underline{L} := \mathsf{Coalg}(L) : \mathsf{Coalg}(\mathcal{B}) \to \mathsf{Coalg}(\mathcal{A})$ such that the diagram on the left-hand side in (3) commutes. Explicitly,

$$\underline{L}\,(C,\Delta,\varepsilon)=\left(LC,\ LC\xrightarrow{L\Delta}L\,(C\otimes C)\xrightarrow{\psi_2(C,C)}LC\otimes LC\ ,\ LC\xrightarrow{L\varepsilon}L\mathbb{I}_{\mathcal{B}}\xrightarrow{\psi_0}\mathbb{I}_{\mathcal{A}}\right).$$

The vertical arrows in the two diagrams below are the obvious forgetful functors.

$$\begin{array}{ccc} \mathsf{Coalg}(\mathcal{B}) & \xrightarrow{\underline{L}=\mathsf{Coalg}(L)} & \mathsf{Coalg}(\mathcal{A}) \\ {\scriptstyle \mho'=\mho_{\mathcal{B}}}\downarrow & & \downarrow{\scriptstyle \mho=\mho_{\mathcal{A}}} \\ \mathcal{B} & \xrightarrow{L} & \mathcal{A} \end{array} \qquad \begin{array}{ccc} \mathsf{Alg}(\mathcal{A}) & \xrightarrow{\overline{R}=\mathsf{Alg}(R)} & \mathsf{Alg}(\mathcal{B}) \\ {\scriptstyle \Omega=\Omega_{\mathcal{A}}}\downarrow & & \downarrow{\scriptstyle \Omega'=\Omega_{\mathcal{B}}} \\ \mathcal{A} & \xrightarrow{R} & \mathcal{B} \end{array} \tag{3}$$

Definition 2.1 ([18 *Definition 2.3*]) Suppose $\mathcal{A}$ and $\mathcal{B}$ are monoidal categories and $R:\mathcal{A}\to\mathcal{B}$ is a lax monoidal functor with a left adjoint L. The pair (L,R) is called *liftable* if the induced functor $\overline{R}=\mathsf{Alg}(R):\mathsf{Alg}(\mathcal{A})\to\mathsf{Alg}(\mathcal{B})$ has a left adjoint[1], say $\overline{L}$, and the induced functor $\underline{L}=\mathsf{Coalg}(L):\mathsf{Coalg}(\mathcal{B})\to\mathsf{Coalg}(\mathcal{A})$ has a right adjoint, say $\underline{R}$.

Notice that being liftable really *is* a condition: there exist examples of lax monoidal functors R between monoidal categories that have a left adjoint L, but for which $\overline{R}$ does not have a left adjoint. For instance, let k be a field and set $S:=\frac{k[X]}{(X^2)}$. Consider the functor

$$R^f:\mathsf{Vec}^{\mathrm{f}}\to\mathsf{Vec}^{\mathrm{f}};\ V\mapsto S\otimes_k V.$$

In [7, Example 4.2], it is shown that $\overline{R^f}$ has no left adjoint.

Liftability for braided monoidal categories. Recall that when a monoidal category is *braided*, its algebras and coalgebras inherit the monoidal structure, see e.g. [4, 1.2.2]. Let $\mathcal{A}$ and $\mathcal{B}$ now be *braided* monoidal categories and let $R:\mathcal{A}\to\mathcal{B}$ be a braided lax monoidal functor having a left adjoint L. By e.g. [4, Proposition 3.80], the functor $\overline{R}$ is lax monoidal too. Explicitly, the lax monoidal functors (R,ϕ_2,ϕ_0) and $(\overline{R},\overline{\phi}_2,\overline{\phi}_0)$ are connected by the following equalities, for every $\overline{A}=\big(A,m_A,u_A\big),\overline{B}=\big(B,m_B,u_B\big)\in\mathsf{Alg}(\mathcal{A})$

$$\Omega_{\mathcal{B}}\bar{R}=R^{\circ}\Omega_{\mathcal{A}},\qquad \Omega_{\mathcal{B}}\big(\bar{\phi}_2\big(\bar{A},\bar{B}\big)\big)=\phi_2(A,\ B),\qquad \Omega_{\mathcal{B}}(\bar{\phi}_0)=\phi_0.$$

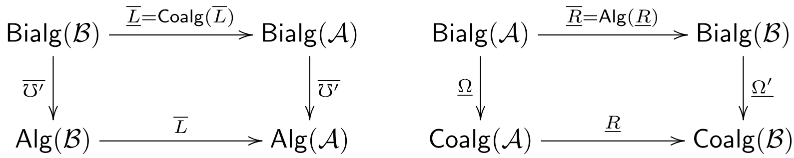
Note that R is a braided lax monoidal functor if and only if L is a braided colax monoidal functor, see e.g. [4, Proposition 3.85]. Moreover, if L is a braided colax monoidal functor one shows in a similar fashion as above that $\underline{L}$ is colax monoidal. The colax monoidal functors (L,ψ_2,ψ_0) and $(\underline{L},\underline{\psi}_2,\underline{\psi}_0)$ are connected by the following equalities for every $\underline{C}=\big(C,\Delta_C,\varepsilon_C\big),\underline{D}=\big(D,\Delta_D,\varepsilon_D\big)\in\mathsf{Coalg}(\mathcal{B})$

$$\mho_{\mathcal{A}}\circ\underline{L}=L\circ\mho_{\mathcal{B}},\qquad \mho_{\mathcal{A}}\Big(\underline{\psi}_2\big(\underline{C},\underline{D}\big)\Big)=\psi_2(C,D),\qquad \mho_{\mathcal{A}}(\underline{\psi}_0)=\psi_0.$$

As [7, Example 4.2] shows, a pair (L,R), where $R:\mathcal{A}\to\mathcal{B}$ is a (braided) lax monoidal functor between (braided) monoidal categories $\mathcal{A}$ and $\mathcal{B}$, having a left adjoint L, needs not to be liftable, *a priori*. But, in case $\mathcal{A}$ and $\mathcal{B}$ are braided monoidal categories and

[1] In general the left adjoint of $\overline{R}$ is not assumed to be of form $\mathsf{Alg}(L)$. In fact we can't even consider $\mathsf{Alg}(L)$ as L needs not to be lax monoidal. A similar observation holds for the right adjoint of $\underline{\mathrm{L}}$.

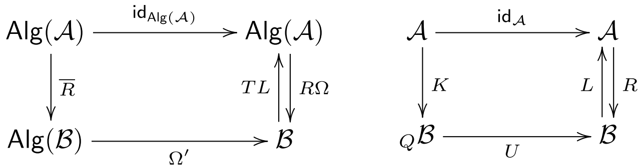
$R : \mathcal{A} \to \mathcal{B}$ is a braided lax monoidal functor having a left adjoint L such that the pair (L, R) *is* liftable, then, by [18, Lemma 2.4 and Theorem 2.7], there is an adjunction $\left(\underline{\overline{L}}, \underline{\overline{R}}\right)$ that fits into the following commutative diagrams (and explains the choice of the -perhaps somewhat fuzzy- term “liftable”)



In this diagram, all vertical arrows are forgetful functors.

2.3 An approach to a result by Tambara, inspired by Dubuc

In this subsection, we provide sufficient conditions (Theorem 2.5 together with Proposition 2.9) for $\overline{R}$ to have a left adjoint. Under relatively mild assumptions, this is obtained by considering a result by Dubuc [16, 17] and by using it to provide a result in the spirit of Tambara’s [34, Remark 1.5], cf. [6, Theorem 2.2.8] for an unpublished proof of this result (Tambara does not provide his own proof). More precisely, let us compare the following two diagrams.



- **The diagram on the left-hand side**. Here $\mathcal{A}$ and $\mathcal{B}$ are monoidal categories and it is assumed that the forgetful functor $\Omega : \mathsf{Alg}(\mathcal{A}) \to \mathcal{A}$ has a left adjoint T. Also, $R : \mathcal{A} \to \mathcal{B}$ is supposed to be a lax monoidal functor having a left adjoint L. If we are moreover given that $\mathcal{A}$ has colimits and the tensor product commutes with them, [34, Remark 1.5] states that $\overline{R}$ has a left adjoint, too.
- **The diagram on the right-hand side**. Here we are in the setting of [16, Theorem 1] (see also [17, Theorem A.2]) where, in case the category $\mathcal{A}$ has reflexive coequalizers, the functor K has a left adjoint (U denotes the forgetful functor from the Eilenberg-Moore category of algebras over the monad Q on $\mathcal{B}$ while K is the functor having a derivable adjoint triangle).

Thus, although the forgetful $\Omega' : \mathsf{Alg}(\mathcal{B}) \to \mathcal{B}$ (as above) is neither right adjoint[2] nor, equivalently, monadic (cf. [9, Theorem A.6]), it is still possible to produce a left adjoint

[2] It does if $\mathcal{B}$ has denumerable coproducts and they are preserved by the tensor products, cf. [21, Theorem 2, page 172].

for $\overline{R}$ like in the diagram on the right-hand side, where instead U is both right adjoint and monadic. Moreover, on the right-hand side, just the existence of reflexive coequalizers is required not of all colimits.

Inspired by Dubuc's work, we now present a result in the spirit of Tambara's. Note that there is no requirement on Ω here (no monadicity, neither left adjoint). As a particular case we do, however, require that $\mathcal{A}$ has all coequalizers (not just reflexive coequalizers).

Proposition 2.2 *Consider the following diagram*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathsf{id}_{\mathcal{A}}} & \mathcal{A} \\ \Lambda \uparrow\downarrow K & & L \uparrow\downarrow R \\ \mathsf{Alg}(\mathcal{B}) & \xrightarrow[\Omega]{} & \mathcal{B} \end{array} \tag{4}$$

where $\mathcal{B}$ is a monoidal category, $\Omega K = R$ and (L, R) is an adjunction with unit η and counit ϵ. Given any algebra $\overline{B} := (B, m_B, u_B)$ in $\mathcal{B}$, write KLB in the form (RLB, m_{RLB}, u_{RLB}) and assume that the diagram

$$L(B \otimes B) \underset{\epsilon_{LB} \circ Lm_{RLB} \circ L(\eta_B \otimes \eta_B)}{\overset{Lm_B}{\rightrightarrows}} LB \underset{\epsilon_{LB} \circ Lu_{RLB}}{\overset{Lu_B}{\leftleftarrows}} L\mathbf{1} \tag{5}$$

has a colimit $\left(\Lambda\overline{B}, \kappa_{\overline{B}} : LB \to \Lambda\overline{B}\right)$ (e.g. the category $\mathcal{A}$ has coequalizers). This yields a functor Λ which is a left adjoint of the functor K. Moreover the morphisms $\kappa_{\overline{B}}$ define a natural transformation $\kappa : L\Omega \to \Lambda$ such that $\kappa = \epsilon\Lambda \circ L\Omega\overline{\eta}$, where $\overline{\eta}$ denotes the unit of (Λ, K)

.

Proof Let $\overline{B} := (B, m_B, u_B)$ be an algebra in $\mathcal{B}$. From $\Omega K = R$, we can write KLB in the form (RLB, m_{RLB}, u_{RLB}). By hypothesis, the diagram (5) has a colimit $\left(\Lambda\overline{B}, \kappa_{\overline{B}} : LB \to \Lambda\overline{B}\right)$ (in case $\mathcal{A}$ has coequalizers, it is obtained by taking the coequalizer $\left(\Lambda'\overline{B}, \kappa'_{\overline{B}} : LB \to \Lambda'\overline{B}\right)$ of the left-hand side pair and then computing the coequalizer of the pair $\left(\kappa'_{\overline{B}} \circ Lu_B, \kappa'_{\overline{B}} \circ \epsilon_{LB} \circ Lu_{RLB}\right)$). Let $f : \overline{B} \to \overline{E}$ be a morphism in $\mathsf{Alg}(\mathcal{B})$. Then

$$\begin{aligned} L\Omega f \circ \epsilon_{LB} \circ Lm_{RLB} \circ L(\eta_B \otimes \eta_B) &= \epsilon_{LE} \circ LRL\Omega f \circ Lm_{RLB} \circ L(\eta_B \otimes \eta_B) \\ &= \epsilon_{LE} \circ L\left[\Omega KL\Omega f \circ m_{RLB} \circ (\eta_B \otimes \eta_B)\right] \\ &= \epsilon_{LE} \circ L\left[m_{RLE} \circ (RL\Omega f \otimes RL\Omega f) \circ (\eta_B \otimes \eta_B)\right] \\ &= \epsilon_{LE} \circ Lm_{RLE} \circ L(\eta_E \otimes \eta_E) \circ L(\Omega f \otimes \Omega f), \\ L\Omega f \circ Lm_B &= L(\Omega f \circ m_B) = L\left(m_E \circ (\Omega f \otimes \Omega f)\right) = Lm_E \circ L(\Omega f \otimes \Omega f), \\ L\Omega f \circ \epsilon_{LB} \circ Lu_{RLB} &= \epsilon_{LE} \circ LRL\Omega f \circ Lu_{RLB} = \epsilon_{LE} \circ L\left(\Omega KL\Omega f \circ u_{RLB}\right) = \epsilon_{LE} \circ Lu_{RLE}, \\ L\Omega f \circ Lu_B &= L(\Omega f \circ u_B) = Lu_E \end{aligned}$$

so that the following diagram serially commutes.

$$\begin{array}{ccccc}
L(B\otimes B) & \underset{\epsilon_{LB}\circ Lm_{RLB}\circ L(\eta_B\otimes\eta_B)}{\overset{Lm_B}{\rightrightarrows}} & LB & \underset{\epsilon_{LB}\circ Lu_{RLB}}{\overset{Lu_B}{\leftleftarrows}} & L\mathbf{1}\ . \\
\downarrow{\scriptstyle L(\Omega f\otimes\Omega f)} & & \downarrow{\scriptstyle L(\Omega f)} & & \downarrow{\scriptstyle \mathsf{Id}_{L\mathbb{I}}} \\
L(E\otimes E) & \underset{\epsilon_{LE}\circ Lm_{RLE}\circ L(\eta_E\otimes\eta_E)}{\overset{Lm_E}{\rightrightarrows}} & LE & \underset{\epsilon_{LE}\circ Lu_{RLE}}{\overset{Lu_E}{\leftleftarrows}} & L\mathbf{1}
\end{array}$$

As a consequence, there is a unique morphism $\Lambda f : \Lambda\overline{B} \to \Lambda\overline{E}$ such that

$$\Lambda f\circ\kappa_{\overline{B}} = \kappa_{\overline{E}}\circ L\Omega f. \tag{6}$$

Since $\kappa_{\overline{B}}$ is an epimorphism, one easily checks that $\Lambda\left(f\circ f'\right) = \Lambda f\circ\Lambda f'$ for all morphisms f, f' in $\mathsf{Alg}(\mathcal{B})$ that can be composed. We thus get a functor $\Lambda : \mathsf{Alg}(\mathcal{B}) \to \mathcal{A}$ and (6) means that $\kappa_{\_}$ is natural in the lower argument. Let us check that (Λ, K) is an adjunction. For every $A \in \mathcal{A}$, consider the diagram (5) for $\overline{B} = KA$ (hence $B = RA$) i.e.

$$L(RA\otimes RA) \underset{\epsilon_{LRA}\circ Lm_{RLRA}\circ L(\eta_{RA}\otimes\eta_{RA})}{\overset{Lm_{RA}}{\rightrightarrows}} LRA \underset{\epsilon_{LRA}\circ Lu_{RLRA}}{\overset{Lu_{RA}}{\leftleftarrows}} L\mathbf{1}\ .$$

Then it is easily verified that $\epsilon_A\circ\left(\epsilon_{LRA}\circ Lm_{RLRA}\circ L\left(\eta_{RA}\otimes\eta_{RA}\right)\right) = \epsilon_A\circ Lm_{RA}$ and that $\epsilon_A\circ\left(\epsilon_{LRA}\circ Lu_{RLB}\right) = \epsilon_A\circ Lu_{RA}$, so there exists a unique morphism $\overline{\epsilon}_A : \Lambda KA \to A$ such that

$$\overline{\epsilon}_A\circ\kappa_{KA} = \epsilon_A.$$

If $h : A \to A'$ is a morphism in $\mathcal{A}$, we have

$$\overline{\epsilon}_{A'}\circ\Lambda Kh\circ\kappa_{KA} \overset{(6)}{=} \overline{\epsilon}_{A'}\circ\kappa_{KA'}\circ L\Omega Kh = \epsilon_{A'}\circ LRh = h\circ\epsilon_A = h\circ\overline{\epsilon}_A\circ\kappa_{KA}$$

and hence $\overline{\epsilon}_{A'}\circ\Lambda Kh = h\circ\overline{\epsilon}_A$ which means that $\overline{\epsilon}_{\_}$ is natural in the lower argument. We have

$$\begin{aligned}
R\kappa_{\overline{B}}\circ\eta_B\circ m_B &= R\left(\kappa_{\overline{B}}\circ Lm_B\right)\circ\eta_{(B\otimes B)} = R\left[\kappa_{\overline{B}}\circ\epsilon_{LB}\circ Lm_{RLB}\circ L\left(\eta_B\otimes\eta_B\right)\right]\circ\eta_{(B\otimes B)} \\
&= R\left[\epsilon_{\Lambda}\overline{B}\circ LR\kappa_{\overline{B}}\circ Lm_{RLB}\circ L\left(\eta_B\otimes\eta_B\right)\right]\circ\eta_{(B\otimes B)} \\
&= R\epsilon_{\Lambda\overline{B}}\circ\eta_{R\Lambda\overline{B}}\circ R\kappa_{\overline{B}}\circ m_{RLB}\circ\left(\eta_B\otimes\eta_B\right) \\
&= R\kappa_{\overline{B}}\circ m_{RLB}\circ\left(\eta_B\otimes\eta_B\right) \\
&= m_{R\Lambda\overline{B}}\circ\left(R\kappa_{\overline{B}}\otimes R\pi_{\overline{B}}\right)\circ\left(\eta_B\otimes\eta_B\right), \\
R\kappa_{\overline{B}}\circ\eta_B\circ u_B &= R\left(\kappa_{\overline{B}}\circ RLu_B\right)\circ\eta_{\mathbb{I}} = R\left(\kappa_{\overline{B}}\circ\epsilon_{LB}\circ Lu_{RLB}\right)\circ\eta_{\mathbb{I}} \\
&= R\kappa_{\overline{B}}\circ R\epsilon_{LB}\circ\eta_{RLB}\circ u_{RLB} = \Omega K\kappa_{\overline{B}}\circ u_{RLB} = u_{R\Lambda\overline{B}}
\end{aligned}$$

so that $R\kappa_{\overline{B}}\circ\eta_B$ induces an algebra map $\overline{\eta}_{\overline{B}} : \overline{B} \to K\Lambda\overline{B}$ such that

$$\Omega\overline{\eta}_{\overline{B}} = R\kappa_{\overline{B}}\circ\eta_B.$$

Given a morphism $f : \overline{B} \to \overline{E}$ in $\mathsf{Alg}(\mathcal{B})$, we get

$$\begin{aligned}\Omega\left(\overline{\eta}_{\overline{E}}\circ f\right) &= R\kappa_{\overline{E}}\circ\eta_E\circ\Omega f = R\left(\kappa_{\overline{E}}\circ L\Omega f\right)\circ\eta_B\\ &\overset{(6)}{=} R\left(\Lambda f\circ\kappa_{\overline{B}}\right)\circ\eta_B = R\Lambda f\circ\Omega\overline{\eta}_{\overline{B}} = \Omega\left(K\Lambda f\circ\overline{\eta}_{\overline{B}}\right)\end{aligned}$$

and hence $\overline{\eta}_{\overline{E}}\circ f = K\Lambda f\circ\overline{\eta}_{\overline{B}}$ which means that $\overline{\eta}_-$ is natural in the lower argument. We have that

$$\begin{aligned}\overline{\epsilon}_{\Lambda\overline{B}}\circ\Lambda\overline{\eta}_{\overline{B}}\circ\kappa_{\overline{B}} &\overset{(6)}{=} \overline{\epsilon}_{\Lambda\overline{B}}\circ\kappa_{K\Lambda\overline{B}}\circ L\Omega\overline{\eta}_{\overline{B}}\\ &= \epsilon_{\Lambda\overline{B}}\circ L\left(R\kappa_{\overline{B}}\circ\eta_B\right) = \kappa_{\overline{B}}\circ\epsilon_{LB}\circ L\eta_B = \kappa_{\overline{B}},\end{aligned}$$

so that $\overline{\epsilon}_{\Lambda\overline{B}}\circ\Lambda\overline{\eta}_{\overline{B}} = \mathrm{Id}_{\Lambda\overline{B}}$. Moreover,

$$\Omega\left(K\overline{\epsilon}_A\circ\overline{\eta}_{KA}\right) = R\overline{\epsilon}_A\circ\Omega\overline{\eta}_{KA} = R\overline{\epsilon}_A\circ R\kappa_{KA}\circ\eta_{RA} = R\epsilon_A\circ\eta_{RA} = \mathrm{Id}_{RA} = \Omega\mathrm{Id}_{KA}.$$

Since Ω is faithful, we get that $K\overline{\epsilon}_A\circ\overline{\eta}_{KA} = \mathrm{Id}_{KA}$. Thus (Λ, K) is an adjunction. We compute

$$\epsilon_{\Lambda\overline{B}}\circ L\Omega\overline{\eta}_{\overline{B}} = \epsilon_{\Lambda\overline{B}}\circ LR\kappa_{\overline{B}}\circ L\eta_B = \kappa_{\overline{B}}\circ\epsilon_{LB}\circ L\eta_B = \kappa_{\overline{B}}.$$

□

Remark 2.3 We already observed that, if $R : \mathcal{A}\to\mathcal{B}$ is a lax monoidal functor having a left adjoint and if $\mathcal{A}$ has colimits and the tensor product commutes with them, then $\overline{R}$ is a right adjoint too. It can be shown that the pair

$$L(B\otimes B) \underset{\epsilon_{LB}\circ Lm_{RLB}\circ L(\eta_B\otimes\eta_B)}{\overset{Lm_B}{\rightrightarrows}} LB$$

is reflexive if we assume that $\eta_{\mathbb{I}} : \mathbb{I}\to RL\mathbb{I}$ is multiplicative.

Let us fix the following setting we will frequently work in.

Setting 2.4 Let $\mathcal{A}$ and $\mathcal{B}$ be monoidal categories and let $R : \mathcal{A}\to\mathcal{B}$ be a lax monoidal functor with a left adjoint L, unit $\eta : \mathrm{id}_{\mathcal{B}}\to RL$ and counit $\epsilon : LR\to\mathrm{id}_{\mathcal{A}}$. Assume that the forgetful functor $\Omega : \mathsf{Alg}(\mathcal{A})\to\mathcal{A}$ has a left adjoint T, with unit $\alpha : \mathrm{id}_{\mathcal{A}}\to\Omega T$ and counit $\gamma : T\Omega\to\mathrm{id}_{\mathsf{Alg}(\mathcal{A})}$.

Given an algebra $\overline{B} = \left(B, m_B, u_B\right)$ in $\mathcal{B}$, write

$$\left(R\Omega TLB, m_{R\Omega TLB}, u_{R\Omega TLB}\right) = \overline{R}TLB, \tag{7}$$

and set $\mu := \gamma_{TLB}\circ T\epsilon_{\Omega TLB}\circ TLm_{R\Omega TLB}\circ TL\left(R\alpha_{LB}\circ\eta_B\otimes R\alpha_{LB}\circ\eta_B\right)$. Consider the diagram

$$TL(B\otimes B) \underset{\mu}{\overset{TLm_B}{\rightrightarrows}} TLB \underset{\gamma_{TLB}\circ T\epsilon_{\Omega TLB}\circ TLu_{R\Omega TLB}}{\overset{TLu_B}{\leftleftarrows}} TL\mathbf{1}\ . \tag{8}$$

The following result provides a sufficient condition for $\overline{R} = \mathsf{Alg}(R)$ to have a left adjoint for a lax monoidal functor R with a left adjoint.

Theorem 2.5 *In the Setting 2.4, assume that for every algebra $\overline{B}$ in $\mathcal{B}$ the diagram (8) has a colimit $\left(\overline{L}\,\overline{B}, \kappa_{\overline{B}} : TLB \to \overline{L}\,\overline{B}\right)$ (e.g. the category $\mathsf{Alg}(\mathcal{A})$ has coequalizers). This yields a functor $\overline{L}$ which is a left adjoint of the functor $\overline{R} = \mathsf{Alg}(\underline{R}) : \mathsf{Alg}(\mathcal{A}) \to \mathsf{Alg}(\mathcal{B})$ and the morphisms $\kappa_{\overline{B}}$ define a natural transformation $\kappa : TL\Omega \to \overline{L}$.*

Proof Let (L, R, η, ϵ) and $(T, \Omega, \alpha, \gamma)$ be the adjunctions as in Setting 2.4. Their composition yields the adjunction

$$\left(TL, R\Omega, \mathrm{id}_{\mathcal{B}} \xrightarrow{\eta} RL \xrightarrow{R\alpha L} R\Omega TL, TLR\Omega \xrightarrow{T\epsilon\Omega} T\Omega \xrightarrow{\gamma} \mathrm{id}_{\mathsf{Alg}(\mathcal{A})}\right).$$

Then diagram (5) becomes (8) in our setting as the role of diagram (4) is played by the following diagram

$$\begin{array}{ccc} \mathsf{Alg}(\mathcal{A}) & \xrightarrow{\quad \mathsf{id} \quad} & \mathsf{Alg}(\mathcal{A}) \\ \big\downarrow{\scriptstyle \overline{R}} & & {\scriptstyle TL}\big\uparrow\big\downarrow{\scriptstyle R\Omega} \\ \mathsf{Alg}(\mathcal{B}) & \xrightarrow[\quad \Omega' \quad]{} & \mathcal{B} \end{array} \tag{9}$$

where $\Omega'\overline{R} = R\Omega$. The conclusion follows by Proposition 2.2. □

Remark 2.6 Theorem 2.5 provides sufficient conditions for the functor $\overline{R}$ to have a left adjoint. This result can be dualized to obtain as well sufficient conditions for the functor $\underline{L}$ to have a right adjoint and hence the pair (L, R) to be liftable. In fact, given a colax monoidal functor $L : \mathcal{B} \to \mathcal{A}$ with a right adjoint R, it suffices to apply the original theorem to the functor $L^{\mathrm{op}} : \mathcal{B}^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}}$ to get a left adjoint for $\overline{L^{\mathrm{op}}} = \mathsf{Alg}(L^{\mathrm{op}}) = (\mathsf{Coalg}(L))^{\mathrm{op}} = (\underline{L})^{\mathrm{op}}$ namely a right adjoint for $\underline{L}$. The corresponding assumptions on $\mathcal{B}$ include the existence of a right adjoint for the forgetful functor $\mho : \mathsf{Coalg}(\mathcal{B}) \to \mathcal{B}$ and of (suitable) equalizers in $\mathsf{Coalg}(\mathcal{B})$.

As a consequence of the previous remark we get the following corollaries.

Corollary 2.7 *For monoidal categories $\mathcal{A}$ and $\mathcal{B}$, suppose a lax monoidal functor $(R, \phi_2, \phi_0) : \mathcal{A} \to \mathcal{B}$ has a left adjoint L. If the forgetful functor $\Omega : \mathsf{Alg}(\mathcal{A}) \to \mathcal{A}$ has a left adjoint, the category $\mathsf{Alg}(\mathcal{A})$ has coequalizers, the forgetful functor $\mho : \mathsf{Coalg}(\mathcal{B}) \to \mathcal{B}$ has a right adjoint and $\mathsf{Coalg}(\mathcal{B})$ has equalizers, then (L, R) is liftable.*

Corollary 2.8 *For a monoidal category $\mathcal{C}$, suppose a lax monoidal functor $(R, \phi_2, \phi_0) : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ has a left adjoint L. If the forgetful functor $\mho : \mathsf{Coalg}(\mathcal{C}) \to \mathcal{C}$ has a right adjoint and $\mathsf{Coalg}(\mathcal{C})$ has equalizers, then (L, R) is liftable.*

The next result collects sufficient conditions for Theorem 1.5 to be applied.

Proposition 2.9 *For a monoidal category $\mathcal{A}$, assume that $\mathsf{Alg}(\mathcal{A})$ is complete, well-powered and it has a cogenerating family. Then the category $\mathsf{Alg}(\mathcal{A})$ has coequalizers. Moreover, the forgetful functor $\Omega : \mathsf{Alg}(\mathcal{A}) \to \mathcal{A}$ has a left adjoint T.*

Proof By [14, Proposition 3.3.8], the category $\mathsf{Alg}(\mathcal{A})$ is cocomplete. In particular, $\mathsf{Alg}(\mathcal{A})$ has coequalizers. Moreover, since Ω creates limits (cf. [26, Proposition 2.5]), it also preserves them so that by the special adjoint functor theorem (cf. [14, Theorem 3.3.4]) it has a left adjoint T. □

Example 2.10 Let k be a commutative ring. Let $\mathcal{A} = k\text{-}\mathsf{Mod}^{\mathrm{op}}$ be the opposite of the category k-Mod. Thus $\mathsf{Alg}(\mathcal{A}) = k\text{-}\mathsf{Coalg}^{\mathrm{op}}$ is the opposite of the category k-$\mathsf{Coalg} = \mathsf{Coalg}(k\text{-}\mathsf{Mod})$ of k-coalgebras and their morphisms. By the proof of [12, Theorem 4.1], the category $\mathsf{Alg}(\mathcal{A})$ is complete, well-powered and it has a cogenerating family. Thus Proposition 2.9 applies. As a consequence, by Theorem 2.5, for every lax monoidal functor $R : k\text{-}\mathsf{Mod}^{\mathrm{op}} \to \mathcal{B}$ with a left adjoint, the functor $\overline{R} = \mathsf{Alg}(R) : k\text{-}\mathsf{Coalg}^{\mathrm{op}} \to \mathsf{Alg}(\mathcal{B})$ has a left adjoint too.

In Theorem 2.5, the existence of a colimit for diagram (8) plays a crucial role. In Sect. 3 we will reinterpret this colimit as a suitable initial object obtaining an explicit description for $\overline{L}\,\overline{B}$, for every algebra $\overline{B}$ in $\mathcal{B}$, under relevant assumptions. The aforementioned reinterpretation is based on the notions of relative weak coreflections and fibrations, which is our next topic of investigation.

3 Relative weak coreflections and fibrations

In this section we consider the notion of weakly coreflective subcategory and a relative version of it as a tool to compare the initial objects in the involved categories, obtaining Corollary 3.3 and Proposition 3.5. Then we study pullbacks of a relative weakly coreflective subcategory along relative fibrations, see Proposition 3.7. These results will be used in Sect. 4 in order to prove Proposition 4.12, Proposition 4.15 and Proposition 4.23.

3.1 Relative weak coreflections

Consider a full subcategory $\mathcal{C}$ of a category $\mathcal{D}$. By a *weak coreflection* of an object $D \in \mathcal{D}$ in $\mathcal{C}$ we mean a morphism $r_D : D^\star \to D$ in $\mathcal{D}$ with $D^\star \in \mathcal{C}$ and such that the function $\mathrm{Hom}_{\mathcal{D}}(C, r_D) : \mathrm{Hom}_{\mathcal{D}}(C, D^\star) \to \mathrm{Hom}_{\mathcal{D}}(C, D)$ is surjective for all $C \in \mathcal{C}$. Given a class $\mathcal{E}$ of morphisms in $\mathcal{D}$, then $\mathcal{C}$ is said to be *weakly $\mathcal{E}$-coreflective* if each object in $\mathcal{D}$ has a weak coreflection $r_D \in \mathcal{E}$. If $\mathcal{E}$ is the whole class of morphisms in $\mathcal{D}$ we will just say *weakly coreflective*, see e.g. [11, Definition 4.5]. Clearly, a weakly $\mathcal{E}$-coreflective subcategory is in particular weakly coreflective.

Remark 3.1 When $\mathcal{E}$ is the class of monomorphisms (resp. epimorphism) one could speak about weakly mono-coreflective (epi-coreflective), in analogy to the non-weak case, see e.g. [19]. Anyway, we will not deal with these cases.

Consider a weakly coreflective subcategory $\mathcal{C}$ of a category $\mathcal{D}$ and let $J : \mathcal{C} \to \mathcal{D}$ be the canonical embedding. Then the object function $(-)^\star : \mathsf{Obj}(\mathcal{D}) \to \mathsf{Obj}(\mathcal{C})$, $D \mapsto D^\star$, yields a *weak right adjoint* to the functor J, see [20] (where it is called a right adjoint system). The item 1) in the following result, proved under the further assumption that the category $\mathcal{C}$ is posetal, is an analogue, for this particular weak right adjoint, of the well-known fact that a right adjoint preserves limits.

Proposition 3.2 *Let $\mathcal{C}$ be a posetal weakly coreflexive full subcategory of a category $\mathcal{D}$ and let $J : \mathcal{C} \to \mathcal{D}$ be the canonical embedding. Given a functor $F : \mathcal{A} \to \mathcal{C}$, the following assertions holds.*

1) *If $(L, (l_A)_{A\in\mathcal{A}})$ is a limit of JF, then $(L^\star, (l_A \circ r_L)_{A\in\mathcal{A}})$ is a limit of F.*
2) *If $(L, (l_A)_{A\in\mathcal{A}})$ is a colimit of JF, then $(L^\star, (l_A^\star)_{A\in\mathcal{A}})$ is a colimit of F, for a unique morphism $l_A^\star : FA \to L^\star$ in $\mathcal{C}$ such that $r_L \circ l_A^\star = l_A$, for every $A \in \mathcal{A}$.*

Proof 1) Set $l_A^\star := l_A \circ r_L : L^\star \to FA$ which is clearly a morphism in $\mathcal{C}$. Given a morphism $a : A \to A'$ in $\mathcal{A}$, we have that $Fa \circ l_A^\star = JFa \circ l_A \circ r_L = l_{A'} \circ r_L = l_{A'}^\star$ so that $(L^\star, (l_A^\star)_{A\in\mathcal{A}})$ is a cone on F. Given another cone $(C, (c_A)_{A\in\mathcal{A}})$ on F, it is in particular a cone on JF so that, since $(L, (l_A)_{A\in\mathcal{A}})$ is a limit of JF, there is a unique morphism $c : C \to L$ in $\mathcal{D}$ such that $l_A \circ c = c_A$. Since $\mathrm{Hom}_{\mathcal{D}}(C, r_L) : \mathrm{Hom}_{\mathcal{D}}(C, L^\star) \to \mathrm{Hom}_{\mathcal{D}}(C, L)$ is surjective, there is $c' : C \to L^\star$ in $\mathcal{D}$ such that $r_L \circ c' = c$. Thus $l_A^\star \circ c' = l_A \circ r_L \circ c' = l_A \circ c = c_A$. Finally, since C and $L^\star$ are in $\mathcal{C}$, the morphism c' is in fact a morphism in $\mathcal{C}$ and hence it is unique as $\mathcal{C}$ is posetal.

2) Since $\mathrm{Hom}_{\mathcal{D}}(FA, r_L) : \mathrm{Hom}_{\mathcal{D}}(FA, L^\star) \to \mathrm{Hom}_{\mathcal{D}}(FA, L)$ is surjective, there is $l_A^\star : FA \to L^\star$ in $\mathcal{D}$ such that $r_L \circ l_A^\star = l_A$. Since FA and $L^\star$ are in $\mathcal{C}$, the morphism $l_A^\star$ is in fact a morphism in $\mathcal{C}$ and hence it is unique as $\mathcal{C}$ is posetal. Clearly $(L^\star, (l_A^\star)_{A\in\mathcal{A}})$ is automatically a cocone on F as $\mathcal{C}$ is posetal. Given another cocone $(C, (c_A)_{A\in\mathcal{A}})$ on F, it is in particular a cocone on JF so that, since $(L, (l_A)_{A\in\mathcal{A}})$ is a colimit of JF, there is a unique morphism $c : L \to C$ in $\mathcal{D}$ such that $c \circ l_A = c_A$. Set $c^\star := c \circ r_L : L^\star \to C$. This is a morphism in $\mathcal{C}$ whence it is unique as $\mathcal{C}$ is posetal. Moreover $c^\star \circ l_A^\star = c \circ r_L \circ l_A^\star = c \circ l_A = c_A$. □

The following result will be useful in constructing explicitly an initial object in a posetal weakly coreflexive full subcategory.

Corollary 3.3 *Let $\mathcal{C}$ be a posetal weakly coreflexive full subcategory of a category $\mathcal{D}$. Assume there is a set $\mathcal{S}$ consisting of objects in $\mathcal{C}$ such that each object in $\mathcal{C}$ is isomorphic to an element of $\mathcal{S}$. If there exists the product $\prod_{S\in\mathcal{S}} S$ in $\mathcal{D}$, then $(\prod_{S\in\mathcal{S}} S)^\star$ is an initial object in $\mathcal{C}$.*

Proof Set $D := \prod_{S\in\mathcal{S}} S \in \mathcal{D}$. By Proposition 3.2, we have that $D^\star \in \mathcal{C}$ is the product of the elements of $\mathcal{S}$ in $\mathcal{C}$ so that we can consider the canonical projection $p_S : D^\star \to S$ in $\mathcal{C}$, for every $S \in \mathcal{S}$. Given $C \in \mathcal{C}$, there is $S \in \mathcal{S}$ and an isomorphism $f : S \to C$ in $\mathcal{C}$. Since $\mathcal{C}$ is posetal we get $\mathrm{Hom}_{\mathcal{C}}(D^\star, C) = \{f \circ p_S\}$ and hence $D^\star$ is an initial object in $\mathcal{C}$. □

Lemma 3.4 *Let $\mathcal{C}$ be a posetal weakly coreflective full subcategory of a category $\mathcal{D}$. Then, for every $C \in \mathcal{C}$ and $D \in \mathcal{D}$, there can be a unique morphism $C \to D$ in $\mathcal{D}$.*

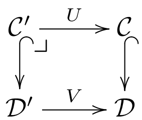
Proof Since $\mathcal{C}$ is weakly coreflective, there is a morphism $r_D : D^\star \to D$ with $D^\star \in \mathcal{C}$ and such that the function $\mathrm{Hom}_{\mathcal{D}}(C, r_D) : \mathrm{Hom}_{\mathcal{D}}(C, D^\star) \to \mathrm{Hom}_{\mathcal{D}}(C, D)$ is surjective. Since $\mathcal{C}$ is a posetal full subcategory of a category $\mathcal{D}$, we have that $\mathrm{Hom}_{\mathcal{D}}(C, D^\star) = \mathrm{Hom}_{\mathcal{C}}(C, D^\star)$ has at most one element so that $\mathrm{Hom}_{\mathcal{D}}(C, D)$ has at most one element. □

Proposition 3.5 *Let $\mathcal{C}$ be a replete posetal weakly coreflective full subcategory of a category $\mathcal{D}$. Then $\mathcal{C}$ and $\mathcal{D}$ have the same initial objects, if any.*

Proof Assume that I is an initial object in $\mathcal{C}$ and let $D \in \mathcal{D}$. By Lemma 3.4, the set $\mathrm{Hom}_{\mathcal{D}}(I, D)$ has at most one element. Since $\mathcal{C}$ is weakly coreflective, there is a morphism $r_D : D^\star \to D$ with $D^\star \in \mathcal{C}$ and since I is initial in $\mathcal{C}$ there is a morphism $i : I \to D^\star$ so that $r_D \circ i \in \mathrm{Hom}_{\mathcal{D}}(I, D)$ and hence $\mathrm{Hom}_{\mathcal{D}}(I, D) = \{r_D \circ i\}$ so that I is initial in $\mathcal{D}$.

Conversely, assume I is an initial object in $\mathcal{D}$. Since I is a particular instance of colimit, by Proposition 3.2, we have that $I^\star$ is an initial object in $\mathcal{C}$. By the foregoing, $I^\star$ is an initial object also in $\mathcal{D}$. By uniqueness we get $I \cong I^\star$ as objects in $\mathcal{D}$. Since $\mathcal{C}$ is replete and $I^\star \in \mathcal{C}$, we get that $I \in \mathcal{C}$. Thus I is an initial object also in $\mathcal{C}$. □

Let $\mathcal{C}$ be a full subcategory of a category $\mathcal{D}$. We can consider the pullback of $\mathcal{C}$ along a functor $V : \mathcal{D}' \to \mathcal{D}$ i.e. the full subcategory $\mathcal{C}'$ of $\mathcal{D}'$ whose objects are the $D' \in \mathcal{D}'$ such that $VD' \in \mathcal{C}$. Clearly V induces the functor $U : \mathcal{C}' \to \mathcal{C}, C' \mapsto VC', f' \mapsto Vf'$ which makes commute the diagram



The instance of this situation we are interested in is the diagram in Remark 4.7.

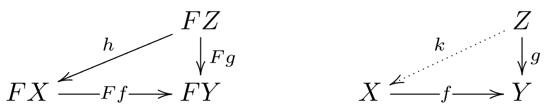
Lemma 3.6 *Let $\mathcal{C}$ be a replete posetal full subcategory of a category $\mathcal{D}$. Consider the pullback $\mathcal{C}'$ of $\mathcal{C}$ along a faithful functor $V : \mathcal{D}' \to \mathcal{D}$. Then $\mathcal{C}'$ is a replete posetal full subcategory of $\mathcal{D}'$.*

Proof Let $C' \in \mathcal{C}'$ and $D' \in \mathcal{D}'$. Given an isomorphism $h' : C' \to D'$ in $\mathcal{D}'$, then we get an isomorphism $Vh' : VC' \to VD'$ in $\mathcal{D}$. Since $VC' \in \mathcal{C}$, and $\mathcal{C}$ is replete, we get that $VD' \in \mathcal{C}$ and hence $D' \in \mathcal{C}'$. Thus $\mathcal{D}'$ is a replete full subcategory of $\mathcal{C}'$.

Given morphisms $f, g : C' \to C''$ in $\mathcal{C}'$, we get that $Vf, Vg : VC' \to VC''$ are morphisms in $\mathcal{C}$. Since $\mathcal{C}$ is posetal we get $Vf = Vg$. Since V is faithful, we get $f = g$ and hence $\mathcal{C}'$ is posetal. □

3.2 Relative fibrations

Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. Recall that a morphism $f \in \mathcal{A}$ is *cartesian* (with respect to F) over a morphism $f' \in \mathcal{B}$ whenever $Ff = f'$ and, when given $g \in \mathcal{A}$ and $h \in \mathcal{B}$ such that $Ff \circ h = Fg$, there exists a unique morphism $k \in \mathcal{A}$ such that $Fk = h$ and $f \circ k = g$.



Let $\mathcal{E}$ be a class of morphisms in $\mathcal{B}$. We say that F is an *$\mathcal{E}$-fibration* if every morphism $f' : B \to FA$ in $\mathcal{E}$ there is $f : A' \to A$which is cartesian over f', see [10, Definition 5.1].

Proposition 3.7 *Let $\mathcal{C}$ be a weakly $\mathcal{E}$-coreflexive full subcategory of a category $\mathcal{D}$. Consider the pullback $\mathcal{C}'$ of $\mathcal{C}$ along an $\mathcal{E}$-fibration $V : \mathcal{D}' \to \mathcal{D}$. Then $\mathcal{C}'$ is a weakly coreflexive full subcategory of $\mathcal{D}'$.*

Proof Given $X \in \mathcal{D}'$, since $VX \in \mathcal{D}$, we can consider $r_{VX} : (VX)^\star \to VX$ in $\mathcal{E}$. Since V is an $\mathcal{E}$-fibration, there is a morphism $r'_X : X^\star \to X$ in $\mathcal{D}'$ which is cartesian over r_{VX}. In particular $V(r'_X) = r_{VX}$ so that $V(X^\star) = (VX)^\star$ and hence $X^\star \in \mathcal{C}'$. We have to check that $\mathrm{Hom}_{\mathcal{D}'}(C', r'_X)$ is surjective for all $C' \in \mathcal{C}'$.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}'}(C', X^\star) & \xrightarrow{\mathrm{Hom}_{\mathcal{D}'}(C', r'_X)} & \mathrm{Hom}_{\mathcal{D}'}(C', X) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{D}}(VC', V(X^\star)) & \xrightarrow{\mathrm{Hom}_{\mathcal{D}}(VC', V(r'_X))} & \mathrm{Hom}_{\mathcal{D}}(VC', VX) \end{array}$$

Let $f \in \mathrm{Hom}_{\mathcal{D}'}(C', X)$. Then $Vf \in \mathrm{Hom}_{\mathcal{D}}(VC', VX)$. Since $\mathrm{Hom}_{\mathcal{D}}(VC', r_{VX})$ is surjective, there is $g \in \mathrm{Hom}_{\mathcal{D}}(VC', (VX)^\star)$ such that $r_{VX} \circ g = Vf$. Since r'_X is cartesian over r_{VX}, there is a unique morphism $g' \in \mathrm{Hom}_{\mathcal{D}'}(C', X^\star)$ such that $Vg' = g$ and $r'_X \circ g' = f$. Thus $\mathrm{Hom}_{\mathcal{D}'}(C', r'_X)$ is surjective. □

4 The crucial colimit as an initial object

In Theorem 2.5 it is shown that, in the Setting 2.4, the existence of a colimit for the diagram (8), for every algebra $\overline{B}$ in $\mathcal{B}$, yields a functor $\overline{L}$ which is a left adjoint of the functor $\overline{R} = \mathsf{Alg}(R) : \mathsf{Alg}(\mathcal{A}) \to \mathsf{Alg}(\mathcal{B})$. The first aim of this section is to reinterpret this colimit as an initial object in the category $\mathsf{IndAlg}(\overline{B})$ of induced algebras of $\overline{B}$. This will lead us to rewrite Theorem 2.5 as Theorem 4.5. Then, under suitable assumptions, in several steps we will see in Theorem 4.21 that $\mathsf{IndAlg}(\overline{B})$ can be replaced by three other categories that can be more easy to handle in practice, among them the category $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$ of epi-induced objects. Then we will provide a construction of an initial object in $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$. Putting together these results we will provide Proposition 4.24 giving an explicit description for $\overline{L}\overline{B}$. By taking $\mathcal{C}^{\mathrm{op}}$ instead of $\mathcal{A}$ we will get Proposition 4.26 that will be used together with Remark 5.6 in the Sect. 4 for our main example.

4.1 Induced objects and algebras

Our aim here is to characterize a colimit for (8) as an initial object in the category $\mathsf{IndAlg}(\overline{B})$ of induced algebras of $\overline{B}$.

Definition 4.1 Let $\left(L : \mathcal{B} \to \mathcal{A}, \psi_2, \psi_0\right)$ be a colax monoidal functor and let $\overline{B} := \left(B, m_B, u_B\right)$ be an algebra in $\mathcal{B}$. We say that $\left(\overline{A}, q\right)$ is an *induced object of* $\overline{B}$ (by L) whenever $\overline{A} = \left(A, m_A, u_A\right)$ consists of an object A and morphisms $m_A : A \otimes A \to A$, $u_A : \mathbb{I} \to A$ and $q : LB \to A$ in $\mathcal{A}$ such that

$$\begin{array}{ccccc} L(B \otimes B) & & \xrightarrow{\quad Lm_B \quad} & & LB \\ \downarrow{\scriptstyle \psi_2(B,B)} & & & & \downarrow{\scriptstyle q} \\ LB \otimes LB & \xrightarrow{q \otimes q} & A \otimes A & \xrightarrow{m_A} & A \end{array} \qquad \begin{array}{ccc} L\mathbb{I} & \xrightarrow{Lu_B} & LB \\ \downarrow{\scriptstyle \psi_0} & & \downarrow{\scriptstyle q} \\ \mathbb{I} & \xrightarrow{u_A} & A \end{array}.$$

A *morphism* $h : (\overline{A}, q) \to (\overline{A'}, q')$ of induced objects of $\overline{B}$ is a morphism $h : A \to A'$ such that $h \circ q = q'$. In this way we have defined the category $\mathsf{IndObj}(\overline{B})$ of induced objects of $\overline{B}$ and their morphisms. Given $(\overline{A}, q)$ in $\mathsf{IndObj}(\overline{B})$, if the triple $\overline{A} = \left(A, m_A, u_A\right)$ is an algebra in $\mathcal{A}$ then $\left(\overline{A}, q\right)$ is called an *induced algebra of* $\overline{B}$ (by L). Note that $\left(\overline{A}, q\right)$ is an object in the comma category $(LB \downarrow \Omega)$, see [21, page 47], where $\Omega : \mathsf{Alg}(\mathcal{A}) \to \mathcal{A}$ is the forgetful functor. Thus we can define a *morphism* $\overline{h} : (\overline{A}, q) \to (\overline{A'}, q')$ of induced algebras of $\overline{B}$ to be an algebra morphism $\overline{h} : \overline{A} \to \overline{A'}$ such that $\Omega\overline{h} \circ q = q'$. In this way we have defined the category $\mathsf{IndAlg}(\overline{B})$ of induced algebras of $\overline{B}$ an their morphisms.

Remark 4.2 The two above notions of induced object and algebra already appeared in [31, Definition 11] with a slightly different terminology.

We now turn to the Setting 1.4. It is well-known that the colimit of a diagram D is the initial object in the category formed by cocones on D. In particular, we get that a colimit for diagram (8) is an initial object in the category $\mathsf{Cocone}(\overline{B})$ whose objects are pairs $(\overline{A}, \xi)$, where $\xi : TLB \to \overline{A}$ is an algebra morphism that coequalizes the pairs in (8) and whose morphisms $(\overline{A}, \xi) \to (\overline{A'}, \xi')$ are algebra morphisms $\overline{f} : \overline{A} \to \overline{A'}$ such that $\overline{f} \circ \xi = \xi'$.

The next aim is to use the adjunction $(T, \Omega, \alpha : \mathsf{id}_{\mathcal{A}} \to \Omega T, \gamma : T\Omega \to \mathsf{id}_{\mathsf{Alg}(\mathcal{A})})$ to show that the category $\mathsf{Cocone}(\overline{B})$ is isomorphic to the category $\mathsf{IndAlg}(\overline{B})$ so that the respective initial objects are in bijective correspondence.

Proposition 4.3 *In the Setting 2.4, let $\overline{B} := \left(B, m_B, u_B\right)$ be an algebra in $\mathcal{B}$. Then an algebra morphism $\xi : TLB \to \overline{A}$ coequalizes the pairs in (8) if and only if $\left(\overline{A}, \Omega\xi \circ \alpha_{LB} : LB \to A\right)$ is an induced algebra of $\overline{B}$. As a consequence we get the category isomorphism*

$$F : \mathsf{Cocone}(\overline{B}) \to \mathsf{IndAlg}(\overline{B}), \quad (\overline{A}, \xi) \mapsto (\overline{A}, \Omega\xi \circ \alpha_{LB}), \quad \overline{f} \mapsto \overline{f}$$

whose inverse F^{-1} is given by $F^{-1}(\overline{A}, q) := (\overline{A}, \gamma_{\overline{A}} \circ Tq)$.

Proof Recall that the morphisms $m_{R\Omega TLB}$ and $u_{R\Omega TLB}$ are determined by the equality (7) so that

$$m_{R\Omega TLB} = Rm_{\Omega TLB} \circ \phi_2(\Omega TLB, \Omega TLB) \text{ and } u_{R\Omega TLB} = Ru_{\Omega TLB} \circ \phi_0$$

where $TLB = \left(\Omega TLB, m_{\Omega TLB}, u_{\Omega TLB}\right)$. By using this fact, we want to rewrite some of the morphisms in (8) . We have

$$\begin{aligned}
&\gamma_{TLB} \circ T\epsilon_{\Omega TLB} \circ TLm_{R\Omega TLB} \circ TL\left(R\alpha_{LB} \circ \eta_B \otimes R\alpha_{LB} \circ \eta_B\right) \\
&\quad = \gamma_{TLB} \circ Tm_{\Omega TLB} \circ T\epsilon_{\Omega TLB \otimes \Omega TLB} \circ TL\phi_2(\Omega TLB, \Omega TLB) \circ TL\left(R\alpha_{LB} \otimes R\alpha_{LB}\right) \circ TL\left(\eta_B \otimes \eta_B\right) \\
&\quad = \gamma_{TLB} \circ Tm_{\Omega TLB} \circ T\left(\alpha_{LB} \otimes \alpha_{LB}\right) \circ T\epsilon_{LB \otimes LB} \circ TL\phi_2(LB, LB) \circ TL\left(\eta_B \otimes \eta_B\right)
\end{aligned}$$

and

$$\begin{aligned}
\gamma_{TLB} \circ T\epsilon_{\Omega TLB} \circ TLu_{R\Omega TLB} &= \gamma_{TLB} \circ T\epsilon_{\Omega TLB} \circ TLRu_{\Omega TLB} \circ TL\phi_0 \\
&= \gamma_{TLB} \circ Tu_{\Omega TLB} \circ T\epsilon_{\mathbb{I}_{\mathcal{A}}} \circ TL\phi_0.
\end{aligned}$$

Now, let $\xi : TLB \to \overline{A}$ be some algebra morphism. Then ξ coequalizes at the same time both pairs in (8) if and only if

$$\begin{aligned}
\xi \circ TLm_B &= \xi \circ \gamma_{TLB} \circ Tm_{\Omega TLB} \circ T\left(\alpha_{LB} \otimes \alpha_{LB}\right) \circ T\epsilon_{LB \otimes LB} \circ TL\phi_2(LB, LB) \circ TL\left(\eta_B \otimes \eta_B\right), \\
\xi \circ TLu_B &= \xi \circ \gamma_{TLB} \circ Tu_{\Omega TLB} \circ T\epsilon_{\mathbb{I}_{\mathcal{A}}} \circ TL\phi_0.
\end{aligned}$$

These are equalities in $\mathrm{Hom}_{\mathsf{Alg}(\mathcal{A})}\left(TL(B \otimes B), \overline{A}\right)$ and $\mathrm{Hom}_{\mathsf{Alg}(\mathcal{A})}\left(TL\mathbb{I}, \overline{A}\right)$, respectively. Note that, using the adjunction (T, Ω), one has that the map

$$\mathrm{Hom}_{\mathsf{Alg}(\mathcal{A})}(TX, Y) \xrightarrow{\Phi(X,Y)} \mathrm{Hom}_{\mathcal{A}}(X, \Omega Y) : f \mapsto \Omega f \circ \alpha_X$$

has inverse

$$\mathrm{Hom}_{\mathcal{A}}(X, \Omega Y) \xrightarrow{\Phi(X,Y)^{-1}} \mathrm{Hom}_{\mathsf{Alg}(\mathcal{A})}(TX, Y) : g \mapsto \gamma_Y \circ Tg$$

By applying $\Phi(X, Y)$, the equalities above reduce to

$$\begin{aligned}
\Omega\xi \circ \alpha_{LB} \circ Lm_B &= \Omega\xi \circ m_{\Omega TLB} \circ \left(\alpha_{LB} \otimes \alpha_{LB}\right) \circ \epsilon_{LB \otimes LB} \circ L\phi_2(LB, LB) \circ L\left(\eta_B \otimes \eta_B\right), \\
\Omega\xi \circ \alpha_{LB} \circ Lu_B &= \Omega\xi \circ u_{\Omega TLB} \circ \epsilon_{\mathbb{I}_{\mathcal{A}}} \circ L\phi_0
\end{aligned}$$

i.e.

$$\begin{aligned}
\Omega\xi \circ \alpha_{LB} \circ Lm_B &= \Omega\xi \circ m_{\Omega TLB} \circ \left(\alpha_{LB} \otimes \alpha_{LB}\right) \circ \psi_2(B, B), \\
\Omega\xi \circ \alpha_{LB} \circ Lu_B &= \Omega\xi \circ u_{\Omega TLB} \circ \psi_0
\end{aligned}$$

Since $\xi : TLB \to \overline{A}$ is an algebra morphism, $\Omega\xi \circ m_{\Omega TLB} = m_A \circ (\Omega\xi \otimes \Omega\xi)$ and $\Omega\xi \circ u_{\Omega TLB} = u_A$ so that, if we set $q_\xi := \Omega\xi \circ \alpha_{LB} : LB \to A$, the last displayed equalities above can be rewritten as

$$\begin{aligned}
q_\xi \circ Lm_B &= m_A \circ \left(q_\xi \otimes q_\xi\right) \circ \psi_2(B, B), \\
q_\xi \circ Lu_B &= u_A \circ \psi_0.
\end{aligned}$$

Since, from the very beginning, $\overline{A} = \left(A, m_A, u_A\right)$ is an algebra, the last displayed equalities mean that $\left(\overline{A}, q_\xi\right)$ is an induced algebra of $\overline{B}$. More precisely, an algebra morphism

$\xi : TL B \to \overline{A}$ coequalizes the pairs in (8) if and only if $\left(\overline{A}, q_\xi : LB \to A\right)$ is an induced algebra of $\overline{B}$. By the foregoing, we have that F is well-defined on objects. Moreover If $\overline{f} : (\overline{A}, \xi) \to (\overline{A'}, \xi')$ is a morphism in $\mathsf{Cocone}(\overline{B})$, then $\Omega\overline{f}\circ q_\xi = \Omega\overline{f}\circ\Omega\xi\circ\alpha_{LB} = \Omega(\overline{f}\circ\xi)\circ\alpha_{LB} = \Omega\xi'\circ\alpha_{LB} = q_{\xi'}$ so that $\overline{f} : (\overline{A}, q_\xi) \to (\overline{A'}, q_{\xi'})$ is a morphism in $\mathsf{IndAlg}(\overline{B})$ and hence F is well-defined on morphisms too. Let now $(\overline{A}, q)$ be an object in $\mathsf{IndAlg}(\overline{B})$. Via the adjunction $(T, \Omega, \alpha, \gamma)$, we have that $q = \Omega\xi_q\circ\alpha_{LB} : LB \to A$ where $\xi_q := \gamma_{\overline{A}}\circ Tq$. By the first part of the statement, we have that ξ_q coequalizes (8) so that $(\overline{A}, \xi_q)$ is an object in $\mathsf{Cocone}(\overline{B})$. Thus we can define $G : \mathsf{IndAlg}(\overline{B}) \to \mathsf{Cocone}(\overline{B})$, $(\overline{A}, q) \mapsto (\overline{A}, \xi_q)$, $\overline{f} \mapsto \overline{f}$. Note that G is well-defined on morphisms as, given $\overline{f} : (\overline{A}, q) \to (\overline{A'}, q')$, we have $\overline{f}\circ\xi_q = \overline{f}\circ\gamma_{\overline{A}}\circ Tq = \gamma_{\overline{A'}}\circ T\Omega\overline{f}\circ Tq = \gamma_{\overline{A'}}\circ T(\Omega\overline{f}\circ q) = \gamma_{\overline{A'}}\circ Tq' = \xi_{q'}$. Since $(T, \Omega, \alpha, \gamma)$ is an adjunction, it is clear that $\xi = \xi_{q_\xi}$ and $q = q_{\xi_q}$ so that we get that $F\circ G$ and $G\circ F$ act as the identity functors on objects. Since they also act as the identity on morphisms, we get $F\circ G = \mathsf{Id}$ and $G\circ F = \mathsf{Id}$. □

As a consequence of Proposition 4.3 and of the observation we made that a colimit for (8) is nothing but an initial object in the category $\mathsf{Cocone}(\overline{B})$, we get the following characterization.

Proposition 4.4 *In the Setting 2.4, the following assertions are equivalent for any algebra* $\overline{B} := \left(B, m_B, u_B\right)$ *in* $\mathcal{B}$.

(1) $\left(\overline{P}, p : LB \to \Omega\overline{P}\right)$ is an initial object in the category $\mathsf{IndAlg}(\overline{B})$ of induced algebras of $\overline{B}$.
(2) $\left(\overline{P}, \kappa : TLB \to \overline{P}\right)$ is a colimit for (8).

The morphisms p and κ correspond to each other through the adjunction $(T, \Omega, \alpha, \gamma)$ i.e. $p := \Omega\kappa\circ\alpha_{LB}$ and $\kappa := \gamma_P\circ Tp$.

By using Proposition 2.4 we are now able to rewrite Theorem 2.5 in a different form.

Theorem 3.5 *In the Setting 2.4, assume that for any algebra* $\overline{B} := \left(B, m_B, u_B\right)$ *in* $\mathcal{B}$ *there is an initial object* $\left(\overline{P_{\overline{B}}}, p_{\overline{B}} : LB \to \Omega\overline{P_{\overline{B}}}\right)$ *in the category* $\mathsf{IndAlg}(\overline{B})$ *of induced algebras of* $\overline{B}$.

Then $\overline{R} = \mathsf{Alg}(R) : \mathsf{Alg}(\mathcal{A}) \to \mathsf{Alg}(\mathcal{B})$ *has a left adjoint* $\overline{L}$ *defined by* $\overline{L}\,\overline{B} := \overline{P_{\overline{B}}}$ *for any* $\overline{B}$ *as above. Moreover the morphisms* $p_{\overline{B}}$ *define a natural transformation* $p : L\Omega \to \Omega\overline{L}$ *whose naturality completely determines how* $\overline{L}$ *acts on morphisms.*

Proof Since condition (1) in Proposition 4.4 is satisfied, we know there is a morphism $\kappa_{\overline{B}} : TLB \to \overline{P_{\overline{B}}}$ such that $\left(\overline{P_{\overline{B}}}, \kappa_{\overline{B}}\right)$ is a colimit for (8) . Moreover we have that $p_{\overline{B}} = \Omega\kappa_{\overline{B}}\circ\alpha_{LB}$.

By Theorem 2.5, this colimit yields a functor $\overline{L}$ which is a left adjoint of the functor $\overline{R} : \mathsf{Alg}(\mathcal{A}) \to \mathsf{Alg}(\mathcal{B})$ and the morphisms $\kappa_{\overline{B}}$ define a natural transformation $\kappa : TL\Omega \to \overline{L}$. Explicitly $\overline{L}\,\overline{B} := \overline{P_{\overline{B}}}$ and the action of $\overline{L}$ on morphisms is uniquely determined by the naturality of κ. From $p_{\overline{B}} = \Omega\kappa_{\overline{B}}\circ\alpha_{LB}$ we get that the morphisms $p_{\overline{B}}$ define a natural transformation $p : L\Omega \to \Omega\overline{L}$ such that $p = \Omega\kappa\circ\alpha L\Omega$. Indeed, since we also have $\kappa_{\overline{B}} = \gamma_{\overline{LB}}\circ Tp_{\overline{B}}$, the naturality of κ is equivalent to the naturality of p and hence the latter completely determines the action of $\overline{L}$ on morphisms as well. $\square$

Theorem 4.5 shows how central is the role played by an initial object in the category $\mathsf{IndAlg}(\overline{B})$ of induced algebras of $\overline{B}$. Under suitable assumptions, we will see that this category can be replaced by three other categories that can be more easy to handle in practice. One of them is $\mathsf{IndObj}(\overline{B})$ while the remaining two, namely $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$ and $\mathsf{IndAlg}^{\mathsf{e}}(\overline{B})$, are introduced in Sect. 4.2.

4.2 Epi-induced objects and algebras and initial objects

Here we introduce the categories $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$ and $\mathsf{IndAlg}^{\mathsf{e}}(\overline{B})$ and, in Theorem 4.21, we show that, under the proper assumptions, the four categories $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$, $\mathsf{IndObj}(\overline{B})$, $\mathsf{IndAlg}^{\mathsf{e}}(\overline{B})$ and $\mathsf{IndAlg}(\overline{B})$ have the same initial object, if any. This will be done by exploiting the results on relative weak coreflections and fibrations of Sect. 2.

Definition 4.6 By an *epi-induced object (or algebra) of* $\overline{B}$ we mean an induced object (or algebra) $\left(\overline{A}, q\right)$ of $\overline{B}$ such that q is an epimorphism.

We denote by $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$ the full subcategory of $\mathsf{IndObj}(\overline{B})$ formed by epi-induced objects and by $\mathsf{IndAlg}^{\mathsf{e}}(\overline{B})$ the full subcategory of $\mathsf{IndAlg}(\overline{B})$ formed by epi-induced algebras.

Remark 4.7 Clearly the forgetful functor $\Omega : \mathsf{Alg}(\mathcal{A}) \to \mathcal{A}$ induces the faithful functors

$$U : \mathsf{IndAlg}^{\mathsf{e}}(\overline{B}) \to \mathsf{IndObj}^{\mathsf{e}}(\overline{B}), \quad (\overline{A}, q) \mapsto (\overline{A}, q), \quad \overline{h} \mapsto \Omega\overline{h},$$
$$V : \mathsf{IndAlg}(\overline{B}) \to \mathsf{IndObj}(\overline{B}), \quad (\overline{A}, q) \mapsto (\overline{A}, q), \quad \overline{h} \mapsto \Omega\overline{h},$$

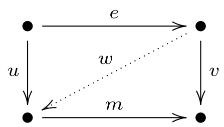
that make commute the following diagram of functors

$$\begin{array}{ccc} \mathsf{IndAlg}^{\mathsf{e}}(\overline{B}) & \xrightarrow{U} & \mathsf{IndObj}^{\mathsf{e}}(\overline{B}) \\ \big\downarrow & \lrcorner & \big\downarrow \\ \mathsf{IndAlg}(\overline{B}) & \xrightarrow{V} & \mathsf{IndObj}(\overline{B}) \end{array}$$

where the vertical arrows are the canonical full embeddings. Note that the category $\mathsf{IndAlg}^{\mathsf{e}}(\overline{B})$ is the pullback of $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$ along V meaning that it is the full subcategory of $\mathsf{IndAlg}(\overline{B})$ consisting of objects whose image through V belongs to $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$.

The assumptions we will use, include the notion of (Epi, StrongMono)-factorization. Let us recall the definition of a strong monomorphism in a category.

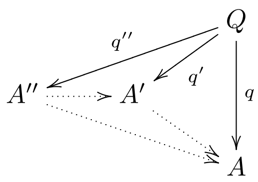
Definition 4.8 A monomorphism m is called *strong* if for every commutative square



where e **is an epimorphism, there is a unique morphism** w **such that** $w \circ e = u$ **and** $m \circ w = v$.

For instance, one can easily verify that a regular monomorphism is always strong.

Remark 4.9 Following [23, page 12], recall that a *coimage* of a morphism $q : Q \to A$ in an arbitrary category $\mathcal{C}$ is a pair $\mathrm{Coim}(q) := \left(A', q'\right)$ where $q' : Q \to A'$ is an epimorphism such that q factors through q' and, if there is another epimorphism $q'' : Q \to A''$ such that q factors through q'', then q' factors through q''. In other words $\mathrm{Coim}(q)$ is the biggest epi-induced object of Q that q factors through.



Now, consider a morphism $q : Q \to A$ in $\mathcal{C}$ that admits an (Epi, StrongMono)-factorization i.e. q factors as an epimorphism $q' : Q \to A'$ followed by a strong monomorphism $h' : A' \to A$, so that $q = h' \circ q'$. Then $\left(A', q'\right) = \mathrm{Coim}(q)$.

4.3 Comparing the initial objects in $\mathsf{IndObj}(\overline{\boldsymbol{B}})$ and $\mathsf{IndObj}^{\mathsf{e}}(\overline{\boldsymbol{B}})$

We are going to prove Proposition 4.12 which compares the initial objects in $\mathsf{IndObj}(\overline{B})$ and $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$. First we need two lemmata.

Lemma 4.10 *Let* $(L : \mathcal{B} \to \mathcal{A}, \psi_2, \psi_0)$ *be a colax monoidal functor and let* $\left(\overline{E}, q\right) \in \mathsf{IndObj}(\overline{B})$ *be such that*

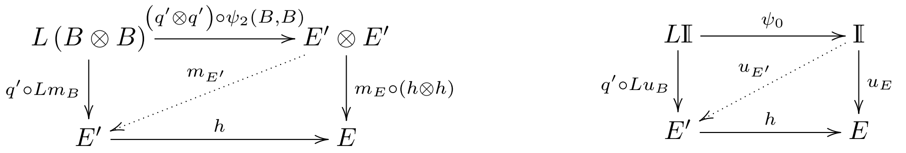
- q factors as $q = h \circ q'$, where $h : E' \to E$ is a strong monomorphism and $q' : LB \to E'$ is an epimorphism;
- the morphisms $\left(q' \otimes q'\right) \circ \psi_2(B, B)$ and ψ_0 are epimorphisms.

Then E' becomes an $\left(\overline{E'}, q'\right) \in \mathsf{IndObj}(\overline{B})$ and h induces a morphism $h : (\overline{E'}, q') \to (\overline{E}, q)$ in $\mathsf{IndObj}(\overline{B})$ such that $h \circ m_{E'} = m_E \circ (h \otimes h)$ and $h \circ u_{E'} = u_E$.

Proof We have

$$m_E \circ (h \otimes h) \circ \left(q' \otimes q'\right) \circ \psi_2(B, B) = m_E \circ (q \otimes q) \circ \psi_2(B, B) = q \circ L m_B = h \circ q' \circ L m_B$$

and $u_E \circ \psi_0 = q \circ L u_B = h \circ q' \circ L u_B$. Hence we have the following commutative squares



Since the morphisms $\left(q' \otimes q'\right) \circ \psi_2(B, B)$ and ψ_0 are epimorphisms, and h is a strong monomorphism, there is a unique morphism $m_{E'}$ such that $h \circ m_{E'} = m_E \circ (h \otimes h)$ and $m_{E'} \circ \left(q' \otimes q'\right) \circ \psi_2(B, B) = q' \circ L m_B$, and there is a unique morphism $u_{E'}$ such that $h \circ u_{E'} = u_E$ and $u_{E'} \circ \psi_0 = q' \circ L u_B$. Thus $\left(\overline{E'}, q'\right) \in \mathsf{IndObj}(\overline{B})$, where we set $\overline{E'} := \left(E', m_{E'}, u_{E'}\right)$. □

Lemma 4.11 $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$ *is a replete posetal full subcategory of* $\mathsf{IndObj}(\overline{B})$.

Proof Let $(\overline{A}, p) \in \mathsf{IndObj}^{\mathsf{e}}(\overline{B})$ and $(\overline{A'}, p') \in \mathsf{IndObj}(\overline{B})$.

- Given an isomorphism $h : (\overline{A}, p) \to (\overline{A'}, p')$, we have $p' = h \circ p$ so that p' is an epimorphism as p. Thus $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$ is a replete full subcategory of $\mathsf{IndObj}(\overline{B})$.
- Given morphisms $f, g : (\overline{A}, p) \to (\overline{A'}, p')$ in $\mathsf{IndObj}(\overline{B})$, we get $f \circ p = p' = g \circ p$. Since p is an epimorphism we get $f = g$. In particular $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$ is posetal.

□

Denote by $\mathcal{E}(\overline{B})$ the class of morphisms $h : (\overline{A'}, q') \to (\overline{A}, q)$ in $\mathsf{IndObj}(\overline{B})$ such that $h : A' \to A$ is a monomorphism, $h \circ m_{A'} = m_A \circ (h \otimes h)$ and $h \circ u_{A'} = u_A$.

Proposition 4.12 *In the Setting* 4.4, *assume that*

- If $\left(\overline{A}, q\right) \in \mathsf{IndObj}(\overline{B})$, then q admits an (Epi, StrongMono)-factorization $q = h \circ q'$,
- the morphisms $\left(q' \otimes q'\right) \circ \psi_2(B, B)$ and ψ_0 are epimorphisms.

Then, $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$ is a weakly $\mathcal{E}(\overline{B})$-coreflective subcategory of $\mathsf{IndObj}(\overline{B})$. Explicitly, given $\left(\overline{A}, q\right) \in \mathsf{IndObj}(\overline{B})$, we have that $(\overline{A}, q)^\star = (\overline{A'}, q')$ where $\left(A', q'\right) = \mathrm{Coim}(q)$. As a consequence $\mathsf{IndObj}(\overline{B})$ and $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$ have the same initial objects.

Proof Let $\left(\overline{A}, q : LB \to A\right)$ be an induced object of $\overline{B}$ in $\mathcal{A}$. By hypothesis, q admits an (Epi, StrongMono)-factorization $q = h \circ q'$ where $h : A' \to A$ is a strong monomorphism and $q' : LB \to A'$ is an epimorphism. Moreover the morphisms $\left(q' \otimes q'\right) \circ \psi_2(B, B)$ and ψ_0 are epimorphisms. Note that, by Remark 4.9, we have that $\left(A', q'\right) = \mathrm{Coim}(q)$. We can apply Lemma 4.10 to deduce that A' becomes an epi-induced object $\left(\overline{A'}, q'\right)$ of $\overline{B}$ and h induces a morphism $h : (\overline{A'}, q') \to (\overline{A}, q)$ of induced objects such that $hom_{A'} = m_A \circ (h \otimes h)$ and $hou_{A'} = u_A$. Thus $h : (\overline{A'}, q') \to (\overline{A}, q)$ is in $\mathcal{E}(\overline{B})$.

We have so proved that, for any $\left(\overline{A}, q\right)$ in $\mathsf{IndObj}(\overline{B})$, there is $\left(\overline{A'}, q'\right)$ in $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$ and a morphism $h : (\overline{A'}, q') \to (\overline{A}, q)$ in $\mathcal{E}(\overline{B})$. Given $(\overline{E}, p)$ in $\mathsf{IndAlg}^{\mathsf{e}}(\overline{B})$, let us check that

$$\mathrm{Hom}_{\mathsf{IndObj}(\overline{B})}((\overline{E}, p), h) : \mathrm{Hom}_{\mathsf{IndObj}(\overline{B})}((\overline{E}, p), (\overline{A'}, q')) \to \mathrm{Hom}_{\mathsf{IndObj}(\overline{B})}((\overline{E}, p), (\overline{A}, q))$$

is surjective. Given a morphism $f : (\overline{E}, p) \to (\overline{A}, q)$ in $\mathsf{IndObj}(\overline{B})$, we have that $f \circ p = q = h \circ q'$ so that $f \circ p = h \circ q'$. Since h is a strong monomorphism and p is an epimorphism, there is a unique morphism $w : E \to A$ such that $h \circ w = f$ and $w \circ p = q'$. These equalities say we have a morphism $w : (\overline{E}, p) \to (\overline{A'}, q')$ whose image through $\mathrm{Hom}_{\mathsf{IndAlg}(\overline{B})}((\overline{E}, p), h)$ is exactly the starting morphism f. Thus $\mathrm{Hom}_{\mathsf{IndAlg}(\overline{B})}((\overline{E}, p), h)$ is surjective. Hence $\mathsf{IndAlg}^{\mathsf{e}}(\overline{B})$ is a weakly $\mathcal{E}(\overline{B})$-coreflective subcategory of $\mathsf{IndAlg}(\overline{B})$. In particular $\mathsf{IndAlg}^{\mathsf{e}}(\overline{B})$ is a weakly coreflective subcategory of $\mathsf{IndAlg}(\overline{B})$. This, together with Lemma 4.11, implies that we can apply Proposition 3.5 to conclude. □

4.4 Comparing the initial objects in $\mathsf{IndAlg}(\overline{B})$ and $\mathsf{IndAlg}^{\mathsf{e}}(\overline{B})$

Next aim is proving Proposition 4.15, which compares the initial objects of $\mathsf{IndAlg}(\overline{B})$ and $\mathsf{IndAlg}^{\mathsf{e}}(\overline{B})$. We first need the following lemmata.

Lemma 4.13 $\mathsf{IndAlg}^{\mathsf{e}}(\overline{B})$ *is a replete posetal full subcategory of* $\mathsf{IndAlg}(\overline{B})$.

Proof It follows by Remark 4.7 and Lemma 3.6. □

Lemma 4.14 *The functor* $V : \mathsf{IndAlg}(\overline{B}) \to \mathsf{IndObj}(\overline{B})$ *of Remark 4.7 is an* $\mathcal{E}(\overline{B})$*-fibration.*

Proof Let $\left(\overline{A}, q\right) \in \mathsf{IndAlg}(\overline{B})$, $\left(\overline{A'}, q'\right) \in \mathsf{IndObj}(\overline{B})$ and let $h : (\overline{A'}, q') \to V(\overline{A}, q)$ be a morphism in $\mathcal{E}(\overline{B})$. Thus $h : A' \to A$ is a monomorphism such that $hom_{A'} = m_A \circ (h \otimes h)$ and $hou_{A'} = u_A$. Since h is a monomorphism, one easily checks that $\overline{A'} = (A', m_{A'}, u_{A'})$ is an algebra, by using the fact that $\overline{A} = (A, m_A, u_A)$ is an algebra. Then h induces an algebra morphism $\overline{h} : \overline{A'} \to \overline{A}$ such that $\Omega\overline{h} = h$ and we get a morphism $\overline{h} : (\overline{A'}, q') \to (\overline{A}, q)$ in $\mathsf{IndAlg}(\overline{B})$ whose image through V is $h : (\overline{A'}, q') \to V(\overline{A}, q)$. It remains to check that $\overline{h}$ is cartesian over h. Given a morphism $\overline{g} : (\overline{A''}, q') \to (\overline{A}, q)$ in $\mathsf{IndAlg}(\overline{B})$ and a morphism $l : V(\overline{A''}, q'') \to V(\overline{A'}, q')$ such that $h \circ l = V\overline{g} =: g$, we have $h \circ l \circ m_{A''} = g \circ m_{A''} = m_A \circ (g \otimes g) = m_A \circ (h \otimes h) \circ (l \otimes l) = h \circ m_{A'} \circ (l \otimes l)$ so that $l \circ m_{A''} = m_{A'} \circ (l \otimes l)$ as h is a monomorphism. Similarly $h \circ l \circ u_{A''} = g \circ u_{A''} = u_A = h \circ u_{A'}$ and hence $l \circ u_{A''} = u_{A'}$. Therefore there is an algebra morphism $\overline{l} : \overline{A''} \to \overline{A'}$ such that $\Omega\overline{l} = l$. Thus $\overline{l} : (\overline{A''}, q'') \to (\overline{A'}, q')$ is a morphism whose image through V is $l : V(\overline{A''}, q'') \to V(\overline{A'}, q')$ and such that $\overline{h} \circ \overline{l} = \overline{g}$. □

Proposition 4.15 *In the Setting 2.4, assume that*

- If $\left(\overline{A}, q\right) \in \mathsf{IndObj}(\overline{B})$, then q admits an (Epi, StrongMono)-factorization $q = h \circ q'$,
- the morphisms $\left(q' \otimes q'\right) \circ \psi_2(B, B)$ and ψ_0 are epimorphisms.

Then, $\mathsf{IndAlg}^{\mathsf{e}}(\overline{B})$ is a weakly coreflective subcategory of $\mathsf{IndAlg}(\overline{B})$. As a consequence $\mathsf{IndAlg}(\overline{B})$ and $\mathsf{IndAlg}^{\mathsf{e}}(\overline{B})$ have the same initial objects.

Proof Our hypotheses guarantee that we can apply Proposition 4.12 to get that $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$ is a weakly $\mathcal{E}(\overline{B})$-coreflective subcategory of $\mathsf{IndObj}(\overline{B})$. Moreover, by Lemma 4.14, the functor $V : \mathsf{IndAlg}(\overline{B}) \to \mathsf{IndObj}(\overline{B})$ is an $\mathcal{E}(\overline{B})$-fibration. Therefore, we can apply Proposition 3.7 to the diagram in Remark 4.7 to get that $\mathsf{IndAlg}^{\mathsf{e}}(\overline{B})$ is a weakly coreflective subcategory of $\mathsf{IndAlg}(\overline{B})$. This, together with Lemma 4.13 imply that we can apply Proposition 3.5 to conclude. □

4.5 Comparing all of the initial objects

Next aim is to obtain Theorem 4.21, where we compare the initial objects in the categories $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$, $\mathsf{IndObj}(\overline{B})$, $\mathsf{IndAlg}^{\mathsf{e}}(\overline{B})$ and $\mathsf{IndAlg}(\overline{B})$ altogether. First we need some lemmata.

Given an induced algebra $\left(\overline{E}, q\right)$ of $\overline{B}$, the next lemma shows that, under mild assumptions, $\mathrm{Coim}(q) = \left(E', q'\right)$ becomes an induced algebra of $\overline{B}$.

Lemma 4.16 *Let $\left(L : \mathcal{B} \to \mathcal{A}, \psi_2, \psi_0\right)$ be a colax monoidal functor. Let $q : LB \to A$ me a morphism that admits two (Epi, StrongMono)-factorizations $q = h \circ q'$ and $q = h' \circ q''$. We have that*

- $\left(q' \otimes q'\right) \circ \psi_2(B, B)$ is an epimorphism if and only if so is $\left(q'' \otimes q''\right) \circ \psi_2(B, B)$;
- $\left(q' \otimes q' \otimes q'\right) \circ \left(LB \otimes \psi_2(B, B)\right) \circ \psi_2(B, B \otimes B)$ is an epimorphism if and only if so is $\left(q'' \otimes q'' \otimes q''\right) \circ \left(LB \otimes \psi_2(B, B)\right) \circ \psi_2(B, B \otimes B)$.

Proof Denote by P' the domain of h and by P'' the domain of h'. By uniqueness of the (Epi, StrongMono)-factorizations, we have an isomorphism $w : P' \to P''$ such that $w \circ q' = q''$. Hence $\left(q'' \otimes q''\right) \circ \psi_2(B, B) = (w \otimes w) \circ \left(q' \otimes q'\right) \circ \psi_2(B, B)$ from which the first item follows. Similarly one treats the second one. □

Lemma 4.17 *In the Setting 4.4, assume that ψ_0 is an epimorphism and let $\left(\overline{A}, q\right) \in \mathsf{IndAlg}^{\mathsf{e}}(\overline{B})$ be such that q admits an (Epi,StrongMono)-factorization $q = h \circ q'$.*

1) *If $\left(q' \otimes q'\right) \circ \psi_2(B, B)$ is an epimorphism, then so is $(q \otimes q) \circ \psi_2(B, B)$.*
2) *If $\left(q' \otimes q' \otimes q'\right) \circ \left(LB \otimes \psi_2(B, B)\right) \circ \psi_2(B, B \otimes B)$ is an epimorphism, then so is $(q \otimes q \otimes q) \circ \left(LB \otimes \psi_2(B, B)\right) \circ \psi_2(B, B \otimes B)$.*

Proof We just prove 1), the argument for 2) being similar. Since q is an epimorphism, then $q = \mathrm{Id}\circ q$ is (Epi, StrongMono)-factorization. Since $\left(q' \otimes q'\right)\circ\psi_2(B,B)$ is an epimorphism, by Lemma 3.16, so is $(q \otimes q)\circ\psi_2(B,B)$. □

Lemma 4.18 *Let* $\left(L : \mathcal{B} \to \mathcal{A}, \psi_2, \psi_0\right)$ *be a colax monoidal functor and let* $\overline{B} \in \mathsf{Alg}(\mathcal{B})$. *Let* $\left(\overline{A}, q\right)$ *and* $\left(\overline{A'}, q'\right)$ *be in* $\mathsf{IndAlg}(\overline{B})$. *Assume that* $(q \otimes q)\circ\psi_2(B,B)$ *and* ψ_0 *are epimorphisms. Then any morphism* $h : A \to A'$ *such that* $h\circ q = q'$ *becomes a morphism* $\overline{h} : (\overline{A}, q) \to (\overline{A'}, q')$ *in* $\mathsf{IndAlg}(\overline{B})$.

Proof We compute

$$
\begin{aligned}
m_{A'}\circ(h \otimes h)\circ(q \otimes q)\circ\psi_2(B,B) &= m_{A'}\circ\left(q' \otimes q'\right)\circ\psi_2(B,B) = q'\circ Lm_B \\
&= h\circ q\circ Lm_B = h\circ m_A\circ(q \otimes q)\circ\psi_2(B,B) \\
h\circ u_A\circ\psi_0 &= h\circ q\circ Lu_B = q'\circ Lu_B = u_{A'}\circ\psi_0
\end{aligned}
$$

so that, in view of the assumptions, we deduce that $m_{A'}\circ(h \otimes h) = h\circ m_A$ and $h\circ u_A = u_{A'}$ i.e. that h becomes an algebra morphism $\overline{h} : \overline{A} \to \overline{A'}$ such that $\Omega\overline{h} = h$. Since $\Omega\overline{h}\circ q = q'$ we get that $\overline{h}$ is a morphism in $\mathsf{IndAlg}(\overline{B})$. □

Lemma 4.19 *In the Setting* 1.4, *assume that*

- If $\left(\overline{A}, q\right) \in \mathsf{IndAlg}^{\mathsf{e}}(\overline{B})$, then q admits an (Epi, StrongMono)-factorization $q = h\circ q'$,
- the morphisms $\left(q' \otimes q'\right)\circ\psi_2(B,B)$ and ψ_0 are epimorphisms.

Then the functor $U : \mathsf{IndAlg}^{\mathsf{e}}(\overline{B}) \to \mathsf{IndObj}^{\mathsf{e}}(\overline{B})$ of Remark 3.7 is fully faithful.

Proof By construction U is faithful. Let $(\overline{A}, q), (\overline{A'}, q') \in \mathsf{IndAlg}^{\mathsf{e}}(\overline{B})$ and let $h : (\overline{A}, q) \to (\overline{A'}, p)$ be a morphism in $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$. Then $h\circ q = p$. By Lemma 4.17 1), we have that $(q \otimes q)\circ\psi_2(B,B)$ is an epimorphism so that we can apply Lemma 4.18 to get an algebra morphism $\overline{h}$ such that $\Omega\overline{h} = h$. Therefore $\Omega\overline{h}\circ q = p$ and hence we have a morphism $\overline{h} : (\overline{A}, q) \to (\overline{A'}, p)$ in $\mathsf{IndAlg}^{\mathsf{e}}(\overline{B})$ whose image through U is h. □

The next aim is to reduce to the case where epi-induced object are epi-induced algebras.

Lemma 4.20 *Let* $\left(L : \mathcal{B} \to \mathcal{A}, \psi_2, \psi_0\right)$ *be a colax monoidal functor and let* $\overline{B} \in \mathsf{Alg}(\mathcal{B})$.

Let $\left(\overline{A}, q\right) \in \mathsf{IndObj}^{\mathsf{e}}(\overline{B})$ *be such that* $(q \otimes q \otimes q)\circ\left(LB \otimes \psi_2(B,B)\right)\circ\psi_2(B, B \otimes B)$ *is an epimorphism. Then* $\left(\overline{A}, q\right) \in \mathsf{IndAlg}^{\mathsf{e}}(\overline{B})$.

Proof Let $\left(\overline{E}, q : LB \to E\right) \in \mathsf{IndObj}^{\mathsf{e}}(\overline{B})$. One easily verifies that

$$
m_E\circ\left(m_E \otimes E\right)\circ(q \otimes q \otimes q)\circ\left(\psi_2(B,B) \otimes LB\right)\circ\psi_2(B \otimes B, B) = q\circ Lm_B\circ L\left(m_B \otimes B\right) \text{ and}
$$
$$
m_E\circ\left(E \otimes m_E\right)\circ(q \otimes q \otimes q)\circ\left(LB \otimes \psi_2(B,B)\right)\circ\psi_2(B, B \otimes B) = q\circ Lm_B\circ L\left(B \otimes m_B\right).
$$

Since m_B is associative and $\left(\psi_2(B,B) \otimes LB\right)\circ\psi_2(B \otimes B, B) = \left(LB \otimes \psi_2(B,B)\right)\circ\psi_2(B, B \otimes B)$ and $(q \otimes q \otimes q)\circ\left(LB \otimes \psi_2(B,B)\right)\circ\psi_2(B, B \otimes B)$ is an epimorphism, we deduce that m_E is associative too. Note that

$$\begin{aligned} l_E \circ (\psi_0 \otimes q) \circ \psi_2(\mathbb{1}, B) =& l_E \circ (\mathbb{1} \otimes q) \circ (\psi_0 \otimes LB) \circ \psi_2(\mathbb{1}, B) \\ =& q \circ l_{LB} \circ (\psi_0 \otimes LB) \circ \psi_2(\mathbb{1}, B) = q \circ L l_B \end{aligned}$$

and hence, since q is an epimorphism, we deduce that $(\psi_0 \otimes q) \circ \psi_2(\mathbb{1}, B)$ is an epimorphism too. Using naturality of ψ_2, we have

$$\begin{aligned} m_E \circ (u_E \otimes E) \circ (\psi_0 \otimes q) \circ \psi_2(\mathbb{1}, B) &= m_E \circ (q \otimes q) \circ (L u_B \otimes LB) \circ \psi_2(\mathbb{1}, B) \\ &= q \circ L m_B \circ L(u_B \otimes B) \\ &= q \circ L(l_B) \\ &= q \circ l_{LB} \circ (\psi_0 \otimes LB) \circ \psi_2(\mathbb{1}, B) \\ &= l_E \circ (\psi_0 \otimes q) \circ \psi_2(\mathbf{1}, B) \end{aligned}$$

so that $m_E \circ (u_E \otimes E) = l_E$. Similarly one proves that $m_E \circ (E \otimes u_E) = r_E$. Then $\overline{E}$ is an algebra so that $(\overline{E}, q) \in \mathsf{IndAlg}^{\mathsf{e}}(\overline{B})$. □

We are now able to prove the announced result.

Theorem 3.21 *In the Setting 2.4, assume that*

- if $(\overline{A}, q) \in \mathsf{IndObj}(\overline{B})$, then q admits an (Epi, StrongMono)-factorization $q = h \circ q'$,
- the morphisms $(q' \otimes q') \circ \psi_2(B, B)$ and ψ_0 are epimorphisms,
- the morphism $(q' \otimes q' \otimes q') \circ (LB \otimes \psi_2(B, B)) \circ \psi_2(B, B \otimes B)$ is an epimorphism.

The following assertions are equivalent.

(1) $(\overline{P}, p)$ *is an initial object in* $\mathsf{IndObj}(\overline{B})$.
(2) $(\overline{P}, p)$ *is an initial object in* $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$.
(3) $(\overline{P}, p)$ *is an initial object in* $\mathsf{IndAlg}(\overline{B})$.
(4) $(\overline{P}, p)$ *is an initial object in* $\mathsf{IndAlg}^{\mathsf{e}}(\overline{B})$.

Proof (1) ⇔ (2). This follows from is Proposition 4.12.

(3) ⇔ (4). This follows from is Proposition 4.15.

(2) ⇔ (4). By Lemma 4.19, the functor $U : \mathsf{IndAlg}^{\mathsf{e}}(\overline{B}) \to \mathsf{IndObj}^{\mathsf{e}}(\overline{B})$ of Remark 4.7 is fully faithful. By construction U is also injective on objects. In order to conclude we check that it is also surjective on objects whence a category isomorphism. Let $(\overline{A}, q) \in \mathsf{IndObj}^{\mathsf{e}}(\overline{B})$. By Lemma 4.17 and the assumptions, we have that $(q \otimes q \otimes q) \circ (LB \otimes \psi_2(B, B)) \circ \psi_2(B, B \otimes B)$ is an epimorphism. By Lemma 4.20, we have $(\overline{A}, q) \in \mathsf{IndAlg}^{\mathsf{e}}(\overline{B})$. □

4.6 Constructing the initial object in $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$

By Theorem 4.21, under the relevant assumptions, to have an initial object in $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$ is equivalent to having an initial object in $\mathsf{IndAlg}(\overline{B})$. By Proposition 4.4, this is equivalent to having a colimit for (8), yielding then an explicit description $\overline{L}\,\overline{B}$. For this reason it is worthwhile to provide a construction of an initial object in $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$. To this aim we first need to prove the following result.

Lemma 4.22 *Let I be a set and let $\left(\overline{E_i}, q_i\right)_{i\in I}$ be a family of objects in* $\mathsf{IndObj}(\overline{B})$. *Assume that the family $\left(E_i\right)_{i\in I}$ has a product $\left(E, \left(p_t\right)_{t\in I}\right)$ in $\mathcal{A}$ and let $q : LB \to E$ be the unique morphism such that $q_i = p_i \circ q$ for every i. Then E induces a tern $\overline{E} = (E, m_E, u_E)$ such that $\left(\left(\overline{E}, q\right), (p_t)_{t\in I}\right)$ is the product of the family $\left(\overline{E_i}, q_i\right)_{i\in I}$ in* $\mathsf{IndObj}(\overline{B})$.

Proof By the universal property of the product, there are unique morphisms $q : LB \to E$, $m_E : E \otimes E \to E$ and $u_E : \mathbb{I} \to E$ such that $p_i \circ q = q_i$, $p_i \circ m_E = m_{E_i} \circ (p_i \otimes p_i)$ and $p_i \circ u_E = u_{E_i}$, for every $i \in I$. Set $\overline{E} = (E, m_E, u_E)$. We have

$$\begin{aligned} p_i \circ q \circ L m_B &= q_i \circ L m_B = m_{E_i} \circ \left(q_i \otimes q_i\right) \circ \psi_2(B,B) = m_{E_i} \circ \left(p_i \otimes p_i\right) \circ (q \otimes q) \circ \psi_2(B,B) \\ &= p_i \circ m_E \circ (q \otimes q) \circ \psi_2(B,B), \\ p_i \circ q \circ L u_B &= q_i \circ L u_B = u_{E_i} \circ \psi_0 = p_i \circ u_E \circ \psi_0. \end{aligned}$$

By the uniqueness in the universal property of the product, we get that

$$q \circ L m_B = m_E \circ (q \otimes q) \circ \psi_2(B,B), \qquad q \circ L u_B = u_E \circ \psi_0.$$

This proves that $\left(\overline{E}, q\right)$ belongs to $\mathsf{IndObj}(\overline{B})$. Let us check it defines the desired product. From the equality $q_t = p_t \circ q$, we get that $p_t : E \to E_t$ yields the projection $p_t : \left(\overline{E}, q\right) \to \left(\overline{E_t}, q_t\right)$ in $\mathsf{IndObj}(\overline{B})$. Given, for every $t \in I$, a morphism $h_t : (\overline{A}, p) \to (\overline{E_t}, q_t)$ in $\mathsf{IndObj}(\overline{B})$, by the universal property of $\left(E, \left(p_t\right)_{t\in I}\right)$, there is a unique morphism $h : A \to E$ such that $p_t \circ h = h_t$. Since $p_t \circ h \circ p = h_t \circ p = q_t$, the uniqueness implies $h \circ p = q$ so that we get a morphism $h : (\overline{A}, p) \to (\overline{E}, q)$ that composed by the projection yields $h_t : (\overline{A}, p) \to (\overline{E_t}, q_t)$. Its uniqueness follows from the universal property of $\left(E, \left(p_t\right)_{t\in I}\right)$. □

Proposition 4.23 *In the Setting 2.4, assume that*

- if $\left(\overline{A}, q\right) \in \mathsf{IndObj}(\overline{B})$, then q admits an (Epi, StrongMono)-factorization $q = h \circ q'$;
- the morphisms $\left(q' \otimes q'\right) \circ \psi_2(B,B)$ and ψ_0 are epimorphisms;
- there is set $\mathcal{S}_B$ of objects of $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$ such that each object in $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$ is isomorphic to an element in $\mathcal{S}_B$;
- in $\mathcal{A}$ there exists the product E of the family $(D)_{(\overline{D},\delta_D)\in\mathcal{S}_B}$.

Then there is $\left(\overline{C}, \delta_C\right) \in \mathcal{S}_B$ which is an initial object in $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$ such that $\left(C, \delta_C\right) = \mathrm{Coim}(\delta)$ where $\delta : LB \to E$ is the diagonal morphism of the family $\left(\delta_D\right)_{(\overline{D},\delta_D)\in\mathcal{S}_B}$.

Proof By Lemma 3.11 and Proposition 3.12, $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$ is a replete posetal weakly coreflective subcategory of $\mathsf{IndObj}(\overline{B})$. Since the elements in $\mathcal{S}_B$ are, in particular objects in $\mathsf{IndObj}(\overline{B})$, by Lemma 3.22, the object $E := \prod_{(\overline{D},\delta_D)\in\mathcal{S}_B} D$ induces a tern $\overline{E} = (E, m_E, u_E)$ such that $(\left(\overline{E}, \delta\right), (p_D)_{(\overline{D},\delta_D)\in\mathcal{S}_B})$ is the product of the objects of $\mathcal{S}_B$ in $\mathsf{IndObj}(\overline{B})$. By Corollary 2.3 applied to the set $\mathcal{S}_B$, we get that $(\overline{E}, \delta)^\star$ is an initial object in $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$. By Proposition 3.12, we know that $(\overline{E}, \delta)^\star = (\overline{E'}, \delta')$ where $\left(E', \delta'\right) = \mathrm{Coim}(\delta)$. Since $(\overline{E'}, \delta') = (\overline{E}, \delta)^\star \in \mathsf{IndObj}^{\mathsf{e}}(\overline{B})$, there is $(\overline{C}, \delta_C) \in \mathcal{S}_B$ such that $(\overline{E'}, \delta') \cong (\overline{C}, \delta_C)$ as objects $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$. Thus also $(\overline{C}, \delta_C)$ is an initial object in $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$ and $\left(C, \delta_C\right) = \mathrm{Coim}(\delta)$. □

Proposition 4.24 *In the Setting 2.4, assume that*

- the tensor products in $\mathcal{A}$ preserve epimorphisms;
- ψ_0 and the components of ψ_2 are epimorphisms in $\mathcal{A}$;
- if $(\overline{E}, q) \in \mathsf{IndObj}(\overline{B})$, then q admits an (Epi,StrongMono)-factorization;
- there is set $\mathcal{S}_B$ of objects of $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$ such that each object in $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$ is isomorphic to an element in $\mathcal{S}_B$;
- in $\mathcal{A}$ there exists the product E of the family $(D)_{(\overline{D},\delta_D)\in\mathcal{S}_B}$.

Then there is $\left(\overline{C}, \delta_C\right) \in \mathcal{S}_B$ which is an initial object in $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$ such that $\left(C, \delta_C\right) = \mathrm{Coim}(\delta)$ where $\delta : LB \to E$ is the diagonal morphism of the family $\left(\delta_D\right)_{(\overline{D},\delta_D)\in\mathcal{S}_B}$.

Finally, if the above assumptions hold for every algebra $\overline{B}$ in $\mathcal{B}$, then $\overline{R}$ has a left adjoint $\overline{L}$ explicitly given by $\overline{L}\,\overline{B} = \overline{C}$.

Proof Proposition 3.23 ensures that there is $\left(\overline{C}, \delta_C\right) \in \mathcal{S}_B$, as in the statement, which is an initial object in $\mathsf{IndObj}^{\mathsf{e}}(\overline{B})$. By Theorem 3.21, this is also an initial object in $\mathsf{IndAlg}(\overline{B})$. By Theorem 3.5, we conclude. □

Setting 4.25 For our purposes it is convenient to write Proposition 3.24 in case $\mathcal{A} = \mathcal{C}^{\mathrm{op}}$ for a covariant functor $L : \mathcal{B} \to \mathcal{C}^{\mathrm{op}}$ regarded as a contravariant functor $(-)^\diamond : \mathcal{B} \to \mathcal{C}$ such that $LB = (B^\diamond)^{\mathrm{op}}$ and $Lf = (f^\diamond)^{\mathrm{op}}$, for a morphism f. To this aim let us rewrite in $\mathcal{C}$ the notion of induced object in $\mathcal{A} = \mathcal{C}^{\mathrm{op}}$ of an algebra $\overline{B} = \left(B, m_B, u_B\right)$ in $\mathcal{B}$. It consists of a pair $\left(\underline{E}, e : E \to B^\diamond\right)$, where $\underline{E} = \left(E, \Delta_E, \varepsilon_E\right)$ with E an object in $\mathcal{C}$ and $\Delta_E : E \to E \otimes E, \varepsilon_E : E \to \mathbb{I}$ and e morphisms in $\mathcal{C}$ such that

$$(m_B)^\diamond \circ e = \varphi_2(B,B) \circ (e \otimes e) \circ \Delta_E, \tag{10}$$

$$(u_B)^\diamond \circ e = \varphi_0 \circ \varepsilon_E. \tag{11}$$

where $\varphi_2(B,B) : B^\diamond \otimes B^\diamond \to (B \otimes B)^\diamond$ and $\varphi_0 : \mathbb{I} \to \mathbb{I}^\diamond$ are determined by $\varphi_2(B,B)^{\mathrm{op}} = \psi_2(B,B)$ and $\varphi_0^{\mathrm{op}} = \psi_0$ respectively. In this case we will say that $\left(\underline{E}, e : E \to B^\diamond\right)$ is a *good object* of $B^\diamond$ in $\mathcal{C}$.

Note that the induced object of $\overline{B}$ in $\mathcal{C}^{\mathrm{op}}$ corresponding to $\left(\underline{E}, e : E \to B^\diamond\right)$ is an epi-induced object if and only if e^{op} is an epimorphism in $\mathcal{C}^{\mathrm{op}}$ that is e is a monomorphism in $\mathcal{C}$. In this case we will say that $\left(\underline{E}, e : E \to B^\diamond\right)$ is a *good subobject* of $B^\diamond$ in $\mathcal{C}$. A morphism of good (sub)objects $h : \left(\underline{E}, e : E \to B^\diamond\right) \to \left(\underline{E'}, e' : E' \to B^\diamond\right)$ is a morphism $h : E \to E'$ such that $e' \circ h = e$. This way we get the category of good (sub)objects of $B^\diamond$ in $\mathcal{C}$ which turns out to be anti-isomorphic to the category of (epi-)induced objects of $\overline{B}$ in $\mathcal{C}^{\mathrm{op}}$.

Proposition 4.26 *Let $\mathcal{C}$ and $\mathcal{B}$ be monoidal categories and let $R : \mathcal{C}^{\mathrm{op}} \to \mathcal{B}$ be a lax monoidal functor with a left adjoint L, unit η and counit ϵ. In the Setting 4.25, assume that the functor $\mho : \mathsf{Coalg}(\mathcal{C}) \to \mathcal{C}$ has a right adjoint and that*

- the tensor products in $\mathcal{C}$ preserve monomorphisms;
- ψ_0 and the components of ψ_2 are monomorphisms in $\mathcal{C}$;
- if $\left(\underline{E}, e : E \to B^\diamond\right)$ is a good object of $B^\diamond$ in $\mathcal{C}$, then the morphism e admits an (StrongEpi,Mono)-factorization in $\mathcal{C}$;
- there is a set $\mathcal{S}_B$ of good subobjects of $B^\diamond$ in $\mathcal{C}$ such that each good subobjects of $B^\diamond$ in $\mathcal{C}$ is isomorphic to an element in $\mathcal{S}_B$;
- in $\mathcal{C}$ there exists the coproduct of the family $(D)_{(\underline{D},e_D)\in\mathcal{S}_B}$.

Then there is $(\underline{B^\blacklozenge}, \theta_B : B^\blacklozenge \to B^\diamond) \in \mathcal{S}_B$ which is a terminal good subobject of $B^\diamond$ in $\mathcal{C}$ such that $(B^\blacklozenge, \theta_B)$ is the sum of the family of subobjects $(D, e_D)_{(\underline{D},e_D)\in\mathcal{S}_B}$ of $B^\diamond$.

Finally, if the above assumptions hold for every algebra $\overline{B}$ in $\mathcal{B}$, then $\overline{R}$ has a left adjoint $\overline{L}$ explicitly given by $\overline{L}\,\overline{B} = (\underline{B^\blacklozenge})^{\mathrm{op}}$.

In the next section, we put all of the above developed theory to work to explicitly compute lifted auto-adjunctions on categories of so-called "color bialgebras".

5 Application: the group-graded case

This section is devoted to investigate the case of group-graded vector spaces. To this aim we first need to recall some auxiliary results connected to the notion of pre-rigid category.

5.1 Pre-rigid monoidal categories

In order to discuss the examples of liftable functors of our concern, we recall the following notion which appeared in [18, 4.1.3].

Definition 5.1 A braided monoidal category $(\mathcal{C}, \otimes, \mathbb{I})$ is called *pre-rigid* if for every object X there exists an object X^* and a morphism $\mathrm{ev}_X : X^* \otimes X \to \mathbb{I}$ (the *evaluation at* X) with the following universal property: For every morphism $t : T \otimes X \to \mathbb{I}$ there is a unique morphism $t^\dagger : T \to X^*$ such that $t = \mathrm{ev}_X \circ \left(t^\dagger \otimes X\right)$. Equivalently the map

$$\mathrm{Hom}_{\mathcal{C}}(T, X^*) \to \mathrm{Hom}_{\mathcal{C}}(T \otimes X, \mathbb{I}), \qquad u \mapsto \mathrm{ev}_X \circ (u \otimes X)$$

is bijective for every object T in $\mathcal{C}$.

One has that a (right) closed braided monoidal category is pre-rigid (cf. [7, Proposition 2.5]). Notice that the converse is not true: the symmetric cartesian monoidal category Top of topological spaces is pre-rigid monoidal [7, Examples 2.17], but not closed.

Given a pre-rigid braided monoidal category, we can construct on it an adjunction that under relevant assumptions results in a liftable pair of functors. More precisely we have the following result, which we record here for further use.

Proposition 5.2 (cf. [7, Proposition 4.4]) *When $\mathcal{C}$ is a pre-rigid braided monoidal category, the assignment $X \mapsto X^*$ induces a functor $R = (-)^* : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ with a left adjoint $L = R^{\mathrm{op}} = (-)^* : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$. Moreover there are ϕ_2, ϕ_0 such that (R, ϕ_2, ϕ_0) is lax monoidal and, the induced colax monoidal structure on L by (1) and (2) is specifically $(\phi_2^{\mathrm{op}}, \phi_0^{\mathrm{op}})$. Explicitly, $\phi_0 : \mathbb{I} \to \mathbb{I}^*$ is uniquely defined by $\mathrm{ev}_{\mathbb{I}} \circ (\phi_0 \otimes \mathbb{I}) = m_{\mathbb{I}}$ and $\phi_2(X^{\mathrm{op}}, Y^{\mathrm{op}}) := \varphi_2(X, Y) : X^* \otimes Y^* \to (X \otimes Y)^*$ by*

$$\mathrm{ev}_{X \otimes Y} \circ (\varphi_2(X, Y) \otimes X \otimes Y) = (\mathrm{ev}_X \otimes \mathrm{ev}_Y) \circ (X^* \otimes (c_{X,Y^*})^{-1} \otimes Y). \tag{12}$$

Moreover, for every X in $\mathcal{C}$, the unit η_X and the counit $\epsilon_{X^{\mathrm{op}}} = (j_X)^{\mathrm{op}}$ of the adjunction (L, R) are uniquely defined by the equalities

$$\mathrm{ev}_X \circ c_{X,X^*} = \mathrm{ev}_{X^*} \circ (\eta_X \otimes X^*), \tag{13}$$

$$\mathrm{ev}_X \circ (c_{X^*,X})^{-1} = \mathrm{ev}_{X^*} \circ (j_X \otimes X^*). \tag{14}$$

Remark 5.3 Let $\mathcal{A}$ be a pre-rigid braided monoidal category. Then Proposition 5.2 establishes that the functor $(-)^* : \mathcal{A}^{\mathrm{op}} \to \mathcal{A}$ is self-adjoint on the right. One can asks whether this situation imply that if $\mathcal{A}$ satisfies the sufficient conditions of Theorem 2.5, then also $\mathcal{A}^{\mathrm{op}}$ does. For instance, it seems unlikely to us that the existence of a left adjoint for the forgetful functor $\Omega : \mathsf{Alg}(\mathcal{A}) \to \mathcal{A}$ would imply the existence of a right adjoint for the forgetful functor $\mho : \mathsf{Coalg}(\mathcal{A}) \to \mathcal{A}$, or that the existence of coequalizers in $\mathsf{Alg}(\mathcal{A})$ would imply the existence equalizers in $\mathsf{Coalg}(\mathcal{A})$. However we could not find explicit counterexamples.

Remark 5.4 We have noticed in Sect. 2.2 that a liftable pair induces an adjunction at the level of bialgebras in case the right adjoint is also braided. In case of Proposition 4.2, the lax monoidal functor (R, ϕ_2, ϕ_0) is braided if and only if the following diagram commutes

$$\begin{array}{ccc} R(X^{\mathrm{op}}) \otimes R(Y^{\mathrm{op}}) & \xrightarrow{c_{R(X^{\mathrm{op}}),R(Y^{\mathrm{op}})}} & R(Y^{\mathrm{op}}) \otimes R(X^{\mathrm{op}}) \\ \downarrow{\scriptstyle \phi_2(X^{\mathrm{op}},Y^{\mathrm{op}})} & & \downarrow{\scriptstyle \phi_2(Y^{\mathrm{op}},X^{\mathrm{op}})} \\ R(X^{\mathrm{op}} \otimes Y^{\mathrm{op}}) & \xrightarrow{R c_{X^{\mathrm{op}},Y^{\mathrm{op}}}} & R(Y^{\mathrm{op}} \otimes X^{\mathrm{op}}) \end{array}$$

that is if and only if the following diagram commutes

$$\begin{array}{ccc} X^*\otimes Y^* & \xrightarrow{c_{X^*,Y^*}} & Y^*\otimes X^* \\ \downarrow{\scriptstyle \varphi_2(X,Y)} & & \downarrow{\scriptstyle \varphi_2(Y,X)} \\ (X\otimes Y)^* & \xrightarrow{(c^{-1}_{X,Y})^*} & (Y\otimes X)^* \end{array}$$

Set $f := (c^{-1}_{X,Y})^* \circ \varphi_2(X,Y)$. We compute

$$\begin{aligned} \mathrm{ev}_{Y\otimes X}\circ(f\otimes Y\otimes X) &= \mathrm{ev}_{Y\otimes X}\circ((c^{-1}_{X,Y})^*\otimes Y\otimes X)\circ(\varphi_2(X,Y)\otimes Y\otimes X) \\ &= \mathrm{ev}_{X\otimes Y}\circ((X\otimes Y)^*\otimes c^{-1}_{X,Y})\circ(\varphi_2(X,Y)\otimes Y\otimes X) \\ &= \mathrm{ev}_{X\otimes Y}\circ(\varphi_2(X,Y)\otimes X\otimes Y)\circ(X^*\otimes Y^*\otimes c^{-1}_{X,Y}) \\ &\overset{(12)}{=} \left(\mathrm{ev}_X\otimes \mathrm{ev}_Y\right)\circ(X^*\otimes c^{-1}_{X,Y^*}\otimes Y)\circ(X^*\otimes Y^*\otimes c^{-1}_{X,Y}) \\ &= \mathrm{ev}_X\circ(X^*\otimes \mathrm{ev}_Y\otimes X) \\ &= (\mathrm{ev}_Y\otimes \mathrm{ev}_X)\circ(Y^*\otimes c^{-1}_{Y,X^*}\otimes X)\circ(c^{-1}_{Y^*,X^*}\otimes Y\otimes X) \\ &\overset{(12)}{=} \mathrm{ev}_{Y\otimes X}\circ(\varphi_2(Y,X)\otimes Y\otimes X)\circ(c^{-1}_{Y^*,X^*}\otimes Y\otimes X) \end{aligned}$$

so that $f = \varphi_2(Y,X)\circ c^{-1}_{Y^*,X^*}$ i.e. $(c^{-1}_{X,Y})^*\circ\varphi_2(X,Y) = \varphi_2(Y,X)\circ c^{-1}_{Y^*,X^*}$.

As a consequence, $\left(R,\phi_2,\phi_0\right)$ is a braided monoidal functor if and only if $\varphi_2(Y,X)\circ c_{X^*,Y^*} = \varphi_2(Y,X)\circ c^{-1}_{Y^*,X^*}$ for all objects X, Y in $\mathcal{C}$. Equivalently one has to ask that $\varphi_2(Y,X)\circ c_{X^*,Y^*}\circ c_{Y^*,X^*} = \varphi_2(Y,X)$ for all objects X, Y in $\mathcal{C}$. In particular, if φ_2 is a monomorphism on components, this is equivalent to ask that $c_{X^*,Y^*}\circ c_{Y^*,X^*} = 1_{Y^*\otimes X^*}$ which is quite close to requiring that $\mathcal{C}$ is symmetric.

Proposition 4.2 suggests a suitable context to obtain examples of liftable pairs of functors, as the following result shows.

Proposition 5.5 [7, *Proposition* 4.6] *For a monoidal category $\mathcal{C}$, suppose a lax monoidal functor $\left(R,\phi_2,\phi_0\right) : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ has a left adjoint $L = R^{\mathrm{op}}$. If the induced colax monoidal structure on L by (1) and (2) is specifically $(\phi_2^{\mathrm{op}},\phi_0^{\mathrm{op}})$, then $\overline{R} = \left(\underline{L}\right)^{\mathrm{op}}$. Moreover, if $\overline{R}$ has a left adjoint, then (L,R) is liftable.*

Remark 5.6 Keeping the hypotheses of Proposition 5.5 and assuming that $\mathcal{A} = \mathcal{C}^{\mathrm{op}}$, the functor $\mho : \mathsf{Coalg}(\mathcal{C}) \to \mathcal{C}$ has a right adjoint and $\mathsf{Coalg}(\mathcal{C})$ has equalizers, then, by Theorem 1.5, the functor $\overline{R}$ has a left adjoint $\overline{L}$ which we will now describe. We have

$$\mathcal{A} = \mathcal{C}^{\mathrm{op}} \underset{L}{\overset{R}{\rightleftarrows}} \mathcal{B} = \mathcal{C}.$$

By assumption, $\left(R,\phi_2,\phi_0\right)$ is lax monoidal and it has a left adjoint $L = R^{\mathrm{op}}$. Moreover, the induced colax monoidal structure $\left(\psi_2,\psi_0\right)$ on L by (1) and (2) is required to be specifically $(\phi_2^{\mathrm{op}},\phi_0^{\mathrm{op}})$. As in the Setting 3.25 we can regard L as a contravariant functor $(-)^{\diamond} : \mathcal{C} \to \mathcal{C}$ and define $\varphi_2(B,B) : B^{\diamond}\otimes B^{\diamond} \to (B\otimes B)^{\diamond}$ and $\varphi_0 : \mathbb{I} \to \mathbb{I}^{\diamond}$ by setting

$\varphi_2(B,B)^{\mathrm{op}} = \psi_2(B,B)$ and $\varphi_0^{\mathrm{op}} = \psi_0$ respectively. Note that, in view of the requirement $\left(\psi_2,\psi_0\right) = \left(\phi_2^{\mathrm{op}},\phi_0^{\mathrm{op}}\right)$, we also have $\varphi_2(B,B) := \phi_2(B^{\mathrm{op}},B^{\mathrm{op}})$ and $\varphi_0 = \phi_0$.

Assume further that

- the tensor products preserve monomorphisms in $\mathcal{C}$;
- $\phi_0 : \mathbb{I} \to \mathbb{I}^{\diamond}$ is invertible and the components of $\varphi_2$are monomorphisms;
- for every $X, Y \in \mathcal{C}$, any morphism $X \to Y^{\diamond}$ in $\mathcal{C}$ has a (StrongEpi,Mono)-factorization;
- for every algebra $\overline{B}$ in $\mathcal{B}$, there is a set $\mathcal{S}_B$ of good subobjects of $B^{\diamond}$ in $\mathcal{C}$ such that each good subobjects of $B^{\diamond}$ in $\mathcal{C}$ is isomorphic to an element in $\mathcal{S}_B$.

Since ϕ_0 is invertible, (11) rewrites as $\varepsilon_E = \phi_0^{-1} \circ (u_B)^{\diamond} \circ e$, so that ε_E is completely determined and can be ignored in the definition of good object. Thus it suffices to consider terns $\left(E, \Delta_E, e\right)$ such that (10) is fulfilled.

All the above assumptions permit to apply Proposition 3.26.

Then there is $(\underline{B^{\blacklozenge}}, \theta_B : B^{\blacklozenge} \to B^{\diamond}) \in \mathcal{S}_B$ which is a terminal good subobject of $B^{\diamond}$ in $\mathcal{C}$ such that $(\underline{B^{\blacklozenge}}, \theta_B)$ is the sum of the family of subobjects $(D, e_D)_{(\underline{D},e_D)\in \mathcal{S}_B}$ of $B^{\diamond}$.

Finally $\overline{R}$ has a left adjoint $\overline{L}$ explicitly given by $\overline{L}\,\overline{B} = (\underline{B^{\blacklozenge}})^{\mathrm{op}}$.

The following proposition will be applied to $\mathcal{C} = \mathsf{Vec}_G$.

Proposition 5.7 *Let $\mathcal{C}$ be a pre-rigid braided monoidal category. Assume that the forgetful functor $\mho : \mathsf{Coalg}(\mathcal{C}) \to \mathcal{C}$ has a right adjoint. Assume also that $\mathsf{Coalg}(\mathcal{C})$ has equalizers.*

Then $\left((-)^* : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}, (-)^* : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}\right)$ is a liftable pair of adjoint functors.

Proof Let $R = (-)^* : \mathcal{A} \to \mathcal{B}$ where $\mathcal{A} := \mathcal{C}^{\mathrm{op}}$ and $\mathcal{B} := \mathcal{C}$. By the assumptions on $\mathcal{C}$, the category $\mathcal{A} = \mathcal{C}^{\mathrm{op}}$ fulfills the requirements of Theorem 1.5 and hence $\overline{R} = \mathsf{Alg}(R) : \mathsf{Alg}(\mathcal{A}) \to \mathsf{Alg}(\mathcal{B})$ has a left adjoint $\overline{L}$. We conclude by [7, Corollary 4.7]. □

The following result is a pre-rigid version of [31, Proposition 8]. Note that, given a closed monoidal category $(\mathcal{C}, \otimes, \mathbb{I})$, the right adjoint $[-, \mathbb{I}]_r$ of the functor $(-) \otimes \mathbb{I}$ induces the functor $R = [-, \mathbb{I}]_r : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$. The authors therein call $\overline{R} = \mathsf{Alg}(R)$ the *dual monoidal functor* and prove it has a left adjoint in case $\mathcal{C}$ is locally presentable. Note also that a closed braided monoidal category is in particular pre-rigid with pre-dual given by $(-)^* := [-, \mathbb{I}]_r$.

Corollary 5.8 *Let $\mathcal{C}$ be a pre-rigid braided monoidal category. Assume that $\mathcal{C}$ is locally presentable and that the tensor products preserve directed colimits.*

Then $\left((-)^* : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}, (-)^* : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}\right)$ is a liftable pair of adjoint functors.

Proof Since $\mathcal{C}$ is monoidal and locally presentable and since the tensor products preserve directed colimits, by the proof of [28, page 8] (which does not use the symmetry assumption present in the definition of admissible category), we have that $\mathsf{Coalg}(\mathcal{C})$ is locally presentable and comonadic over $\mathcal{C}$. In particular the functor $\mho : \mathsf{Coalg}(\mathcal{C}) \to \mathcal{C}$ has a right adjoint. By [11, Corollary 1.28], the category $\mathsf{Coalg}(\mathcal{C})$ is complete so that it has equalizers. We conclude by Proposition 4.7. □

Remark 5.9 In the setting of Corollary 4.8, since $\mathcal{C}$ is a locally presentable category, it has (StrongEpi, Mono)-factorization of morphisms. Thus, part of the conditions of Remark 4.6 are automatically satisfied.

Vec_G is an example of a locally presentable category by [27, Theorem 10]. Since the tensor product in Vec_G is $\otimes_k$, it preserves directed colimits. Thus one can also apply Corollary 4.8 to this category. Indeed locally presentability is even too much to gain the liftability of the adjunction induced by the pre-dual in Vec_G since in the next subsection we will be able to directly apply the hypotheses of the Proposition 4.7.

5.2 A group-graded version of Sweedler's finite dual

We now take a closer look at two examples, by first taking $\mathcal{C} = \mathsf{Vec}$, then by taking $\mathcal{C} = \mathsf{Vec}_G$ which is our case of main interest.

The vector space case. Let us consider the case of vector spaces, putting $\mathcal{C} = \mathsf{Vec}$, which is a pre-rigid braided monoidal category, the pre-dual of a vector space V being given by the linear dual $V^* := \mathrm{Hom}_k(V, k)$. By Proposition 4.2, we have a functor $R = (-)^* : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}, X \mapsto X^*$ with left adjoint $L = R^{\mathrm{op}}$. The maps φ_2 and ϕ_0 of Proposition 4.2 are defined by

$$\varphi_2(X, Y) : X^* \otimes Y^* \to (X \otimes Y)^*, \quad f \otimes g \mapsto m_{\Bbbk}(f \otimes g),$$
$$\phi_0 : k \to k^*, \quad a \mapsto a1_k.$$

Note that all the requirements of Remark 4.6 are satisfied (in particular all epimorphisms are regular whence strong), where, given an algebra $\overline{B} = \left(B, m_B, u_B\right)$ in $\mathcal{C}$, we let $\mathcal{S}_B$ be the set of all good subspaces of B^* (we will use the word subspace when the monomorphism is an inclusion). As a consequence, $\overline{L}\,\overline{B} = (\underline{B^\circ})^{\mathrm{op}}$ where B° is the sum of all good subspaces of B^*. By [22, pages 19-20] we know that B° is exactly the Sweedler's finite dual of B.

Remark 5.10 We point out that the functor $L = (-)^* : \mathsf{Vec} \to \mathsf{Vec}^{\mathrm{op}}$ has not a left adjoint. Otherwise there should be a contravariant functor $(-)^\flat : \mathsf{Vec} \to \mathsf{Vec}$ such that $\mathrm{Hom}_k(U^*, V) \cong \mathrm{Hom}_k(V^\flat, U)$ for all U, V in Vec. By choosing $U = k$, we would get that $V \cong \mathrm{Hom}_k(k^*, V) \cong \mathrm{Hom}_k(V^\flat, k) = (V^\flat)^*$. Thus, by choosing as V a vector space with a countable basis, we would be led to a contradiction as the dual of an infinite-dimensional vector space never has a countable basis. Similarly one gets that $R = (-)^* : \mathsf{Vec}^{\mathrm{op}} \to \mathsf{Vec}$ has not a right adjoint.

The group-graded case. Let G be an abelian group, with neutral element e and let Vec_G be the category whose objects are vector spaces (over a field k) graded by the group G. For objects $V = \oplus_{g\in G} V_g, W = \oplus_{g\in G} W_g \in \mathsf{Vec}_G$, the set of morphisms in Vec_G (i.e. degree-preserving k-linear maps) will be denoted as $\mathrm{Hom}(V, W)$. The category Vec_G admits a monoidal structure, which we now briefly recall. If $V, W \in \mathsf{Vec}_G$, then $V \otimes W := \bigoplus_g (\oplus_{xy=g} V_x \otimes_k W_y)$ becomes an object in Vec_G. The unit object is $k = k_e$. Taking associativity and unit constraints to be trivial, $(\mathsf{Vec}_G, \otimes, k)$ indeed becomes a monoidal category.

Note that the monoidal category Vec_G is (right) closed. In fact we can consider the right adjoint to the endofunctor $(-) \otimes V$ of Vec_G (tensor product of graded vector spaces) and denote this adjoint by $\mathrm{HOM}(V, -)$. Since G is abelian, we have that $\mathrm{HOM}(V, W) = \oplus_{g\in G} \mathrm{HOM}(V, W)_g$ where

$$\mathrm{HOM}(V,W)_g = \left\{ f \in \mathrm{Hom}_k(V,W) \mid f\left(V_h\right) \subseteq W_{hg} \text{ for every } h \in G \right\},$$

for any $V, W \in \mathsf{Vec}_G$. As we already mentioned that a (right) closed monoidal category is pre-rigid, we get that Vec_G is pre-rigid. To avoid confusion with the usual (non-graded) linear dual of a vector space, the pre-dual of a G-graded vector space $V = \oplus_g V_g$ in Vec_G will be denoted by $V^\diamond$. Thus $V^\diamond := \mathrm{HOM}(V,k)$. We can write explicitly the graduation of $V^\diamond$ as

$$\begin{aligned}\left(V^\diamond\right)_g =& \mathrm{HOM}(V,k)_g = \left\{ f \in \mathrm{Hom}_k(V,k) \mid f\left(V_h\right) \subseteq k_{hg} \text{ for every } h \in G \right\} \\ =& \left\{ f \in \mathrm{Hom}_k(V,k) \mid f\left(V_h\right) = 0 \text{ for every } h \in G, h \neq g^{-1} \right\} \cong \mathrm{Hom}_k\left(V_{g^{-1}}, k\right) = (V_{g^{-1}})^*.\end{aligned}$$

In order to discuss braided structures on Vec_G, recall that a bicharacter on G is a map $\alpha : G \times G \to k \setminus \{0\}$ such that

$$\alpha(gh,l) = \alpha(g,l)\alpha(h,l) \qquad \text{and} \qquad \alpha(g,hl) = \alpha(g,h)\alpha(g,l), \qquad \text{for all } g,h,l \in G.$$

Letting α be a bicharacter, we can define a braiding c^α on Vec_G, given on homogeneous objects by

$$c^\alpha_{V_g,W_h} : V_g \otimes W_h \to W_h \otimes V_g, \qquad v \otimes w \mapsto \alpha(g,h) w \otimes v.$$

We notice that, in order for c^α to be a morphism in Vec_G, we need that G is abelian. Remark also that c^α is a symmetry if and only if moreover holds that $\alpha(g,h)\alpha(h,g) = 1$, $\forall g,h \in G$. In this case we say that α is *skew-symmetric*. We shall denote the thus-obtained braided monoidal category as Vec^α_G.

Since Vec^α_G is a braided and pre-rigid monoidal category, we can use Proposition 4.2 to get that

$$R := (-)^\diamond : (\mathsf{Vec}^\alpha_G)^{\mathrm{op}} \to \mathsf{Vec}^\alpha_G$$

is a self-adjoint lax monoidal functor, for any bicharacter α.

Now, using [3, Corollary 4.6] (note that Vec_G can be regarded as the category of comodules over the group-algebra kG), the forgetful functor $\mho : \mathsf{Coalg}(\mathsf{Vec}_G) \to \mathsf{Vec}_G$ has a right adjoint. Moreover any parallel pair $f, g : C \to D$ in $\mathsf{Coalg}(\mathsf{Vec}_G)$ has equalizer given by

$$\{c \in C \mid \sum c_1 \otimes_k f(c_2) \otimes_k c_3 = \sum c_1 \otimes_k g(c_2) \otimes_k c_3\},$$

where we are using Sweedler's notation for the comultiplication[3].

As a consequence, by Proposition 4.7 we can conclude that the adjoint pair of functors (L, R) introduced above is liftable.

Although it does not seem to appear in literature, the left adjoint $\overline{L}$ of $\overline{R} = \mathsf{Alg}(R)$ -whose existence is part of the definition of a liftable pair of adjoint functors- can be described explicitly. It is our purpose here to do so. Indeed, it is shown below that, given an algebra $\overline{B} = \left(B, m_B, u_B\right)$ in Vec_G, the object $\overline{L}\,\overline{B}$ can be identified with the biggest "good" G-graded subspace of $B^\diamond$. More precisely, $\overline{L}\,\overline{B} = \left(\underline{B^\blacklozenge}\right)^{\mathrm{op}}$ where

$$B^\blacklozenge = \left\{ f \in B^\diamond \mid \xi_B(f) \text{ vanishes on some } I \in \mathcal{I}^f_{\overline{B}} \right\},$$

$\xi_B : B^\diamond = \mathrm{HOM}(B,k) \to \mathrm{Hom}_k(B,k) = B^*$ denoting the canonical injection and $\mathcal{I}^f_{\overline{B}}$ being the set of finite-codimensional G-graded ideals of $\overline{B}$.

3 Note that it coincides with the equalizer of the same pair in $\mathsf{Coalg}(\mathsf{Vec})$, see e.g. [2, Remark 1.2].

Remark 5.11 Note that, if we take G to be the trivial group in the above discussion, we recover the case of vector spaces. When taking $G = \langle g | g^2 = e\rangle$, the cyclic group of order two, and α trivial everywhere except for $\alpha(g, g) = -1$, one obtains the super vector space case; this incorporates [18, Remark 3.1].

Let us proceed with the details of the computation of $\overline{L}\,\overline{B}$.

Note that, given objects T and X in Vec_G, for every morphism $t : T \otimes X \to k$, the map $t^\dagger : T \to X^\diamond$ of Definition 4.1 is uniquely determined by the equality $t = \mathrm{ev}_X \circ (t^\dagger \otimes X)$ i.e. $t^\dagger(a) = t(a \otimes -)$ for $a \in T$. Let us start by describing explicitly some of the maps given in Proposition 4.2.

Lemma 5.12 *The map $\varphi_2(X, Y) : X^\diamond \otimes Y^\diamond \to (X \otimes Y)^\diamond$ is given, for $f \in \left(X^\diamond\right)_a, g \in \left(Y^\diamond\right)_b$, by $\varphi_2(X, Y)(f \otimes g) := \alpha(a, b)m_k(f \otimes g)$. The map $\phi_0 : k \to k^\diamond$ is given, for $\lambda \in k$, by the equality $\phi_0(\lambda) = \lambda 1_k$. In particular ϕ_0 is invertible and the components of φ_2 are monomorphisms.*

Proof By Proposition 4.2, $\varphi_2(X, Y)$ is given, for $f \in \left(X^\diamond\right)_a, g \in \left(Y^\diamond\right)_b$, by the equality

$$\varphi_2(X, Y)(f \otimes g) = \left(\mathrm{ev}_X \otimes \mathrm{ev}_Y\right)\Big(X^\diamond \otimes \left(c_{X,Y^\diamond}\right)^{-1} \otimes Y\Big)(f \otimes g \otimes -).$$

Given $x \in X_c, y \in Y_d$, we have $f(x) = \delta_{ca,e} f(x)$ and $g(y) = \delta_{db,e} g(y)$ so that

$$\begin{aligned}\varphi_2(X, Y)(f \otimes g)(x \otimes y) =& \Big[\left(\mathrm{ev}_X \otimes \mathrm{ev}_Y\right)\circ\Big(X^\diamond \otimes \left(c_{X,Y^*}\right)^{-1} \otimes Y\Big)\Big](f \otimes g \otimes x \otimes y)\\ =& \alpha(c, b)^{-1} f(x)g(y) = \delta_{ca,e}\delta_{db,e}\alpha(c, b)^{-1} f(x)g(y)\\ =& \alpha(a, b)f(x)g(y) = \alpha(a, b)m_k(f \otimes g)(x \otimes y).\end{aligned}$$

Thus $\varphi_2(X, Y)(f \otimes g) := \alpha(a, b)m_k(f \otimes g)$. On the other hand ϕ_0 is uniquely determined, for $\lambda \in k$, by the equality $\phi_0(\lambda) = m_k(\lambda \otimes -) = \lambda 1_k$. □

Remark 5.13 We already noticed that the lax monoidal functor $R := (-)^\diamond : (\mathsf{Vec}_G^\alpha)^{\mathrm{op}} \to \mathsf{Vec}_G^\alpha$ induced by the pre-dual is part of a liftable adjoint pair of functors (L, R). Moreover, by Lemma 4.12 we know that the components of φ_2 are monomorphisms. Thus, in view of Remark 4.4, we have that R is braided if and only if $c^\alpha_{X^*,Y^*} \circ c^\alpha_{Y^*,X^*} = 1_{Y^* \otimes X^*}$ for all objects X, Y in $\mathcal{C}$. In particular this holds if c^α is a symmetry, which happens if and only if α is skew-symmetric. In this case, we get an induced auto-adjunction $(\underline{L}, \underline{R})$ on the category of bialgebras in Vec_G^α, i.e. “color bialgebras” (in the sense of [5, Section 1.4] e.g.), for any such a bicharacter α.

Note that all the requirements of Remark 4.6 are satisfied, by the discussion above where we chose $\mathcal{S}_B$ to be the set of all good G-graded subspaces of $B^\diamond$ (as in case of Vec, we will use the word subspace when the monomorphism is an inclusion). Thus, given an algebra $\overline{B} = \left(B, m_B, u_B\right)$ in $\mathcal{C} = \mathsf{Vec}_G$, we get that $\overline{L}\,\overline{B}$ can be identified with the biggest good G-graded subspace of $B^\diamond$. Explicitly, there is $(\underline{B^\blacklozenge}, \theta_B : B^\blacklozenge \to B^\diamond) \in \mathcal{S}_B$, where $\theta_B : B^\blacklozenge \to B^\diamond$ is the canonical inclusion, which is a terminal good G-graded subspace of

$B^\diamond$ such that $(B^\blacklozenge, \theta_B)$ is the sum of the family of G-graded subspaces $(D, e_D)_{(\underline{D}, e_D)\in\mathcal{S}_B}$ of $B^\diamond$. Finally $\overline{L}$ is given by $\overline{L}\,\overline{B} = (\underline{B^\blacklozenge})^{\mathrm{op}}$.

To round off this example, let us further refine the description of $(B^\blacklozenge, \theta_B)$. To this aim denote by $\mathcal{S}_B^f$ the set of finite-dimensional good G-graded subspaces of $B^\diamond$.

Lemma 5.14 *$(B^\blacklozenge, \theta_B)$ is the sum of the family of finite-dimensional G-graded subspaces $(D, e_D)_{(\underline{D}, e_D)\in\mathcal{S}_B^f}$ of $B^\diamond$.*

Proof Let $\left(E, \Delta_E, e\right)$ be a good G-graded subspace of $B^\diamond$, where $e : E \to B^\diamond$ denotes the canonical inclusion. Then the right-hand side square in the following diagram commutes by (10).

$$\begin{array}{ccccccc} C & \xrightarrow{\gamma} & E & \xrightarrow{e} & B^\diamond & \xrightarrow{(m_B)^\diamond} & (B\otimes B)^\diamond \\ \downarrow{\scriptstyle \Delta_C} & & \downarrow{\scriptstyle \Delta_E} & & & & \downarrow{\scriptstyle 1_{(B\otimes B)^\diamond}} \\ C\otimes C & \xrightarrow{\gamma\otimes\gamma} & E\otimes E & \xrightarrow{e\otimes e} & B^\diamond\otimes B^\diamond & \xrightarrow{\varphi_2(B,B)} & (B\otimes B)^\diamond \end{array}$$

Consider a subcoalgebra $\left(C, \Delta_E, \varepsilon_E, \gamma : C \to E\right)$. Hence the external diagram above commutes. This means that $\left(C, \Delta_E, e\circ\gamma\right)$ is a good G-graded subspace of $B^\diamond$. This proves that a subcoalgebra of a good G-graded subspace of $B^\diamond$ is a good subspace of $B^\diamond$.

Given $(\underline{D}, e_D) \in \mathcal{S}_B$, we know that $\underline{D}$ becomes a G-graded coalgebra, cf. the dual result of Lemma 3.20. Hence we can apply [3, Theorem 4.5] to get that each $\underline{D}$ is sum of finite-dimensional G-graded subcoalgebras. Since a subcoalgebra of a good subspace of $B^\diamond$ is again a good subspace of $B^\diamond$, we can write $(B^\blacklozenge, \theta_B)$ as the desired sum. □

In order to describe the elements in $\mathcal{S}_B^f$, we first need the following lemma, which further specifies other maps involved in Proposition 4.2.

Lemma 5.15 *The morphisms $\eta_X, j_X : X \to X^{\diamond\diamond}$ are given, for $x \in X_a$, $f \in \left(X^\diamond\right)_b$, by*

$$\eta_X(x)(f) = \alpha(a,a)^{-1} f(x), \text{ for } x \in X_a, f \in X^\diamond,$$
$$j_X(x)(f) = \alpha(a,a) f(x), \text{ for } x \in X_a, f \in X^\diamond.$$

Moreover these η_X and j_X are both injective.

Proof By Proposition 4.2, for $x \in X_a$, $f \in \left(X^\diamond\right)_b$, we have

$$\eta_X(x)(f) \overset{(13)}{=} \mathrm{ev}_X c_{X,X^\diamond}(x\otimes f) = \alpha(a,b)f(x) = \delta_{ab,e}\alpha(a,b)f(x) = \alpha(a,a)^{-1}f(x),$$
$$j_X(x)(f) \overset{(14)}{=} \mathrm{ev}_X\left(c_{X^\diamond,X}\right)^{-1}(x\otimes f) = \alpha(b,a)^{-1}f(x) = \delta_{ab,e}\alpha(b,a)^{-1}f(x) = \alpha(a,a)f(x).$$

Let us check the injectivity. Let $x \in X_a$ be nonzero. If we complete x to a basis of X we can consider the map $\lambda_x \in \mathrm{Hom}_k(X, k)$ such that $\lambda_x(x) = 1$ and λ_x vanishes on the other elements of the basis. By construction $\lambda_x \in \mathrm{HOM}(X, k)_{a^{-1}}$. Thus we can compute

$\eta_X(x)\left(\lambda_x\right) = \alpha(a,a)^{-1}\lambda_x(x) = \alpha(a,a)^{-1} \neq 0$. This proves that $\left(\eta_X\right)_a$ is injective and hence η_X is injective. Similarly one gets that j_X is injective. □

We are now ready to provide the promised description of $B^{\blacklozenge}$.

Proposition 5.16 *We have that* $B^{\blacklozenge} = \left\{ f \in B^{\lozenge} \mid \xi_B(f) \text{ vanishes on some } I \in \mathcal{I}_B^f \right\}$, *where* $\mathcal{I}_B^f$ *denotes the set of finite-codimensional* G*-graded ideals of* $\overline{B}$ *and* $\xi_B : B^{\lozenge} \to B^*$ *is the canonical injection.*

Proof Let $\left(E, \Delta_E, e\right)$ be a finite-dimensional good G-graded subspace of $B^{\lozenge}$. Since E is finite-dimensional we have that $E^{\lozenge} = \mathrm{HOM}(E,k) = \mathrm{Hom}(E,k) = E^*$ (see [24, Lemma 3.3.2]). Thus in this case $\eta_E, j_E : E \to E^{\lozenge\lozenge} = E^{**}$. Since, by Lemma 4.15, these maps are injective maps between spaces with the same dimension, we deduce that η_E, j_E are invertible.

Since $(\underline{B^{\blacklozenge}}, \theta_B : B^{\blacklozenge} \to B^{\lozenge}) \in \mathcal{S}_B$ is a terminal good G-graded subspace of $B^{\lozenge}$, there is a G-graded coalgebra map $\underline{f} : \underline{E} \to \underline{B^{\blacklozenge}}$ such that $\theta_B \circ f = e$. Thus we get the algebra morphism $(\underline{f})^{\mathrm{op}} : (\underline{B^{\blacklozenge}})^{\mathrm{op}} \to (\underline{E})^{\mathrm{op}}$. If we regard $(\underline{E}, e)$ as the epi-induced object $\left(\overline{E^{\mathrm{op}}} = \underline{E}^{\mathrm{op}}, e^{\mathrm{op}}\right)$ of $\overline{B}$, we can rewrite this morphism as $\overline{f^{\mathrm{op}}} : \overline{L}\,\overline{B} \to \overline{E^{\mathrm{op}}}$ where $\overline{f^{\mathrm{op}}} = (\underline{f})^{\mathrm{op}}$. Then, if we recall that the canonical projection $p : LB \to \Omega \overline{L}\,\overline{B}$ is just $p = \theta_B^{\mathrm{op}}$ and we apply to $\overline{f^{\mathrm{op}}} : \overline{L}\,\overline{B} \to \overline{E^{\mathrm{op}}}$ the following adjunction, where $\mathcal{A}^{\mathrm{op}} = \mathcal{B} = \mathsf{Vec}_G^{\alpha}$,

$$\mathrm{Hom}_{\mathsf{Alg}(\mathcal{A})}\left(\overline{L}\,\overline{B}, \overline{E^{\mathrm{op}}}\right) \cong \mathrm{Hom}_{\mathsf{Alg}(\mathcal{B})}\left(\overline{B}, \overline{R}\,\overline{E^{\mathrm{op}}}\right) : h \mapsto \overline{R}h \circ \overline{\eta}_{\overline{B}},$$

we get the algebra morphism $\overline{\tau_B} := \overline{R}\,\overline{f^{\mathrm{op}}} \circ \overline{\eta}_{\overline{B}} : \overline{B} \to \overline{R}\,\overline{E^{\mathrm{op}}}$. To get a better description of this morphism, recall that the unit $\overline{\eta}$ of $(\overline{L}, \overline{R})$ is given by $\Omega \overline{\eta}_{\overline{B}} = R' \kappa_{\overline{B}} \circ \eta'_B$ where $R' := R\Omega$ and $\eta' := R\alpha L \circ \eta$ (see the proofs of Proposition 2.2 and Theorem 2.5). Moreover, by Proposition 2.4, the morphism $p : LB \to \Omega \overline{L}\,\overline{B}$ can be written in terms of $\kappa_{\overline{B}} : TLB \to \overline{L}\,\overline{B}$ as $p = \Omega \kappa_{\overline{B}} \circ \alpha_{LB}$. Therefore we get

$$\Omega \overline{\eta}_{\overline{B}} = R\Omega \kappa_{\overline{B}} \circ R\alpha_{LB} \circ \eta_B = Rp \circ \eta_B = \theta_B^{\lozenge} \circ \eta_B : B \to R\Omega \overline{L}\,\overline{B} = \Omega \overline{R}\overline{L}\,\overline{B} = B^{\blacklozenge\lozenge},$$

and hence

$$\tau_B = \Omega \overline{\tau_B} = R(f^{\mathrm{op}}) \circ \Omega \overline{\eta}_{\overline{B}} = f^{\lozenge} \circ \theta_B^{\lozenge} \circ \eta_B = \left(\theta_B \circ f\right)^{\lozenge} \circ \eta_B = e^{\lozenge} \circ \eta_B : B \to E^{\lozenge}.$$

Since $\overline{\tau_B}$ is a G-graded algebra map, then the kernel of the map τ_B, say I, is obviously a G-graded ideal of $\overline{B}$. Consider the following exact sequence

$$0 \to I \xrightarrow{i_I} B \xrightarrow{p_I} \frac{B}{I} \to 0.$$

Since it is a sequence in Vec_G -which is a semisimple category- applying the contravariant functor $(-)^{\lozenge}$, we get the exact sequence

$$0 \to \left(\frac{B}{I}\right)^{\lozenge} \xrightarrow{p_I^{\lozenge}} B^{\lozenge} \xrightarrow{i_I^{\lozenge}} I^{\lozenge} \to 0. \tag{15}$$

Since $I = \mathrm{Ker}\left(\tau_B\right)$, there is a G-graded algebra injection $\lambda_I : \frac{B}{I} \to E^{\lozenge}$ such that

$$\lambda_I \circ p_I = \tau_B = e^{\Diamond} \circ \eta_B.$$

Since E is finite-dimensional, so is $E^{\Diamond}$ and hence B/I is, too. This shows that I has finite codimension whence $I \in \mathcal{I}_B^f$. Define the map

$$\chi_B := \left(E \xrightarrow{j_E} E^{\Diamond\Diamond} \xrightarrow{(\lambda_I)^{\Diamond}} \left(\frac{B}{I}\right)^{\Diamond} \right).$$

Note that χ_B is surjective as j_E is invertible and $\left(\lambda_I\right)^{\Diamond}$ is surjective. We compute

$$\left(p_I\right)^{\Diamond} \circ \chi_B = \left(p_I\right)^{\Diamond} \circ \left(\lambda_I\right)^{\Diamond} \circ j_E = \left(\eta_B\right)^{\Diamond} \circ e^{\Diamond\Diamond} \circ j_E = \left(\eta_B\right)^{\Diamond} \circ j_{B^{\Diamond}} \circ e = e$$

so that the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow[\cong]{\chi_B} & \left(\frac{B}{I}\right)^{\Diamond} \\ & \searrow^{e} & \downarrow{\scriptstyle p_I^{\Diamond} = (p_I)^{\Diamond}} \\ & & B^{\Diamond} \end{array}$$

From this diagram, since e is an inclusion, we deduce that χ_B is injective. Since we already know that χ_B is surjective, we get that χ_B is invertible and hence $E = \mathrm{Im}(e) = \mathrm{Im}\big(p_I^{\Diamond}\big)$. This proves, in view of Lemma 4.14, that $B^{\blacklozenge} \subseteq \sum_{I \in \mathcal{I}_B^f} \mathrm{Im}\big(p_I^{\Diamond}\big)$.

Conversely, let $I \in \mathcal{I}_B^f$ and let us check that $\mathrm{Im}\big(p_I^{\Diamond}\big)$ belongs to $\mathcal{S}_B^f$. Note that $\varphi_2(B/I, B/I) : (B/I)^{\Diamond} \otimes (B/I)^{\Diamond} \to (B/I \otimes B/I)^{\Diamond}$ is an injective map between spaces with the same dimension as $(B/I)^{\Diamond} = (B/I)^*$ and $(B/I \otimes B/I)^{\Diamond} = (B/I \otimes B/I)^*$, B/I being finite-dimensional. As a consequence $\varphi_2(B/I, B/I)$ is invertible. Thus we can define a unique $\Delta_{(B/I)^{\Diamond}}$ such that the following diagram commutes

$$\begin{array}{ccc} \left(\frac{B}{I}\right)^{\Diamond} & & \\ \downarrow{\scriptstyle \Delta_{\left(\frac{B}{I}\right)^{\Diamond}}} & \searrow^{\left(m_{\frac{B}{I}}\right)^{\Diamond}} & \\ \left(\frac{B}{I}\right)^{\Diamond} \otimes \left(\frac{B}{I}\right)^{\Diamond} & \xrightarrow{\varphi_2\left(\frac{B}{I}, \frac{B}{I}\right)} & \left(\frac{B}{I} \otimes \frac{B}{I}\right)^{\Diamond} \end{array}$$

By using the definition of $\Delta_{(B/I)^{\Diamond}}$ and the naturality of φ_2, one obtains that

$$\begin{array}{ccccc} \left(\frac{B}{I}\right)^{\Diamond} & \xrightarrow{(p_I)^{\Diamond}} & B^{\Diamond} & \xrightarrow{(m_B)^{\Diamond}} & (B \otimes B)^{\Diamond} \\ \downarrow{\scriptstyle \Delta_{(B/I)^{\Diamond}}} & & & & \downarrow{\scriptstyle 1_{(B \otimes B)^{\Diamond}}} \\ \left(\frac{B}{I}\right)^{\Diamond} \otimes \left(\frac{B}{I}\right)^{\Diamond} & \xrightarrow{(p_I)^{\Diamond} \otimes (p_I)^{\Diamond}} & B^{\Diamond} \otimes B^{\Diamond} & \xrightarrow{\varphi_2(B,B)} & (B \otimes B)^{\Diamond} \end{array}$$

This means that $\left((B/I)^{\diamond}, \Delta_{(B/I)^{\diamond}}, p_I^{\diamond}\right)$ is a good G-graded vector space of $B^{\diamond}$ and hence $\mathrm{Im}\left(p_I^{\diamond}\right)$ is a good G-graded subspace of $B^{\diamond}$. Thus $\mathrm{Im}\left(p_I^{\diamond}\right)$ becomes an object in $\mathcal{S}_B^f$. Summing up we proved that $B^{\blacklozenge} = \sum_{I\in\mathcal{I}_B^f} \mathrm{Im}\left(p_I^{\diamond}\right)$.

In order to arrive at our goal, we now give another description of $\mathrm{Im}\left(p_I^{\diamond}\right)$. Let $\xi_X : X^{\diamond} \to X^*$ denote the canonical injection. Note that, by the commutativity of the following diagram

$$\begin{array}{ccc} B^{\diamond} & \xrightarrow{i_I^{\diamond}} & I^{\diamond} \\ {\scriptstyle \xi_B}\downarrow & & \downarrow{\scriptstyle \xi_I} \\ B^* & \xrightarrow{i_I^*} & I^* \end{array}$$

and the injectivity of ξ_I, we get the following alternative description

$$\mathrm{Im}\left(p_I^{\diamond}\right) \overset{(15)}{=} \mathrm{Ker}\left(i_I^{\diamond}\right) = \mathrm{Ker}\left(\xi_I \circ i_I^{\diamond}\right) = \mathrm{Ker}\left(i_I^* \circ \xi_B\right) = \xi_B^{-1}\left(\mathrm{Ker}\left(i_I^*\right)\right).$$

Therefore

$$\begin{aligned} B^{\blacklozenge} &= \sum_{I\in\mathcal{I}_B^f} \mathrm{Im}\left(p_I^{\diamond}\right) = \sum_{I\in\mathcal{I}_B^f} \xi_B^{-1}\left(\mathrm{Ker}\left(i_I^*\right)\right) \overset{(*)}{=} \xi_B^{-1}\left(\sum_{I\in\mathcal{I}_B^f} \mathrm{Ker}\left(i_I^*\right)\right) \\ &\overset{(*)}{=} \xi_B^{-1}\left(\bigcup_{I\in\mathcal{I}_B^f} \mathrm{Ker}\left(i_I^*\right)\right) = \left\{f \in B^{\diamond} \mid \xi_B(f) \in \bigcup_{I\in\mathcal{I}_B^f} \mathrm{Ker}\left(i_I^*\right)\right\} \end{aligned}$$

where in $(*)$ we are using that $\{\mathrm{Ker}\left(i_I^*\right) \mid I \in \mathcal{I}_B^f\}$ is a direct set of subobjects of B^* i.e., given $I, J \in \mathcal{I}_B^f$, there is $K \in \mathcal{I}_B^f$ such that $\mathrm{Ker}\left(i_I^*\right) \subseteq \mathrm{Ker}\left(i_K^*\right) \supseteq \mathrm{Ker}\left(i_J^*\right)$, namely $K = I \cap J$, see [30, Theorem 8.6(4)] .

In conclusion, noting that $\xi_B(f) \in \mathrm{Ker}\left(i_I^*\right)$ if and only if $\xi_B(f)$ vanishes on I, we get $B^{\blacklozenge} = \left\{f \in B^{\diamond} \mid \xi_B(f) \text{vanishes on some } I \in \mathcal{I}_B^f\right\}$. □

In conclusion, we got an explicit analogue of Sweedler's finite dual in Vec_G. More generally, having in mind that Vec_G can be regarded as the category of comodules over the group-algebra kG, we expect that one could carry out computations as in Vec_G for the category of comodules over a coquasi-bialgebra.

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References

1. Abe, E.: Hopf algebras. Cambridge Tracts in Mathematics, **74**. Cambridge University Press, New York, (1980)
2. Agore, A.L.: Limits of coalgebras, bialgebras and Hopf algebras. Proc. Amer. Math. Soc. **139**(3), 855–863 (2011)
3. Abdulwahid, A., Iovanov, M.: Generators for comonoids and universal constructions. Arch. Math. **106**, 21–33 (2016)
4. Aguiar, M., Mahajan, S.: Monoidal functors, species and Hopf algebras. CRM Monograph Series, **29**. American Mathematical Society, Providence, RI, (2010)
5. Andruskiewitsch, N., Angiono, I., Bagio, D.: Examples of pointed color Hopf algebras. J. Algebra Appl. **13**, 1350098 (2014)
6. Ardizzoni, A., El Kaoutit, L., Menini, C.: Coendomorphism Bialgebroid and Chain Complexes, https://arxiv.org/abs/1004.4572v1
7. Ardizzoni, A., Goyvaerts, I., Menini, C.: Pre-rigid Monoidal Categories, http://arxiv.org/abs/2201.03952
8. Abuhlail, J.W., Gómez-Torrecillas, J., Wisbauer, R.: Dual coalgebras ofalgebras over commutative rings. J. Pure Appl. Algebra **153**, 107–120 (2000)
9. Ardizzoni, A., Menini, C.: Milnor-Moore categories and monadic decomposition. J. Algebra **448**, 488–563 (2016)
10. Ardizzoni, A., Menini, C.: Monadic vs adjoint decomposition. J. Pure Appl. Algebra **226**, 106945 (2022)
11. Adámek, J., Rosický, J.: Locally presentable and accessible categories. London Mathematical Society Lecture Note Series, **189**. Cambridge University Press, Cambridge, (1994)
12. Barr, M.: Coalgebras over a commutative ring. J. Algebra **32**(3), 600–610 (1974)
13. Bénabou, J.: Introduction to bicategories. 1967 Reports of the Midwest Category Seminar p. 1-77. Springer, Berlin
14. Borceux, F.: Handbook of categorical algebra. 1. Basic category theory. Encyclopedia of Mathematics and its Applications, **50**. Cambridge University Press, Cambridge, (1994)
15. Chen, C.Y., Nichols, W.D.: A duality theorem for Hopf module algebras over Dedekind rings. Comm. Algebra **18**, 3209–322 (1990)
16. Dubuc, E. J.: Adjoint triangles. 1968 Reports of the Midwest Category Seminar, II p. 69-91. Springer, Berlin
17. Dubuc, E.J.: Kan extensions in enriched category theory. Lecture Notes in Mathematics, vol. 145. Springer-Verlag, Berlin-New York (1970)
18. Goyvaerts, I., Vercruysse, J.: On the duality of generalized Lie and Hopf algebras. Adv. Math. **258**, 154–190 (2014)
19. Herrlich, H., Strecker, G.E.: Coreflective subcategories. Trans. Amer. Math. Soc. **157**, 205–226 (1971)
20. Maranda, J.-M.: Injective structures. Trans. Amer. Math. Soc. **110**, 98–135 (1964)
21. Mac Lane, S.: Categories for the working mathematician. Second edition. Graduate Texts in Mathematics, **5**. Springer-Verlag, New York, (1998)
22. Michaelis, W.: Lie coalgebras. Adv. Math. **38**, 1–54 (1980)
23. Mitchell, B.: Theory of categories. Pure and Applied Mathematics, Academic Press, USA (1965)
24. Nastasescu, C., Van Oystaeyen, F.: Graded and filtered modules. Lecture Notes in Mathematics, vol. 758. Springer, Berlin Heidelberg (1979)
25. Pareigis, B.: Categories and functors . Translated from the German. Pure and Applied Mathematics, Vol. **39** . Academic Press, New York-London (1970)
26. Pareigis, B.: Non-additive Ring and Module Theory I. General theory of monoids. Publ. Math. Debrecen **24**, p. 189-204 (1977)
27. Porst, H.-E.: On corings and comodules. Arch. Math. (Brno) **42**(4), 419–425 (2006)
28. Porst, H.-E.: On Categories of Monoids, Comonoids, and Bimonoids. Quaestiones Math. **31**, 127–139 (2008)
29. Porst, H.-E.: Dual adjunctions between algebras and coalgebras. Arab. J. Sci. Eng. Sect. C **33**, 407–411 (2008)
30. Popescu, N.: Abelian categories with applications to rings and modules. London Mathematical Society Monographs, No. **3**. Academic Press, London-New York, (1973)
31. Porst, H.-E., Street, R.: Generalizations of the Sweedler dual. Appl. Categ. Structures **24**(5), 619–647 (2016)
32. Saavedra Rivano, N.: Catégories Tannakiennes. (French) Lecture Notes in Mathematics, Vol. **265**. Springer-Verlag, Berlin-New York, (1972)

33. Sweedler, M.E.: Hopf algebras. Mathematics Lecture Note Series., W. A. Benjamin Inc, New York (1969)
34. Tambara, D.: The coendomorphism bialgebra of an algebra. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **37**, 425–456 (1990)

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