

# Monomials and polynomials: the long march towards a definition

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**The process of pairing a name with representations or peculiar properties permeates many mathematics classroom situations. In school, many practices go under the label ‘definition’, even though they can be very different from what mathematicians conceive as a formal definition, and in fact there are substantial differences between these different kinds of practices. This can create difficulties for pupils, since they may encounter non-coherent definitions of the same thing. Features of these schemes are exemplified through the analysis of some algebra high school textbooks, looking at the proposed ‘definitions’ of monomials and polynomials. We show how all these definition schemes are used in textbooks. Starting from the observed differences, we highlight some educational research problems.**

## I. Introduction

Even in very early grades, the approach to mathematical discourse requires a knowledge about mathematical ‘objects’, which are involved in the discourse, and their names.<sup>1</sup> Definitions of mathematical terms are provided but defining is a process ruled by specific mathematical norms. Furthermore, the act of introducing names plays a double role in the construction of a formal theory: one can introduce the basic names used in the theory (as was the case for *primitive concepts* such as point and straight line in Euclid’s setting or in the late XIX century debate that led to the turning point of Hilbert’s *Grundlagen der Geometrie*, see f.i. Bolondi, 2002) or one can use those basic names to create derived ones, and so on. Hence, knowledge of one kind of definition seems fundamental for the construction of a new one within the same mathematical theory. This problem has been described by Sfard (2008) as a paradox:

*For a future mathematicist the self-generating nature of mathematical discourse creates a paradoxical situation: one’s familiarity with what the discourse is all about seems to be a precondition for participation in the discourse. But, at the same time, such familiarity can only emerge from this participation!* (p.130)

<sup>1</sup> When we use the locution ‘mathematical object’ we do not assume any explicit epistemological or philosophical position, f.i. a platonic one, as will be later clarified.

The role of ‘starting points’ for the chain of formal definitions, which in Euclid was performed by the anomalous ‘definitions’ of point and line (see f.i. a discussion with educational implications in [Arzarello et al., 2012](#)), is assumed in a modern axiomatic setting by the *implicit definitions*. A classical non-technical discussion is contained in the work by [Enriques \(1906\)](#) and in a more modern contribution by [Freudenthal \(1962\)](#).

The process of pairing a name with representations or properties occurs at all levels of mathematical education, even though ‘definition’ in a strictly mathematical sense is often considered appropriate only in higher grades. In the literature, it is possible to find several terms that refer to the dichotomy between the formal and rigorous definitions belonging to scientific mathematical discourse and knowledge about the essence of a mathematical object (in the sense of [Borasi, 1991](#)) shared in a mathematics classroom, often based on a pupil’s perceptual experiences, a teacher’s knowledge about mathematics and textbook contents. [Vinner \(1976\)](#) juxtaposes a *lexical* approach with a *formal* one, while [Tall & Vinner \(1981\)](#) refer to *concept definition* and *concept image*, assigning them the adjectives *personal* (belonging to the pupil) and *formal* (belonging to the mathematical scientific community). Another distinction is made by [Freudenthal \(1973\)](#) between *descriptive* and *constructive* definitions. In descriptive defining, the concept image precedes the eventual concept definition, whereas in constructive defining the concept image is developed or explored after the concept definition.

[Mariotti & Fischbein \(1997\)](#) talk about *figural* and *conceptual* aspects in the particular case of geometrical shapes. In this sense, we can assume that, in mathematics education research into definitions, there is an agreement about the presence of a ‘starting point’ and a ‘goal’.

Despite these significant differences in terminology, the mathematics education community agrees on the characterizing features and criteria of a formal definition. For instance, some commonly accepted requirements for mathematical definitions are non-contradictory, non-circularity, precision in terminology, essentiality ([Borasi, 1991](#)), unambiguity, logical equivalence to other definitions and invariance under changes of representation ([Zaslavsky & Shir, 2005](#); [Harel et al., 2006](#)).

The authors agree on these criteria but, looking at this list, a question arises: How can we describe and characterize the possible actions intended to introduce a mathematical term that does not conform to these criteria? Many examples of these actions, as we will show, already exist in mathematical educational practice.

While asking this question, we are referring not just to personal meanings or images but also to those definitions that acquire a ‘formal status’ because they are given by teachers or textbooks. The aim of this paper is to develop a framework to characterize these actions.

The proposed framework is then used to analyse definitions drawn from some textbooks. In contrast with the work by [Morgan \(2005\)](#), who used linguistic tools, we focus on logical aspects such as the nature of *definiendum* and *definiens* (‘defined’ and ‘definer’) and the use of representations and properties that might be considered, in a more formal setting, necessary and/or sufficient.

The choice of *polynomials* as a case study is motivated mainly, as we will discuss, by their particular status in the school curriculum and in the advanced mathematical curriculum. Indeed, in school practices they lean heavily on existing knowledge of properties of numbers and on the use of analogies and—not always explicit—extensions. In school, the introduction of polynomials is often based on implicit facts. In advanced mathematical teaching, polynomials are *formal* objects, hence explicitly recalling in their definition both an abstract form of representation and formal properties.

## 2. Theoretical lenses

In the previous section we used words such as *name*, *properties*, *representation* or *object*, exploiting the meaning that these words can have in common usage. Here we try to specify the operational

meaning of each of the key terms of our framework. The adopted theoretical lenses do not constitute a theoretical framework as a whole. We follow the research trend for innovation as defined by [Arzarello & Bussi \(1998\)](#). Indeed, we do not claim to define a global coherent theoretical framework; rather, we borrow instruments from various theoretical approaches and we apply them as a kit of tools. Local coherence of the framework is assumed as necessary and is tested through the analysis, but global coherence is considered as difficult to attain at this moment of the research. In fact, these theoretical tools are here tested and this process of testing is part of the construction of the theoretical framework itself. In our particular case, the analysis concerns the text, without addressing their cognitive counterpart and this assures that there are not incompatibilities between the selected theoretical tools.

Let us move on to examine these theoretical tools in detail: we consider ‘*name*’ to refer to any noun. A ‘*property*’ is a sentence containing a name related to an adjective and/or to other names (potentially just one). We class ‘*representation*’ as any visual mediator (in the sense of [Sfard, 2008](#)), meaning both symbols and icons.

By classroom ‘*mathematical object*’ we refer to a name related to one or more representations, associated to one or more properties and/or other names.

It should be noted that contemporary debate on the nature of mathematical *entes*, to use a term from [Benacerraf \(1973\)](#), developed quite independently in the mathematics education community and in the mathematics community *stricto sensu* (e.g. [Thom, 1973](#)). In the first, a semiotic approach (with many possible variants, see f.i. [Eco, 1973](#), for a synoptic table) is adopted, and it is not generally acceptable to speak about the *Objekt* (Peirce), the *Denotatum* (Morris) or the *Extension* (Carnap) without referring to the other vertices of the semiotic triangle. Despite this fact, the authors of school textbooks seem to share a strong ontological faith. In the textbooks we examined, the authors have (and try to communicate) an implicit platonic attitude. The *objects* ‘polynomials’ and ‘monomials’ to which they refer are (in their intentions) those existing in the super-celestial world of Academic Mathematics. The didactic problem is that in that world the *objects* emerge through a shared and (historically) established process where (formal) definitions play a crucial role in giving sense and legitimacy to representations and practices. A polynomial is therefore an element of a ring of formal expressions and the ‘name-thing’ pairing develops at a higher level of formalization and with a higher order of logical organization. Monomials are particular cases of polynomials. This level is of course inapproachable in a school classroom. Hence, most authors start with monomials and then build with operations what they hope will become a formal structure. Our analysis tries to describe how this evolves. In synthesis, what is associated to the word ‘polynomial’ (and what can be done with it) in an advanced mathematical setting is ruled by a definition as it is used in a ‘standard’ logical organization of a mathematical theory. Textbook authors seems to refer to this ‘object’ implicitly, but the final taught ‘object’ is in fact dependent on complex dynamics in which the definition schemes used may play an important role.

In analogy with the work of [Harel & Sowder \(1996\)](#) on *proof schemes*, we describe four different kinds of *definition schemes*. With this term, we refer to a process of pairing a name to a representation, a set of properties and/or other names. Hence, as for proof schemes, also a definition scheme ‘is idiosyncratic and naturally can vary from time to time and from context to context, even within mathematics’ (*ibidem*).

Working on the distinction made by [D’Amore & Fandiño \(2012\)](#)<sup>2</sup> between *designation*, *denotation*, *description* and *denomination* we characterize, for each of these schemes, the peculiar interplay among representations, properties and names Fig. 1.

<sup>2</sup> The cited authors offer a list of six terms, including *definition* and *proof*. In this paper we are considering just four of these terms .

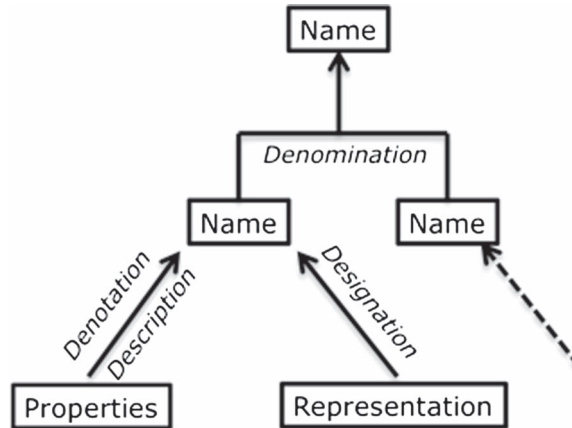


FIG. 1. Relationships between name, properties or representation in different definition schemes.

### 2.1 Designation

We say that we are in presence of a *designation* when the name is associated to a representation. Designation does not need explicit reference to properties; hence, the understanding of this kind of scheme is context-dependent. Briefly, we can say that we *designate* a representation whenever a name is linked to the representation itself. For instance, Fig. 2 is a designation because a drawn quadrilateral is associated with the word ‘rectangle’.

### 2.2 Denotation

We use the term *denotation* when one or more properties are associated to the same name; these properties are necessary but not sufficient, hence it is not a definition because it does not characterize the object as conceived by mathematicians. We can say that a denotation states what the defined object is not: if something has not these properties, then we cannot use this name to label it. For instance, we have a denotation in the following sentence about a rectangle:

‘A rectangle is a quadrilateral whose diagonals divide themselves in two equal parts.’

That means that a quadrilateral without this property about its diagonals cannot be called a rectangle.

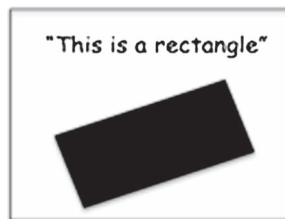


FIG. 2. Example of designation.

### 2.3 Description

A *unenumerate* is conceived as a list of properties, which may include non-necessary properties; they must be sufficient but may also be non-necessary, even superfluous. We can find an example of description in the following sentence:

‘A rectangle is a parallelogram whose angles are right angles and that has congruent diagonals.’

So, a description is not a formal definition because it is not essential (Borasi, 1991).

Hence, both denotation and description are lists of properties, but they are different according to mathematics experts’ interpretation in terms of being sufficient and necessary.

### 2.4 Denomination

‘An equiangular quadrilateral is called rectangle.’

We consider this last sentence as a *denomination*, which refers to two, or more, already-known names (in this case equiangular polygons and quadrilateral polygons) that are baptized with one shared new name. It has to be observed that, in denominations, properties can be condensed into single words. A denomination can be a definition in the formal sense.

Sometimes it is not possible to clearly discern between two definition schemes; in our analysis we consider prevalent aspects.

## 3. Analysis of textbooks

In order to understand how definition schemes operate in practice, we focus our attention on definitions presented in some high school textbooks, written in different periods, for grade 9–11 students. We have selected popular recent and past Italian textbooks and, for a comparison with textbooks from other countries, we have chosen to analyse a Spanish and a US textbook because these languages are understandable to the authors. Textbooks have been analysed in original languages; excerpts were then translated in English for this paper. We want to stress that any textbook would be suitable for our purpose. Indeed, we do not want to generalize our observations. We are not looking for general results, but choose to test the relevance of the framework.

Each textbook of our case study has been labelled using a code as follows:

- (1) **IT1** Bergamini, Trifone, Barozzi—Matematica blu 1—Zanichelli: 2010.
- (2) **IT2** Fraschini, Grazi—nuova ALGEBRA 1—Atalas: 2005.
- (3) **IT3** Bernardo, Mocchetti, Carboncini—Matematica C3 Algebra 1—Matematicamente.it: 2012.
- (4) **IT4** De Marco, Leso—Algebra—Poseidonia: 1979.
- (5) **IT5** Re Fraschini M., Grazi G.—I principi della matematica 1—Atlas: 2011.
- (6) **IT6** Sasso L.—Nuova Matematica a colori 1—Petrini: 2011.
- (7) **IT7** Oriolo P., Coda A.—Algebra 1—Bruno Mondadori: 1983.
- (8) **USA** Gantert—Integrated Algebra 1—AMSCO school publications: 2007
- (9) **SPA** Colera, Garcia, Gaztelu, Oliveira—Matemáticas 3—Anaya: 2005

In order to circumscribe the object of analysis, we decide to look at definition schemes used for a particular content: ‘*polynomials*’. However, to define polynomials, many terms are used, for instance *variable*, *algebraic expression*, *literal equation*, *algebraic sum* and *monomials*. For this reason we decided also to analyse the definition schemes used in these other cases.

TABLE 1. Classification of definition schemes for terms under examination in our sample of textbooks. ‘Des’ stays for ‘designation’, ‘Denom’ for ‘denomination’, ‘Denot’ for ‘denotation’ and ‘Descr’ for ‘description’

	IT1	IT2	IT3	USA	IT4	IT5	IT6	IT7	SPA
<b>Variable</b>	Des		Des	Des			Des		Denom + Des
<b>Algebraic sum</b>	Denom	Denom	Denom	Denom	Denom	Denom	Denom	Denom	Denom
<b>Literal (algebraic) expression</b>	Des	Denot	Descr	Descr	Des	Descr	Denot		Descr
<b>Monomial</b>	Denot	Denot	Denot + Descr	Denot + Descr + Des	Denot + Descr	Denot + Descr + Des	Des + Descr + Denot	Des +	Descr
<b>Polynomial</b>	Denot + Descr	Des + Descr	Denot + Descr	Denom	Descr	Descr	Des + Descr + Denot	Des	Des + Descr

Definition schemes are coded according to the criteria outlined in the previous section. Coding was performed separately by the three authors. There was an agreement on 90% of codes; the other cases were discussed until consensus was reached between all three authors. A review of the whole analysis is shown in Table 1; as can be seen, some of the terms cited above are not included in some textbooks.

A paradigmatic example for each of the categories is provided below. In order to define a *variable*, designation is generally used. Indeed, a variable is almost always introduced starting from a representation. We can find an instance in IT1, which firstly gives an example of usage of the letter  $n$  as a variable to refer to a generic natural number, before declaring that

*when we want to refer to a generic number, we hence use a letter from the alphabet. To this letter is given the name of a **numerical variable** (briefly, **variable**); in the previous example  $n$  is a variable.* (IT1, p.3)<sup>3</sup>

For *Algebraic expression* we found the widest variety of names and properties involved in the definition schemes. The first substantial difference is the name: sometimes it is called ‘algebraic literal expression’ but is also referred to as simply ‘algebraic expression’ or ‘literal expression’. Although the names vary, we can see from the given examples in our sample of textbooks that they refer to the same representation.

<sup>3</sup> Translations from Italian and Spanish are made by this paper’s authors.

Other elements of disagreement can be found in the primitive terms that are used in the different definition schemes (*calculation scheme, writing, variable, set . . .*).

For instance, in IT6 an algebraic expression is denoted as follows:

*every writing in which numbers and letters appear (possibly in brackets) linked by operations symbols.*

It is a denotation because given conditions are necessary, but not sufficient, to define an algebraic literal expression. As a matter of fact, the expression

$$A + ) K - n 9 + : M f , (7)$$

might be an algebraic expression according to this denotation, but would not be considered as an algebraic expression by a mathematician. By the definition scheme used by IT6, the student can understand that an absence of letters and numbers means you are not looking at an algebraic expression, but nothing is said about how operations or symbols should link such letters and numbers when they do exist.

*Monomials* are introduced mostly through lists of properties that are both necessary and sufficient. In some textbooks, only a description or a denomination is used. In the Spanish textbook it is written:

*Monomial is the indicated product between a number and one or more letters.* (SPA, p. 88)

In this textbook, a description is used. It allows identification of some kinds of monomials because it provides a condition that is sufficient but not necessary. According to this definition both ‘2’ and ‘a’ cannot be considered as monomials, without further specifications. As we will observe below, numbers and single letters have to be considered as monomials in order to have accordance between the definitions of polynomials and the given examples. In IT1 a denotation is used to introduce monomials:

*A monomial is a literal expression in which, between letters, only multiplications and powers appear. The exponents of letter are natural numbers.* (IT1, p. 291)

This definition scheme is a denotation because it is necessary that only multiplications appear between letters, and also between letters and numbers, otherwise  $2 + ab$  would be considered as a monomial. We underline that this textbook explicitly specifies the fact that exponents are natural numbers; hence, division is explicitly excluded. Indeed, we notice that the possibility of considering negative exponents is not always discussed or explained.

All the definition schemes for *algebraic sum* that are given in our sample of textbooks are of the ‘denomination’ kind:

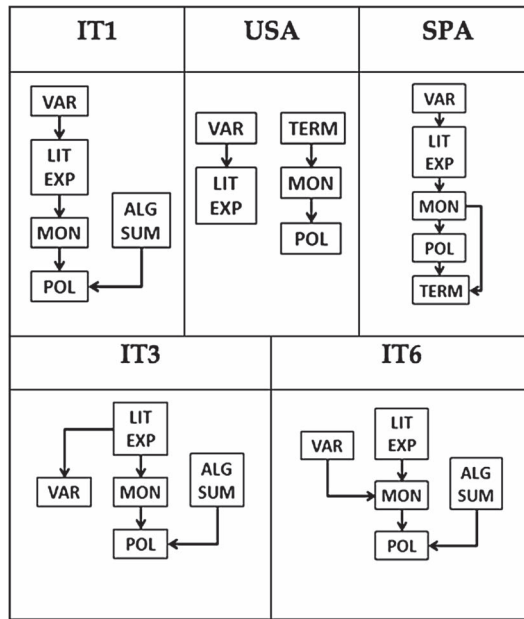
*A subtraction can always be transformed into an addition; hence, to refer to sum or difference between two monomials, we generally speak about algebraic sum of monomials.* (IT2, p. 283)

As can be noticed, algebraic sum is defined by starting from two previously known operations (addition and subtraction), which are unified under the same name. To use Anna Sfard’s expression, we can say that it is a process of *saming*, i.e.

*by assigning one signifier (giving one name) to a number of things that, so far, have not been considered as in any way ‘the same’ but are mutually replaceable in a certain closed set of narratives.* (Sfard, 2008, p. 170)



TABLE 2. Examples of definition paths from analysed textbooks



Finally, many textbooks introduce polynomials with both necessary and sufficient conditions but that depend on terms previously defined. We report an example from the US textbook:

*A monomial or the sum of monomials is called a **polynomial**. A polynomial may have one or more terms.* (USA, p. 169) (Here ‘terms’ means the monomials.)

This definition scheme differs from those in Italian textbook because it specifies that monomials are also polynomial.<sup>4</sup> In other textbooks, a polynomial is defined as the algebraic sum of monomials, but the classification of monomials as polynomials is justified by considering the sum of a monomial and the null monomial.

The presence of blank spaces in Table 1 suggests that textbooks use not only different definition schemes but also different *definition paths* in the construction of *polynomial*. Even where we find the same *definiendum*, the *definiens* can be different. A second analysis has been conducted, looking for names used in the different definition schemes. Some results of said analysis are summarized in Table 3, in which two names are linked by an arrow if the first one is one of *definientes* (definers) of the second one. No arrow is included when a definition scheme does not recall any of the other names that we analysed.

As can be seen from the paths in the textbooks analysed, the construction process of polynomials is not straightforward: relationships between *definiens* and *definiendum* are not always the same, and the terms used are not the same throughout all our textbooks. For instance in IT4 there is a distinction between

<sup>4</sup> It can be observed that also in this case a process of ‘saming’ is used, as highlighted for algebraic sum.



**ESEMPL. Ecco alcuni monomi interi e alcuni monomi frazionari:**

$3ab^2$	$a^5m^3$	$-xy$	$\frac{2}{5}ab^3c$	$\frac{x}{3y}$	$\frac{9a^2}{4bx}$
monomio intero	monomio intero	monomio intero	monomio intero	monomio frazionario	monomio frazionario

FIG. 3. Examples of what a monomial is, in IT4.

**ESEMPIO**

**Non sono monomi:**

$3\frac{x}{y}$ ,  $2(a+b)$ ,  $4a^3 - b^2$ ,  $\frac{x-y}{2a}$

FIG. 4. Examples of what a monomial is not, in IT1.

‘integer monomial’ and ‘monomial’. Furthermore, the same word is used with different meanings in different textbooks. This kind of ambiguity can be identified even in textbooks from the same country. For instance, in IT4 a monomial is defined as follows:

*A literal expression is called a monomial when only operations of multiplication, division and power are indicated. (p.75)*

However, in the majority of Italian textbooks, the operation of division is excluded from the definition of monomials; what in IT4 is used as an example of monomial is used in IT1 as an example of what is not a monomial (Figs 3–4). We recall that, in a previously examined example, powers in monomials were explicitly required to have natural exponents.

The same problem can be observed in the definition schemes of algebraic sum, a term which is sometimes used to talk about the sum between the coefficients of monomials (referring to sum in  $\mathbb{Z}$ ) and sometimes as an operation on the set of monomials.

Our analysis sheds light on some peculiarities of definition schemes and paths for polynomials and some other related terms. In different textbooks there are several analogies but also substantial differences; in the next section, some possible didactic implications are discussed.

#### 4. Conclusions and issues

We presented four different types of definition schemes and, through the analysis of some textbooks, we showed that all these definition schemes are present and that it is possible to distinguish between them. The particular case of polynomials allowed us to exemplify features that characterize each of the definition schemes presented in the framework section of this paper.

The fact that different practices are presented in textbooks raises several questions of interest for the mathematics education field. In particular, we can ask whether the different approaches (both in terms of

definition schemes and paths) have an impact on teacher practice and/or on student learning. Do different definitions schemes influence the process of *definition-construction* (Ouvrier-Bufferet, 2004)?

In accordance with foundational works by Freudenthal (1973), different authors stress the importance of ‘teaching to define’ instead of ‘teaching definitions’ (De Villiers, 1998). How far is students’ definition construction influenced by the definition schemes offered by teacher and textbooks? As Sfard (2008) remarks, the mathematical discourse (as with any other discourse) begins with a passive use of others’ sentences. If that which is termed ‘definition’ in textbooks is something different from what a mathematician would indicate with the same word, how can students really produce definitions? Zandieh & Rasmussen (2010) propose the *definition game* which consists of using unfamiliar definitions in familiar contexts or vice versa. For instance, Borasi (1991) described a lesson in which student had to use a familiar definition of circle in the unfamiliar mathematical setting of taxicab geometry. Ouvrier-Bufferet (2004) outlines different *Situations of Definition Construction*: starting from representation, re-defining in new contexts, and starting from wrong definitions. Can these different situations lead to the development of different definition schemes?

As we can read in De Villiers (1998),

*From the constructivist assumption that meaningful knowledge needs to be actively (re)- constructed by the learner, it also follows that students should be engaged in the activity of defining and allowed to choose their own definitions at each Van Hiele level. (p. 5)*

The author argues, in the case of geometry, that each different definition procedure may be significant at different Van Hiele levels and that, according to Van Hiele theory, Level 1 (Visualization) is often crucial to comprehension. Thus, a definition scheme such as designation (which appears to be ‘less formal’) also seems to be important from an educational point of view.

We can also notice that most educators’ work on definitions is related to geometry (Ouvrier-Bufferet 2004). In this paper we applied the same classification to geometrical definitions (in the framework section) and to algebraic ones. But, from our analysis of algebraic ‘definitions’ taken from textbooks, a question arises: Is it possible for students to ‘construct’ such definitions?

A reflection about what kind of definition schemes are used in the classroom appears to be useful for many reasons. If we let pupils use only one of the defining schemes, we run the risk of establishing certain conventions/customs proper to this part of the process of teaching/learning, specific sociomathematical norms (Yackel & Cobb, 1996).

For example, Sanchez & Garica (2014) show how drawing upon description can generate a kind of socio-mathematical norm that consists in assuming that in the mathematics classroom ‘the “good” response is that which contains more information, although this information may be superfluous or unnecessary’ (p. 314).

Moreover, let us imagine the scenario of a student who moves to a class in which definition schemes and paths differ from the ones she/he met in the previous class. Will she/he have difficulties in actively participating in the classroom discourse? If so, can examples and exercises (i.e. operational practices) help avoid these difficulties? How much does the chosen definition scheme affect the construction of concept image, compared with the formulation of exercises and problems on the same mathematical content? It is in the relation between practice and theory that Sfard (2008) finds a solution for the paradox presented in the introduction; briefly, we can summarize that solution with the expression ‘practice before theory’.

In this paper we give examples of different definition schemes from textbooks: the schemes used by teachers, their relationship with those of textbooks and their impact on students’ learning are all questions open to further studies.

Each of these issues should be investigated through an *ad hoc* experimental apparatus and we hope that our analysis can help in setting the framework needed for such investigations.

Finally, we wish to underline that we agree with Paola (2008) who, in using a metaphor by Eco (2014), describes knowledge as an encyclopaedia in which each object is not simply defined (as in a dictionary) but is related to other objects. Hence, the process of learning cannot be perceived comparably as simply a movement between the branches of a tree (as it may seem, looking at Table 3) but rather as a path through a labyrinth, influenced by the personal experiences of each student. As a consequence, even though ‘good’ definitions are very important, the importance of the teacher’s role as a guide through such a labyrinth is undeniable.

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