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Sobolev classes and bounded variation functions on domains of Wiener spaces, and applications

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Contents

Introduction

In this work, we present some new results: their main thread is the bounded variation functions in abstract Wiener space (which we will usually call Wiener space).

Here, by Wiener space, we mean a separable Banach space with a centered non-degenerate Gaussian measure γ: the canonical example is the classical Wiener measure on $C([0,1])$, which represents the standard Brownian motion (see e.g. [**14**] for more information); a Wiener space admits a particular subspace H, the Cameron-Martin space, such that, for every $h \in H$, the measure defined as $\gamma_h := \gamma(\cdot - h)$ is absolutely continuous with respect to γ; on H is well defined an inner product $\langle \cdot, \cdot \rangle_H$ which gives on H an Hilbert structure. We can define the spaces L^p, the Sobolev spaces $W^{1,p}$ with the H-derivatives ∂_h and a H-gradient ∇_H; by ∂_h^* we denote the formal adjoint of the h-derivative ∂_h and, on $W^{1,p}$, it is verified an integration by parts formula

$$\int_X \partial_h f g \, d\gamma = -\int_X f \partial_h^* g \, d\gamma.$$

By functions of bounded variation (BV), we mean a function f on (X, γ) which admits a vector measure $D_\gamma f$ such that an integration by parts formula is verified for every g regular

$$\int_X f \partial_h^* g \, d\gamma = -\int_X g \, d \left\langle D_\gamma f, h \right\rangle_H$$

(we refer to [**8, 7**] and al.); equivalently, a function f is in BV if there exists a sequence of functions $f_n \in W^{1,1}$ s.t. $f_n \to f$ in L^1 and

$$\limsup_{n \to +\infty} \int_X |\nabla_H f_n|_H \, d\gamma < \infty.$$

A set $A \subset X$ is said of finite perimeter if its characteristic function $1\!\!1_A$ is BV.

The topic of BV (bounded variation) functions in Wiener space has been studied for instance in [**42, 45, 7, 8, 18, 17, 51**]; we widely use the survey [**54**].

There are different possible definition of bounded variation on subsets of X; in this work we will follow a definition of functions of bounded variation on an open convex $O \subset X$ for X Wiener space, given in [**17**].

In this work, we also deal with the problem of defining Sobolev spaces on open subsets O of Wiener spaces: following for instance [**26**], [**51**], the main idea is to define the weak gradients on Lipschitz functions, and then to define $W^{1,p}(O)$ as the completion of the set of Lipschitz functions with respect to a norm $\|\cdot\|_{W^{1,p}(O)}$. In the case of O convex, an equivalent definition is that f is in $W^{1,p}(O)$ if it is absolutely continuous along the lines (almost everywhere) and if the weak gradient defined in this way is L^p. The proof of this fact is in [**14**] for $W^{1,2}(O)$; in this work (Proposition 3.2.23, Subsection 3.3.2,) the proof is reconstructed for $W^{1,p}$ with p generic.

A greater problem is to define $W_0^{1,p}(O)$, the functions with zero trace on O: we define a function in $W_0^{1,p}(O)$ as the limit of a sequence of regular functions which are null out of O. In [**26**], a particular kind of sets allows to define the trace as an operator from $W^{1,p}(O)$ to $L^p(\partial O)$ (for a measure on the boundary called Feyel-de La Pradelle), so in this setting a possible definition of $W_0^{1,p}(O)$ is the space of functions with null trace on the boundary; in this work (Chapter 6, Theorem 6.2.2) we prove, under certain conditions stronger than that of [**26**], that for $p=2$ the two definitions are equivalent.

The most of Chapter 7 is devoted to the extension in the Wiener spaces setting of a result (due to H. Brezis) given in the section 8 of the paper [**12**] by Barbu and Röckner. about the resolvent of the Laplacian in finite dimension: if O is a convex bounded set with regular boundary in $X=\mathbb{R}^d$ and L is the Laplace operator in O with null Dirichlet boundary conditions, if $\sigma>0$, $y\in W_0^{1,1}(O,\mathscr{L}^d)\cap L^2(O,\mathscr{L}^d)$ ($\mathscr{L}^d$ is the Lebesgue measure), and $u:=(I-\sigma L)^{-1}y$, then

$$\int_O |\nabla u(x)|\,dx \le \int_O |\nabla y(x)|\,dx. \tag{0.0.1}$$

We give some extensions. (0.0.1) is true also if L is the Laplace operator in O with null Neumann boundary condition (Section7.4) and $y\in W^{1,1}(O,\mathscr{L}^d)\cap L^2(O,\mathscr{L}^d)$. Moreover, if we replace the Lebesgue measure with a Gaussian measure γ (in $\mathbb{R}^d$ or in a infinite dimensional space X) and we substitute to L the Ornstein-Uhlenbeck operator (which takes the place of Laplace operator in Wiener spaces in many respects), we can find an equivalent of (0.0.1) both for null Dirichlet boundary conditions, and for null Neumann boundary conditions.

For Ornstein-Uhlenbeck operator with null Dirichlet boundary conditions (Section 7.2), we set a particular condition on O (Chapter 6), which we use to get Theorem 6.2.2 on $W_0^{1,2}(O)$; moreover, we impose a condition, that we could name *Gaussian convexity;* under this hypothesis, for $y\in W_0^{1,p}(O)\cap L^2(O)$ for some $p>1$, and $u:=(I-\sigma L)^{-1}y$ we have Theorem 7.3.7

$$\int_O |\nabla_H u|\,d\gamma \le \int_O |\nabla y|\,d\gamma. \tag{0.0.2}$$

For Ornstein-Uhlenbeck operator with null Neumann boundary conditions (Section 7.4), we can impose that O is convex, and we obtain Theorem 7.4.7 for every $y\in W^{1,1}(O)\cap L^2(O)$ the inequality (0.0.2).

We make use of this last result in Section 7.5; in that section, we want to find a characterization of bounded variation function on O which is equivalent to that in [**17**]; we get Theorem 7.5.11 that, under the hypothesis that $y\in L^2(O)$, it is BV if and only if (for $J_\sigma:=(I-\sigma L)^{-1}$)

$$\limsup_{\sigma\to 0}\int_O |\nabla_H J_\sigma(y)|_H\,d\gamma < +\infty$$

and if and only if there exists a sequence of functions $f_n\in W^{1,1}$ s.t. $f_n\to f$ in L^1 and

$$\limsup_{n\to+\infty}\int_O |\nabla_H f_n|_H\,d\gamma < \infty;$$

to prove this result, we need the equivalent definitions of $W^{1,p}(O)$ given in Proposition 3.2.23.

In Chapter 8, we consider $X:=C([0,1],\mathbb{R}^d)$ and $X_*:=C_*([0,1],\mathbb{R}^d)$ (continuous functions with starting point in 0) with the measure given by the Brownian motion with starting point in

$0 \in \mathbb{R}^d$, hence it is represented by a Gaussian measure. For every $\Omega \subseteq X$, we define $\Xi^*_\Omega := \{\omega \in X_* | \omega(t) \in \Omega \, \forall t \in [0,1]\}$.

In ([**46**], Thm. 5.1) it is proved that, if $d \geq 2$ and $\Omega \subset \mathbb{R}^d$ is an open set which satisfies a uniform outer ball condition then Ξ^*_A has finite perimeter in the sense of Gaussian measure.

In Chapter 8 we give a weaker condition on Ω (in dimension sufficiently large) such that Ξ_Ω has finite perimeter: in particular, Ω can be the complement of a symmetric cone.

In the first part of this work, we present the known results which are used to prove the above described results.

In Chapter 1, we recall some well-known notions about operators, semigroups and forms, and notions of measure theory, in particular probability (with a great attention for Markov process, which will be used in Chapter 8).

In Chapter 2, we recall a great part of the theory of Gaussian measures, which allows us to define Wiener spaces. In Chapter 3 we deal with derivatives and Sobolev spaces in Wiener spaces, and we also introduce the Ornstein-Uhlenbeck semigroup.

The topic of Sobolev spaces in Wiener spaces is reserved for Chapter 3: this Chapter contains the treatment about $W^{1,p}(O)$, and in particular an assertion (Proposition 3.2.23): we present a more general extension of it; we also introduce $W^{2,2}(X)$ and the Ornstein-Uhlenbeck semigroup and operator; it is also recalled the theory of traces contained in [**26**].

In Chapter 4 we introduce the topic of BV functions, as stated above.

In Chapter 5 we recall a particular kind of convergence of forms, introduced by U. Mosco in [**56**], which implies the convergence of the semigroups and of the associated resolvents: we use it extensively in Chapter 7.

In the second part of the thesis (Chapters 6, 7, 8) we present our results, as described above.

In the Appendix we recall several notations and definitions used through the above chapters, like: Banach spaces and complexifications (used in Chapter 1), holomorphic functions, convolutions (used especially in the proof of Proposition 3.2.23), a version of Riesz-Thorin theorem, absolute continuity, Banach-Alaoglu theorem; particularly important is the Hölderianity of the solution of elliptic problems, used in Chapter 7.

I want to especially thank Michael Röckner, who supervised my work when I was visiting the the University of Bielefeld in fall 2016 and proposed the problem of the contraction of the gradients which I afford in Chapter 7.

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Basic notations

$\arg z$	argument of the complex number z.
Id	identity function.
A^c	complement of a set A
f^+	positive part of a function f.
f^-	negative part of a function f.
$\{G = k\}$	$\{x \in X \| G(x) = k\}$, for sets X,Y a function $f : X \to Y$, and $k \in Y$.
$\{G \in A\}$	$\{x \in X \| G(x) \in Ak\}$, for sets X,Y a function $f : X \to Y$, and $A \subseteq Y$.
I	identity.
$\bar{A}$	closure of a set A.
A°	interior of a set A.
diam	diameter of a metric space.
$C(X)$	set of continuous functions from the topological space X in $\mathbb{R}$.
$C_b(X)$	space of continuous bounded functions from the metric space X in $\mathbb{R}$.
$C(X,Y)$	set of continuous functions from the topological space X in the topological space Y.
$C_b(X,Y)$	set of continuous bounded functions from the metric space X in the metric space Y.
$C_c(X)$	space of continuous functions with compact support from the metric space X in $\mathbb{R}$.
$C_c^1(\mathbb{R}^d)$	subspace of $C_c(\mathbb{R}^d)$ of functions differentiable, with gradient continuous in $\mathbb{R}^d$.
$\frac{\partial \cdot}{\partial x_i}$, ∂_{x_i}	partial derivative along i.
$\frac{\partial^k \cdot}{\partial x_i^k}$	$k-$th partial derivative (if a basis is fixed).
$f^{-1}(A)$	inverse image of a set A with respect to a function f.
$\bar{\mathbb{R}}$	$\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.
X'	algebraic dual of the Banach space X, i.e. the set of linear functions $l : X \to \mathbb{R}$.
X^*	topological dual of the Banach space X, i.e. the set of continuous linear functions $l : X \to \mathbb{R}$.
$\mathcal{L}(X,Y)$	given the metric spaces X,Y, space of bounded linear function from X in Y.
$\mathscr{L}(X,\mathcal{L}(X,Y))$	given the metric spaces X,Y, space of bounded linear function from X in $\mathcal{L}(X,Y)$.
$\operatorname{supp} f$	the support of a Y valued function (Y vector space) on a topological space X, i.e. $\overline{f^{-1}(Y \setminus 0)}$.
$\mathscr{L}^d$	Lebesgue measure on $\mathbb{R}^d$.
$L^p(O,\mathscr{L}^d)$	L^p space with respect to $\mathscr{L}^d$ on O subset of $\mathbb{R}^d$ for $p \in [1,+\infty]$.
$\mathrm{Lip}(X,Y)$	the set of Y-valued Lipschitz functions on X (where X and Y are metric spaces).
$\mathrm{Lip}_b(X,Y)$	the set of Y-valued Lipschitz bounded functions on X (where X and Y are metric spaces).
$\mathrm{Lip}(X)$	$\mathrm{Lip}(X,\mathbb{R})$.
$\mathrm{Lip}_b(X)$	$\mathrm{Lip}_b(X,\mathbb{R})$.

Part 1

Basics facts

CHAPTER 1

Notations and miscellaneous of basics facts

In this Chapter we recall some basics notions.

The Section 1.1 recalls some standard notions about operators and semigroups, which is used in Section 3.4 to define the Ornstein-Uhlenbeck operator; moreover, we introduce Proposition 1.1.21, which will be used in Chapter 8.

In Section 1.2 we recall standard notions of measure theory; in particular, Hausdorff measures are introduced. Moreover, we present the concept of vector measure, in particular a technical result about their total variation (Lemma 1.2.36): they will be used in Chapters 4 and 7.

In Section 1.3, some notions of probability are introduced, with a great attention for Markov processes, and their link with symmetric forms; this topics are used in Subsection 1.3.4 to explain properties of d-dimensional Brownian motion (as a Markov process), and absorbing Brownian motion; in Subsection 1.3.5 it is introduced the Bessel process; these topics will be used in Chapter 8.

1.1. Closable operators, resolvents, semigroups and symmetric forms

1.1.1. Operators, resolvents, semigroups and generators. For this subsection we refer to [**37**], [**11**], [**27**] and [**41**].

We recall the definition of operator, of closable operator, and of the closure of a closable operator.

In this section we deal with real and complex Banach spaces; in this work, by Banach space we mean a real Banach space.

DEFINITION 1.1.1. Let E, F be Banach spaces (both real or both complex); a (unbounded) *operator* is a couple $(D(L), L)$ where $D(L)$ is a subspace of E and L is a linear operator $L : D(L) \to F$; $D(L)$ is said *domain* of the operator.

If $E = F$, we say that L is an operator on E.

We say that L is *closed* if its graph is closed in $E \times F$.

A bounded operator L on E is said *contractive*, or a *contraction*, if $\|L(x)\|_X \leq \|x\|_X$.

For $x \in D(L)$, sometimes we will write Lx instead of $L(x)$.

DEFINITION 1.1.2. Let E, F be (both real or both complex) Banach spaces and let $L : D(L) \subset E \to F$ be a linear operator. L is called *closable* (in E) if there exists a linear operator $\overline{L} : D(\overline{L}) \subset E \to F$ whose graph is the closure of the graph of L in $E \times F$.

REMARK 1.1.3. An operator L is closable if: for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset D(L)$ which satisfies $\lim_{n\to\infty} x_n = 0$, if $\lim_{n\to\infty} Lx_n = z$ then $z = 0$.

If L is closable, the domain of the closure $\overline{L}$ of L is the set

$$D(\overline{L}) = \left\{ x \in E : \ \exists (x_n) \subset D(L), \ \lim_{n\to\infty} x_n = x, \ \ Lx_n \text{ converges in } F \right\}$$

and for $x \in D(\overline{L})$ we have

$$\overline{L}x = \lim_{n\to\infty} Lx_n,$$

for every sequence $(x_n) \subset D(L)$ such that $\lim_{n\to\infty} x_n = x$. By the closability of L, we have that $\lim_{n\to\infty} Lx_n$ is independent of the sequence (x_n). Since $\overline{L}$ is a closed operator, its domain is a Banach space with the graph norm $x \mapsto \|x\|_E + \|\overline{L}x\|_F$.

A linear operator L on X is said *bounded* if $D(L) = X$ and

$$\|L\| := \sup_{x\in X} \frac{\|Lx\|_X}{\|x\|_X} = \sup_{x\in X, \|x\|_X \le 1} \|Lx\|_X < \infty;$$

the set of bounded operator on X will be denoted with $\mathcal{L}(X)$. in this setting, $\|\cdot\|$ will be called *operator norm*, and we denote it also with $\|\cdot\|_{\mathcal{L}(X)}$.

We recall the concept of complexification of a real Banach space (see Appendix).

If X is a (complex or real) Banach space, then $\mathcal{L}(X)$ is a (complex or real) Banach space with the norm $\|\cdot\|_{\mathcal{L}(X)}$.

REMARK 1.1.4. Let X be a (real) Banach space, and L be an operator; we can define $L_{\mathbb{C}}$ as operator on the complexifixation $X_{\mathbb{C}}$ as

$$D(L_{\mathbb{C}}) = \{x_1 + \mathrm{i}x_2 \in X_{\mathbb{C}} | x_1, x_2 \in D(L)\},$$

$$L_{\mathbb{C}}(x_1 + \mathrm{i}x_2) = Lx_1 + \mathrm{i}Lx_2,$$

we have that $L_{\mathbb{C}}$ is a linear operator; we have that if L is closed (closable, bounded) then $L_{\mathbb{C}}$ is closed (closable, bounded).

Now we will give some notions of spectral theory; we will do it both for real and complex Banach spaces.

DEFINITION 1.1.5. Let $(D(L), L)$ be an operator on a real (or complex) Banach space X, and $\lambda \in \mathbb{R}$ (or $\lambda \in \mathbb{C}$); an operator $R \in \mathcal{L}(X)$ is said a *resolvent operator* for (L, λ) if it is the inverse of$\lambda I - L$ as a function; which means, $R(X) = D(L)$ and

$$(\lambda I - L) \circ R = I$$

on X, and

$$R \circ (\lambda I - L) = I$$

on $D(L)$.

We have that there exists at most one of this operator; if there exists, we call it also the *resolvent*, and we denote it by $(\lambda I - L)^{-1}$ or $R(\lambda, L)$.

The *resolvent set* $\rho(L)$ and the *spectrum* $\sigma(L)$ of L are defined by

$$\rho(L) = \{\lambda \in \mathbb{K} | \exists\, (\lambda I - L)^{-1} \in \mathcal{L}(X)\}, \;\; \sigma(L) = \mathbb{K} \backslash \rho(L). \tag{1.1.1}$$

where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

The numbers $\lambda \in \sigma(L)$ such that $\lambda I - L$ is not injective are the *eigenvalues*, and the vectors $x \in D(L)$ such that $Lx = \lambda x$ are the *eigenvectors* (or *eigenfunctions*, when X is a function space). The set $\sigma_p(L)$ whose elements are all the eigenvalues of L is the *point spectrum*.

If $\lambda \in \rho(L)$, by definition we have

$$(\lambda x - Lx) \circ R(\lambda, L)(x) = x$$

we have also

$$R(\lambda, L)(\lambda x - Lx) = x$$

for every $x \in D(L)$.

We have that, if the resolvent set $\rho(L)$ is not empty, then L is a closed operator.

We have the following equality, known as the *resolvent identity*

$$R(\lambda, L) - R(\mu, L) = (\mu - \lambda) R(\lambda, L) R(\mu, L), \ \forall \lambda, \mu \in \rho(L). \tag{1.1.2}$$

If X is a (real) Banach space and L is an operator, $\rho(L) = \rho(L_{\mathbb{C}}) \cap \mathbb{R}$, and, for $\lambda \in \rho(L)$ the operator $R(\lambda, L)$ can be extended to $X_{\mathbb{C}}$ as operator and it is $R(\lambda, L_{\mathbb{C}})$ (see e.g. [**11**]).

For the next definition see e.g. ([**52**], Def. I.1.4).

DEFINITION 1.1.6. A family $\{G_\lambda\}_{\lambda>0} \subset \mathcal{L}(X)$ is said *strongly continuous contraction resolvent* if:

i) $\lim_{\lambda \to 0^+} \lambda G_\lambda x = x$ for every $x \in X$;
i) λG_λ is contractive for every λ;
i) $\lambda G_\lambda - \mu G_\mu = (\mu - \lambda) G_\lambda G_\mu, \ \forall \lambda, \mu > 0$.

A *semigroup of operators* on a (real or complex) Banach space X is a family $\{T_t\}_{t \in [0,+\infty)}$ (we will write also simply T_t) of linear bounded operators $T_t : X \to X$, s.t. T_0 is the identity and for every $t, s \geq 0$,

$$T_{t+s}(x) = T_t(T_s(x))$$

for every $x \in X$. A semigroup T_t on X is said *strongly continuous* if, for every $x \in X$

$$\lim_{t \to 0} \|T_t x - x\|_X = 0;$$

a semigroup T_t on X is said *contractive* (or *contraction semigroup*) if, for every $x \in X$, $t \geq 0$

$$\|T_t x\|_X \leq \|x\|_X .$$

DEFINITION 1.1.7. The *infinitesimal generator* (or, shortly, the *generator*) of the strongly continuous semigroup $\{T_t\}_{t \geq 0}$ is the operator defined by

$$D(L) = \left\{ x \in X : \exists \lim_{t \to 0^+} \frac{T_t - I}{t} x \right\}, \quad Lx = \lim_{t \to 0^+} \frac{T_t - I}{t} x.$$

We have that, if $(D(L), L)$ is the generator of a strongly continuous semigroup T_t, then $D(L)$ is dense in X, and $T_t(D(L)) \subseteq D(L)$ for every $t \geq 0$. One operator can be the generator of at most one semigroup.

REMARK 1.1.8. If X is a Banach space and T_t is an semigroup on X, then T_t can clearly be extended to a semigroup $T_t^{\mathbb{C}}$ on the complexification $X_{\mathbb{C}}$ as operator, if T_t is strongly continuous then $T_t^{\mathbb{C}}$ is strongly continuous, if the generator of T_t is L then the generator of $T_t^{\mathbb{C}}$ is $L_{\mathbb{C}}$.

We recall that if H is a Hilbert space then for $x, y \in H$ we have

$$|x + iy|_{H_{\mathbb{C}}} = \sqrt{|x|_H^2 + |y|_H^2};$$

so if T_t is a contractive semigroup on H, then its extension $T_t^{\mathbb{C}}$ is contractive on $H_{\mathbb{C}}$.

A self-adjoint operator L on a (real or complex) Hilbert space H is said *accretive* if

$$\Re(\langle Lh,h\rangle_H)\geq 0$$

for every $h\in D(L)$.

An operator L is said *dissipative* if $-L$ is accretive.

By the fact that $|\cdot|_H^2=\langle\cdot,\cdot\rangle_H$, the following Lemma follows immediately.

LEMMA 1.1.9. *A self-adjoint operator L on H is accretive iff, for every $\lambda>0$, $h\in D(A)$*

$$|\lambda h+Lh|_H\geq\lambda|h|_H.$$

We have this result.

PROPOSITION 1.1.10. *If L is an accretive operator, then $\lambda\in\rho(L)$ for every $\lambda<0$. If L is dissipative, then $\lambda\in\rho(L)$ for every $\lambda>0$.*

An accretive operator L is said *m-accretive* if, $I+L$ is surjective (here *m*- stands for maximal).

Clearly, if H is a real Hilbert space and L is an operator on H, we have that if L is accretive (m-accretive) then $L_{\mathbb{C}}$ is accretive (m-accretive).

We have this Proposition (see e.g. [**11**], Thm. 3.1).

PROPOSITION 1.1.11. (Lumer-Phillips theorem) *Let H be a (real or complex) Hilbert space and L an operator on H; then L is the generator of a strongly continuous contractive semigroup on H iff $-L$ is m-accretive.*

In particular, if L generates a strongly continuous contractive semigroup, then L admits for every $\lambda>0$ a resolvent $G_\lambda:=R(\lambda,L)$, and it is a strongly continuous contraction resolvent (see e.g. [**52**] Prop. I.1.10); we call G_λ the strongly continuous contraction resolvent of L.

If $\theta\in[0,+\frac{\pi}{2})$, then in $\mathbb{C}$ we can define the set $\Sigma_\theta:=\{z\in\mathbb{C}\,|\,|\arg z|<\theta\}$.

We introduce the concepts of bounded holomorphic semigroup (we follow [**57**]).

DEFINITION 1.1.12. Let $\theta\in(-\frac{\pi}{2},+\frac{\pi}{2})$. Let X be a complex Banach space (so $\mathcal{L}(X)$ is a complex Banach space). A strongly continuous semigroup $\{T_t\}_{t\in(0,+\infty)}$ on X is called a bounded holomorphic semigroup on the sector Σ_θ if it can be extended to a holomorphic function $\{T_z\}_{z\in\Sigma_\theta}$ on Σ_θ s.t., for every $0<\theta_1<\theta$, $\{T_z\}_{z\in\Sigma_{\theta_1}}$ is uniformly bounded on $\mathcal{L}(X)$.

1.1.2. Forms and associated operators and semigroups.

In the sequel of this subsection, $\overline{z}$ denotes the conjugate of z in $\mathbb{C}$.

We recall that a *symmetric form* on a real (*sesquilinear form* on a complex) Hilbert space H is a couple $(D(a),a)$ where $D(a)$, said *domain*, is a dense linear subspace of H and a is a function $D(a)\times D(a)\to\mathbb{K}$ for $\mathbb{K}=\mathbb{R}$ (or $\mathbb{K}=\mathbb{C}$) s.t. $a(h_1,h_2)=\overline{a(h_2,h_1)}$ and

$$a(\alpha h_1+\beta h_2,h_3)=\alpha a(h_1,h_3)+\beta a(h_2,h_3)$$

for every $h_1,h_2,h_3\in D(a)$ and $\alpha,\beta\in\mathbb{K}$; we have that on $D(a)$ we can define the norm $|\cdot|_{a_1}$ as

$$|h|_{a_1}:=\sqrt{a(h,h)+\langle h,h\rangle_H}$$

for every $h\in D(a)$; if $D(a)$ endowed with the norm $|\cdot|_{a_1}$ is a Banach space then a is said *closed*.

If there exists M s.t.

$$|a(h_1,h_2)|\leq M|h_1|_{a_1}|h_2|_{a_1},$$

for every $h_1,h_2\in D(a)$, then we say that a is continuous.

If

$$\Re(a(h,h)) \geq 0$$

for every $h \in H$.

REMARK 1.1.13. If H is a real Hilbert space and $(D(a),a)$ is a symmetric form on H, then there exists an extension $(D(a_{\mathbb{C}}),a_{\mathbb{C}})$ on the complexification $H_{\mathbb{C}}$: $D(a_{\mathbb{C}}) := D(a) + \mathrm{i}D(a)$,

$$a_{\mathbb{C}}(h_1+\mathrm{i}h_2, h_3+\mathrm{i}h_4) := a(h_1,h_3) + \mathrm{i}a(h_2,h_3) - \mathrm{i}a(h_1,h_4) + a(h_2,h_4);$$

it is sesquilinear, and if a is closed (or continuous) then $a_{\mathbb{C}}$ is closed (or continuous).

We have this result (see e.g. [**48**],VI, Theo 2.1), due to K. Friedrichs.

PROPOSITION 1.1.14. *Given a closed symmetric form $(D(a),a)$ on H, there exists exactly one linear operator A on H associated to a s.t. $-A$ is an m-accretive operator, $D(A) \subseteq D(a)$, and, for every $f \in D(A), g \in D(a)$, we have*

$$a(f,g) = -\langle Af, g\rangle_H.$$

DEFINITION 1.1.15. Given a closed symmetric form $(D(a),a)$ on H, the operator A on H introduced in Proposition 1.1.14 is said *associated to a*.

By what we said, given a continuous closed symmetric form $(D(a),a)$ on H, and the operator L associated to it, $-L$ is m-accretive, so by Lumer-Phillips theorem L generates a strongly continuous contractive semigroup T_t, and $G_\lambda := R(\lambda, L)$ is a strongly continuous contraction resolvent.

We call the above defined T_t the strongly continuous semigroup associated to a, and G_λ the strongly continuous contraction resolvent associated to a.

REMARK 1.1.16. Let H be a real Hilbert space, a a closed symmetric form on H and A the operator associated to a; then, on the complexification $H_{\mathbb{C}}$ the operator associated to the extension $a_{\mathbb{C}}$ is the extension $A_{\mathbb{C}}$.

DEFINITION 1.1.17. Let H be a (real or complex) Hilbert space; a semigroup T_t on H is said *symmetric* or *self-adjoint* if , for every $t \geq 0$, $h_1,h_2 \in H$

$$\langle T_t h_1, h_2\rangle_H = \overline{\langle h_1, T_t h_2\rangle_H}.$$

We have that a strongly continuous semigroup is self-adjoint if and only if its generator is self-adjoint.

We have that if a closed symmetric form a Let T_t be a self-adjoint strongly continuous contractive semigroup on H; for every $t > 0$ we define the form with domain H

$$a^{(t)}(h_1,h_2) := t^{-1}\langle h_1 - T_t h_1, h_2\rangle_H$$

for $h_1,h_2 \in H$; we have that $(H, a^{(t)})$ is a closed symmetric form. We define a form a in this way

$$D(a) := \{h \in H | \lim_{t\to 0} |a^{(t)}(h,h)| < \infty\}$$

$$a(h_1,h_2) := \lim_{t\to 0} a^{(t)}(h_1,h_2)$$

for $h_1,h_2 \in D(a)$. This is a closed symmetric form.

DEFINITION 1.1.18. The form $(a, D(a))$ defined above is the *form associated to the semigroup T_t*.

We have this result ([**57**], Thm. 1.52).

PROPOSITION 1.1.19. *If H is a complex Hilbert space, and a is a continuous closed sesquilinear form on H, if T_t is the strongly continuous semigroup associated to a, then T_t is bounded holomorphic.*

1.1.3. Kernel of a semigroup. Let $\Omega \subseteq \mathbb{R}^d$ be an open set; T_t be a semigroup on $L^2(\Omega, \mathscr{L}^d)$ (it is a Hilbert space); following ([**57**], Sec. 6.1), if there exists a Lebesgue-measurable function $p : \Omega \times \Omega \times \mathbb{R} \to \mathbb{R}$ s.t.: for some C, $|p(x,y,t)| \leq Ct^{-d/2}$ for every $x, y \in \Omega$, $t > 0$, and, for every $f \in L^2(\Omega, \mathscr{L}^d)$, $t > 0$ we have

$$T_t f(x) = \int_\Omega f(y) p(x,y,t)\, dy$$

(for $\mathscr{L}^d$-almost every $x \in \Omega$), then we say that p is the *kernel* of T_t.

REMARK 1.1.20. By the properties of the semigroups we have, for all $0 < s < t$,

$$\begin{aligned}
\int_\Omega f(y) p(x,y,t)\, dy &= T_t f(x) = T_{t-s}(T_s f)(x) = \\
&= \int_\Omega p(x,y,t-s) \left(\int_\Omega f(z) p(y,z,s)\, dz\right) dy \\
&= \int_\Omega \int_\Omega f(z) p(x,y,t-s) p(y,z,s)\, dz\, dy
\end{aligned} \tag{1.1.3}$$

for $\mathscr{L}^d$-almost every x, for every $f \in L^2(\Omega, \mathscr{L}^d)$; from this we can deduce, for every B bounded Borel subset of Ω

$$\int_B p(x,y,t)\, dy = \int_\Omega \int_B p(x,y,t-s) p(y,z,s)\, dz\, dy$$

Let X be the complexification of $L^2(\Omega, \mathscr{L}^d)$; let T_t be a semigroup on X, if there exists $p : \Omega \times \Omega \times \mathbb{R} \to \mathbb{C}$ s.t., for every $f \in X$ (writing $f = f_1 + \mathrm{i} f_2$ where $f_1, f_2 \in L^2(\Omega, \mathscr{L}^d)$)

$$T_t^{\mathbb{C}} f(x) = \int_\Omega f_1(y) p(x,y,t)\, dx + \mathrm{i} \int f_2(y) p(x,y,t)\, dx \tag{1.1.4}$$

(for $\mathscr{L}^d$-almost every $x \in \Omega$), then we say that p is called *kernel* of T_t

Now, if T_t on $L^2(\Omega, \mathscr{L}^d)$ has kernel p, we have that its extension $T_t^{\mathbb{C}}$ on X has kernel p.

About kernel of semigroups associated to forms, we have this result (see e.g. [**57**] Thm. 6.17 for a more general result).

PROPOSITION 1.1.21. *Let $\Omega \subseteq \mathbb{R}^d$, X be the complexification of $L^2(\Omega, \mathscr{L}^d)$. Let $\{S_t\}_{t \geq 0}$ be a bounded holomorphic semigroup on X, and $p : \Omega \times \Omega \times \mathbb{R} \to \mathbb{C}$ be the kernel associated to S_t; if there exist $C, c > 0$ s.t.*

$$|p(x,y,t)| \leq Ct^{-\frac{d}{2}} \exp\left(-\frac{c|x-y|^2}{t}\right)$$

for every $x, y \in \Omega$ and $t > 0$, then for every $k \in \mathbb{N}$ we have that p is k-times differentiable in t and

$$\left|\frac{\partial^k}{\partial t^k} p(x,y,t)\right| \leq Ct^{-\frac{d}{2}-k} \exp\left(-\frac{c|x-y|^2}{t}\right)$$

1.2. Abstract Measure Theory

We recall some definitions and results about positive measures and vector measures. A complete treatment about real measures can be found in [**15**], [**16**]. For vector measures we follow [**30**], for further reading see [**22**] Sec. 4, and [**31**].

We recall also that positive measures generalizes the well-known case of Lebesgue measure.

1.2.1. Measure spaces.

DEFINITION 1.2.1. [σ-algebras and measure spaces] Let X be a nonempty set and let $\mathscr{F}$ be a collection of subsets of X.

i) We say that $\mathscr{F}$ is a an *algebra* (or also a *ring of sets* or a *clan*) if $\varnothing \in \mathscr{F}$, $E_1 \cup E_2 \in \mathscr{F}$ and $X \setminus E_1 \in \mathscr{F}$ whenever $E_1, E_2 \in \mathscr{F}$.
ii) We say that an algebra $\mathscr{F}$ is a σ*-algebra* if for any sequence $E_{nn} \in \mathbb{N} \subset \mathscr{F}$ its union $\bigcup_{n\in\mathbb{N}} E_n$ belongs to $\mathscr{F}$.
iii) For any collection $\mathscr{G}$ of subsets of X, the σ-algebra generated by $\mathscr{G}$ is the smallest σ-algebra containing $\mathscr{G}$. If (X,τ) is a topological space, we denote the σ-algebra generated by the open subsets of X by $\mathfrak{B}(X)$; an element of $\mathfrak{B}(X)$ is said a Borel set in X; the σ-algebra generated by the set of the form $f^{-1}(B)$ where $f \in C(X)$ and $B \in \mathfrak{B}(\mathbb{R})$ is denoted with $\mathfrak{Ba}(X)$; an element of $\mathfrak{Ba}(X)$ is said a Baire set in X. Clearly $\mathfrak{Ba}(X) \subseteq \mathfrak{B}(X)$. If X is a metric space, we have $\mathfrak{Ba}(X) = \mathfrak{B}(X)$.
iv) If $\mathscr{F}$ is a σ-algebra in X, we call the pair $(X,\mathscr{F})$ a *measurable space*.

REMARK 1.2.2. With the De Morgan laws, it is easy to prove that algebras are closed under finite intersections, and σ-algebras are closed under countable intersections.

If $\mathscr{F}$ is a σ-algebra in X, if $Y \in \mathscr{F}$ then we can define the restriction $\mathscr{F}_{|Y} := \{A \in \mathscr{F} | A \subseteq Y\}$; we have that $\mathscr{F}_{|Y}$ is a σ-algebra in Y; it can be proved that the restriction of $\mathfrak{B}(X)$ is always $\mathfrak{B}(Y)$.

The intersection of any family of σ-algebras is a σ-algebra; the set of all subsets of X is a σ-algebra; hence, the definition of generated σ-algebra is well posed.

DEFINITION 1.2.3. [Finite measures] Let $(X,\mathscr{F})$ be a measure space and $\mu : \mathscr{F} \to [0,+\infty)$ (positive finite set function). We say that μ is additive if

$$\mu(A \cup B) = \mu(A) + \mu(B) \tag{1.2.1}$$

for all $A, B \in \mathscr{F}$, s.t. $A \cap B = \varnothing$.

We say that μ is countably additive if $\mu(\varnothing) = 0$ and μ is σ-additive on $\mathscr{F}$, i.e., for any sequence (E_h) of pairwise disjoint elements of $\mathscr{F}$ the equality

$$\mu\left(\bigcup_{h=0}^{\infty} E_h\right) = \sum_{h=0}^{\infty} \mu(E_h) \tag{1.2.2}$$

is verified. A positive finite set function that is countably additive is said a *positive finite measure*; in this case, $(X,\mathscr{F},\mu)$ is said a measure space.

If μ is a positive measure on $(X,\mathscr{F})$ and there exists a countable sequence of sets $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathscr{F}$ s.t. $\bigcup_{i=1}^{+\infty} A_i = X$ and $\mu(A_i) < \infty$ for every i, then μ is said σ*-finite*.

We say that μ is a *probability measure* if $\mu(X) = 1$; in this case $(X,\mathscr{F},\mu)$ is said a probability space.

We say that $\mu : \mathscr{F} \to \mathbb{R}$ is a (finite) *real measure* if $\mu = \mu_1 - \mu_2$, where μ_1 and μ_2 are positive finite measures.

A measure on $(X, \mathfrak{B}(X))$ is said a Borel measure. A measure space $(X, \mathfrak{B}(X), \mu)$ is said Borel measure space (Borel probability space if μ is a probability); we denote it also as (X, μ).

If $(X, \mathfrak{B}(X), \mu)$ is a probability space, X is said *event space*.

If $(X, \mathscr{F}, \mu)$ is a measure space, if $Y \in \mathscr{F}$, then we can define the *restriction* $\mu_{|Y} := \mu_{|\mathscr{F}_{|Y}}$ (restriction as set function); we have that $(X, \mathscr{F}_{|Y}, \mu_{|Y})$ is a measure space; clearly for what we have said about restriction of Borel σ-algebras, the restriction of a Borel measure is a Borel measure.

If μ, ν are finite real measures on $(X, \mathscr{F})$, we will define the sum $\mu + \nu$ on $(X, \mathscr{F})$ in this way:

$$(\mu + \nu)(A) := \mu(A) + \nu(A), \tag{1.2.3}$$

for all $A \in \mathscr{F}$. It is obvious that $\mu + \nu$ is a measure.

REMARK 1.2.4. If $(X, \mathscr{F}, \mu)$ is a measure space, if $A \in \mathscr{F}$, then we can consider $\mathscr{F}_{|A} = \{B \in \mathscr{F} | B \subseteq A\}$, we have that $(X, \mathscr{F}_{|A})$ is a σ-algebra and that $(A, \mathscr{F}_{|A}, \mu_{|\mathscr{F}_A})$ is a measure space; in this case we will use μ to say $\mu_{|\mathscr{F}_{|A}}$.

A positive measure μ on $(X, \mathscr{F})$ has some properties:

i) μ is increasing, i.e., for all $A, B \in \mathscr{F}$, we have $\mu(A) \leq \mu(B)$ if $A \subseteq B$;

ii) μ is continuous along monotone sequences, i.e., if A_n is an increasing sequence in $\mathscr{F}$ then

$$\mu\Big(\bigcup_{i=1}^{\infty} A_n\Big) = \lim_{n \to \infty} \mu(A_n) = \sup_{n \in \mathbb{N}} \mu(A_n), \tag{1.2.4}$$

and, if A_n is a decreasing sequence in $\mathscr{F}$ then

$$\mu\Big(\bigcap_{i=1}^{\infty} A_n\Big) = \lim_{n \to \infty} \mu(A_n) = \inf_{n \in \mathbb{N}} \mu(A_n); \tag{1.2.5}$$

iii) μ is countably subadditive, i.e., if A_n is a sequence in $\mathscr{F}$ then

$$\mu\Big(\bigcup_{i=1}^{\infty} A_n\Big) \leq \sum_{i=1}^{\infty} \mu(A_i). \tag{1.2.6}$$

Hereafter, for a topological space X, if not otherwise specified, a positive measure μ is always a positive measure on $(X, \mathscr{F})$ where $\mathscr{F}$ is the σ-algebra of Borel subsets of X.

We usually say that μ is a measure on X, meaning that there exists $\mathscr{F}$ s.t. μ is a measure on $(X, \mathscr{F})$; in this setting we will say that A is measurable on μ or that it is μ-measurable to mean that $A \in \mathscr{F}$.

If μ is a set function on $(X, \mathscr{F})$, we define (following [**30**]) its *total variation* $|\mu|$ for every $E \in \mathscr{F}$ as follows:

$$|\mu|(E) := \sup \left\{ \sum_{h=0}^{\infty} |\mu(E_h)| : E_h \in \mathscr{F} \text{ pairwise disjoint, } E = \bigcup_{h=0}^{\infty} E_h \right\}. \tag{1.2.7}$$

Clearly, it is a set function on $(X, \mathscr{F})$. If $\mathscr{F}$ is a σ-algebra and μ is a measure, then $|\mu|$ is a measure (i.e. is countably additive) (see e.g. [**30**], Proposition I.3.11).

The measure $|\mu|$ is called the total variation measure defined by μ.

A real measure μ on $(X, \mathscr{F})$ with finite total variation is said *bounded variation measure* .

We will usually consider only bounded variation measure (with the exception of Lebesgue measure).

If μ is a measure on $X, \mathscr{F}$, then there exists a couple (X^+, X^-) of disjoint sets in $\mathscr{F}$ s.t. $\mu(A \cap X^+) \geq 0$ and $\mu(A \cap X^-) \leq 0$ for all $A \in \mathscr{F}$. (see e.g. [**15**] Thm. 3.1.1). We can define μ^+ as $\mu^+(A) := \mu(A \cap X^+)$ and $\mu^-(A) := -\mu(A \cap X^-)$, we have that these are positive measures and $\mu = \mu^- - \mu^-$ (*Jordan decomposition*); μ^+, μ^-, are uniquely defined; we can give an alternative definition of total variation as $|\mu| := \mu^+ + \mu^-$, this definition coincides with the above.

DEFINITION 1.2.5. [Radon measures] A Borel measure μ on a topological space X is called a *real Radon measure* if for every $B \in \mathfrak{B}(X)$ and $\varepsilon > 0$ there is a compact set $K \subset B$ such that $|\mu|(B \setminus K) < \varepsilon$.

For the following result see e.g. [**16**], Theo 7.1.7.

PROPOSITION 1.2.6. *If (X, d) is a separable complete metric space then every real measure on $(X, \mathfrak{B}(X))$ is Radon.*

1.2.2. Integrals. In the previous subsections we recalled the main concepts of measure theory; in this subsection we recall some well-known concepts about integration, referring to [**15**] and other basic books of measure theory for details and more.

1.2.2.1. *Measurable functions and integrals.*

DEFINITION 1.2.7. [Measurable functions] Let $(X, \mathscr{F})$, $(Y, \mathscr{G})$ be measurable spaces. A function $f : X \to Y$ is said to be $(\mathscr{F} - \mathscr{G})$-*measurable* (or simply measurable) if $f^{-1}(A) \in \mathscr{F}$ for every $A \in \mathscr{G}$. If Y is a topological space, a function $f : X \to Y$ is said to be $\mathscr{F}$-*measurable* if $f^{-1}(A) \in \mathscr{F}$ for every open set $A \subset \mathfrak{B}(Y)$, i.e. f is $(\mathscr{F} - \mathfrak{B}(Y))$-measurable.

A function $X \to Y$ which is $\mathfrak{B}(X)$ measurable is also said *Borel measurable*.

If $(X, \mathscr{F}, \mu)$ is a measure space and $(Y, \mathscr{G})$ is a measurable space, and f is $(\mathscr{F} - \mathscr{G})$-*measurable*, we will also say that it is μ-measurable.

We will say that a function f is $\mathscr{F}$-*measurable* (or μ-measurable) on $A \in \mathscr{F}$ if it is measurable with respect to $(A, \mathscr{F}_A, \mu_{|\mathscr{F}_A})$.

LEMMA 1.2.8. *Let X a topological space. If $f : X \to \mathbb{R}$ is a lower semicontinuous (or continuous) function, then f is $\mathfrak{B}(X)$-measurable.*

PROOF. Let f be lower semicontinuous: $f^{-1}((r, +\infty])$ is an open set for every $r \in \mathbb{R}$, hence $f^{-1}((r_1, r_2]) \in \mathfrak{B}(X)$ for every r_1, r_2, and $\mathfrak{B}(\mathbb{R})$ is generated by the set of the form $(r_1, r_2]$, hence $f^{-1}(A) \in \mathfrak{B}(X)$ for every $A \in \mathfrak{B}(\mathbb{R})$; so f is $\mathfrak{B}(X)$-measurable. If f is continuous, then it is lower semicontinuous, and we can conclude. $\square$

We have that the set of measurable functions is a $\mathbb{R}$-vector space contained in the space of functions with real values.

In particular, if Y is a topological space and f is $\mathscr{F}$-measurable then $f^{-1}(B) \in \mathscr{F}$ for every $B \in \mathfrak{B}(Y)$.

Hereafter, we will suppose that $(X, \mathscr{F}, \mu)$ is a measure space, with $\mu|$ positive measure.

We say that a property is true for *almost every* (a.e.) x (or μ-almost every x) if it is true in a set A where $A \in \mathscr{F}$ and $\mu(X \setminus A) = 0$; in particular, if f, g are functions, we will write $f \equiv g$ to say that $f = g$ almost everywhere. Sometimes, we will define classes of functions which are equal almost everywhere: to say that a function f is in a class g we usually say $f \equiv g$, and sometimes we will call functions these classes of functions.

A set $A \in \mathscr{F}$ is said negligible or μ-negligible if $\mu(A) = 0$.

We say that a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ converges almost everywhere to a function f if $f_n(x) \to f(x)$ for almost every x.

We recall a generalization of a part of Lusin theorem (see e.g. [**16**] Thm. 7.1.13).

LEMMA 1.2.9. *If X is a topological space, if μ is a Radon measure on X, if Y is a Banach space and $f : X \to Y$ is a μ-measurable function, then for every $\varepsilon > 0$ there exists $f_\varepsilon : X \to Y$ continuous s.t. $\mu(\{x \in X \text{ s.t. } f(x) \neq f_\varepsilon(x)\}) < \varepsilon$.*

For $E \subset X$ we define the *indicator function* (or *characteristic function*) of E, denoted by $1\!\!1_E$, by

$$\chi_E(x) := 1\!\!1_E(x) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E. \end{cases}$$

We say that $f : X \to \mathbb{R}$ is a *simple function* if the image of f is finite, i.e., if f belongs to the vector space generated by the indicator functions of measurable sets: clearly, the set of simple functions is a $\mathbb{R}$-linear space, and, if f is a simple function then $|f|$ is a simple function.

We recall the definition of the integral of a simple function $f = \sum_{i=1}^{n} c_i 1\!\!1_{E_i}$ (where $c_i \in \mathbb{R}$ and E_i is μmeasurable for every i) as, for each A measurable set,

$$\int_A f \, d\mu = \sum_{i=1}^{\infty} c_i \mu(A \cap E_i);$$

this is well defined.

In this setting, we say that a sequence of simple functions $\{f_n\}_{n\in\mathbb{N}}$ is mean fundamental with respect to μ if, for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ s.t. for all $n, m > N$,

$$\int_X |f_n - f_m| \, d\mu \leq \varepsilon.$$

We have that, if two sequences $\{f_n\}_{n\in\mathbb{N}}, \{g_n\}_{n\in\mathbb{N}}$ of simple functions are mean fundamental and they both converges almost everywhere to one function f, then the integrals of f_n, g_n converges to the same value (see e.g. [**15**] Lem. 2.4.2).

DEFINITION 1.2.10. We recall that a function f finite and definite up to a negligible set is μ-*integrable* if there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of simple functions which is mean fundamental s.t. f_n converges a.e. to f. In this case we write, for each A, measurable set, the integral of f on A

$$\int_A f(x) \, d\mu(x) := \int_A f \, d\mu := \lim_{n\to\infty} \int_A f_n \, d\mu;$$

for what we said, this is a good definition, which does not depend on the sequence f_n.

If $f_{|A}$ is μ-measurable on $A \in \mathscr{F}$ (where $\mu = \mu_{|\mathscr{F}_A}$, see Definition 1.2.7) and the integral of $f_{|A}$ is defined on A then we will write

$$\int_A f(x) \, d\mu(x) := \int_A f \, d\mu := \int_A f_{|A}(x) \, d\mu(x).$$

If $A_1 \cap A_2 = \varnothing$ then

$$\int_{A_1 \cup A_2} f \, d\mu = \int_{A_1} f \, d\mu + \int_{A_2} f \, d\mu .$$

Given a σ-algebra, we have defined the class of measurable functions. Conversely, given a family of functions, it is possible to define a suitable σ-algebra.

DEFINITION 1.2.11. Given a family F of functions $f : X \to \mathbb{R}$, let us define the σ-algebra $\mathscr{E}(X,F)$ generated by F on X as the smallest σ-algebra such that all the functions $f \in F$ are measurable, i.e., the σ-algebra generated by the sets $\{f < t\}$, with $f \in F$ and $t \in \mathbb{R}$.

DEFINITION 1.2.12. A sequence of measurable functions f_n *converges in measure* μ to a measurable function f if, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu(\{x \in X : |f(x) - f_n(x)| \geq \varepsilon\}) = 0.$$

If μ is finite, a sequence of functions which converges μ-a.e. converges in measure μ.

1.2.2.2. *L^p Spaces* . In this subsection we recall the definition of L^p spaces in a measure space, and some results about them.

Herefter, X will be always a metric space.

For μ positive, we define the L^p (semi)-norms and spaces as follows,

$$\|u\|_{L^p(X,\mu)} := \left(\int_X |u|^p \, d\mu \right)^{1/p}$$

if $1 \leq p < \infty$, and

$$\|u\|_{L^\infty(X,\mu)} := \inf \{C \in [0,+\infty] \colon |u(x)| \leq C \text{ for } \mu\text{-a.e. } x \in X\} .$$

DEFINITION 1.2.13. For $p \in [1,+\infty]$ we define the space $L^p(X,\mu)$ as the space of equivalence classes of measurable functions agreeing μ-a.e. such that $\|u\|_{L^p(X,\mu)} < \infty$. If $f : X \to \mathbb{R}$ is a function, sometimes we will write $f \in L^p(X,\mu)$ to mean that it is in a class in $L^p(X,\mu)$; if A is a set of functions $X \to \mathbb{R}$, sometimes we will write $A \subseteq L^p(X,\mu)$ to mean that each element of A is in a class which is element of $L^p(X,\mu)$.

We define the set of locally L^p functions, $L^p_{loc}(X,\mu)$, of the equivalence classes of measurable functions f agreeing μ-a.e. s.t., for every $x \in X$, there is a ball B centered in x s.t. $f_{|B} \in L^p(B,\mu)$ (a priori it is a set, not a topological space).

REMARK 1.2.14. We will usually treat the elements of L^p and L^p_{loc} spaces as functions, and sometimes we will say that $f \in L^p(X,\mu)$ to mean that f is an element of $L^p(X,\mu)$ or that it is a representative of an element of f; if f is a function on X and $g \in L^p(X,\mu)$, we will say that $f = g$ almost everywhere to mean that f is an element of the class g; if A is a set of functions on X, we will say that $A \subseteq L^p(X,\mu)$ to mean that each element of A is element of an element of $L^p(X,\mu)$.

All this remains true if we substitute L^p with L^p_{loc}.

In this space, the operations of sum and product by a scalar are coherent, so $L^p(X,\mu)$ is a real vector space. We have that, in this space, $\|\cdot\|_{L^p(X,\mu)}$ is a norm and $L^p(X,\mu)$ is a Banach space, see e.g. [**15**, Theorem 4.1.3]. When the measure space is obvious by the setting, we will use also the notation $\|\cdot\|_p$.

We have that $L^2(X,\mu)$ is a Hilbert space with the inner product

$$\langle f,g\rangle := \int_X fg\, d\mu.$$

We recall that for integral and L^p spaces in a measure space we have many properties and results, analogously to the case of the Lebesgue measure: linearity of the integral,Hölder inequality, dominated convergence theorem, monotone convergence theorem.

We write explicitly only a version of the Lebesgue-Vitali theorem (see e.g. [**15**], Thm. 4.5.4), and the Jensen theorem (see e.g. [**15**], Thm. 2.12.19).

THEOREM 1.2.15. Lebesgue-Vitali theorem *Let $(X,\mathscr{F})$ be a measure space, let μ be a positive finite measure on it and let (f_k) be a sequence of measurable functions and f be a measurable function: we have that $f \in L^1(X,\mu)$ and f_n converges to f if and only if*

$$\lim_{M\to\infty} \sup_{k\in\mathbb{N}} \int_{\{|f_k|>M\}} |f_k|\, d\mu = 0.$$

and $f_k \to f$ in measure, i.e.,

$$\lim_{k\to\infty} \mu(\{x\in X:\ |f_k(x)-f(x)|>\varepsilon\}) = 0 \qquad \textit{for every } \varepsilon>0. \tag{1.2.8}$$

THEOREM 1.2.16 (Jensen). (Jensen Inequality) *Let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space, let $\mathscr{G}\subset\mathscr{F}$ be a sub-σ-algebra, let $X\in L^1(\Omega,\mathscr{F},\mathbb{P})$ be a real random variable, and let $\varphi:\mathbb{R}\to\mathbb{R}$ be a convex C^1 function such that $\varphi(X)\in L^1(\Omega,\mathscr{F},\mathbb{P})$.*

$$\mathbb{E}(\varphi\circ X|\mathscr{G}) \geq \varphi\circ\mathbb{E}(X|\mathscr{G}). \tag{1.2.9}$$

We recall also the next Corollary.

COROLLARY 1.2.17. *If μ is a finite measure and $p,q\in[1,+\infty]$ with $q\leq p$, then $L^p(X,\mu)\subseteq L^q(X,\mu)$ the inclusion is a continuous embedding $L^p(X,\mu)\hookrightarrow L^q(X,\mu)$.*

Moreover, we recall that sequence of functions which converges in sense L^p converges in measure.

We shortly recall also some concepts about Bochner integral (see e.g. [**32**], Sec. III.2 and also [**31**], Sec. III.2).

Let $(X,\mathscr{F},\mu)$ a measure space, Y a Banach space, therefore a definition of simple functions from X to Y is possible ([**32**] Def. III.2.9), hence we can define the integral for these functions; hence a function is Bochner μ-integrable if it satisfies (in the vector case) a condition expressed in the same way of the μ-integrability, and we define the μ-integral in the same way. The L^p spaces of Y-valued functions are defined as in the scalar case (see e.g. [**32**], Sec. III.3); for every $1\leq p<\infty$, $L^p(X,\mu,Y)$ is the space of the equivalence classes of Bochner integrable functions $F:\to Y$ such that

$$\|F\|_{L^p(,\mu,Y)} := \left(\int_X \|F(x)\|_Y^p \mu(dx)\right)^{1/p} < \infty.$$

$L^p(X,\mu,Y)$ is a Banach space with the above norm. If $p=2$ and Y is a Hilbert space, $L^p(X,\mu,Y)$ is a Hilbert space with the scalar product

$$\langle F,G\rangle_{L^2(X,\mu;Y)} := \int_X \langle F(x),G(x)\rangle_Y \mu(dx).$$

We define

$$L^\infty(X,\mu,Y) := \Big\{F : X \to Y \text{ measurable s.t. } \|F\|_{L^\infty(X,\mu;Y)} < \infty\Big\},$$

where

$$\|F\|_{L^\infty(X,\mu,Y)} := \inf\Big\{M > 0 : \mu(\{x : \|F(x)\|_Y > M\}) = 0\Big\}.$$

1.2.3. Other properties and definitions about real valued measure.

Weak* convergence of measures. Given a topological space X, the set of real Borel measures μ is a vector space in an obvious way. All continuous and bounded functions are in $L^1(X,\mu)$ and we define the *weak* convergence of measures* by

$$\mu_j \rightharpoonup^* \mu \quad \Longleftrightarrow \quad \int_X f\,d\mu_j \to \int_X f\,d\mu \qquad \forall f \in C_b(X); \tag{1.2.10}$$

if we consider the set of the measures (or of finite measures), let the weakest topology on it s.t. the sequence which weak* converges converges also in the topology, this is said weak* topology *weak* topology of measures*.

The particular case K compact justifies the term weak* convergence, as seen in the next theorem.

THEOREM 1.2.18. (Riesz-Markov or Riesz representation Theorem) *If K is a compact Hausdorff space, then $C(K)$ is a Banach space, the set $\mathscr{M}(K)$ of Radon measures on K can be seen as $(C(K))^*$ by the isometry*

$$i : \mathscr{M}(K) \to (C(K))^* \; \mu \mapsto (f \mapsto \int_K f\,d\mu);$$

in this setting, $\mu_j \rightharpoonup^* \mu$ iff $i(\mu_j) \rightharpoonup^* i(\mu)$ (i.e. we have the convergence in $(C(K))^*$ with the weak* topology).

We have this result by A. D. Alexandroff, sometimes called Portmanteau Theorem (see e.g. [**16**] Cor. 8.2.10).

THEOREM 1.2.19. *Let X be a metric space; let $\{\mu_i\}_{i\in\mathbb{N}}$ a sequence of probability measures on $(X,\mathfrak{B}(X))$, and let μ be a measure on $(X,\mathfrak{B}(X))$. The following are equivalent.*

- i) $\mu_j \rightharpoonup^* \mu$;
- ii) $\limsup_{j\to+\infty} \mu_j(F) \le \mu(F)$ *for every closed set F;*
- iii) $\liminf_{j\to+\infty} \mu_j(\Omega) \ge \mu(\Omega)$ *for every open set Ω.*

Absolute continuity and singularity of measures. Let μ be a positive finite measure and ν a real measure on a measurable space $(X,\mathscr{F})$. We say that ν is *absolutely continuous* with respect to μ, and write $\nu \ll \mu$, if for every $B \in \mathscr{F}$ s.t. $\mu(B) = 0$ we have $|\nu|(B) = 0$. If μ, ν are real measures, we say that they are *mutually singular*, and write $\nu \perp \mu$, if there exists $E \in \mathscr{F}$ such that $|\mu|(E) = 0$ and $|\nu|(X \setminus E) = 0$.

If μ,ν are mutually singular measures, the equality $|\mu+\nu| = |\mu| + |\nu|$ holds. If $\mu \ll \nu$ and $\nu \ll \mu$ we say that μ and ν are *equivalent* and write $\mu \approx \nu$. If μ is a positive measure and

$f \in L^1(X,\mu)$, then the measure $\nu := f\mu$ defined below is absolutely continuous with respect to μ and the following integral representations hold:

$$\nu(B) = \int_B f\,d\mu, \quad |\nu|(B) = \int_B |f|\,d\mu \qquad \forall B \in \mathscr{F}. \tag{1.2.11}$$

We recall the Radon-Nikodym theorem.

THEOREM 1.2.20 (Radon-Nikodym). *Let μ be a positive σ-finite measure and let ν be a real measure. Then there is a unique pair of real measures ν^a, ν^s such that $\nu^a \ll \mu$, $\nu^s \perp \mu$ and $\nu = \nu^a + \nu^s$. Moreover, there is a unique function $f \in L^1(X,\mu)$ such that $\nu^a = f\mu$. The function f is called the density (or Radon-Nikodym derivative) of ν with respect to μ and is denoted by $d\nu/d\mu$ or $\frac{d\nu}{d\mu}$.*

Since trivially each real measure μ is absolutely continuous with respect to $|\mu|$, from the Radon-Nikodym theorem the *polar decomposition* of μ follows: there exists a unique real valued function $f \in L^1(X,|\mu|)$ such that $\mu = f|\mu|$ and $|f| = 1$ $|\mu|$-a.e.

Image measure. We recall the notions of *push-forward* of a measure (or *image measure*) and the constructions and main properties of the *product measure*. The push-forward of a measure generalises the classical change of variable formula.

DEFINITION 1.2.21. [Push-forward] Let $(X,\mathscr{F})$ and $(Y,\mathscr{G})$ be measurable spaces, and let $f : X \to Y$ be such that $f^{-1}(F) \in \mathscr{F}$ whenever $F \in \mathscr{G}$. For any positive or real measure μ on $(X,\mathscr{F})$ we define the push-forward measure or the law of μ under f, that is the measure $\mu \circ f^{-1}$, sometimes denoted by $f_\#\mu$, in $(Y,\mathscr{G})$ by

$$\mu \circ f^{-1}(F) := \mu\left(f^{-1}(F)\right) \qquad \forall F \in \mathscr{G}.$$

By the previous definition we have the *change of variables formula*. If $u \in L^1(Y,\mu \circ f^{-1})$, then $u \circ f \in L^1(X,\mu)$ and we have the equality

$$\int_Y u\,d(\mu \circ f^{-1}) = \int_X (u \circ f)\,d\mu. \tag{1.2.12}$$

Product measure. We consider now two measure spaces and describe the natural resulting structure on their Cartesian product.

DEFINITION 1.2.22. [Product σ-algebra] Let $(X_1,\mathscr{F}_1)$ and $(X_2,\mathscr{F}_2)$ be measure spaces. The *product σ-algebra* of $\mathscr{F}_1$ and $\mathscr{F}_2$, denoted by $\mathscr{F}_1 \times \mathscr{F}_2$, is the σ-algebra generated in $X_1 \times X_2$ by

$$\mathscr{G} = \{E_1 \times E_2 \colon E_1 \in \mathscr{F}_1, E_2 \in \mathscr{F}_2\}.$$

REMARK 1.2.23. Let $E \in \mathscr{F}_1 \times \mathscr{F}_2$; then for every $x \in X_1$ the section $E_x := \{y \in X_2 \colon (x,y) \in E\}$ belongs to $\mathscr{F}_2$, and for every $y \in X_2$ the section $E^y := \{x \in X_1 \colon (x,y) \in E\}$ belongs to $\mathscr{F}_1$. In fact, the families

$$\mathscr{G}_x := \{F \in \mathscr{F}_1 \times \mathscr{F}_2 \colon F_x \in \mathscr{F}_2\}, \quad \mathscr{G}^y := \{F \in \mathscr{F}_1 \times \mathscr{F}_2 \colon F^y \in \mathscr{F}_1\}$$

are σ-algebras in $X_1 \times X_2$ and contain $\mathscr{G}$.

THEOREM 1.2.24. *Let $(X_1,\mathscr{F}_1,\mu_1)$, $(X_2,\mathscr{F}_2,\mu_2)$ be measure spaces with μ_1, μ_2 positive and finite. Then, there is a unique positive finite measure μ on $(X_1 \times X_2, \mathscr{F}_1 \times \mathscr{F}_2)$, denoted also by $\mu_1 \otimes \mu_2$, such that*

$$\mu(E_1 \times E_2) = \mu_1(E_1) \cdot \mu_2(E_2) \qquad \forall E_1 \in \mathscr{F}_1, \forall E_2 \in \mathscr{F}_2.$$

Furthermore, for any μ-measurable function $u : X_1 \times X_2 \to [0,\infty]$ the functions

$$x \mapsto \int_{X_2} u(x,y)\,\mu_2(dy) \quad \textit{and} \quad y \mapsto \int_{X_1} u(x,y)\,\mu_1(dx)$$

are respectively μ_1-measurable and μ_2-measurable and

$$\begin{aligned}\int_{X_1\times X_2} u\,d\mu &= \int_{X_1}\left(\int_{X_2} u(x,y)\,\mu_2(dy)\right)\mu_1(dx)\\ &= \int_{X_2}\left(\int_{X_1} u(x,y)\,\mu_1(dx)\right)\mu_2(dy).\end{aligned}$$

For $n \in \mathbb{N}$, if, for every $i \in (1,\dots,n)$, $(X_i, \mathscr{F}_i, \mu_i)$, is a probability space, the product σ-algebra $\mathscr{F}_1 \otimes \cdots \otimes \mathscr{F}_n$ is that generated by the family of sets of the form

$$B_1 \times \cdots \times B_n, \quad B_i \in \mathscr{F}_i.$$

We have that if X_i are all normed vector spaces, the product of the Borel σ-algebras is the Borel σ-algebra on $X_1 \times \ldots \times X_n$.

We have that there exists a unique measure μ on $(X_1 \times \ldots \times X_n, \mathscr{F}_1 \otimes \ldots \otimes \mathscr{F}_n)$ s.t.

$$\mu(B_1 \times \ldots \times B_n) = \mu_1(B_1) \cdot \ldots \cdot \mu_n(B_n)$$

where $B_i \in \mathscr{F}_i$ for every $i \in \mathbb{N}$.

DEFINITION 1.2.25. In the above hypothesis, we say that μ is the *product measure* of $\mu_{t_1}, \dots, \mu_{t_n}$, and we denote it with $\mu_1 \otimes \ldots \otimes \mu_n$.

Fourier transforms of measures. Another important concept is that of *Fourier transform* of measures. Let X be a separable Banach space; given a probability μ on X, we define its Fourier transform $\hat{\mu} : X^* \to \mathbb{C}$ by setting

$$\hat{\mu}(\xi) := \int_X e^{i\langle x,\xi\rangle_{X,X^*}}\,\mu(dx); \tag{1.2.13}$$

if X is a Hilbert space, we can canonically define $\hat{\mu}$ on X.

We list the main elementary properties of Fourier transforms.

(1) $\hat{\mu}$ is uniformly continuous on X;
(2) $\hat{\mu}(0) = \mu(X)$;
(3) if $\hat{\mu}_1 = \hat{\mu}_2$ then $\mu_1 = \mu_2$;
(4) if $\mu_j \to \mu$ in the sense of (1.2.10), then $\hat{\mu}_j \to \hat{\mu}$ uniformly on compact sets;
(5) if (μ_j) is a sequence of probability measures and there is $\phi : X \to \mathbb{C}$ continuous in $\xi = 0$ such that $\hat{\mu}_j \to \phi$ pointwise, then there is a probability measure μ such that $\hat{\mu} = \phi$.

Lebesgue measure.

DEFINITION 1.2.26. For $d \in \mathbb{N}$ there is only a Borel measure $\mathscr{L}^d$ on $\mathbb{R}^d$ s.t. for every $x \in \mathbb{R}^d$, $a > 0$, we have

$$\mathscr{L}^d(x + a([0,1])^d) = a^d.$$

This measure is called the *Lebesgue measure* .

1.2.4. Hausdorff measures . Let $A \subset \mathbb{R}^d$; $\mathcal{C}$ the set of the countable coverings of A (i.e. the set of countable sequences of sets which union contains A), each element of $\mathcal{C}$ can be indicated as $\{O_i\}_{i\in\mathbb{N}}$; for $\delta > 0$ we denote

$$\mathcal{C}_\delta := \{\{O_i\}_{i\in\mathbb{N}} \in \mathcal{C} | \operatorname{diam}(O_j) < \delta\}.$$

DEFINITION 1.2.27. Let $A \in \mathbb{R}^d$, $n \in \mathbb{N}$ e $0 < \delta \leq \infty$.

We define the set function

$$\mathscr{H}^n_\delta(A) := \inf_{\{O_i\}_{i\in\mathbb{N}} \in \mathcal{C}_\delta} \left\{ \sum_{j=1}^{\infty} \alpha(n) \left(\frac{\operatorname{diam}(O_j)}{2} \right)^n \right\}$$

where $\alpha(n)$ is the n-Lebesgue measure of a unit ball in $\mathbb{R}^n$.

We define the measure

$$\mathscr{H}^n(A) := \sup_{\delta>0} \mathscr{H}^n_\delta(A) = \lim_{\delta\to 0} \mathscr{H}^n_\delta(A).$$

We have that it is well defined as a positive Borel measure (if restricted to Borel set), and it is called n-Hausdorff measure.

We have these properties:

(1) if $n > d$ then $\mathscr{H}^n(A) = 0$ for every $A \in \mathfrak{B}(R^d)$;
(2) if $n < d$ then $\mathscr{H}^n(A) = +\infty$ for every A open, $A \neq \varnothing$;
(3) if $n = d$ then $\mathscr{H}^n(A) = \mathscr{L}^d(A)$ for every $A \in \mathfrak{B}(R^d)$;
(4) if $\mathscr{H}^n(A) < +\infty$ for some $0 \leq n \leq d$ then for every $t \in \mathbb{N}$, $t < n$ we have $\mathscr{H}^n(A) = +\infty$ and for every $t > s$ we have $\mathscr{H}^t(A) = +\infty$

We also introduce the *spherical Hausdorff measure,* $\mathcal{S}^n$: the idea is that we define $\mathcal{C}^S$ as the set of the countable open coverings of A made by balls, and similarly to the definition of Hausdorff measure,

$$\mathcal{C}^S_\delta := \{\{O_i\}_{i\in\mathbb{N}} \in \mathcal{C}^S | \operatorname{diam}(O_j) < \delta\}.$$

$$S^n_\delta(A) := \inf_{\{O_i\}_{i\in\mathbb{N}} \in \mathcal{C}_\delta} \left\{ \sum_{j=1}^{\infty} \alpha(s) \left(\frac{\operatorname{diam}(O_j)}{2} \right)^n \right\}$$

$$\mathcal{S}^n(A) := \sup_{\delta>0} \mathcal{S}^n_\delta(A) = \lim_{\delta\to 0} \mathcal{S}^n_\delta(A).$$

1.2.5. Vector measures in X. Now, we briefly introduce the vector measures (see e.g. [**22**] Sec. 4, [**30**], [**31**]).

Hereafter, H is always a separable Hilbert space with a basis $\{h_i\}_{i\in\mathbb{N}}$; for every $n \in \mathbb{N}$ we define $F_n := < h_1, \ldots, h_n >$.

DEFINITION 1.2.28. Let $(X, \mathscr{F})$ be an algebra and H a Hilbert space. A set function μ from $\mathscr{F}$ in H, is said to be a *(finitely additive) vector measure* with values in H if, for every $E_1, E_2 \in \mathscr{F}$ with $E_1 \cap E_2 = \varnothing$, we have $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$. If $E \in \mathscr{F}$ we will also say that E is μ-measurable.

We will say that μ is a *countably additive vector measure* with values in H if, for every sequence $\{E_n\}_{n\in\mathbb{N}}$ of pairwise disjoint sets in $\mathscr{F}$ we have$\mu(\bigcup_{i=1}^{+\infty} E_i) = \sum_{i=1}^{+\infty} \mu(E_i)$.

If $\mathscr{F} = \mathscr{B}(X)$ we will say that μ is a Borel countably additive vector measure.

If μ is a (finitely or countable) additive vector measure on $(X, \mathscr{F})$ and $A \in \mathscr{F}$, we can define the *restriction* $\mu_{|A} := \mu_{|\mathscr{F}_{|A}}$ (as restriction of set function), and we have that $\mu_{|A}$ is a (finitely or countable) additive vector measure on $(H, \mathscr{F}_{|A})$.

For a vector measure μ with values in H, for $h \in H$ we will write $\langle \mu, h \rangle_H$ to mean the real-valued measure defined as

$$\langle \mu, h \rangle_H (A) = \langle \mu(A), h \rangle_H$$

for all A $\mu-$measurable.

DEFINITION 1.2.29. Let H be a Hilbert space, and Ω be a topological space, μ a vector (Borel) measure on Ω with values in H.

Let $\mathcal{A}$ the set of all the finite sequences $\{A_1, \ldots, A_n\}$ of μ-measurable sets s.t. $A_i \cap A_j = \varnothing$ if $i \neq j$. Given A that is μ-measurable,

$$\mathcal{A}_A := \{\{A_1, \ldots, A_n\} \in \mathcal{A} | A_i \subseteq A \text{ for every } i \ A = \bigcup_{i=1}^{n} A_i\}.$$

The set function on $\mathbb{R}^+ \cup \{+\infty\}$ defined as

$$|\mu|(A) = \sup\{\sum_{i=1}^{n} |\mu(A_i)|_H \ |\{A_i\}_{i \in \{1,\ldots,n\}} \in \mathcal{A}_A\},$$

for all A that is μ-measurable is a finitely additive (see (see e.g. [**22**] Sec. 4, [**30**])), and it is called *variation measure* of μ.

We say that μ has bounded variation if $|\mu|(\Omega) < +\infty$.

PROPOSITION 1.2.30. *μ is a countably additive vector measure if and only if $|\mu|$ is a (countably additive) positive finite measure.*

Clearly, each component of μ is absolutely continuous respect to $|\mu|$.

We have this easy extension of the Radon-Nikodym theorem (see e.g. [**22**] Theo 4.4, [**30**] Thm. 13.4).

LEMMA 1.2.31. *If μ is a countably additive vector measure on Ω with values in a Hilbert space H, then there exists a function σ s.t. $|\sigma|_H = 1$ $|\mu|$-almost everywhere and $\mu = \sigma|\mu|$, i.e. for all $h \in H$, A $|\mu|$-measurable set*

$$\langle \mu(A), h \rangle_H = \int_A \langle \sigma(x), h \rangle_H \ |\mu|(dx).$$

Hereafter, $\mu = \sigma|\mu|$ will be called the polar decomposition of μ.

We recall a generalization of a part of Lusin Theorem (see e.g. [**16**] Thm. 7.1.13).

LEMMA 1.2.32. *If Ω is a metric space, μ is a Radon measure on Ω, Y is a separable Banach space and $f : \Omega \to Y$ is a μ-measurable function, then for every $\varepsilon > 0$ there exists $f_\varepsilon : \Omega \to Y$ continuous s.t. $\mu(\{x \in \Omega \text{ s.t. } f(x) \neq f_\varepsilon(x)\}) < \varepsilon$.*

We recall also this known fact (see e.g. [**8**], 3.1 for a finite dimensional version) and we prove it.

LEMMA 1.2.33. *If Ω is a metric space, μ is a Radon measure on $\Omega$and H is a separable Hilbert space, then*

$$|\mu|(\Omega) := \sup\{\int_X d\langle f, \mu \rangle_H | f \in C_b(\Omega, H), \ |f|_H \leq 1 \text{ everywhere}\}.$$

PROOF. Let $\sigma|\mu|$ a polar decomposition of μ (see Lemma 1.2.31). Clearly

$$\int_\Omega d\langle f,\mu\rangle_H = \int_\Omega \langle f,\sigma\rangle_H \; d|\mu| \leq |\mu|(\Omega)$$

for all measurable f s.t.$|f|_H \leq 1$ everywhere.

By Lemma 1.2.32, for each $\varepsilon > 0$ the function σ can be approximated by a continuous function σ_ε s.t. $|\mu|(\{x \in \Omega \text{ s.t. } \sigma(x) \neq \sigma_\varepsilon(x)\}) < \varepsilon$; now we can define the continuous function

$$f_\varepsilon := \begin{cases} \frac{\sigma_\varepsilon(x)}{|\sigma_\varepsilon(x)|_H} & \text{if } |\sigma_\varepsilon|_H(x) > 1 \\ \sigma_\varepsilon(x) & \text{otherwise} \end{cases};$$

we have that $\|f_\varepsilon\| \leq 1$ $|\mu|$-almost everywhere, and $f_\varepsilon(x) \neq \sigma(x)$ only if $\sigma_\varepsilon \neq \sigma$ or if $x \in A$ where

$$A := \{x \in \Omega | \sigma(x) = \sigma_\varepsilon(x) \text{ and } |\sigma_\varepsilon(x)|_H > 1\} \subseteq \{x \in \Omega | \, \|\sigma\|(x) > 1\},$$

A is Borel (σ is measurable and σ_ε is continuous) and $|\mu|(A) = 0$ because $\|\sigma\| = 1$ $|\mu|$-almost everywhere. So we have

$$|\mu|(\{x \in \Omega \text{ s.t. } \sigma(x) \neq f_\varepsilon(x)\}) < \varepsilon$$

and

$$\left| |\mu|(\Omega) - \int_X \langle f,\sigma\rangle \; d|\mu| \right| \leq 2\varepsilon$$

and we can conclude. □

DEFINITION 1.2.34. Given $\Omega \subseteq X$ open set, for each $h \in H$, the set $\mathrm{Lip}_{0,h}(\Omega)$ will be the set of the restrictions to Ω of the measurable functions f on X s.t., there exists $c > 0$ s.t., for every $x \in X$, the function $f_x : t \mapsto f(x+th)$ is Lipschitz with Lipschitz constant less than c, and $f = 0$ everywhere out of Ω.

DEFINITION 1.2.35. For each $m \in \mathbb{N}$, for $\Omega \subset X$ open set, we define $\mathrm{Lip}_{0,m}(\Omega, H)$ as the set of functions $f : \Omega \to < h_1, \dots, h_m >$, s.t. $f_i \in \mathrm{Lip}_{0,h_i}(\Omega)$ for each $i \in \{1, \dots, m\}$.

The next Lemma is inspired by ([**51**], Lem. 2.3). We recall that, by Lemma 1.2.31, there exists σ measurable s.t. $|\sigma|_H = 1$ and $\mu = \sigma|\mu|$ (polar decomposition).

LEMMA 1.2.36. *If Ω is a open set in X, if μ is a countable additive vector measure on Ω with values in H and bounded total variation and $\sigma|\mu|$ is the polar decomposition, then*

$$|\mu|(\Omega) = \sup\{\int_\Omega \langle \sigma, f\rangle_H \; d|\mu| : \; m \in \mathbb{N}, \; f \in Lip_{0,m}(\Omega, H), \; \sup_{x\in\Omega} |f(x)|_H \leq 1\}.$$

PROOF. It is obvious that

$$\int_\Omega \langle \sigma, f\rangle_H \; d|\mu| \leq \int_\Omega |\sigma|_H \, d|\mu| = |\mu|(\Omega)$$

for all $m \in \mathbb{N}$, $f \in \mathrm{Lip}_{0,m}(\Omega, H)$, $\sup_{x\in\Omega} |f(x)|_H \leq 1$.

Let $\varepsilon > 0$.

By the Fatou lemma, there exists m s.t.

$$|\mu|(\Omega) - \int_\Omega \sqrt{\sum_{i=1}^m |\sigma_i|^2} \, d|\mu| \leq \varepsilon;$$

so, we can consider μ_m, a measure with values in $\mathbb{R}^m$ s.t., for every $i \in 1,\dots,m$,

$$\langle \mu, h_i \rangle_H = \langle \mu_m, e_i \rangle_{\mathbb{R}^d}$$

where e_i is the versor in the direction of the i-th axis, and we have $\langle \mu_m, e_i \rangle_{\mathbb{R}^d} = \sigma_i |\mu|$; we have $|\mu_m| = \sqrt{\sum_{i=1}^m \sigma_i^2} |\mu|$ (so $|\mu_m|(A) \le |\mu|(A)$ for every A which is μ-measurable) and if $\sigma_{(m)} = \sum_{i=1}^m \left(\sum_{i=1}^m \sigma_i^2\right)^{-\frac{1}{2}} \sigma_i e_i$ we can rewrite $\mu_m = \sigma_{(m)} |\mu_m|$ (polar decomposition). By what we said, we have

$$|\mu|(\Omega) - |\mu_m|(\Omega) \le \varepsilon. \tag{1.2.14}$$

Now, we can apply Lemma 1.2.33, so there exists a function $l \in C(\Omega, \mathbb{R}^m)$, $\sup_{x \in \Omega}(|l(x)|) \le 1$ s.t.

$$|\mu_m|(\Omega) \le \int_\Omega \left\langle \sigma_{(m)}, l \right\rangle_{\mathbb{R}^m} \, d|\mu_m| + \varepsilon. \tag{1.2.15}$$

$|\mu|$ is Radon (because X is separable), hence we can suppose that there exists K compact in Ω s.t. $|\mu|(\Omega \backslash K) \le \varepsilon$, and hence also $|\mu_m|(\Omega \backslash K) \le \varepsilon$; hereafter $a := |\mu_m|(K)$; now, recalling that $|\sigma_{(m)}|_{\mathbb{R}^m}, |l|_{\mathbb{R}^m} \le 1$ we have

$$\int_\Omega \left\langle \sigma_{(m)}, l \right\rangle_{\mathbb{R}^m} \, d|\mu_m| \le \int_K \left\langle \sigma_{(m)}, l \right\rangle_{\mathbb{R}^m} \, d|\mu_m| + \varepsilon. \tag{1.2.16}$$

Now K is compact; we consider $\mathrm{Lip}_b(X)$ (Lipschitz bounded functions); it is a lattice, and, if we consider $x, y \in X$ ($x \ne y$ and $a, b \in \mathbb{R}$, there exists a function $\psi \in \mathrm{Lip}_b(X)$ s.t. $\psi(x) = a$, $\psi(y) = b$; so we can apply the Stone-Weierstrass theorem, for each i there exists $g_i \in \mathrm{Lip}_b(X)$ s.t. g_i approximate l_i on K (where $l_i := \langle l, e_i \rangle_{\mathbb{R}^m}$) in such a way that

$$\sup\{g_i(x) - l_i(x) | x \in K\} \le \varepsilon a^{-1} m^{-1},$$

and we introduce $g := \sum_{i=1}^m g_i e_i$, we have that

$$\sup\{|(g-l)(x)| | x \in K\} \le \varepsilon a^{-1}$$

and in particular

$$\sup\{|g(x)| | x \in K\} \le 1 + \varepsilon a^{-1}$$

(because $|l|_{\mathbb{R}^m} \le 1$); moreover if we define on $\mathbb{R}^m$ the function

$$F(x) := \begin{cases} x & \text{if } |x| \le 1 \\ \frac{x}{|x|} & \text{if } |x| > 1, \end{cases}$$

if $|x| \le 1 + \varepsilon$ then $|F(x) - x| \le \varepsilon$; if $G := F \circ g$, then we have $|G|_{\mathbb{R}^m} \le 1$ everywhere and

$$\sup\{|(G-l)(x)| | x \in K\} \le \sup\{|(G-g)(x)| | x \in K\} + \varepsilon a^{-1} \le 2\varepsilon a^{-1}$$

so by definition of a,

$$\int_K \left\langle \sigma_{(m)}, l \right\rangle_{\mathbb{R}^m} \, d|\mu_m| \le \int_K \left\langle \sigma_{(m)}, G \right\rangle_{\mathbb{R}^m} \, d|\mu_m| + 2\varepsilon \tag{1.2.17}$$

Now, we have that $\mathrm{dist}(K, X \backslash \Omega) > 0$, that K is compact and $X \backslash \Omega$ is closed, hence there exists $\theta \in \mathrm{Lip}_{0,m}(\Omega, \mathbb{R}^m)$ s.t. it is equal to 1 on K and to 0 on $X \backslash \Omega$, and $|\theta| \le 1$ everywhere; hence if

$\psi := G\theta$, then $\psi \in \mathrm{Lip}_{0,m}(\Omega, \mathbb{R}^m)$, $|\psi|_{\mathbb{R}^m} \leq 1$ everywhere (because $|G|_{\mathbb{R}^m} \leq 1$ everywhere) and we have $g_{|K} \equiv \psi_{|K}$, so

$$\int_K \left\langle \sigma_{(m)}, g \right\rangle_{\mathbb{R}^m} \, d|\mu_m| = \int_K \left\langle \sigma_{(m)}, \psi \right\rangle_{\mathbb{R}^m} \, d|\mu_m|.$$

So, by (1.2.14), (1.2.15), (1.2.16), (1.2.17), we have

$$|\mu|(\Omega) \leq \int_K \left\langle \sigma_{(m)}, \psi \right\rangle_{\mathbb{R}^m} \, d|\mu_m| + 5\varepsilon,$$

now, if we define $f_\varepsilon := \sum_{i=1}^m \langle \psi, e_i \rangle_{\mathbb{R}^m} h_i$, we have that $f_\varepsilon \in \mathrm{Lip}_{0,m}(\Omega, H)$,

$$\sup\{|f_\varepsilon(x)|_H | x \in \Omega\} \leq 1$$

and hence

$$\int_{\Omega \setminus K} \langle \sigma, f_\varepsilon \rangle_H \, d|\mu| \leq \varepsilon$$

(because $|\mu|(\Omega \setminus K) \leq \varepsilon$); therefore

$$|\mu|(\Omega) \leq \int_\Omega \langle \sigma, f_\varepsilon \rangle_H \, d|\mu| + 5\varepsilon;$$

and by the arbitrariness of ε we concluded. □

1.3. Notions of probability theory

1.3.1. General probability, random variables and random processes. In this chapter we refer to [**16**] (particularly for conditional expectations) and to [**27**] (particularly for Markov processes).

For us, a probability space is a measure space $(\Omega, \mathscr{F}), \mu)$ where μ is a positive measure with $\mu(\Omega) = 1$.

We recall this definitions.

DEFINITION 1.3.1. A *random variable* Y on a probability space $(\Omega, \mathscr{F}, \mu)$ is a $\mathscr{F}$-measurable function $Y : \Omega \to \mathbb{R}$; if $Y \in L^1(\Omega, \mu)$ we define the *expectation*, or *mean*, of Y

$$\mathbb{E}(Y) := \int_\Omega Y \, d\mu;$$

we will write also $\mathbb{E}_\mu$ to mean $\mathbb{E}$; if $Y \in L^2(\Omega, \mu)$ we define the *variance* of Y

$$\mathrm{Var}(Y) := \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 \geq 0.$$

If $Y_1, Y_2 \in L^2(\Omega, \mu)$, we define their *covariance*

$$\mathrm{cov}(Y_1, Y_2) = \mathrm{cov}(Y_2, Y_1) := \mathbb{E}(Y_1 Y_2) - \mathbb{E}(Y_1)\mathbb{E}(Y_2).$$

Let E a topological space. A *E-valued random variable* Y on a probability space $(\Omega, \mathscr{F}, \mu)$ is a function $Y : \Omega \to E$ which is $\mathscr{F}$-measurable (with respect to $\mathfrak{B}(X)$).

For a real valued random variable Y on $(\Omega, \mathscr{F}, \mu)$ we define its law as the measure $\mu \circ Y^{-1}$ (measure on $\mathbb{R}$).

We remark that, if Y is a E-valued random variable and $B \in \mathfrak{B}(E)$, then $1\!\!1_B \circ Y$ is a random variable.

For simplicity, given a E-valued random variable Y on $(\Omega, \mathscr{F}, \mu)$ we will write, for $B \in \mathfrak{B}(E)$

$$\mu(Y \in B) := \mu(\{\omega \in \Omega | Y(\omega) \in B\};$$

when $Y(\omega) \in B$ for every ω, we say that $Y \in B$ surely; when $\mu(Y \in B) = 1$ we say that $Y \in B$ *almost surely* (a.s.).

We can also consider n topological spaces $E_1, \ldots, E_n$, and for every $i \in \{1, \ldots, n\}$ a E_i-valued random variable Y_i on $(\Omega, \mathscr{F}, \mu)$ (it is equivalent to consider a random variable $Y = (Y_1, \ldots, Y_n)$ in $E_1 \times \ldots \times E_n$), and we write, if $B_i \in \mathfrak{B}(E_i)$ for every i,

$$\mu(Y_1 \in B_1, \ldots, Y_n \in B_n) := \mu(\{\omega \in \Omega | Y_1(\omega) \in B_1, \ldots, Y_n(\omega) \in B_n\}).$$

DEFINITION 1.3.2. Let E be a topological space. A (E-valued) *stochastic* (or *random*) *process* $\{Y_t\}_{t \in I}$ on a probability space $(\Omega, \mathscr{F}, \mu)$, indexed on the interval $I = [a, b] \subseteq \overline{\mathbb{R}}$ is a function $Y : I \times \Omega \to E$ such that for any $t \in I$ the function $Y_t(\cdot) = Y(t, \cdot)$ is a random variable; we can also say that a random process is a quadruplet $Y = \{\Omega, \mathscr{F}, \{Y_t\}_{t \in I}, \mu\}$. A $\mathbb{R}$-valued stochastic process will be called a *real stochastic* (or *random*) *process*

A *d-dimensional random variable* Y on a probability space $(\Omega, \mathscr{F}, \mu)$ is a $\mathscr{F}$-measurable function $Y : \Omega \to \mathbb{R}^d$.

A $\mathbb{R}^d$-valued stochastic process will be called *d-dimensional stochastic* (or *random*) *process* $(Y_t)_{t \in I}$ on a probability space $(\Omega, \mathscr{F}, \mu)$, indexed on an interval $I \subseteq \bar{\mathbb{R}}$ is a function $Y : I \times \Omega \to \mathbb{R}^d$ such that for any $t \in I$ the function $Y_t(\cdot) = Y(t, \cdot)$ is a d-dimensional random variable on $(\Omega, \mathscr{F}, \mu)$.

DEFINITION 1.3.3. Given two stochastic processes Y_t, Z_t on a same probability space $(\Omega, \mathscr{F}, \mu)$, indexed on the interval $I = [a, b] \subseteq \overline{\mathbb{R}}$, we say that Z_t is a *version* of Y_t if, given

$$A = \{\omega \in \Omega | Y_t(\omega) = Z_t(\omega) \text{ for every } t \in [a, b]\}$$

then $A \in \mathscr{F}$ and $\mu(A) = 1$.

The typical example of stochastic process will be the standard Brownian motion: it is a stochastic process which admits various models, all respecting the above conditions.

DEFINITION 1.3.4. A real valued *standard Brownian motion* on $[0, 1]$ is a stochastic process $B_{t\,t \in [0,1]}$ on a probability space $(\Omega, \mathscr{F}, \mu)$ such that:

i) $B_0 = 0$ almost surely;
ii) for any $t, s \in [0, 1]$, $s < t$, both random variables $B_t - B_s$ and B_{t-s} have the law

$$\frac{1}{\sqrt{2\pi(t-s)}} \exp - \frac{|x|^2}{t-s} \mathscr{L}^1(dx);$$

iii) for any $0 \leq t_0 \leq t_1 \leq \ldots \leq t_n$ the random variables $B_{t_0}, B_{t_1} - B_{t_0}, \ldots, B_{t_n} - B_{t_{n-1}}$ are independent.

Let us introduce the notion of *conditional expectation*.

THEOREM 1.3.5. *We consider a probability space $(\Omega, \mathscr{F}, \mu)$, a sub-$\sigma$-algebra $\mathscr{G} \subset \mathscr{F}$. Let $X \in L^1(\Omega, \mathscr{F}, \mu)$ a random variable, there exists a random variable $Y \in L^1(\Omega, \mathscr{G}, \mu)$ such that*

$$\int_A Y \, d\mu = \int_A X \, d\mu, \qquad \forall A \in \mathscr{G}; \tag{1.3.1}$$

two random variables with this property are equal almost surely. We denote the class of this Y as $\mathbb{E}(X|\mathscr{G})$.

In addition, $|\mathbb{E}(X|\mathscr{G})| \leq \mathbb{E}(|X|\,|\mathscr{G})$ almost surely.

We call $\mathbb{E}(X|\mathscr{G})$ the *expectation of X conditioned by $\mathscr{G}$* (when $\mathscr{G}$ is obvious for the setting we call Y also *conditional expectation of X*); we will indicate it also as $\mathbb{E}_\mu(X|\mathscr{G})$.

REMARK 1.3.6. Using approximations by simple functions, we have that (1.3.1) implies

$$\int_\Omega gX d\mu = \int_\Omega g\,\mathbb{E}(X|\mathscr{G}) d\mu$$

for any bounded $\mathscr{G}$–measurable functions $g : \Omega \to \mathbb{R}$.

PROPOSITION 1.3.7. *The conditional expectation satisfies the following properties.*

i) *If $\mathscr{G} = \{\emptyset, \Omega\}$, then $\mathbb{E}(X|\mathscr{G}) = \mathbb{E}[X]$ almost surely.*
ii) $\mathbb{E}[\mathbb{E}(X|\mathscr{G})] = \mathbb{E}[X]$.
iii) *For any X, Y and $\alpha, \beta \in \mathbb{R}$, $\mathbb{E}(\alpha X + \beta Y|\mathscr{G}) = \alpha\mathbb{E}(X|\mathscr{G}) + \beta\mathbb{E}(Y|\mathscr{G})$ almost surely.*
iv) *For any countable sequence $\{X_i\}_{i\in\mathbb{N}}$, if $\sum_{i=1}^\infty \mathbb{E}(|X_i||\mathscr{G}) < \infty$ almost surely then $\sum_{i=1}^\infty \mathbb{E}(X_i|\mathscr{G}) = \mathbb{E}(\sum_{i=1}^\infty X_i|\mathscr{G})$ almost surely.*
iv) *If $X \leq Y$, then $\mathbb{E}(X|\mathscr{G}) \leq \mathbb{E}(Y|\mathscr{G})$ almost surely; in particular, if $X \geq 0$, then $\mathbb{E}(X|\mathscr{G}) \geq 0$ almost surely.*
v) *If $\mathscr{H} \subset \mathscr{G}$ is a sub-σ-algebra of $\mathscr{G}$, then almost surely*

$$\mathbb{E}(\mathbb{E}(X|\mathscr{G})|\mathscr{H}) = \mathbb{E}(X|\mathscr{H}).$$

vi) *If X is $\mathscr{G}$-measurable, then $\mathbb{E}(X|\mathscr{G}) = X$ almost surely.*
vii) *If $X, Y, X \cdot Y \in L^1(\Omega, \mathscr{F}, \mathbb{P})$ and X is $\mathscr{G}$-measurable, then*

$$\mathbb{E}(X \cdot Y|\mathscr{G}) = X \cdot \mathbb{E}(Y|\mathscr{G})$$

almost surely.

DEFINITION 1.3.8. Let Y be a E-valued variable on $(\Omega, \mathscr{F}, \mu)$, let $B \in \mathfrak{B}(E)$ and $\mathscr{G}$ a sub-σ-algebra of $\mathscr{F}$; we define the *conditional probability* of B with respect to $\mathscr{G}$ (or probability conditioned by $\mathscr{G}$). as the function $\mu(Y \in B|\mathscr{G})$: $\Omega \to \mathbb{R}$ defined by

$$\mu(Y \in B|\mathscr{G}) := \mathbb{E}_\mu(1\!\!1_B \circ Y|\mathscr{G})(\omega);$$

we have that $\mu(Y \in B|\mathscr{G})$ is a random variable on $(\Omega, \mathscr{G}, \mu)$.

PROPOSITION 1.3.9. *The conditional probability satisfies the following properties.*

i) *If $\mathscr{G} = \{\varnothing, \Omega\}$, then $\mu(X \in B|\mathscr{G}) = \mu(X \in B)$ almost surely.*
ii) *if Y is $\mathscr{G}$ measurable, then*

$$\mu(Y \in B|\mathscr{G}) = 1\!\!1_B \circ Y$$

almost surely.
iii) $\mathbb{E}[\mu(X \in B|\mathscr{G})] = \mu(X \in B)$.
iv) *For any countable sequence of $\mathscr{F}$-measurable subsets $\{B_i\}_{i\in\mathbb{N}}$ with mutually null intersection, then*

$$\sum_{i=1}^\infty \mu(X \in B_i|\mathscr{G}) = \mu(X \in \bigcup_{i=1}^\infty B_i|\mathscr{G})$$

almost surely.
v) *If $\mathscr{H} \subset \mathscr{G}$ is a sub-σ-algebra of $\mathscr{G}$, then*

$$\mathbb{E}(\mu(X \in B|\mathscr{G})|\mathscr{H}) = \mu(X \in B|\mathscr{H})$$

almost surely.

vi) *If Y_1, is a E_1-valued random variable and Y_2 is a E_2-valued random variable, if Y_1 is $\mathscr{G}$-measurable then*

$$\mu((Y_1,Y_2) \in B_1 \times B_2|\mathscr{G}) = (1\!\!1_{B_1} \circ Y_1) \cdot \mu(Y_2 \in B_2|\mathscr{G})$$

almost surely.

1.3.2. Markov processes. Let $(\Omega, \mathscr{F})$ be a measurable space; a *filtration* $\{\mathscr{F}_t\}_{t\in[0,+\infty]}$ in $(\Omega, \mathscr{F})$ is a family of σ-algebras s.t. $\mathscr{F}_t \subset \mathscr{F}$ for every $t \in \mathbb{R}^+$, and, $\mathscr{F}_s \subset \mathscr{F}_t$ for $0 \leq s < t$. A filtration $\{\mathscr{F}_t\}_{t\in[0,+\infty]}$ is said *adapted* to a E-valued stochastic process $\{Y_t\}_{t\in\mathbb{R}^+}$ (or Y) if $Y_t : \Omega \to E$ is $\mathscr{F}_t$-measurable for every $t \in I$.

DEFINITION 1.3.10. In the above setting, a function $\tau : \Omega \to [0, +\infty]$ is called a *stopping time* if it is a random variable on $(\Omega, \mathscr{F})$ s.t. $\{\omega \in \Omega | \tau(\omega) \leq t\} \in \mathscr{F}_t$ for every $t \in \mathbb{R}^+$(hence $\tau \wedge t$ is real variable in $(\Omega, \mathscr{F}_t)$).

For a stopping time τ we define the σ-algebra $\mathscr{F}_\tau$ generated by

$$\{A \cap \{\omega \in \Omega | \tau(\omega) \leq t\} | t \geq 0, A \in \mathscr{F}_t\};$$

clearly $\mathscr{F}_\tau$ is a sub-σ-algebra of $\mathscr{F}$.

We give a definition of the Markov processes, which are linked to stochastic processes (we base on [**27**]). Hereafter, E is a topological space; we define E_δ as a set given by E and a point ∂ (cemetery point): $E_\partial = E \cup \partial$, with a topology s.t. E is a open subspace of E_∂, so the Borel algebra of E_∂ is

$$\mathfrak{B}(E_\partial) = \mathfrak{B}(E) \cup \{B \cup \{\partial\} | B \in \mathfrak{B}(E)\}.$$

DEFINITION 1.3.11. We define *Markov process* on $(E, \mathfrak{B}(E))$ is a quintuplet

$$Y = (\Omega, \mathscr{F}, \{Y_t\}_{t\in[0,+\infty]}, \{\mu_x\}_{x\in E_\partial}, \{\mathscr{F}_t\}_{t\in[0,+\infty]})$$

(where $(\Omega, \mathscr{F})$ is a measurable space, $\{\mathscr{F}_t\}_{t\in[0,+\infty]}$ is a filtration on $(\Omega, \mathscr{F})$, Y_t is a function $\Omega \to E$ for every $t \in [0, +\infty]$, and μ_x is a probability on $(\Omega, \mathscr{F})$ for every x) which satisfies these conditions.

i) For each $x \in E_\partial$ we have that $(\Omega, \mathscr{F}, \{Y_t\}_{t\in[0,+\infty]}, \mu_x)$ is a E_∂-valued stochastic process of E_∂ s.t. $Y_{+\infty}(\omega) = \partial$ for every $\omega \in \Omega$.

ii) For each $t \geq 0$, $B \in \mathfrak{B}(E_\partial)$, the function on E_∂ defined as

$$x \mapsto \mu_x(\{\omega \in \Omega | Y_t(\omega) \in B)\}$$

is $\mathfrak{B}(E_\partial)$-measurable.

iii) Y_t is $\mathscr{F}_t$-measurable for every $t \in [0, +\infty]$ and, for every $x \in E_\partial$, s, $t \geq 0$, $B \in \mathfrak{B}(E_\partial)$ we have that

$$\mu_x(Y_{s+t} \in B|\mathscr{F}_t)(\omega) = \mu_{(Y_t(\omega))}(Y_s \in B) \text{ for } \mu_x\text{-almost every } \omega \in \Omega.$$

iv) $\mu_\partial(Y_t = \partial) = 1$ for every $t \geq 0$.

v) $\mu_x(Y_0 = x) = 1$ for every $x \in E_\partial$.

In this setting, for every $\omega \in \Omega$ we define the *sample path* of ω as the map $[0, +\infty) \to E_\partial$, $t \mapsto Y_t(\omega)$.

Clearly, a Markov process Y can be seen as a stochastic process, if we don't consider the filtration $\{\mathscr{F}_t\}_{t\in[0,+\infty]}$.

For a Markov process

$$Y = (\Omega, \mathscr{F}, \{Y_t\}_{t\in[0,+\infty]}, \{\mu_x\}_{x\in E_\partial}, \{\mathscr{F}_t\}_{t\in[0,+\infty]}),$$

if τ is a stopping time on the filtration $\{\mathscr{F}_t\}_{t\in[0,+\infty]}$, and μ is a probability on $(\Omega,\mathscr{F})$, we define for every $t\in[0,+\infty]$ the random variable $Y_{\tau+t}$ on $(\Omega,\mathscr{F},\mu)$ as

$$Y_{\tau+t}(\omega) := Y_{\tau(\omega)+t}(\omega);$$

if ν is a probability on $(E_\partial, \mathfrak{B}(E_\partial))$, we define μ_ν as the measure on $(\Omega,\mathscr{F})$ defined as

$$\mu_\nu(B) := \int_{E_\partial} \mu_x(B)\, d\nu(x), \tag{1.3.2}$$

we have that μ_ν is a probability.

DEFINITION 1.3.12. Let

$$Y = (\Omega, \mathscr{F}, \{Y_t\}_{t\in[0,+\infty]}, \{\mu_x\}_{x\in E_\partial}, \{\mathscr{F}_t\}_{t\in[0,+\infty]})$$

be a Markov process on $(E,\mathfrak{B}(E))$. We will say that it is a *strong Markov process* if it satisfies the *strong Markov property*, i.e. for every ν probability measure on $(E_\partial,\mathfrak{B}(E_\partial))$, $t\geq 0$, $B\in\mathfrak{B}(E_\partial)$, and for every stopping time τ on $\{\mathscr{F}_t\}_{t\in[0,+\infty]}$ we have

$$\mu_\nu(Y_{\tau+t}\in B|\mathscr{F}_\tau)(\omega) = \mu_{(Y_{\tau(\omega)}(\omega))}(Y_t\in B) \text{ for } \mu_\nu\text{-almost every } \omega\in\Omega.$$

In particular, the strong Markov property implies that for every $x\in E_\partial$, $t\geq 0$, $B\in\mathfrak{B}(E_\partial)$, and for every stopping time τ on $\{\mathscr{F}_t\}_{t\in[0,+\infty]}$ we have

$$\mu_x(Y_{\tau+t}\in B|\mathscr{F}_\tau)(\omega) = \mu_{(Y_{\tau(\omega)}(\omega))}(Y_t\in B) \text{ for } \mu_x\text{-almost every } \omega\in\Omega.$$

Hereafter, we will always suppose that E is a separable metric space (in [**27**] there is a weaker hypothesis, that E is a Lusin space).

DEFINITION 1.3.13. Let E be a separable metric space. A strong Markov process

$$Y = (\Omega, \mathscr{F}, \{Y_t\}_{t\in[0,+\infty]}, \{\mu_x\}_{x\in E_\partial}, \{\mathscr{F}_t\}_{t\in[0,+\infty]})$$

on $(E,\mathfrak{B}(E))$ is called a Borel right process if it satisfies this properties:

i) for every $\omega\in\Omega$, the set $I := \{t\in[0,+\infty]|Y_t(\omega)=\partial\}$ is a closed interval and $+\infty\in I$;
ii) for every $t\geq 0$ there exists a map $\theta_t:\Omega\to\Omega$ s.t. $Y_{s+t} = Y_s\circ\theta_t$ for every $s\geq 0$; this map is called *shift operator;*
iii) for every $\omega\in\Omega$, the sample path $t\mapsto Y_t(\omega)$ is right continuous on $[0,+\infty]$.

Let Y be a Borel right process on $(E,\mathfrak{B}(E))$, let $B\in\mathfrak{B}(E)$, we call *exit time* from B the random variable given by

$$\omega\mapsto \inf\{t\in[0,+\infty]|Y_t(\omega)\notin B\}$$

and *hitting time* in B the random variable given by

$$\omega\mapsto \inf\{t\in(0,+\infty]|Y_t(\omega)\in B\}$$

We have this proposition (see e.g. [**27**], Appendix, Theo A.1.19).

PROPOSITION 1.3.14. *Let E be a metric separable space, $B\in\mathfrak{B}(E)$; if Y is a Borel right process on $(E,\mathfrak{B}(E))$, the exit time from B and the hitting time B are stopping times with respect to the filtration of Y.*

1.3.3. Markov processes and symmetric forms. Let Y be a Markov process,

$$Y = (\Omega, \mathscr{F}, \{Y_t\}_{t\in[0,+\infty]}, \{\mu_x\}_{x\in E_\partial}, \{\mathscr{F}_t\}_{t\in[0,+\infty]})$$

we define for every $t \geq 0$ the *kernel measure* on $\mathfrak{B}(E) \times E$ (or kernel)

$$K_t^x(B) := \mu_x(Y_t \in B);$$

for every $t > 0$, $x \in E$, the set function K_t^x is a positive bounded measure with $K_t^x(E) \leq 1$; it is a probability if $\mu_x(Y_t = \partial) = 0$. Moreover, we can define the *transition function* P_t, a function on the set of bounded measurable real functions on E: if f is such a function

$$(P_t f)(x) = \int_E f(y)\, dK_t^x(y);$$

P_t satisfies the Chapman-Kolmogorov condition (see e.g. [**27**], Def. 1.1.13 (t.1)) i.e. if $0 < s < t$

$$P_t f = P_{t-s}(P_s f) \tag{1.3.3}$$

for every f measurable bounded function.

REMARK 1.3.15. The Chapman-Kolmogorov condition is an equality of functions, not of class of functions; the equality (1.3.3) is verified in every x, not only almost everywhere.

Now, it can be proved (see [**27**], Sec. 1.1) that, if there is a positive measure m on E, to such P_t we can associate the semigroup $\{T_t\}_{t\in[0,+\infty)}$ on $L^2(E,m)$ defined as

$$T_t(f)(x) := \int_E f(y)\, dK_t^x(y):$$

it is a strongly continuous contractive symmetric semigroup.

If Y is a Markov process on E, and if m is a positive measure on E, then we can associate to Y a self-adjoint strongly continuous contractive semigroup T_t on $L^2(E,m)$, and to T_t we can associate a closed symmetric form a.

DEFINITION 1.3.16. The form $(a, D(a))$ defined above is the *form associated to the Markov process* Y.

Let Y be a Markov process on $E \in \mathfrak{B}(\mathbb{R}^d)$, $m := \mathscr{L}^d$, to Y is associated the semigroup T_t on $L^2(E, \mathscr{L}^d)$; we suppose that a kernel p is associated to T_t; so the kernel measure is

$$K_t^x(B)(x) = \mu_x(Y_t \in B) = \int_B p(x,y,t)\, dy$$

for every $x \in E$, $B \in \mathfrak{B}(E)$ and $t > 0$; the transition function is

$$P_t f(x) = \int_E f(y) p(x,y,t)\, dy$$

for every $x \in E$, $t > 0$, f bounded measurable real functions on E, and we have also this version of the Chapman-Kolmogorov property: for every $x \in E$, $0 < s < t$, f bounded measurable real functions on E

$$\int_E f(y) p(x,y,t)\, dy = \int_E \int_E f(y) p(z,y,t-s) p(x,z,s)\, dy\, dz \tag{1.3.4}$$

1.3.4. Example: Brownian motion and absorbing Brownian motion. For this subsection we refer to ([**27**], Exa. 3.5.9).

For $d \geq 1$, there exists a Markov process on $(\mathbb{R}^d, \mathscr{L}^d)$ such that its semigroup on $L^2(\mathbb{R}^d, \mathscr{L}^d)$ has kernel

$$p(x,y,t) := (2\pi t)^{-\frac{d}{2}} \exp(-\frac{\|x-y\|^2}{2t})$$

for $x, y \in \mathbb{R}^d$ and $t > 0$); it is called *heat semigroup* (in this subsection we will indicate it as T_t), and we have that the form associated if $(W^{1,2}(\mathbb{R}^d, \mathscr{L}^d), \mathbb{D})$ where, for $f, g \in W^{1,2}(\mathbb{R}^d, \mathscr{L}^d)$,

$$\mathbb{D}(f,g) := \frac{1}{2} \int_{\mathbb{R}^d} \nabla f(x) \cdot \nabla g(x) \, dx. \tag{1.3.5}$$

The generator of T_t is the operator associated to $(W^{1,2}(\mathbb{R}^d, \mathscr{L}^d), \mathbb{D})$, it is called *Laplace operator*, and it is indicated as Δ.

We have that there exists a Borel right process with the above properties (see also [**27**] Theo 1.5.1); such a Markov semigroup Y is called a d-standard Brownian motion (the stochastic process associated to Y_0 is a d-dimensional centered Brownian motion, which we will define in the sequel).

In particular, such a process has the strong Markov property; we have that the sample path of such process is almost surely continuous in $[0, +\infty)$.

Now, for a d-standard Brownian motion Y on $(\mathbb{R}^d, \mathscr{L}^d)$

$$Y = (\Omega, \mathscr{F}, \{Y_t\}_{t \in [0,+\infty]}, \{\mu_x\}_{x \in \mathbb{R}^d_\partial}, \{\mathscr{F}_t\}_{t \in [0,+\infty]})$$

given D open set in $\mathbb{R}^d$, we can define the exit time τ_D from D: it is a stopping time by Proposition 1.3.14; we define the *absorbing Brownian motion* Y^D on $(D, \mathscr{L}^d)$ as

$$Y^D = (\Omega, \mathscr{F}, \{Y^D_t\}_{t \in [0,+\infty]}, \{\mu_x\}_{x \in D_\partial}, \{\mathscr{F}_t\}_{t \in [0,+\infty]})$$

($D_\partial = D \cup \partial$) where, for every $t \geq 0$, $\omega \in \Omega$,

$$Y^D_t(\omega) = \begin{cases} Y_t(\omega) & \text{if } t < \tau_D(\omega) \\ \partial & \text{otherwise} \end{cases}.$$

This process Y^D is also associated to a strongly continuous semigroup S_t on $L^2(D, \mathscr{L}^d)$ and to a form $(W^{1,2}_0(D, \mathscr{L}^d), \mathbb{D})$, where

$$W^{1,2}_0(D, \mathscr{L}^d) = \{f \in L^2(\mathbb{R}^d, \mathscr{L}^d) | \exists g \in W^{1,2}(\mathbb{R}^d, \mathscr{L}^d) \text{ s.t. } g_{|D} \equiv f \text{ and } g_{|\mathbb{R}^d \setminus D} \equiv 0\}$$

and $\mathbb{D}$ has the same formula of (1.3.5) on $W^{1,2}_0(\mathbb{R}^d, \mathscr{L}^d)$; it is a closed, continuous and symmetric form.

The operator associated on L^2 to this form and which generates S_t is called *Laplace operator with Dirichlet boundary condition* and is indicated as Δ_D ($D(\Delta_D) \subset W^{1,2}_0(D, \mathscr{L}^d)$).

REMARK 1.3.17. Clearly, for every $r \geq 0$ and $x \in D$ we have $\mu_x(\tau_D \leq r) = \mu_x(Y^D_r = \partial)$.

Now, we consider the transition functions: the kernel measure of Y can be written as

$$P_t(B)(x) = \int_B p(x,y,t) \, dy = \mu_x(Y_t \in B)$$

for every $x \in \mathbb{R}^d$, $B \in \mathfrak{B}(\mathbb{R}^d)$ and $t > 0$.

Let P_t^D the kernel measure of Y^D and $x \in D$: we have that for every $B \in \mathfrak{B}(D)$

$$P_t^D(B)(x) = \mu_x(Y_t^D \in B) = \mu_x(Y_t \in B, \tau_D \geq t) \leq \mu_x(Y_t \in B);$$

now $P_t^D(\cdot)(x)$ is a positive measure absolutely continuous with respect to $P_t(\cdot)(x)$, which is absolutely continuous respect to $\mathscr{L}^d$; so given $x \in \Omega$ and $t \geq 0$, there exists $q_t^x : D \to \mathbb{R}$, the density of $P_t^D{}_{|\Omega}$ with respect to $\mathcal{L}_{|\Omega}$, and we have, for every $t > 0$, for $\mathscr{L}^d$-a.e. $x, y \in D$

$$q(x,y,t) := q_t^x(y) \leq p(x,y,t) = (2\pi t)^{-\frac{d}{2}} \exp(-\frac{\|x-y\|^2}{2t}) \tag{1.3.6}$$

(so we can suppose that $0 \leq q(x,y,t) \leq (2\pi t)^{-\frac{d}{2}}$ everywhere) and

$$P_t^D(B)(x) = \int_B q(x,y,t)\, dy,$$

for every $x \in \mathbb{R}^d$, $B \in \mathfrak{B}(\mathbb{R}^d)$ and $t > 0$; so in particular we have that the associated semigroup has the form

$$S_t f(x) = \int_\Omega f(y) q(x,y,t)\, dy$$

for every $f \in L^2(D, \mathscr{L}^d)$, $t > 0$; hence the q is the kernel associated to this absorbing Brownian motion.

Now, we have that S_t can be extended to a semigroup on the complexification Z of $L^2(D, \mathscr{L}^d)$, and it has the same kernel q; $(W_0^{1,2}(D, \mathscr{L}^d), \mathbb{D})$ can be easily extended to a closed sesquilinear form $(D(b), b)$ on Z; it is a closed continuous sesquilinear form, so we can apply the Proposition 1.1.19, and we have that S_t is bounded holomorphic; hence, by (1.3.6) we are in the hypotheses of Proposition 1.1.21, and we can deduce this result.

PROPOSITION 1.3.18. *In this setting, for $D \subseteq \mathbb{R}^d$, D open, the transition function has a kernel q which, for every $k \in \mathbb{N}$, is k-times differentiable in t and there exists $C, c > 0$ s.t.*

$$\left|\frac{\partial^k}{\partial t^k} q(x,y,t)\right| \leq C t^{-\frac{d}{2}-k} \exp\left(-\frac{c|x-y|^2}{t}\right)$$

for every $x, y \in D$ and $t > 0$.

1.3.5. Bessel process. Let

$$B = (\mathcal{A}, \mathscr{F}, \{B_t\}_{t\in[0,+\infty]}, \{\mu_x\}_{x\in\mathbb{R}^d_\partial}, \{\mathscr{F}_t\}_{t\in[0,+\infty]})$$

be a d-dimensional standard Brownian motion (as a Markov process). We have that the sample path is almost surely continuous for every starting point x, so it is not restrictive to suppose that the sample path are continuous for every $\omega \in \mathcal{A}$.

Fixed x, we can define $\{B_t\}_{t\in[0,+\infty]}$ a Brownian motion (stochastic process) in d dimension with starting point $x \in \mathbb{R}^d$ on a probability space $(\mathcal{A}, \mathscr{F}, \mu)$ $(\mu = \mu_x)$. The Bessel process R_t in dimension d with starting point $r = \|x\|$ associated to B_t is the stochastic process on the same probability space with values in $\mathbb{R}^+$ given by $R_t(a) = \|B_t(a)\|$ for every $a \in \mathcal{A}$.

We have this Lemma (see e.g. [**47**], Prop. 3.21) .

LEMMA 1.3.19. *Let $d \in \mathbb{N}$, $d \geq 2$, $r > 0$. If R_t is the Bessel process in dimension d on $(\Omega, \mathscr{F}, \mu)$ with starting point r, there exists a standard one dimensional Brownian motion S_t on $(\Omega, \mathscr{F}, \mu)$ with starting point 0 s.t. almost surely*

$$R_t := r + \int_0^t \frac{d-1}{2R_s}\,ds + S_t.$$

In particular, let $M, t > 0$, and $\omega \in \Omega$ s.t. $R_s \geq M$, for every $s \in [0,t]$, we have

$$R_t(\omega) \leq r + t\frac{d-1}{2M} + S_t(\omega).$$

We have this result, which proof is a modification of the proof of ([**46**], Proposition 3.1).

LEMMA 1.3.20. *In the above setting, if x is the starting point of the Brownian motion, if A is a ball of center $y \in \mathbb{R}^d$ (with $|x-y| = r$) and radius $a < 1$, then, for some $c > 0$ independent of x, y, a and u,*

$$\mu(E_u) \leq c\left(\frac{r}{\sqrt{u}} + \frac{r}{a}\right)$$

for every $u > 0$, where $E_u := \{\omega \in \mathcal{A} | \nexists t \in [0,u]$ s.t. $B_t(\omega) \in A\}$ and $r := \|x-y\| - a$.

PROOF. Let τ be the hitting time of A (we consider the Markov process); by 1.3.14, τ is a stopping time, in particular

$$E_u = \{\omega \in \mathcal{A} | \tau(\omega) < u\} \in \mathscr{F}.$$

If B_t is the Brownian process with starting point in x, if we define $R_t = \|B_t - y\|$, it is a d-dimensional Bessel process with starting point $\|x-y\|$; now (by arguing as in [[**46**], Lemma 3.1] with $\|x-y\| - a$ instead of $q(x)$ and $a < 1$ instead of δ) we have that Lemma 1.3.19 yields the existence of a 1-dimensional Brownian process S_t starting at 0 s.t.

$$E_u = \{\omega \in \mathcal{A} | \inf_{t\in[0,u]} |R_t(\omega)| \geq a] \subseteq \{\omega \in \Omega | R_t(\omega) \leq \|x-y\| + \frac{d-1}{2a}t + S_t(\omega) \text{ for all } t \in [0,u]\},$$

hence

$$\{\omega \in \mathcal{A} | \inf_{t\in[0,u]} R_t(\omega) \geq a\} \subseteq \{\omega \in \Omega | a \leq \|x-y\| + \frac{d-1}{2a}t + S_t(\omega) \text{ for all } t \in [0,u\} =: E_u^1.$$

We have that $E_u^1 = \bigcap_{E_i \in I} E_i$ where I is the collections of sets s.t., for some $\{t_1, \ldots, t_n\} \subset \mathbb{Q} \cap [0,u]$

$$a \leq \|x-y\| + \frac{d-1}{2a}t_n + S_{t_n}(\omega)$$

for every n: this because $\mathcal{A}$ was chosen s.t. the sample path are always continuous; so $I \subset \mathscr{F}$ and hence $E_u^1 \in \mathscr{F}$.

We have $r > 0$ because $x \in \Omega$; we consider the hitting time defined as

$$\tau_{-r}(\omega) := \inf_{t\in(0,\infty]} \left\{\left(\frac{d-1}{2a}t + S_t(\omega)\right) \leq -r\right\}$$

for every $\omega \in \Omega$; $\frac{d-1}{2a}t + S_t$ is a process called 1-dimensional Brownian motion with drift; by the formulas [**19**], Part II, Section 2,(2.0.2(and (2.0.2)(1) we have

$$\mu(E_u^1) = \mu(\tau_{-r} \in (u, +\infty)) + \mu(\tau_{-r} = +\infty) =$$

$$= \int_u^\infty \frac{r}{\sqrt{2\pi t^3}} \exp\left(-\frac{\left(r + \frac{d-1}{2a}t\right)^2}{2t}\right) dt + 1 - \exp\left(-\frac{d-1}{2a}r - |\frac{d-1}{2a}r|\right) \leq$$

(by $\exp z \leq 1$ for $z \leq 0$, and by $1 - \exp(-s) < s$ for $s > 0$)

$$\leq \frac{r}{\sqrt{2\pi}} \int_u^\infty t^{-\frac{3}{2}}\, dt + \frac{d-1}{a} r \leq$$
$$\leq c\left(\frac{r}{\sqrt{u}} + \frac{r}{a}\right)$$

for some $c > 0$ independent on r, u, a. □

LEMMA 1.3.21. *In the above setting, if $d > 2$, if x is the starting point of the Brownian motion, if A is a ball of center $y \in \mathbb{R}^d$ (with $\|x - y\| = r$) and radius $a < 1$, then,*

$$\mu_x(E) = \frac{a^{2-d}}{r^{2-d}}$$

where $E := \{\omega \in \mathcal{A} | \exists t \in [0, +\infty)$ s.t. s.t. $Z_t(\omega)) \in A\}$ and $r := \|x - y\| - a$.

PROOF. It is not restrictive to suppose $y = 0$; so, $\|B_t(a) - y\|$ is, as we said, a Bessel process with starting point $\|x - y\|$. So, let τ the hitting time for R of the ball centered in 0 with radius a. We have

$$E = [\tau < +\infty],$$

so, we can apply the formula for the hitting time of Bessel process ([**19**], Appendix 1, Part II, Cap 4, formula 2.0.2 (1)), and we immediately conclude. □

CHAPTER 2

Wiener spaces

In this Chapter 2, we recall a great part of the theory of Gaussian measures, which allows to define Wiener spaces; knowing the measure theory, this Chapter is self-contained. The topic of differentiation in Wiener spaces is left to Chapter 3.

The main reference here is [**14**].

Sections 2.1 and 2.2 define the basic concepts and properties of Gaussian measure; in Section 2.3, we introduce the concept of (abstract) Wiener space which can be seen as a generalization of the classical Wiener space, and, in this work, it is actually a separable Banach space endowed with a Gaussian measure; hence we recap the concepts which will be used later; in Section 2.4 we introduce the cylindrical functions and the cylindrical approximations, in the setting of Gaussian measures.

Section 2.5 introduce a particular case, that of a Wiener space which is a Hilbert space; it will be used in Subsection 2.6.3 and later in the Example 2 in Subsection 7.3.2.1.

Section 2.6 introduces the Brownian motion, not as Markov process, but as classical Wiener space, which is essential in Chapter 8; by analogy we introduce, in Subsection 2.6.3, the concept of Brownian bridge, which will be used in the Example 2 in Subsection 7.3.2.1.

2.1. Gaussian measures

2.1.1. Gaussian measures in finite dimension. We recall that, for every $a \in \mathbb{R}$ and $\sigma > 0$

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} \exp\Big\{-\frac{(x-a)^2}{2\sigma^2}\Big\}\, dx = 1. \tag{2.1.1}$$

DEFINITION 2.1.1. Gaussian measures on $\mathbb{R}$ *A probability measure γ on $(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$ is called* Gaussian *if it is either a Dirac measure δ_a at a point a (in this case, we put $\sigma = 0$), or a measure absolutely continuous with respect to the Lebesgue measure λ_1 with density*

$$\frac{1}{\sigma\sqrt{2\pi}} \exp\Big\{-\frac{(x-a)^2}{2\sigma^2}\Big\}.$$

In this case we call a the mean*, $\sigma > 0$ the* mean-square deviation *and σ^2 the* variance *of γ and we say that γ is* cenetred *or* symmetric *if $a = 0$ and* standard *if in addition $\sigma = 1$.*

By elementary computations we get

$$a = \int_{\mathbb{R}} x\,\gamma(dx), \qquad \sigma^2 = \int_{\mathbb{R}} (x-a)^2\,\gamma(dx).$$

REMARK 2.1.1. For every $a, \sigma \in \mathbb{R}$ we have $\hat{\gamma}(\xi) = e^{ia\xi - \frac{1}{2}\sigma^2\xi^2}$. Conversely, a probability measure on $\mathbb{R}$ is Gaussian iff its Fourier transform has this form.

DEFINITION 2.1.2. *Gaussian measures on* $\mathbb{R}^d$ A probability measure γ on $\mathbb{R}^d$ is said to be Gaussian if for every linear functional l on $\mathbb{R}^d$ the measure $\gamma \circ l^{-1}$ is Gaussian on $\mathbb{R}$.

$\gamma^d := (2\pi)^{-d/2} e^{-\frac{|x|^2}{2}} \mathscr{L}^d$ is called the *standard Gaussian measure*. We denote by G_d the function defined as

$$G_d(x) := (2\pi)^{-d/2} e^{-\frac{|x|^2}{2}}$$

the *standard Gaussian density* in $\mathbb{R}^d$, i.e., the density of γ^d with respect to $\mathscr{L}^d$.

REMARK 2.1.3. If $d = h + k$ $(d, h, k \in \mathbb{N})$ then $\gamma^d = \gamma^h \otimes \gamma^k$.

PROPOSITION 2.1.4. *A measure γ on $\mathbb{R}^d$ is Gaussian if and only if its Fourier transform is*

$$\hat{\gamma}(\xi) = \exp\Big\{ia \cdot \xi - \frac{1}{2} Q\xi \cdot \xi\Big\} \tag{2.1.2}$$

for some $a \in \mathbb{R}^d$ and Q a nonnegative symmetric $d \times d$ matrix. Moreover, γ is absolutely continuous with respect to the Lebesgue measure λ_d if and only if Q is nondegenerate. In this case, the density of γ is

$$\frac{1}{\sqrt{(2\pi)^d \det Q}} \exp\Big\{-\frac{1}{2}\big(Q^{-1}(x-a)\cdot(x-a)\big)\Big\}. \tag{2.1.3}$$

REMARK 2.1.5. If γ is a Gaussian measure and (2.1.2) holds, we call a the *mean* and Q the *covariance* of γ. If $a = 0$ we say that γ is centered. If the matrix Q is invertible then the Gaussian measure is said to be *nondegenerate*. Its density, given by (2.1.3), is denoted $G_{a,Q}$. The nondegeneracy is equivalent to the fact that $\gamma \circ l^{-1} \ll \mathscr{L}^1$ for every $l \in \big(\mathbb{R}^d\big)^*$.

PROPOSITION 2.1.6. *Every centered Gaussian measure γ on $\mathbb{R}^d$ is invariant under the rotation map ϕ defined, for every $\theta \in \mathbb{R}$, by $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ by $\phi(x,y) := x\sin\theta + y\cos\theta$; then, the image measure $(\gamma \otimes \gamma) \circ \phi^{-1}$ in $\mathbb{R}^d$ is γ.*

REMARK 2.1.7. The property stated in Proposition 2.1.6 is not the invariance of γ under rotations in $\mathbb{R}^d$; the last is true only if the covariance of γ is λO where $\lambda > 0$ and O is an orthogonal matrix.

2.1.2. Gaussian measures in infinite dimension.

DEFINITION 2.1.8. For X vector normed space, the σ-algebra $\mathscr{E}(X)$ is that generated by the *cylindrical* sets, i.e., the sets of the form

$$C = \Big\{x \in X : \ (f_1(x), \dots, f_n(x)) \in C_0\Big\},$$

where $f_1, \dots, f_n \in X^*$ and $C_0 \in \mathfrak{B}(\mathbb{R}^n)$, called a *base* of C.

THEOREM 2.1.9. *If X is a separable Banach space, then $\mathscr{E}(X) = \mathfrak{B}(X)$. Moreover, there is a countable family $F \subset X^*$ such that for every pair of points $x \neq y \in X$ there is $f \in F$ such that $f(x) \neq f(y)$ and $\mathscr{E}(X) = \mathscr{E}(X, F)$.*

DEFINITION 2.1.10. [Gaussian measures on X] Let X be a Banach space. A probability measure γ on $(X, \mathfrak{B}(X))$ is said to be Gaussian if $\gamma \circ f^{-1}$ is a Gaussian measure in $\mathbb{R}$ for every $f \in X^*$. The measure γ is called *centred* (or *symmetric*) if all the measures $\gamma \circ f^{-1}$ are centered and it is called *nondegenerate* if for any $f \neq 0$ the measure $\gamma \circ f^{-1}$ is nondegenerate.

REMARK 2.1.11. We will use only separable Banach space in this work, but the definition of Gaussian measure usually is given for more general spaces, locally convex or Fréchet (see e.g. [**14**]).

Gaussian measures in infinite dimensions can be defined in terms of Fourier transforms, analogously to the case $\mathbb{R}^d$, (Proposition 2.1.4).

Notice that if $f \in X^*$ then $f \in L^p(X,\gamma)$ for every $p \geq 1$: indeed, the integral

$$\int_X |f(x)|^p \, \gamma(dx) = \int_{\mathbb{R}} |t|^p \, (\gamma \circ f^{-1})(dt)$$

is finite because $\gamma \circ f^{-1}$ is Gaussian in $\mathbb{R}$. Therefore, we can give the following definition.

DEFINITION 2.1.12. We define the *mean* a_γ and the *covariance* B_γ of γ by

$$a_\gamma(f) := \int_X f(x)\, \gamma(dx), \tag{2.1.4}$$

$$B_\gamma(f,g) := \int_X [f(x) - a_\gamma(f)]\,[g(x) - a_\gamma(g)]\, \gamma(dx), \tag{2.1.5}$$

$f, g \in X^*$.

Observe that $f \mapsto a_\gamma(f)$ is linear and $(f,g) \mapsto B_\gamma(f,g)$ is bilinear in X^*. Moreover, $B_\gamma(f,f) = \|f - a_\gamma(f)\|^2_{L^2(X,\gamma)} \geq 0$ for every $f \in X^*$.

THEOREM 2.1.13. *A Borel probability measure γ on X is Gaussian if and only if its Fourier transform is given by*

$$\hat{\gamma}(f) = \exp\Big\{ia(f) - \frac{1}{2}B(f,f)\Big\}, \qquad f \in X^*, \tag{2.1.6}$$

where a is a linear functional on X^* and B is a nonnegative symmetric bilinear form on X^*.

As in the finite dimensional case, we say that γ is *centered* if $a_\gamma = 0$; in this case, the bilinear form B_γ is nothing but the restriction of the inner product in $L^2(X,\gamma)$ to X^*,

$$B_\gamma(f,g) = \int_X f(x)g(x)\, \gamma(dx), \qquad B_\gamma(f,f) = \|f\|^2_{L^2(X,\gamma)}. \tag{2.1.7}$$

PROPOSITION 2.1.14. *Let X be a Banach space and let γ be a Gaussian measure on X.*

i) *If μ is a Gaussian measure on a locally convex space Y, then $\gamma \otimes \mu$ is a Gaussian measure on $X \times Y$.*

ii) *If μ is a Gaussian measure on X, then the convolution measure $\gamma * \mu$, defined as the image measure in X of $\gamma \otimes \mu$ on $X \times X$ under the map $(x,y) \mapsto x+y$ is a Gaussian measure and is given by*

$$(\gamma * \mu)(B) = \int_X \mu(B-x)\gamma(dx) = \int_X \gamma(B-x)\mu(dx). \tag{2.1.8}$$

iii) *If γ is centered, then for every $\theta \in \mathbb{R}$, given $R_\theta : X \times X \to X \times X$, $R_\theta(x,y) := (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$ the image measure $(\gamma \otimes \gamma) \circ R_\theta^{-1}$ in $X \times X$ is $\gamma \otimes \gamma$.*

iv) *If γ is centred, then for every $\theta \in \mathbb{R}$ the image measures $(\gamma \otimes \gamma) \circ \phi_i^{-1}$, $i = 1,2$ in X under the maps $\phi_i : X \times X \to X$,*

$$\phi_1(x,y) := x\cos\theta + y\sin\theta, \quad \phi_2(x,y) := -x\sin\theta + y\cos\theta$$

are again γ.

We recall that $\mathscr{E}(X) = \mathfrak{B}(X)$; we give a definition of measurable seminorm inspired by ([**14**], Def. 2.8.1).

DEFINITION 2.1.15. If X is a Banach space and γ is a Gaussian measure on X, a $\mathcal{E}(X)_\gamma$-*measurable seminorm* is a function f Borel measurable on X s.t. there exists a linear subsets $X_0 \in \mathfrak{B}(X)$ s.t. $\gamma(X_0) = 1$ and $f_{|X_0}$ is a seminorm on X_0.

We have this important result, the Fernique Theorem ([**14**], Thm. 2.8.5; Fernique theorem was introduced in [**40**]).

THEOREM 2.1.16. *Let γ be a centered Gaussian measure on a Banach space X, and let $|\cdot|$ be a $\mathcal{E}(X)_\gamma$-measurable seminorm of X. Then there exists $\alpha > 0$ such that*

$$\int_X \exp\{\alpha|x|^2\}\,\gamma(dx) < \infty.$$

If X is a Banach space, it is easily seen that the space X^* is contained in $L^2(X,\gamma)$ and the inclusion map $j : X^* \to L^2(X,\gamma)$,

$$j(f) = f - a_\gamma(f), \qquad f \in X^* \tag{2.1.9}$$

is continuous.

DEFINITION 2.1.17. [Reproducing kernel] The *reproducing kernel* is defined by the closure of $j(X^*)$ in $L^2(X,\gamma)$, we denote it by X^*_γ.

We have defined the functions a_γ in X^* and the function B_γ in $X^* \times X^*$; the extension of a_γ to X^*_γ is trivial, since the mean value of every element of X^*_γ is zero. The extension of B_γ to $X^*_\gamma \times X^*_\gamma$ is continuous ($X^*_\gamma \times X^*_\gamma$ is endowed with the $L^2(X,\gamma) \times L^2(X,\gamma)$ norm), and since $a_\gamma \equiv 0$ on X^*_γ,

$$B_\gamma(f,g) = \int_X f(x)g(x)\gamma(dx) = \langle f,g\rangle_{L^2(X,\gamma)}, \quad f,\ g \in X^*_\gamma.$$

If γ is nondegenerate then two different elements of X^* define two different elements of X^*_γ.
We have also this result.

PROPOSITION 2.1.18. *If γ is a Gaussian measure on a separable Banach space X, then $a_\gamma : X^* \to \mathbb{R}$ and $B_\gamma : X^* \times X^* \to \mathbb{R}$ are continuous. In addition, there exists $a \in X$ representing a_γ, i.e., such that*

$$a_\gamma(f) = f(a), \qquad \forall f \in X^*.$$

If γ centered then $a = 0$.

For $f \in X^*_\gamma$, we have that $|f|$ is a $\mathcal{E}(X)_\gamma$-measurable seminorm by e.g. [**14**] Thm. 2.10.9, so by the Fernique Theorem (Theorem 2.1.16) we have this Corollary.

COROLLARY 2.1.19. *If $f \in X^*_\gamma$, then there exists $c > 0$ s.t.*

$$\int_O \exp\left(c|g|^2\right)\, d\gamma < \infty.$$

2.2. The Cameron-Martin space

Hereafter, we assume that X is a separable Banach space.
We define the operator $R_\gamma : X^*_\gamma \to (X^*)'$.

$$R_\gamma f(g) := \int_X f(x)(g(x) - a_\gamma(g))\,\gamma(dx), \qquad f \in X^*_\gamma,\ g \in X^*. \tag{2.2.1}$$

i.e.

$$R_\gamma f(g) = \langle f, g - a_\gamma(g)\rangle_{L^2(X,\gamma)}. \tag{2.2.2}$$

We recall that R_γ maps X^*_γ into X (see [**14**, Theorem 3.2.3]).

PROPOSITION 2.2.1. *If X is a separable Banach space, the range of R_γ is contained in X, i.e., for every $f \in X^*_\gamma$ there is $y \in X$ such that $R_\gamma f(g) = g(y)$ for all $g \in X^*$.*

REMARK 2.2.2. By Proposition 2.2.1, we can identify $R_\gamma f$ with the element $y \in X$ representing it, i.e. we shall write

$$R_\gamma f(g) = g(R_\gamma f), \qquad \forall g \in X^*.$$

DEFINITION 2.2.3. [Cameron-Martin space] For every $h \in X$ set

$$|h|_H := \sup\Big\{f(h) :\ f \in X^*,\ \|j(f)\|_{L^2(X,\gamma)} \le 1\Big\}, \tag{2.2.3}$$

where $j : X^* \to L^2(X,\gamma)$ is the inclusion defined in (2.1.9). The *Cameron-Martin space* is defined by

$$H := \Big\{h \in X :\ |h|_H < \infty\Big\}. \tag{2.2.4}$$

If X is a Banach space, calling c the norm of $j : X^* \to L^2(X,\gamma)$, we have

$$\|h\|_X = \sup\{f(h) :\ \|f\|_{X^*} \le 1\} \le \sup\{f(h) :\ \|j(f)\|_{L^2(X,\gamma)} \le c\} = c|h|_H, \tag{2.2.5}$$

and then H is continuously embedded in X.

This embedding is even compact and the norms $\|\cdot\|_X$ and $|\cdot|_H$ are not equivalent in H, in general; they are equivalent only if X is finite dimensional and γ is non-degenerate.

The Cameron-Martin space inherits a natural Hilbert space structure from the space X^*_γ through the $L^2(X,\gamma)$ Hilbert structure.

PROPOSITION 2.2.4. *An element $h \in X$ belongs to H if and only if there is $\hat{h} \in X^*_\gamma$ such that $h = R_\gamma \hat{h}$. In this case,*

$$|h|_H = \|\hat{h}\|_{L^2(X,\gamma)}. \tag{2.2.6}$$

Therefore $R_\gamma : X^*_\gamma \to H$ is an isometry and H is a Hilbert space with the inner product

$$[h,k]_H := \langle \hat{h}, \hat{k}\rangle_{L^2(X,\gamma)}$$

whenever $h = R_\gamma \hat{h}$, $k = R_\gamma \hat{k}$.

For every $h \in R_\gamma(X^*_\gamma)$ there is only one $\hat{h} \in X^*_\gamma$ which satisfies the above condition; hereafter, we will always use the formalism $\hat{\cdot}$ in this way.

The space $L^2(X,\gamma)$ (hence its subspace X^*_γ as well) is separable, because X is separable. Therefore, H, being isometric to a separable space, is separable.

REMARK 2.2.5. The map $R_\gamma : X^*_\gamma \to X$ can be defined directly using the Bochner integral through the formula

$$R_\gamma f := \int_X (x-a) f(x)\, \gamma(dx),$$

where a is the mean of γ.

Now we describe the finite dimensional case $X=\mathbb{R}^d$. If $\gamma=\mathscr{N}(a,Q)$ then for $f\in\mathbb{R}^d$ we have

$$\|j(f)\|^2_{L^2(\mathbb{R}^d,\gamma)}=\int_{\mathbb{R}^d}\langle x-a,f\langle,\mathscr{N}(a,Q)(dx)=\langle Qf,f\langle$$

and therefore $|h|_H$ is finite if and only if $h\in Q(\mathbb{R}^d)$ and, as a consequence, $H=Q(\mathbb{R}^d)$ is the range of Q. According to the notation introduced in Proposition 2.2.4, if γ is nondegenerate, namely Q is invertible, $h=R_\gamma\hat{h}$ iff $\hat{h}(x)=\langle Q^{-1}h,x\rangle_{\mathbb{R}^d}$. if γ is nondegenerate the measures γ_h defined by $\gamma_h(B)=\gamma(B-h)$ are all equivalent to γ, and an elementary computation shows that we have

$$\rho_h(x):=\exp\Big\{(Q^{-1}h)\cdot x-\frac{1}{2}|h|^2\Big\}=\exp\Big\{\hat{h}(x)-\frac{1}{2}|h|^2\Big\},$$

where $\gamma_h:=\rho_h\gamma$. In the next theorem, we consider the infinite dimensional case.

THEOREM 2.2.6. *For $h\in X$, define the measure $\gamma_h(B):=\gamma(B-h)$. If $h\in H$ the measure γ_h is equivalent to γ and $\gamma_h=\rho_h\gamma$, with*

$$\rho_h(x):=\exp\Big\{\hat{h}(x)-\frac{1}{2}|h|^2_H\Big\},\tag{2.2.7}$$

where $\hat{h}=R_\gamma^{-1}h$. If $h\notin H$ then $\gamma_h\perp\gamma$. Hence, $\gamma_h\approx\gamma$ if and only if $h\in H$.

From now on, we denote by $B^H(x,r)$ the open ball of center $x\in H$ and radius r in H and by $\overline{B}^H(0,r)$ its closure in H. We denote by $x+\overline{B}^H(0,r)$ for $x\in X$,

$$\overline{B}_H(x,r)=\{y\in X||y-x|_H\leq r\}.$$

THEOREM 2.2.7. *Let γ be a Gaussian measure in a separable Banach space X, and let H be its Cameron-Martin space. The following statements hold.*

- i) *The unit ball $B^H(0,1)$ of H is relatively compact in X and hence the embedding $H\hookrightarrow X$ is compact.*
- ii) *If γ is centered then H is the intersection of all the Borel subspaces of X with measure 1.*
- iii) *If γ is centered and X^*_γ is infinite dimensional then $\gamma(H)=0$.*
- iv) *There exists an orthonormal basis of H that is contained in $R_\gamma(X^*)$.*

PROPOSITION 2.2.8. *Let γ be a Gaussian measure on a Banach space X. Let us assume that X is continuously embedded in another Banach space Y, i.e., there exists a continuous injection $i:X\to Y$. Then the image measure $\gamma_Y:=\gamma\circ i^{-1}$ in Y is Gaussian and the Cameron–Martin space H associated with the measure γ is isomorphic to the Cameron–Martin space H_Y associated with the measure γ_Y in Y.*

2.3. Notations about Wiener spaces

Hereafter, we will call *abstract Wiener space*, or *Wiener space*, a space (X,γ) where X is a Banach space and γ is a centered nondegenerate Gaussian measure.

In the sequel, we will write $L^p(X)$ to mean $L^p(X,\gamma)$; if Y is a normed space, we will write $L^p(X,Y)$ to mean $L^p(X,\gamma,Y)$.

By using Theorem 2.2.7 iv) we can consider a basis $\{h_i\}_{i\in\mathbb{N}}$ of H s.t. for every $j\in\mathbb{N}$ we have $\hat{h}_j\in X^*$, and $h_j\in R_\gamma(X^*)$. We define

$$\pi_d x:=\sum_{j=1}^{d}\hat{h}_j(x)h_j,\quad n\in\mathbb{N},\ x\in X.\tag{2.3.1}$$

Note that every π_d is a projection, since by (2.2.1) $\hat{h}_j(h_i) = \delta_{ij}$. Moreover, if $x \in H$, then $\hat{h}_j(x) = [x, h_j]_H$ so $\pi_d x$ is an extension to X of the orthogonal projection of H on $< h_1, \ldots, h_d >$.

More in general for every $F \subseteq R_\gamma(X^*) \subseteq H$, with $\dim(F) = d < \infty$; for every $h \in F$ we can consider $\hat{h} \in X^*$; in this setting, we define $\hat{h}$ as the bounded linear function (defined everywhere, and non only almost everywhere) which corresponds to h.

If $(h_1, \ldots, h_d)$ is a an orthonormal basis of F, we define

$$\pi_F(x) := \sum_{j=1}^{d} \hat{h}_j(x) h_j$$

function from X to F, and we that this function is linear and continuous and it does not depend on the orthonormal basis; we can consider it a the extension to X of the projection from H to F.

In this setting, we consider the measure $\gamma_F := \gamma \circ \pi_F^{-1}$; we have that γ_F is a centered nondegenerate Gaussian measure on F; if we identify F with $\mathbb{R}^d$, by identifying an orthonormal basis of F (inner product inherited from H) with the canonical basis of $\mathbb{R}^d$, then γ_F is the standard Gaussian measure γ^d. For such a F we can define

$$F^\perp := \pi_F^{-1}(0) = (I - \pi_F)^{-1}(X),$$

it is a closed set (hence it is a Banach space). We remark that $F^\perp$ is a subset of X, not of H; we remark also that $F^\perp$ is well defined as close linear subset of X for $F \subseteq R_\gamma(X^*$, but not in general for $F \subseteq H$.

If $F \subset R_\gamma(X^*)$, if we consider $I - \pi_F$ as a linear map from X to X_F, then we can define

$$\gamma_F^\perp := \gamma \circ (I - \pi_F)^{-1}$$

as a measure on $F^\perp$; it is a centered nondegenerate Gaussian measure, and the Cameron-Martin space of $(X_F, \gamma_F^\perp)$ is the orthogonal $F^\perp$ to F in H; on $F \times F^\perp$, if $F \times F^\perp$ corresponds to X through the function $(v, w) \mapsto v + w$, we have that in this setting γ is given by the product measure $\gamma_F \otimes \gamma_{F^\perp}$.

When $F =< h >$, we will write $h^\perp$ to mean $< h >^\perp$, $\gamma_h := \gamma_{<h>}$, $\gamma_{h^\perp} := \gamma_{<h>^\perp}$.

For $h \in R_\gamma(X^*)$, $y \in X_h^\perp$, we will usually define $O_y := \{t \in \mathbb{R} | y + th \in O\}$, and the function f_y on O_y defined as $f_y(t) := f(y + th)$.

2.4. Cylindrical approximations in Wiener spaces

DEFINITION 2.4.1. [Cylindrical functions] We say that $\varphi : X \to \mathbb{R}$ is a *cylindrical function* if there are $d \in \mathbb{N}$, $l_1, \ldots, l_d \in X^*$ and a function $\psi : \mathbb{R}^n \to \mathbb{R}$ such that $\varphi(x) = \psi(l_1(x), \ldots, l_d(x))$ for all $x \in X$. For $k \in \mathbb{N}$, we write $\varphi \in \mathcal{F}C_b^k(X)$ (resp. $\varphi \in \mathcal{F}C_b^\infty(X)$), and we say that φ is a cylindrical k times (resp. infinitely many times) boundedly differentiable function, if, with the above notation, $\psi \in C_b^k(\mathbb{R}^d)$ (resp. $\psi \in C_b^\infty(\mathbb{R}^d)$).

In the notations of 2.3 we have the following theorem, see e.g. [**14**] Thm. 3.5.1, Cor. 3.5.8.

THEOREM 2.4.2. *For every* $p \in [1, +\infty)$,

$$\|\pi_d x - x\|_X \xrightarrow[d \to +\infty]{L^p(X)} 0.$$

For γ*-a.e.* $x \in X$, $\lim_{n \to \infty} \pi_d x = x$.

We define

$$(\mathbb{E}_d f)(x) := \int_X f(\pi_d x + (I - \pi_d)y)\gamma(dy), \quad x \in X,$$

clearly if $f \in C_b^k(X)$ then $\mathbb{E}_d f \in \mathcal{F}C_b^k(X)$; the functions$\mathbb{E}_d f$ is called *cylindrical approximation* of f and it is the conditional expectation of f with respect to the σ-algebra $\pi_F^{-1}(\mathfrak{B}(F))$.

We have this result (see e.g. [**14**] Cor. 3.5.2).

PROPOSITION 2.4.3. *For every $1 \le p < \infty$ and $f \in L^p(X,\gamma)$ the sequence $\mathbb{E}_n f$ converges to f in $L^p(X,\gamma)$ and γ-a.e. in X.*

It has the following corollary.

COROLLARY 2.4.4. *For every $1 \le p < \infty$ the space $\mathcal{F}C_b^\infty(X)$ is dense in $L^p(X,\gamma)$.*

Now, if $F \subseteq R_\gamma(X^*)$, for every $h \in F$ the function $\hat{h} \in X_\gamma^*$ in general is not in X^*; we have that it is not continuous, but it has a (not unique) representative which is measurable and linear on X (but, in general, not bounded); so, for a sequence F_d generated by an orthonormal basis we can define π_d analogously as a measurable linear functional, and $\mathbb{E}_d f$; also in this case for γ-a.e. $x \in X$, $\lim_{d\to\infty} \pi_d x = x$, while $\mathbb{E}_d f$ converges to f in $L^p(X,\gamma)$ and γ-a.e. in X.

2.5. Hilbert space case

Let X be an infinite dimensional separable Hilbert space, with norm $\|\cdot\|_X$ and inner product $\langle\cdot,\cdot\rangle_X$. We identify X^* with X via the Riesz representation.

We say that an operator $L \in \mathcal{L}(X)$ is nonnegative if $\langle Lx, x\rangle_X \ge 0$ for all $x \in X$; an operator $L \in \mathcal{L}(X)$ is compact if the image of every bounded set is relatively compact.

We also recall that an operator $L \in \mathcal{L}(X)$ is compact if (and only if) L is the limit in the operator norm of a sequence of finite rank operators.

Let us recall that if L is a compact self-adjoint operator on X, the spectrum of L is at most countable and if the spectrum is infinite it consists of a sequence of eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$ which converges in 0. If L is compact and self-adjoint, there is an orthonormal basis of eigenvectors of X. Moreover, L has the representation

$$Lx = \sum_{k=1}^{\infty} \lambda_k \langle x, e_k\rangle_X, \qquad x \in X, \tag{2.5.1}$$

where $\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of eigenvectors and $Le_k = \lambda_k e_k$ for any $k \in \mathbb{N}$. If in addition L is nonnegative, then its eigenvalues are nonnegative.

We may define the *square root of* L by

$$L^{1/2}x = \sum_{k=1}^{\infty} \lambda_k^{1/2} \langle x, e_k\rangle_X e_k.$$

The operator $L^{1/2}$ is obviously self-adjoint, and also compact.

It can be proved that

$$T := \sum_{k=1}^{\infty} \langle Le_k, e_k\rangle_X = \sum_{k=1}^{+\infty} \lambda_k < +\infty$$

(where $\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of eigenvectors) is well defined for every L compact and self-adjoint; therefore, we can give the following definition.

DEFINITION 2.5.1. [Trace-class operators] A nonnegative self-adjoint operator $L \in \mathcal{L}(X)$ is of *trace-class* or *nuclear* if there is an orthonormal basis $\{e_k : \ k \in \mathbb{N}\}$ of X such that

$$\sum_{k=1}^{\infty} \langle Le_k, e_k \rangle_X < \infty$$

and the *trace of L* is

$$\mathrm{tr}(L) := \sum_{k=1}^{\infty} \langle Le_k, e_k \rangle_X \tag{2.5.2}$$

for any orthonormal basis $\{e_k : \ k \in \mathbb{N}\}$ of X.

For a complete treatment of the present matter we refer e.g. to [**32**, VI.5], [**33**, XI.6, XI.9].

Let γ be a Gaussian measure in X. According to Theorem 2.1.13 and (2.1.4), (2.1.5) we have

$$\hat{\gamma}(f) = \exp\Big\{ia_\gamma(f) - \frac{1}{2}B_\gamma(f,f)\Big\}, \quad f \in X^*,$$

where the linear mapping $a_\gamma : X^* \to \mathbb{R}$ and the bilinear symmetric mapping $B_\gamma : X^* \times X^* \to \mathbb{R}$ are continuous by Proposition 2.1.18. Then, there are $a \in X$ and a self-adjoint $Q \in \mathcal{L}(X)$ such that $a_\gamma(f) = \langle f, a \rangle_X$ and $B_\gamma(f,g) = \langle Qf, g \rangle_X$ for every f, $g \in X^* = X$. So,

$$\langle Qf, g \rangle_X = \int_X \langle f, x-a \rangle_X \langle g, x-a \rangle_X \gamma(dx), \quad f, \ g \in X, \tag{2.5.3}$$

and

$$\hat{\gamma}(f) = \exp\Big\{i\langle f, a \rangle_X - \frac{1}{2}\langle Qf, f \rangle_X\Big\}, \quad f \in X. \tag{2.5.4}$$

We denote by $\mathcal{N}(a,Q)$ the Gaussian measure γ whose Fourier transform is given by (2.5.4). As in finite dimension, a is called the mean and Q is called the covariance of γ.

The following theorem is analogous to Theorem 2.1.13, but there is an important difference. In Theorem 2.1.13 a measure is given and we give a criterion to see if it is Gaussian. Instead, in Theorem 2.5.2 we characterize all Gaussian measures in X.

THEOREM 2.5.2. *If γ is a Gaussian measure on X then its Fourier transform is given by* (2.5.4)*, where $a \in X$ and Q is a self-adjoint nonnegative trace-class operator. Conversely, for every $a \in X$ and for every nonnegative self-adjoint trace-class operator Q, the function $\hat{\gamma}$ in* (2.5.4) *is the Fourier transform of a Gaussian measure with mean a and covariance operator Q.*

REMARK 2.5.3. Since in infinite dimensions the identity is not a trace-class operator, the function $x \mapsto \exp\{-\frac{1}{2}\|x\|_X^2\}$ cannot be the Fourier transform of any Gaussian measure on X.

PROPOSITION 2.5.4. *Let $\gamma = \mathcal{N}(a,Q)$ be a Gaussian measure on X and let (λ_k) be the sequence of the eigenvalues of Q. If γ is not a Dirac measure, the integral*

$$\int_X \exp\{\alpha\|x\|_X^2\}\,\gamma(dx)$$

is finite if and only if

$$\alpha < \inf\Big\{\frac{1}{2\lambda_k} : \ \lambda_k > 0\Big\}. \tag{2.5.5}$$

Let us characterize X^*_γ and the Cameron-Martin space H. By definition, X^*_γ is the closure of $j(X^*)$ in $L^2(X,\gamma)$. Hereafter, if $\gamma = \mathscr{N}(a,Q)$ we fix an orthonormal basis $\{e_k : k \in \mathbb{N}\}$ of eigenvectors of Q such that $Qe_k = \lambda_k e_k$ for any $k \in \mathbb{N}$ and for every $x \in X$, $k \in \mathbb{N}$, we set $x_k := \langle x, e_k\rangle_X$.

THEOREM 2.5.5. *Let $\gamma = \mathscr{N}(a,Q)$ be a nondegenerate Gaussian measure in X. The space X^*_γ is*

$$X^*_\gamma = \Big\{ f : X \to \mathbb{R} :\ \exists z \in X \ \ s.t.\ f(x) = \sum_{k=1}^\infty (x_k - a_k) z_k \lambda_k^{-1/2} \Big\} \tag{2.5.6}$$

and the Cameron-Martin space is the range of $Q^{1/2}$, i.e.,

$$H = \Big\{ x \in X :\ \sum_{k=1}^\infty x_k^2 \lambda_k^{-1} < \infty \Big\}. \tag{2.5.7}$$

For $h = Q^{1/2} z \in H$, we have

$$\hat{h}(x) = \sum_{k=1}^\infty (x_k - a_k) z_k \lambda_k^{-1/2} \tag{2.5.8}$$

and

$$[h,k]_H = \langle Q^{-1/2} h, Q^{-1/2} k\rangle_X \qquad \forall\, h,k \in H. \tag{2.5.9}$$

2.6. Brownian motion and classical Wiener space.

2.6.1. One-dimensional Brownian motion. We consider $C([0,1])$ as a metric space with the sup norm. It is possible to prove (see [**14**], Sec 2.3, and [**54**]) that there exists a centered Gaussian probability γ^W on $C([0,1])$ s.t. the family of functions $B_t : C([0,1]) \to \mathbb{R}$ defined by

$$B_t(\omega) = \omega(t), \qquad t \in [0,1]$$

represents a real valued standard Brownian motion if we consider each B_t as a random variable on $(C([0,1]), \gamma^W)$.

Clearly γ^W concentrates on $C_*([0,1]) = \{\omega \in C([0,1]) | \omega(0) = 0\}$.

γ^W is called classical Wiener measure on $C_*([0,1])$, it is nondegenerate and centered. The (Borel) measure space $(C_*([0,1]), \gamma^W)$ is called *classical Wiener space*.

Every standard Brownian motion B_t on $(\Omega, \mathscr{F}, \mu)$ has a version $\tilde{B}_t$ that is γ-Hölder continuous for every $\gamma < \frac{1}{2}$; hence, there exists a set $A \in \mathscr{F}$ with $\mu(A) = 1$ and a version $(\widetilde{B}_t)_{t\in[0,1]}$ such that the map $t \mapsto \widetilde{B}_t(\omega)$ is continuous for any $\omega \in A$.

γ^W is a Gaussian measure with mean zero and covariance operator

$$B_{\gamma^W}(\mu,\nu) = \int_{[0,1]^2} \min\{t,s\} (\mu \otimes \nu)(d(t,s)), \qquad \mu,\nu \in \mathscr{M}([0,1]). \tag{2.6.1}$$

We consider the embedding $\imath : C([0,1]) \to L^2(0,1)$, $\imath(f) = f$, which is a continuous injection since

$$\|\imath(f)\|_{L^2(0,1)} \le \|f\|_\infty.$$

We consider the image measure $\gamma_W := \gamma^W \circ \iota^{-1}$ on $L^2(0,1)$; clearly γ_W concentrates on the space Z_2 of elements of $L^2([0,1])$ which have a continuous representative; we have that the Cameron-Martin space on $(C([0,1]),\gamma^W)$, and on $(L^2(0,1),\gamma_w)$ is the same in the sense of Proposition 2.2.8, and it is given by

$$H_0^1([0,1]) := \{f \in L^2(0,1) : f' \in L^2(0,1) \text{ and } f(0)=0\}.$$

$L^2([0,1])$ is a Hilbert space, so we can consider eigenvalues and eigenvectors: the eigenvalues are

$$\lambda_k = \frac{1}{\pi^2\left(k+\frac{1}{2}\right)^2}, \qquad k \in \mathbb{N} \tag{2.6.2}$$

and the eigenvectors are

$$e_k(x) = \sqrt{2}\sin\Big(\frac{x}{\sqrt{\lambda_k}}\Big) = \sqrt{2}\sin\Big(\frac{2k+1}{2}\pi x\Big).$$

2.6.2. d-dimensional Brownian motion. Let $X := \{\omega \in C([0,1],\mathbb{R}^d) | \omega(0)=0\}$, we can see it as $X = C_*([0,1]) \times \ldots \times C_*([0,1])$ d times, if we define $\gamma^{W,d} = \gamma^W \otimes \ldots \otimes \gamma^W$ d times (where γ^W is the measure of classical Wiener space, associated to the standard Brownian motion), we have that it is a nondegenerate Gaussian measure (recalling that the product of Borel σ-algebras is the Borel σ-algebra).

DEFINITION 2.6.1. A *d-dimensional Brownian motion* on $[0,1]$ is a d-dimensional stochastic process $(B_t)_{t\in[0,1]}$ on a probability space $(\Omega,\mathscr{F},\mu)$ such that, for every $i \in (1,\ldots,d)$, the process $(B_t)_i$ is a standard real valued Brownian motion.

It is immediate that given $\gamma^{W,d}$ on X, the family of functions $B_\cdot(\cdot) : [0,1]\times X \to \mathbb{R}^d$ defined by

$$B_t(\omega) = \omega(t), \qquad t \in [0,1]$$

is a d-dimensional Brownian motion on $(X,\gamma^{W,d})$.

2.6.3. Brownian bridge. Let B_t a real valued standard Brownian motion $t \in [0,1]$ on the space $(\Omega,\mathscr{F},\mu)$; then, the process given by $B_t^0 = B_t - tB_1$ on the space $(\Omega,\mathscr{F},\mu)$ is called *Brownian bridge* ; we have that $B_0^0 = 0$ and $B_1^0 = 0$ almost surely i.e. if

$$A = \{\omega \in \Omega | B_0^0(\omega) = B_1^0(\omega) = 0\}$$

then $A \in \mathscr{F}$ and $\mu(A) = 1$. It is well known that, if $t \in [0,1]$ then

$$\mathbb{E}(B_t^0) = 0,$$

and if $s,t \in [0,1]$ then

$$\operatorname{cov}(B_t^0, B_s^0) = (s \wedge t) - st.$$

We consider the case of $(C_*([0,1]),\gamma^W)$, classical Wiener space; we define the bounded linear function $\psi : C([0,1]) \to C([0,1])$,

$$\psi(f)(t) := f(t) - tf(1),$$

and we define the pinned Wiener measure $\gamma^{Wp} := \gamma^W \circ \psi^{-1}$ that is Gaussian, centered and it imposes that the values on 0 and 1 are 0: clearly the family of functions $B_t^0 : C([0,1]) \to \mathbb{R}$ defined by

$$B_t^0(\omega) = \omega(t), \qquad t \in [0,1]$$

corresponds a Brownian bridge, and clearly it concentrates on the closed linear subspace $C_0([0,1])$ (so γ^{Wp} is degenerate as a measure on $C([0,1])$, because $C([0,1]) \neq C_0([0,1])$).

If $Y = C([0,1])$ then $Y^* = \mathscr{M}([0,1])$, where $\mathscr{M}([0,1])$ is the set of real finite Borel measures on $[0,1]$, with the weak* topology (by the Riesz-Markov Theorem 1.2.18).

Now, if $s,t \in [0,1]$, if δ_s, δ_t are Dirac probabilities concentrated respectively in s and t, then we have

$$B_{\gamma^{Wp}}(\delta_t,\delta_s) = \int_X \delta_t(\omega)\delta_s(\omega)\, d\gamma^{Wp}(\omega) = \int_X \omega(t)\omega(s)\, d\gamma^{Wp}(\omega) =$$

$$= \mathbb{E}(B_t^0 B_s^0) = \mathbb{E}(B_t^0 B_s^0) - \mathbb{E}(B_t^0)\mathbb{E}(B_s^0) = \operatorname{cov}(B_t^0, B_s^0) = (s \wedge t) - st.$$

Hence, by the continuity of $B_{\gamma^{Wp}}$ we can write that, if μ, ν are linear combinations of Dirac measures,

$$B_{\gamma^{Wp}}(\mu,\nu) = \int_0^1 \int_0^1 ((s\wedge t) - st)\mu(ds)\nu(dt); \tag{2.6.3}$$

but these measures are dense in the weak* topology (see e.g. [**16**] Exa. 8.1.6 i), by recalling that Borel sets and Baire sets are the same in a metric space) and $B_{\gamma^{Wp}}$ is continuous, so (2.6.3) is verified also for generic elements of $\mathscr{M}([0,1])$.

We remark that two functions $f_1, f_2 \in L^1([0,1])$ can be seen as density of measures (with respect to Lebesgue measure $\mathscr{L}^1$), and in that case we can write

$$B_{\gamma^{Wp}}(f_1\mathscr{L}^1, f_2\mathscr{L}^1) = \int_0^1 \int_0^1 ((s\wedge t) - st) f_1(s) f_2(t)\, dt\, ds. \tag{2.6.4}$$

As in the above case, we can consider the embedding $\iota : C([0,1]) \to L^2(0,1)$, $\iota(f) = f$; we can define on $L^2(0,1)$ the measure $\gamma_{Wp} = \gamma^{Wp} \circ \iota^{-1}$.

Let $X = L^2([0,1])$ (hence a Hilbert space, see Section 2.5). Clearly, γ_{Wp} is well defined in X and concentrates on the space Z of elements of $L^2([0,1])$ which have a continuous representative; we already know that γ_W is well defined in X (and it concentrates on Z_2) and that is a nondegenerate centered Gaussian measure with Cameron-Martin space

$$H_W = \{f \in W^{1,2}([1,2]) | f \text{ has a continuous representative } \hat{f} \text{ with } \hat{f}(0) = 0, \hat{f}(1) = 0\},$$

dense in X (see [**14**] Lem. 2.3.14); moreover, ψ is well defined in Z_2 and $\gamma_{Wp} = \gamma_W \circ \psi^{-1}$, so γ_{Wp} on X is a centered Gaussian measure with Cameron-Martin space

$$H_{Wp} = \psi(H_W) = W_0^{1,2}((0,1))$$

(it is clear, because $\psi(H_{Wp}) = H_{Wp}$ and $\psi(H_W) \subseteq H_{Wp}$) hence γ is nondegenerate because γ_{Wp} is dense in X.

We have that $X^* = L^2([0,1])$ and we can consider the function $i^* : X^* \to Y^*$ as $i^*(f) = f\mathscr{L}^1$. Now, for what we said, by using (2.6.4) we can calculate the covariance $B_{\gamma_{Wp}}$ for $f_1, f_2 \in X$: it is given by

$$B_{\gamma_{Wp}}(f_1, f_2) = \int_0^1 \int_0^1 ((s\wedge t) - st) f_1(s) f_2(t)\, dt\, ds;$$

if λ is an eigenvalue of Q and f_λ is an eigenvector, then

$$B_{\gamma^{Wp}}(f,g) = \langle \lambda f_\lambda, g\rangle_{L^2([0,1])}$$

for every $g \in L^2([0,1])$, so

$$\lambda f_\lambda(t) = (1-t)\int_0^t s f_\lambda(s)\, ds + t\int_t^1 (1-s) f_\lambda(s)\, ds,$$

hence

$$\lambda f_\lambda'(t) = -\int_0^t s f_\lambda(s)\, ds + \int_t^1 (1-s) f_\lambda(s)\, ds$$

and

$$\lambda f_\lambda''(t) = f_\lambda(t) \tag{2.6.5}$$

and clearly

$$f_\lambda(0) = f_\lambda(1) = 0; \tag{2.6.6}$$

if we take $\lambda_k = (\pi k)^{-2}$ and $e_k := f_{\lambda_k} = \sqrt{2}\sin(k\pi\cdot)$, clearly $\{e_k\}_{k\in\mathbb{N}}$ is a basis of eigenvectors of Q: in fact, each solution of the ODE (2.6.5) is of the form

$$f_\lambda = \alpha \sin(\lambda\pi\cdot) + \beta\cos(\lambda\pi\cdot)$$

and if we want to satisfy the boundary conditions (2.6.6), we need $\beta = 0$, and $\lambda = (\pi k)^{-2}$ for some $k \in \mathbb{N}$; this is to prove that only the elements of $\{e_k\}_{k\in\mathbb{N}}$ are eigenvectors; now, it is clear that $\{e_k\}_{k\in\mathbb{N}}$ is a basis of H_{Wp}, hence that set is the set of eigenvectors of Q.

Now, for each e_k, we have $\frac{\|e_k'\|_{L^2([0,1])}}{\|e_k\|_{L^2([0,1])}} = \sqrt{\lambda_k}$; this yields that $H = W_0^{1,2}((0,1))$ and, for every $h \in H$, that $|h|_H = \|h'\|_{L^2([0,1])}$.

In particular, a orthogonal basis of eigenvector of H is $\{\sqrt{2}k^{-1}\pi^{-1}\sin(k\pi\cdot)\}_{k\in\mathbb{N}}$.

CHAPTER 3

Sobolev space in Wiener spaces

This long Chapter recalls several topics about differentiation in Wiener space: except Section 3.3, they are all well-known.

(X,γ) is always a Wiener space: we use the definitions and properties introduced in Chapter 2 (especially Section 2.3).

In Section 3.1 it is presented the notion of Sobolev spaces of the first order in a Wiener space X (see [**14**]).

In Section 3.2 we give a definition of Sobolev space $W^{1,p}(O)$ as completion of Lipschitz functions (it is one of the possible definitions, it is used in [**26**]); we define an alternative Sobolev space $W_*^{1,p}(O)$, and we recall a result in [**44**] which allows to state that, for O convex, $W^{1,2}(O) = W_*^{1,2}(O)$; Proposition 3.2.23 allows to introduce Corollary 3.2.24, which states that each element of $\left(W_*^{1,p} \cap L^q\right)(O)$ can be approximated by regular functions; Proposition 3.2.23 and Corollary 3.2.24 are used in Chapter 7.

In Section 3.3, Proposition 3.2.23 is proved, following the same steps of the result in [**44**].

In Section 3.4, we define concepts linked to second derivatives: the space $W^{2,2}$ (which is used in Section 3.5) and the Gaussian divergence div_γ. The definition of Ornstein-Uhlenbeck operators and semigroups (in various settings) is recalled in Subsection 3.4.3 (see Section 1.1 for operators and semigroups); these concept will be used in Section 4.1, and also in Subsection 7.5.3.

In Section 4.1 we recall the theory of infinite dimensional Hausdorff measures (introduced by D. Feyel and A. de La Pradelle) following [**39**]; we recall that, by following [**26**], a particular kind of sets (which satisfies Hypotheses 3.5.8 and 3.5.10) allows to define the trace operator on ∂O, so in this setting a possible definition of $W_0^{1,p}(O)$ is the space of functions with null trace; this will be used in Chapter 6. To define the necessary conditions on the set O, we use the concepts of Section 3.4.

3.1. Sobolev spaces $W^{1,p}(X)$

3.1.1. Differentiable functions. For this subsection we refer to [**14**]. We will use the notations of 2.3.

In the following, given a Wiener space $X,\gamma)$, we denote $L^p(X,\gamma)$ with $L^p(X)$ and $L^p(X,\gamma,H)$ with $L^p(X,H)$.

DEFINITION 3.1.1. Let X, Y be normed spaces. Let $x_0 \in X$ and let Ω be a neighbourhood of x_0. A function $f:\Omega \to Y$ is called Fréchet differentiable (or simply differentiable) at x_0 if there exists $l \in \mathcal{L}(X,Y)$ such that

$$\|f(x_0+h) - f(x_0) - l(h)\|_Y = o(\|h\|_X) \quad \text{as } h \to 0 \text{ in } X.$$

In this case, l is unique, and we set $f'(x_0) := l$.

$C^1_b(X)$ will be the set of all the Fréchet differentiable functions bounded s.t. f' is bounded and continuous as a function $X \to H$.

If f is Fréchet differentiable at x_0 it is continuous at x_0. Moreover, for every $v \in X$ the directional derivative

$$\frac{\partial f}{\partial v}(x_0) := Y - \lim_{t\to 0} \frac{f(x_0+tv)-f(x_0)}{t}$$

exists and is equal to $f'(x_0)(v)$.

If $Y = \mathbb{R}$ and $f : X \to \mathbb{R}$ is differentiable at x_0, $f'(x_0)$ is an element of X^*. In particular, if $f \in X^*$ then f is differentiable at every x_0 and f' is constant, with $f'(x_0)(y) = f(y)$ for every x_0, $y \in X$. If $f \in \mathcal{F}C^1_b(X)$, $f(x) = \varphi(l_1(x),\dots,l_n(x))$ with $l_k \in X^*$ for every $k \in \{1,\dots,n\}$, $\varphi \in C^1_b(\mathbb{R}^n)$, f is differentiable at every x_0 and

$$f'(x_0)(y) = \sum_{k=1}^n \frac{\partial \varphi}{\partial \xi_k}((l_1(x_0),\dots,l_n(x_0))l_k(y), \quad x_0,\ y \in X.$$

Let f be differentiable at x for every x in a neighbourhood of x_0: if the function $X \to \mathcal{L}(X,Y)$, $x \mapsto f'(x)$ is differentiable at x_0, then the derivative is denoted by $f''(x_0)$, and it is an element of $\mathscr{L}(X,\mathcal{L}(X,Y))$.

The higher order derivatives are defined recursively, in the same way.

If $f : X \to \mathbb{R}$ is twice differentiable at x_0, $f''(x_0)$ is an element of $\mathscr{L}(X,X^*)$, which is canonically identified with the space of the continuous bilinear forms $\mathscr{L}^{(2)}(X)$: indeed, if $v \in \mathscr{L}(X,X^*)$, the function $X^2 \to \mathbb{R}$, $(x,y) \mapsto v(x)(y)$, is linear both with respect to x and with respect to y and it is continuous, so that it is a continuous bilinear form; conversely every bilinear continuous form $a : X^2 \to \mathbb{R}$ gives rise to the element $v \in \mathscr{L}(X,X^*)$ defined by $v(x)(y) = a(x,y)$. Moreover,

$$\|v\|_{\mathscr{L}(X,X^*)} = \sup_{x\neq 0, y\neq 0} \frac{|v(x)(y)|}{\|x\|_X\,\|y\|_X} = \sup_{x\neq 0, y\neq 0} \frac{|a(x,y)|}{\|x\|_X\,\|y\|_X} = \|a\|_{\mathscr{L}^{(2)}(X)}.$$

Similarly, if $f : X \to \mathbb{R}$ is k times differentiable at x_0, $f^{(k)}(x_0)$ is identified with an element of the space $\mathscr{L}^{(k)}(X)$ of the continuous k-linear forms.

DEFINITION 3.1.2. Let $k \in \mathbb{N}$. We denote by $C^k_b(X)$ the set of bounded and k times continuously differentiable functions $f : X \to \mathbb{R}$, with bounded norm $\sup_{x\in X} \|f^{(j)(x)}\|_{\mathscr{L}^{(j)}(X)}$ for every $j = 1,\dots,k$. It is normed by

$$\|f\|_{C^k_b(X)} = \sum_{j=0}^k \sup_{x\in X} \|f^{(j)}(x)\|_{\mathscr{L}^{(j)}(X)},$$

where we set $f^{(0)}(x) = f(x)$. Moreover we set

$$C^\infty_b(X) = \bigcap_{k\in\mathbb{N}} C^k_b(X).$$

Let X,Y be Banach spaces. A function $F : X \to Y$ is said *Gâteaux differentiable* in a point $x \in X$ if there exists a bounded linear mapping from X to Y, called Gâteaux differential and denoted by $DF(x)$ s.t. for every $h \in X$

$$\lim_{t\to 0} \frac{F(x+th)-F(x)}{t} = (DF(x))(h).$$

Given a linear subspace Z of X, a function $F : X \to Y$ is said Gâteaux differentiable with respect to Z or Z-Gâteaux differentiable in a point $x \in X$ if there exists a bounded linear mapping from Z to Y, called Z-Gâteaux differential and denoted by $D_Z F(x)$ s.t. for every $h \in Z$

$$\lim_{t\to 0} \frac{F(x+th)-F(x)}{t} = (D_Z F(x))(h).$$

From now on, X is a separable Banach space endowed with a norm $\|\cdot\|_X$ and with a Gaussian centered non degenerate measure γ, and H is its Cameron-Martin space.

DEFINITION 3.1.3. A function $f : X \to \mathbb{R}$ is called *H-differentiable* at $\bar{x} \in X$ if there exists $l_0 \in H^*$ such that

$$|f(\bar{x}+h) - f(\bar{x}) - l_0(h)| = o(|h|_H) \quad \text{as } h \to 0 \text{ in } H.$$

If f is H-differentiable at $\bar{x}$, the operator l_0 in the definition is called H-derivative of f at x_0, and there exists a unique $y \in H$ such that $l_0(h) = \langle h, y \langle_H$ for every $h \in H$. We set

$$\nabla_H f(x_0) := y.$$

LEMMA 3.1.4. *If f is H-Gâteaux differentiable at x_0, then it is H-differentiable at x_0, with H-derivative given by $h \mapsto f'(x_0)(h)$ for every $h \in H$. Moreover, we have*

$$\nabla_H f(x_0) = R_\gamma D_H(x_0). \tag{3.1.1}$$

If f is just H-differentiable at $\bar{x}$, the directional derivative $\frac{\partial f}{\partial v}(\bar{x})$ exists for every $v \in H$, and it is given by $[\nabla_H f(\bar{x}), v]_H$. Fixed any orthonormal basis $\{h_n : n \in \mathbb{N}\}$ of H, we set

$$\partial_i f(\bar{x}) := \frac{\partial f}{\partial h_i}(\bar{x}), \quad i \in \mathbb{N}.$$

So, we have

$$\nabla_H f(\bar{x}) = \sum_{i=1}^{\infty} \partial_i f(\bar{x}) h_i, \tag{3.1.2}$$

where the series converges in H.

3.1.2. $W^{1,p}(X)$.

LEMMA 3.1.5. *For every $1 \le p < \infty$, the operator $\nabla_H : D(\nabla_H) = \mathcal{F}C_b^\infty(X) \to L^p(X,\gamma,H)$ is closable as an operator from $L^p(X,\gamma)$ to $L^p(X,\gamma,H)$.*

DEFINITION 3.1.6. For every $1 \le p < \infty$, the *Sobolev space* $W^{1,p}(X,\gamma)$ is the domain of the closure of $\nabla_H : \mathcal{F}C_b^\infty(X) \to L^p(X,\gamma,H)$ in $L^p(X,\gamma)$ (still denoted by ∇_H). Therefore, an element $f \in L^p(X,\gamma)$ belongs to $W^{1,p}(X,\gamma)$ iff there exists a sequence of functions $f_n \in \mathcal{F}C_b^\infty(X)$ such that $f_n \to f$ in $L^p(X,\gamma)$ and $\nabla_H f_n$ converges in $L^p(X,\gamma,H)$, and in this case, $\nabla_H f = \lim_{n\to\infty} \nabla_H f_n$.
We will usually denote $W^{1,p}(X,\gamma)$ with $W^{1,p}(X)$.

For more details on th prevous definition we refer to [**14**], Section 5.2.

REMARK 3.1.7. If a subspace E is dense in $W^{1,p}(X)$, and we can define ∇_H on E in a coherent way, then we can give the above definition with E instead of $\mathcal{F}C_b^\infty(X)$, and we get an equivalent definition.

For instance, it is immediate that to $\mathcal{F}C_b^1$ is associable ∇_H , that it is dense in $W^{1,p}(X)$ (because it contains $\mathcal{F}C_b^\infty(X)$), so we can substitute it in Definition 3.1.6.

We have an *integration by part* formula: for every $f \in W^{1,p}(X)$, $g \in C_b^1(X)$ and $h \in H$ we have

$$\int_X \frac{\partial f}{\partial h} g\, d\gamma = -\int_X \frac{\partial g}{\partial h} f\, d\gamma + \int_X f\, g\, \hat{h}\, d\gamma. \tag{3.1.3}$$

In the sequel, we will often define the $*$-partial derivatives

$$\frac{\partial^* g}{\partial h} := \partial_h^* g := \frac{\partial g}{\partial h} - g\hat{h},$$

and we have that the integration by part formula can be written as

$$\int_X \frac{\partial f}{\partial h} g\, d\gamma = -\int_X \frac{\partial^* g}{\partial h} f\, d\gamma \tag{3.1.4}$$

which corresponds to the usual integration by parts formula.

The next Lemma is contained in ([**14**], Lem. 5.7.7).

LEMMA 3.1.8. *If $p > 1$ and $f \in W^{1,p}$, then $|\nabla_H f|_H = 0$ γ-almost everywhere in the set $f^{-1}(0)$.*

The next proposition is contained in ([**14**], Prop. 5.4.5).

PROPOSITION 3.1.9. *Let $1 \le p < \infty$ and let $f \in W^{1,p}(X)$. Then, $\mathbb{E}_n f \in W^{1,p}(X)$ for all $n \in \mathbb{N}$ and:*

i) *for every $j \in \mathbb{N}$*

$$\partial_j(\mathbb{E}_n f) = \begin{cases} \mathbb{E}_n(\partial_j f) & \text{if } j \le n, \\ 0 & \text{if } j > n; \end{cases} \tag{3.1.5}$$

ii) $\|\mathbb{E}_n f\|_{W^{1,p}(X,\gamma)} \le \|f\|_{W^{1,p}(X)}$;

iii) $\lim_{n\to\infty} \mathbb{E}_n f = f$ *in* $W^{1,p}(X)$.

We will use the following definition.

DEFINITION 3.1.10. A real function on $\Omega \subseteq X$ is said *H-Lipschitz* with constant $c > 0$ if for every $x \in \Omega$, for every $h \in H$ s.t. $x + h \in \Omega$,

$$|f(x+h) - f(x)| \le c|h|_H.$$

Clearly a Lipschitz function on Ω is H-Lipschitz (because for some $c_1 > 0$ we have $\|h\|_X \le c_1|h|_H$ for every $h \in H$ by (2.2.5)).

We recall this result ([**14**], Thm. 5.11.2).

THEOREM 3.1.11. *Let (X, γ) be a Wiener space with Cameron Martin space H; if $\Omega \subseteq \mathbb{R}$ is open and $F : \Omega \to X$ is H-Lipschitz with constant $c > 0$, then almost everywhere it is Gâteaux H-differentiable and H-differentiable; moreover γ-almost everywhere,*

$$|D_H F|_H \le c.$$

REMARK 3.1.12. In [**14**], Thm. 5.11.2, the argument is done for all X, but it can easily extended to the case $\Omega \subset X$ open.

We recall the Lemma (see [**14**], Prop. 5.4.6 and Def. 5.2.4).

LEMMA 3.1.13. *A measurable function F is in $W^{1,p}(X)$, iff these conditions are satisfied:*

i) *for every $h \in H$, for $\gamma_{h^\perp}$-a.e. $y \in h^\perp$ there exists f locally absolutely continuous s.t. $F_y = f$ γ_h-almost everywhere ;*

ii) *there exists a function* $\nabla_H F \in L^p(X,H)$ *s.t.*

$$\frac{F(x+th)-F(x)}{t} - \langle \nabla_H F, h\rangle_H$$

tends to 0 in measure γ *for* $t \to 0$.

REMARK 3.1.14. In [**14**] the expression 'absolutely continuous' is equivalent to our 'locally absolutely continuous' (it follows [**3**] and [**20**] II.4.1.5).

By putting together Lemma 3.1.13 and Theorem 3.1.11, we have this Corollary.

COROLLARY 3.1.15. *If F is H-Lipschitz with constant $c > 0$, then it is in $W^{1,p}(X)$ with gradient equal to its Gâteaux differentiable γ-almost everywhere, $D_H F = \nabla_H F \le c$. If F is Lipschitz on O with constant c, we have the same property.*

Now, by Lemma 3.1.13 and Corollary 3.1.15, we have that $\mathrm{Lip}_b(X)$ is dense in $W^{1,p}(X)$ (because it contains $\mathcal{F}C_b^\infty(X)$), so by Remark 3.1.7, we have this Corollary.

COROLLARY 3.1.16. *The operator $\nabla_H : Lip_b(X) \to L^p(X,H)$ defined by the Gâteaux derivative is closable, the domain of its closure is (isomorphic to) $W^{1,p}(X)$, and the closure defines $\nabla_H : W^{1,p}(X) \to L^p(X,H)$.*

The above Corollary yields an alternative definition of the Sobolev space $W^{1,p}$; following [**26**] et al. we can use this idea to define $W^{1,p}(O)$ on an open set $O \subset X$, see Section 3.2.

3.1.3. The divergence operator. Let us recall the definition of adjoint operator. If X_1, X_2 are real Hilbert spaces and $T : D(T) \subset X_1 \to X_2$ is a densely defined linear operator, an element $v \in X_2$ belongs to $D(T^*)$ iff the function $D(T) \to \mathbb{R}$, $f \mapsto \langle Tf, v\rangle_{X_2}$ has a linear continuous extension to the whole X_1, namely iff there exists $g \in X_1$ such that

$$\langle Tf, v\rangle_{X_2} = \langle f, g\rangle_{X_1}, \quad f \in D(T).$$

In this case g is unique (because $D(T)$ is dense in X_1) and we set

$$g = T^* v.$$

Now, let (X,γ) be a Wiener space, as usual.

We consider $X_1 := L^2(X)$, $X_2 := L^2(X,H)$ and $T := \nabla_H$. For $f \in W^{1,2}(X)$, $v \in L^2(X,H)$ we have

$$\langle Tf, v\rangle_{L^2(X,H)} = \int_X \langle \nabla_H f(x), v(x)\rangle_H \, d\gamma(x)$$

so that $v \in D(T^*)$ if and only if there exists $g \in L^2(X,\gamma)$ such that

$$\int_X \langle \nabla_H f(x), v(x)\rangle_H \, \gamma(dx) = \int_X f(x) g(x)\, d\gamma(x), \quad f \in W^{1,2}(X,\gamma). \tag{3.1.6}$$

In analogy to the finite dimensional case, we can set

$$\mathrm{div}_\gamma v := -g$$

and we call $\mathrm{div}_\gamma v$ the *divergence* or *Gaussian divergence* of v. As $\mathcal{F}C_b^1(X)$ is dense in $W^{1,2}(X)$, (3.1.6) is equivalent to

$$\int_X \langle \nabla_H f(x), v(x)\rangle_H \, d\gamma(x) = \int_X f(x) g(x)\, d\gamma(x)$$

for every $f \in \mathcal{F}C_b^1(X)$.

Clearly, for every $m \in \mathbb{N}$, the set $\mathrm{Lip}_{0,m}(X,H)$ is contained in the domain of div_γ, and for $v \in \mathrm{Lip}_{0,m}(X,H)$ we have the formula

$$\mathrm{div}_\gamma v(x) = \sum_{n=1}^{\infty} \big(\partial_n v_n(x) - v_n(x)\hat{h}_n(x)\big) = \sum_{n=1}^{m} \big(\partial_n v_n(x) - v_n(x)\hat{h}_n(x)\big) \tag{3.1.7}$$

(we recall that, if $v \in \mathrm{Lip}_{0,m}(X,H)$ then $v_i = 0$ for $i > m$).

REMARK 3.1.17. Let $g \in \mathrm{Lip}_{0,m}(O,H)$ (see Definition 1.2.35), hence g can be extended to 0 out of O, which implies that $\mathrm{div}_\gamma g_{|X\setminus O} = 0$, so $g, \mathrm{div}_\gamma g \in L^p(X)$ for every $p \in [1,+\infty)$; if $f \in \mathrm{Lip}_b(O)$, then it can be extended out of X (for example with McShane extension, see Appendix) and by the formulas (3.1.4) and (3.1.7)

$$\int_O \langle \nabla_H f, g\rangle_H \, d\gamma = -\int_O \mathrm{div}_\gamma g f \, d\gamma;$$

clearly, by the definition of $W^{1,p}$, the above equation is true for every $f \in W^{1,p}(O)$ for every $p \in [1,+\infty)$.

3.2. Sobolev spaces on $O \subset X$ open

3.2.1. $L\log^{\frac{1}{2}} L$. We recall the theory of Orlicz spaces in infinite dimension (see [**42**] for the particular case: it is done for the whole space X, but it remains true for an open subset O; see [**1**] for the general case).

We introduce the function on $\mathbb{R}^+$

$$A_{\frac{1}{2}}(t) = \int_0^t (\log(1+s))^{\frac{1}{2}} \, ds,$$

this function is called an N-function.

We say that $A_{\frac{1}{2}}$ satisfies the Δ_2-condition near infinity: i.e. for some $k > 0$, $t_0 > 0$, for all $t \in (t_0,+\infty)$ we have $A_{\frac{1}{2}}(2t) \le kA_{\frac{1}{2}}(t))$ as $t \to 0$. This is true because $\frac{d}{dt} t(\log(1+t))^{\frac{1}{2}} \sim (\log(1+t))^{\frac{1}{2}}$.

For every $p > 1$ there exists $C > 0$ and $R \in (0,1)$ s.t. $A_{\frac{1}{2}}(t) \le R \vee Ct^p$ for every $t > 0$ because there exists R $(0 < R < 1)$ s.t. if $t > R$ then $(\log(1+t))^{\frac{1}{2}} < Ct^{p-1}$.

Given $O \subseteq X$ open set, we introduce the space

$$\begin{aligned} L\log^{\frac{1}{2}} L(O) := L(\log L)^{\frac{1}{2}}(O) &:= \{f \text{ measurable on } O \,|A_{1/2}(c|f|) \in L^1(O) \text{ for some } c > 0\} = \\ &= \{f \text{ measurable on } O \,|A_{1/2}(|f|) \in L^1(O)\} = \\ &= \{f \text{ measurable on } O \,|f(\log|f| \vee 0)^{\frac{1}{2}} \in L^1(O)\} \end{aligned}$$

to see this equality, we can compare the derivative of $|f(\log|f|\vee 0)^{\frac{1}{2}}$ with $(\log(1+s))^{\frac{1}{2}}$. $L\log^{\frac{1}{2}} L(O)$ is an Orlicz class, see [**1**]; it is a Banach space (by the Δ_2-condition and γ finite, see ([**1**]) with the norm

$$\|f\|_{L(\log L)^{\frac{1}{2}}(O,\gamma)} := \inf\{\alpha > 0 | \int_O A_{1/2}(|f|/\alpha) \, d\gamma \le 1\};$$

we introduce on $\mathbb{R}^+$ the function

$$\Psi(x) = \int_0^x \exp(t^2 - 1) dt;$$

we can do the same construction, and, given $O \subseteq X$ open set, the space

$$L^{\Psi}(O) := \{f \text{ measurable on } O \,|\exp(c|f|^2) \in L^1(O) \text{ for some } c > 0\}$$

is a Banach space with the norm

$$\|f\|_{L^{\Psi}(O)} = \inf\{\alpha > 0| \int_O \Psi(|f|/\alpha)\, d\gamma \leq 1\}.$$

$A_{1/2}$ and Ψ are complementary functions in the sense of [**1**] (see for instance [**42**]), and we have that, if $f \in L(\log L)^{\frac{1}{2}}(O)$, $g \in L^{\Psi}(O)$ then we have this generalized Hölder inequality

$$\|fg\|_{L^1(O)} \leq 2\,\|f\|_{L(\log L)^{\frac{1}{2}}(O)}\,\|g\|_{L^{\Psi}(O)};$$

we have that, if $g \in X^*_\gamma$, hence there exists $c > 0$ s.t.

$$\int_O \exp\left(c|g|^2\right)\, d\gamma < \infty$$

by Corollary 2.1.19 (particular case of the Fernique theorem), so for all $h \in H$, $f \in L(\log L)^{\frac{1}{2}}(O)$, the integral $\int_O f\hat{h}\, d\gamma$ is well defined and finite.

We want to prove

$$L^p(O) \subseteq L\log^{\frac{1}{2}} L(O) \subseteq L^1(O);$$

the first step is to prove for some $c > 0$ that $\|\hat{h}\|_{L^{\Psi}(O)} \leq c|h|_H$ for every $h \in H$.

If $h \in H$, then we can define the measure $\gamma \circ \hat{h}^{-1}$; by [**14**], Lem. 2.2.8, we have that $\gamma \circ \hat{h}^{-1}$ is a centered Gaussian measure with variance $|h|_H^2 = \|\hat{h}\|^2_{L^2(X)}$.

We want to estimate

$$\|\hat{h}\|_{L^{\Psi}(O)} \leq \|\hat{h}\|_{L^{\Psi}(X)} = \inf\{\alpha > 0| \int_X \Psi\left(\frac{\hat{h}}{\alpha}\right)\, d\gamma \leq 1\};$$

now, for $\alpha > 0$,

$$a_\alpha := \int_X \Psi\left(\frac{|\hat{h}|}{\alpha}\right)\, d\gamma = \int_X \int_0^{|\hat{h}(x)|/\alpha} \exp(t^2 - 1)\, dt\, d\gamma(x);$$

we have

$$\exp(s^2 - 1) \leq 1$$

if $|s| \leq 1$, and

$$\exp(s^2 - 1) \leq s \exp s^2$$

if $s > 1$, therefore for every $s > 0$

$$\exp(s^2 - 1) \leq 1 + s \exp s^2,$$

hence

$$\int_0^t \exp(s^2 - 1)\, ds \leq \int_0^t (1 + s\exp s^2)\, ds = t + \frac{\exp t^2 - 1}{2}$$

and so

$$a_\alpha = \int_X \int_0^{|\hat{h}(x)|/\alpha} \exp(t^2 - 1)\, dt d\gamma(x) \leq$$

$$\leq \alpha^{-1}\int_X |\hat{h}(x)|\,d\gamma(x)+\frac{1}{2}\int_X \exp(|\hat{h}(x)|^2/\alpha^2)\,d\gamma(x)-\frac{1}{2}\leq$$

(because $\gamma\circ\hat{h}^{-1}$ is a centered Gaussian measure with variance $|h|_H^2$ and by Hölder inequality)

$$\leq \alpha^{-1}|\hat{h}|_{L^2(X)}+\frac{1}{2|h|_H\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp\left(-\frac{t^2}{2|h|_H^2}\right)\exp\frac{t^2}{\alpha^2}\,dt-\frac{1}{2}=$$

$$=\alpha^{-1}|h|_H+\frac{1}{2|h|_H\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp(t^2(\alpha^{-2}-\frac{1}{2|h|_H^2}))\,dt-\frac{1}{2}$$

so for every $\alpha>2|h|_H$

$$\int_X \Psi\left(\frac{|\hat{h}|}{\alpha}\right)\,d\gamma\leq\frac{1}{2}+\frac{|h|_H\sqrt{4\pi}}{2|h|_H\sqrt{2\pi}}-\frac{1}{2}\leq\frac{\sqrt{2}}{2}<1;$$

hence

$$\left\|\hat{h}\right\|_{L^\Psi(O)}\leq 2|h|_H.$$

In particular, we have that if $f\in L\log^{\frac{1}{2}}L(O)$ then

$$\int f\hat{h}\,d\gamma\leq 2\,\|f\|_{L\log^{\frac{1}{2}}L(O)}\left\|\hat{h}\right\|_{L^\Psi(O)}\leq 4\,\|f\|_{L\log^{\frac{1}{2}}L(O)}|h|_H.$$

We have that $L\log^{\frac{1}{2}}L(O)$ is continuously embedded in $L^1(O)$, because

$$\int_O |f|\,d\gamma\leq 2\,\|\chi_O\|_{L^\Psi(O)}\|f\|_{L\log^{\frac{1}{2}}L(O)}.$$

We notice that, for all $p>1$, there exists $C>0$ and R ($0<R<1$) s.t. $A_{1/2}(t)\leq R+Ct^p$ for every $t>0$, hence, for $\|f\|_{L^p(O)}^p<C^{-1}(1-R)$ we have

$$\int_O A_{1/2}(|f(x)|)\,d\gamma(x)\leq R+C\int_O |f|^p\,d\gamma\leq 1,$$

i.e. for every $f\in L^p$

$$\int_O A_{1/2}\left(\frac{(1-R)^{\frac{1}{p}}|f|}{\|f\|_{L^p(O)}C^{\frac{1}{p}}}\right)\,d\gamma\leq 1$$

so

$$\|f\|_{L\log^{\frac{1}{2}}L(O)}\leq\|f\|_{L^p(O)}\left(\frac{C}{1-R}\right)^{\frac{1}{p}}$$

and $L^p(O)$ is continuously embedded in $L\log^{\frac{1}{2}}L(O)$.

Hence we have

$$L^p(O)\subseteq L\log^{\frac{1}{2}}L(O)\subseteq L^1(O)$$

and the embeddings are continuous.

3.2.2. Definition of $W^{1,p}(O)$. We will always assume that $O \subseteq X$ is an open set; we want to define the Sobolev space $W^{1,p}(O)$ for $p \in [1,+\infty]$. There are different possible definitions; we use the definition as the closure of Lipschitz functions in norm $W^{1,p}$.

We recall that Lipschitz functions are Gâteaux differentiable and H- Gâteaux differentiable γ-almost everywhere by Theorem 3.1.11.

DEFINITION 3.2.1. If $O \subseteq X$ is open, we define the operator ∇_H from $\mathrm{Lip}(O)$ to $L^p(O,H)$.

The next Lemma is a generalization of [**26**], Lem. 2.2.

LEMMA 3.2.2. *For every $p \in [1,+\infty)$, the operator ∇_H is closable as operator $L^p(O) \to L^p(O,H)$.*

PROOF. We have to prove that, if there are two sequences f_n, g_n in $\mathrm{Lip}(O)$ s.t. $f_n \to f$ and $g_n \to f$ in $L^p(O)$ while $\nabla_H f_n \to \psi_1$ and $\nabla_H g_n \to \psi_2$ in $L^p(O,H)$ then $\psi_1 \equiv \psi_2$.

By linearity, the above is equivalent to prove that given a sequence f_n in $\mathrm{Lip}(O)$ s.t. $f_n \to 0$ in $L^p(O)$ and $\nabla_H f_n \to \psi_1$ in $L^p(O,H)$ then $\psi_1 \equiv 0$.

It is not restrictive to suppose that $p = 1$. We consider $h \in H, |h|_H = 1$, we define for every $y \in h^\perp$ the open set $O_y \subseteq \mathbb{R}$ where $O_y := \{t \in \mathbb{R} | y + th \in O\}$ and the functions $f_{n,y} \in \mathrm{Lip}(O_y)$ where $f_{n,y}(t) = f_n(y+th)$; clearly $f_{n,y} \to 0$ in $L^1(O_y, \gamma^1)$ for $n \to +\infty$ for $\gamma_{h^\perp}$-a.e. $y \in h^\perp$; we also have that $f_{n,y}$ is absolutely continuous and

$$f'_{n,y}(t) \equiv \langle \nabla_H f_n(y+th), h \rangle_H$$

(for $\gamma_{h^\perp}$-a.e. $y \in h^\perp$). Now, $\langle \nabla_H f_n, h \rangle_H$ converges to $\langle \psi, h \rangle_H$ in $L^1(O)$, hence, for $\gamma_{h^\perp}$-a.e. $y \in h^\perp$ we have that $f'_{n,y}$ converges in $L^1(O_y, \gamma^1)$. We consider such y: in every open bounded interval $(a,b) \subseteq O_y$, we have that $f_{n,y}$ is a sequence which converges in $W^{1,1}((a,b), \gamma^1)$, and hence also in $W^{1,1}((a,b))$ with the Lebesgue measure (because (a,b) is bounded); so $f_{n,y}$ converges to 0 in $W^{1,1}(a,b)$ and $\langle \psi(y+\cdot h), h \rangle_H \equiv 0$; so $\langle \psi(y+\cdot h), h \rangle_H \equiv 0$ on O_y (that is open) and this for $\gamma_{h^\perp}$-a.e. $y \in h^\perp$, hence $\langle \psi(\cdot), h \rangle_H = 0$ γ-almost everywhere on O; this for every $h \in H, |h|_H = 1$, so $\psi \equiv 0$ and we conclude. □

By the above Lemma, we can introduce this Definition.

DEFINITION 3.2.3. (Sobolev Spaces) For every $p \in [1,+\infty)$, $W^{1,p}(O)$ will be the domain of the closure of $\nabla_H : \mathrm{Lip}(O) \to L^p(O,H)$, this closure will always be denoted by ∇_H.

By Corollary 3.1.16, for $O = X$ the above definition is coherent with the Definition 3.1.6.

REMARK 3.2.4.
- i) The Sobolev space $W^{1,p}(O)$ can be defined in other ways for $O = X$ and $p = 1$, other definition are in [**14**]). An equivalent definition of $W^{1,2}(O)$ with O convex is in the next subsection 3.2.3.
- ii) If f is Lipschitz in O, it can be extended to a Lipschitz function $\bar{f}$ in X, hence $\bar{f} \in W^{1,p}(X)$; we know that $\mathscr{F}C_b^\infty(X)$ is dense in $W^{1,p}(X)$, so $\bar{f}$ can be approximated by functions in $\mathscr{F}C_b^\infty(X)$; hence, by definition, the restrictions to O of functions of $\mathscr{F}C_b^\infty(X)$ are dense in $W^{1,p}(O)$.

DEFINITION 3.2.5. For $p,q \in [1,+\infty)$, $q > p$, we define $W^{1,p}(O) \cap L^q(O)$ or $\left(W^{1,p} \cap L^q\right)(O)$ as a space given by the norm $\|\cdot\|_{W^{1,p}} + \|\cdot\|_{L^q}$; a sequence converges in this space if it converges both in $W^{1,p}(O)$ and in $L^q(O)$.

REMARK 3.2.6. We have that the restrictions to O of $\mathscr{F}C_b^\infty$ are also dense in the space $W^{1,2}(O)$: it suffices to consider the extensions of Lipschitz functions, and to approximate those with elements of $\mathscr{F}C_b^\infty$.

REMARK 3.2.7. We recall that a Lipschitz function f on a subset of a metric space can always be extended to a function Lipschitz in all the space, with the same Lipschitz constant l, for example with the McShane extension (see Appendix).

REMARK 3.2.8. By the definition, it is easy to deduce that some versions of the chain rule and of the Leibniz rule are true for the H-gradient ∇_H: if $\varphi \in \mathrm{Lip}_b(O)$ (Lipschitz and bounded) and $f \in W^{1,p}(O)$ for $p \geq 1$, then $\varphi f \in W^{1,p}(O)$, and

$$\nabla_H(\varphi f) = \nabla_H \varphi f + \varphi \nabla_H f;$$

if $\varphi \in \mathrm{Lip}(\mathbb{R})$ and $f \in W^{1,p}(O,\gamma)$ for $p \geq 1$, then $\varphi \circ f \in W^{1,p}(O)$, and

$$\nabla_H(\varphi \circ f) = (\varphi' \circ f)\nabla_H f.$$

In particular, if f^+ is the positive part of a function $f \in W^{1,p}(O)$ (i.e. $f^+ = f \vee 0$), we have that $f^+ \in W^{1,p}(O)$ and, if $A = \{x \in O | f(x) > 0\}$ then $\nabla_H f^+_{|A} \equiv \nabla_H f_{|A}$ and $\nabla_H f^+_{|X \setminus A} \equiv 0$; in fact $f^+ := g \circ f$ where g is the positive part of the identity on $\mathbb{R}$, g is Lipschitz, $g'_{|\mathbb{R}^+} \equiv 0$ and $g'_{|\mathbb{R}^-} \equiv 1$, so we can conclude by the chain rule.

REMARK 3.2.9. We have this result ([**42**], Prop. 3.2).

LEMMA 3.2.10. *The space $W^{1,1}(X)$ is continuously embedded in $L\log^{\frac{1}{2}} L(X)$.*

We recall that $C_b^1(X)$ is the set of all the Fréchet differentiable functions bounded s.t. f' is bounded and continuous as a function $X \to H$.

DEFINITION 3.2.11. $C_0^1(O)$ will be the set of the restrictions to O of all the functions $f \in C_b^1(X)$ s.t. $f = 0$ out of O.

We have that each function in $C_b^1(X)$ is Lipschitz. We could consider $C_0^1(O)$ as a subspace of $W^{1,2}(X)$ or of $W^{1,2}(O)$ (with the same topology).

DEFINITION 3.2.12. In this setting, for all $p \geq 1$, the set $W_0^{1,p}(O)$ will be the closure in $W^{1,p}(O)$ of $C_0^1(O)$ (equivalently, we can consider it as the closure in $W^{1,p}(O)$ or as the completion in the topology $W^{1,p}$).

3.2.3. $W_*^{1,p}(O)$ **for** O **open and convex.** We recall the concept of locally absolute continuous function (see Appendix).

We will consider $O \subseteq X$ open set.

We recall that for every $h \in R_\gamma(X^*)$, $|h|_H = 1$ we can decompose (X,γ) as $X = \mathbb{R} \times X_h^\perp$ (where $X_h^\perp$ is the closure of the orthogonal of h) and $\gamma = \gamma_1 \otimes \gamma_{h^\perp}$ (γ_1 is the standard Gaussian measure on $\mathbb{R}$ and $\gamma_{h^\perp}$ is a nondegenerate centered Gaussian measure on $X_h^\perp$).

We fix a basis $\{h_1,\dots\}$ of H s.t. , $h_i \in R_\gamma(X^*)$ for all $i \in \mathbb{N}$.

Fixed $h \in R_\gamma(X^*)$, $|h|_H = 1$ and $y \in X_h^\perp$ we recall the definition of $O_y = \{t \in \mathbb{R} | y + th \in O\}$, interval in $\mathbb{R}$, and the function f_y on O_y defined as $f_y := f(y+th)$.

DEFINITION 3.2.13. For every $h \in R_\gamma(X^*)$, we define D_h^O as the set of γ-class of $\gamma-$measurable functions s.t. for $\gamma_{h^\perp}$-almost every $y \in X_h^\perp$, the function f_y on O_y has γ_1-representative $\tilde{f}_y$ (i.e. $f_y(t) = \tilde{f}_y(t)$ for γ_1-a.e. t) that is locally absolutely continuous; hence, for such y we define

$$\partial_h f(y+th) := f_y'(t),$$

that is well defined for $\gamma_{h^\perp}$-almost every $y \in X_h^\perp$ for γ_1-almost every $t \in O_y$.

It is clear that, if $f \in D_h^O$ then f has a representative $\tilde{f}$ s.t. for $\gamma_{h^\perp}$-almost every $y \in X_h^\perp$ the function $\tilde{f}_y$ is locally absolutely continuous.

DEFINITION 3.2.14. Given $p \in [1,+\infty)$, we say that $f \in W_*^{1,p}(O)$ if $f \in L^p(O)$ and $f \in D_h^O$ for all $h \in R_\gamma(X^*)$, and there exists $\nabla_H f \in L^p(O,H)$ s.t. $\partial_h f = \langle \nabla_H f, h \rangle_H$.

LEMMA 3.2.15. *$W_*^{1,p}(O)$ is always a Banach space with the norm given by*

$$\|\cdot\|_{W_*^{1,p}(O)} = \|\cdot\|_{L^p(O)} + \|\nabla_H \cdot\|_{L^p(O,H)}.$$

PROOF. The only thing to prove is that, if f_n is a Cauchy sequence in $W_*^{1,p}(O)$, then there exists $f \in W_*^{1,p}(O)$ s.t. $f_n \to f$ in $L^p(O)$ and $\|\nabla_H(f_n - f)\|_{L^p(O,H)} \to 0$.

We already know that $L^p(O)$ and $L^p(O,H)$ are complete, hence $f_n \to f$ and $\nabla_H f_n \to \psi$ for some $f \in L^p(O)$ and $\psi \in L^p(O,H)$, we want to prove that $f \in W^{1,p}(O)$ and $\psi = \nabla_H f$.

For every $h \in H$, for every $y \in X_h^\perp$ we define

$$O_y := \{t \in \mathbb{R} | y + th \in O\}$$

and $f_{y,n}, f$ functions on O_y defined as

$$f_{y,n}(t) := f_n(y+th), \;\; f_y(t) := f(y+th);$$

clearly for $\gamma_{h^\perp}$-almost every $y \in X_h^\perp$, we have that the convergences $f_{y,n} \xrightarrow[n\to+\infty]{} f_y$ and $\partial_h f_y \xrightarrow[n\to+\infty]{}$ $(\langle \psi, h \rangle_H)_y$ in $L^p(O_y, \gamma^1)$.

For every interval $(a,b) \subseteq \mathbb{R}$, for such y we have that $f_{n,y}$ converges to f_y in $W^{1,p}((a,b),\mathscr{L}^1)$ (therefore f_y has γ^1-representative that is locally absolutely continuous), and $\partial_h f = \langle \psi, h \rangle_H$ γ-almost everywhere; so we can conclude. $\square$

REMARK 3.2.16.
- i) The norm $\|\cdot\|_{W_*^{1,p}(O)}$ is the same of $\|\cdot\|_{W^{1,p}(O)}$ for every $p \in [1,+\infty)$. Lipschitz functions are clearly in $W_*^{1,p}(O)$, so $W^{1,p}(O) \subseteq W_*^{1,p}(O)$.
- ii) Obviously if $p \leq q$ then $W_*^{1,q}(O) \subseteq W_*^{1,p}(O)$.
- iii) The definition 3.2.14 is inspired by the definition of $W^{1,2}(X)$ in [**44**] (that is linked to that of weak Sobolev space in [**35**]); in [**44**] the expression 'absolutely continuous' is equivalent to our 'locally absolutely continuous' (they follow [**3**] and [**20**] II.4.1.5).

The next definition is taken from ([**14**], Def. 5.2.3).

DEFINITION 3.2.17. A measurable function $f : X \to \mathbb{R}$ is said stochastically Gâteaux differentiable if there exists a measurable function $D_H f : X \to H$ (called stochastic derivative) s.t. for every $h \in H$, $\frac{f(\cdot+th)-f(\cdot)}{t}$ converges to $\langle \nabla_H f(\cdot), h \rangle_H$ in measure γ for $t \to 0$.

LEMMA 3.2.18. *Given $p \in [1,+\infty)$, if $f \in W_*^{1,p}(X)$, then f is stochastically Gâteaux differentiable and $D_H f = \nabla_H f$.*

PROOF. It is not restrictive to suppose $p=1$. Let $h \in H$, $|h|_H = 1$, $\pi_{h^\perp}$ the projection on $h^\perp$ (given by $I-\hat{h}h$).

For every $h \in H$, $t \in [0,+\infty)$, let $f_{h,t} := \frac{f(\cdot+th)-f(\cdot)}{t}$; we have to prove that $f_{h,t}$ converges to $\partial_h f$ in measure γ for $t \to 0$.

Let $\varepsilon > 0$; let

$$A_{t,\varepsilon} := \{x \in X \,|\, |f_{h,t}(x) - \partial_h f(x)| > \varepsilon\},$$

we want to prove that χ_{A_ε} converges to 0 in $L^1(X,\gamma)$.

For every $h \in H$, f has a representative $\tilde{f}$ s.t. for $\gamma_{h^\perp}$-almost every $y \in X_h^\perp$ the function $\tilde{f}_y$ is locally absolutely continuous and $\tilde{f}_y{}'(t) = \partial_h f(y+th)$; we have that $\partial_h f \in L^1(X,\gamma)$, so $(\partial_h f)_y \in L^1(\mathbb{R},\gamma_1)$ for $\gamma_{h^\perp}$-almost every $y \in X_h^\perp$, for such y we have that $(\partial_h f)_y \equiv \tilde{f}_y{}'$ is $L^1_{\text{loc}}(\mathbb{R},\mathscr{L}^1)$ (locally L^1 with respect to the Lebesgue measure) hence γ_1-almost every point of $\mathbb{R}$ is a Lebesgue point for $\tilde{f}_y{}'$; if t_0 is a Lebesgue point, we have that

$$\frac{\tilde{f}_y(t_0+t)-\tilde{f}_y(t_0)}{t} - \tilde{f}_y{}'(\cdot) = \fint_{t_0}^{t} \tilde{f}_y{}'(s)\,ds - \tilde{f}_y{}'(t) \xrightarrow[t\to t_0]{} 0$$

hence we have that $f_{h,t}$ converges to $\partial_h f$ almost everywhere in measure γ_1 for $t \to 0$; therefore χ_{A_ε} converges to 0 γ-almost everywhere.

We can apply the dominated convergence theorem, so χ_{A_ε} converges to 0 in L^1 and we conclude. $\square$

REMARK 3.2.19. The problem to find an identity between $W^{1,p}(O)$ and $W^{1,p}_*(O)$ can be seen as an extension to the Wiener space case of the Meyers-Serrin theorem (see [**55**]).

We recall this result ([**14**], Prop. 5.4.6 ,iii)).

PROPOSITION 3.2.20. $W^{1,p}_*(X) = W^{1,p}(X)$.

REMARK 3.2.21. In [**14**], Prop. 5.4.6, it is used a space $D^{p,1}(X)$ defined in Def. 5.2.4: the elements of $D^{p,1}(X)$ are functions in $W^{1,p}_*(X)$ which are Gâteaux differentiable with $D_H f = \nabla_H f$; hence, by Lemma 3.2.18 $W^{1,p}_*(X)$ coincides with $D^{p,1}(X)$.

The next proposition is an immediate consequence of the results in [**44**].

PROPOSITION 3.2.22. *If $O \subseteq X$ is open and convex, then the set of the restrictions of the functions in $W^{1,2}(X) = W^{1,2}_*(X)$ is dense in $W^{1,2}_*(O)$.*

In Proposition 3.2.23, we will generalize the above result, by adapting the proof in [**44**].

For $p,q \in [1,+\infty)$, $p \le q$, we can consider $W^{1,p}_*(O) \cap L^q(O)$ as a linear normed space with norm

$$\|\cdot\|_{W^{1,p}_*(O)\cap L^q(O)} := \|\cdot\|_{W^{1,p}_*(O)} + \|\cdot\|_{L^p(O)}.$$

Clearly it is a Banach space.

We will prove this generalization of Proposition 3.2.20.

PROPOSITION 3.2.23. *For $p,q \in [1,+\infty)$, $p \le q$, if $O \subseteq X$ is open and convex, then the set of the restrictions of the functions in $W^{1,p}(X) \cap L^q(X)$ is dense in $W^{1,p}_*(O) \cap L^q O)$.*

Section 3.3 will be dedicated to the proof of the above Proposition, modelled on that in [**44**].

By the above Proposition, we have in particular that the set of restrictions of the functions in $W^{1,p}(X)$ is dense in $W^{1,p}_*(O)$.

Now, we recall that $\mathscr{F}C_b^\infty(X)$, $\mathscr{F}C_b^1(X)$ are dense subspace of $W^{1,p}(X)$ (see Definition 3.1.6, and Remark 3.2.16); hence the spaces

$$\mathscr{F}C_b^\infty(O) := \{f : O \to \mathbb{R} | f = g_{|O} \text{ where } g \in \mathscr{F}C_b^\infty(X)\},$$

$$\mathscr{F}C_b^1(O) := \{f : O \to \mathbb{R} | f = g_{|O} \text{ where } g \in \mathscr{F}C_b^1(X)\},$$

are dense subspaces of $W^{1,p}(O)$.

We have also that $\mathrm{Lip}_b(O)$ is a dense subspace of $W^{1,p}(O)$ by Corollary 3.1.15.

Now, bounded Lipschitz function in O can be extended to bounded Lipschitz functions of X; hence we have this Corollary of Proposition 3.2.23.

COROLLARY 3.2.24. *For $p,q \in [1,+\infty)$, $p \le q$, if O is a open convex set, then $Lip_b(O), \mathscr{F}C_b^\infty(O)$, $\mathscr{F}C_b^1(O)$ are dense subspaces of $W_*^{1,p}(O)$ and of $W_*^{1,p}(O) \cap L^q(O)$; moreover $W_*^{1,p}(O) = W^{1,p}(O)$.*

3.3. Proof of Proposition 3.2.26

In this section we adapt the argument of [**44**] to prove Proposition 3.2.23.

In Subsection 3.3.1 we recall some concepts, and we define, given a function f and a finite subspace $F \subset H$, a function f_ε which is in some sense, an approximation of f by convolution in the directions of F; we give also some properties of f_ε. In Subsection 3.3.2 the proof is given, by using these properties.

3.3.1. Preliminaries. In the sequel, as usually,(X,γ) is a Wiener space, H is the Cameron-Martin space, $\{h_i\}_{i\in\mathbb{N}}$ is an orthonormal basis, and, for every $n \in \mathbb{N}$, $F_n :=< h_1,\dots,h_n >$. For each F_n, we will define π_n as $\pi_n(x) := \sum_{i=1}^n \hat{h}_i(x)h_i$ and $P_n := I - \pi_n$; for $F \subseteq H$, F finite dimensional, we consider $F^\perp$ as in Section 2.3. $B(a,r)$ will be the ball of center a and radius r in the metric of X, $\bar{B}(a,r)$ is the closure of $B(a,r)$, $\overline{B}(s)$ is the closed ball with center 0 and radius s; $B_H(a,r)$ is the set of all the points $x \in X$ s.t. $x-a \in H$ and $|x-a|_H < r$, $\overline{B}_H(s)$ is the closed ball in H (as a metric space) with center 0 and radius s. If F is a subspace of H, $B_F(a,r)$ will be the set of all the points $x \in X$ s.t. $x-a \in F$ and $|x-a|_H < r$, and $B_F(r) := B_F(0,r)$.

If A is a set in X, we will write $\mathrm{co}(A)$ to mean its *convex hull*, i.e. the smaller convex set contained in X which contains A.

We recall that there is $c > 0$ s.t., for every $h \in H$, $\|h\|_X \le c|h|_H$ by (2.2.5). We recall also that, if f is H-Lipschitz on X, then it is $W^{1,p}(X)$ for every $p \in [1,+\infty]$ by Corollary 3.1.15. For every $F \subseteq H$, as usual we define γ_F measure on F and $\gamma_{F^\perp}$ measure on $F^\perp$ as in Section 2.4; for $h \in H$ we will write $\gamma_{h^\perp}$ to mean $\gamma_{<F>^\perp}$. Let O be an open subset of X, and f a measurable function on O: O_y and f_y are defined as usually.

DEFINITION 3.3.1. If $O \subseteq X$ we will say that O is *moderate* if, for every $h \in H$, for $\gamma_{h^\perp}$-a.e. y the boundary in $\mathbb{R}$ of O_y has null Lebesgue measure (in particular, if O is convex it is clearly moderate).

Let F be a finite dimensional subspace of $R_\gamma(X^*) \subseteq H$ of dimension N; with the inner product of H, F has an orthonormal basis and it is isomorphic to $\mathbb{R}^N$, so on F we can define the N-dimensional Lebesgue measure $\mathscr{L}^N$; let $\Psi \in C^\infty(F)$ be a nonnegative function such that $\mathrm{supp}\,\Psi \subset B_F(1)$ and $\|\Psi\|_{L^1(F,\mathscr{L}^N)} = 1$. For $\varepsilon > 0$ we define on F the function $\Psi_\varepsilon(y) := \Psi(x\varepsilon^-1)$.

Let O be an open convex (hence moderate) subset of X. We consider an element $f \in L^p(O)$ as an element of $L^p(X)$ setting $f \equiv 0$ on $X \setminus O$. Moreover, assume that $f \equiv 0$ on $X \setminus A + B_F(R)$, where $A \subset F^{\perp}$ is a compact set. Then,

$$\infty > \int_A \int_{B_F(R)} |f(x+y)|^p d\gamma_F(y)\ d\gamma_{F^\perp}(x) =$$

$$= (2\pi)^{-\frac{N}{2}} \int_A \int_{B_F(R)} |f(x+y)|^p \exp\left(-\frac{1}{2}|y|^2\right) dy\ d\gamma_{F^\perp}(x) \geq$$

$$\geq (2\pi)^{-\frac{N}{2}} \int_A \int_{B_F(R)} |f(x+y)|^p \exp\left(-\frac{1}{2}R^2\right) dy\ d\gamma_{F^\perp}(x),$$

and so $y \mapsto f(x+y) \in L^p_{loc}(F, \mathscr{L}^N)$ for $\gamma_{F^\perp}$-a.e. $x \in F^\perp$. Now for such x, for $y \in F$, we can define

$$f_\varepsilon(x) := \int_F f(x+y-z)\Psi_\varepsilon(z)dz = \int_F f(x+z)\Psi_\varepsilon(x+y-z)dz. \tag{3.3.1}$$

In other words, for $\gamma_{F^\perp}$-a.e. $x \in F^\perp$ the function $f_{\varepsilon,x}$ on F is the convolution of f_x with Ψ_ε, where it is defined (by identifying F with $\mathbb{R}^N$).

For almost $\gamma_{F^\perp}$-a.e. $x \in F^\perp$ we can introduce the section $f_{\varepsilon,x}:\ F \to \mathbb{R}$, $h \mapsto f_\varepsilon(x+h)$.

For $\gamma_{F^\perp}$-a.e. $x \in F^\perp$ we have, for some $C > 0$ depending only on R,

$$\|f_{\varepsilon,x}\|^p_{L^p(F)} \leq C\|f_x\|^p_{L^p(F)}, \tag{3.3.2}$$

hence $f_\varepsilon \in L^p(X)$.

In fact

$$\|f_{\varepsilon,x}\|^p_{L^p(F)} = \int_F \left|\int_F f(x+y-y')\Psi_\varepsilon(y')dy'\right|^p \gamma_F(dy) \leq$$

(by the Jensen inequality, by remarking that $\psi\mathscr{L}^1$ is a probability)

$$\leq \int_F \int_F |f(x+y-y')|^p \Psi_\varepsilon(y')dy'\gamma_F(dy) =$$

(with a change of variables)

$$= \int_F \Psi_\varepsilon(y') \int_{B_F(R)} |f(x+y'')|^p \exp\left(-\frac{1}{2}|y'+y''|^2 + \frac{1}{2}|y''|^2\right) \gamma_F(dy'')dy' \leq$$

(by $y'' \in B_F(R)$)

$$\leq \exp\left(\frac{R^2}{2}\right) \int_F \Psi_\varepsilon(y')\|f_x\|^p_{L^p(F)}dy' = \exp\left(\frac{R^2}{2}\right) |f_x\|^p_{L^p(F)}.$$

For every $x \in F^\perp$ s.t. f_x is $L^1_{loc}(F, \gamma_{F^\perp})$ (so, for $\gamma_{F^\perp}$ almost every x) we have that the function $f_{\varepsilon,x}$ is a continuous function (by the properties of convolution in finite dimension).

Moreover for such x, for any $h \in F \setminus \{0\}$, we have, by differentiating under the integral sign, that $f_{\varepsilon,x} \in C^1(\mathbb{R})$ for $\gamma_F^\perp$-a.e. $x \in F^\perp$, and for such a x, for every $y, h \in F$,

$$\partial_h f_\varepsilon(x+y) = \int_F f(x+z)\partial_h\Psi_\varepsilon(x+y-z)dz,. \tag{3.3.3}$$

Let $f \in W^{1,p}_*(X)$; then for $\gamma_F^\perp$-a.e. $x \in F^\perp$, we have $f_x \in W^{1,p}(F)$, so $f_x \in W^{1,p}_{loc}$ in the Lebesgue sense; in this case, by the properties of convolutions, for every $y, h \in F$ we have

$$\partial_h f_\varepsilon(x+y) = \int_F \partial_h f(x+y-z)\Psi_\varepsilon(z)dz, \tag{3.3.4}$$

(in particular $f_\varepsilon \in D_h^O$)): hence $\partial_h f_{\varepsilon,x}$ is the convolution of $\partial_h f_x$ with Ψ, and arguing as above we obtain that for γ-a.e. $x \in X$, $\partial_h f_\varepsilon(x+\cdot) \in L^p(X)$.

We have this Lemma.

LEMMA 3.3.2. *Let $f \in L^p(X)$ and let f_ε be defined as above, for any $\varepsilon > 0$.*

i) *$f_\varepsilon \to f$ in $L^p(X)$ as $\varepsilon \to 0$.*

ii) *Let $U \subseteq X$ be an open; let $f \in W^{1,p}_*(U)$ and let $U' \subset X$ be an open set such that $U' + B_F(r) \subset U$ for some $r > 0$. Then, $\partial_h f_\varepsilon = (\partial_h f)_\varepsilon$ in U' and it converges to $\partial_h f$ in $L^p(U')$ as $\varepsilon \to 0$ for any $h \in F \setminus \{0\}$.*

PROOF. i) At first, we notice that $f_{\varepsilon,x} \to f_x$ in $L^p(F)$ by the properties of convolutions: in addition $\|f_{\varepsilon,x}\|^p_{L^p(F)} \le \|f_{\varepsilon,x}\|^p_{L^p(F)}$), and we get

$$\|f_\varepsilon - f\|^p_{L^p(X)} = \int_A \|f_{\varepsilon,x} - f_x\|^p_{L^p(F)} \gamma_{F^\perp}(dx) \to 0,$$

as $\varepsilon \to 0$ and we can conclude by dominated convergence theorem.

ii) As we seen for 3.3.4, $\partial_h f_\varepsilon(z) = (\partial_h f)_\varepsilon(z)$ if $z + B_F(\varepsilon) \subset U$, and so $\partial_h f_\varepsilon = (\partial_h f)_\varepsilon$ in U' for any $\varepsilon < r$. Therefore, the same argument of i) implies that $(\partial_h f)_\varepsilon \to \partial_h f$ in $L^p(U')$. □

3.3.2. Proof of Proposition 3.2.23. Let $p \ge 1$. We define:

$W_1(O) := W^{1,p}_*(O) \cap L^\infty(O)$.

$W_2(O)$ as the set of elements $f \in W_1(O)$ s.t. there exist some $F \in \{F_n\}_{n\in\mathbb{N}}$, a compact convex $V \subset F^\perp$, $a \in F$ and $s > 0$ s.t. $V + B(a,s) \subseteq O$ and $f_{|O\setminus(V+F)} = 0$ γ-a.e..

$W_3(O)$ as the set of elements $f \in W_1(O)$ s.t. there exist some $F \in \{F_n\}_{n\in\mathbb{N}}$, a compact convex $V \subset F^\perp$, $a \in F$, $s > 0$ and $R > 0$ s.t. $V + B(a,s) \subseteq O$ and $f_{|O\setminus(V+\overline{B_F(R)})|} = 0$ γ-a.e..

Proof of Proposition 3.2.23.

The proof will be in four steps.

Step 1.

We prove that if $f \in W_1(O)$ then it can be approximated in sense $W^{1,p}_*$ by functions in $W_2(O)$ which are uniformly bounded in $L^\infty(O)$ by $\|f\|_{L^\infty(O)}$.

Let $f \in W_1(O)$; hence, for any $\varepsilon > 0$ there exists $\delta \in (0,\varepsilon)$ such that

$$\int_A |\nabla_H f|^p d\gamma \le \varepsilon, \qquad \text{for every } A \in \mathfrak{B}(X) \text{ s.t. } \gamma(A) \le \delta;$$

in fact, if by contradiction such a δ does not exist, then there is a sequence of Borel sets A_n s.t. $\gamma(A_n) \to 0$ but $\int_{A_n} |\nabla_H f|^p d\gamma$ does not converge to 0, but this contradicts the absolute continuity of the integral.

Fix ε and δ as above. We define the following sets.

- $V_1 := B(a_0, s)$ such that $B(a_0, 3s) \subset O$.
- $V_2 \subset V_1 + F$ be a compact set such that $\gamma(V_2) \ge 1 - \delta$, for some $F \in \{F_n\}_{n\in\mathbb{N}}$; we prove the existence of V_2. We define $F_\infty := \bigcup_{n=1}^{+\infty} F_n$; we know that it is dense in X, so $V_1 + F_\infty = X$ by definition of V_1. Now, $\gamma(V_1 + F_\infty) = \sup_{n\in\mathbb{N}} \gamma(F_n)$, so there exists $F := F_n$ (we will write $P_F := P_n$ and $\pi_F := \pi_n$) such that $\gamma(V_1 + F) \ge 1 - \delta/2$. We can conclude by recalling that $V_1 + F$ is an open set, and so γ is a Radon measure on $V_1 + F$ (by Proposition 1.2.6).

- $V_3 := P_F(V_2)$. We stress that $V_3 = P_F(V_2) \subset P_F(V_1+F) = P_F(V_1)$. V_3 is clearly a compact subset of $F^\perp$.
- $V_4 := \overline{\mathrm{co}V_3}$ is a convex compact set in X (the closure of a convex hull of a compact in a Banach space is compact, see e.g. [[**14**], Prop. A.1.6]), and therefore in $F^\perp$.
- $V := V_4 + (F^\perp \cap \overline{B}_H(s))$, and $a := \pi_F(a_0)$. Clearly, also V is compact in $F^\perp$ since $\overline{B}_H(s)$ is compact in X (because the inclusion of H in X is compact).

We have $V_2 \subset V_4 + F$. Indeed, for any $v \in V_2$ we obtain $v = P_F v + \pi_F v \in V_3 + F \subset V_4 + F$. Moreover, $V + B(a,s) \subset O$. To prove this inclusion, we note that

$$V + B(a,s) \subset V_4 + \overline{B}_H(s) + B(a,s) \subset V_4 + \overline{B}(s) + B(a,s) \subset V_4 + B(a,2s). \tag{3.3.5}$$

Hence, it remains to prove that V_4 is contained in a suitable set. To this aim, from the above definitions we have

$$V_4 = \overline{\mathrm{co}V_3} = \overline{\mathrm{co}P_F(V_2)} \subset \overline{\mathrm{co}P_F(V_1+F)} = \overline{\mathrm{co}P_F V_1} \subset \overline{\mathrm{co}P_F B(a_0,s)} \subset \overline{B}_E(a_0,s). \tag{3.3.6}$$

The last inclusion follows from the fact that $P_F B(a_0,s) \subset B(P_F a_0, s)$, and that $\mathrm{co}P_F B(a_0,s)$ is the smallest convex set which contains $P_F B(a_0,s)$. Indeed, putting together (3.3.5) and (3.3.6) we get

$$V_4 + B(a,s) \subset B(P_F a_0, s) + B(a,s) \subset B(a_0, 3s).$$

The last inclusion is quite easy to prove. Let $x \in B(P_F a_0, s)$ and $y \in B(a,s)$. Then,

$$\|x + y - a_0\| = \|x + y - a - P_F a_0\| \le \|x - P_F a_0\| + \|y - a\| < 3s,$$

and since $B(a_0, 3s) \subset O$ the proof is complete.

Now, we set

$$\varphi(z) := (1 - s^{-1}\mathrm{dist}(z, V_4 + F))^+$$

where dist is the distance in the space X. Clearly φ is $2s^{-1}$-Lipschitz, hence $\varphi \in W^{1,p}(O)$ and $|\nabla_H \varphi|_H \le 2s^{-1}$ γ-a.e.. Moreover, $\varphi(z) \in [0,1]$ everywhere, $\varphi(z) = 1$ if $z \in V_4 + F$, $\varphi \equiv 0$ on $O \setminus (V+F)$ and

$$\|\nabla_H \varphi\|_{L^p(O)} = \|\nabla_H \varphi\|_{L^p(O \setminus (V_4+)F)} \le 2s^{-1}\gamma(O \setminus V_2)^{1/p} \le \frac{2\varepsilon^{1/p}}{s}.$$

We prove $\varphi f \in W_1(O)$. We get

$$\|f - \varphi f\|_{L^p(O)} \le M\|1 - \varphi\|_{L^p(O)} \le M\gamma(O \setminus (V_4+F))^{1/p} \le M\gamma(X \setminus V_2)^{1/p} \le M\delta^{1/p},$$

where $M := \|f\|_\infty$. As far as the L^p-norm of the gradient has concerned, we obtain

$$\begin{aligned}\|\nabla_H f - \nabla_H(\varphi f)\|_{L^p(O)} \le& \|\nabla_H f\|_{L^p(O \setminus V_4+F)} + M\|\nabla_H \varphi\|_{L^2(O \setminus V_4+F)} \\ \le& \varepsilon + \frac{2M\varepsilon^{1/p}}{s} \le \varepsilon^{1/p}(\varepsilon^{1/p'} + 2Ms^{-1}).\end{aligned}$$

Obviously $\|\varphi f\|_{L^\infty(O)} \le M$.Therefore, $W_2(O)$ is dense in $W^1(O)$.

Step 2.

We prove that if $g \in W_1(O)$ then it can be approximated in $W^{1,p}_*$ by functions in $W_3(O)$ which are uniformly bounded in $L^\infty(O)$ by $\|g\|_{L^\infty(O)}$.

By the first step, we know that a function in $W_1(O)$ can be approximated by functions in $W_2(O)$ which are uniformly bounded in $L^\infty(O)$ with its L^∞ constant; we prove that a function $f \in W_3(O)$ can be approximated by functions in $W_2(O)$ which are uniformly bounded in $L^\infty(O)$. To this aim, we consider $f \in W_2(O)$, F as in the definition of $W_2(O)$ and for any $R > 0$ we take a smooth function $\Phi_R : [0,+\infty) \longrightarrow [0,1]$ such that $\Phi_R = 1$ on $[0,R/3)$, $\Phi_R = 0$ on $[R,+\infty)$ and $|\Phi_R'| \le 2/R$. We set $\varphi_R(z) := \Phi_R(|\pi_F z|_H)$. Clearly, $f\varphi_R \in W_3(O)$ and

$$\|f-\varphi_R f\|_{L^p(O)} \le M\gamma_F(B^F(R/3)^c)^{1/p} \to 0,$$
$$\|\nabla_H(f-\varphi_R f)\|_{L^p(O)} \le \|\nabla_H f\|_{L^p(O)}\gamma_F(B_F(R/3)^c)^{1/p} + 2R^{-1}M \to 0,$$

as $R \to +\infty$, where $M := \|f\|_\infty$. It is also clear that $\|\varphi f\|_{L^\infty(O)} \le M$.

Step 3.

We prove that if $f \in W_1(O)$ then it can be approximated in $W^{1,p}_*$ by functions in $W^{1,p}(X)_{|O}$ which are uniformly bounded in $L^\infty(O)$ by $|f|_{L^\infty(O)}$.

By the second step, it is not restrictive to suppose that $f \in W_3(O)$. Let V, F, a, s and R be as in the definition of $W_3(O)$, and let R be large enough such that

$$V + B(a,s) \subset F^\perp + B(a,s) \subset F^\perp + \overline{B}(R).$$

Let $\alpha \in (0,1/2]$ and let us consider the homeomorphism T_α on X defined by

$$T_\alpha(z) := P_F(z) + (1-\alpha)\pi_F(z) + \alpha a = z + \alpha(a - \pi_F(z)),$$

for any $z \in X$. If $z = x+g$, with $g \in F$, then $T_\alpha(z) = z + \alpha(a-\pi_F(z)) = x + g + \alpha(a - \pi_F(z)) = a + g'$, where $g' = g + \alpha(a - \pi_F(z)) \in F$. Hence, $x+F$ is invariant under T_α, for any $x \in F^\perp$. Moreover, it is easy to see that $T_{\alpha|x+F}$ is the homothety centered in $P_F(x)+a$ with ratio $1-\alpha$. Therefore, if we consider $z \in V + \overline{B}(s/2)$, we get

$$\begin{aligned} T_\alpha^{-1}(O)\cap(z+F) =& T_\alpha^{-1}(O\cap(z+F)) \\ =& T_\alpha^{-1}((O\cap(z+F)) \\ \supset& \overline{O\cap(z+F)} = \overline{O}\cap(z+F). \end{aligned}$$

Now, we define

$$Y_\alpha := T_\alpha^{-1}((V + B(s/2) + F)\cap O),$$

for any $\alpha \in (0,1/2]$.

Each Y_α is convex, $Y_\beta \subset Y_\alpha$ if $\beta < \alpha$ and we can define $f_\alpha := f \circ T_\alpha$, on Y_α, since if $z \in Y_\alpha$ then $T_\alpha(z) \in O$.

Obviously $f_\alpha \in L^\infty(Y_\alpha)$ and $\|f_\alpha\|_{L^\infty(Y_\alpha)} \le \|f\|_{L^\infty(O)}$. By Lemma 3.3.3, we have that $f_\alpha \in W^{1,p}_*(Y_{\alpha/2})$ and it converges to $f_{|Y_{\alpha/2}}$ in that space if α goes to 0.

Now we set $O_1 := (V + \overline{B}_F(R'))\cap\overline{O}$ and $O_2 := (V + B(s/4) + B_F(R'+1))\cap T^{-1}_{\alpha/4}(O^\circ)$. Clearly, O_1 is a compact set, O_2 is an open set and we have the following chain of inclusions:

$$Y \subset O_1 \subset O_2 \subset \overline{O}_2 \subset Y^\circ_{\alpha/2} = Y_{\alpha/2}.$$

We introduce the function

$$\rho(z) := \frac{\operatorname{dist}(z, O_2^c)}{\operatorname{dist}(z,O_1) + \operatorname{dist}(z,O_2^c)}$$

where dist is the distance in X. $0 \le \rho \le 1$, $\rho \equiv 1$ on O_1 and $\rho \equiv 0$ on O_2^c.

We define

$$g(z) := \begin{cases} f_\alpha(z)\rho(z), & z \in O_2, \\ 0 & z \notin O_2. \end{cases}$$

ρ is bounded and Lipschitz continuous, hence $g \in W^{1,p}(X)$. Moreover, $g \equiv f_\alpha$ on Y and $g \equiv 0$ on $O \setminus Y$. Indeed, $f_\alpha \equiv 0$ on $(Y_\alpha \cap O) \setminus Y$ and $\rho \equiv 0$ on $(O \setminus Y_\alpha) \setminus Y \subset (O \setminus Y_{\alpha/2}) \subset O_2^c$. Then, we have

$$\|g - f\|_{W_*^{1,p}(O)} = \|f_\alpha - f\|_{W_*^{1,p}(Y)} \leq \varepsilon. \tag{3.3.7}$$

Hence, g approximates f in $W_*^{1,p}(O)$, and clearly

$$\|g\|_{L^\infty(O)} \leq \|f_\alpha\|_{L^\infty(O_2)} \leq \|f\|_{L^\infty(O)}.$$

Step 4.

Let $f \in W_*^{1,p}(O)$, for $M \in \mathbb{N}$ we define the function $f_M := f \wedge M \vee (-M)$; for every $h \in H$, it suffices to consider f_y on O_y for $\gamma^\perp$-almost every $y \in h^\perp$ to prove that $f_M \in D_h^O$ and that (up to a γ-representative)

$$\partial_h f_M(x) = \begin{cases} \partial_h f(x) & \text{if } |f(x)| < M \\ 0 & \text{otherwise} \end{cases};$$

hence, it is immediate that $f \in W_*^{1,p}(O)$ with

$$\nabla_H f = \begin{cases} \nabla_H f(x) & \text{if } |f(x)| < M \\ 0 & \text{otherwise} \end{cases};$$

moreover $f_M \to f$ in $W_*^{1,p}(O)$ as $M \to +\infty$, for any $p \geq 1$ (by dominated convergence theorem).

Then, let $f \in W_*^{1,p}(O) \cap L^q(O)$, we have that for $f_M = f \wedge M \vee (-M)$, $f_M \to f$ in $W_*^{1,p}(O)$ by what we said, and $f_M \to f$ in $L^q(O)$ by the dominated convergence theorem and clearly $f_M \in W_*^{1,p}(O) \cap L^\infty(O)$ for every M. By the above steps, each function in $W_*^{1,p}(O) \cap L^\infty(O)$ can be approximated in $W^{1,p}$ by a sequence of restrictions on O of functions f_n in $W^{1,p}(X) \cap L^\infty(X)$ which are uniformly bounded in $L^\infty(O)$; so, by the dominated convergence Theorem (because γ is finite and f_n converges pointwise), f_n converges also in $L^q(O)$ for every $q < +\infty$; therefore, we conclude the proof.

□

Now we prove Lemma 3.3.3 which we used in the step 3.

LEMMA 3.3.3. *Let $f \in W_3$, f_α defined in the step 3; then $f_\alpha \in W_*^{1,p}(Y_{\alpha/2})$ and it converges to $f_{|Y_{\alpha/2}}$ in that space if α goes to* 0.

PROOF. We recall that $f \in W_3$ satisfies the hypotheses of Subsection 3.3.1: we can define f_ε, and we have $f_\varepsilon \in L^p(X)$, $\partial_h f_\varepsilon \in L^p(X)$ because $f \in W^{1,p}(X)$.

We recall that

$$Y_\alpha = T_\alpha^{-1}((V + B(s/2) + F) \cap O)$$

for $\alpha \in (0, 1/2]$.

At first, we notice that $f_\alpha \equiv 0$ on $Y_\alpha \setminus T_\alpha^{-1}(V + \overline{B}_F(R))$, since $f \equiv 0$ on $O \setminus (V + \overline{B}_F(R))$. Hence, there exists $R' > R$ independent of α such that $f_\alpha \equiv 0$ on $Y_\alpha \setminus (V + \overline{B}_F(R'))$ for any $\alpha \in (0, 1/2]$ (it is enough to take $R' > 2R$).

We define

$$Y := (V + \overline{B}_F(R')) \cap O \subseteq (V + F) \cap O \subseteq Y_\alpha,$$

it is a set relatively compact (because V is compact) and convex.

We prove that $\gamma\circ T_\alpha^{-1}$ is absolutely continuous with respect to γ: in fact for any bounded Borel function g on X we have

$$\int_X g(z)d\gamma\circ T_\alpha^{-1}(dz)=\int_X g(T_\alpha(z))\,d\gamma(z)=$$

$$=\int_{F^\perp}\int_F g(x+(1-\alpha)y+\alpha a)\,d\gamma_F(y)\,d\gamma_{F^\perp}(x)=$$

$$=(2\pi)^{-\frac{n}{2}}\int_{F^\perp}\int_F g(x+y')\exp\left(-\frac{|y'-\alpha a|^2}{2(1-\alpha)}\right)(1-\alpha)^{-n}dy'\,d\gamma_{F^\perp}(x)=$$

$$=(2\pi)^{-\frac{n}{2}}\int_{F^\perp}\int_F g(x+y')\exp\left(-\frac{|y'-\alpha a|^2}{2(1-\alpha)}+\frac{1}{2}|y'|^2\right)(1-\alpha)^{-n}\,d\gamma_F(y')d\gamma_{F^\perp}(x).$$

Hence, $\gamma\circ T_\alpha^{-1}$ is absolutely continuous with respect to γ and its Radon-Nikodym density

$$\frac{d(\gamma\circ T_\alpha^{-1})}{d\gamma}(y)=\exp\left(-\frac{|y'-\alpha a|^2}{2(1-\alpha)}+\frac{1}{2}|y'|^2\right)(1-\alpha)^{-n}$$

is uniformly bounded with respect to $\alpha\in(0,\frac{1}{2})$ on any compact of X, hence also on Y.

$f_\alpha=0$ on $(Y_\alpha\cap O)\setminus Y$ and $f\equiv 0$ on $O\setminus Y$ (because $f\in W_3$ and $R'>R$). We have

$$\int_{Y_\alpha} f_\alpha^p d\gamma=\int_{T_\alpha(Y_\alpha)} f^p d(\gamma\circ T_\alpha^{-1})=\int_{T_\alpha(Y_\alpha)\cap Y} f^p\frac{d(\gamma\circ T_\alpha^{-1})}{d\gamma}d\gamma<+\infty,$$

since the Radon-Nikodym derivative of $\gamma\circ T_\alpha^{-1}$ is bounded on compact set. Therefore, $f_\alpha\in L^p(Y_\alpha)$.

For every $f\in L^p(Y)$ (f not necessarily bounded) we have

$$\|f_\alpha-f\|^p_{L^p(Y)}\le\|f\circ T_\alpha-f_\varepsilon\circ T_\alpha\|_{L^p(Y)}+\|f_\varepsilon\circ T_\alpha-f_\varepsilon\|_{L^p(Y)}+\|f_\varepsilon-f\|_{L^p(Y)}\tag{3.3.8}$$

$$\le\|f_\varepsilon-f\|_{L^p(Y)}\left(\left\|\frac{d(\gamma\circ T_\alpha^{-1})}{d\gamma}\right\|^{1/p}_{L^\infty(Y)}+1\right)+\|f_\varepsilon\circ T_\alpha-f_\varepsilon\|_{L^p(Y)}.\tag{3.3.9}$$

The last term goes to 0 as $\alpha\to 0$: in fact, f_ε is continuous in direction h for every $h\in F$, we have $T_\alpha(x+F)\subseteq x+F$ for every $\alpha\in(0,1)$ and $x\in X$, and T_α converges to the identity; moreover (3.3.2) yields

$$\|f_\varepsilon\circ T_\alpha-f_\varepsilon\|\le g$$

where, for every $y\in F^\perp$, $h\in F$,

$$g(y+k):=\alpha R\|f\|_{L^1(Y_y,\mathscr{L}^N)}\|\Psi_\varepsilon\|_{C^1};$$

now

$$\int_Y g^p\,d\gamma\le\varepsilon^N\alpha^p\|\Psi_\varepsilon\|^p_{C^1}\int_{F^\perp}\int_{(Y_\alpha)_y}|f(y+k)|^p\,dk\,d\gamma_{F^\perp}(y)\le$$

(if ρ_F is the density of γ_F with respect to the Lebesgue measure)

$$\le\alpha^p\|\Psi_\varepsilon\|^p_{C^1}\int_{F^\perp}\int_{(Y_\alpha)_y}|f(y+k)|^p\rho_F^{-1}(k)\,d\gamma_F(k)\,d\gamma_{F^\perp}(y)\le$$

$$\le\alpha^p\|\Psi_\varepsilon\|^p_{C^1}\varepsilon^{N(p-1)}\sup\{\rho_F(y,k)|y+k\in Y\}\int_X|f|^p d\gamma$$

($\sup\{\rho_F^{-1}(y,k)|y+k\in Y\}<\infty$ because Y is compact) hence $g^p\in L^1(Y)$ and we can apply the dominated convergence Theorem to conclude (we remark that in this argument we did not use the boundedness of f). Hence, we get

$$\limsup_{\alpha\to 0}\|f_\alpha-f\|_{L^p(Y)}\le\|f_\varepsilon-f\|_{L^p(Y)}\left(\sup_{\alpha\in(0,1/2]}\left\|\frac{d(\gamma\circ T_\alpha^{-1})}{d\gamma}\right\|_{L^\infty(Y)}^{1/p}+1\right),$$

and letting ε approaches 0, by Lemma [?] we obtain that f_α converges to f in $L^p(Y)$ as $\alpha\to 0$.

Analogously we can prove that $\nabla_H f\circ T_\alpha$ converges to $\nabla_H f$ in L^p (in the above paragraphs we used the fact that f is L^p and 0 out of the compact Y, not that it is bounded).

To prove the convergence of the derivatives, let us consider $h\in H\setminus\{0\}$ and let us define $\tilde F:=\mathrm{span}(F+h)$. We denote by $\{h_1,\dots,h_m\}$ an orthonormal basis of $\tilde F$. Clearly, for all $z\in X$ the maps $y\mapsto T_\alpha(z+y)$ maps $\tilde F$ to $\tilde F+T_\alpha(z)$ and it is smooth. Now, we define f_ε with F replaced by $\tilde F$ and we notice that f is 0 on $X\setminus(\tilde V+B_{\tilde F}(\tilde R)$, where $\tilde V\subset\tilde F^\perp$. Indeed, f is zero on $O\setminus Y$ and Y is a compact set. Moreover, the map $f_\varepsilon\circ T_\alpha$ is differentiable along any direction of $\tilde F$ γ-a.e. and

$$\partial_h(f_\varepsilon\circ T_\alpha)=\sum_{i=1}^m\partial_{h_i}f_\varepsilon\circ T_\alpha\langle h-\alpha\pi_F(h),h_i\rangle_H. \tag{3.3.10}$$

By Lemma 3.3.2, with $\tilde F=F$, $O=X$ and $T_\alpha(Y_{\alpha/2})=U'$, we deduce that $\partial_{h_i}f_\varepsilon\to\partial_{h_i}f$ in $L^p(T_\alpha(Y_{\alpha/2}))$ as $\varepsilon\to 0$. Then,

$$\begin{aligned}\|\partial_{h_i}f_\varepsilon\circ T_\alpha-\partial_{h_i}f\circ T_\alpha\|_{L^p(Y_{\alpha/2})}&=\int_{T_\alpha(Y_{\alpha/2})}|\partial_{h_i}f_\varepsilon-\partial_{h_i}f|^p d(\gamma\circ T_\alpha^{-1})\\&=\int_{T_\alpha(Y_{\alpha/2})\cap(Y+B^{\tilde F}(1))}|\partial_{h_i}f_\varepsilon-\partial_{h_i}f|^p\frac{d(\gamma\circ T_\alpha^{-1})}{d\gamma}d\gamma\\&\to 0,\qquad \varepsilon\to 0.\end{aligned}\tag{3.3.11}$$

Hence, putting together (3.3.10) and (3.3.11) we conclude that there exists the $L^p(T_\alpha(Y_{\alpha/2}))$-limit of $\partial_h(f_\varepsilon\circ T_\alpha)$ as $\varepsilon\to 0$ and

$$\lim_{\varepsilon\to 0}\partial_h(f_\varepsilon\circ T_\alpha)=\sum_{i=1}^m\partial_{h_i}f\circ T_\alpha\langle h-\alpha\pi_F(h),h_i\rangle_H=\langle\nabla_H f\circ T_\alpha,h-\alpha\pi_F(h)\rangle_H=langle\psi_\alpha,h\rangle_H,$$

in $L^p(T_\alpha(Y_{\alpha/2}))$, where

$$\psi_\alpha:=\nabla_H f\circ T_\alpha-\alpha\pi_F\circ\nabla_H f\circ T_\alpha.$$

Since $f_\varepsilon\circ T_\alpha$ converges to f_α in $L^p(Y_\alpha)$, we conclude that $f_\alpha\in D_h^{Y_{\alpha/2}}$ (see the definition in the above subsection) and $\nabla_H f_\alpha=\psi_\alpha\in L^p(Y_{\alpha/2})$. Hence, $f_\alpha\in W_*^{1,p}(Y_{\alpha/2})$ and

$$\begin{aligned}\|\nabla_H f_\alpha-\nabla_H f\|_{L^p(Y)}^p&=\int_Y|\psi_\alpha-\nabla_H f|^p d\gamma=\int_{Y\cap O^\circ}|\psi_\alpha-\nabla_H f|^p d\gamma\\&\le\int_{Y\cap O^\circ}|\nabla_H f\circ T_\alpha-\nabla_H f|^p d\gamma+2^{p-1}\alpha^{p-1}\int_{Y\cap O^\circ}|\pi_F\circ\nabla_H f\circ T_\alpha|^p d\gamma\\&=:I_1(\alpha)+I_2(\alpha).\end{aligned}$$

By arguing as in (3.3.9), we deduce that $I_1(\alpha)$ vanishes as $\alpha \to 0$. Moreover,

$$I_2(\alpha) \leq \alpha^{p-1} 2^{p-1} \int_{T_\alpha(Y\cap O^\circ)} |\nabla_H f|^p \frac{d(\gamma\circ T_\alpha)}{d\gamma} d\gamma \leq \alpha^{p-1}2^{p-1} \|\nabla_H f\|_{L^p(Y)} \left\| \frac{d(\gamma\circ T_\alpha)}{d\gamma} \right\|_{L^\infty(Y)}$$
$$\to 0,$$

as $\alpha \to 0$. Hence, $\nabla_H f_\alpha$ converges to $\nabla_H f$ in $L^p(Y)$ and therefore for any $\varepsilon > 0$ there exists $\alpha \in (0,1/2]$ such that $\|f_\alpha - f\|_{W*^{1,p}(Y)} < \varepsilon$.

□

3.4. Second derivatives in Wiener spaces

3.4.1. Second derivatives and Hilbert-Schmidt norm. We recall the definition of Hilbert–Schmidt operators, see e.g. [**33**, S XI.6]; for H-derivative we refer to [**14**], Chap. 5.

DEFINITION 3.4.1. Let H_1, H_2 be separable Hilbert spaces. A linear operator $A \in \mathcal{L}(H_1,H_2)$ is called a *Hilbert-Schmidt operator* if there exists an orthonormal basis $\{h_j :\ j \in \mathbb{N}\}$ of H_1 such that

$$\sum_{j=1}^{\infty} \|Ah_j\|^2_{H_2} < \infty. \tag{3.4.1}$$

If A is a Hilbert-Schmidt operator and $\{e_j :\ j \in \mathbb{N}\}$ is any orthonormal basis of H_1, $\{y_j :\ j \in \mathbb{N}\}$ is any orthonormal basis of H_2, then

$$\|Ae_j\|^2_{H_2} = \sum_{k=1}^{\infty} \langle Ae_j, y_k\rangle^2_{H_2} = \sum_{k=1}^{\infty} \langle e_j, A^* y_k\rangle^2_{H_2}$$

so that

$$\sum_{j=1}^{\infty} \|Ae_j\|^2_{H_2} = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \langle e_j, A^* y_k\rangle^2_{H_2} = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \langle e_j, A^* y_k\rangle^2_{H_2} = \sum_{k=1}^{\infty} \|A^* y_k\|^2_{H_1}.$$

So, the convergence of the series (3.4.1) and the value of its sum are independent of the basis of H_1. We denote by $\mathcal{H}(H_1,H_2)$ the space of the Hilbert-Schmidt operators from H_1 to H_2, and we set

$$\|A\|_{\mathcal{H}(H_1,H_2)} = \left(\sum_{j=1}^{\infty} \|Ah_j\|^2_{H_2} \right)^{1/2},$$

for any fixed orthonormal basis $\{h_j :\ j \in \mathbb{N}\}$ of H_1; we call $\|\cdot\|_{\mathcal{H}(H_1,H_2)}$ the Hilbert-Schmidt norm. When the setting is clear we denote this norm simply by $|\cdot|_{HS}$. Notice that if $H_1 = \mathbb{R}^n$, $H_2 = \mathbb{R}^m$, the Hilbert–Schmidt norm of any linear operator coincides with the Euclidean norm of the associated matrix.

The norm (3.4.1) comes from the inner product

$$\langle A,B\rangle_{\mathcal{H}(H_1,H_2)} = \sum_{j=1}^{\infty} \langle Ah_j, Bh_j\rangle_{H_2};$$

for every couple of Hilbert-Schmidt operators A, B, the series converges for every orthonormal basis $\{h_j :\ j \in \mathbb{N}\}$ of H_1, and its value is independent of the basis. The space $\mathcal{H}(H_1,H_2)$ is a separable Hilbert space with the above inner product.

If $H_1 = H_2 = H$, where H is the Cameron–Martin space of (X,γ), we set $\mathcal{H} := \mathcal{H}(H,H)$.

We want to define $W^{k,p}(X)$ for $k>1$. To this aim, we define the cylindrical E-valued functions as follows, where E is any normed space.

DEFINITION 3.4.2. For $k\in\mathbb{N}$ we define $\mathcal{F}C_b^k(X,E)$ (respectively, $\mathcal{F}C_b^\infty(X,E)$) as the linear span of the functions $x\mapsto v(x)y$, with $v\in\mathcal{F}C_b^k(X)$ (respectively, $v\in\mathcal{F}C_b^\infty(X)$) and $y\in E$.

Therefore, every element of $\mathcal{F}C_b^k(X,E)$ may be written as

$$v(x)=\sum_{j=1}^n v_j(x)y_j \tag{3.4.2}$$

for some $n\in\mathbb{N}$, and $v_j\in\mathcal{F}C_b^k(X)$, $y_j\in E$. Such functions are Fréchet differentiable at every $x\in X$, with $v'(x)\in\mathcal{L}(X,E)$ given by

$$(v'(x))(h)=\sum_{j=1}^n (v_j'(x))(h)y_j$$

for every $h\in X$.

Similarly to the scalar case, we introduce the notion of H-differentiable function.

DEFINITION 3.4.3. A function $f:X\to E$ is called *H-differentiable* at $\overline{x}\in X$ if there exists $L\in\mathscr{L}(H,E)$ such that for every

$$\|f(\overline{x}+h)-f(\overline{x})-L(h)\|_E=o(|h|_H)\quad \text{for } h\in H.$$

i.e.

$$\lim_{r\to 0^+}\sup_{h\in B^H(r)}\frac{\|f(\overline{x}+h)-f(\overline{x})-L(h)\|_E}{|h|}=0$$

where $B^H(r)$ is the ball in H cantered in 0 with radius r.

In this case we set $L=:D_Hv(\overline{x})$.

If $f\in\mathcal{F}C_b^1(X,E)$ is given by $f(\cdot)=\psi(\cdot)y$ with $\psi\in\mathcal{F}C_b^1(X)$ and $y\in E$, then f is H-differentiable at every $\overline{x}\in X$, and

$$D_Hf(\overline{x})(h)=[\nabla_H\psi(\overline{x}),h]_H\,y.$$

In particular, if $E=H$ and $\{h_j:\ j\in\mathbb{N}\}$ is any orthonormal basis of H we have

$$|D_Hv(\overline{x})(h_j)|_H^2\le|\left\langle\nabla_H\psi(\overline{x}),h_j\right\rangle_H|^2\,|y|_H^2$$

so that $D_Hv(\overline{x})$ is a Hilbert-Schmidt operator, and we have

$$\begin{aligned}|D_Hv(\overline{x})|_{\mathcal{H}}^2&=\sum_{j=1}^\infty|D_Hv(\overline{x})(h_j)|^2=\sum_{j=1}^\infty|\left\langle\nabla_H\psi(\overline{x}),h_j\right\rangle_H|^2|y|_H^2\\&=|\nabla_H\psi(\overline{x})|_H^2|y|_H^2.\end{aligned}$$

Moreover, $x\mapsto\nabla_H\psi(x)$ is continuous and bounded. In addition, the operator $J:H\to\mathcal{H}$,

$$(Jk)(h):=\langle k,h\rangle_H\,y,\qquad k,h\in H$$

is bounded since

$$|Jk|_{\mathcal{H}}^2=\sum_{j=1}^\infty|\left\langle k,h_j\right\rangle_H y|_H^2=|k|_H^2|y|_H^2.$$

Then $x\mapsto D_Hv(x)=J(\nabla_H\psi(x))$ is continuous and bounded from X to $\mathcal{H}$. In particular, it belongs to $L^p(X,\mathcal{H})$ for every $1\le p<\infty$.

We have this Lemma

LEMMA 3.4.4. *For every $1 \le p < \infty$, the operator $D_H : \mathcal{F}C_b^1(X,H) \to L^p(X,\gamma;\mathcal{H})$ is closable in $L^p(X,H)$.*

DEFINITION 3.4.5. For every $1 \le p < \infty$ we define $W^{1,p}(X,H)$ as the domain of the closure of the operator $D_H : \mathcal{F}C_b^1(X,H) \to L^p(X,\mathcal{H})$ (still denoted by D_H) in $L^p(X,H)$.

We get that $W^{1,p}(X,H)$ is a Banach space with the graph norm

$$\begin{aligned}\|V\|_{W^{1,p}(X,H)} &= \Big(\int_X |V(x)|_H^p d\gamma\Big)^{1/p} + \Big(\int_X |D_H V(x)|_{\mathcal{H}}^p d\gamma\Big)^{1/p}\\ &= \Big(\int_X\Big(\sum_{j=1}^{\infty}\langle V(x),h_j\rangle_H^2\Big)^{p/2} d\gamma\Big)^{1/p} + \Big(\int_X\Big(\sum_{i,j=1}^{\infty}\langle D_H V(x)(h_i),h_j\rangle_H^2\Big)^{p/2} d\gamma\Big)^{1/p}.\end{aligned}$$

3.4.2. The Sobolev spaces $W^{2,p}(X,\gamma)$. If $f : X \to \mathbb{R}$ is H-differentiable at any $x \in X$ (hence the operator ∇_H is everywhere defined), we say that f is twice H-differentiable at $\bar{x}$ if $\nabla_H$and there exists a linear operator $L_H \in \mathscr{L}(H)$ such that

$$|\nabla_H f(\bar{x}+h) - \nabla_H f(\bar{x}) - L_H h|_H = o(|h|_H) \quad \text{as } h \to 0 \text{ in } H.$$

The operator L_H is denoted by $D_H^2 f(\bar{x})$, and by Definition 3.4.3, we have that $D_H^2 f(\bar{x}) = D_H \nabla_H f(\bar{x})$.

If $f \in \mathcal{F}C_b^2(X)$, $f(x) = \varphi(l_1(x),\dots,l_n(x))$ with $\varphi \in C_b^2(\mathbb{R}^n)$, $l_k \in X^*$, then f is twice differentiable at any $\bar{x} \in X$ and

$$(f''(\bar{x})v)(w) = \sum_{i,j=1}^{n} \partial_i\partial_j\varphi(l_1(\bar{x}),\dots,l_n(\bar{x})l_i(v)l_j(w), \quad v,\, w \in X$$

so that

$$\langle D_H^2 f(\bar{x})h,k\rangle_H = \sum_{i,j=1}^{n} \partial_i\partial_j\varphi(l_1(\bar{x}),\dots,l_n(\bar{x})\,\langle R_\gamma l_i,h\rangle_H\,\langle R_\gamma l_j,k\rangle_H, \quad h,\, k \in H.$$

$D_H^2 f(\bar{x})$ is a Hilbert–Schmidt operator. We have this Lemma (see [**14**], Sec. 5.2).

LEMMA 3.4.6. *For every $1 \le p < \infty$, the operator*

$$(\nabla_H, D_H^2) : \mathcal{F}C_b^2(X) \to L^p(X,\gamma;H) \times L^p(X,\gamma;\mathcal{H})$$

is closable in $L^p(X,\gamma)$.

REMARK 3.4.7. In [**14**] the space of $\mathcal{F}C_b^\infty(X)$ is used instead of $\mathcal{F}C_b^2(X)$, but it is equivalent, because each element of $\mathcal{F}C_b^2(X)$ can be approximated by an element of $\mathcal{F}C_b^\infty(X)$ by convolutions.

DEFINITION 3.4.8. For every $1 \le p < \infty$, $W^{2,p}(X,\gamma)$ is the domain of the closure of

$$(\nabla_H, D_H^2) : \mathcal{F}C_b^2(X) \to L^p(X,\gamma;H) \times L^p(X,\gamma;\mathcal{H})$$

in $L^p(X,\gamma)$. Therefore, $f \in L^p(X,\gamma)$ belongs to $W^{2,p}(X,\gamma)$ iff there exists a sequence $(f_n) \subset \mathcal{F}C_b^2(X)$ such that $f_n \to f$ in $L^p(X,\gamma)$, $\nabla_H f_n$ converges in $L^p(X,\gamma;H)$ and $D_H^2 f_n$ converges in $L^p(X,\gamma;\mathcal{H})$. In this case we set $D_H^2 f := \lim_{n\to\infty} D_H^2 f_n$.

$W^{2,p}(X,\gamma)$ is a Banach space with the graph norm

$$\begin{aligned}(3.4.3)\qquad \|f\|_{W^{2,p}} &:= \|f\|_{L^p(X,\gamma)} + \|\nabla_H f\|_{L^p(X,\gamma;H)} + \|D^2_H f\|_{L^p(X,\gamma;\mathcal{H})}\\ &= \left(\int_X |f|^p d\gamma\right)^{1/p} + \left(\int_X |\nabla_H f|^p_H d\gamma\right)^{1/p} + \left(\int_X |D^2_H f|^p_{\mathcal{H}} d\gamma\right)^{1/p}.\end{aligned}$$

Fixed any orthonormal basis $\{h_j : \ j\in\mathbb{N}\}$ of H, for every $f\in W^{2,p}(X,\gamma)$ we set

$$\partial_{ij} f(x) = \left\langle D^2_H f(x) h_j, h_i\right\rangle_H .$$

For every sequence of approximating functions f_n we have

$$\left\langle D^2_H f_n(x) h_j, h_i\right\rangle_H = \left\langle D^2_H f_n(x) h_i, h_j\right\rangle_H , \quad x\in X,\ i,j\in\mathbb{N},$$

then the equality

$$\partial_{ij} f(x) = \partial_{ji} f(x),$$

holds a.e.. Therefore, the $W^{2,p}$ norm may be rewritten as

$$\left(\int_X |f|^p d\gamma\right)^{1/p} + \left(\int_X \left(\sum_{j=1}^{\infty} (\partial_j f)^2\right)^{p/2} d\gamma\right)^{1/p} + \left(\int_X \left(\sum_{i,j=1}^{\infty} (\partial_{ij} f)^2\right)^{p/2} d\gamma\right)^{1/p}.$$

3.4.3. Ornstein-Uhlenbeck operator. The concepts of strongly continuous semigroup on a Banach space, of its generator and of form associated to semigroup on Hilbert space (see Subsection 1.1).

We recall that if an operator L is associated to a dissipative form, then its spectrum is contained in $(-\infty,0]$, and, for $\sigma>0$ we can define $(\sigma I - L)^{-1}$ as a self-adjoint operator in $L^2(O)$.

We introduce, in this setting

$$J_\sigma := (I-\sigma L)^{-1} = \sigma(\sigma I - L)^{-1} = \sigma R(\sigma, L).$$

Let (E,μ) be a measure space; in the sequel, L will always be an operator on $H = L^2(E,\mu)$, a a form on H and T_t the strongly continuous semigroup on H generated by L, G_λ be the strongly continuous contractive resolvent associated to L.

For the following definitions see e.g. [[**52**], Def. I.4.1].

DEFINITION 3.4.9. T_t (G_λ) is said *sub-Markovian* if, for every f s.t. $0\le f\le 1$ then $0\le T_t(f)\le 1$ for every $t>0$ ($0\le T_\lambda(f)\le 1$ for every $\lambda>0$).

L is said a *Dirichlet operator* if $\langle Lu,(u-1)^+\rangle_H \le 0$ for every $u\in H$.

We also recall this result about operators and generators (see [**52**], Prop. I.4.3).

PROPOSITION 3.4.10. *The following are equivalent:*

i) *L is a Dirichlet operator;*
ii) *T_t is sub-Markovian;*
iii) *G_λ is sub-Markovian.*

In particular we recall what that the *heat semigroup* on $L^2(\mathbb{R}^d,\mathscr{L}^d)$ is associated to the form $(W^{1,2}(\mathbb{R}^d,\mathscr{L}^d),\mathbb{D})$ where, for $f,g\in W^{1,2}(\mathbb{R}^d,\mathscr{L}^d)$,

$$\mathbb{D}(f,g) := \frac{1}{2}\int_{\mathbb{R}^d} \nabla f(x)\cdot\nabla g(x)\,dx.$$

The generator of T_t is called *Laplace operator*, and it is denote by Δ.

We have that $D(\Delta) = W^{2,2}(\mathbb{R}^d, \mathscr{L}^d)$, and

$$\Delta f = \sum_{i=1}^{d} \frac{\partial_i^2 f}{\partial x_i^2}$$

and we can say that Δf is the divergence of the gradient of f.

The fact that Δ is associated to $\mathbb{D}$ implies this formula

$$\int_{\mathbb{R}^d} \Delta f(x) g(x)\, dx = -\int_{\mathbb{R}^d} \nabla f(x) \cdot \nabla g(x)\, dx$$

for every $f \in W^{2,2}(\mathbb{R}^d, \mathscr{L}^d)$, $g \in W^{1,2}(\mathbb{R}^d, \mathscr{L}^d)$.

We can find make equivalent construction in infinite dimensional space. Let (X, γ) be a Wiener space. We will consider the Hilbert space $L^2(X)$.

We can define the form $(W^{1,2}(X), \mathbb{D})$ where

$$\mathbb{D}(f,g) := \frac{1}{2} \int_X \langle \nabla_H f, \nabla_H g \rangle_H \, d\gamma(x);$$

the semigroup associated to this form is the *Ornstein-Uhlenbeck semigroup on* X, and it can be expressed by the Mehler formula

$$T_t f(x) := \int_X f(e^{-t}x + \sqrt{1-e^{-2t}}y)\gamma(dy); \tag{3.4.4}$$

the operator which generates this semigroup is said Ornstein-Uhlenbeck operator on X. We have that its domain is $W^{2,2}(X)$ and it can be represented as

$$Lf(x) = \sum_{i=1}^{+\infty} \left(\partial_{h_i}^2 f(x) - \hat{h}_i(x)\partial_{h_i} f(x)\right)$$

where $\{h_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of H; for what we said about the Gaussian divergence we have

$$Lf = \mathrm{div}_\gamma(\nabla_H f).$$

The fact that L is associated to $\mathbb{D}$ implies this formula

$$\int_X (Lf)\, g\, d\gamma = -\int_X \langle \nabla_H f, \nabla_H g \rangle_H \, d\gamma$$

for every $f \in W^{2,2}(X)$, $g \in W^{1,2}(X)$.

In the case $\mathbb{R}^N$ with standard Gaussian measure γ^N, L has the form

$$Lf(\xi) = \Delta f(\xi) - \sum_{i=1}^{N} \xi_i D_i f(\xi)$$

where Δ is the Laplace operator, for $f \in W^{q,2}$ for some $q \geq 1$; If X is infinite dimensional we have that

$$Lf(x) = \sum_{i=1}^{\infty} \left(\left\langle D_H^2 f(x) h_i, h_i \right\rangle_H - \hat{h}_i(x)\partial_{h_i} f(x)\right) \tag{3.4.5}$$

or $f \in W^{q,2}$ for some $q \geq 1$.

Now, let $O \subset X$ be an open set. We have that we can define on $L^2(O)$ the form given by $(W^{1,2}(O), \mathbb{D})$ where $\mathbb{D}$ is expressed as

$$\mathbb{D}(f,g) := \frac{1}{2}\int_O \langle \nabla_H f, \nabla_H g\rangle_H \, d\gamma(x); \tag{3.4.6}$$

it is a closed symmetric form (the closure is a consequence of the definition of $W^{1,2}(O)$). We call the semigroup associated to Ornstein-Uhlenbeck semigroup with Neumann boundary conditions on O or Neumann Ornstein-Uhlenbeck semigroup, and the operator L_N will be called Neumann Ornstein-Uhlenbeck operator ; also in this case we have

$$\int_O (L_N f)\, g \, d\gamma = -\int_O \langle \nabla_H f, \nabla_H g\rangle_H \, d\gamma \tag{3.4.7}$$

for every $f \in D(L_N)$, $g \in W^{1,2}(O)$ (from an heuristic point of view, the above formula is a version of an integration by parts formula without the part of the boundary; so, $D(L_N)$ imposes that f satisfy, in some weak sense, a Neumann boundary conditions).

If we consider the form given by $(W_0^{1,2}(O), \mathbb{D})$ where $\mathbb{D}$ is expressed as in (3.4.6), it is a closed symmetric form; it is associated to a semigroup called Ornstein-Uhlenbeck semigroup with Dirichlet boundary conditions on O or Dirichlet Ornstein-Uhlenbeck semigroup, and the generator L_D will be called Dirichlet Ornstein-Uhlenbeck operator; by definition $D(L_D) \subseteq W_0^{1,2}(O)$ (so it satisfies a Dirichlet boundary condition) and L_D satisfies

$$\int_O (L_D f)\, g \, d\gamma = -\int_O \langle \nabla_H f, \nabla_H g\rangle_H \, d\gamma \tag{3.4.8}$$

for every $f \in D(L_D)$, $g \in W_0^{1,2}(O)$.

In all cases, $L \in \{L_D, L_N\}$ is a Markov operator, in fact, if $u \in D(L)$, then (by applying (3.4.7), (3.4.8))

$$\int_O Lu(u-1)^+ \, d\gamma = -\int_O \left\langle \nabla_H u, \nabla_H (u-1)^+ \right\rangle_H \, d\gamma \le 0;$$

hence in all of these cases, T_t is a contractive strongly continuous semigroup (by Proposition 3.4.10), and G_σ is a contractive resolvent semigroup on $L^2(X)$, which implies (by definition of contractive resolvent semigroup) that $J_\sigma = \sigma G_\sigma(\cdot)$ is contractive in $L^2(O)$ and that T_t is a sub-Markovian semigroup i.e. if $f \le 1$ then $T_t(f) \le 1$ and it is L^∞-contractive; we also have that G_σ has the same property; so we can restrict T_t and J_σ to $L^\infty(O)$, and they are contraction operators.

We see that J_σ (and T_t) can be defined as operator from $L^p(O)$ to $L^p(O)$ for every $p \in [1, +\infty)$ (but we do not know if they is regularizing).

We have now that J_σ (and T_t) in L^2 is contractive with respect to the metric L^1: in fact, for all $y \in L^2$, for all test functions φ, by the self-adjointness of J_σ,

$$\int_O J_\sigma(y)\varphi \, d\gamma = \int_O y J_\sigma(\varphi)\, d\gamma \le \|y\|_{L^1} \|J_\sigma(\varphi)\|_{L^\infty} \le \|y\|_{L^1}\|\varphi\|_{L^\infty}, \tag{3.4.9}$$

hence $\|J_\sigma(y)\|_{L^1} \le \|y\|_{L^1}$; so by the density of L^2 in L^1, we can define J_σ as a contractive (and hence continuous) operator in L^1.

Hence, we have that J_σ (and T_t) can be defined as a contractive operator $L^p(O) \to L^p(O)$, for all $p \in [1, \infty)$, by the Riesz-Thorin interpolation theorem (see e.g. [**58**], Sub. 1.3.18), because L^2 is dense in L^p (see the Appendix for one of the several statements of the Riesz-Thorin theorem).

As in (3.4.9) we can see that T_t is contractive in L^p for every $p > 2$ (by using the duality with respect to $L^{p'}$).

So, J_σ and T_t can be defined in every L^p for every $p \in [1, +\infty)$ and they are contractive.

REMARK 3.4.11. Exactly the same argument can be used for the Laplace operators Δ_N and Δ_D in $L^2(O, \mathscr{L}^N)$.

We consider $O = X$; for $p \in [1, +\infty)$ the Ornstein-Uhlenbeck semigroup $(T_t)_{t\geq 0}$ on $L^p(X)$ can be defined pointwise by *Mehler's formula*:

$$T_t u(x) = \int_X u\left(e^{-t}x + \sqrt{1-e^{-2t}}y\right) d\gamma(y) \tag{3.4.10}$$

for all $u \in L^p(X,\gamma)$, $t \geq 0$; we remark that the formula does not depend on p.

Moreover, T_t is a strongly continuous semigroup on $L^p(X)$; in the rest of this subsection, we call L_p the generator of the Ornstein-Uhlenbeck semigroup in $L^p(X)$.

3.5. Feyel-de La Pradelle measures and traces

3.5.1. Feyel-de La Pradelle measures. For this Chapter the main reference is [**39**].

Let (X,γ) be a Wiener space.

Let $F \subset R_\gamma(X^*)$ be an m-dimensional subspace of H; as usual we identify F with $\mathbb{R}^m$, choosing an orthonormal basis. We recall the concept, for $k \in \mathbb{N}$, of spherical k-dimensional Hausdorff measure $\mathcal{S}^k$.

We denote by $z = \pi_F(x)$ the canonical projection as in Section 2.4, and $F^\perp$, γ_F and $\gamma^\perp$, for $y \in \mathrm{Ker}(\pi_F)$, by B_y we denote the section

$$B_y = \{z \in F : \ y + z \in B\}. \tag{3.5.1}$$

as in Section 2.4.

We can now define spherical $(\infty-1)$-dimensional Hausdorff measures in X relative to F by

$$\mathcal{S}_F^{\infty-1}(B) = \int_{\mathrm{Ker}(\pi_F)} \int_{B_y} G_m(z)\, d\mathcal{S}^{m-1}(z)\, d\gamma^\perp(y) \qquad \forall B \subset X. \tag{3.5.2}$$

for $B \in \mathfrak{B}(F)$.

LEMMA 3.5.1. *$\mathcal{S}_F^{\infty-1}$ is a σ-additive Borel measure on $\mathcal{B}(X)$. In addition, for all Borel sets B the map $y \mapsto \int_{B_y} G_m\, d\mathcal{S}^{m-1}$ is $\gamma^\perp$-measurable in $F^\perp$.*

A remarkable fact is the monotonicity of $\mathcal{S}_F^{\infty-1}$ with respect to F, which crucially depends on the fact that we are considering spherical Hausdorff measures.

LEMMA 3.5.2. *$\mathcal{S}_F^{\infty-1} \leq \mathcal{S}_G^{\infty-1}$ on $\mathcal{B}(X)$ whenever $F \subset G$.*

It follows from Lemma 3.5.2 that the following definition of *spherical* $(\infty-1)$*-Hausdorff measure* or *Feyel-de La Pradelle measure* $\mathcal{S}^{\infty-1}$ in is well-posed; we set

$$\mathcal{S}^{\infty-1}(B) = \sup_F \mathcal{S}_F^{\infty-1}(B) = \lim_F \mathcal{S}_F^{\infty-1}(B), \tag{3.5.3}$$

the limits being understood in the directed set of finite-dimensional subspaces of QX^*; we have that it is actually a measure.

LEMMA 3.5.3. *If $\mathcal{S}^{\infty-1}(B) < +\infty$ then $\gamma(S) = 0$.*

3.5.2. Traces. We recall some results from [**26**].

As usual, (X,γ) is a Wiener space. We recall that it is defined the Ornstein-Uhlenbeck semigroup on $L^p(X)$; its generator can be denoted as L_p.

For $k=1,2$, let $(I-L_p)^{-\frac{k}{2}}$ be the operator on $L^p(X)$ defined as

$$(I-L_p)^{-\frac{k}{2}} f := \Gamma(k)^{-1}\int_0^\infty t^{\frac{k}{2}-1}e^{-t}T_t f\, dt$$

where T_t is the Ornstein-Uhlenbeck semigroup on $L^p(X)$ and

$$\Gamma(k)=\int_0^\infty t^{k-1}e^{-t}\, dt$$

(see [**14**], Sec. 5.3). We have that for $p>1$, the image of $(I-L_p)^{-\frac{k}{2}}$ is $W^{1,p}(X)$ (see [**14**], Thm. 5.7.2). Following [**26**] and also ([**14**], Sec. 5.9), we recall a particular kind of capacity $C_{k,p}$ for $k\in\mathbb{N}$ and $p\in[1,+\infty)$; for an open set U,

$$C_{k,p}(U) := \inf\{\|f\|_{L^p(X)} \,|\, f\in L^p(X),\ (I-L_p)^{-\frac{k}{2}} f \geq 1\ \gamma-\text{a.e. in } U\};$$

for a general set A,

$$C_{k,p}(A) := \inf\{C_{k,p}(U) | A\subseteq U,\ A \text{ is open}\}.$$

DEFINITION 3.5.4. A function f is called $C_{1,p}$-quasicontinuous if for each $\varepsilon>0$ there is an open set $A\subseteq X$ such that $C_{1,p}(A)\leq\varepsilon$ and $\tilde{f}_{|X\setminus A}$ is continuous.

Always following [**26**], we have the next Lemma which is an immediate consequence of [**14**], Thm. 5.9.6.

LEMMA 3.5.5. *Let $p\in(1,+\infty)$Let $f\in W^{1,p}(X)$ (f considered as a class of functions): there exists a version $\tilde{f}$ of f (i.e. $\tilde{f}$ is a function element of f) that is Borel measurable and $C_{1,p}$-quasicontinuous; moreover for every $r>0$*

$$C_{1,p}(x\in X : |\tilde{f}(x)|>r) \leq \frac{1}{r}\left\|(I-L_p)^{-\frac{1}{2}}\tilde{f}\right\|_{L^p(X)}.$$

Such $\tilde{f}$ is called a *precise version* of f. It is easy to deduce, from the above Lemma, that two precise versions of f differs only in a set with 0 $C_{1,p}$-capacity: in fact their difference is $C_{1,p}$-quasicontinuous, and it is a version of the null function, so it is 0 but in a set with 0 $C_{1,p}$-capacity.

We remark that, if G is the precise version of a function in $W^{1,p}(X)$, then $G^{-1}(0)$ is a Borel set.

Hereafter, we denote by ρ the measure $\mathcal{S}^{\infty-1}$.

We have this result ([**26**], Prop. 2.1)

LEMMA 3.5.6. *If A is a Borel set s.t. $C_{1,p}(A)=0$ for some $p>1$, then $\rho(A)=0$.*

We consider $G\in W^{1,q}(X)$ for some $q>1$ and $O:=G^{-1}((-\infty,0))$; in this setting, following [**26**] we can give a definition of $W^{1,p}(O)$ for $p\geq q'$ also if O is not open. Firstly, given $f\in \mathrm{Lip}(O)$ we can consider an extension to a function Lipschitz on X, and its gradient: by Lemma 3.1.8 this gradient is uniquely defined in γ-almost every point of O. So, we can define the gradient ∇_H as an operator from $\mathrm{Lip}(O)$ in $L^p(O,H)$; we have that this operator is closable by ([**26**], Lem. 2.2). Hence we can give this definition.

DEFINITION 3.5.7. Let $G \in W^{1,q}(X)$ for some $q > 1$ and $O := G^{-1}((-\infty,0)$; for $p \geq q'$ we define the Sobolev space $W^{1,p}(O)$ as the domain of the closure of ∇_H (defined on Lip(O)), with the graph norm; it is a Banach space.

In this subsection we always use the above definition of $W^{1,p}(O)$; in Chapter 6, we will make stronger hypotheses on G (continuity) which imply that O is open; clearly in this case the definition of $W^{1,p}(O)$ is equivalent to that of Section 3.1.

The next Hypothesis correspond to ([**26**], Hyp. 3.1)

HYPOTHESIS 3.5.8. We consider a function G on X and a $\delta > 0$ such that:

i) $G \in W^{2,q}(X)$ for every $q \in [1,\infty)$ (so $\nabla_H G$ is well defined almost everywhere);
ii) $\gamma(G^{-1}((-\infty,0)) > 0$;
iii) $|\nabla_H G|_H^{-1} \in L^q(X)$ for every $q \in [1,\infty)$.

In the above hypothesis, we define the measurable set $O := G^{-1}((-\infty,0))$; on O we can define the space $W^{1,p}(O)$ for every $q > 1$, by following Definition 3.5.7.

By $G \in W^{2,q}(X)$ for every $q \in (1,+\infty)$, we can consider LG where L is the Ornstein-Uhlenbeck operator on $L^q(X)$ for every $q \in (1,+\infty)$.

REMARK 3.5.9. Thanks to iii), we can define

$$\mathrm{div}_\gamma \frac{\nabla_H G}{|\nabla_H G|_H} = \frac{LG}{|\nabla_H G|_H} - \frac{\left\langle D_H^2 G(\nabla_H G), \nabla_H G\right\rangle_H}{|\nabla_H G|_H^3},$$

which is used in the proof of [**26**], Prop. 4.2, which corresponds here to Lemma 3.5.15; for div_γ see Subsection 3.1.3.

By $G \in W^{2,q}(X)$ and $|\nabla_H G|_H^{-1} \in L^q(X)$ for every $q \in [1,+\infty)$, we have that $|\nabla_H G|_H \in W^{1,p}(X)$ for every $p \in [1,+\infty)$ and

$$\nabla_H |\nabla_H G|_H = \frac{D_H^2 G(\nabla_H G)}{\nabla_H G|_H};$$

in particular, when we will write $|\nabla_H G|_H$, we will usually intend a precise version of $|\nabla_H G|_H$ considered as a class of functions; we recall that two precise version are equal everywhere except in a set with 0 $C_{1,p}$-capacity (and 0 Feyel-de La Pradelle measure by Lemma 3.5.6).

We consider some additional hypotheses, that we will add in some situation.

HYPOTHESIS 3.5.10. For the function G above defined, we add these properties:

i) $|\nabla_H G|_H^{-1}$ (considered as a precise version) is well defined in $G^{-1}(0)$ and it is $L^\infty(G^{-1}(0),\rho)$;
ii) $|\nabla_H G|_H \in L^\infty(G^{-1}(-\delta,0))$;
iii) $|LG| \in L^\infty(G^{-1}(-\delta,0))$.

The next Lemma corresponds to ([**26**], Cor. 3.2) (see also Remark 3.5.12 here).

LEMMA 3.5.11. *Under Hypothesis 3.5.8, let $\delta_0 > 0$ and $O_{\delta_0} := G^{-1}(-\delta_0,\delta_0)$; if f is a Borel functions that is in $L^1(O_\delta)$, then the function*

$$q_f(\xi) := \int_{G^{-1}(\xi)} \frac{f}{|\nabla_H G|_H}\, d\rho$$

is well defined for almost every $\xi \in (-\delta_0,\delta_0)$ (in the Lebesgue sense), and it is in $L^1((-\delta_0,\delta_0),\mathscr{L}^1)$ (where $\mathscr{L}^1$ is the 1-dimensional Lebesgue measure); moreover the measure $f\gamma \circ G^{-1}$ id absolutely continuous with respect to $\mathscr{L}^1$ and q_f is its density.

REMARK 3.5.12. In ([**26**], Cor. 3.2), it is used a measure $\rho_{\mathscr{V}}$, which in this setting coincides with ρ by ([**26**], Cor. 3.6).

The next Lemma corresponds to the first part of ([**26**], Prop. 4.1).

LEMMA 3.5.13. *Under Hypothesis 3.5.8, let $p>1$, $f\in W^{1,p}(X)$, and $\tilde{f}$ be a precise version of f*

$$\int_{\{G=0\}} |f|^q |\nabla_H G|_H \, d\rho =$$

$$= q\int_{G^{-1}(-\infty,0)} |f|^{q-2} f \langle \nabla_H f, \nabla_H G\rangle_H \, d\gamma + \int_{G^{-1}(-\infty,0)} LG|f|^q \, d\gamma$$

where L is the Ornstein-Uhlenbeck operator on L^p.

The next Lemma corresponds to the second part of ([**26**], Prop. 4.1).

LEMMA 3.5.14. *Under Hypothesis 3.5.8, let $p>1$, $f\in W^{1,p}(X)$, and $\tilde{f}$ be a precise version of f*

$$\int_{\{G=0\}} |\tilde{f}|^q \, d\rho =$$

$$= q\int_{G^{-1}(-\infty,0)} |f|^{q-2} f \frac{\langle \nabla_H f, \nabla_H G\rangle_H}{|\nabla_H G|_H} \, d\gamma + \int_{G^{-1}(-\infty,0)} div_\gamma\left(\frac{\nabla_H G}{|\nabla_H G|_H}\right) |f|^q \, d\gamma.$$

The next Lemma corresponds to ([**26**], Prop. 4.2).

LEMMA 3.5.15. [Trace] *Under Hypothesis 3.5.8, let $p>1$; for every $\varphi\in W^{1,p}(O)$, there exists exactly one $\psi\in\bigcap_{q<p} L^q(\{G=0\},\rho)$ with the following property: if $\{\varphi_n\}_{n\in\mathbb{N}}\subset Lip(X)$ is a sequence s.t. $\varphi_{n|O}$ converge to φ in $W^{1,p}(O)$, then the sequence $\varphi_{n|\{G=0\}}$ converges to ψ in $L^q(\{G=0\},\rho)$ for every $q<p$.*

For $\varphi=1$ everywhere, we have the following Corollary (see also [**26**], Rem. 4.9 (i)).

COROLLARY 3.5.16. *Under Hypothesis 3.5.8, $\rho(\{G=0\})<\infty$.*

By the above Lemma we can give the following Definition (it corresponds to [**26**], Def. 4.3).

DEFINITION 3.5.17. *[Trace]* For each $\varphi\in W^{1,p}(O)$, we define the trace $\mathrm{Tr}\varphi$ as the element of $\varphi\in W^{1,p}(O)$ defined in Lemma 3.5.15; we have that $\mathrm{Tr}\varphi\in L^q(G^{-1}(0),\rho)$ for every $q\in[1,+\infty)$. In particular,under Under Hypothesis 3.5.10 $\mathrm{Tr}\varphi\in L^p(G^{-1}(0),\rho)$ (by the Lemma 3.5.18).

The next Lemma corresponds to ([**26**], Lem. 4.6).

LEMMA 3.5.18. *Under Hypotheses 3.5.8, 3.5.10, Tr is a bounded operator $W^{1,p}(O)\to L^p(G^{-1}(0),\rho)$ for every $p>1$.*

The next Lemma corresponds to ([**26**], Prop. 4.10).

LEMMA 3.5.19. *Under Hypotheses 3.5.8, for all $p>1$, let $f\in W^{1,p}(O)$, we have that $Trf\equiv 0$ iff the extension of f to 0 out of O is in $W^{1,p}(X)$.*

CHAPTER 4

BV functions

The topic of BV (bounded variation) functions in Wiener space has been studied for instance in [**42, 45, 7, 8, 4, 5, 18, 17, 51**]; we widely used the survey [**54**].

In the finite dimensional case, the total variation of a function in an open set $A \subset \mathbb{R}^d$ is

$$|Du|(A) := \sup\Big\{ \int_A u(x)\mathrm{div}\phi(x)\,dx : \phi \in C^1_c(A,\mathbb{R}^d), \|\phi\|_\infty \le 1\Big\};$$

in Theorem 4.1.1 is stated that this is equivalent to the existence of a countably additive vector measure with bounded variation (see Subsection 1.2.5) which satisfies to an integration by part formula.

In the case of a Wiener space X, it can be given similar definitions (see in particular [**8**]); this is recalled in Theorem 4.1.3.

A set $A \subset X$ is said of finite perimeter if its characteristic function $1\!\!1_E$ is BV; this concept will be especially used in Chapter 8.

All this is in Section 4.1.

A possible definition of functions of bounded variation on $O \subset X$ for X Wiener space is given in [**17**]; the idea is that a function f must be BV along almost every line, hence a weak derivative can be defined as a vector measure; if we impose that this measure has bounded variation, we have the definition of $BV(O)$; the idea of [**17**] is based upon the concept of Skorohod differentiability (see e.g. [**13**]). This is recalled in Section 4.2.

In Section 7.5 we will introduce an equivalent definition of $BV(O)$ (only for function which are in $L^2(O)$): it is inspired by one of the equivalences in Theorem 4.1.3.

In [**17**], the definition of $BV(O)$ is actually done for sets O which are H-convex a condition weaker that convexity. We remark that the definition in [**17**] make sense for every open subset of X.

4.1. Definition of BV functions in $\mathbb{R}^d$ and in X

Let μ be a countably additive vector measure with values in H and bounded variation; if $h \in H$, $h \neq 0$ we define $\pi_h\mu$ as the real measure with bounded variation defined as

$$\pi_h\mu(A) := \pi_h(\mu(A)),$$

and for every $i \in \mathbb{N}$ we define π_i projection of H in $F_i =< h_1,\dots,h_n >$, and we define $\pi_i\mu$ as the countably additive vector measure with values in F_i and bounded variation defined as

$$\pi_i\mu(A) := \pi_i(\mu(A)),$$

We recall that there exists C_1 s.t. if $L\log^{\frac{1}{2}}L(O)$ then $\int_O f\hat{h}\,d\gamma \le C\|f\|_{L\log^{\frac{1}{2}}L(O)}|h|_H$ for every $h \in H$ (see subsection 3.2.1).

There are several ways of defining BV functions on $\mathbb{R}^d$, which are useful in different contexts.

For the general concept see [**6**], [**53**]. For the next theorem see e.g [**8**] Thm. 2.1.

THEOREM 4.1.1. *Let $u \in L^1(\mathbb{R}^d)$. The following are equivalent:*

i) *there exist real finite measures $\mu_j,\ j = 1,\dots,d$, on $\mathbb{R}^d$ such that*

$$\int_{\mathbb{R}^d} u(x)\partial_j\phi(x)\,dx = -\int_{\mathbb{R}^d}\phi(x)\,d\mu_j(x),\ \ \forall\phi\in C^1_c(\mathbb{R}^d), \tag{4.1.1}$$

i.e., the distributional gradient $Du = \{\mu_j\}_{j\in\mathbb{N}}$ is an $\mathbb{R}^d$-valued measure with finite total variation $|Du|(\mathbb{R}^d)$;

ii) *the quantity*

$$V(u) = \sup\left\{\int_{\mathbb{R}^d} u(x)\mathrm{div}\phi(x)\,dx : \phi\in C^1_c(\mathbb{R}^d,\mathbb{R}^d), \|\phi\|_\infty \le 1\right\}$$

is finite;

iii) *the quantity*

$$L(u) = \inf\left\{\liminf_{n\to\infty}\int_{\mathbb{R}^d}|\nabla u_n(x)|\,dx|\{u_n\}_{n\in\mathbb{N}}\subset \mathrm{Lip}(\mathbb{R}^d),\ u_n \xrightarrow{L^1} u\right\}$$

is finite;

iv) *if $(T_t)_{t\ge 0}$ denotes the heat semigroup in $\mathbb{R}^d$, then*

$$\mathscr{W}[u] = \lim_{t\to 0}\int_{\mathbb{R}^d}|\nabla T_t u|\,dx < \infty.$$

Moreover, $|Du|(\mathbb{R}^d) = V(u) = L(u) = \mathscr{W}[u]$.

If one of (hence all) the conditions in Theorem 4.1.1 holds, we say that $u\in BV(\mathbb{R}^d)$. . $V(u)$ is called total variation of u.

If $E\subset\mathbb{R}^d$ and $|D\chi_E|(\mathbb{R}^d)$ is finite, we say that E is a set with finite perimeter, use the notation $P(E)$ (*perimeter* of E) for the total variation of the measure $D\chi_E$ and write $P(E,\cdot)$ for $|D\chi_E|(\cdot)$.

We say that $E\subset\mathbb{R}^d$ has density $\alpha\in[0,1]$ at $x\in\mathbb{R}^d$ if

$$\lim_{\rho\to 0}\frac{\mathscr{L}^d(E\cap B_r(x))}{\mathscr{L}^d(B_r(x))} = \alpha \tag{4.1.2}$$

($B_r(x)$ is the ball of radius r and center x) and in this case we write $x\in E^\alpha$.

We introduce the *essential boundary*

$$\partial^* E := \mathbb{R}^d\setminus(E^0\cup E^1).$$

Let E be a set with finite perimeter, we define the *reduced boundary* $\mathcal{F}E$ in this way: $x\in\mathcal{F}E$ if the following conditions hold:

$$|D\chi_E|(B_\rho(x)) > 0\ \forall\rho>0 \text{ and } \exists\ \nu_E(x) = \lim_{r\to 0}\frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))} \tag{4.1.3}$$

with $|\nu_E(x)| = 1$; in this case, the perimeter measure coincide with the $d-1$-Hausdorff measure on $\mathcal{F}E$ (De Giorgi structure theorem)

$$P(E,\cdot) = \mathscr{H}^{d-1}_{|\mathcal{F}E}(\cdot). \tag{4.1.4}$$

We have that

$$\mathcal{F}E\subset E^{1/2}\subset\partial^* E,$$

and

$$\mathscr{H}^{d-1}(\mathbb{R}^d \setminus (E^0 \cup E^1 \cup E^{1/2})) = 0,$$

and in particular $\mathscr{H}^{d-1}(\partial^* E \setminus \mathfrak{F}E) = 0$.

We say that B is *countably $\mathscr{H}^s$-rectifiable* (for $s \in \mathbb{N}$) if there are countably many Lipschitz functions $f_j : \mathbb{R}^s \to \mathbb{R}^d$ such that

$$\mathscr{H}^s\Big(B \setminus \bigcup_{j=0}^{\infty} f_j(\mathbb{R}^s)\Big) = 0. \tag{4.1.5}$$

DEFINITION 4.1.2. Let $A \subseteq \mathbb{R}^d$ be an open set. Given a function $f : A \to \mathbb{R}$, we say that it is *locally BV* if, for every $x \in A$, there is a B neighbourhood of x s.t.

$$V_B(f) := \sup\Big\{ \int_B u(x\mathrm{div}\phi(x)\, dx : \phi \in C^1_c(B,\mathbb{R}^d), \|\phi\|_\infty \leq 1\Big\} < +\infty;$$

equivalently, it can be defined a countably additive vector measure Du (with variation not necessarily bounded) s.t. for every open B s.t. $|Du|(B) < +\infty$ we have for every $\phi \in C^1_c(B,\mathbb{R}^d)$ that (4.1.1) is verified.

We recall that for every $j \in \mathbb{N}$, the operator $\partial^*_{h_j}$ can be defined as $\partial^*_{h_j}\phi(x) = \partial_{h_j}\phi(x) - \hat{h}_j(x)\phi(x)$.

Now, the concept of BV function can be generalized for the case of a Wiener space (X,γ) see [**8**] Thm. 4.5.

THEOREM 4.1.3. *Let $u \in L\log^{\frac{1}{2}} L(X)$. The following are equivalent:*

i) *there exists a countably additive vector measure $D_\gamma u$ with values in H and bounded variation such that*

$$\int_X u\partial^*_{h_j}\phi d\gamma = -\int_X \phi d\left\langle dD_\gamma u, h_j\right\rangle_H, \ \ \forall\phi \in C^1_b(X), \tag{4.1.6}$$

i.e., the distributional gradient $D_\gamma u$ is a H-valued measure with finite total variation $|D_\gamma u|(X)$;

ii) *the quantity*

$$V(u) = \sup\Big\{ \int_X u div_\gamma \phi\ d\gamma | \phi \in \mathscr{F}C^\infty_b(X,H), \|\phi\|_\infty \leq 1\Big\}$$

is finite;

iii) *the quantity*

$$L(u) = \inf\Big\{ \liminf_{n\to\infty} \int_X |\nabla_H u_n|_H\, d\gamma | \{u_n\}_{n\in\mathbb{N}} \in \mathrm{Lip}(X),\ u_n \xrightarrow{L^1} u\Big\}$$

is finite;

iv) *if $(T_t)_{t\geq 0}$ denotes the Ornstein-Uhlenbeck semigroup in X, then*

$$\mathscr{W}[u] = \lim_{t\to 0}\int_X |\nabla_H T_t u|_H\, d\gamma < \infty.$$

Moreover, $|D_\gamma u|(X) = V(u) = L(u) = \mathscr{W}[u]$.

If one of (hence all) the conditions in Theorem 4.1.3 holds, we say that $u \in BV(X)$. $V(u)$ is called total variation of u.

Similarly to the finite dimensional case, if $E \subset X$ and $|D_\gamma \chi_E|(X)$ is finite, we say that E is a set with finite perimeter, use the notation $P_\gamma(E)$ (γ *-perimeter* of E) for the total variation of the measure $D_\gamma \chi_E$ and write $P_\gamma(E,\cdot)$ for $|D_\gamma \chi_E|(\cdot)$.

LEMMA 4.1.4. *Let $X=\mathbb{R}^d$, $(\mathbb{R}^d,\gamma^d)$ be a Wiener space; if f is of bounded variation in the Gaussian sense with weak gradient $D_\gamma f$, then it is locally BV in sense Lebesgue; in particular f is $BV(O)$ in sense Lebesgue on every bounded set and Df is absolutely continuous with respect to $D_\gamma f$ with Radon-Nikodym derivative $g(x)=\exp(-\frac{|x|^2}{2})$.*

PROOF. It is an immediate calculation that, for every $\phi \in C_c^1(O)$,

$$\int_{\mathbb{R}^d} f\partial_{x_j}^*\phi \; d\gamma = \int_{\mathbb{R}^d} f(x)\partial_{x_j}(\phi(x)/g(x)) \; dx = \int_{\mathbb{R}_d} \frac{\phi}{g} \; d(Df)_j$$

and so, by the definitions, we can conclude. □

We recall this result from [**25**], Prop. 4.2.

PROPOSITION 4.1.5. *If $O \subseteq X$ is an open convex set, then $\gamma(\partial O)=0$ and O has finite perimeter.*

REMARK 4.1.6. In [**25**] there is indeed an example of compact set with infinite perimeter.

DEFINITION 4.1.7. [Essential boundary relative to F] If we write $X=F\oplus \mathrm{Ker}(\pi_F)$, we recall by (3.5.1) the definition of the slice of E in direction F

$$E_y=\{z\in F : y+z\in E\}\subset F;$$

the essential boundary of E relative to F is then defined as

$$\partial_F^* E = \{x=y+z : z\in \partial^*(E_y)\}.$$

DEFINITION 4.1.8. [Cylindrical essential boundary] Let $\mathcal{F}$ be a countable set of finite-dimensional subspaces of H stable under finite union, with $\cup_{F\in\mathcal{F}}F$ dense in H. Then, we define cylindrical essential boundary $\partial_{\mathcal{F}}^* E$ along $\mathcal{F}$ the set

$$\partial_{\mathcal{F}}^* E := \bigcup_{F\in\mathcal{F}} \bigcap_{G\in\mathcal{F},\, G\supset F} \partial_G^* E.$$

The cylindrical essential boundary depends on $\mathcal{F}$.

By [**45**] and [**9**], we get a representation of the perimeter measure as follows.

THEOREM 4.1.9. *Let $E\in\mathfrak{B}(X)$ be a set with finite γ-perimeter in X, let $\mathcal{F}$ be as in Definition 4.1.8 and let $\partial_{\mathcal{F}}^* E$ be the corresponding cylindrical essential boundary. Then, if $\{F_n\}_{n\in\mathbb{N}}\subset\mathcal{F}$ is an increasing sequence s.t. $\bigcup_{n\in\mathbb{N}}F_n$ is dense in H, hence for every $B\in\mathfrak{B}(X)$*

$$\mathscr{S}_{F_n}^{\infty-1}(B\cap\partial_{\mathcal{F}}^* E) \xrightarrow{n\to+\infty} |D_\gamma\chi_E|(B).$$

REMARK 4.1.10. The above result is similar to the De Giorgi structure theorem (see (4.1.4), but we can say that $\partial_{\mathcal{F}}^* E$ corresponds more to the essential boundary than to the reduced boundary.

The generalization of the concept of BV in a set $O\subset X$ is more complicated; we consider it in the next section, for O convex, by using some concepts from [**17**].

4.2. Definition of BV functions in convex subsets of Wiener spaces

4.2.1. Fundamental notions. We consider an orthonormal basis $\{h_i,\dots\}_{i\in\mathbb{N}}$ of H s.t. $h_i \in R_\gamma(X^*)$ for all $i\in\mathbb{N}$; for f function with values in H, we will write $f_i := \langle f,h_i\rangle_H$.

As said in Subsection 2.3, for each $h\in R_\gamma(X^*)$, the set $X_{h^\perp}$ will be the closure in X of $h^\perp$(in H), and we recall that on it a centered nondegenerate Gaussian measure $\gamma_{h^\perp}$ is uniquely defined, s.t. $\gamma = \gamma_{h^\perp}\otimes\gamma_1$ (where γ_1 is the standard centered Gaussian measure on $\mathbb{R}$).

The next definition is taken from [**13**].

DEFINITION 4.2.1. A Radon measure μ on X is said Skorohod differentiable along $h\in X$, if, for every $f\in C_b(X)$, the function on $\mathbb{R}$

$$t\mapsto \int_X f(x-th)\,d\mu(x)$$

is differentiable. If μ is Skorohod differentiable then there exists exactly a measure ν, said Skorohod derivative of μ along h s.t. for every $f\in C_b(X)$

$$\lim_{t\to 0}\int_X \frac{f(x-th)-f(x)}{t}\,d\mu(x) = \int_X f\,d\nu$$

(see [**13**] Subsec. 3.1).

REMARK 4.2.2. In ([**13**] Subsec. 3.1) the measures are supposed to be defined in the Baire σ-algebra, but it coincides with Borel σ-algebra for metric spaces.

The next results are contained in [**13**], Thm. 3.6.5. and Cor. 3.6.7

PROPOSITION 4.2.3. *A Radon measure μ on X is Skorohod differentiable along $h\in X$ with derivative ν if and only if, for every $\varphi\in\mathcal{F}C_b^\infty(X)$, for every $t\in\mathbb{R}$,*

$$\int_X (\varphi(x-th)-\varphi(x))\,d\mu(x) = \int_0^t\int_X \varphi(x-sh)\,d\nu(x)\,ds.$$

PROPOSITION 4.2.4. *If a Radon measure μ on X is Skorohod differentiable along $h\in X$ with derivative ν then, for every $\varphi\in C_b^1(X)$,*

$$\int_X \partial_h\varphi\,d\mu = -\int_X \varphi\,d\nu.$$

From the above two propositions and the Fubini theorem we can deduce this Corollary.

COROLLARY 4.2.5. *μ is Skorohod differentiable along $h\in X$ if and only if, for every $\varphi\in\mathcal{F}C_b^\infty(X)$,*

$$\int_X \partial_h\varphi\,d\mu = -\int_X \varphi\,d\nu.$$

Instead of $\mathcal{F}C_b^\infty(X)$ we can equivalently use $C_b^1(X)$, or $\mathcal{F}C_b^1(X)$.

Now we can give a definition of $BV(X)$ which is equivalent to that of [**17**], of [**7**], of [**51**] and others (the difference is only in the set of test functions chosen).

DEFINITION 4.2.6. A function $f \in L\log^{\frac{1}{2}} L(O)$ is $BV(O)$ if $f\gamma$ is Skorohod differentiable along every $h \in H\backslash\{0\}$ and if there exists a countably additive vector measure $D_\gamma f$ with values in H and bounded variation ($D_\gamma f$ is called weak gradient), s.t. for every $h \in H\backslash\{0\}$ the real measure $\pi_h D_\gamma f - f\hat{h}\gamma$ is the Skorohod derivative of $f\gamma$; equivalently (see Corollary 4.2.5), for every $\varphi \in \mathcal{F}C_b^\infty(X)$,

$$\int_X f\partial_h\varphi \, d\gamma = -\int_X f \, d\left(\pi_h D_\gamma f\right) + \int_X f\varphi\hat{h} \, d\gamma,$$

which equivales to

$$\int_X f\partial_h^*\varphi \, d\gamma = -\int_X f \, d\left(\pi_h D_\gamma f\right).$$

REMARK 4.2.7. If instead of $\mathcal{F}C_b^\infty(X)$ we use $\mathcal{F}C_b^1(X)$ or $C_b^1(X)$ as set of test functions, the definition is equivalent.

We will give a definition of BV functions taken by ([**17**]).

DEFINITION 4.2.8. Let $O \subseteq X$ a convex open set, a function $f \in L\log^{\frac{1}{2}} L(O)$ is $BV(O)$ if there exists a countably additive vector measure $D_\gamma f$ with values in H and bounded variation ($D_\gamma f$ is called weak gradient), s.t., for all $h \in H\backslash\{0\}$ we have: if for $y \in X_{h^\perp}$ we define the set $O_y := \{t \in \mathbb{R} | y+th \in O\}$, then for $\gamma_{h^\perp}$-almost every y the function f_y^* defined as

$$f_y^* : t \mapsto f(y+th)\exp(-t^2/2),$$

is well defined and BV on O_y (with Lebesgue measure), and, for each A Borel subset of O,

$$\pi_h D_\gamma f(A) = \int_{X_{h^\perp}} d\gamma^\perp(y)(\int_{A_y} dDf_y^* - \int_{A_y} tf_y^*(t) \, dt),$$

where Df_y^* is the weak derivative (with respect to the Lebesgue measure) of f_y^*, and $A_y := \{t \in \mathbb{R} | y+th \in A\}$.

REMARK 4.2.9. The above definition is equivalent to that of ([**17**], Def. 3.4) in the Gaussian case, if we add the requirement $f \in L\log^{\frac{1}{2}} L(O)$.

In particular we can define the BV norm

$$\|f\|_{BV(O)} = |f|_{L^1(O)} + |D_\gamma f|(O) + \sup\{\int_O f\hat{h} \, d\gamma | h \in H, \ |h|_H\}.$$

and we have that there exists $C > 0$ s.t. $\|f\|_{BV(O)} \leq |D_\gamma f|(O) + \|f\|_{L\log^{\frac{1}{2}} L(O)}$.

In particular, for $O = X$, it is equivalent to the Skorohod differentiability of $f\gamma$ for every $h \in H$ with derivative $\pi_h D_\gamma f - f\hat{h}\gamma$ (see [**13**], Theorem 3.5.1.(iii)), so it is equivalent to Definition 4.2.6.

4.2.2. Further properties.

LEMMA 4.2.10. *If y is $BV(O)$, then for all $n \in \mathbb{N}$, $v_n := n \wedge y \vee (-n)$ is in $BV(O)$ and $|D_\gamma v_n|(O) \leq |D_\gamma y|(O)$.*

PROOF. By the Definition 4.2.8, and by $|v_n| \leq |y_n|$ a.e., it suffices to prove that, if a function f on $(a,b) \subseteq \mathbb{R}$ is BV with Lebesgue norm, its truncation $f_n := n \wedge y \vee (-n)$ is BV with Lebesgue norm and

$$|Df|(a,b) \leq |Df_n|(a,b);$$

(where $D\cdot$ is the weak gradient, with the Lebesgue measure); clearly this is true if $f \in C^\infty(a,b)$: in fact in this case the weak derivative coincides with the derivative multiplied by the Lebesgue measure, so we observe that $f_n' = f'$ where $|f| < n$ and $f_n' = 0$ where $|f| > n$.

Now it suffices to consider a sequence of functions $\{f_i\} \subset C^\infty(a,b)$ s.t. $f_i \to f$ in L^1 and $\|f_i'\|_{L^1(a,b)} \to |Df|(a,b)$ (see e.g. [**6**], Thm. 3.9), we have that the truncation $f_{n,i} := n \wedge f_i \vee (-n)$ converges to f_n in L^1 and $|Df_{n,i}|((a,b)) \le |Df_i|((a,b))$, so by the lower semicontinuity of total variation in L^1 (see e.g. [**6**], Rem. 3.5) we conclude. □

REMARK 4.2.11. It is immediate that $f \in L\log^{\frac{1}{2}} L(O)$ is $BV(O)$ if and only if there exists a countably additive vector measure $D_\gamma f$ with values in H and bounded variation s.t., for all $h \in H$, defined for $y \in X_{h^\perp}$ the set $O_y := \{t \in \mathbb{R} | y + th \in O\}$, then for $\gamma_{h^\perp}$-almost every y the function f_y defined as

$$f_y : t \mapsto f(y+th),$$

is well defined and BV on O_y (with Gaussian measure γ^1), and, for each A Borel subset of O,

$$\pi_h D_\gamma f(A) = \int_{X_{h^\perp}} (D_{\gamma^1} f_y)(A_y)\, d\gamma_h^\perp(y),$$

where $D_{\gamma^1} f_y$ is the weak derivative (with respect to the Gaussian measure γ^1) of f_y.

A kind of Leibniz rule can be defined for this weak derivative:

LEMMA 4.2.12. *Let O an open convex set, if $f \in L^p(O)$ for some $p > 1$, $f \in BV(O)$ and g is Lipschitz and bounded, then $fg \in BV(X)$ and*

$$D_\gamma(fg) = gD_\gamma f + f\nabla_H g\gamma. \tag{4.2.1}$$

PROOF. Clearly $fg \in L^p(X)$. If X is one-dimensional, then f is locally BV in Lebesgue sense and

$$Df = (2\pi)^{-\frac{1}{2}} e^{\frac{|\cdot|^2}{2}} D_\gamma f$$

(see the proof of Lemma 4.1.4); it is a simple calculation that fg is locally BV in Lebesgue sense and

$$D(fg) = gDf + f\nabla_H g\mathscr{L}^1;$$

and in this case it is immediate that (4.2.1) is verified.

The general case is a consequence of the Definition 4.2.8. □

REMARK 4.2.13. We have that, if $f \in \mathrm{Lip}_{0,h}(O)$, then for every x the function f_x is differentiable $\mathscr{L}^1$ almost everywhere on $\mathbb{R}$; we remark that, for all $t \in \mathbb{R}$, for all $x \in X$ the function f_{x+th} is differentiable in 0 iff f_x is differentiable in 0. We recall that for each h, the measure γ can be decomposed $\gamma = \gamma_h^\perp \otimes \gamma_1$ where γ_1 is the 1-dimensional standard Gaussian measure and $\gamma_h^\perp$ is a Gaussian measure on $h^\perp := \ker(\hat{h})$ (see Subsection 2.3), and γ_1 is continuous with respect to the Lebesgue measure; hence, for almost every $x \in X$, the function f_x is differentiable in 0.

DEFINITION 4.2.14. If $f \in \mathrm{Lip}_{0,h}(O)$ (see Definition 1.2.34), we define

$$\partial_h f(x) := f_x'(0)$$

whenever the derivative $f_x'(0)$ exists (clearly it is well defined for almost every $x \in X$ by the above Remark), $\partial_h f$ is defined in $L^\infty(X)$ and $\|\partial_h f\|_{L^\infty(X)} \le c$ and

$$\partial_h^* f(x) := \partial_h f - f\hat{h}.$$

The next Lemma is a slight modification of ([**17**], Lem. 3.5).

LEMMA 4.2.15. *Let O be an open convex subset of X. A function $f \in L\log^{\frac{1}{2}} L(O)$ is $BV(O)$ iff there exists μ s.t. for all $h \in H$, for all $\varphi \in Lip_{0,h}(O)$,*

$$\int_O \partial_h^* \varphi \ f \ d\gamma = -\int_O \varphi \ d\langle h, \ \mu \rangle_H , \tag{4.2.2}$$

and, in this case, $D_\gamma f = \mu$.

REMARK 4.2.16. There are some remarks about the proof of the above lemma.

In ([**17**], Lem. 3.5), the set of test function is a particular subset $\mathcal{D}_h(O)$ of $\mathrm{Lip}_{0,h}(O)$: if (4.2.2) is satisfied for all $\varphi \in \mathrm{Lip}_{0,h}(O)$, then it will be satisfied for all $\varphi \in \mathcal{D}_h(O)$, and $f \in BV(O)$ by ([**17**], Lem. 3.5); if $f \in \mathrm{BV}(O)$, then (4.2.2) is a consequence of the definition of $\mathrm{BV}(O)$.

In fact, if $f \in \mathrm{BV}(O)$, for all $\varphi \in \mathrm{Lip}_{0,h}(O)$ (so $\varphi_{|X\setminus O} \equiv 0$) if

$$f_y^*(t) := f(y+th)(2\pi)^{-\frac{1}{2}} \exp(-t^2/2),$$

and

$$\varphi_y(t) := \varphi(y+th)$$

and $O_y := \{t \in \mathbb{R} | y+th \in O\}$, then, recalling that $\varphi_{y|\mathbb{R}\setminus O_y} \equiv 0$ by definition of $\mathrm{Lip}_{0,h}(O)$ and the Definition 4.2.8

$$\int_O \partial_h^* \varphi \ f \ d\gamma = \int_{X_{h^\perp}} d\gamma^\perp(y) \int_{\mathbb{R}} (\varphi_y'(t) f_y^*(t) - t\varphi_y(t) f_y^*(t)) \ dt =$$

$$= \int_{X_{h^\perp}} d\gamma^\perp(y) (\int_{O_y} \varphi_y(t) \ dDf_y^* - \int_{O_y} \varphi_y(t) t f_y^*(t) \ dt) = -\int_O \varphi \ d\left\langle h, \ D_\gamma f \right\rangle_H .$$

For the proof in the particular case $O = X$, see for instance [**13**], Thm. 3.6.5.

REMARK 4.2.17. In the case $O = X$, the condition of equation (4.2.2) is equivalent if we use $\mathcal{F}C_b^\infty(X)$ as the set of test functions (see [**17**], Sec. 3).

REMARK 4.2.18. It is clear that, if $f \in W^{1,2}(O)$, then $f \in BV(O)$ and $D_\gamma f$ coincides with $\nabla_H f \ \gamma$ (we recall that $L^2(O) \subseteq L(\log L)^{\frac{1}{2}}(O)$).

We Remark that, if $f \in \mathrm{BV}(O)$ and we decompose $D_\gamma f$ as $\sigma|\mu|$ then we can write $\sigma_i|\mu| = \left\langle h_i, \ D_\gamma f \right\rangle_H$ for all $i \in \mathbb{N}$.

We have, by Lemma 1.2.36, if Ω is an open set in O, for each function $f \in BV(O)$

$$|D_\gamma f|(\Omega) =$$

$$= \sup\{\sum_{i=1}^m \int_\Omega \varphi_i \ d\left\langle h_i, \ D_\gamma f \right\rangle_H : \ m \in \mathbb{N}, \varphi \in \mathrm{Lip}_{0,m}(\Omega, H), \ \ \|\varphi\|_{L^\infty(\Omega,H)} \le 1\}$$

and, by Lemma 4.2.15, we can deduce this Corollary (recalling the Definition 1.2.35 of $\mathrm{Lip}_{0,m}(\Omega, H)$).

COROLLARY 4.2.19. *If Ω is a open set in O, for each function $f \in BV(O)$*

$$|D_\gamma f|(\Omega) =$$

$$= \sup\{\sum_{i=1}^{m}\int_{\Omega} f\partial_{h_i}^{*}\varphi_i\, d\gamma :\ m\in\mathbb{N}, \varphi\in \mathrm{Lip}_{0,m}(\Omega,H),\ \|\varphi\|_{L^\infty(\Omega,H)}\leq 1\}. \tag{4.2.3}$$

REMARK 4.2.20. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions and f a function s.t. $f_n(x)\to f(x)$ for γ-almost every $x\in X$ and

$$V := \liminf_{n\to\infty}\|f_n\|_{L\log^{\frac{1}{2}}L(O)} < \infty,$$

then, by the Fatou Lemma and the properties of $L\log^{\frac{1}{2}}L(O)$ spaces we have $f\in L\log^{\frac{1}{2}}L(O)$ and $\|f\|_{L\log^{\frac{1}{2}}L(O)}\leq V$.

By ([**17**], Thm. 3.8) and Remarks 4.2.9, 4.2.20 we have the following result:

PROPOSITION 4.2.21. *If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of BV functions s.t. $f_n(x)\to f(x)$ for γ-almost every $x\in X$ and*

$$V_1 := \liminf_{n\to\infty}|D_\gamma f_n|(O) < \infty,$$
$$V_2 := \liminf_{n\to\infty}\|f_n\|_{L\log^{\frac{1}{2}}L(O)} < \infty$$

then $f\in BV(O)$ and $|D_\gamma f|(O)\leq V_1$, $\|f\|_{L\log^{\frac{1}{2}}L(O)}\leq V_2$.

From the above Proposition follows the lower semicontinuity of the BV norm respect to the $L\log^{\frac{1}{2}}L(O)$ convergence, and we can deduce the following Corollary.

COROLLARY 4.2.22. *If $\{f_n\}$ is a sequence of BV functions s.t. $f_n\to f$ in $L\log^{\frac{1}{2}}L(O)$ and*

$$V := \liminf_{n\to\infty}|D_\gamma f_n|(O) < \infty$$

then $f\in BV(O)$ and $|D_\gamma f|(O)\leq V_1$.

We will use the two following results, the first corresponds to a part of ([**7**], Theorem 4.5), we can consider it in our case by the Remark 4.2.9; the second is an easy extension of ([**51**], Cor. 2.5).

PROPOSITION 4.2.23. *In our hypothesis T_t be the Ornstein-Uhlenbeck semigroup in $L^2(X)$: if $f\in BV(X)$, then $\int_X|\nabla_H T_t f|_H\, d\gamma \xrightarrow[t\to 0]{} |D_\gamma f|(X)$.*

LEMMA 4.2.24. *Let $\Omega\subseteq O$ be open: if $f\in BV(O)$, and $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in $BV(O)$ s.t. $f_n\rightharpoonup f$ in $L\log^{\frac{1}{2}}L(\Omega)$, then $|D_\gamma f|(\Omega)\leq \liminf_{n\to\infty}|D_\gamma f_n|(\Omega)$.*

PROOF. We argue as in the proof of ([**51**], Cor. 2.5).

Let $c := \liminf_{n\to\infty}|D_\gamma f_n|(\Omega)$. Up to a subsequence, $|D_\gamma f_n|(\Omega)\xrightarrow[n\to\infty]{} c$; hence, by Corollary 4.2.19, for all $\varepsilon>0$ there exists n_ε s.t. $\sum_{i=1}^m\int_O f_n\partial_{h_i}^*\varphi\, d\gamma\leq c+\varepsilon$ for all $m\in\mathbb{N}$, $n>n_\varepsilon$ and $\varphi\in\mathrm{Lip}_m(\Omega,H)$, $\|\varphi\|_{L^\infty(\Omega,H)}\leq 1$. Now for each $i\in\mathbb{N}$, we recall that $\partial_{h_i}^*\psi_i\in L^\Psi$, and hence in particular $\partial_{h_i}^*\psi_i\in(L\log^{\frac{1}{2}}L)'$ (see Section 3.2.1), so

$$\sum_{i=1}^{m_0}\int_\Omega f_n\partial_{h_i}^*\psi_i\, d\gamma \xrightarrow[n\to\infty]{} \sum_{i=1}^{m_0}\int_\Omega f\partial_{h_i}^*\psi_i\, d\gamma$$

by the weak convergence; hence

$$\sum_{i=1}^{m_0} \int_\Omega f \partial_{h_i}^* \psi_i \, d\gamma \le c + \varepsilon$$

for every $\varepsilon > 0$, therefore, always by Corollary 4.2.19,

$$|D_\gamma f|(\Omega) \le \sum_{i=1}^{m_0} \int_\Omega f \partial_{h_i}^* \psi_i \, d\gamma \le c-$$

□

We can deduce the following result.

COROLLARY 4.2.25. *Let T_t be the Ornstein-Uhlenbeck semigroup in $L^2(X)$: if $f \in BV \cap L^2(X)$, then $|\nabla_H T_t f|_H \gamma$ weakly converges to $|D_\gamma f|$ as a measure.*

PROOF. We have that $T_t f \xrightarrow[t\to 0]{L^2(\Omega)} f$ for each open $\Omega \subseteq X$, and recalling that $L^2(X)$ is embedded in $L \log^{\frac{1}{2}} L(X)$, we can apply Lemma 4.2.24 and we have that

$$|D_\gamma f|(\Omega) \le \liminf_{n\to\infty} \int_\Omega |\nabla_H T_t f|_H \, d\gamma \tag{4.2.4}$$

(because $|D_\gamma T_t f| = |\nabla_H T_t f|_H \gamma$, see Remark 4.2.18).

Now, by Proposition 4.2.23 we have $\int_X |\nabla_H T_t f|_H \, d\gamma \xrightarrow[t\to 0]{} |Df|(X)$.

If $|D_\gamma f|(X) = 0$ then $D_\gamma f$ is the null measure and $|\nabla_H T_t f|_H \gamma$ converges to it (because it is positive).

We consider the case $|D_\gamma f|(X) > 0$; hence we can define

$$\mu := (|D_\gamma f|(X))^{-1} |D_\gamma f|,$$

and, for t sufficiently small, since $\int_X |\nabla_H T_t f|_H \, d\gamma > 0$

$$\mu_t := (\int_X |\nabla_H T_t f|_H \, d\gamma)^{-1} |\nabla_H T_t f|_H \gamma;$$

μ, μ_t are probability measure.

Applying (4.2.4) we have that

$$\mu(\Omega) \le \liminf_{t\to 0} \mu_t(\Omega)$$

for each open set $\Omega \subseteq X$. Now, to prove $\mu_t \rightharpoonup^* \mu$, we can use Theorem 1.2.19 (the 'Portmanteau Theorem'); hence we can conclude because

$$\int_X |\nabla_H T_t f|_H \, d\gamma \xrightarrow[t\to 0]{} |D_\gamma f|(X).$$

□

CHAPTER 5

Mosco Convergence

We recall a particular kind of convergence of forms, introduced by U. Mosco in [**56**], which implies the convergence of the semigroups and of the associated resolvents (see Section 1.1). We apply this concepts in some lemmas, in which we prove that, if a sequence of sets $\{O_n\}_{n\in N}$ converges in some sense to O (in $(\mathbb{R}^d, \mathscr{L}^d)$ or in (X,γ)) then also the corresponding Dirichlet forms converges: so we have also a convergence of resolvents. This concepts and results are used extensively in Chapter 7.

The Mosco convergence is a topic currently well known, and the concepts has been extended for instance in [**49**].

5.1. General concepts

As usual, $\rightharpoonup$ represents the weak convergence.

We recall the following easy result.

LEMMA 5.1.1. i) *Let O be an open subset of $\mathbb{R}^d$. If $f \in W^{1,2}(O,\mathscr{L}^d)$ and $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in $W^{1,2}(O,\mathscr{L}^d)$ which weakly converges to f, then $f_n \rightharpoonup f$ in $L^2(O,\mathscr{L}^d)$ and $\nabla f_n \rightharpoonup \nabla f$ in $L^2(O,\mathscr{L}^d,\mathbb{R}^N)$.*

ii) *Let O be an open subset of X. If $f \in W^{1,2}(O)$ and $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in $W^{1,2}(O)$ which weakly converges to f, then $f_n \rightharpoonup f$ in $L^2(O)$ and $\nabla_H f_n \rightharpoonup \nabla_H f$ in $L^2(O,H)$.*

PROOF. Case i): to prove that $f_n \rightharpoonup f$ in $L^2(O,\mathscr{L}^d)$, i.e. $f_n - f \rightharpoonup 0$, it suffices to consider that, for all $g \in L^2(O,\mathscr{L}^d)$, we have that $f \mapsto \langle f,g\rangle_{L^2(O,\mathscr{L}^d)}$ is in the dual of $W^{1,2}(O,\mathscr{L}^d)$, so there exists $\varphi \in W^{1,2}(O,\mathscr{L}^d)$ s.t.

$$\langle f,g\rangle_{L^2(O,\mathscr{L}^d)} = \langle \varphi, f\rangle_{W^{1,2}(O,\mathscr{L}^d)},$$

hence we have $\langle f_n - f, g\rangle_{L^2(O,\mathscr{L}^d)} \to 0$ for every $g \in L^2(O,\mathscr{L}^d)$. Analogously, we can prove $\nabla f_n \rightharpoonup \nabla f$ in $L^2(O,\mathscr{L}^d,\mathbb{R}^d)$.

Case ii) is similar. □

Let X be a separable metric space and μ a σ-finite measure on $(X,\mathfrak{B}(X))$. In this setting a form a on $L^2(X,\mu)$ will always be a nonnegative bilinear symmetric form defined on a subspace $D(a)$ of $L^2(X)$ s.t. $a(u,u) \geq 0$ for all $u \in D(a)$; we set $a(v,v) := +\infty$ if $v \notin D(a)$.

DEFINITION 5.1.2. Let X be a separable metric space and μ a σ-finite measure on $(X,\mathfrak{B}(X))$. A sequence of forms a_n defined on $L^2(X,\mu)$ is Mosco-convergent to a form a defined on $L^2(X,\mu)$ if:

i) For every sequence f_n s.t. $f_n \rightharpoonup f$ in $L^2(X)$ we have $\liminf a_n(f_n,f_n) \geq a(f,f)$.
ii) For every $f \in L^2(X)$, there exists $f_n \to f$ in $L^2(X)$ s. t. $\limsup a_n(f_n,f_n) \leq a(f,f)$.

For us, a closed form a is a form s.t. $a(u,u)$ as a functional of u is lower semicontinuous with respect to $L^2(X)$.

For each $\lambda > 0$, given a closed form a' on $L^2(X)$, we define the resolvent G_λ as the operator that to each $y \in L^2(X)$ associates the only element of $D(a')$ s.t., for all $v \in D(a')$,

$$a'(G_\lambda(y),v) + \lambda \langle G_\lambda(y),v\rangle_{L^2(X)} = \langle y,v\rangle_{L^2(X)}.$$

We will use this known theorem ([**56**] Thm. 2.4.1).

THEOREM 5.1.3. *Let X be a separable topological space and $(X,\mathscr{F})$ be a measurable space with μ a σ-finite measure on $(X,\mathfrak{B}(X))$; let a be a form, $\{a_n\}_{n\in\mathbb{N}}$ a sequence of closed forms, for all $\sigma > 0$ let $G_{\sigma,n}$ be the resolvent associated to a_n for σ, let G_σ be the resolvent associated to a for σ.*

$\{a_n\}_{n\in\mathbb{N}}$ converges to a in the Mosco sense, if and only if, for all $\sigma > 0$ for every $u \in L^2(X)$ we have that $G_{\sigma,n}(u) \to G_\sigma(u)$.

REMARK 5.1.4. In [**56**] the hypotheses on $G_{\sigma,n}$ is that they are the resolvents associated to the relaxed of a_n: but if a_n is closed then it coincides with a_n itself(see [**56**] Subsec. 1.e).

5.2. Applications

We recall this result (see e.g. [**29**], Prop. 2.70, in that book the hypothesis is that the set is uniformly Lipschitz).

PROPOSITION 5.2.1. *Let $O \subseteq \mathbb{R}^d$ be a bounded open set with Lipschitz boundary.*
If $f \in W^{1,2}(O,\mathscr{L}^d)$ then f can be extended to a function $g \in W^{1,2}(\mathbb{R}^d,\mathscr{L}^d$

For O open set in $\mathbb{R}^d$ we will say say that a is the Dirichlet form associated to $W^{1,2}(O,\mathscr{L}^d)$ if $D(a) = \{f | f_{|O} \in W^{1,2}(O,\mathscr{L}^d)\}$,

$$a(f,g) = \int_O \nabla f(x)\cdot\nabla g(x)\,dx;$$

it will be always a closed form in $L^2(X,\mathscr{L}^d)$ (i.e. $a(u,u)$ as function of u will be lower semicontinuous respect to $L^2(X,\mathscr{L}^d)$).

Now we can state the next Lemma.

LEMMA 5.2.2. *Let O a bounded open set with Lipschitz boundary, let $\{O_n\}_{n\in\mathbb{N}}$ a decreasing sequence of open sets in $\mathbb{R}^d$ s.t. $\mathscr{L}^d(O_n \backslash O) \to 0$. If a_n is the sequence of the Dirichlet forms in $W^{1,2}(O_n,\mathscr{L}^d)$ and a is the Dirichlet form in $W^{1,2}(O,\mathscr{L}^d)$, then a_n converges to a in the sense of Mosco.*

PROOF. We consider the first condition of the Mosco convergence: so, let f_n a sequence of functions s.t. $f_n \rightharpoonup f$ in $L^2(\mathbb{R}^d,\mathscr{L}^N)$.

By contradiction, we suppose $\liminf a_n(f_n,f_n) < a(f,f)$: then, up to a subsequence, $f_n \in W^{1,2}(O_n)$ for all $n \in \mathbb{N}$ and the sequence $\nabla f_{n|O}$ is uniformly bounded in $L^2(O,\mathscr{L}^d,\mathbb{R}^d)$ and $f_{n|O}$ is uniformly bounded in $W^{1,2}(O,\mathscr{L}^d)$, so, by the Banach-Alaoglu theorem (see Appendix) up to a subsequence, $f_{n|O} \rightharpoonup g$ in $W^{1,2}(O,\mathscr{L}^d)$ for some $g \in W^{1,2}(O)$, hence $f_{n|O} \rightharpoonup g$ in $L^2(O,\mathscr{L}^d)$, and $\nabla f_{n|O} \rightharpoonup \nabla g$ in $L^2(O,\mathscr{L}^d,\mathbb{R}^d)$ by Lemma 5.1.1; clearly, $g = f_{|O}$, so $f_{|O} \in W^{1,2}(O,\mathscr{L}^d)$ and

$$\int_{\mathbb{R}^d} |\nabla f|^2(x)\,dx \le \liminf a_n(f_n,f_n),$$

contradiction.

Now we consider the second condition of the Mosco convergence: let $f \in L^2(\mathbb{R}^d, \mathscr{L}^d)$, we look for a sequence f_n s.t. $f_n \to f$ in $L^2(\mathbb{R}^d, \mathscr{L}^d)$ and $\limsup a_n(f_n, f_n) \le a(f, f)$.

If $f_{|O} \notin W^{1,2}(O, \mathscr{L}^d)$, we can simply take $f_n := f$ for all $n \in \mathbb{N}$. If $f_{|O} \in W^{1,2}(O, \mathscr{L}^d)$, then we can extend $f_{|O}$ to some g out of O keeping the condition $W^{1,2}$ by Proposition 5.2.1; we will have $g \in W^{1,2}(\mathbb{R}^d, \mathscr{L}^d)$, and we can define

$$f_n(x) := \begin{cases} g(x) & \text{if } x \in O_n \\ f(x) & \text{otherwise,} \end{cases}$$

and it is obvious that the condition is satisfied. □

For O convex set in $\mathbb{R}^d$ we will say say that a is the Dirichlet form associated to $W_0^{1,2}(O, \mathscr{L}^d)$ if $D(a) = \{f | f \in W^{1,2}(O, \mathscr{L}^d), f_{|\mathbb{R}^d \setminus O} \equiv 0\}$,

$$a(f,g) = \int_O \nabla f(x) \cdot \nabla g(x)\, dx;$$

it will always be a closed form in $L^2(\mathbb{R}^d, \mathscr{L}^d)$ (i.e. $a(u,u)$ as function of u will be lower semicontinuous respect to $L^2(\mathbb{R}^d, \mathscr{L}^d)$).

LEMMA 5.2.3. *Let $O \subseteq \mathbb{R}^d$ be a convex open set, let $\{O_n\}_{n\in\mathbb{N}}$ a decreasing sequence of convex open sets in $\mathbb{R}^d$ s.t. $O \subseteq \bigcap_{n\in\mathbb{N}} O_n$ and $\mathscr{L}^d(O_n \setminus O) \to 0$ where $\mathscr{L}^d$ is the Lebesgue measure. If a_n is the sequence of the Dirichlet forms in $W_0^{1,2}(O_n, \mathscr{L}^d)$ and a is the Dirichlet form in $W_0^{1,2}(O, \mathscr{L}^d)$, then a_n converges to a in the sense of Mosco.*

PROOF. We consider the first condition of the Mosco convergence: so, let f_n a sequence of functions s.t. $f_n \rightharpoonup f$ in $L^2(\mathbb{R}^d, \mathscr{L}^d)$.

By contradiction, we suppose $\liminf a_n(f_n, f_n) < a(f, f)$: then, up to a subsequence, $f_{n|O_n} \in W_0^{1,2}(O_n)$ for all $n \in \mathbb{N}$; so, each f_n can be extended as 0 out of O_n an $f_n \in W^{1,2}(\mathbb{R}^d, \mathscr{L}^d)$; the sequence ∇f_n is uniformly bounded in $L^2(\mathbb{R}^d, \mathscr{L}^d, \mathbb{R}^d)$ and f_n is uniformly bounded in $W^{1,2}(\mathbb{R}^d, \mathscr{L}^d)$, so, by the Banach-Alaoglu theorem (see Appendix) up to a subsequence, $f_n \rightharpoonup g$ in $W^{1,2}(\mathbb{R}^d, \mathscr{L}^d)$ for some g; clearly $g_{|O_n^c} \equiv 0$ for every n, therefore $g_{|O^c} \equiv 0$ because $\mathscr{L}^d(O_n \setminus O) \to 0$, so $g_{|O} \in W_0^{1,2}(O, \mathscr{L}^d)$ hence $f_n \rightharpoonup g$ in $L^2(\mathbb{R}^d, \mathscr{L}^d)$, and $\nabla f_{n|O} \rightharpoonup \nabla g$ in $L^2(O, \mathscr{L}^d, \mathbb{R}^d)$ by Lemma 5.1.1; clearly, $g = f_{|O}$, so $f_{|O} \in W^{1,2}(O, \mathscr{L}^d)$ and

$$\int_O |\nabla f|^2(x)\, dx \le \liminf a_n(f_n, f_n),$$

contradiction.

Now we consider the second condition of the Mosco convergence: let $f \in L^2(\mathbb{R}^d, \mathscr{L}^d)$, we look for a sequence f_n s.t. $f_n \to f$ in $L^2(\mathbb{R}^d, \mathscr{L}^d)$ and $\limsup a_n(f_n, f_n) \le a(f, f)$.

If $f_{|O} \notin W_0^{1,2}(O, \mathscr{L}^d)$, we can simply take $f_n := f$ for all $n \in \mathbb{N}$. If $f_{|O} \in W_0^{1,2}(O, \mathscr{L}^d)$, then we can extend $f_{|O}$ to a function g which is 0 out of O we will have $g_{|O_n} \in W_0^{1,2}(O_n, \mathscr{L}^d)$, and we can define $f_n := g$, and it is obvious that the condition is satisfied. □

Now let (X, γ) a Wiener space. For O open set in X s.t. $O = G^{-1}((-\infty, 0))$ for $G \in L^q$ for $q < 2$ we will say that a is the Dirichlet form associated to $W^{1,2}(O, \gamma)$ if $D(a) = \{f | f_{|O} \in W^{1,2}(O, \gamma)\}$,

$a(f,g) = \int_O \langle \nabla_H f, \nabla_H g\rangle \; d\gamma$; it will be always a closed form in $L^2(X,\gamma)$ (i.e. $a(u,u)$ as function of u will be lower semicontinuous with respect to $L^2(X,\gamma)$).

Let O an open set and $\{O_n\}_{n\in\mathbb{N}}$ a sequence of open sets in X; hence we can define $W^{1,2}(O,\gamma)$ and $W^{1,2}(O_n,\gamma)$ for every $n\in\mathbb{N}$.

LEMMA 5.2.4. *In the above hypothesis, we suppose $O \subseteq O_n$ for every $n\in\mathbb{N}$ and $\gamma(O_n\backslash O)\to 0$. If a_n is the sequence of the Dirichlet forms in $W^{1,2}(O_n,\gamma)$ and a is the Dirichlet form in $W^{1,2}(O,\gamma)$, then a_n converges to a in the sense of Mosco.*

PROOF. We consider the first condition of the Mosco convergence: it can be done in same way of Lemma 5.2.2.

Now we consider the second condition of the Mosco convergence: let $f\in L^2(X)$, we look for a sequence f_n s.t. $f_n \to f$ in $L^2(X,\gamma)$ and $\limsup a_n(f_n,f_n) \le a(f,f)$.

If $f_{|O} \notin W^{1,2}(O,\gamma)$, we can simply take $f_n := f$ for all $n\in\mathbb{N}$. If $f_{|O}\in W^{1,2}(O,\gamma)$, then there exists a sequence of Lipschitz functions g_m which approximates f in $W^{1,2}(O,\gamma)$, s.t. $\|g_m - f\|_{W^{1,2}(O,\gamma)} \le m^{-1}$; each g_m can be extended out of O with the same Lipschitz constant (for example with McShane extension, see Appendix), and $g_m \in W^{1,2}(X,\gamma)$, hence

$$\int_{O_n\backslash O} |\nabla_H g_m|_H{}^2 \; d\gamma \xrightarrow[n\to\infty]{} 0$$

and

$$\int_{O_n\backslash O} |g_m|^2 \; d\gamma \xrightarrow[n\to\infty]{} 0,$$

(because $O = \bigcap_{n=1}^{\infty} O_n$).

We Remark that we cannot use a simple diagonal argument to conclude, because we have to define f_n for every n, not only up to a subsequence.

For each m, the set

$$A_m := \{a\in\mathbb{N} | a > m, \int_{O_i\backslash O} |\nabla_H g_m|_H^2 \; d\gamma \le m^{-1}, \int_{O_i\backslash O} |g_m|^2 \; d\gamma \le m^{-1} \text{ for all } i \ge a\}$$

is not empty (because $\gamma(O_n\backslash O)\to 0$), and we can define $a_m := \min A_m$, we have that $a_m > m$ and $A_m = \{a\in\mathbb{N}, a > a_m\}$; for $n\in\mathbb{N}$, $n > a_1$, the set

$$B_n := \{b\in\mathbb{N} | n > a_b\}$$

is not empty; for each n, B_n is bounded by n (because $b < a_b < n$ for every $b\in B_n$), and we can define $b_n := \max B_n$. For $n > a_1$, we have $b_n < n$, moreover $b_m \xrightarrow[m\to\infty]{} \infty$: in fact, for every $c\in\mathbb{N}$, if $n > a_c$ we have $c\in B_n$ (by definition of B_n) and so $b_n \ge c$.

Let $n\in\mathbb{N}$, $n > a_1$; $b_n\in B_n$ by definition of b_n, so by definition of B_n we have $n > a_{b_n}$, so, recalling that O_n are decreasing,

$$\int_{O_n\backslash O} |\nabla_H g_{b_n}|_H^2 \; d\gamma \le \int_{O_{a_{b_n}}\backslash O} |\nabla_H g_{b_n}|_H^2 \; d\gamma$$

and

$$\int_{O_n\backslash O} |g_{b_n}|^2 \; d\gamma \le \int_{O_{a_{b_n}}\backslash O} |g_{b_n}|^2 \; d\gamma,$$

hence, by $a_{b_n} \in A_{b_n}$ and by the definition of A_{b_n},

$$\int_{O_n \setminus O} |\nabla_H g_{b_n}|_H^2 \, d\gamma \leq b_n^{-1},$$

$$\int_{O_n \setminus O} |g_{b_n}|^2 \, d\gamma \leq b_n^{-1},$$

we already know that $\|g_{b_n} - f\|_{W^{1,2}(O)} \leq b_n^{-1}$, so, if

$$f_n(x) := \begin{cases} g_{b_n}(x) & x \in O_n \\ f(x) & x \notin O_n \end{cases},$$

by $b_n \xrightarrow[n\to\infty]{} \infty$ we have that $f_n \to f$ in $L^2(X,\gamma)$ and

$$\int_{O_n} |\nabla_H f_n|_H^2 \, d\gamma \to \int_O |\nabla_H f|_H^2 \, d\gamma,$$

hence we can conclude. $\square$

Part 2

Main Results

CHAPTER 6

Classes $W_0^{1,p}$ in Wiener spaces

We deal with the problem of defining $W_0^{1,p}(O)$, the functions with zero trace on ∂O: we define a function in $W_0^{1,p}(O)$ as the limit of a sequence of regular functions which are null out of O. In [**26**], a particular kind of sets allows us to define the trace as an operator, so in this setting a possible definition of $W_0^{1,p}(O)$ is the space of functions with null trace; in Theorem 6.2.2 (which is the main result of the Chapter) we prove, under certain conditions stronger than that of [**26**] (Hypothesis 6.1.3 that for $p = 2$ the two definitions are equivalent. We also apply the concept of Mosco convergence (Chapter 5)

We extensively make use of Section 3.5.

6.1. Setting

We will suppose (X,γ) is a separable Wiener space (so X is separable): H will be the Cameron-Martin space, R_γ the isometry from X_γ^* in H; for all $h \in R_\gamma(X^*)$, we will set $\hat{h} = R_\gamma^{-1}(h)$; we introduce a basis $\{h_n\}_{n\in\mathbb{N}}$ of H, contained in $R_\gamma(X^*)$; for each $n \in \mathbb{N}$,let F_n be the space generated by $h_1,\dots,h_n$. For each F_n we define $\pi_{F_n}(x) = \sum_{i=1}^n \hat{h}_i(x)h_i$.

L is always the Ornstein-Uhlenbeck operator defined pointwise for regular functions by

$$LG(x) = \sum_{i=1}^{\infty} \left(\left\langle D_H^2 G(x)h_i, h_i \right\rangle_H - \hat{h}_i(x)\partial_{h_i} G(x) \right)$$

where $\nabla_H G$ and the H-second derivative D_H^2 of G are everywhere defined (for bounded regular functions, it can be considered as the Ornstein-Uhlenbeck operator in L^p, for $p \in [1,+\infty]$).

We recall that ρ is the Feyel-de La Pradelle measure.

DEFINITION 6.1.1. $C^1_{b,H}(X)$ will be the set of all the continuous functions (not necessarily bounded) s.t. $\nabla_H f$ is bounded and continuous as a function $X \to H$.

REMARK 6.1.2. If $G \in C^1_{b,H}(X)$ s.t. the H-second derivative D_H^2 of G is everywhere defined and $|D_H^2 G|_{HS}$ is bounded in a set O, if $|\nabla_H G|_H^{-1} \in L^\infty(O)$, then $|\nabla_H G|_H^{-1}$ is H-differentiable in O and

$$\nabla_H(|\nabla_H G|_H^{-1}) = -\frac{D_H^2 G(\nabla_H G)}{|\nabla_H G|_H^3};$$

if in addition $LG \in L^\infty(O)$ we have that $\mathrm{div}_\gamma \frac{\nabla_H G}{|\nabla_H G|_H}$ is well defined in every L^p for $p < \infty$ and

$$\mathrm{div}_\gamma \frac{\nabla_H G}{|\nabla_H G|_H} = \frac{LG}{|\nabla_H G|_H} - \frac{\left\langle D_H^2 G(\nabla_H G), \nabla_H G \right\rangle_H}{|\nabla_H G|_H^3}.$$

Hereafter we will suppose: O is the sublevel $G^{-1}((-\infty,0))$ of a function G with the following properties, which implies Hypotheses 3.5.8 and 3.5.10.

HYPOTHESIS 6.1.3. We consider a function G on X and a $\delta > 0$ such that:

i) $G \in C^1_{b,H}(X)$ (hence $\nabla_H G$ exists everywhere);
ii) $\nabla_H G$ is everywhere H-differentiable, with derivative $D^2_H G$ such that $|D^2_H G|_{HS}$ is uniformly bounded (where $|\cdot|_{HS}$ is the Hilbert-Schmidt norm);
iii) $G^{-1}(0) \neq \varnothing$;
iv) $|\nabla_H G|_H^{-1} \in L^\infty(X)$;
v) LG is bounded on $G^{-1}(-\delta,\delta)$.

In the above hypotheses, we define the open set $O := G^{-1}((-\infty,0))$.

REMARK 6.1.4. i) In the above hypothesis, $\partial O = G^{-1}(0)$; we prove it. It is clear that $\partial O \subseteq G^{-1}(0)$ by the continuity of G. To prove that $G^{-1}(0) \subseteq \partial O$ we remark that $\nabla_H G \neq 0$ on $G^{-1}(0)$ by iv); so, let $x \in G^{-1}(0)$, for $h = \frac{\nabla_H(G)}{|\nabla_H(G)|_H}$ the function $g : \mathbb{R} \to \mathbb{R}$, $t \mapsto G(x+th)$ is strictly monotone increasing, hence it is null only for $t = 0$, therefore $x \in \partial O$.

The point iii) implies $\partial O \neq \varnothing$ for what we said above, and hence $O \neq \varnothing$ and $\gamma(O) > 0$ because it is an open.

ii) The above hypotheses imply Hypotheses 3.5.8.
iii) The points i), iv) and v) imply the Hypothes is3.5.10, hence we can define a bounded trace operator $W^{1,p}(O) \to L^p(G^{-1}(O))$ for all $p > 1$. To have this result we could also substitute iv) with: $|\nabla_H G|_H^{-1} \in L^\infty(G^{-1}(-\delta,0))$ and $|\nabla_H G|_H^{-1} \in L^q(G^{-1}(0,\delta))$ for all $q > 1$.
iv) By the points ii) and iv) we have that $\frac{\nabla_H G}{|\nabla_H G|_H} \in W^{1,2}(G^{-1}(-\delta,0),H)$. By adding the point v) we have also that $\mathrm{div}_\gamma\left(\frac{\nabla_H G}{|\nabla_H G|_H}\right) \in L^\infty(G^{-1}(-\delta,\delta))$ (see Remark 6.1.2).
v) The point iv) is very restrictive, for example it is not satisfied by $\|\cdot\|_X^2$ in Hilbert spaces, which would allow to consider the ball (conversely [**26**] consider this case).
vi) For $-\delta < \varepsilon < \delta$ the Hypothesis remains true if we substitute G with $G+\varepsilon$ or with $-G+\varepsilon$, value δ substituted with $\delta' := \delta - |\varepsilon|$.
viii) By Hypotheses i), ii), we have that $LG \in L^p(X)$ for all $p < \infty$.

We also remark that we can apply all the results in Subsection 3.5.2, but there $W^{1,p}(O)$ was defined in the sense of Definition 3.5.7, here O is open (because $G \in C^1_{b,H}(X)$), so $W^{1,p}(O)$ is clearly that of our usual definition (because in both cases we define the Sobolev space as the domain of the closure of gradient ∇_H on Lipschitz functions).

EXAMPLE 6.1.5. If $h \in H$, $\tilde{X} := h^\perp$, π_h projection on $\tilde{X}$, we can consider a function F on $\tilde{X}$ that is twice differentiable with bounded Hilbert-Schmidt norm and s.t. $LF \in L^\infty(\tilde{X})$, then we have that $G(x) = \hat{h}(x) - F(\pi_h(x))$ satisfies all the properties (see also [**26**], 5.2).

We recall Definition 3.2.3 for $W^{1,p}(O)$; also in this case we have Remark 3.2.7: hence, if $f \in W^{1,p}(O)$, it can be approximated by a sequence of smooth cylindrical functions.

As we said, we are in the situation of Hypotheses 3.5.8 and 3.5.10. This allows us to (see Subsection 3.5.2)

- define, for all $\varphi \in L^1$, densities of the measure $\varphi\gamma \circ G^{-1}$ (restricted to $G^{-1}(-\delta,\delta)$) with respect to the Lebesgue measure on $\mathbb{R}$;
- define bounded trace operators $\mathrm{Tr} : W^{1,p}(O) \to L^p(G^{-1}(0),\rho)$ for $p > 1$.

In the Hypothesis 6.1.3, we can apply Lemma 3.5.18 and for all $p>1$, and then for all $t\in(-\delta,0)$ there exists a bounded trace operator

$$\mathrm{Tr}_t : W^{1,p}(G^{-1}(-\infty,t)) \to L^p(G^{-1}(t),\rho)$$

s.t. if φ is a Lipschitz function in X hence $\mathrm{Tr}\varphi = \varphi_{|G^{-1}(t)}$.

We have this two Lemmas, which are derived respectively by Lemma 3.5.11 and 3.5.19.

LEMMA 6.1.6. *Under Hypothesis 6.1.3 points i)-iv), let $\delta_0>0$ and $O_{\delta_0}:=G^{-1}(-\delta_0,\delta_0)$; if f is a Borel function that is in $L^1(O_\delta)$, then the function*

$$q_f(\xi) := \int_{G^{-1}(\xi)} \frac{f}{|\nabla_H G|_H}\, d\rho$$

is well defined for almost every $\xi\in(-\delta_0,\delta_0)$, and it is in $L^1((-\delta_0,\delta_0),\mathscr{L}^1)$; moreover q_f is a density, with respect to $\mathscr{L}^1$, of the measure $f\gamma\circ G^{-1}$.

LEMMA 6.1.7. *Under Hypothesis 6.1.3 points i)-iv), for all $p>1$, for all $t\in(-\delta,\delta)$, for all $f\in W^{1,p}(G^{-1}(-\infty,t))$, we have that $Tr_t f\equiv 0$ iff the extension of f to 0 out of $G^{-1}(-\infty,t)$ is in $W^{1,p}(X)$.*

REMARK 6.1.8. In Lemma 3.5.19 it is supposed $t=0$; but because of Remark 6.1.4 vi), we can apply the result to $t\in(-\delta,\delta)$.

Now, we recall that for $t\in G(\mathbb{R})$, $p>1$, $q\in(1,p)$, we have that $W^{1,p}(G^{-1}(-\infty,t))$ is continuously embedded in $W^{1,q}(G^{-1}(-\infty,t))$, and, under our hypotheses, the trace is defined as a bounded operator from $W^{1,q}(G^{-1}(-\infty,t))$ to $L^q(G^{-1}(t))$ and so also as a bounded operator from $W^{1,q}(G^{-1}(-\infty,t))$ to $L^q(G^{-1}(t))$.

We have this Lemma, which is an easy consequence of Lemma 3.5.13.

LEMMA 6.1.9. *Under Hypothesis 6.1.3 points i)-iv), for all $p>1$, for all $t\in(-\delta,\delta)$ for all $f\in W^{1,p}(X)$, if $q\in[1,p)$ we have*

$$\int_{\{G=t\}} |Tr_t f|^q |\nabla_H G|_H\, d\rho = \tag{6.1.1}$$

$$= q\int_{G^{-1}(-\infty,t)} |f|^{q-2} f\, \langle \nabla_H f, \nabla_H G\rangle_H\, d\gamma + \int_{G^{-1}(-\infty,t)} LG|f|^q\, d\gamma =$$

$$= -q\int_{G^{-1}(t,+\infty)} |f|^{q-2} f\, \langle \nabla_H f, \nabla_H G\rangle_H\, d\gamma - \int_{G^{-1}(t,+\infty)} LG|f|^q\, d\gamma.$$

PROOF. From Lemma 3.5.13 it follows the first equality of (6.1.1) for $t=0$ and f precise; by Remark 6.1.4 vi), we can prove (6.1.1) in general by considering the precise version of f (because only Hypothesis 6.1.3 i)-iv) are used, equivalent to Hypothesis 3.5.8).

We prove the case $f\in W^{1,p}$: if f Lipschitz then it is precise, and then we have only to observe that Lip is dense in $W^{1,p}$, that Tr_t is continuous as operator in $L^p(G^{-1}(t),\rho)$, that $\nabla_H G$ is bounded and $LG\in L^{p'}$ for all $p'<\infty$ (see Remark 6.1.4 vii)). □

Now we prove some Lemmas.

LEMMA 6.1.10. *Under Hypothesis 6.1.3 (all the points), for all $p>1$, for all $\delta_0 \in (0,\delta)$ there exists $C>0$ s.t. for all $t \in (-\delta_0, 0)$, if $f \in W^{1,p}(X)$ then we have,*

$$\|Tr_t f\|^p_{L^p(G^{-1}(t),\rho)} \le C(\|f\|^{p-1}_{L^p(O_t)}\|\nabla_H f\|_{L^p(G^{-1}(t,\delta),H)} + \|f\|^p_{L^p(G^{-1}(t,\delta))}). \tag{6.1.2}$$

PROOF. Hereafter we will write $O_t := G^{-1}(t,\delta)$.

By density, it isn't restrictive to assume f Lipschitz.

Arguing as in the proof of ([**26**], Prop. 4.1), we can introduce a function $\theta \in C_b^\infty(\mathbb{R},[0,1])$ that is 1 in $(-\infty,t]$ and 0 in $[t+(\delta-\delta_0),+\infty)$ (hence, in particular $\theta_{|(\delta,+\infty)} \equiv 0$) and $\theta' \le 2(\delta-\delta_0)^{-1}$; we introduce $\psi = f\cdot(\theta\circ G)$; we have that $\psi \in W^{1,p}(X)$ for all $p<\infty$ (because f Lipschitz and $G \in C^1_{b,H}$), that $\theta\circ G_{|G^{-1}(\delta,+\infty)} \equiv 0$ and that, for

$$C_1 = 2(\delta-\delta_0)^{-1}\|\nabla_H G\|_{L^\infty(X,H)} + 1$$

we have

$$\|\psi\|_{L^p(G^{-1}(t,+\infty))} \le \|f\|_{L^p(O_t)}$$

and

$$\|\nabla_H \psi\|_{L^p(G^{-1}(t,+\infty),H)} \le C_1\|\psi\|_{L^p(O_t,H)}$$

(recalling the Hypothesis 6.1.3 i)); we have also $\psi_{|G^{-1}(t)} \equiv f$ and so, we can apply Lemma 6.1.9 and recalling Hypothesis 6.1.3 i) and iv) and that $\psi_{|G^{-1}(\delta,+\infty)} \equiv 0$ we have

$$\int_{G^{-1}(t)} |\mathrm{Tr}_t f|^p |\nabla_H G|_H \, d\rho = \int_{G^{-1}(t)} |\mathrm{Tr}_t \psi|^p |\nabla_H G|_H \, d\rho \le$$

$$\le p\|\nabla_H G\|_{L^\infty(O_t,H)} \int_{G^{-1}(t,+\infty)} |\psi|^{p-1}|\nabla_H \psi|_H \, d\gamma + \|LG\|_{L^\infty(O_t)}\|\psi\|^p_{L^p(G^{-1}(t,+\infty))} \le$$

(by Hölder inequality)

$$\le p\|\nabla_H G\|_{L^\infty(O_t,H)}\|\psi\|^{p-1}_{L^p(O_t)}\|\nabla_H \psi\|_{L^p(O_t,H)} + \|LG\|_{L^\infty(O_t)}\|\psi\|^p_{L^p(O_t)} =$$

$$\le pC_1\|\nabla_H G\|_{L^\infty(O_t,H)}\|f\|^{p-1}_{L^p(O_t)}\|\nabla_H f\|_{L^p(O_t,H)} + \|LG\|_{L^\infty(O_t)}\|f\|^p_{L^p(O_t)}.$$

So recalling that $|\nabla_H G|_H \in L^\infty(O_t)$ by Hypothesis 6.1.3 i) and $LG \in L^\infty(O_t)$ by Hypothesis 6.1.3 v), for some $C>0$ independent on f, t we have (6.1.2). □

REMARK 6.1.11. In the above Lemma, we used for the first time Hypothesis 6.1.3 v); we needed $LG \in L^\infty(G^{-1}(-\delta,\delta))$ and not only $LG \in L^\infty(G^{-1}(-\delta,0))$.

6.2. Results about $W_0^{1,p}(O)$

LEMMA 6.2.1. *Under Hypothesis 6.1.3, for all $p>1$ there exist $C'>0$ and $\delta_0>0$ s.t if $f \in W^{1,p}(O)$ and $Tr_0 f \equiv 0$ (on $G^{-1}(0)$) then for all $t \in (0,\delta_0)$ with $\delta_0 < \delta$ we have*

$$\int_{G^{-1}(-t,0)} f^p \, d\gamma \le C' t^2 \|\nabla_H f\|^p_{L^p(G^{-1}(t,0),H)}.$$

PROOF. By Lemma 6.1.7, f can be extended out of O with 0 value, and $f \in W^{1,p}(X)$; we consider a $\delta' < \delta$ and we apply Lemma 6.1.10 for such a δ', hence for all $s \in (-\delta_0, 0)$, for some $C>0$ independent on s,

$$\|\mathrm{Tr}_s f\|^p_{L^p(G^{-1}(s),\rho)} = C(\|f\|^{p-1}_{L^p(G^{-1}(s,0))}\|\nabla_H f\|_{L^p(G^{-1}(s,0),H)} + \|f\|^p_{L^p(G^{-1}(s,0))}) \le$$

$$\leq C(\|f\|^{p-1}_{L^p(G^{-1}(s,0))}\|\nabla_H f\|_{L^p(G^{-1}(s,0),H)}+\|f\|^p_{L^p(G^{-1}(s,0))})$$

by recalling that $f\equiv 0$ in $G^{-1}(0,+\infty)$.

We have also, by Lemma 6.1.6 and 6.1.3 iv) that we can define

$$q(s):=\int_{G^{-1}(s)}\mathrm{Tr}_s f^p|\nabla_H G|_H^{-1}\,d\rho\leq C_0\|\mathrm{Tr}_s(f)\|^p_{L^p(G^{-1}(t),\rho)};$$

for

$$C_0:=\||\nabla_H G|_H^{-1}\|_{L^\infty(G^{-1}(-\delta,\delta))};$$

always by Lemma 6.1.6 we have for all $t\in(0,\delta')$

$$\|f\|^p_{L^p(G^{-1}(-t,0))}\leq\int_{-t}^0 q(s)ds\leq C_0t\sup_{s\in(0,t)}\|\mathrm{Tr}_s(f)\|^p_{L^p(G^{-1}(s),\rho)}\leq$$

(for some $C_2=CC_0$ independent on f, t)

$$\leq C_2t(\|f\|^{p-1}_{L^p(G^{-1}(t,0))}\|\nabla_H f\|_{L^p(G^{-1}(t,0),H)}+\|f\|^p_{L^p(G^{-1}(t,0))});$$

for $\delta_0\leq\delta'$ sufficiently small ($\delta_0=\frac{1}{2}C_2^{-1}$ independent on f), for all t s.t. $0\leq t<\delta_0$ we have

$$|f|^p_{L^p(G^{-1}(-t,0))}\leq C_2t\|f\|^{p-1}_{L^p(G^{-1}(t,0))}\|\nabla_H f\|_{L^p(G^{-1}(t,0),H)}+\frac{1}{2}\|f\|^p_{L^p(G^{-1}(t,0))},$$

and hence

$$\|f\|^p_{L^p(G^{-1}(-t,0))}\leq 2C_2t\|f\|^{p-1}_{L^p(G^{-1}(t,0))}\|\nabla_H f\|_{L^p(G^{-1}(t,0),H)}\leq$$

(by the Young inequality, for all $\varepsilon>0$)

$$\leq 2C_2\varepsilon tp'\|f\|^p_{L^p(G^{-1}(t,0))}+2C_2\varepsilon^{-1}tp\|\nabla_H f\|^p_{L^p(G^{-1}(t,0),H)};$$

with $\varepsilon:=(4C_2tp')^{-1}$, for some $C_3>0$,

$$\|f\|^p_{L^p(G^{-1}(-t,0))}\leq\frac{1}{2}\|f\|^p_{L^p(G^{-1}(-t,0))}+C_3t^2\|\nabla_H f\|^p_{L^p(G^{-1}(t,0),H)}$$

and we can conclude. □

The next result (the main result of the Chapter) is the infinite-dimensional version of a well-known theorem (see e.g. [**38**] Thm. 5.5.2).

THEOREM 6.2.2. *Let O satisfies Hypotheses 6.1.3, and $f\in W^{1,2}(O)$: the following claims are equivalent:*

i) $f\in W_0^{1,2}(O)$*;*
ii) $Trf\equiv 0$*;*
iii) *given the extension of f that is* 0 *out of O, we have $f\in W^{1,2}(X)$.*

PROOF. The points ii) and iii) are equivalent by Lemma 6.1.7.

It is obvious that, if $f\in W_0^{1,2}(O)$, then it is the limit of a sequence $f_n\in C_0^1(O)$; clearly f_n, can be extended as 0 out of O, and these extended functions $\bar f_n$ converges to some $\bar f\in W^{1,2}(X)$ that is 0 out of O, and f in O; so i)⇒iii).

We will prove ii)⇒i).

Let $f\in W^{1,2}(O)$ be s.t. $\mathrm{Tr}(f)\equiv 0$: we will prove that $f\in W_0^{1,2}(O)$ by finding a sequence $g_n\in C_0^1(O)$ which converges to f in $W^{1,2}(O)$.

Let η a smooth function s.t. $0 \leq \eta \leq 1$ everywhere, and η is 1 in $(-\infty,-1]$ and 0 in $[-1/2,+\infty)$, and $\eta' \leq 0$. In our hypotheses, for each $m \in \mathbb{N}$, we can define $\chi_m \in C_0^1(O)$ as

$$\chi_m(x) = \begin{cases} \eta(mG(x)) & \text{if } x \in O \\ 0 & \text{if } x \notin O \end{cases},$$

and clearly, χ_m is 1 on $G^{-1}(-\infty,-\frac{1}{m}]$ and it is 0 on $G^{-1}(0,+\infty]$; in $O_m := G^{-1}(-\frac{1}{m},0)$ we have

$$\nabla_H \chi_m = (m\eta' \circ (mG))\nabla_H G, \tag{6.2.1}$$

(hence $\nabla_H \chi_m$ is continuous and bounded) while $\nabla_H \chi_{m|X\setminus O_m} = 0$ and, for some $C > 0$ independent on m

$$\|\nabla_H \chi_m\|_{L^\infty(X,H)} \leq Cm. \tag{6.2.2}$$

$\{G^{-1}(-\frac{1}{m},0)\}_{m\in\mathbb{N}}$ is a sequence of open decreasing sets s.t.

$$\bigcap_{m\in\mathbb{N}} G^{-1}(-m^{-1},0) = \varnothing. \tag{6.2.3}$$

For each $\varphi \in \mathrm{Lip}(G^{-1}(-\delta,0))$ or all $\varphi \in W^{1,2}(O)$ s.t. $\mathrm{Tr}\varphi \equiv 0$,

$$\int_O \varphi^2 |\nabla_H \chi_m|_H^2 \, d\gamma \leq Cm^2 \int_{O_m} \varphi^2 \, d\gamma;$$

hence, by Lemma 6.2.1, for C' defined in such Lemma, for $C_2 := CC' > 0$ independent on m, φ,

$$\int_O \varphi^2 |\nabla_H \chi_m|_H^2 \, d\gamma \leq C_2 \|\varphi\|^2_{W^{1,2}(O_m)}. \tag{6.2.4}$$

Now, we consider our $f \in W^{1,2}(X)$ s.t. $f = 0$ out of O; there exists a sequence g_n of bounded smooth cylindrical functions s.t. $g_n \to f$ in $W^{1,2}(X)$; fixed g_n we consider for $m \in \mathbb{N}$ the function $f_m = g_n\chi_m$, clearly it is in $C_0^1(O)$. Now,

$$\int_O |g_n\chi_m - g_n|^2 \, d\gamma = \int_O g_n^2 |\chi_m - 1|^2 \, d\gamma$$

converges to 0 if $m \to \infty$, because $\chi_m \xrightarrow[m\to\infty]{L^2(X)} 1\!\!1_O$ and g_n is bounded. By (6.2.1) we have $|\nabla_H \chi_m| \leq Cm$ for some $C > 0$ independent on m; we have that, for each $n \in \mathbb{N}$, for some $c_1, c > 0$ independent on m,

$$\int_O |\nabla_H(g_n\chi_m) - \nabla_H f|_H^2 \, d\gamma =$$

$$= \int_O |g_n \nabla_H \chi_m + \chi_m \nabla_H g_n - \nabla_H f - f\nabla_H \chi_m + f\nabla_H \chi_m - \chi_m \nabla_H f + \chi_m \nabla_H f|_H^2 \, d\gamma \leq$$

(by reordering the terms)

$$\leq c_1 \Big(\int_O (g_n - f)^2 |\nabla_H \chi_m|_H^2 \, d\gamma + \int_O \chi_m^2 |\nabla_H g_n - \nabla_H f|_H^2 \, d\gamma +$$

$$+ \int_O (\chi_m - 1)^2 |\nabla_H f|_H^2 \, d\gamma + \int_O f^2 |\nabla_H \chi_m|_H^2 \, d\gamma \Big) \leq$$

(recalling (6.2.1))

$$\leq c(m^2 \|g_n - f\|^2_{L^2(O)} + \|\nabla_H g_n - \nabla_H f\|^2_{L^2(O,H)} +$$

$$+\int_{G^{-1}(O_m)}|\nabla_H f|_H^2 d\gamma+\int_O f^2|\nabla_H\chi_m|_H^2 d\gamma)\leq$$

(by (6.2.4))

$$\leq c(m^2\|g_n-f\|^2_{L^2(O)}+\|\nabla_H g_n-\nabla_H f\|^2_{L^2(O,H)}+(1+C_2)|f|^2_{W^{1,2}(O_m)}).$$

Now, recalling that $g_n\to f$ in $W^{1,2}(O)$ we have that, fixed $m\in\mathbb{N}$, there exists n_m s.t.

$$\int_O|\nabla_H(g_{n_m}\chi_m)-\nabla_H f|_H^2\ d\gamma\leq c(m^{-1}+(1+C_2)\|f\|^2_{W^{1,2}(O_m)}),$$

and, by (6.2.3), the last term converges to 0 for $m\to\infty$; so, if $f_m:=g_{n_m}\chi_m$, it converges to f in $W^{1,2}$ for $m\to\infty$ and $f_m\in C_0^1(O)$ and we can conclude. □

REMARK 6.2.3. Thank to the above proposition, we have that, if O satisfies Hypothesis 6.1.3, then $W_0^{1,2}(O)$ can be equivalently defined as the closure in $W^{1,2}(O)$ of $\mathrm{Lip}_0(O)$.

We return to consider the situation of the previous subsection: O will be a set which satisfies Hypothesis 6.1.3, we will define for all n, $G_n:=G\circ\pi_{F_n}$; we define $O_n:=G_n^{-1}((-\infty,0))$. We remark that, for each n, G_n satisfies the Hypothesis 6.1.3, and it is a cylindrical set.

Here, given the set $O=G^{-1}((-\infty,0))$ and the sequence of sets $O_n=G_n^{-1}((-\infty,0))$, we will consider for each $n\in\mathbb{N}$ the Dirichlet form a_n on $W_0^{1,2}(O_n)$, i.e.

$$D(a_n)=\{f\in L^2(X)|f_{|X\setminus O_n}\equiv 0,\ f_{|O_n}\in W_0^{1,2}(O_n)\},$$

$$a_n(f,g)=\int_{O_n}\langle\nabla_H f,\nabla_H g\rangle\ d\gamma;$$

we consider a the Dirichlet form in $W_0^{1,2}(O)$, i.e.

$$D(a)=\{f\in L^2(X)|f_{|X\setminus O}\equiv 0,\ f_{|O}\in W_0^{1,2}(O)\},$$

$$a(f,g)=\int_O\langle\nabla_H f,\nabla_H g\rangle\ d\gamma.$$

By Theorem 6.2.2, $D(a),D(a_n)\subseteq W^{1,2}(X)$ for all $n\in\mathbb{N}$.

LEMMA 6.2.4. *If $f\in C_b^1(X)$, we have, for all $p\in[1,+\infty)$,*

$$f\circ\pi_n\xrightarrow[n\to\infty]{W^{1,p}(X)}f.$$

PROOF. If $f_n:=f\circ\pi_{F_n}$ we have $f_n\to f$ almost everywhere by the continuity of f and Theorem 2.4.2, and

$$\nabla_H f_n(x)=\pi_n(\nabla_H f\circ\pi_{F_n}(x));$$

we recall that for each $F_n\subset H$, for each $h\in H$ we have $|\pi_n(h)|_H\leq|h|_H$ (π_{F_n} is a projection in H) and $\pi_n(h)\xrightarrow[n\to\infty]{}h$ for all $h\in H$; so

$$|\nabla_H f_n(x)-\nabla_H f(x)|_H\leq|\pi_n(\nabla_H f\circ\pi_n(x))-\pi_n(\nabla_H f(x))|_H+$$

$$+|\pi_n(\nabla_H f(x))|_H-\nabla_H f(x)|_H\leq$$

$$\leq|\nabla_H f(\pi_n(x))-\nabla_H f(x)|_H+|\pi_n(\nabla_H f(x))-\nabla_H f(x)|_H,$$

the first term converges to 0 for almost every x by $f \in C_b^1$ and Theorem 2.4.2, the second converges to 0 because of the convergence of π_n in H. By the dominated convergence theorem, we can conclude □

Now, we want to prove the following result.

PROPOSITION 6.2.5. *Let G be a function which satisfies Hypothesis 6.1.3, O and O_n defined as above: if a_n is the sequence of the Dirichlet forms in $W_0^{1,2}(O_n)$ and a is the Dirichlet form in $W_0^{1,2}(O)$, then a_n converges to a in the sense of Mosco.*

PROOF. We have that, $\rho(G^{-1}(0)) < \infty$, see ([**26**], Rem. 4.9 (i)) taking into account Remark 6.1.4 iii), hence $\gamma(G^{-1}(0)) = 0$ (see Lemma 3.5.3).

So in our hypotheses $\gamma(G^{-1}(0)) = 0$ i.e. $\gamma(\partial O) = 0$.

We consider the first condition of the Mosco convergence: so, let f_n a sequence of functions s.t. $f_n \rightharpoonup f$ in $L^2(X)$.

If $\liminf a_n(f_n, f_n) = +\infty$, there is nothing to prove; so, we suppose that $\liminf a_n(f_n, f_n) < +\infty$, in particular it is not restrictive to suppose that, up to a subsequence, $f_n \in D(a_n)$ for every n. Firstly, we prove that $f_{|X\setminus\bar{O}} \equiv 0$. We define $U_n := \bigcup_{i=n}^{\infty} O_i$, it is a decreasing sequence of open sets. $f_{m|X\setminus O_m} \equiv 0$ for each $m \in \mathbb{N}$; for each $n \in \mathbb{N}$ we have $f_{m|X\setminus U_n} \equiv 0$ for all $m \geq n$, and hence $f_{|X\setminus U_n} \equiv 0$ for all $n \in \mathbb{N}$, because $f_m \rightharpoonup f$ in $L^2(X)$, so $f_{|\bigcup_{i=1}^{\infty}(X\setminus U_i)} \equiv 0$.

We have also that

$$\gamma((X\setminus\bar{O})\setminus\bigcup_{i=1}^{\infty}(X\setminus U_i)) = 0:$$

in fact

$$X\setminus\bar{O} = \{x \in X | G(x) > 0\},$$

hence, for almost every $x \in X\setminus\bar{O}$ there exists $n_0 \in \mathbb{N}$ s.t. $G_n(x) = G \circ \pi_n(x) > 0$ for all $n > n_0$ (because G is continuous and $\pi_n(x) \xrightarrow[n\to\infty]{} x$ for almost every $x \in X$ by Theorem 2.4.2, so, for such a x, we have, for all $n > n_0$, that $x \notin O_n$, hence $x \notin U_{n_0}$; this yields $x \in \bigcup_{i=1}^{\infty}(X\setminus U_i)$.

Hence, $f_{|X\setminus\bar{O}} \equiv 0$; for $\gamma(\partial O) = 0$, we have that $f_{|X\setminus O} \equiv 0$.

Now, we consider the sequence f_n; if $\liminf a_n(f_n, f_n) = +\infty$, there is nothing to prove; otherwise, up to a subsequence, $f_n \in D(a_n)$ for all $n \in \mathbb{N}$ (hence for what we said f_n is in $W^{1,2}(X)$ and $f_{n|X\setminus O_n} \equiv 0$), and it is uniformly bounded in $W^{1,2}(X)$, so, up to a subsequence, $f_n \rightharpoonup g$ in $W^{1,2}(X)$ for some $g \in W^{1,2}(X)$; clearly, $f_n \rightharpoonup g$ also in $L^2(X)$ so $g = f$, therefore $f \in W^{1,2}(X)$ and

$$\int_X |\nabla_H f|_H^2 \, d\mu \leq \liminf a_n(f_n, f_n);$$

we already know $f_{|X\setminus O} \equiv 0$, then $f_{|O} \in W_0^{1,2}$ because $f \in W^{1,2}(X)$ and Theorem 6.2.2, and in that case $f \in D(a)$) and $a(f, f) \leq \liminf a_n(f_n, f_n)$.

Now we consider the second condition of the Mosco convergence: let $f \in L^2(X)$, we look for a sequence f_n s.t. $f_n \to f$ in $L^2(X)$ and $\limsup a_n(f_n, f_n) \leq a(f, f)$.

If $f \notin D(a)$, we can simply take $f_n := f$ for all $n \in \mathbb{N}$. If $f \in D(a)$, then $f_{|O} \in W_0^{1,2}(O)$ and $f_{|X\setminus O} \equiv 0$, then by definition there exists a sequence of functions $\tilde{g_m} \in C_0^1(O)$ which approximates f in $W^{1,2}(O)$, s.t. $\|\tilde{g_m} - f\|_{W^{1,2}(O)} \leq m^{-1}$; we will extend each $\tilde{g_m}$ to a function g_m that is 0 on $X\setminus O$, and $g_m \in C_b^1(X)$ by the definition of $C_0^1(O)$, hence $\|\tilde{g_m} - f\|_{W^{1,2}(X)} \leq m^{-1}$; for each g_m we can define $g_{m,n} := g_m \circ \pi_{F_n}$ which approximate g_m in $W^{1,2}(X)$ by Lemma 6.2.4 (because $g_m \in C_b^1(X)$), while $g_{m,n|X\setminus O_n} \equiv 0$, hence $g_{m,n|O_n} \in C_0^1(O_n) \subseteq W_0^{1,2}(O_n)$.

We Remark that we cannot use a simple diagonal argument, because we have to define f_n for every n, not only up to a subsequence; we conclude with an argument similar to that of the last part of Lemma 5.2.4.

For each $m \in \mathbb{N}$ the set

$$A_m := \{a \in \mathbb{N} | a > m, \|g_{m,i} - g_m\|_{W^{1,2}(X)} \leq m^{-1} \text{ for all } i \geq a\},$$

is not empty (because $g_{m,n} \to g_m$), and we can define $a_m := \min A_m$, we have that $a_m > m$ and $A_m = \{a \in \mathbb{N}, a > a_m\}$; for $n \in \mathbb{N}$ with $n > a_1$ the set

$$B_n := \{b \in \mathbb{N} | n > a_b\}$$

is not empty; for each n, B_n is bounded by n (because $b < a_b < n$ for every b), so for every $n > a_1$ we can define $b_n := \max B_n$. For such n, we have $b_n < n$, moreover $b_n \xrightarrow[n\to\infty]{} \infty$: in fact, for every $c \in \mathbb{N}$, for every $n > a_c$ we have $c \in B_n$ (by definition of B_n) and so $b_n \geq c$.

For every $n > a_1$, $b_n \in B_n$ by definition of b_n, so by definition of B_n we have $n > a_{b_n}$, hence, by $a_{b_n} \in A_{b_n}$ and by the definition of A_{b_n},

$$\|g_{b_n,n} - g_{b_n}\|_{W^{1,2}(X)} \leq b_n^{-1}$$

we already know that $\|g_{b_n} - f\|_{W^{1,2}(X)} \leq b_n^{-1}$, so, if $f_n := g_{b_n,n}$, by $b_n \xrightarrow[n\to\infty]{} \infty$ we have that $f_n \to f$ in $W^{1,2}(X)$, hence we concluded. $\square$

We recall the definition of J_σ with zero Dirichlet boundary condition from $L^2(O)$ in $W_0^{1,2}(O)$. We can give an equivalent definition: if $G_{\sigma^{-1}}$ is the σ^{-1}-resolvent of *a* Dirichlet boundary conditions on O we can always extend $f \in L^2(O)$ as $\tilde{f}$ that is 0 out of O and

$$J_\sigma(f) = \sigma G_{\sigma^{-1}}(\tilde{f})_{|O}.$$

In the same way we can define $J_{\sigma,n}$.

By Theorem 5.1.3 and Proposition 6.2.5 we can easily deduce this Corollary.

COROLLARY 6.2.6. *In our hypotheses about O, and O_n, let A_n be the Ornstein-Uhlenbeck operator with zero Dirichlet boundary conditions in $L^2(O_n)$, A the Ornstein-Uhlenbeck operator with zero Dirichlet boundary conditions in $L^2(O)$, $\sigma > 0$, $J_{\sigma,n} := (I - \sigma A_n)^{-1}$, $J_\sigma := (I - \sigma A)^{-1}$; then for every $x \in L^2(X)$, we have that $J_{\sigma,n}(x) \to J_\sigma(x)$ in $L^2(X)$.*

REMARK 6.2.7. We have collected in Subsection 7.3.2.1 some examples of sets satisfying Hypothesis 6.1.3.

CHAPTER 7

Gradient contractivity of operators

This is the main Chapter of the thesis: a description of its content is in the Introduction; the concepts of Sobolev spaces in Wiener spaces (Chapter 3), Mosco convergence (Chapter 5) and that of resolvent (Section 1.1) will be largely used.

In Section 7.1 the setting is defined, together with concepts and results which are used in Sections 7.2 and 7.4; it is based upon Section 8 of the paper [**12**] (due to H. Brezis) about the resolvent of the Laplacian in finite dimension: if O is a convex bounded set with regular boundary in $X=\mathbb{R}^d$ and L is the Laplace operator in O with null Dirichlet boundary conditions, if $\sigma>0$, if $y\in W_0^{1,1}(O,\mathscr{L}^d)\cap L^2(O,\mathscr{L}^d)$, and $u:=(I-\sigma L)^{-1}y$ (in the sense of resolvents), then

$$\int_O |\nabla u(x)|\,dx \le \int_O |\nabla y(x)|\,dx. \tag{7.0.1}$$

This result is recalled (with some minimal modifications) in Proposition 7.2.4. In this section we use regularity of solutions of elliptic equations (see Appendix).

In Section 7.2 we extend (7.0.1) to the Gaussian case in finite dimension (with null Dirichlet boundary condition), adapting the results in section 8 of the paper [**12**]; here, L is the Ornstein-Uhlenbeck operator (with null Dirichlet boundary conditions), see Subsection 3.4.1.

In Section 7.3 we use the concepts and the results of Chapter 6, together with the above Section and Mosco convergence (Chapter 5: we pose a particular condition on O seen in Chapter 6, which we use to have the Theorem 6.2.2 about $W_0^{1,2}(O)$; moreover, we impose a condition, that we could name *Gaussian convexity;* under this hypotheses (Hypothesis 7.3.5), for $y\in W_0^{1,p}(O)\cap L^2(O)$ for some $p>1$, and $u:=(I-\sigma L)^{-1}y$ (L Ornstein-Uhlenbeck operator with null Dirichlet boundary conditions) we have this inequality (Theorem 7.3.7)

$$\int_O |\nabla_H u|\,d\gamma \le \int_O |\nabla y|\,d\gamma. \tag{7.0.2}$$

which is the most important result of the section.

In Subsection 7.3.2.1 we provide two applications; one is an epigraph, the other one is in (X,γ) which represents the Brownian Bridge, seen in Section 2.6.

Section 7.4 is divided in two sections, with two important results.

In Subsection 7.4.1 we extend (7.0.1) to the case where L is the Laplace operator in O with null Neumann boundary conditions and $y\in W^{1,1}(O,\mathscr{L}^d)\cap L^2(O,\mathscr{L}^d)$, in Theorem 7.4.4 (main result of the Section). In Subsection 7.4.2, we extend the result to the case where L is Ornstein-Uhlenbeck operator on O (open convex) with null Neumann boundary conditions and $y\in\left(W^{1,1}\cap L^2\right)(O)$, Gaussian measure (Theorem 7.4.7); in this section we use Lemma 7.4.2.

In Section 7.5, we want to find a definition of bounded variation function on O condition which is equivalent to that in [**17**] (see Section 4.2); in Proposition 7.5.9 we get that, if $f\in$

$BV(O) \cap L^2(O)$ then there exists a sequence of approximations in variation in terms of functions of $W^{1,1}(O) \cap L^2(O)$ (we also use Proposition 3.2.23); in Theorem 7.5.11 we give a version of 7.0.2 for BV functions which are $L^2(O)$:

$$D_\gamma(J_\sigma(y))(O) \leq |D_\gamma y|(O).$$

Moreover, with Corollary 7.5.13 (which resumes the main results of the Sections) we have (by using Theorem 7.4.7) that, under the hypothesis that $y \in L^2(O)$, it is BV if and only if

$$\limsup_{\sigma \to 0} \int_O |\nabla_H J_\sigma(y)| \, d\gamma < +\infty$$

and if and only if there exists a sequence of functions $f_n \in W^{1,1}$ s.t. $f_n \to f$ in L^1 and

$$\limsup_{n \to +\infty} \int_O |\nabla_H f_n|_H \, d\gamma < \infty.$$

Moreover, by using a result in [**24**](thesis) and [**23**](submitted paper), we can find a result similar to Corollary 7.5.13 but with the T_t Ornstein-Uhlenbeck semigroup with null Neumann boundary conditions instead of J_σ (Corollary 7.5.16).

7.1. Preliminaries about gradient contractivity of operators

In this Chapter we will consider:

i) $\mathbb{R}^d$ endowed with the Lebesgue measure;
ii) $X = \mathbb{R}^d$ with the standard Gaussian measure γ^d;
iii) a finite or infinite dimensional separable Banach space X with non-degenerate Gaussian measures γ.

In case i), we consider an open set $U \subset \mathbb{R}^d$, in the other cases, an open set $O \subset X$ with some properties.

In the setting of the Lebesgue measure, we will consider the Laplace operator with Dirichlet boundary condition Δ_D on U, the Laplace operator with Neumann boundary condition Δ_d on U.

In the setting of Gaussian measure, we recall the results about Ornstein-Uhlenbeck operator with Dirichlet boundary condition L_D (Subsection 3.4.3) and Ornstein-Uhlenbeck operator with Neumann boundary condition L_d.

For what we said, the operators Δ_D, Δ_d are associated to contractive strongly continuous semigroups and to contractive resolvent semigroups on $L^2(U, \mathscr{L}^d)$; in the same way, L_D, L_d are associated to contractive strongly continuous semigroups and to contractive resolvent semigroups on $L^2(O)$.

In each of thise cases, we define for $\sigma > 0$ an operator $J_\sigma = (I - \sigma A)^{-1}$ (where $A = \Delta_D, \Delta_d, L_D, L_d$); in all cases we have that for $y \in L^2$, $J_\sigma(y) \in W^{1,2}$ (it is regularizing) and we can prove that, for $y \in W^{1,1} \cap L^2$

$$\|\nabla J_\sigma(y)\|_{L^1} \leq \|\nabla y\|_{L^1};$$

the case for Δ_D is proved in ([**12**], Section 8); we will prove the other cases.

Hereafter $u :== J_\sigma y$, so $u - \sigma L u = y$.

Our strategy will be, at first, to prove the result for the finite dimensional case, then for the infinite dimensional case.

REMARK 7.1.1. For an open set O, if $y \in W^{1,1}(O)$ then $\nabla_H y\gamma$ is obviously a countably additive vector measure with bounded variation. We have that the functional $y \mapsto \int_O |\nabla_H y|_H \, d\gamma$ defined from $W^{1,1}(O)$ in $[0,+\infty)$, is lower semicontinuous with respect to the $L\log^{\frac{1}{2}} L(O)$ norm; analogously, $y \mapsto \int_U |\nabla y| \, d\mathscr{L}^d$ is lower semicontinuous with respect to the $L^1(U,\mathscr{L}^d)$ norm.

We write the simple proof for the Gaussian case (the Lebesgue case is analogous): if

$$y_n \xrightarrow[n\to\infty]{L(\log L)^{\frac{1}{2}}(O)} y$$

and $y \in W^{1,1}(O)$, then (by Lemma 1.2.36)

$$\int_O |\nabla_H y|_H \, d\gamma =$$

$$= \sup_{m\in\mathbb{N},\varphi\in \mathrm{Lip}_{0,m}(\Omega,H),\|\varphi\|_{L^\infty}\le 1} \int_O \langle \nabla_H y, \varphi\rangle_H \, d\gamma =$$

(by Remark 3.1.17)

$$= \sup_{m\in\mathbb{N},\varphi\in \mathrm{Lip}_{0,m}(\Omega,H),\|\varphi\|_{L^\infty}\le 1} \int_O y \mathrm{div}_\gamma \varphi \, d\gamma =$$

$$= \sup_{m\in\mathbb{N},\varphi\in \mathrm{Lip}_{0,m}(\Omega,H),,\|\varphi\|_{L^\infty}\le 1} \lim_{n\to\infty} \int_O y_n \mathrm{div}_\gamma \varphi d\gamma \le$$

$$\le \liminf_{n\to\infty} \sup_{m\in\mathbb{N},\varphi\in \mathrm{Lip}_{0,m}(\Omega,H),,\|\varphi\|_{L^\infty}\le 1} \int_O y_n \mathrm{div}_\gamma \varphi d\gamma =$$

(by Remark 3.1.17)

$$= \liminf_{n\to\infty} \sup_{m\in\mathbb{N},\varphi\in \mathrm{Lip}_{0,m}(\Omega,H),\|\varphi\|_{L^\infty}\le 1} \int_O \langle \nabla_H y_n, \varphi\rangle_H \, d\gamma = \liminf_{n\to\infty} \int_O |\nabla_H y_n|_H \, d\gamma.$$

If O is bounded, we have that the functional is lower semicontinuous with respect to the $L^1(O)$ norm (because $\mathrm{div}_\gamma \varphi \in L^\infty(O)$ if $\varphi \in \mathrm{Lip}_0(O,H)$).

We introduce $u = J_\sigma y$, so $u - \sigma L u = y$.

Our strategy will be, at first, to prove the result for the finite dimensional case, then for the infinite dimensional case.

In the finite dimensional case (with measure $\mu = \mathscr{L}^d$ or γ^d) the general idea will be to prove the result for y smooth, and we can conclude because of the density of smooth function in $W^{1,p}$, of the continuity of J_σ and of the lower semicontinuity of the functional $\int_O |\nabla \cdot| \, d\mu$.

So, in $O \subset \mathbb{R}^d$, we consider $A = \Delta = \sum_{i=1}^d \partial^2_{x_i x_i}$, Laplace operator in C^2, for $y \in C^\infty(O)$, let $u \in C^2(O)$ s.t. $u - \sigma A u = y$, and

$$\varphi(\xi) = |\nabla u(\xi)|, \quad \varphi_\varepsilon(\xi) = \sqrt{\varepsilon^2 + |\nabla u(\xi)|^2}.$$

In [**12**], Lem. 8.2 it is proved that if O is a bounded convex domain with boundary C^2 then

$$\frac{\varphi^2}{\varphi_\varepsilon} - \sigma \varphi_\varepsilon \le |\nabla y|$$

in each point.

Our first goal is to prove the same estimate for $A = L = \Delta - \sum_{i=1}^{d} x_i \partial_{x_i}$.

So, let y be $C^\infty(O)$; let $u \in C^2(O)$ s.t. $u - \sigma A u = y$, then $u \in C^\infty(U)$ in each bounded set U in O due to the regularity and ellipticity of the operator $I + \sigma A$ (see e.g. [**38**], 6.3.1, Theorem 3) hence $u \in C^\infty(O)$. In this hypothesis, we introduce φ and $\varphi_\varepsilon \in C^\infty(O)$ by setting:

$$\varphi(\xi) = |\nabla u(\xi)|, \quad \varphi_\varepsilon(\xi) = \sqrt{\varepsilon^2 + |\nabla u(\xi)|^2}.$$

Firstly, we prove the equivalent of Lemma 8.2 in [**12**].

LEMMA 7.1.2. *In this setting, we have, in every point of the domain O,*

$$\frac{\varphi^2}{\varphi_\varepsilon} - \sigma L \varphi_\varepsilon \leq |\nabla y|. \tag{7.1.1}$$

PROOF. We will use D_i for $\frac{\partial}{\partial x_i}$ and D^2_{ij} for $\frac{\partial^2}{\partial x_i \partial x_j}$.

In the same way of [**12**], we have

$$\varphi_\varepsilon D_j \varphi_\varepsilon = \sum_{i=1}^{d} D_i u D^2_{ij} u, \tag{7.1.2}$$

and

$$|\nabla \varphi_\varepsilon|^2 \leq \frac{\sum_{j=1}^{d} \left(\sum_{i=1}^{d} D_i u D^2_{ij} u \right)^2}{\varphi_\varepsilon^2} \leq \frac{\sum_{j=1}^{d} \left(\sum_{i=1}^{d} (D_i u)^2 \sum_{i=1}^{d} (D^2_{ij} u)^2 \right)}{\varphi_\varepsilon^2} \leq$$

$$\leq \frac{\sum_{i=1}^{d} (D_i u)^2}{\varphi_\varepsilon^2} \sum_{i,j=1}^{d} (D^2_{ij} u)^2 \leq \sum_{i,j=1}^{d} |D^2_{ij} u|^2, \tag{7.1.3}$$

by using (7.1.2) we have

$$D^2_{jj} \varphi_\varepsilon = D_j \frac{\sum_{i=1}^{d} D_i u D^2_{ij} u}{\varphi_\varepsilon} =$$

$$= \frac{\sum_{i=1}^{d} |D^2_{ij} u|^2}{\varphi_\varepsilon} + \frac{\sum_{i=1}^{d} D_i u D^3_{jji} u}{\varphi_\varepsilon} - \frac{\sum_{i=1}^{d} D_j \varphi_\varepsilon D_i u D^2_{ij} u}{\varphi_\varepsilon^2}. \tag{7.1.4}$$

We recall that, for each $f \in C^\infty$,

$$D_i L f(\xi) = \Delta D_i f(\xi) - \sum_{j=1}^{d} D^2_{ij} f(\xi) \xi_j - D_i f(\xi)$$

and, for all ξ,

$$L D_i f(\xi) = \Delta D_i f(\xi) - \sum_{j=1}^{d} D^2_{ij} f(\xi) \xi_j = D_i L f(\xi) + D_i f(\xi);$$

so,

$$D_i f L D_i f \geq D_i f \, D_i L f. \tag{7.1.5}$$

Now, by using 7.1.4 and 7.1.2, we can do the calculation, for all $\xi \in \mathbb{R}^d$,

$$\varphi_\varepsilon L \varphi_\varepsilon(\xi) = (\varphi_\varepsilon \Delta \varphi_\varepsilon)(\xi) - \varphi_\varepsilon(\xi) \nabla \varphi_\varepsilon(\xi) \cdot \xi =$$

$$= \left(\varphi_\varepsilon \left(\frac{\sum_{i,j=1}^{d} |D^2_{ij} u|^2}{\varphi_\varepsilon} + \frac{\sum_{i=1}^{d} D_i u \Delta D_i u}{\varphi_\varepsilon} - \frac{\sum_{i,j=1}^{d} D_j \varphi_\varepsilon D_i u D^2_{ij} u}{\varphi_\varepsilon^2} \right) \right)(\xi) +$$

$$-\varphi_\varepsilon(\xi)\nabla\varphi_\varepsilon(\xi)\cdot\xi =$$

$$= \left(\varphi_\varepsilon\left(\frac{\sum_{i,j=1}^d |D^2_{ij}u|^2}{\varphi_\varepsilon} + \frac{\sum_{i=1}^d D_iu\Delta D_iu}{\varphi_\varepsilon} - \frac{\sum_{j=1}^d D_j\varphi_\varepsilon\left(\sum_{i=1}^d D_iuD^2_{ij}u\right)}{\varphi_\varepsilon^2}\right)\right)(\xi)+$$

$$-\sum_{i,j=1}^d \xi_j\left(D_iuD^2_{ij}u\right)(\xi) =$$

(by using one more time (7.1.2) and reordering the terms)

$$= \sum_{i,j=1}^d |D^2_{ij}u|^2(\xi) + \sum_{i=1}^d D_iu(\xi)\left(\Delta D_iu(\xi) - \sum_{j=1}^d \xi_j D^2_{ij}u(\xi)\right)+$$

$$-\frac{\varphi_\varepsilon^2\sum_{j=1}^d D_j\varphi_\varepsilon D_j\varphi_\varepsilon}{\varphi_\varepsilon^2}(\xi) =$$

$$= \left(\sum_{i,j=1}^d |D^2_{ij}u|^2 + \sum_{i=1}^d D_iu\mathscr{L}D_iu\right)(\xi) - |\nabla\varphi_\varepsilon|^2(\xi) \geq$$

(by (7.1.5))

$$\geq \sum_{i,j=1}^d |D^2_{ij}u|^2(\xi) + \sum_{i=1}^d D_iu(\xi)D_i\mathscr{L}u(\xi) - |\nabla\varphi_\varepsilon|^2(\xi) =$$

(recalling (7.1.3))

$$= \left(\sum_{i,j=1}^d |D^2_{ij}u|^2 + \sum_{i=1}^d D_iu\left(\sigma^{-1}D_iu - \sigma^{-1}D_iy\right)\right)(\xi) - |\nabla\varphi_\varepsilon|^2(\xi) =$$

$$= \left(\sum_{i,j=1}^d |D^2_{ij}u|^2 + \sigma^{-1}|\nabla u|^2 - \sigma^{-1}\nabla u\cdot\nabla y\right)(\xi) - |\nabla\varphi_\varepsilon|^2(\xi) \geq$$

(recalling $|\nabla\varphi_\varepsilon|^2 \leq \sum_{i,j=1}^d |D^2_{ij}u|^2$)

$$\geq \left(\sigma^{-1}\varphi^2 - \sigma^{-1}\varphi|\nabla y|\right)(\xi);$$

then

$$-\sigma\varphi_\varepsilon L\varphi_\varepsilon + \varphi^2 \leq \varphi|\nabla y|,$$

and, since $\frac{\varphi}{\varphi_\varepsilon} \leq 1$ we can conclude. □

DEFINITION 7.1.3. Let O be a domain with a C^2-regular boundary ∂O i.e. for each point $x \in \partial O$ there is a ball B centered in x, s.t. $\partial O \cap B$ is locally the graph of a C^2 function Ψ_x s.t. $\nabla_H\Psi(x) = 0$; we say that ∂O has positive mean curvature in x if $\Delta\Psi_x$ is positive in the point corresponding to x with the positive direction of the axis entering in ∂O).

It is easy to see that a regular convex set has positive mean curvature in every point of its boundary.

This result is ([**12**], Prop. 8.1) rewritten:

PROPOSITION 7.1.4. *If $O \subseteq \mathbb{R}^d$ is an open, bounded set with $C^{2,\alpha}$ boundary with positive mean curvature; if $J_\sigma = (I - \sigma\Delta)^{-1}$ with Δ Laplacian operator with Dirichlet null boundary conditions then*

$$\int_O |\nabla J_\sigma(y)|(x)\, dx \le \int_O |\nabla y|(x)\, dx$$

for all $y \in (W_0^{1,1}(O, \mathscr{L}^d)$.

REMARK 7.1.5. We imposed $C^{2,\alpha}$ to have that $J_\sigma(y)$ is $C^2(\bar{O})$ (see Appendix); this condition is necessary in the proof of ([**12**], Prop. 8.1).

In Prop. 8.1 in [**12**], O must be convex, but in the proof it is used only the fact that the boundary has positive mean curvature.

The proof of ([**12**], Prop. 8.1) is very similar to that of the Proposition 7.2.4 in the next section.

7.2. Case Dirichlet

7.2.1. Case Dirichlet $O \neq X$, finite dimension. We will consider $\mathbb{R}^d$ with the standard Gaussian measure γ^d. We will suppose $O \subseteq \mathbb{R}^d$ is an open $C^{2,\alpha}_{loc}$ regular set. which means that $O = \{g < 0\}$ with $g \in C^{2,\alpha}_{loc}(\mathbb{R}^2)$ with $\alpha > 0$). We will impose on the boundary a kind of condition of positive Gaussian mean curvature.

Under such conditions, we define L Ornstein-Uhlenbeck operator with zero Dirichlet boundary condition, that is, L is associated to the Dirichlet form in $W_0^{1,2}(O)$ in the sense of Definition 3.2.12). We want to prove that, if $J_\sigma = (I - \sigma L)^{-1}$ (where L is the O), then

$$\int_O |\nabla J_\sigma(y)|\, d\gamma^d \le \int_O |\nabla y|\, d\gamma^d,\ \forall y \in (W_0^{1,1} \cap L^2)(O). \tag{7.2.1}$$

We define the (inner) mean curvature of ∂O at a point as $\Delta\psi$ where ψ is the function of the graph with the axis oriented inside O.

Equivalently, if $O = \{g < 0\}$ with $g \in C^2(\mathbb{R}^2)$, we have that the mean curvature on the point of ∂O is

$$H_{\partial O} = \frac{\Delta g}{|\nabla g|} - \frac{D^2 g(\nabla g, \nabla g)}{|\nabla g|^3} \tag{7.2.2}$$

considered as a bilinear operator.

REMARK 7.2.1. To be more precise, $H_{\partial O}$ is the sum of the principal curvatures and the mean curvature is given by $\frac{1}{d-1} H_{\partial O}$.

We recall the Definition of $W_0^{1,p}(O)$ (Definition 3.2.12): let $f \in W_0^{1,p}(O)$, then it is a limit of functions in $C_0^1(O)$; we have that it can be approximated by a sequence of functions f_n in $C_0^1(O)$ with bounded support (because γ^d is a finite measure). This means that $f_n \in W_0^{1,p}(O, \mathscr{L}^d)$; so, we can construct a sequence of approximating functions which are in $C_c(O)$, such that they approximate f also in $f \in W_0^{1,p}(O)$.

DEFINITION 7.2.2. If $O \subseteq \mathbb{R}^d$ is a set with boundary $C^{2,\alpha}_{loc}$-regular, the Gaussian mean curvature in a point $x \in \partial O$ is $H'_{\partial O}(x) = H_{\partial O}(x) - x \cdot \nu_{\partial O}$ where $H_{\partial O}$ is the mean curvature and $\nu_{\partial O}$ is the outer normal to ∂O.

We will suppose that O is bounded and that it has a $C^{2,\alpha}_{\mathrm{loc}}$ boundary for some $\alpha > 0$, with everywhere non negative Gaussian mean curvature.

Now, we define L as the Ornstein-Uhlenbeck operator in O with zero Dirichlet boundary condition; we introduce J_σ in $L^1(O)$, and, for each $y \in L^2(O)$, the function $u = J_\sigma(y)$ is well defined and belongs to $W^{1,2}_0(O)$.

Now, we suppose that $y \in C^\infty_c(\bar{O})$, by Remark A.0.1 (in the Appendix) we have $u \in C^2(\bar{O})$.

We can define $\varphi = |\nabla u|$ and $\varphi_\varepsilon = \sqrt{\varepsilon + \varphi^2}$, we will find the equivalent of the Lemma 7.1.2.

We will change the coordinates in a way such that $0 \in \partial O$, and that $\partial\Omega$ is the graph of a function $\psi : \mathbb{R}^{d-1} \to \mathbb{R}$, with $\psi(0) = 0$, $\nabla\psi(0) = 0$; the graph is oriented with the outer normal downward. We redefine u, φ and φ_ε in this case (the main difference will be that now γ is not centered; we denote the new center by x).

LEMMA 7.2.3. *In the above hypotheses,*

$$D_d\varphi_\varepsilon(0) = (D_du(0))^2(\varepsilon^2 + (D_du(0))^2)^{-\frac{1}{2}}\,(\Delta\psi(0) - x_d)\,.$$

PROOF. We repeat the argument of [[**12**], Sec. 8]. Arguing as in the proof of [**12**], (8.11), we consider that $u(\xi', \psi(\xi')) = 0$ where $\xi' \in \mathbb{R}^{d-1}$, and then we have

$$D_iu + D_duD_i\psi = 0,$$

$$D^2_{ii}\psi + 2D^2_{id}uD_i\psi + D^2_{dd}(D_i\psi)^2 + D_duD^2_{ii}\psi = 0,$$

and $D_i\psi = 0$; hence we have $D_iu(0) = 0$ and

$$D^2_{ii}u(0) + D_du(0)D^2_{ii}\psi(0) = 0$$

for all $i \le n-1$: so (recalling that L can be applied to u also on the boundary because $u \in C^2_{loc}(\bar{O})$), we can write

$$Lu(0) = D^2_{dd}u(0) - D_du(0)\Delta_{\xi'}\psi(0) - D_du(0)(0 - x_d),$$

clearly $Lu(0) = 0$ because u, y are 0 on 0, so

$$D^2_{dd}u(0) = D_du(0)(\Delta_{\xi'}\psi(0) - x_d);$$

by (7.1.2) we have

$$\varphi_\varepsilon D_d\varphi_\varepsilon = \sum_{i=1}^{d} D_iuD^2_{di}u$$

(this is true also on the boundary, because $u \in C^2_{loc}(\bar{O})$) so, using the above equalities, we have

$$\varphi_\varepsilon D_d\varphi_\varepsilon(0) = D_du(0)D^2_{dd}u(0) = (D_du)^2(0)\left(\Delta_{\xi'}\psi(0) - x_d\right),$$

and we can conclude. $\square$

Now we can argue as in the proof of ([**12**],Proposition 8.1); we assume that the outward normal to ∂O is $\eta = (0, \ldots, 0, -1)$, then $\frac{\partial\varphi_\varepsilon}{\partial\eta}(0) = -D_d\varphi_\varepsilon(0)$, and that $|D_du(0)| = |\nabla u(0)| = \varphi(0)$, since $u_{|\partial O} = 0$ we have

$$\frac{\partial\varphi_\varepsilon}{\partial\eta}(0) = -\frac{(D_du(0))^2}{\varphi_\varepsilon(0)}(\Delta\psi(0) - x_d) = -\frac{\varphi^2(0)}{\varphi_\varepsilon(0)}(\Delta\psi(0) - x_d);$$

but $\Delta_{\xi'}\psi(0)-x_d=H'_S(0)$, and, by hypothesis, $H'_S\geq 0$. Now,

$$\int_O L\varphi_\varepsilon\, d\gamma^d=\frac{1}{\sqrt{2\pi}}\int_{\partial O}\frac{\partial\varphi_\varepsilon}{\partial n}\varphi\, dS\leq 0,$$

where $g(x):=(2\pi)^{-\frac{d}{2}}\exp-\frac{\|x\|^2}{2}$ and S is the measure given by the area of ∂O.

Now, integrating 7.1.1, we have that

$$\int_O\frac{\varphi^2}{\varphi_\varepsilon}\, d\gamma^d-\sigma\int_O L\varphi_\varepsilon\, d\gamma^d\leq\int_O|\nabla y|\, d\gamma^d,$$

hence

$$\int_O\frac{\varphi^2}{\varphi_\varepsilon}\, d\gamma^d\leq\int_O|\nabla y|\, d\gamma^d;$$

by letting $\varepsilon\to 0$, we obtain the inequality 7.2.1.

The previous computations has been done for $y\in C_c^\infty(\bar{O})$, but for the density of $C_c^\infty(\bar{O})$ in $W_0^{1,p}(O)$, by the Remark 7.1.1 and the continuity of J_σ in L^2, we have proved the next Proposition.

PROPOSITION 7.2.4. *In $\mathbb{R}^d$, if $O\subseteq X$ is a $C^{2,\alpha}_{loc}$-regular set for some $\alpha>0$ s.t. $H'_{\partial O}(x)\geq 0$ at each point $x\in\partial O$, if $\sigma>0$, L is the Ornstein-Uhlenbeck operator on $L^2(O)$ with zero Dirichlet boundary conditionn and $J_\sigma:=(I-\sigma L)^{-}1$,then*

$$\int_O|\nabla J_\sigma(y)|\, d\gamma^d\leq\int_O|\nabla y|\, d\gamma^d$$

for all $y\in W_0^{1,2}(O)$.

REMARK 7.2.5. The above Proposition could be easily extended to the case $y\in(W_0^{1,2}\cap L^2)(O)$, but we are more interested to the infinite dimensional case, in which the extension is impossible with our instruments.

7.3. Case Dirichlet $O\neq X$, infinite dimension

We will use the notations of the above chapters.

7.3.1. Cylindrical case. For some $\alpha>0$, let O be a cylinder, $O=O_1\times X_m^\perp$ where $O_1\subseteq F_m$ ($F_m=<h_1,\ldots,h_m>$ subspace of $R_\gamma(X^*)$), $X_m^\perp$ is the closure in X of $F^\perp$ (orthogonal of F in H), O_1 is a regular set in F with $C^{2,\alpha}_{\mathrm{loc}}$-regular boundary: i.e. there exists $g\in C^{2,\alpha}_{loc}(F)$ s.t. $O_1=g^{-1}((-\infty,0))$; if we define $G(x):=g(\hat{h}_1(x),\ldots,\hat{h}_m(x))$, we have that $O=G^{-1}((-\infty,0))$, and we can define the space $W^{1,p}(O)$ and $W_0^{1,p}(O)$.

DEFINITION 7.3.1. $\mathscr{F}C_0^1(O)$ will be the set of all C^1 cylindrical functions that are 0 on ∂O: $y\in\mathscr{F}C_0^1(O)$ if $y(\cdot)=v(\hat{h_1}(\cdot),\ldots,\hat{h_m}(\cdot))$ for some $v\in C^1_{0,b}(O_1)$ i.e. $v\in C_b^1(\mathbb{R}^n)$ with support in O_1.

REMARK 7.3.2. For $W_0^{1,p}(O)$ we follow the definition of 3.2.12; it is equivalently defined as the closure in norm $W^{1,p}$ of $\mathscr{F}C_0^1(O)$: in fact, if $f\in C_0^1(O)$, and if $f_n:=\mathbb{E}_n f$ is the cylindrical approximation relative to F_n (see Section 2.4), clearly f_n will be 0 out of O and $f_n\to f$ in $W^{1,p}(X)$, and we can write $f_n(x)=v(\hat{h}_i(x),\ldots,\hat{h}_n(x))$ where $h_1,\ldots,h_n\in H$ and $v\in C_b^1(F_n)$ (we have $\mathrm{supp}(f_n)\subseteq\bar{O}\times F^\perp$).

The following Proposition will be generalized in Theorem 7.3.7.

PROPOSITION 7.3.3. *In the above setting, if* $H'_{\partial O}(x) \geq 0$ *at each point* $x \in \partial O$, *if* $\sigma > 0$, L *and* J_σ *are the operators associated to the zero Dirichlet boundary conditions, then* $\int_O |\nabla_H J_\sigma(y)|\, d\gamma \leq \int_O |\nabla_H y|\, d\gamma$ *for all* $y \in W_0^{1,2}(O)$.

PROOF. For each $n > m$, we define $O_n := \pi_{F_n}(O)$, we have that O is isometric to $O_n \times X_n^\perp$, and O_n is isometric to $O_m \times \mathbb{R}^{n-m}$.

At first, we suppose that $y \in \mathscr{F}C_0^1(O)$ and that there exists $n > m$ s.t. $y(\cdot) = v(\hat{h}_1(\cdot), \dots, \hat{h}_n(\cdot))$ for some $v \in C_0^\infty(O_n)$. On $L^2(F_n)$ we can define the Ornstein-Uhlenbeck operator L_n and the bounded operator $J_{\sigma,n}$ with respect to O_n, in the case Dirichlet; we will have that

$$(J_{\sigma,n} v) \circ \pi_{F_n} = J_\sigma y,$$

moreover

$$\|\nabla_H y\|_{L^1(O,H)} = \|\nabla v\|_{L^1(O_n,\mathbb{R}^n)}$$

and

$$\|\nabla_H J_\sigma y\|_{L^1(O,H)} = \|\nabla J_{\sigma,n} v\|_{L^1(O_n,\mathbb{R}^n)}.$$

Hence, by using Proposition 7.2.4 we have

$$\int_O |\nabla_H J_\sigma(y)|_H\, d\gamma = \int_{O_n} |\nabla J_{\sigma,n} v|\, d\gamma^n \leq \int_{O_n} |\nabla v|\, d\gamma^d = \int_O |\nabla_H y|\, d\gamma,$$

and we have concluded, in the case $\mathscr{F}C_0^1(\bar{O})$.

Now, let y be in $W_0^{1,2}(O)$; it can be approximated in $W^{1,2}$ by a sequence of function $y_n \in \mathscr{F}C_0^1(O)$ by Remark 7.3.2; we already know that $J_\sigma y_n$ converges to $J_\sigma y$ in L^2; we have that, for a couple of function $y_n, y_m \in W_0^{1,2}(O)$

$$\int_O |\nabla_H J_\sigma(y_n - y_m)|_H d\gamma \leq \int_O |\nabla_H (y_n - y_m)|_H d\gamma,$$

so, by the linearity of J_σ and of ∇_H, we have that $J_\sigma y_n$ converges to $J_\sigma y$ in $W_0^{1,2}(O)$. So, if we extend those functions to 0 out of O, we have a convergence in $W^{1,2}(X)$ and so also in $L^2(X)$ and $L\log^{\frac{1}{2}} L(X)$; and by lower semicontinuity of the functional $\int_O |\nabla_H \cdot|_H\, d\gamma^d$ with respect to $L\log^{\frac{1}{2}} L(X)$, we can conclude. $\square$

7.3.2. Generalization to non-cylindrical case. We recall a technical remark that will be used later.

REMARK 7.3.4. If $\{a_{i,n}\}_{i \in I, n \in \mathbb{N}}$ for some set I is a sequence, then

$$\sup\{\limsup_{n\to\infty} a_{i,n} | i \in I\} \leq \limsup_{n\to\infty} \sup\{a_{i,n} | i \in I\};$$

in fact, for each $j \in I$, $n \in \mathbb{N}$

$$a_{j,n} \leq \sup\{a_{i,n} | i \in I\},$$

so

$$\limsup_{n\to\infty} a_{j,n} \leq \limsup_{n\to\infty} \sup\{a_{i,n} | i \in I\},$$

hence we can conclude.

We will use the hypotheses and notations of Chapter 6.

In this setting, for a regular function f on X, we can write

$$Lf(x) := \sum_{i=1}^{\infty} \partial^2_{h_i,h_i} f(x) - \sum_{i=1}^{\infty} \partial_{h_i} f(x)\hat{h}_i(x).$$

π_n will be the projection on $F_n = \langle h_1, \dots, h_n \rangle$. For a function g on $\mathbb{R}^n$, we define, for all $n \in \mathbb{N}$

$$L_n g(x) := \sum_{i=1}^{n} \partial^2_{i,i} g(x) - \sum_{i=1}^{n} \partial_i g(x) x_i.$$

We recall that if $O = G^{-1}(-\infty, 0)$, $\partial O = G^{-1}(0)$; we have that ∂O has in each point an outer H-normal that plays the role of the outer normal to ∂O.

The next Remarks are inspired also by [**28**].

We define, for all $n \in \mathbb{N}$, the cylindrical function $G_n = G \circ \pi_n$, and

$$\mathcal{H}_n(x) = \frac{LG_n(x)}{|\nabla_H G_n(x)|_H} - \frac{D^2_H G_n(x)(\nabla_H G_n(x), \nabla_H G_n(x))}{|\nabla_H G_n(x)|^3_H}.$$

For $n \in \mathbb{N}$, we will call $O_n := G_n^{-1}((-\infty,0))$, it will be a cylindrical approximation of O; then O_n will be a cylinder $C_n \times X_0$ for some Banach space X_0 and $C_n = g_n^{-1}((-\infty,0))$ where $g_n : \mathbb{R}^n \to \mathbb{R}$, $g_n = G \circ I$ where I is the identification of $\mathbb{R}^n$ with O_n; we introduce on $\mathbb{R}^n$ the function

$$H_n(x) = \frac{L_n g_n(x)}{|\nabla g_n(x)|} - \frac{D^2 g_n(x)(\nabla g_n(x), \nabla g_n(x))}{|\nabla g_n(x)|^3}$$

(where L_n is the Ornstein Uhlenbeck operator for smooth functions in F_n with Gaussian standard measure γ^n); clearly we have

$$\mathcal{H}_n = H_n \circ \pi_n.$$

HYPOTHESIS 7.3.5. We suppose that Hypothesis 6.1.3 is verified, and moreover that for some $k \in \mathbb{N}$, we have $\pi_k(\nabla_H G) \neq 0$ on ∂O and that $G \circ \pi_{n|F_n}$ is $C^{2,\alpha}_{\text{loc}}$ in F_n for all $n \in \mathbb{N}$.

We suppose that, for some $n_0 \in \mathbb{N}$, $\mathcal{H}_n(x) \geq 0$ for all $x \in \partial O_n$, for all $n > n_0$.

We remark that, in the above hypotheses, we can apply Corollary 6.2.6 because Hypothesis 7.3.5 contains Hypothesis 6.1.3.

REMARK 7.3.6. If $n > k$, then g_n is $C^{2,\alpha}$ and satisfies the hypothesis of the implicit function theorem (because $\pi_k(\nabla_H G) \neq 0$) and C_n is an open set with $C^{2,\alpha}$ boundary (see e.g [**36**], Appendix A, Corollary A.4).

$$C_n = g_n^{-1}((-\infty,0)),$$

and on $\partial C_n = g_n^{-1}(0)$ we have $\nu_{\partial C_n} = \frac{\nabla g_n}{|\nabla g_n|}$. On the boundary, $\mathcal{H}_n(x)$ coincides with the Gaussian mean curvature defined in Definition 7.2.2: in fact for $x \in \partial C_n$, recalling (7.2.2),

$$H'_{\partial C_n}(x) = H_{\partial C_n}(x) - x \cdot \nu_{\partial C_n} = \frac{\Delta g_n}{|\nabla g_n|} - \frac{D^2 g_n(x)(\nabla g_n(x), \nabla g_n(x))}{|\nabla g_n(x)|^3} - x \cdot \frac{\nabla g_n}{|\nabla g_n|} = \mathcal{H}_n(x).$$

In particular, if $\mathcal{H}_n(x)$ is always positive, for all $n \in \mathbb{N}$ the set O_n satisfies the hypotheses of Theorem 7.3.3.

As usual, $u := J_\sigma(y)$.

THEOREM 7.3.7. *Let O be an open set which satisfies to Hypothesis 7.3.5, if $\sigma > 0$, L is the Ornstein-Uhlenbeck operator with zero Dirichlet boundary condition on O, and $J_\sigma := (I - \sigma L)^{-1}$, then $\int_O |\nabla_H u|_H \, d\gamma \leq \int_O |\nabla_H y|_H \, d\gamma$ for all $y \in W_0^{1,2}(O)$.*

PROOF. By Remark 7.3.6, and by Proposition 7.3.3, we have that, for all $n > n_0$, (n_0 defined in Hypothesis 7.3.5) if $y \in W_0^{1,2}(O_n)$, if L_n is the Ornstein-Uhlenbeck operator with zero Dirichlet condition in O_n, if $J_{\sigma,n} = \sigma^{-1} R(\sigma^{-1}, L_n)$, then $\int_{O_n} |\nabla_H J_{\sigma,n}(y)| \, d\gamma \leq \int_{O_n} |\nabla_H y| \, d\gamma$.

We will suppose $y \in C_0^1(O)$ (hence $y \in L^2(O)$), hence we can define, for every $n \in \mathbb{N}$, the cylindrical C^1 function $y_n = y \circ \pi_n$, then $y_n \in C_0^1(O_n)$ and we can define $u_n = J_{\sigma,n}(y_n)$ in L^2: hence $u_n \in W^{1,2}(O_n)$ and

$$\int_{O_n} |\nabla_H u_n|_H \, d\gamma \leq \int_{O_n} |\nabla_H y_n|_H \, d\gamma$$

by Proposition 7.3.3.

We remark that, by Theorem 6.2.2, y_n can always be extended to X, by setting $y_n = 0$ out of O_n (and y can be extended to X in the same way). Now, by Lemma 6.2.4 we have that

$$\int_{O_n} |\nabla_H y_n|_H \, d\gamma \to \int_O |\nabla_H y|_H \, d\gamma.$$

We have

$$\liminf_{n\to\infty} \int_{O_n} |\nabla_H u_n|_H \, d\gamma \leq \lim_{n\to\infty} \int_{O_n} |\nabla_H y_n|_H \, d\gamma = \int_O |\nabla_H y|_H \, d\gamma. \tag{7.3.1}$$

We remark that $u \in W_0^{1,2}(O)$ and $u_n \in W_0^{1,2}(O_n)$ for every $n \in \mathbb{N}$, hence by Theorem 6.2.2, we can extend u, u_n to functions $\tilde{u}, \tilde{u}_n \in W^{1,2}(X)$ ($\tilde{u}_{|X\setminus O} \equiv 0$, $\tilde{u}_{n|X\setminus O_n} \equiv 0$). By Corollary 6.2.6, we have that $u_n \to u$ in $L^2(X)$ (and then also in $L\log^{\frac{1}{2}} L(O)$). Hence, by Remark 7.1.1

$$\|\nabla_H u\|_{L^1(X,H)} \leq \liminf_{n\to\infty} \|\nabla_H u_n\|_{L^1(X,H)}.$$

Now by (7.3.1), we can conclude

$$\int_O |\nabla_H u|_H \, d\gamma \leq \int_O |\nabla_H y|_H \, d\gamma,$$

in the case of $y \in C_0^1(O)$. For the case $W_0^{1,2}(O)$, we recall that $C_0^1(O)$ is dense in $W_0^{1,2}(O)$ by Definition 3.2.12, and then we can conclude because J_σ is continuous from L^2 in L^2, because L^{p2} is continuously embedded in $L\log^{\frac{1}{2}} L$ and $\int_O |\nabla_H \cdot|_H \, d\gamma$ is lower semicontinuous with respect to $L\log^{\frac{1}{2}} L$. □

7.3.2.1. *Examples* .

Example 1. Epigraphs. Let X be a Hilbert space (see Subsection 2.5). We consider a basis $\{h_i\}_{i\in\mathbb{N}}$ in $R_\gamma(X^*)$. We want that $O = G^{-1}((-\infty, 0))$ is a the epigraph of a function.

For simplicity, let Φ be a function on all X, we suppose $\partial_{h_1}(\Phi) \equiv 0$ everywhere (so, Φ can be seen as a function with domain given by an hyperplane of X, $\Phi(x) = \Phi(x - \pi_1(x))$) and we set

$$G(x) := \hat{h}_1(x) + \Phi(x).$$

Now, on Φ we suppose (in addition to $\partial_{h_1}(\Phi) \equiv 0$ everywhere): $\Phi \circ \pi_n \in C^{2,\alpha}_{\text{loc}}(F_n)$ for every $n \in \mathbb{N}$ (by identifying F_n with $\mathbb{R}^n$), for some $\alpha > 0$; $\nabla_H \Phi \in R_\gamma(X^*)$ everywhere, $|\nabla_H \Phi|_H$ and

$|D_H^2\Phi|_{HS}$ are bounded, and, for some $C, C_1, C_2, C_3 > 0$ independent on $h, n \in H$, we have $\phi(x) \geq C$, $|D_H^2\Phi|_{HS}(x) \leq C_1$, $\widehat{\nabla_H\Phi(x)}(x) \leq C_2$ ($\widehat{\cdot}$ is, as usual the function $R_\gamma(X^*) \to X^*$) and

$$\sum_{i=2}^{\infty} |D_H^2\Phi(y_n(x))(h_i,h_i)|_H \leq C_3$$

for every $x \in X$, with $C_1 + C_2 + C_3 \leq C$.

Now, for $n \geq 1$,

$$G_n(x) = \hat{h}_1(x) - \Phi(\pi_n(x) - \pi_1(x)),$$

(it is $C^{2,\alpha}$ on F_n) hereafter we will write $y_n(x) := \pi_n(x) - \pi_1(x)$, so $G_n(x) = \hat{h}_1(x) - \Phi(y_n(x))$; then

$$\nabla_H G(x) = h_1 + \nabla_H\Phi(x - \pi_1(x)),$$

(clearly $\pi_n(\nabla_H G) \neq 0$ everywhere for every n)

$$LG(x) = -\hat{h}_1(x) + L\Phi(x - \pi_1(x)),$$

$$\nabla_H G_n(x) = h_1 + \pi_n\left(\nabla_H\Phi(y_n(x))\right)$$

$$LG_n(x) = -\hat{h}_1(x) + \sum_{i=2}^{n}\left(-D_H^2\Phi(y_n(x))(h_i,h_i) + \hat{h}_i(x)\left\langle D_H\Phi(y_n(x)), h_i\right\rangle_H\right),$$

and by ([**26**], 5.2) the Hypothesis 6.1.3 is verified; moreover, on $G^{-1}(0)$ we have

$$H_n(x) = \frac{LG_n(x)}{|\nabla_H G_n(x)|} - \frac{D_H^2 G_n(x)(\nabla_H G_n(x), \nabla_H G_n(x))}{|\nabla_H G_n(x)|^3} =$$

(since $\hat{h}_1(x) = -\Phi(x)$ in $G^{-1}(0)$)

$$= \frac{\Phi(x) + \sum_{i=2}^{n}\left(-D_H^2\Phi(y_n(x))(h_i,h_i) + \hat{h}_i(x)\left\langle D_H\Phi(y_n(x)), h_i\right\rangle_H\right)}{\sqrt{1 + \sum_{i=2}^{n}\left\langle \nabla_H\Phi(y_n(x)), h_i\right\rangle_H^2}} +$$

$$-\frac{D_H^2\Phi(y_n(x))(\nabla_H\Phi(y_n(x)), \nabla_H\Phi(y_n(x))}{\left(1 + \sum_{i=2}^{n}\left\langle \nabla_H\Phi(y_n(x), h_i\right\rangle_H^2\right)^{\frac{3}{2}}};$$

hence we have

$$H_n(x) \geq$$

$$\geq \frac{\Phi(x) - \sum_{i=2}^{\infty}|D_H^2\Phi(y_n(x))(h_i,h_i)|_H - \widehat{\nabla_H\Phi(x)}(x)}{\sqrt{1 + \sum_{i=2}^{n}\left\langle \nabla_H\Phi(y_n(x)), h_i\right\rangle_H^2}} +$$

$$+ \frac{-D_H^2\Phi(y_n(x))|\nabla_H\Phi|_H^2}{\sqrt{1 + \sum_{i=2}^{n}\left\langle \nabla_H\Phi(y_n(x)), h_i\right\rangle_H^2}} \geq$$

$$\geq \frac{C - C_1 - C_2 - C_3}{\sqrt{1 + \sum_{i=2}^{n}\left\langle \nabla_H\Phi(y_n(x)), h_i\right\rangle_H^2}} \geq 0.$$

then G satisfies Hypothesis 7.3.5.

In particular, if $\Phi \equiv C \geq 0$ everywhere, it satisfies the above condition: so, the halfspace $\{\hat{h_1} \leq -C\}$ satisfies the condition if $C < 0$.

Example 2. Pinned Wiener space (Brownian bridge). For this example, we refer to ([**28**], Example 5.4).

Let $Y = C([0,1])$. We recall the concept of Brownian Bridge (see Subsection 2.6.3); it is represented by a Wiener space (X, γ_W) where $X = L^2[0,1]$, the Cameron-Martin space is $H = W_0^{1,2}((0,1))$. We remark that, for every $h \in H$,

$$\|h\|_{C([0,1])} \leq \int_0^1 |h'(t)|\, dt \leq |h|_{W_0^{1,2}((0,1))} = |h|_H. \tag{7.3.2}$$

We assume that $g \in C^{2,\alpha}(\mathbb{R})$ is a function with bounded first and second order derivative in $\mathbb{R}$ (let c be the Hölder constant of g'') such that, for some $C > 0$, for every $\xi, \eta \in \mathbb{R}$,

$$|g''(\xi) - g''(\eta)| \leq C|\xi - \eta|(|\xi| + |\eta|), \tag{7.3.3}$$

(so, in particular g'' is $\mathrm{Lip}_{\mathrm{loc}}$) and moreover, for some $a > 0$, $\alpha > 0$ and $\beta_1, \beta_2 \in \mathbb{R}$ it satisfies $|g'(\xi)| \geq a$ (hence $g'(\xi) \neq 0$ for every ξ) and

$$\alpha g(\xi) + \beta_1 \leq \xi g'(\xi) \leq \alpha g(\xi) + \beta_2 \tag{7.3.4}$$

for all $\xi \in \mathbb{R}$.

The above assumptions are satisfied by $g = p/q$ for q positive polynomial of degree $n \in \mathbb{N}$ and p polynomial of degree $n+1$ s.t. $g'(\xi) \neq 0$ for all $\xi \in \mathbb{R}$ (in this case, g has an asymptote l, the angular coefficient of l will be α).

PROPOSITION 7.3.8. *Given r in the range of g and $r < \alpha_1^{-1}(-\beta_1 - \frac{\|g\|_{C_b^2(\mathbb{R})}}{6})$, we define*

$$G(x) := \int_0^1 g(x(s))\, ds - r$$

on X; we have that $O := G^{-1}((-\infty, 0))$ satisfies Hypotheses 7.3.5(see also proof of [**28**]*, Prop. 5.1).*

In fact, G is Fréchet differentiable with gradient given by

$$(DG(x))\,(h) = \int_0^1 g'(x(s))h(s)\, ds,$$

so G is H-differentiable and, for every $h \in H$,

$$\langle \nabla_H G(x), h \rangle_H = \int_0^1 g'(x(s))h(s)\, ds$$

so

$$|\nabla_H G(x)|_H^2 \leq \int_0^1 |g'(x(s))|^2\, ds \leq \|g\|^2{}_{C_b^1(\mathbb{R})}$$

and moreover for $x, y \in X$,

$$|\nabla_H G(x) - \nabla_H G(y)|_H^2 \leq \int_0^1 |g'(x(s)) - g'(y(s))|^2\, ds \leq$$

$$\leq \|g\|^2_{C^2([0,1])}\, |x - y|_X^2$$

hence $G \in C_b^1(X, H)$, $D_H^2 G$ is everywhere defined and, for every $h, k \in H$,

$$D_H^2 G(x)(h,k) = \int_0^1 g''(x(s))h(s)k(s)\, ds.$$

Recalling (7.3.2),

$$|D_H^2 G(x)(h,k)| \leq \|g\|_{C_b^2(\mathbb{R})} \|h\|_{C([0,1])} \|k\|_{C([0,1])} \leq \|g\|_{C_b^2(\mathbb{R})} |h|_H |k|_H,$$

and so

$$|D_H^2 G|_{HS} \leq |g|_{C_b^2(\mathbb{R})}. \tag{7.3.5}$$

Moreover, fixed h,k we have that if $x_1, x_2 \in X$

$$|D_H^2 G(x_1)(h,k) - D_H^2 G(x_2)(h,k)| =$$

$$= \left| \int_0^1 (g''(x_1(s)) - g''(x_2(s)))h(s)k(s)\, ds \right| \leq c \int_0^1 |x_1(s) - x_2(s)|^\alpha h(s)k(s)\, ds \leq$$

(if c is the Hölder constant of g'')

$$\leq c|h|_X |k|_X\, |x_1 - x_2|^\alpha$$

hence G is $C^{2,\alpha}$ on every $F \leq H$ with $\dim(F) < \infty$.

If $l \in g^{-1}(r)$ (l exists because r is in the range of g), if $x(s) = l$ for every $s \in [0,1]$ then $x \in G^{-1}(0)$, so

$$G^{-1}(0) \neq \varnothing. \tag{7.3.6}$$

We recall that in Subsection 2.6.3 is defined an orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ with eigenvalues $\lambda_k = (\pi k)^{-2}$, and H has an orthonormal basis of eigenvector $h_k = \{\sqrt{2}\pi^{-1}k^{-1}\sin(k\pi\cdot)\}_{k\in\mathbb{N}}$.

For $h_1(s) := \sqrt{2}\pi^{-1}\sin \pi s$ ($h_1(s) > 0$ everywhere), by $g' \geq a$ we have that for every $x \in X$

$$|\langle \nabla_H G(x), h_1\rangle_H| = \sqrt{2}\pi^{-1} \int_0^1 g'(x(s)) \sin \pi s\, ds \geq \sqrt{2}\pi^{-1} a \int_0^1 \sin \pi s\, ds = 2\sqrt{2}\pi^{-2} a > 0$$

so $|\pi_n(\nabla_H G)|_H \neq 0$ everywhere for every n and

$$|\nabla_H G|_H^{-1} \leq \pi^2 a^{-1} \frac{\sqrt{2}}{4} \tag{7.3.7}$$

(because $|h_1|_H^2 = 1$).

If we consider the sequence h_k we have that the series $\sum_{k=1}^\infty h_k^2(\cdot)$ converges uniformly to

$$f(s) := s - s^2$$

in $[0,1]$: in fact (by the duplication formula of cosine)

$$h_k^2(s) = \pi^{-2}k^{-2}(1 - \cos 2k\pi s)$$

converges uniformly (because $\sum_{k=1}^{+\infty} k^{-2}$ is a convergent series) and if we expand f in Fourier series of cosines we have

$$f(s) = \frac{1}{6} - \sum_{k=1}^{+\infty} \pi^{-2}k^{-2} \cos 2k\pi s$$

and

$$\sum_{k=1}^{+\infty} k^{-2} = \frac{\pi^2}{6}$$

(the above formula is the solution of the Basel problem, see e.g. [**34**]). Now we have

$$LG(x) = \sum_{i=1}^\infty D_H^2 G(x)(h_i, h_i) - \sum_{i=1}^\infty \langle \nabla_H G(x), h_i\rangle_H =$$

$$= \sum_{i=1}^{\infty} \int_0^1 g''(x(s)) h_i^2(s)\, ds - \int_0^1 g'(x(s)) x(s)\, ds =$$
$$= \int_0^1 g''(x(s))(s-s^2)\, ds - \int_0^1 g'(x(s)) x(s)\, ds,$$

clearly $\int_0^1 g''(x(s))(s-s^2)\, ds$ is bounded because $g \in C_b^2(\mathbb{R})$, and, by (7.3.4)

$$\int_0^1 g'(x(s))x(s)\, ds \leq \int_0^1 (\alpha_1 g(x(s)) + \beta_1)\; ds = \alpha G(x) - \alpha r + \beta_1$$

and

$$\int_0^1 g'(x(s))x(s) \geq \int_0^1 (\alpha_2 g(x(s)) + \beta_2)\; ds = \alpha G(x) - \alpha r + \beta_2;$$

therefore LG is bounded in $G^{-1}(-\delta,\delta)$ (for every $\delta > 0$); so, by (7.3.7) and (7.3.5), we have the Hypothesis 6.1.3.

For $n \in \mathbb{N}$, and $x \in \partial O_n$, we consider $\varphi_n \in H \subset X = L^2([0,1])$ as $\varphi_n := \nabla_H G_n(x)$ (so $|\varphi_n|_H \leq \|g\|^2{}_{C_b^2(\mathbb{R})}$ everywhere), π_n projection from X in $F_n = < e_1, \ldots, e_n >$,

$$f_n(s) := \sum_{k=1}^{n} 2\pi^{-1} k^{-1} \sin^2 k\pi s,$$

we have $f_n > 0$ everywhere, and

$$\int_0^1 f_n\, ds \leq \int_0^1 f\, ds = \frac{1}{6};$$

we have that, for some sequence $\{\mu_k\}_{k\in\mathbb{N}}$,

$$\varphi_n(s) = \sum_{k=1}^{n} \mu_k \sin k\pi s.$$

Hence

$$|\varphi_n|_H^2 = \sum_{i=1}^{n} \lambda_k^{-1} \mu_k^2,$$

so

$$(\varphi_n(s))^2 \leq \left(\sum_{i=1}^{n} \lambda_k^{-1}\mu_k^2\right)\left(\sum_{k=1}^{n} \lambda_k \sin^2 k\pi s\right) \leq$$
$$= |\varphi_n|_H^2 f_n(s),$$

therefore in particular

$$f_n - \frac{\varphi_n^2}{|\varphi_n|_H^2} \geq 0 \tag{7.3.8}$$

and

$$LG_n(x) = \int_0^1 g''(\pi_n(x)(s)) f_n(s)\, ds - \int_0^1 g'(\pi_n(x)(s))\pi_n(x)(s)\, ds$$

therefore

$$\mathcal{H}_n(x) = \frac{LG_n(x)}{|\nabla_H G_n(x)|_H} - \frac{D_H^2 G_n(x)(\nabla_H G_n(x), \nabla_H G_n(x))}{|\nabla_H G_n(x)|_H^3} =$$

$$= \frac{\int_0^1 g''(\pi_n(x)(s)) f_n(s)\, ds - \int_0^1 g'(\pi_n(x)(s))\pi_n(x)(s)\, ds}{|\varphi_n|_H} - \frac{\int_0^1 g''(\pi_n(x)(s))\varphi_n^2(s)\, ds}{|\varphi_n|_H^3} =$$

$$= |\varphi_n|_H^{-1}\Big(\int_0^1 g''(\pi_n(x)(s)) \left(f_n - \frac{\varphi_n^2(s)}{|\varphi_n|_H^2}\right)\, ds +$$

$$- \int_0^1 g'(\pi_n(x)(s))\pi_n(x)(s)\, ds\Big) \geq$$

(by (7.3.8) and by (7.3.4))

$$\geq |\varphi_n|_H^{-1}\Big(\int_{\{s\in[0,1] | g''(\pi_n(x)(s))<0\}} g''(\pi_n(x)(s)) \left(f_n - \frac{\varphi_n^2(s)}{|\varphi_n|_H^2}\right)\, ds +$$

$$- \int_0^1 \alpha_1 g(\pi_n(x)(s)) + \beta_1\, ds \geq$$

(by the fact that, in the first integral, $g''(\pi_n(x)(s)) < 0$ everywhere)

$$\geq |\varphi_n|_H^{-1}\Big(\int_{\{s\in[0,1] | g''(\pi_n(x)(s))<0\}} g''(\pi_n(x)(s)) f_n\, ds +$$

$$- \int_0^1 \alpha_1 g(\pi_n(x)(s)) + \beta_1\, ds\Big) \geq$$

($f_n > 0$ everywhere)

$$\geq |\varphi_n|_H^{-1}\Big(- \|g\|_{C_b^2(\mathbb{R})} \int_0^1 f_n\, ds +$$

$$- \int_0^1 \alpha_1 g(\pi_n(x)(s)) + \beta_1\, ds\Big) \geq$$

(by $\int_0^1 f_n\, ds \leq \frac{1}{6}$)

$$\geq |\varphi_n|_H^{-1} \left(-\frac{\|g\|_{C_b^2(\mathbb{R})}}{6} - \alpha_1 \int_0^1 g(\pi_n(x)(s))\, ds + \beta_1 \right) =$$

$$= |\varphi_n|_H^{-1} \left(-\frac{\|g\|_{C_b^2(\mathbb{R})}}{6} - \alpha_1 G_n(x) - \alpha_1 r - \beta_1 \right).$$

Now, if $x \in G_n^{-1}(0)$

$$\mathcal{H}_n(x) = |\varphi_n|_H^{-1} \left(-\frac{\|g\|_{C_b^2(\mathbb{R})}}{6} - \alpha_1 r - \beta_1 \right) \geq 0$$

in our hypotheses about r. Hence, we have all the Hypotheses of Theorem 7.3.7.

7.4. Case Neumann boundary conditions

7.4.1. Case Neumann boundary conditions, Lebesgue measure. Firstly we consider the case with Lebesgue measure.

We suppose $X=\mathbb{R}^d$ and O is an open, bounded and convex set.

We define $J_\sigma=(I-\sigma\Delta_N^{-1}$ where Δ_N is the Laplacian on O with Neumann condition.

REMARK 7.4.1. If O is bounded has C^∞-boundary, and $y\in C^\infty(\bar{O})$, A is an operator in O which is strictly elliptic on bounded sets (see e.g. [**43**]), then $u:=(I-\sigma A)^{-1}y$ is $u\in C^\infty$ in $\bar{O}$.

In fact, for each $R'>R>0$, we can consider two balls $B_{R'},B_R$ centered in a point of ∂O, and a smooth function θ that is 1 on B_R and 0 out of $B_{R'}$, and a bounded set with smooth boundary C s.t. $C\cap B_{R'}=O\cap B_{R'}$; hence, $v:=\theta u$ will be the solution of a Neumann problem

$$\begin{cases}\sigma Lv-v=g & \text{in } C\\ \nabla v\cdot\nu_{\partial O}=0 & \text{on } \partial C\end{cases}$$

(where $\nu_{\partial O}(x)$ is the normal vector to ∂O in x) for some g that is in $L^2(C,\mathscr{L}^d)$ (because $u\in W^{1,2}(O,\mathscr{L}^d)$) and L is strictly elliptic on C; therefore, $v\in W^{2,2}(O\cap B_{R'},\mathscr{L}^d)$ (e.g. by [**21**], 9, Rem. 24), hence $u\in W^{2,2}(O\cap B_R,\mathscr{L}^d)$ (and this for all $R>0$). By repeating the argument, we can find that $u\in W^{k,2}(O\cap B_R,\mathscr{L}^d)$ for all $k\in\mathbb{N}$ and $R>0$ (at each step, by knowing that $u\in W^{k,2}(O\cap B_R,\mathscr{L}^d)$ we can find that $g\in W^{k-1,2}(C,\mathscr{L}^d)$ and hence $u\in W^{k+1,2}(O\cap B_R,\mathscr{L}^d)$). So, u has a representative in $C^\infty(\bar{O})$ (see e.g. [**21**], Cor. 9.15).

In the same way, we can prove that, if O has C^∞-boundary in a neighbourhood B of a point $x_0\in\partial O$, and $y_{|\bar{O}\cap B}\in C^\infty(\bar{O}\cap B)$, L is an operator in O which is strictly elliptic on bounded sets (see e.g. [**43**]) and $u:=(I-\sigma L)^{-1}y$ we have that $u_{|\bar{O}\cap B}\in C^\infty(\bar{O}\cap B)$.

We Remark that, when L is strictly elliptic on all O (as in the case of Laplace operator), the passages are simpler (e.g. by [**21**], Rem. 24); we will use the case of L locally strictly elliptic in the next section.

We prove, in some steps, that

$$\int_O |\nabla J_\sigma(y)|(x)\,dx\le\int_O |\nabla y|(x)\,dx.$$

LEMMA 7.4.2. *Let O be an open convex set s.t. ∂O is $C^2(O)$, if $u\in C^1(\bar{O})$, and ∇u in x_0 is orthogonal to $\nu_{\partial O}$ (orthogonal to ∂O), and $\varphi_\varepsilon(\xi)=\sqrt{\varepsilon^2+|\nabla u(\xi)|^2}$, then $\nu_{\partial O}\cdot\nabla\varphi_\varepsilon\le 0$ in each regular point x_0 of ∂O.*

PROOF. We have, for 7.1.2

$$\nu_{\partial O}\cdot\nabla\varphi_\varepsilon=\frac{D^2u(\nabla u,\nu_{\partial O}}{\varphi_\varepsilon})$$

where D^2u is the Hessian operator of u, considered as bilinear operator in $\mathbb{R}^d$.

Now, we observe that $\nabla u\cdot\nu_O=0$ (i.e. ∇u is tangent to ∂O in each point) because $u\in D(\Delta_N)$; ∂O is regular, hence we can consider it as the sublevel of a smooth function $g\in C^\infty(\mathbb{R}^d)$, s.t. $O=g^{-1}((-\infty,0))$, $\partial O=g^{-1}(0)$ and $\nu_{\partial O}=\frac{\nabla g}{\|\nabla g\|}$ on ∂O. If we consider the tangent plane T_p in a point $p\in\partial O$, we will have that $g\ge 0$ on the plane T_p, and $g(p)=0$, so p is a local minimum point on T_p for g; hence, the Hessian D^2g in p restricted to the vectors of T_p is positive semidefinite.

Now on ∂O we have

$$\langle\nabla u,\nabla g\rangle=\langle\nabla u,\nu_{\partial O}\rangle=0$$

because ∇u is on the tangent; we can consider $\langle \nabla u, \nabla g\rangle$ as a function on O which is constant on ∂O, hence $\nabla(\langle \nabla u, \nabla g\rangle)$ is normal to ∂O in each point of ∂O; so, recalling that ∇u is tangent to ∂O,

$$0 = \nabla(\langle \nabla u, \nabla g\rangle) \cdot \nabla u = D^2 u \nabla g \cdot \nabla u + D^2 g \nabla u \cdot \nabla u; \tag{7.4.1}$$

$D^2 g$ in p restricted to the vectors of T_p will be positive semidefinite (because of the convexity of O) so

$$D^2 g \nabla u \cdot \nabla u \geq 0,$$

due to ∇u is tangent to ∂O, hence, applying (7.4.1).

$$D^2 u \nabla u \cdot \nu_{\partial O} = D^2 u \nabla u \cdot \nabla g = -D^2 g \nabla u \cdot \nabla u \leq 0,$$

and we have concluded. □

LEMMA 7.4.3. *If $O \subseteq \mathbb{R}^d$ is convex, open and bounded set with C^∞ boundary, then*

$$\int_O |\nabla J_\sigma(y)|(x)\, dx \leq \int_O |\nabla y|(x)\, dx$$

for all $y \in W^{1,1}(O, \mathscr{L}^d) \cap L^2(O, \mathscr{L}^d)$.

PROOF. We recall the lower semicontinuity of the functional $\int_O |\nabla \cdot|(x)\, dx$ with respect to the norm L^1, and that J_σ is a bounded operator $L^2 \to L^2$.

Let $y \in W^{1,1}(O, \mathscr{L}^d) \cap L^2(O, \mathscr{L}^d)$. It is clear, by truncation, that $y \wedge n \vee -n$ converges to y both in $W^{1,1}(O, \mathscr{L}^d)$ and in $L^2(O, \mathscr{L}^d)$. So, it is not restrictive to suppose $y \in W^{1,1}(O, \mathscr{L}^d) \cap L^\infty(O, \mathscr{L}^d)$; now, we know that y can be extended in $W^{1,1}(X)$ (because O is convex) and this extension can be truncated to the same $\|\cdot\|_{L^\infty}$ norm of y; the extension can be approximated by convolutions (which converges both in $W^{1,1}$and in L^2): so, it is not restrictive to suppose that y are restrictions of functions in $C^\infty(\mathbb{R}^d)$.

So, let y be smooth in $\bar{O}$: hence we can apply Lemma 7.1.2; moreover, by Remark 7.4.1 we have in this case $u \in \bar{O}$.

Arguing as in ([**12**], 8) we can suppose that $\sigma = 1$, and we introduce φ and $\varphi_\varepsilon \in C^1(O)$ (with $\varepsilon > 0$) in a similar way

$$\varphi(\xi) = |\nabla u(\xi)|, \quad \varphi_\varepsilon(\xi) = \sqrt{\varepsilon^2 + |\nabla u(\xi)|^2},$$

and we can prove

$$\frac{\varphi^2}{\varphi_\varepsilon} - \Delta \varphi_\varepsilon \leq |\nabla y|$$

exactly in the same way of ([**12**], Lemma 8.2).

If we integrate, we have

$$\int_O \frac{\varphi^2}{\varphi_\varepsilon}(x)\, dx - \int_O \Delta \varphi_\varepsilon(x)\, dx =$$

$$= \int_O \frac{\varphi^2}{\varphi_\varepsilon}(x)\, dx - \int_{\partial O} \nu_{\partial O} \cdot \nabla \varphi_\varepsilon\, dS \leq \int_O |\nabla y|(x)\, dx;$$

by Lemma 7.4.2, we have $\nu_{\partial O} \cdot \nabla \varphi_\varepsilon \leq 0$ in each regular point on ∂O, hence

$$\int_O \frac{\varphi^2}{\varphi_\varepsilon}(x)\, dx \leq \int_O |\nabla y|(x)\, dx,$$

so, due to the convergence $\frac{\varphi^2}{\varphi_\varepsilon} \to \nabla u$, we can conclude. □

In the next result, by L in $O \subseteq \mathbb{R}^d$ we will denote the operator s.t.:

$$D(L) = \{f | f_{|O} \in D(\Delta_N)\}$$

and $L(f) = \Delta_N(f_{|O})$ extended to 0 out of O, where Δ_N is the Laplace operator with Neumann null boundary conditions in O.

THEOREM 7.4.4. *If $O \subseteq \mathbb{R}^d$ is open, bounded and convex, if $J_\sigma = (I - \sigma\Delta_N)^{-1}$, then*

$$\int_O |\nabla J_\sigma(y)|(x)\, dx \le \int_O |\nabla y|(x)\, dx$$

for all $y \in W^{1,1}(O, \mathscr{L}^d) \cap L^2(O, \mathscr{L}^d)$.

PROOF. We consider $y \in (W^{1,1} \cap L^2)(O, \mathscr{L}^d)$; by the convexity of O, and arguing as at the beginning of the proof of Lemma 7.4.3, we can suppose that y is $L^\infty(O)$, and we can extend it out of O.

Hence, we can suppose $y \in W^{1,1}(\mathbb{R}^d, \mathscr{L}^d) \cap L^\infty(\mathbb{R}^d, \mathscr{L}^d)$.

Clearly there exists a decreasing sequence $\{O_n\}_{n\in\mathbb{N}}$ of open bounded convex sets in $\mathbb{R}^d$ with C^∞ boundary s.t. $O \subseteq \bigcap_{n=1}^\infty O_n$ and $\mathscr{L}^d(O_n \backslash O) \to 0$. We consider L_O Laplace operator in O (on $L^2(\mathbb{R}^d, \mathscr{L}^d)$) and L_{O_n} Laplace operator in O_n (on $L^2(\mathbb{R}^d, \mathscr{L}^d)$), for $\sigma > 0$ we consider $J'_\sigma := (I - \sigma L)^{-1}$, $J'_{\sigma,n} := (I - \sigma L_n)^{-1}$ operators in L^2. We can apply Lemma 5.2.2, and we have a Mosco convergence of a_n (form associated to L_{O_n}) to a (form associated to L). It is clear that for all $f \in L^2(\mathbb{R}^d, \mathscr{L}^d)$, we have

$$u := J'_\sigma(y)_{|O} = J_\sigma(y_{|O})$$

and

$$u_n := J'_{\sigma,n}(y)_{|O_n} = J_{\sigma,n}(y)$$

where $J_\sigma := (I - \sigma\Delta_O)^{-1}$ in $L^2(O)$ and $J_{\sigma,n} := (I - \sigma\Delta_{O_n})^{-1}$ in $L^2(O_n, \mathscr{L}^d)$.

It is clear that $\sigma J'_{\sigma,n}$ is associated to a_n,σ where a_n is the Dirichlet form in $W^{1,2}(O_n)$, while $\sigma J'_\sigma$ is associated to a,σ where a is the Dirichlet form in $W_0^{1,2}(O_n, \mathscr{L}^d)$; hence, by using Theorem 5.1.3 and Mosco convergence of a_n to a we have that $J'_{\sigma,n}(y) \to J'_\sigma(y)$ in $L^2(\mathbb{R}^d, \mathscr{L}^d)$; so, each subsequence converges up to a subsequence, hence $J'_{\sigma,n}(y) \to J'_\sigma(y)$ in $L^2(\mathbb{R}^d, \mathscr{L}^d)$, and $u_{n|O} \to u$ in $L^2(O, \mathscr{L}^d)$

By Lemma 7.4.3 we have that

$$\|\nabla_H J'_{\sigma,n}(y_{|O_n})\|_{L^1(O_n,H)} \le \|\nabla_H y\|_{L^1(O_n,H)}$$

and hence

$$\limsup \|\nabla_H J'_{\sigma,n}(y_{|O_n})\|_{L^1(O_n,H)} \le \|\nabla_H y\|_{L^1(O,H)}. \tag{7.4.2}$$

(because $\mathscr{L}^d(O_n \backslash O) \to 0$).

We have that $u_{n|O} \to u$ in $L^2(O, \mathscr{L}^d)$. Now, the functional $\int_O |\nabla \cdot |(x)\, dx$ is lower semicontinuous with respect to L^2convergence, so

$$\int_O |\nabla_H u|_H\, d\gamma \le \int_O |\nabla_H y|_H\, d\gamma.$$

□

7.4.2. Case Neumann boundary conditions, Gaussian measure.

7.4.2.1. *Finite dimensional case.* In the case of the Gaussian measure (in finite dimension), and zero Neumann boundary conditions, the argument is similar to that in the above section: for $O \subseteq \mathbb{R}^d$ convex we define L as the Ornstein-Uhlenbeck operator with zero Neumann boundary conditions on O, and $J_\sigma = (I - \sigma L)^{-1}$.

We recall that, for $p \in [1, +\infty]$, we say $f \in W^{1,p}(O)$ (in the sense that $f \in W^{1,p}(O, \gamma^d)$) if $f \in W^{1,p}_{loc}(O, \gamma^d)$ and $f \in L^p(O)$, $\nabla f \in L^p(O, \gamma^d, \mathbb{R}^d)$. Clearly in this setting $W^{1,2} \subseteq W^{1,1}$ because the measure γ^d is finite.

LEMMA 7.4.5. *If $O \subseteq \mathbb{R}^d$ is convex with C^∞ boundary, then*

$$\int_O |\nabla J_\sigma(y)| \, d\gamma^d \leq \int_O |\nabla y| \, d\gamma^d$$

for all $y \in (W^{1,1} \cap L^2)(O)$.

PROOF. Because of the lower semicontinuity of the functional $\int_O |\nabla \cdot| \, d\gamma$ with respect to L^2 and of the density of $C^\infty(O)$ in $(W^{1,1} \cap L^2)(O)$ (by Corollary 3.2.24), it isn't restrictive to suppose that $y \in C^\infty$.

We recall the definitions of φ_ε in Section 7.2.1.

Remark 7.4.1 implies that $u \in C^\infty(\bar{O})$. Now, we have that ∇u is orthogonal to $\nu_{\partial O}$ because $u \in D(L_N)$; so we can apply Lemma 7.4.2 and we have $\nu_{\partial O} \cdot \nabla \varphi_\varepsilon \leq 0$ in each point of ∂O.

Now, we can apply Lemma 7.1.2, and we have

$$\frac{\varphi^2}{\varphi_\varepsilon} - \sigma L_N \varphi_\varepsilon \leq |\nabla y|,$$

integrating we obtain that

$$\int_O \frac{\varphi^2}{\varphi_\varepsilon} \, d\gamma^d - \int_O A\varphi_\varepsilon \, d\gamma^d =$$

$$= \int_O \frac{\varphi^2}{\varphi_\varepsilon} \, d\gamma^d - \int_O g_d(x) \Delta \varphi_\varepsilon(x) \, dx + \int_O g_d(x) nabla \varphi_\varepsilon(x) \cdot x \, dx =$$

(where $g_d(x) := (2\pi)^{-\frac{d}{2}} \exp \frac{\|x\|^2}{2}$)

$$= \int_O \frac{\varphi^2}{\varphi_\varepsilon} \, d\gamma^d - \int_{\partial O} \nu_{\partial O} \cdot \nabla \varphi_\varepsilon g_d \, dS \leq \int_O |\nabla y| \, d\gamma^d$$

where S is the area measure of ∂O; so

$$\int_O \frac{\varphi^2}{\varphi_\varepsilon} \, d\gamma^d \leq \int_O |\nabla y| \, d\gamma^d$$

and $\frac{\varphi^2}{\varphi_\varepsilon} \to \nabla u$ pointwise, therefore we can conclude by the Fatou Lemma. □

7.4.2.2. *Infinite dimensional case.* From now on, we will suppose X to be a generic Banach separable space,γ a Gaussian measure on X, H the Cameron-Martin space; as above, we will suppose that O is an open convex set in X.

In this setting, we can define the (Neumann) Ornstein-Uhlenbeck operator on the open convex O (see [**51**]).

We suppose $\dim(F_n) = n$ and $F_n =< h_1,\dots,h_n >$ where $\{h_j\}_{j\in\mathbb{N}}$ in $R_\gamma(X^*)$ is an orthonormal base of H, and we define π_n, projection from X in F_n; we define $\gamma_{F_n} = \gamma^n = \gamma\circ\pi_n^{-1}$, it is a non-degenerate Gaussian measure on F_n, and, if we identify F_n with $\mathbb{R}^n$ through the basis $\{h_1,\dots,h_n\}$, then γ^n is a standard Gaussian measure.

LEMMA 7.4.6. *If X is a Banach separable space, if $O \subseteq X$ is open convex cylindrical regular set, and J_σ is defined in the Neumann boundary conditions, then*

$$\int_O |\nabla_H J_\sigma(y)|_H \, d\gamma \le \int_O |\nabla_H y|_H \, d\gamma$$

for all $y \in (W^{1,1}\cap L^2)(O)$.

PROOF. We suppose that O is n-cylindrical. For $m \in \mathbb{N}$ s.t. $n \le m$, we can consider the space F_m with $F_n \le F_m$ and we can define L_n^m as the (Neumann) Ornstein-Uhlenbeck operator in

$$B_{n,m} = C_n \times \mathbb{R}^{m-n} \subseteq \mathbb{R}^m$$

(we can consider $B_{n,m} \subseteq F_m$) with the standard Gaussian measure γ^m; we have that, if $f \in W^{1,p}(B_{n,m})$ then $f\circ\pi_{F_m} \in W^{1,p}(O)$, and we have

$$(L_n^m f)\circ\pi_{F_m|O} = L_n(f\circ\pi_{F_{m|O}});$$

where L_n is the Ornstein-Uhlenbeck operator in O; we can deduce also, that, for every $\lambda > 0$,

$$(R(\lambda,-L_n^m)f)\circ\pi_{F_m|O} = R(\lambda,-L_n)\left(f\circ\pi_{F_m|O}\right); \tag{7.4.3}$$

so, by Lemma 7.4.5, we have that, if $J_{\sigma,n} := \sigma^{-1}R(\sigma^{-1},-L_n)$ then

$$\int_O |\nabla_H J_{\sigma,n}(y)|_H \, d\gamma \le \int_{O_n} |\nabla_H y|_H \, d\gamma$$

for all $y = f\circ\pi_{F_m}$ for some $f \in W^{1,p}(B_{n,m})$.

Now, given $y \in (W^{1,1}\cap L^2)(O)$, we consider for every $m \in \mathbb{N}$, $m > n$ the measure $\gamma_{F_m^\perp}$ (see Part I) and the function on $B_{n,m}$

$$\mathbb{E}_m(y)(x) := \int_{\pi_m^{-1}(x)} y \, d\gamma_{F_m^\perp}$$

that is well defined for γ^m-almost every point of $B_{n,m} \subseteq F_m$ (by identifying F_m with $\mathbb{R}^m$); we have also $\mathbb{E}_m(y) \in W^{1,p}(B_{n,m})$, and if

$$y_m := \mathbb{E}_m(y)\circ\pi_{F_m},$$

then f_m converges to f in $W^{1,p}(O)$ (so also in $L\log^{\frac{1}{2}} L$); for what we said we have

$$\int_{O_n} |\nabla_H J_{\sigma,n}(y_m)|_H \, d\gamma \le \int_{O_n} |\nabla_H y_m|_H \, d\gamma,$$

so, also by the lower semicontinuity of $\int_{O_n} |\nabla_H \cdot|_H \, d\gamma$ with respect to $L\log^{\frac{1}{2}} L$, we can conclude. $\square$

Using the Mosco convergence (see Subsection 5) we can prove this Proposition, which will be extended in Theorem 7.5.11.

THEOREM 7.4.7. *If X is a Banach separable space, if $O \subseteq X$ is open and convex, and J_σ is defined using the zero Neumann boundary condition, then*

$$\int_O |\nabla_H J_\sigma(y)|_H \, d\gamma \leq \int_O |\nabla_H y|_H \, d\gamma$$

for all $y \in (W^{1,1} \cap L^2)(O)$.

PROOF. We recall, by convexity, that $W^{1,1} = W^{1,1}_*$ (see Part I). By Proposition 3.2.23, if $y \in W^{1,1}_*(O) \cap L^p$, there exists a sequence of Lipschitz functions which converges to y both in $W^{1,1}_*(O)$ and in $L^p(O)$; by recalling that J_σ is bounded from L^p to L^p, that L^p is embedded in $L\log^{\frac{1}{2}} L$, and Remark 7.1.1, we have that it if we prove the statement for y Lipschitz, we can conclude.

Now, each $y \in \mathrm{Lip}(O)$ has a Lipschitz extension on X; we will consider one of such extensions: from now on, we will call this extension y (in particular, $y \in W^{1,2}(X)$). u will be $J_\sigma(y_{|O})$.

O is convex, so, $C_n = \pi_n(O)$ is a convex set and $O'_n := \pi_n^{-1}(C_n)$ is a convex n-cylinder (not regular); O'_n is a decreasing sequence of open sets containing O; moreover, if $x \in \bigcap_{n=1}^\infty O'_n \backslash O$ then $x \in \partial O$: by the convexity of O and the density of H, if $x \notin \bar{O}$ there exists $n \in \mathbb{N}$ s.t. $\sup\{\hat{h}_n(x - x_0) | x_0 \in O\} > 0$, and hence $x \notin O'_n$. So, since $\gamma(\partial O) = 0$ (by Proposition 4.1.5) we have that $\gamma(O'_n \backslash O) \to 0$; now, for each C_n, there exists B_n s.t. $C_n \subseteq B_n$, B_n is convex with C^∞ boundary and $\gamma^n(B_n \backslash C_n) \leq n^{-1}$ (see [**51**], Prop. A.4).

Let $O_n := \pi_n^{-1}(B_n)$, we have that $O \subseteq O_n$ for every $n \in \mathbb{N}$ and $\gamma(O_n \backslash O) \to 0$.

Now, if L_n is the Ornstein-Uhlenbeck operator with zero Neumann boundary condition in O_n and $J_{\sigma,n} := (I - \sigma L_n)^{-1}$ is the operator in $L^2(O_n)$ defined in the Neumann boundary conditions, by Lemma 7.4.6 we have that

$$\int_{O_n} |\nabla_H J_{\sigma,n}(y)|_H \, d\gamma \leq \int_{O_n} |\nabla_H y|_H \, d\gamma;$$

let a_n the Dirichlet form in $W^{1,2}(O_n)$ and a the Dirichlet form in $W^{1,2}(O)$ (see Subsection 5), by Lemma 5.2.4 we have that a_n converges to a in the sense of Mosco; so, by Lemma 5.2.4, $J_{\sigma,n}(y)_{|O}$ converges to $J_\sigma(y)$ in $L^2(O)$, and so also in $L\log^{\frac{1}{2}} L$; so by the lower semicontinuity of $\int_{O_n} |\nabla_H \cdot|_H \, d\gamma$ in $L\log^{\frac{1}{2}} L$ we have

$$\int_O |\nabla_H J_\sigma(y)|_H \, d\gamma \leq \liminf_{n\to\infty} \int_O |\nabla_H J_{\sigma,n}(y)|_H \, d\gamma \leq \liminf_{n\to\infty} \int_{O_n} |\nabla_H J_{\sigma,n}(y)|_H \, d\gamma \leq$$

$$\leq \liminf_{n\to\infty} \int_{O_n} |\nabla_H y|_H \, d\gamma = \int_O |\nabla_H y|_H \, d\gamma.$$

and we can conclude. □

REMARK 7.4.8. In the finite dimensional case, the above theorem is an extension of Lemma 7.4.5 to the case of a convex non-regular set.

7.5. BV functions and resolvent contractivity

7.5.1. $BV(O)$ **and approximating sequence.** We will consider the Neumann case (section 7.4.2), with O open convex set. In this Subsection, for a function f on X, we will denote the set $f^{-1}(\mathbb{R}^+)$ with $\mathrm{supp}(f)$, and we will call it support of f.

Now, we recall that let $u \in L\log^{\frac{1}{2}} L(X)$ by Theorem 4.1.3, u is $BV(X)$ with total variation $L(u)$ if and only if the quantity

$$L(u) = \inf\left\{ \liminf_{h\to\infty} \int_X |\nabla_H u_n|_H \, d\gamma | \{u_n\}_{n\in\mathbb{N}} \in \mathrm{Lip}(X),\ u_h \xrightarrow{L^1} u \right\} < \infty,$$

and if and only if

$$L(u) = \lim_{t\to 0} \int_X |\nabla_H T_t u|_H \, d\gamma < \infty.$$

($(T_t)_{t\geq 0}$ denotes the Ornstein-Uhlenbeck semigroup in X); our goal is to find a version of this result for $BV(O)\cap L^2(O)$.

Our first step will be to prove that, if $u \in BV(O)\cap L^2(O)$ then there exists a sequence of functions u_n in $W^{1,1}(O)\cap L^2(O)$ which converges to u in $L^2(O)$ s.t. $\int_X |\nabla_H u_n|_H \, d\gamma$ converges to the total variation of u.

We recall that a covering is locally finite if each point has a neighbourhood which intersects only a finite number of elements of the covering.

DEFINITION 7.5.1. Given an open set $O \subseteq X$, given an open covering $\{U_\alpha\}$ of O, given W linear space of real valued functions on O, we will say that a set of functions $\{\psi_i\}$ in W is a partition of unity of class W subordinated to $\{U_\alpha\}$ if

i) $\psi_i \geq 0$ for all i, and $\sum_i \psi_i(x) = 1$,
ii) there exists a locally finite open covering $\{V_i\}$ of X s.t. each V_i is contained in some U_α, and $\mathrm{supp}(\psi_i) \subseteq V_i$.

We will say that O admits partition of unity of class W, if, for all open covering $\{U_\alpha\}$ of O, there exists a partition of unity of class W subordinated to $\{U_\alpha\}$.

REMARK 7.5.2. O is metric and separable, so it is second countable, and then it has the Lindelöf property: i.e. each open covering has a countable subcovering. So, it is not restrictive to suppose that $\{U_\alpha\}$, $\{V_i\}$, $\{\psi_i\}$ are countable.

We observe that the above definition implies that $\psi_i \leq 1$ for all i.

We have this result of Albeverio, Ma and Röckner ([**2**], Cor. 1.4).

LEMMA 7.5.3. *Let X be a separable metric space, let W be a linear space of real-valued functions on X; moreover, let us assume that the following conditions are satisfied:*

i) *for each $f \in W$, if $\psi \in C_b^\infty$ with $\psi(0) = 0$ then $\psi \circ f \in W$;*
i) *given two open sets A_1, A_2 of X s.t. dist$(A_1, A_2) > 0$, there is a positive element of W that is greater than 1 on A_1 and 0 on A_2.*

Then, X admits partition of unity of class W.

We can deduce this Lemma.

LEMMA 7.5.4. *If X is a separable metric space, if W is the set of bounded Lipschitz functions on O, then O admits partition of unity of class W.*

PROOF. We will use Lemma 7.5.3: we have to prove that W satisfies two conditions: the first one is that, for each $f \in W$, if $\psi \in C_b^\infty$ then $\psi \circ f \in W$, and this is clearly satisfied because ψ is Lipschitz and bounded on $f(O)$ that is a bounded set.

The second condition is: given two open sets A_1, A_2 s.t. $\mathrm{dist}(A_1, A_2) > 0$, there is a positive element of W that is greater than 1 on A_1 and 0 on A_2. This condition is satisfied, for instance by considering it suffices to consider the function

$$f(x) := \frac{\mathrm{dist}(x, A_2)}{\mathrm{dist}(x, A_1) + \mathrm{dist}(x, A_2)};$$

it is bounded and Lipschitz with constant $\mathrm{dist}(A_1, A_2)^{-1}$. □

To proceed, we will need to prove that a BV function can be approximated in a suitable way by $W^{1,2}$ functions.

REMARK 7.5.5. We recall that the Ornstein-Uhlenbeck operator L with zero Neumann boundary conditions is the generator of the Ornstein-Uhlenbeck semigroup that is sub-Markovian (see Subsection 3.4.3).

If O is a convex open set, we fix a point $x_0 \in O$ and we consider, for each r s.t. $0 < r \leq 1$, the shrinking o_r centered in x_0

$$o_r(x) := r(x - x_0) + x_0$$

and the set $O_r := o_r(O)$, that is clearly an open convex set.

We will need a technical result.

LEMMA 7.5.6. *If $O \neq X$, then*

$$dist(\partial O_{r_1}, \partial O_{r_2}) \geq |r_1 - r_2| d$$

where $d := dist(x_0, \partial O_{r_1}) \wedge dist(x_0, \partial O_{r_2})$.

PROOF. By the geometric properties of the functions o_r, it is not restrictive to suppose $r_1 = 1$, we will use r instead to r_2.

Clearly here $d := \mathrm{dist}(x_0, \partial O_r)$; given a plane π s.t. $x_0 \in \pi$, the sets $O_r \cap \pi$ and $O \cap \pi$ are open in O and convex and $\mathrm{dist}(x_0, \pi \backslash O_r) > d$. We have that

$$\mathrm{dist}(\partial O_r, \partial O) = \inf\{\mathrm{dist}(\pi \backslash O, O_r \cap \pi) | \pi \text{ plane s.t. } x_0 \in \pi\} \tag{7.5.1}$$

(it suffices to consider, for each couple (x, y) with $x \in \partial O_r$ and $y \in \partial O$, a plane through x_0, x and y).

So, we can consider the bidimensional case: let $X = \mathbb{R}^2$, O a convex open, $x_0 \in O$ and $O_r = o_r(O)$, we can prove that

$$\mathrm{dist}(x, \partial O) \geq (1 - r)\mathrm{dist}(x_0, \pi \backslash O_r) \geq (1 - r)d$$

and we conclude.

In fact, given $x \in \partial O_r$, clearly there exists $y \in \partial O$ s.t. $\|x - y\|_X = \mathrm{dist}(x, \partial O)$ (by using the local compactness of $\mathbb{R}^2$); for such a y, there must be only a tangent line t to O in y, and t must be orthogonal to $y - x$; then

$$\mathrm{dist}(x, y) = \mathrm{dist}(x, t);$$

we consider $t_r = o_r(t)$, then t_r is parallel to t and tangent to O_r, and it separates x and y, hence

$$\begin{aligned} \mathrm{dist}(x, t) \geq \mathrm{dist}(t_r, t) = (1 - r)\mathrm{dist}(x_0, t') \geq \\ \geq (1 - r)\mathrm{dist}(x_0, \pi \backslash O_r) \end{aligned}$$

so

$$\mathrm{dist}(x, \partial O) \geq (1 - r)\mathrm{dist}(x_0, \pi \backslash O_r).$$

□

We also recall this Remark.

REMARK 7.5.7. If $f \in W^{1,p}(O)$, if g is bounded and Lipschitz and $\mathrm{supp}(g) \subseteq O$, then the function

$$l(x) := \begin{cases} f(x)g(x) & \text{if } x \in O \\ 0 & \text{otherwise} \end{cases}$$

is in $W^{1,p}(X)$ (it can be seen by density).

We also recall this technical calculations.

LEMMA 7.5.8. *Let X be a normed vector, μ be a positive measure on some space Ω, $f,g \in L^1(\Omega,\mu,X)$ and $c \in \mathbb{R}$; then*

$$\int_\Omega \|f+g\|_X \, d\mu + c \leq \int_\Omega \|f\|_X \, d\mu + \int_\Omega \|g\|_X \, d\mu + c \leq$$

$$\leq \int_\Omega \|f\|_X \, d\mu + |\int_\Omega \|g\|_X \, d\mu + c|. \tag{7.5.2}$$

PROOF. By the triangular inequality applied to $L^1(\Omega,\mu,X)$,

$$\int_\Omega \|-g\|_X \, d\mu \leq \int_\Omega \|-f-g\|_X \, d\mu + \int_\Omega \|f\|_X \, d\mu$$

so

$$-\int_\Omega \|-f-g\|_X \, d\mu - c \leq \int_\Omega \|f\|_X \, d\mu - \int_\Omega \|-g\|_X \, d\mu - c \leq$$

$$\leq \int_\Omega \|f\|_X \, d\mu + |\int_\Omega \|g\|_X \, d\mu + c|. \tag{7.5.3}$$

Hence, by (7.5.2), (7.5.3)

$$|\int_\Omega \|f+g\|_X \, d\mu + c| \leq \int_\Omega \|f\|_X \, d\mu + |\int_\Omega \|g\|_X \, d\mu + c|. \tag{7.5.4}$$

□

PROPOSITION 7.5.9. *If O is a convex open set, if $f \in L^2(O) \cap BV(O)$, then there exists a sequence $f_n \in (W^{1,1}_* \cap L^2)(O)$ s.t. $f_n \to f$ in $L^2(O)$ and $\int_O |\nabla_H f_n|_H \, d\gamma$ converges to the total variation of f in O.*

PROOF. Hereafter, T_t will be the Ornstein-Uhlenbeck semigroup in X.

We recall that, if $f \in L^2(X)$, then $T_t f \xrightarrow[t\to 0]{L^2(X)} f$ and, by Corollary 4.2.25 (clearly it can be applied), if $f \in BV(X)$ then $|\nabla_H T_t f|_H \gamma$ weakly converges to $|D_\gamma f|$ as a measure.

If $O = X$, we know that $T_t f \in W^{1,2}(X)$, so for $t_n \to 0$ we can use $T_{t_n} f$ to approximate f, and we conclude.

Hereafter we suppose $O \neq X$.

Let $f \in L^2(O) \cap BV(O)$.

We fix a point $x_0 \in O$ and we consider, for each r s.t. $0 < r < 1$, the set O_r, defined above, that is clearly an open convex set; for each r we can define the function l_r,

$$l_r(x) := \frac{\mathrm{dist}(x, X \backslash O)}{\mathrm{dist}(x, X \backslash O) + \mathrm{dist}(x, O_r)},$$

we have that l_r is bounded and Lipschitz with constant $\mathrm{dist}(O_r, X\backslash O)^{-1}$ which is finite by Lemma 7.5.6; for every x, $0 \le l_r(x) \le 1$, moreover $l_{r|O_r} \equiv 1$, $l_{r|X\backslash O} \equiv 0$.

We have that $f_{|O_r}$ can be extended to a function $f_r \in BV(X)$ in this way:

$$f_r(x) := \begin{cases} f(x)l_r(x) & \text{if } x \in O \\ 0 & \text{otherwise} \end{cases},$$

clearly $f_r \in BV(X)$ and by Lemma 4.2.12

$$D_\gamma f_r = (D_\gamma f)l_r + f(\nabla_H l_r)\gamma \tag{7.5.5}$$

(clearly $l_r \in W^{1,p}(O)$ for all $p > 1$ and $|\nabla_H l_r|_H$ is bounded because l_r it is Lipschitz) and clearly $f_{r|O_r} = f_{|O_r}$; by (7.5.5) we have that

$$D_\gamma f_{r|O_r} = D_\gamma f_{|O_r} \tag{7.5.6}$$

and,

$$|D_\gamma f_r|(\partial O_r) = |D_\gamma f|(\partial O_r). \tag{7.5.7}$$

If $r_1 \neq r_2$ then $\partial O_{r_1} \cap \partial O_{r_2} = \varnothing$ by Lemma 7.5.6, and $|D_\gamma f|$ is a bounded measure: so $|D_\gamma f|(\partial O_r) = 0$ for all $r \in (0,1)$ but a countable subset.

Let $r \in (0,1)$ s.t. $|D_\gamma f|(\partial O_r) = 0$. We define $f_{r,t} := T_t(f_r)$ is $W^{1,2}(O)$, clearly $f_{r,t} \xrightarrow[t\to 0^+]{L^2(O_r)} f$ (hence $f_{r,t} \xrightarrow[t\to 0^+]{L^1(O_r)} f$ because γ is a probability measure) and by Corollary 4.2.25 we have the weak convergence

$$|\nabla_H f_{r,t}|_H \gamma \rightharpoonup^* |D_\gamma f_r|$$

for $t \to 0^+$, hence

$$\int_O \psi |\nabla_H f_{r,t}|_H \, d\gamma \xrightarrow[t\to 0^+]{} \int_O \psi \, d|D_\gamma f_r| \tag{7.5.8}$$

for all ψ continuous bounded functions, and, since $|D_\gamma f|(\partial O_r) = 0$ and (7.5.5) we can deduce

$$\int_{O_r} \psi |\nabla_H f_{r,t}|_H \, d\gamma \to \int_{O_r} \psi \, d|D_\gamma f_r|. \tag{7.5.9}$$

Let $\{r_i\}_{i\in\mathbb{N}}$ be an increasing sequence of positive numbers s.t. $r_i \xrightarrow[i\to\infty]{} 1$ and $|D_\gamma f|(\partial O_{r_i}) = 0$ for every $i \in \mathbb{N}$. We define $U_1 := O_{r_1}$, $U_2 := O_{r_2}$ and $U_i := O_{r_i} \backslash \overline{O_{r_{i-2}}}$ for $i \in \mathbb{N}$, $i > 2$ (we recall that $\overline{O_{r_{i-1}}} \subseteq O_{r_{i+1}}$ because O is open and convex), we have that $\{U_i\}_{i\in\mathbb{N}}$ is an open covering of O, that $U_i \cap U_j = \varnothing$ if $|i-j| > 1$, and $U_i \subseteq O_{r_i}$.

By Lemma 7.5.4, there exists a partition of the unity $\{\psi_i\}_{i\in\mathbb{N}}$ (i.e. $\psi_i(x) \in [0,1]$ and $\sum_{i=1}^\infty \psi_i = 1$ on O) s.t. each ψ_i is Lipschitz and it has support contained in U_i for every $i \in \mathbb{N}$.

Let $\varepsilon > 0$.

There exists $i_\varepsilon \in \mathbb{N}$ s.t.

$$|D_\gamma f|(O\backslash O_{r_i}) \le \varepsilon, \tag{7.5.10}$$

for all $i \ge i_\varepsilon - 1$.

By the convergence of $f_{r_{i_\varepsilon},t}$ in L^2 for $t \to 0$, by (7.5.9) and the convergence in L^2 of T_t there exists t_ε s.t.

$$|\int_{O_{r_{i_\varepsilon}}} f^2_{r_{i_\varepsilon},t_\varepsilon}\, d\gamma - \int_{O_{r_{i_\varepsilon}}} f^2\, d\gamma| \le \varepsilon, \tag{7.5.11}$$

$$\|f_{r_{i_\varepsilon},t_\varepsilon} - f\|_{L^1(O_{r_{i_\varepsilon}})} \le 2^{-i}\varepsilon(\|\nabla_H \psi_i\|_{L^\infty(X,H)}+1)^{-1}, \tag{7.5.12}$$

$$|\int_{O_{r_{(i_\varepsilon-1)}}} |\nabla_H f_{r_{i_\varepsilon},t_\varepsilon}|_H\, d\gamma - |D_\gamma f|(O_{r_{i_\varepsilon}})| \le \varepsilon \tag{7.5.13}$$

and

$$|\int_O \psi_i |\nabla_H f_{r_{i_\varepsilon},t_\varepsilon}|_H\, d\gamma - \int_O \psi_i\, d|D_\gamma f_{r_i}|| \le 2^{-i}\varepsilon \tag{7.5.14}$$

for all i with $0 < i \le i_\varepsilon$ (because $\{1,\dots,i_\varepsilon\}$ is finite); clearly by (7.5.14) we can deduce

$$|\int_O \psi_i |\nabla_H f_{r_{i_\varepsilon},t_\varepsilon}|_H\, d\gamma - \int_O \psi_i\, d|D_\gamma f|| \le 2^{-i}\varepsilon \tag{7.5.15}$$

for all i with $0 < i \le i_\varepsilon$ because ψ_i has support in $U_i \subset O_{r_{i_\varepsilon}}$, and because $D_\gamma f_{r|O_r} = D_\gamma f_{|O_r}$ (7.5.6).

For each $i \in \mathbb{N}$, $i > i_\varepsilon$ there exists $t_{\varepsilon,i} > 0$ s.t.:

$$\|f_{r_i,t_{\varepsilon,i}} - f\|_{L^2(O_{r_i})} \le 2^{-i}\varepsilon, \tag{7.5.16}$$

$$\|f_{r_i,t_{\varepsilon,i}} - f\|_{L^1(O_{r_i})} \le 2^{-i}\varepsilon(\|\nabla_H \psi_i\|_{L^\infty(X,H)}+1)^{-1}, \tag{7.5.17}$$

and, by (7.5.9) and the fact that ψ_i has support in O_{r_i}, we can also suppose that

$$|\int_O \psi_i |\nabla_H f_{r_i,t_{\varepsilon,i}}|_H\, d\gamma - \int_O \psi_i\, d|D_\gamma f_{r_i}|| \le 2^{-i}\varepsilon; \tag{7.5.18}$$

this last one implies

$$|\int_O \psi_i |\nabla_H f_{r_i,t_{\varepsilon,i}}|_H\, d\gamma - \int_O \psi_i\, d|D_\gamma f|| \le 2^{-i}\varepsilon \tag{7.5.19}$$

because ψ_i has support in $U_i \subset O_{r_i}$, and (7.5.6).

For each $i \in \mathbb{N}$, $\varepsilon > 0$, we can choose such a $t_{\varepsilon,i}$ and we define on O,

$$f_{\varepsilon,i}(x) := \begin{cases} \psi_i(x) f_{r_{i_\varepsilon},t_\varepsilon}(x) & \text{if } i \le i_\varepsilon \\ \psi_i(x) f_{r_i,t_{\varepsilon,i}}(x) & \text{if } i > i_\varepsilon \end{cases},$$

clearly $f_{\varepsilon,i} \in W^{1,2}(X)$ and $f_{\varepsilon,i}$ has support in U_i.

We define on O

$$f_\varepsilon := \sum_{i=1}^{\infty} f_{\varepsilon,i}$$

(it is well defined because ψ_i has support in U_i and U_i meets only U_{i+1} and U_{i-1}), clearly we have

$$f_{\varepsilon|O_{r_{(i_\varepsilon-1)}}} \equiv f_{r_{i_\varepsilon},t_\varepsilon|O_{r_{(i_\varepsilon-1)}}}. \tag{7.5.20}$$

As usually, given $h \in R_\gamma(X^))$, we define the set O_y and the function $f_{\varepsilon,y}$ on O_y.

We have that $f_{\varepsilon|O_{r_i}} \in W^{1,2}_*(O_{r_i})$; we recall that $W^{1,2}(X) = W^{1,2}_*(X)$ (see Proposition 3.2.20). Hence, for every $h \in R_\gamma(X^*)$ we have $f_\varepsilon \in D_h^{O_{r_i}}$; this means that, for every $h \in R_\gamma(X^*)$, for $\gamma_{h^\perp}$-almost every $y \in h^\perp$, the function $f_{\varepsilon,y}$ on the section $(O_{r_i})_y$ has γ_1-representative $\tilde{f}_y$ that is locally absolutely continuous (see Definition 3.2.13); now, by considering the countable sequence r_i, we have that for $\gamma_{h^\perp}$-almost every $y \in X_h^\perp$, the function f_y on the section O_y has γ_1-representative $\tilde{f}_y$ that is locally absolutely continuous, therefore $f \in D_h^{O_{r_i}}$ for all $h \in R_\gamma(X^*)$. $\nabla_H f_\varepsilon$ is defined on every U_i as $\nabla_H f_{\varepsilon,i-1} + \nabla_H f_{\varepsilon,i} + \nabla_H f_{\varepsilon,i+1}$; in this sense it is well defined on O.

Now, to prove that $f \in W^{1,1}_*(O_{r_i})$, we need only the finiteness of $\int_O |f_\varepsilon|\, d\gamma$ and of $\int_O |\nabla_H f_\varepsilon|_H\, d\gamma$

We recall that in O we can also write

$$f = \sum_{i=1}^{+\infty} \psi_i f$$

(because $\{\psi_i\}_{i\in\mathbb{N}}$ is a partition of the unity in O).

We have

$$\|f_\varepsilon - f\|_{L^2(O)} = \left\| \sum_{i=1}^{+\infty} f_{\varepsilon,i} - \sum_{i=1}^{+\infty} \psi_i f \right\|_{L^2(O)} \le$$

$$\le \left\| \sum_{i=1}^{i_\varepsilon} \psi_i f_{r_{i_\varepsilon},t_\varepsilon} - \sum_{i=1}^{i_\varepsilon} \psi_i f \right\|_{L^2(O)} + \left\| \sum_{i=i_\varepsilon+1}^{+\infty} \psi_i f_{r_i,t_{\varepsilon,i}} - \sum_{i=i_\varepsilon+1}^{+\infty} \psi_i f \right\|_{L^2(O)} \le$$

(by (7.5.20) and recalling that ψ_i has support in $U_i \subset O_{r_i}$)

$$\le \left\| \sum_{i=1}^{i_\varepsilon} \psi_i f_{r_{i_\varepsilon},t_\varepsilon} - \sum_{i=1}^{i_\varepsilon} \psi_i f \right\|_{L^2(O_{r_{i_\varepsilon}})} + \sum_{i=i_\varepsilon+1}^{+\infty} \left\| \psi_i f_{r_i,t_{\varepsilon,i}} - \psi_i f \right\|_{L^2(U_i)}$$

$$\le \left\| f_{r_{i_\varepsilon},t_\varepsilon} - f \right\|_{L^2(O_{r_{(i_\varepsilon-1)}})} + \sum_{i=i_\varepsilon+1}^{+\infty} \left\| f_{r_i,t_{\varepsilon,i}} - f \right\|_{L^2(O_{r_i})} \le$$

(by (7.5.16) and (7.5.11))

$$\le \sum_{i=1}^{+\infty} \varepsilon 2^{-i} + \varepsilon = 2\varepsilon.$$

We recall that

$$\sum_{i=1}^{+\infty} \nabla_H \psi_i \equiv 0;$$

(in each point, $\nabla_H \psi_i = 0$ for all $i \in \mathbb{N}$ but two) hence

$$\int_O \Big| \sum_{i=i_\varepsilon+1}^{+\infty} \nabla_H \psi_i f_{r_i,t_{\varepsilon,i}} + \sum_{i=1}^{i_\varepsilon} \nabla_H \psi_i f_{r_{i_\varepsilon},t_\varepsilon} \Big|_H\, d\gamma \le$$

$$\le \int_O \Big| \sum_{i=i_\varepsilon+1}^{+\infty} \nabla_H \psi_i (f_{r_i,t_{\varepsilon,i}} - f) + \sum_{i=1}^{i_\varepsilon} \nabla_H \psi_i (f_{r_{i_\varepsilon},t_\varepsilon} - f) \Big|_H\, d\gamma \le$$

(recalling that $\nabla \psi_i$ ha support in U_i)

$$\le \sum_{i=i_\varepsilon+1}^{+\infty} \int_{U_i} |\nabla_H \psi_i|_H |f_{r_i,t_{\varepsilon,i}} - f|\, d\gamma + \sum_{i=1}^{i_\varepsilon} \int_{U_i} |\nabla_H \psi_i|_H |f_{r_{i_\varepsilon},t_\varepsilon} - f|\, d\gamma \le$$

$$\leq \sum_{i=i_\varepsilon}^{+\infty} \|\nabla_H \psi_i\|_{L^\infty(X,H)} \|f_{r_i,t_{\varepsilon,i}} - f\|_{L^1(O_{r_i})} + \sum_{i=1}^{i_\varepsilon} \|\nabla_H \psi_i\|_{L^\infty(X,H)} \|f_{r_{i_\varepsilon},t_\varepsilon} - f\|_{L^1(O_{r_i})} \leq$$

(by (7.5.17) and (7.5.12))

$$\leq \sum_{i=1}^{+\infty} \varepsilon 2^{-i} = \varepsilon;$$

we have found then that

$$\int_O |\sum_{i=i_\varepsilon+1}^{+\infty} \nabla_H \psi_i f_{r_i,t_{\varepsilon,i}} + \sum_{i=1}^{i_\varepsilon} \nabla_H \psi_i f_{r_{i_\varepsilon},t_\varepsilon}|_H \, d\gamma \leq \varepsilon. \tag{7.5.21}$$

So we have that, in our hypotheses,

$$|\int_O |\nabla_H f_\varepsilon|_H \, d\gamma - |D_\gamma f|(O)| =$$

$$= |\int_O |\sum_{i=i_\varepsilon+1}^{+\infty} \nabla_H \psi_i f_{r_i,t_{\varepsilon,i}} + \sum_{i=1}^{i_\varepsilon} \nabla_H \psi_i f_{r_{i_\varepsilon},t_\varepsilon} + \sum_{i=i_\varepsilon+1}^{+\infty} \psi_i \nabla_H f_{r_i,t_{\varepsilon,i}} +$$

$$+ \sum_{i=1}^{i_\varepsilon} \psi_i \nabla_H f_{r_{i_\varepsilon},t_\varepsilon}|_H \, d\gamma - |D_\gamma f|(O)| \leq$$

(by (7.5.4) in Remark 7.3.4)

$$\leq \int_O |\sum_{i=i_\varepsilon+1}^{+\infty} \nabla_H \psi_i f_{r_i,t_{\varepsilon,i}} + \sum_{i=1}^{i_\varepsilon} \nabla_H \psi_i f_{r_{i_\varepsilon},t_\varepsilon}|_H \, d\gamma +$$

$$+ |\int_O |\sum_{i=i_\varepsilon+1}^{+\infty} \psi_i \nabla_H f_{r_i,t_{\varepsilon,i}} + \sum_{i=1}^{i_\varepsilon} \psi_i \nabla_H f_{r_{i_\varepsilon},t_\varepsilon}|_H \, d\gamma - |D_\gamma f|(O)| \leq$$

(by (7.5.21 and (7.5.4) again)

$$\leq \varepsilon + \int_{O \setminus O_{r_{(i_\varepsilon)-1}}} |\sum_{i=i_\varepsilon+1}^{+\infty} \psi_i \nabla_H f_{r_i,t_{\varepsilon,i}} + \sum_{i=1}^{i_\varepsilon} \psi_i \nabla_H f_{r_{i_\varepsilon},t_\varepsilon}|_H \, d\gamma +$$

$$+ |\int_{O_{r_{(i_\varepsilon)-1}}} |\sum_{i=i_\varepsilon+1}^{+\infty} \psi_i \nabla_H f_{r_i,t_{\varepsilon,i}} + \sum_{i=1}^{i_\varepsilon} \psi_i \nabla_H f_{r_{i_\varepsilon},t_\varepsilon}|_H \, d\gamma - |D_\gamma f|(O)| \leq$$

(recalling that, for $i > i_\varepsilon$, $\psi_{i|O_{r_{(i_\varepsilon)-1}}} \equiv 0$ and, for $i \leq i_\varepsilon$, $\psi_{i|O^c_{r_{i_\varepsilon}}} \equiv 0$, and that $\sum_{i=1}^{i_\varepsilon} \psi_i = 1$ on $O_{r_{i_\varepsilon}}$)

$$\leq \varepsilon + \int_{O \setminus O_{r_{(i_\varepsilon)-1}}} |\sum_{i=i_\varepsilon+1}^{+\infty} \psi_i \nabla_H f_{r_i,t_{\varepsilon,i}} + \psi_{i_\varepsilon} \nabla_H f_{r_{i_\varepsilon},t_\varepsilon}|_H \, d\gamma +$$

$$+ |\int_{O_{r_{(i_\varepsilon)-1}}} |\nabla_H f_{r_{i_\varepsilon},t_\varepsilon}|_H \, d\gamma - |D_\gamma f|(O)| \leq$$

$$\leq \sum_{i=i_\varepsilon}^{\infty} (\int_O \psi_i |\nabla_H f_{r_i,t_{\varepsilon,i}}|_H \, d\gamma) + \int_O \psi_{i_\varepsilon} |\nabla_H f_{r_{i_\varepsilon},t_\varepsilon}|_H \, d\gamma + \varepsilon +$$

$$+ \left| \int_{O_{r_{(i_\varepsilon)-1}}} |\nabla_H f_{r_{i_\varepsilon},t_\varepsilon}|_H \, d\gamma - |D_\gamma f|(O_{r_{(i_\varepsilon)-1}}) \right| + ||D_\gamma f|(O_{r_{(i_\varepsilon)-1}}) - |D_\gamma f|(O)| \leq$$

(by (7.5.19), (7.5.15), (7.5.13), and (7.5.10))

$$\leq \sum_{i=i_\varepsilon}^{\infty}\Big(\int_{O\setminus O_{r_{i_\varepsilon-1}}}\psi_i\, d|D_\gamma f|+2^{-i}\varepsilon\Big)+3\varepsilon\leq$$

$$\leq 4\varepsilon+\int_{O\setminus O_{r_{i_\varepsilon-1}}}\sum_{i=i_\varepsilon-1}^{\infty}\psi_i\, d|D_\gamma f|=$$

(recalling that $\sum_{i=i_\varepsilon}^{\infty}\psi_i=1$ on $O\setminus O_{r_{i_{\varepsilon+1}}}$)

$$=4\varepsilon+|D_\gamma f|(O\setminus O_{r_{i_{\varepsilon-1}}})\leq 5\varepsilon$$

by (7.5.10).

So, $|\nabla_H f_\varepsilon|\in L^1(O)$ and $\int_O|\nabla_H f_\varepsilon|_H\, d\gamma \underset{\varepsilon\to 0}{\longrightarrow}|D_\gamma f|(O)$.

Hence, if we take $f_n:=f_{\frac{1}{n}}$, we have that $f_n\in W_*^{1,1}(O),L^2(O)$, $f_n\to f$ in L^2, and $\int_O|\nabla_H f_n|_H\, d\gamma\to$ $|D_\gamma f|(O)$. □

REMARK 7.5.10. The convergence we have found ($f_n\to f$ in L^2 and $\int|\nabla_H f_n|\gamma\to\int|D_\gamma f|\gamma$) is similar to the finite dimensional intermediate convergence (see e.g. [**10**], Def. 10.1.3).

We remark that in the above Theorem we use the convexity only to define the sequence of concentric sets with mutually disjoint boundaries which cover all the domain; this can be done in a more general setting.

7.5.2. Approximation by J_σ. We recall that J_σ can be defined as a contractive operator from $L^p(O)$ in $L^p(O)$ for every $p\in[1,+\infty]$ (see Subsection 3.4.3); in general we don't know if it is regularizing.

We have this Theorem.

THEOREM 7.5.11. *If O is open and convex and $y\in(BV\cap L^p)(O)$ and $y\in L^p(O)$ for some $p>1$, if J_σ is defined in $L^p(O)$ and associated to L Ornstein-Uhlenbeck operator with zero Neumann boundary conditions, then $J_\sigma(y)\in BV(O)$ and*

$$|D_\gamma(J_\sigma(y))|(O)\leq|D_\gamma y|(O).$$

PROOF. We consider a representative of $y\in BV(O)$ finite in each point. We have that, for all $n\in\mathbb{N}$, the function $v_n:=n\wedge y\vee(-n)$ is $L^\infty\cap BV(O)$ and $|D_\gamma v_n|(O)\leq|D_\gamma y|(O)$ for all $n\in\mathbb{N}$ by Lemma 4.2.10; we have also that the sequence v_n converges to y in L^p and so also in $L\log^{\frac{1}{2}}L(O)$, hence, by (4.2.3) (and recalling that, for φ Lipschitz and bounded, $\partial_h^*\varphi\in L^\Psi$ for every $h\in H$, see Subsection 3.2.1)

$$|D_\gamma y|(O)=\sup\{\sum_{i=1}^{m}\int_O y\partial_{h_i}^*\varphi_i\, d\gamma|\ m\in\mathbb{N},\varphi\in\mathrm{Lip}_{0,m}(O,H),\ \|\varphi\|_{L^\infty(O,H)}\leq 1\}=$$

$$=\sup\{\lim_{n\to\infty}\sum_{i=1}^{m}\int_O v_n\partial_{h_i}^*\varphi_i\, d\gamma|\ m\in\mathbb{N},\varphi\in\mathrm{Lip}_{0,m}(O,H),\ \|\varphi\|_{L^\infty(O,H)}\leq 1\}\leq$$

$$\leq\liminf_{n\to\infty}\sup\{\sum_{i=1}^{m}\int_O v_n\partial_{h_i}^*\varphi_i\, d\gamma:\ m\in\mathbb{N},\varphi\in\mathrm{Lip}_{0,m}(O,H),\ \|\varphi\|_{L^\infty(O,H)}\leq 1\}=$$

$$=\liminf_{m\to\infty}|D_\gamma v_m|(O);$$

therefore, $|D_\gamma y|(O) = \lim_{n\to\infty} |D_\gamma v_n|(O)$.

By Proposition 7.5.9 each v_n can be approximated by a sequence of function $y_{m,n} \in W^{1,1}_*(O) \cap L^2(O)$ (which converges to v_n in L^2 and s.t. $|\nabla_H y_{m,n}|_{L^1(O,H)} \xrightarrow[m\to+\infty]{} |D_\gamma v_n|(O)$; we recall that $W^{1,1}_*(O) = W^{1,1}(O)$ by the convexity of O and Corollary 3.2.24.

So, with a diagonal procedure we can find a sequence $y_n = y_{m_n,n}$ in $W^{1,1}(O) \cap L^2(O)$ which converges to y in L^q (for $q := 2 \wedge p$, by recalling that L^p is embedded in L^q because γ is finite) and s.t. $\|\nabla_H y_n\|_{L^1(O,H)} \to |D_\gamma y|(O)$; we recall that J_σ is a bounded operator in L^q, so for all $\sigma > 0$, $J_\sigma(y_n) \xrightarrow[n\to\infty]{L^q} J_\sigma(y)$ and hence the convergence is also in $L(\log L)^{\frac{1}{2}}$; to each y_n we can apply Theorem 7.4.7,

$$\int_O |\nabla_H J_\sigma(y_n)|_H \, d\gamma \leq \int_O |\nabla_H y_n|_H \, d\gamma;$$

hence, by lower semicontinuity of BV norm with respect to $L(\log L)^{\frac{1}{2}}(O)$ convergence (Corollary 4.2.22), we have that $J_\sigma(y) \in BV(O)$ and

$$|D_\gamma(J_\sigma(y))|(O) \leq \liminf_{n\to\infty} \int_O |\nabla_H J_\sigma(y_n)|_H \, d\gamma \leq$$

$$\leq \liminf_{n\to\infty} \int_O |\nabla_H y_n|_H \, d\gamma = |D_\gamma y|(O).$$

□

PROPOSITION 7.5.12. *Let $y \in L^2(O)$; if there exist $c > 0$, and a sequence σ_n s.t. $\sigma_n \to 0$ and $\|\nabla_H J_{\sigma_n}(y)\|_{L^2(X,H)} \leq c$ for all n, then $J_{\sigma_n}(y) \rightharpoonup y$ in $L\log^{\frac{1}{2}} L$, $y \in BV(O)$ and $|D_\gamma y|(O) \leq c$.*

PROOF. We recall that J_σ is a contractive operator from L^2 in L^2 for all $\sigma > 0$. Hence, for a $y \in L^2$ the functions $J_{\sigma_n}(y)$ are uniformly bounded in L^2; we have that up to a subsequences as $\sigma_n \xrightarrow[n\to\infty]{} 0$, s.t. $J_{\sigma_n}(y) \rightharpoonup w$ in L^2 for some $w \in L^2$ by the Banach-Alaoglu theorem (see Appendix); recalling that $J_\sigma = (I - \sigma L)^{-1}$, the definition of L (which is the operator associated to the form $\int_O \langle \nabla_H, \nabla_H \rangle_H \, d\gamma$) and the fact that the image of J_{σ_n} is in the domain of L, we have that, if $\varphi \in \mathcal{F}C^\infty_b$,

$$\int_O (J_{\sigma_n}(y) - y)\varphi \, d\gamma = \sigma_n \int_O L(J_{\sigma_n}(y))\varphi \, d\gamma =$$

$$= -\sigma_n \int_O \langle \nabla_H J_{\sigma_n}(y), \nabla_H \varphi \rangle_H \, d\gamma$$

but $\|\nabla_H \varphi\|_{L^\infty(X,H)}$ is bounded, and $\|\nabla_H J_{\sigma_n}(y)\|_{L^1(X,H)} \leq c$ for every n, hence, for some $C > 0$ independent on n,

$$|\int_O (J_{\sigma_n}(y) - y)\varphi \, d\gamma| \leq \|\nabla_H \varphi\|_{L^\infty(X,H)} c \sigma_n \xrightarrow[n\to\infty]{} 0;$$

hence

$$\int_O (w - y)\varphi \, d\gamma = 0.$$

By the density of $\mathcal{F}C^\infty_b$ in L^2, we have that $w = y$. With this argument, we proved that $J_\sigma y \rightharpoonup y$ in L^2 (because, for every sequence, there is a subsequence which converges). Recalling that $L^2 \subset L\log^{\frac{1}{2}} L$ for all $p > 1$, we have $J_\sigma(y) \rightharpoonup y$ in $L\log^{\frac{1}{2}} L$; so we can conclude by the lower semicontinuity (Lemma 4.2.24). □

COROLLARY 7.5.13. *If O is a convex open set, for $y \in L^2(O)$, these conditions are equivalent.*

i) *y is $BV(O)$;*
ii) *$\int_O |\nabla_H J_\sigma(y)|_H \, d\gamma$ is uniformly bounded with respect to σ;*
iii) *$\int_O |\nabla_H J_\sigma(y)|_H \, d\gamma$ converges for $\sigma \to 0$;*
iv) *there exists a sequence σ_n s.t. $\sigma_n \to 0$ and $\int_O |\nabla_H J_{\sigma_n}(y)|_H \, d\gamma$ converges for $n \to +\infty$.*

In these cases,

$$\lim_{\sigma\to 0}\int_O |\nabla_H J_\sigma(y)|_H \, d\gamma = \sup\{\int_O |\nabla_H J_\sigma(y)|_H \, d\gamma | \sigma > 0\} = |D_\gamma y|(O).$$

PROOF. It is obvious that iii)⇒iv) and ii) ⇒iv). $J_{\sigma_n}(y) \xrightarrow{L^2(O)} y$, hence by the Corollary 4.2.22, iv)⇒i) .

Let us now assume i), i.e. $y \in BV(O)$: we will prove ii), iii) and the last statement.

By Theorem 7.5.11 we know that $\|J_\sigma(y)\|_{W^{1,1}(O)}$ is uniformly bounded

$$\int_O |\nabla_H J_\sigma(y)|_H \, d\gamma \le |D_\gamma y|(O).$$

Therefore, by the Proposition 7.5.12, for each sequence σ_n which converges to 0, we have $J_{\sigma_n}(y) \rightharpoonup y$ in $L^2(O)$ for $\sigma_n \to 0$, so the convergence is also in $L\log^{\frac{1}{2}} L$ and we can apply Lemma 4.2.24 and

$$|D_\gamma y|(O) \le \liminf_{n\to\infty}\int_O |\nabla_H J_{\sigma_n}(y)|_H \, d\gamma$$

so we can conclude that $\int_O |\nabla_H J_\sigma(y)|_H \, d\gamma$ converges to $|D_\gamma y|(O)$ for $\sigma \to 0$. □

7.5.3. Approximation by T_t. We recall that, if T_t is the Ornstein-Uhlenbeck semigroup in $L^2(O)$ then it is analytic, and $T_t f \in W^{1,2}(O)$ for every $f \in L^2(O)$. Moreover T_t is contractive as an operator in $L^p(O)$ for every $p \in [1, +\infty)$ (see Subsection 3.4.3).

We consider also a result from [**23**], and also ([**24**], Thm. 17).

PROPOSITION 7.5.14. *If O is a convex open set and T_t is the Ornstein-Uhlenbeck semigroup with zero Neumann boundary condition on $L^2(O)$, then*

$$|\nabla_H T_t f|_H \le e^{-t} T_t(|\nabla_H f|_H)$$

γ-a. e. on O, for every $f \in W^{1,2}(O)$.

We have these consequences.

COROLLARY 7.5.15. *If O is a convex open set and T_t is the Ornstein-Uhlenbeck semigroup with zero Neumann boundary condition on $L^2(O)$, if $f \in (W^{1,1} \cap L^2)(O)$ then $T_t f \in BV_\gamma(O)$ and*

$$\int_O \nabla_H T_t f|(O) \le e^{-t} \int_O |\nabla_H f|_H \, d\gamma.$$

PROOF. If f is Lipschitz and bounded, then the inequality is verified by the above proposition and by the contractivity of T_t in $L^2(O)$.

If $f \in (W^{1,1} \cap L^2)(O)$, then we can consider a sequence f_n of Lipschitz functions which converges to f in $(W^{1,1} \cap L^2)(O)$ (by Corollary 3.2.24): by the fact that T_t is contractive in $L^1(O)$.

$$\lim_{n\to+\infty}\int_O T_t(|\nabla_H f_n|_H)) \, d\gamma = \int_O T_t(|\nabla_H f|_H)) \, d\gamma \le \int_O |\nabla_H f|_H \, d\gamma$$

by Proposition 7.5.14 we have

$$\liminf_{n\to+\infty}\int_O |\nabla_H T_t f_n|_H \, d\gamma \leq e^{-t} \liminf_{n\to+\infty}\int_O T_t(|\nabla_H f_n|_H)\, d\gamma \leq e^{-t}\int_O |\nabla_H f|_H \, d\gamma;$$

moreover $T_t f_n$ converges to $T_t f$ in $L^p(O)$ and hence in $L\log^{\frac{1}{2}} L$; so by the lower semicontinuity (Corollary 4.2.22), we have that $T_t f \in BV(O)$ and we can conclude. □

COROLLARY 7.5.16. *If $f \in L^2(O)$, then $f \in BV(O)$ iff*

$$\liminf_{t\to 0}\int_O |\nabla_H T_t f|_H \, d\gamma < \infty, \tag{7.5.22}$$

and in this case

$$\lim_{t\to 0} |\int_O \nabla_H T_t f|_H = |D_\gamma f|(O). \tag{7.5.23}$$

PROOF. If (7.5.22) is satisfied, then there exists a sequence $t_n \to 0$ s.t.

$$\lim_{n\to+\infty}\int |\nabla_H T_{t_n} f|_H \, d\gamma = c < \infty,$$

we already know that $T_{t_n} f$ converges to f in $L^2(O)$ and hence in $L\log^{\frac{1}{2}} L(O)$, and therefore $f \in BV(O)$ and $|D_\gamma f|(O) \leq c$ by Corollary 4.2.22.

If $f \in BV(O)\cap L^2(O)$, by truncation there exists a sequence of bounded functions which converges to f in $BV(O)\cap L^2(O)$, and hence by Proposition 7.5.9 there exists a sequence f_n of functions in $\left(W^{1,1}\cap L^2\right)(O)$ s.t. $f_n \xrightarrow{L^p(O)} f$ and $\int_O |\nabla_H f_n|_H \, d\gamma$ converges to $|D_\gamma f|(O)$; therefore we have (recalling that T_t is bounded in $L^p(O)$ for every f_t and Lemma 4.2.24)

$$\limsup_{t\to 0}\int |\nabla_H T_t f|_H \, d\gamma \leq \limsup_{t\to 0}\limsup_{n\to+\infty}\int |\nabla_H T_t(f_n)|_H \, d\gamma \leq$$

(recalling that $J_\sigma f \in W^{1,p}(O)$ and Corollary 7.5.15)

$$\leq \limsup_{t\to 0} e^{-t}\left(\limsup_{n\to\infty}\int_O |\nabla_H f_n|_H \, d\gamma\right) \leq \limsup_{n\to\infty}\int_O |\nabla_H f_n|_H \, d\gamma = |D_\gamma f|(O).$$

Hence, in this case, we have (7.5.23) and we can conclude. □

REMARK 7.5.17. The argument of this section could be reversed: the Proposition 7.5.14 could be used together with Proposition 7.5.9 to prove the Corollary 7.5.16, and this yields Theorem 7.5.11.

CHAPTER 8

A finite perimeter subset of a classical Wiener space

Let $X_* = C_*([0,1], \mathbb{R}^d)$ (continuous functions starting by 0), we consider the measure given by the Brownian motion (see Section 2.6) with starting point in $0 \in X$, hence it is represented by a Gaussian measure P_0. For every $A \in \mathfrak{B}(X_*)$, we define $\Xi^*_A := \{\omega \in X | \omega(t) \in A \ \forall t \in [0,1]\}$.

In [**46**] (see Thm. 5.1) it is proved that, if $d \geq 2$ and $\Omega \subset \mathbb{R}^d$ is an open set which satisfies a uniform outer ball condition then Ξ^*_Ω has finite perimeter in the sense of Gaussian measure (see Section 7.5).

Our aim is to find a weaker condition on Ω (for dimension sufficiently large) such that Ξ^*_Ω has finite perimeter

The main points are these: in Section 8.2 we introduce ρ and δ functions on $\mathbb{R}^d$ s.t. $\rho(x) \in [0,1]$ for every $x \in X$, $\rho_{|O^c} \equiv 0$ and ρ is locally Lipschitz in $\bar{\Omega}$ with a local constant given by δ, except in a set $\partial_s\Omega$ of singular points of $\partial\Omega$.

Hence, in Section 8.3, we impose that Ω satisfies certain conditions (Hypotheses 8.2.1, 8.3.20, 8.3.11, 8.3.19), we define function $\bar{\rho}$ on X and $\bar{\delta}$ on $X \backslash \Theta_{\partial_s\Omega}$ based on ρ and δ, and we prove that $\bar{\rho} \in W^{1,1}(X)$, and we use it to build a sequence of functions which converges to the characteristic function of Ξ^*_Ω; hence we can state Theorem 8.3.21, main result of the Chapter, which asserts that, under our conditions, Ξ^*_Ω has finite perimeter. This result is actually an extension of ([**46**], Thm. 5.1), see Example 8.3.22.

In Section 8.1 we introduce some preliminary results that are used in Section 8.3: among others, we use stochastic concepts (see Section 1.3), the concept of Bessel process (see Subsection 1.3.5) and Proposition 1.3.18.

In Section 8.4, we prove that if Ω is the complementary of a symmetric cone in dimension greater than 6, then it satisfies our conditions and Ξ^*_Ω has finite perimeter (Proposition 8.4.2).

It remains, as conjecture, the possibility to extend this result to sets which satisfy an uniform cone condition.

8.1. Preliminary results

8.1.1. Pseudo-Hausdorff set-function.

DEFINITION 8.1.1. (pseudo-Hausdorff set function) In $\mathbb{R}^d$, given $E \subset \mathbb{R}^d$, we define the family $I_{s,E}$ of finite covering C_i of E with $C_\alpha = (B_{\alpha,1}, \dots, B_{\alpha,n_\alpha})$ where $B_{\alpha,j}$ is a ball with radius s for every $j \in \{1, \dots, n_\alpha\}$ and $E \subseteq \bigcup_{j=1}^{n_\alpha} B_{\alpha,j}$; for $s > 0$ we define the index

$$n_s(E) = \min_{C_i \in I_{s,E}} n_i; \tag{8.1.1}$$

for $m, s > 0$ we define the set function

$$\mathcal{H}^m_s(E) = n_s s^m;$$

we define the spherical pseudo-m-Hausdorff set function

$$\mathcal{H}^m(E) = \liminf_{s\to 0} \mathcal{H}^m_s(E).$$

REMARK 8.1.2. i) We have $\mathcal{H}^m(E) = \mathcal{H}^m(\bar{E})$ for every E (differently from the Hausdorff measure $\mathscr{H}$), because if we substitute the sphere B_j with $\overline{B_j}$, they cover $\bar{E}$. If $\mathcal{H}^m(E) < \infty$ then $\mathscr{H}^m(\bar{E}) < \infty$ (where $\mathscr{H}^m$ is the Hausdorff measure).

ii) If $\mathcal{H}^m(E) = 0$ then $\mathscr{H}^m(\bar{E}) = 0$

iii) If E is a k-manifold then $\dim_{\mathcal{H}}(E) = k$.

In the next Lemma we make use of the Brownian motion

$$Z = (\mathcal{A}, \mathscr{F}, \{Z_t\}_{t\in[0,+\infty]}, \{\mu_x\}_{x\in\mathbb{R}^d_\partial}, \{\mathscr{F}_t\}_{t\in[0,+\infty]})$$

(see Section 1.3.2 for the concept of Markov process and Subsection 1.3.4 for the Brownian motion as a Markov process).

LEMMA 8.1.3. *Let $E \subset \mathbb{R}^d$, $x \in \mathbb{R}^d$ s.t. $r = dist(x,E) > 0$; we consider, for all s s.t. $\frac{r}{2} > s > 0$, the neighbourhood $A_s := B(E,s)$, and the random variable τ_s, the hitting time of A_s; then*

$$\mu_x(\tau_s < \infty) \le \frac{2^{2d-4}}{r^{d-2}} \mathcal{H}^{d-2}_s(E).$$

PROOF. By definition of $\mathcal{H}^{d-2}_s$ we can define a covering $C = (B_1, \dots, B_{n_s})$ of E s.t. B_i has radius s for all i and n_s satisfies (8.1.1); now, we can define, for all i, a ball B'_i with the same centre y_i of B_i and radius $2s$; we remark that $\|x_0 - y_i\| \ge \frac{r}{2}$ because $s < \frac{r}{2}$.

Clearly the balls B'_i cover A_s; we define $U := \bigcup_{i=1}^{n_s} B'_i$, τ_U the hitting time of U (it is surely positive because $r > 2s$), and, for every $i \in \{1, \dots, n_s\}$ the variable $\tau_{B'_i}$ hitting time of B'_i.

It is also clear that $\tau_s \ge \tau_U$ surely, so $\mu_x(\tau_s < \infty) \le \mu_x(\tau_U < \infty)$; we have also that

$$\{a \in \mathcal{A} | \tau_U(a) < \infty\} = \bigcup_{i=1}^{n_s} \{a \in \mathcal{A} | \tau_{B'_i}(a) < \infty\},$$

so

$$\mu_x(\tau_U < \infty) \le \sum_{i=1}^{n_s} \mu_x(\tau_{B'_i} < \infty);$$

now, for some $c_0 > 0$ we have, recalling $\|x_0 - y_i\| \ge \frac{r}{2}$ and that

$$\mu_x(\tau_{B'_i} < \infty) = \frac{(2s)^{d-2}}{\|x_0 - y_i\|^{d-2}}$$

by Lemma 1.3.21, we have, for every i,

$$\mu_x(\tau_{B'_i} < \infty) \le \frac{2^{2d-4} s^{d-2}}{r^{d-2}}$$

so

$$\sum_{j=1}^{n_i} \mu_x(\tau_{B'_i} < \infty) \le \frac{n_j 2^{2d-4} s^{d-2}}{r^{d-2}} = \frac{2^{2d-4}}{r^{d-2}} \mathcal{H}^{d-2}_s(E)$$

and we conclude. □

To control the hitting probability of a set, we will suppose that the next hypothesis is true, so we can apply the above Lemma.

HYPOTHESIS 8.1.4. We suppose that, for some $l > 0$, for $s < 1$, the set E is such that $\mathcal{H}_s^{d-2}(E) \le cs^l$ for some $c > 0$ independent of s.

EXAMPLE 8.1.5. If E is a bounded subset of an affine p-dimensional subspace of $\mathbb{R}^d$, we can consider, for the center of the ball of the coverings, a regular distribution in p-dimensional cells, so $n_s \le cs^{-p}$ with $c > 0$ independent of $s < 1$; then $\mathcal{H}_s^{d-2}(E) \le cs^{d-2-p}$.

REMARK 8.1.6. If the above hypothesis is true, clearly $\mathcal{H}^{d-2}(E) = 0$.

COROLLARY 8.1.7. *If $E \subseteq \mathbb{R}^d$ satisfies Hypothesis 8.1.4, let $x_0 \in \mathbb{R}^d$ (starting point) s.t. $r = dist(x,E) > 0$; we consider, for all s s.t. $\frac{r}{2} \wedge 1 > s > 0$, the neighbourhood $A_s = B(E,s)$, and the random variable τ_s, the hitting time of A_s; then, there exists a constant $c_0 > 0$ independent of E and x s.t. $\mu_x(\tau_s < \infty) \le \frac{c_0}{r^{d-2}} s^l$.*

PROOF. It is an immediate consequence of Lemma 8.1.3. $\square$

8.1.2. A result about exit time. If μ and ν are measures over a X, as usual we will write $\mu << \nu$ to mean that μ is absolutely continuous with respect to ν.

We will need some preliminary results.

LEMMA 8.1.8. *Given a bounded space interval $[a,b]$, for $c := (b-a)^{-2}$ s.t., if $f \in C^1([a,b])$ and $\int_a^b |f(x)|\ dx \le 1$, we have*

$$\sup_{[a,b]} f^2 \le 2 \int_a^b |f(x)|\ dx \left(\sup_{[a,b]} |f'| + c \right).$$

PROOF. It is clear that

$$\inf_{[a,b]} |f| \le (b-a)^{-1} \int_a^b |f|\ dx,$$

and that

$$\sup_{[a,b]} f^2 - \inf_{[a,b]} f^2 \le \int_a^b |\frac{d}{dx} f^2(x)|\ dx = 2 \int_a^b |f(x) f'(x)|\ dx \le 2 \sup_{[a,b]} |f'| \int_a^b |f(x)|\ dx;$$

so (by $\int_a^b |f(x)|\ dx \le 1$)

$$\sup_{[a,b]} f^2 = \sup_{[a,b]} f^2 - \inf_{[a,b]} f^2 + \inf_{[a,b]} f^2 \le 2 \sup_{[a,b]} |f'| \int_a^b |f(x)|\ dx +$$

$$+ \left(\int_a^b |f(x)|\ dx \right)^2 (b-a)^{-2} \le 2 \int_a^b |f(x)|\ dx \left(\sup_{[a,b]} |f'| + c \right).$$

$\square$

We suppose that $X := \{\omega \in C([0,1], \mathbb{R}^d)\}$. We have that there exists a d-standard Brownian motion (as a Markov process, see Section 1.3) on $\mathbb{R}^d$

$$Z = (\mathcal{A}, \mathscr{F}, \{Z_t\}_{t \in [0,+\infty]}, \{\mu_x\}_{x \in \mathbb{R}^d_\partial}, \{\mathscr{F}_t\}_{t \in [0,+\infty]})$$

(where $(\mathcal{A}, \mathscr{F})$ is a measurable space). Fixed $x \in \mathbb{R}^d$, we define the function

$$i_x : \mathcal{A} \to \{\text{measurable functions } \mathbb{R}^+ \to \mathbb{R}^d\}$$

as

$$i_x(a) \mapsto (t \mapsto Z_t(a));$$

we have that i_x is a measurable functions, we can define

$$P_x := i_x^{-1} \circ \mu_x, \tag{8.1.2}$$

it concentrates on X, so we can consider P_x as a measure on X

$\mathcal{F}$ will be the Brownian filtration.

We will suppose $1 \leq p < \infty$.

Given $\Omega \subseteq \mathbb{R}^d$ open, $x \in \Omega$, we consider the absorbing Brownian motion in Ω with starting point x (see Subsection 1.3.4):

$$Z^{\Omega} = (\mathcal{A}, \mathscr{F}, \{Z_t^{\Omega}\}_{t \in [0,+\infty]}, \{\mu_x\}_{x \in \Omega_\partial}, \{\mathscr{F}_t\}_{t \in [0,+\infty]});$$

arguing in a way similar to that above, we define P_x' the probability associated to this motion on $X_\Omega := \{\omega \in C([0,1], \Omega)\}$. For $t > 0$ and $B \in \mathfrak{B}(D)$,

$$P_x'(\{\omega \in \mathcal{A} | \omega(t) \in B\}) = \mu_x(Z_t^D \in B).$$

We recall (see Subsection 1.3.4) that there exists $q \geq 0$ kernel s.t.

$$\mu_x(Z_t^{\Omega} \in B) = \int_B q(x,y,t)\ dy,$$

so $\int_B q(x,y,t)\ dy \leq 1$ for every x.

We can apply Proposition 1.3.18, so q is 2-derivable and

$$\left| \frac{\partial}{\partial t} q(x,y,t) \right| \leq C_1 t^{-\frac{d}{2}-1} \exp(-\frac{C_2 \|x-y\|^2}{t}), \tag{8.1.3}$$

and

$$\left| \frac{\partial^2}{\partial t^2} q(x,y,t) \right| \leq C_3 t^{-\frac{d}{2}-2} \exp(-\frac{C_4 \|x-y\|^2}{t}), \tag{8.1.4}$$

with $C_1, C_2, C_3, C_4 > 0$ independents of x, Ω. for q transition function associated to Z^{Ω}.

Now, we can consider τ is the exit time associated to Ω, and define

$$P_x^{\tau} := \tau^{-1} \circ \mu_x,$$

we have that P_x^{τ} is a probability on $\mathbb{R}^+$.

We will argue in a way similar to [[**46**], Lemma 3.2].

LEMMA 8.1.9. *In this setting, given $\Omega \subseteq \mathbb{R}^d$ open with $x \in \Omega$, let τ be the exit time associated to Ω, and P_x^{τ} the probability associated to τ on $\mathbb{R}^+$ with starting point x: then the function $P_x^{\tau}((0,t))$ is continuous and derivable in t, with non negative derivative $\frac{d}{dt} P_x^{\tau}((0,t)) \leq c_1 t^{-1}$ with $c_1 > 0$ independent of x, Ω; in particular, $P_x[\tau = t] = 0$ for all $t \neq \infty$. Moreover, there exists $c_2 > 0$ independent of x, Ω s.t. $\sup_{t \in [\frac{1}{2}, 1]} \frac{d}{dt} P_x^{\tau}((0,t)) \leq c_2 \sqrt{P_x^{\tau}([\frac{1}{2}, 1])}$.*

PROOF. By recalling the concepts of Markov processes, let ∂ be the cemetery point. We remark that for $t > 0$,

$$\{a \in \mathcal{A} | \tau(a) \in (0,t)\} = \{a \in \mathcal{A} | Z_t^{\Omega}(a) = \partial\},$$

so

$$P_x^{\tau}((0,t)) = \mu_x(Z_t^{\Omega} = \partial) = 1 - \mu_x(Z_t^{\Omega} \in \Omega) =$$

$$= 1 - \int_\Omega q(x,y,t)dy =$$

(by the Chapman-Kolmogorov property for kernels (1.3.4))

$$= 1 - \int\int_{\Omega\times\Omega} q(x,z,t-s)q(z,y,s)\ dy\ dz$$

for all $s < t$.

By (8.1.3), $P_x^\tau((0,t))$ is differentiable with respect to t; clearly it is increasing (by definition of τ), so $\frac{d}{dt}P_x^\tau((0,t)) \geq 0$ for all $x \in \Omega$, $0 < t$; hence, by $q \geq 0$ and (8.1.3), for all $x \in \Omega$, $0 < s < t$

$$|\frac{d}{dt}P_x^\tau([0,t])| = |\int\int_{\Omega\times\Omega} q(x,z,s)\frac{\partial}{\partial t}q(z,y,t-s)\ dy\ dz| \leq$$

$$\leq C_1 \int\int_{\Omega\times\Omega} q(x,z,s)(t-s)^{-\frac{d}{2}-1}\exp\left(-C_2\frac{\|z-y\|^2}{t-s}\right)\ dy\ dz \leq$$

(recalling that $\int_{\mathbb{R}^d} e^{-\sigma|x|^2}dx \leq C\sigma^{-\frac{d}{2}}$ for some $C > 0$, we have for some C_5 independent on x,s,t)

$$\leq C_5(t-s)^{-1}\int_\Omega q(x,z,s)\ dy \leq$$

(by $\int_\Omega q(x,z,s)\ dy \leq 1$)

$$\leq C_1(t-s)^{-1}$$

and, if we choose $s < \frac{t}{2}$, we have $\frac{d}{dt}P_x^\tau((0,t)) \leq c_1 t^{-1}$ for some $c_1 > 0$; we do the same thing for $\frac{d^2}{dt^2}$, we have that $P_x^\tau((0,t))$ is two derivable and

$$|\frac{d^2}{dt^2}P_x^\tau([0,t])| = |\int\int_{\Omega\times\Omega} q(x,z,s)\frac{\partial^2}{\partial t^2}q(z,y,t-s)\ dy\ dz| \leq$$

$$\leq C_3 \int\int_{\Omega\times\Omega} q(x,z,s)(t-s)^{-\frac{d}{2}-2}\exp\left(-C_4\frac{\|z-y\|^2}{t-s}\right)\ dy\ dz \leq$$

(for some $C_5 > 0$ independent on x,s,t)

$$\leq C_5(t-s)^{-2}\int_\Omega q(x,z,s)\ dy \leq C_5(t-s)^{-2}$$

because $\int_B q(x,y,t)\ dy \leq 1$ for every x. Therefore, for some $C_6 > 0$ independent on x

$$\sup_{[\frac{1}{2},1]} |\frac{d^2}{dt^2}P_x^\tau((0,t))| \leq C_6,$$

so, if we define $g(t) = \frac{d}{dt}P_x^\tau((0,t))$, we have $|g'(t)| \leq C_6$ on $\left[\frac{1}{2},1\right]$, and clearly $g \geq 0$ everywhere; so, by Lemma 8.1.8

$$\sup_{t\in\left[\frac{1}{2},1\right]} g^2(t) \leq 2(c+C_6)\int_{\frac{1}{2}}^1 g(t)\ dt = 2(c+C_6)P_x^\tau(\left[\frac{1}{2},1\right]),$$

hence

$$\sup_{t\in\left[\frac{1}{2},1\right]} g(t) \leq c_2\sqrt{P_x^\tau(\left[\frac{1}{2},1\right])}$$

for some c_2 independent on Ω, x. □

REMARK 8.1.10. This proof is done for a d-dimensional standard Brownian motion; if, instead, we consider a d dimensional Brownian motion on the interval $[0,1]$, clearly the inequality of the above propositions remain true because the exit time in this case is $\tau' = \tau \wedge 1$, and so if we define $P_x^{\tau'} := \tau'^{-1} \circ \mu_x$, we have $\frac{d}{dt} P_x^{\tau'}((0,t)) = \frac{d}{dt} P_x^{\tau}((0,t))$ for every $t < 1$.

8.1.3. Construction of a $W^{1,1}$ function which is piecewise Lipschitz. We consider a separable Banach space with centered non-degenerate Gaussian measure $(X;\gamma)$; clearly it is Radon.

We suppose that a centered non-degenerate Gaussian measure γ is defined on $\mathbb{R}^n$.

DEFINITION 8.1.11. We consider an open $\Omega \subseteq X$. F function on Ω is locally Lipschitz if there exists an open covering $\{O_i\}_{i\in\mathbb{N}}$ of Ω s.t. F is Lipschitz in O_i for every $i \in \mathbb{N}$.

F function on Ω is locally H-Lipschitz if there exists an open covering $\{O_i\}_{i\in\mathbb{N}}$ of Ω s.t. F is H-Lipschitz in O_i for every $i \in \mathbb{N}$.

Let $F \in L^\infty(X)$, F locally Lipschitz on Ω, hence it is also locally H-Lipschitz on Ω; then, by Corollary 3.1.15, it is almost everywhere Gâteaux differentiable , and we can define almost everywhere a local H-derivative, $\nabla_H F$ and the partial derivatives $\partial_h F$ for all $h \in H$.

We recall that, for all measures $\mu \ll \mathcal{L}^1$, a function $f \in L^\infty(\mathbb{R})$ is in $W^{1,1}(\mathbb{R})$ iff it is $W^{1,1}_{loc}(\mathbb{R})$ and $|f'|_{L^1(\mathbb{R})} < \infty$; if f is Lipschitz on an interval, then it is absolutely continuous and hence $W^{1,1}$ on that interval; in particular, if f is locally Lipschitz on an open of $\mathbb{R}$, then it is $W^{1,1}_{loc}$ on this open (for the absolute continuity and the local absolute continuity see the Appendix).

We give a definition of 2-capacity of sets of (X,γ), modelled on that in $\mathbb{R}^d$.

DEFINITION 8.1.12. We define the 2-*capacity* of an open $O \subseteq X$ as

$$C_2(O) := \inf_{f\in W^{1,2}(X), f_{|O}\geq 1} \|f\|_{W^{1,2}(X)}.$$

and for a generic set A we define $C_2(A) = \inf_{O\in\mathcal{O}, A\subseteq O} C_2(O)$ where $\mathcal{O}$ is the set of open subsets of X.

It is clear that, if $\gamma(A) > 0$ then $C_2(O) > 0$.

PROPOSITION 8.1.13. *We suppose that there exists a sequence of $\{l_i\}_{i\in\mathbb{N}} \subset [0,+\infty)$, a sequence of mutually disjoint Borel subsets $\{X_i\}_{i\in\mathbb{N}}$, s.t. $\gamma(X \setminus \bigcup_{i=1}^\infty X_i) = 0$ and a closed Θ s.t. $C_2(\Theta) = 0$; we will suppose*

$$\sum_{i=0}^{\infty} l_i^2 \gamma(X_i) =: l < \infty$$

(hence $\sum_{i=0}^\infty l_i \gamma(X_i) < \infty$ due to the finiteness of γ).

Let $F \in L^\infty(X)$, such that F is locally Lipschitz out of Θ, and F is l_i-Lipschitz in X_i for every $i \in \mathbb{N}$: then $F \in W^{1,1}(X)$.

Moreover, F admits almost everywhere H-derivative, and, for almost each point, if $x \in X_i$ then $|\nabla_H F(x)|_H \leq l_i$.

PROOF. Under the hypothesis, F is almost everywhere Gâteaux differentiable and admits almost everywhere H-gradient, moreover, for almost each point, if $x \in X_i$ then $|\nabla_H F(x)|_H \leq l_i$ by Theorem 3.1.11 and Corollary 3.1.15.

F is locally Lipschitz out of Θ, so it is locally absolutely continuous along lines.
$\nabla_H F \in L^1(X,H)$ because

$$\int_X \|\nabla_H F\|_H \, d\gamma \le \sum_{i=1}^{\infty} l_i \gamma(X_i) < \infty.$$

Now, for $n \in \mathbb{N}$ we consider an $O_n \subset X$ open with $\Theta \subset O_n$, and a $g_n \in W^{1,2}(X)$ s.t.

$$\|g_n\|_{W^{1,1}(X)} \le \|g_n\|_{W^{1,2}(X)} \to 0$$

($\|\cdot\|_{W^{1,1}} \le \|\cdot\|_{W^{1,2}}$ due to the fact that γ is a probability), $0 \le g_n \le 1, g_{n|O_n} = 1$.

We consider $F_n := F(1-g_n)$, we prove it is in $W^{1,1}(X)$ with weak gradient

$$\nabla_H F_n := \nabla_H F(1-g_n) - F\nabla_H g_n;$$

by Lemma 3.1.13 it suffices to prove that, for every $h \in H$, for $\gamma_{h^\perp}$-a.e. $y \in h^\perp$ the function $(F_n)_y$ has a representative locally absolutely continuous, and that

$$\frac{F_n(x+th) - F_n(x)}{t} - \langle \nabla_H F_n, h\rangle_H \tag{8.1.5}$$

converges to 0 in measure γ for $t \to 0$.

For every $y \in h^\perp$, we have that $g_n \equiv 0$ on $(O_n)_y \supset \Theta_y$ and $(g_n)_y$ has a representative locally absolutely continuous, while the function F_y is locally absolutely continuous on $\mathbb{R}\backslash\Theta_y$ for $\gamma_{h^\perp}$-a.e. $y \in h^\perp$, so $(F_n)_y$ has a representative locally absolutely continuous in such y.

We have that, for every $h \in H$, by F locally Lipschitz out of Θ, that

$$\frac{F(x+th) - F(x)}{t} - \langle \nabla_H F, h\rangle_H$$

tends to 0 γ-a.e, and that for

$$\frac{g_n(x+th) - g_n(x)}{t} - \langle \nabla_H g_n, h\rangle_H$$

we have the convergence in measure to 0; so we have the convergence (8.1.5) in measure γ.

So $F_n \in W^{1,1}(X)$ for every $n \in \mathbb{N}$ by Lemma 3.1.13.

Now, we prove that $F_n \overset{W^{1,1}(X)}{\to} F$; we have (by using the hypothesis on the sequence $\{l_i\}_{i\in\mathbb{N}}$)

$$\|\nabla_H F g_n\|_{L^1(X,H)} \le \int_X |\nabla_H F|_H g_n \, d\gamma \le$$

(by the Hölder inequality)

$$\le \||\nabla_H F|_H\|_{L^2(X)} \|g_n\|_{L^2(X)} = \left(\sum_{i=1}^{\infty} l_i^2 \gamma(X_i)\, d\gamma\right)^{\frac{1}{2}} \|g_n\|_{L^2(X)} =$$

$$= l\,\|g_n\|_{L^2(X)} \xrightarrow{n\to\infty} 0$$

and therefore

$$\|F - F_n\|_{W^{1,1}(X,H)} = \|F g_n\|_{W^{1,1}(X,H)} \le$$

$$\le \|F\|_{L^\infty} \|1-g_n\|_{L^1} + \|F\|_{L^\infty} \|\nabla g_n\|_{L^1} + \|(1-g_n)\nabla F\|_{L^1(X,H)} \xrightarrow{n\to\infty} 0;$$

hence, F is the limit of the sequence $\{F g_n\}_{n\in\mathbb{N}}$ in $W^{1,1}$, hence it is in $W^{1,1}$. $\square$

REMARK 8.1.14. We recall that for every $h \in H$, $\|h\|_X \leq \|h\|_H$. If we want to prove that $\|\nabla_H F(x)\|_H \leq l_i$ in a.e. $x \in X_i$, it is sufficient to prove that for all $h \in H$, for t sufficiently small

$$\left|\frac{F(x+th)-F(x)}{t}\right| \leq l_i \|h\|_X;$$

then

$$|\partial_h F| = \liminf_{t\to 0}\left|\frac{F(x+th)-F(x)}{t}\right| \leq l_i \|h\|_X \leq l_i \|h\|_H \leq l_i.$$

So, for such a x we can write $\nabla_H F \in H$, and $\|\nabla_H F(x)\| \leq l_i$.

REMARK 8.1.15. We will apply the Proposition 8.1.13 for Brownian motion in $\mathbb{R}^d$, that is a particular case of Gaussian measure; if the starting point is 0, it will be a centered Gaussian measure; however, the property of the Brownian motion doesn't change if we change the starting point, so we can apply this result for all the starting points.

8.2. Some technical lemmas about open sets

Let $\Omega \subseteq \mathbb{R}^d$ be an open set. For every $r > 0$ we define B_r as the ball with radius $r > 0$ centered in 0, and $B(x,r)$ as the ball centered in $x \in \mathbb{R}^d$ and with radius $r > 0$.

Very heuristically, our main goal in this subsection is to define: a set $\partial_s\Omega$ of points of $\partial\Omega$ which do not admits a tangent ball out of Ω; a function δ on Ω which, for every $x \in \Omega$, express the radius of a ball out of Ω which is, in a certain sense, 'near' to x; and a function ρ_1 which, in some sense, substitutes the distance from $\bar{\Omega}$, s.t. it is c^{-1}-Lipschitz in regions of Ω in which $\delta > c$ (so ρ_1 is not regular near the points of $\partial_s\Omega$).

Hereafter we will suppose that the next hypothesis is true.

HYPOTHESIS 8.2.1. There exists $R, \eta > 0$ s.t. for every $x \in \partial\Omega$ there exists a $y \in \Omega^c$ s.t. $\mathrm{dist}(y,x) \leq R$ and $\mathrm{dist}(y,\Omega) \geq \eta$.

We define $q(x) := \mathrm{dist}(x,\Omega^c)$.We consider for some $0 < r < 1$ the open set

$$\Omega_r := \{x \in \Omega | q(x) > r\} \subseteq \Omega; \tag{8.2.1}$$

so in particular

$$\Omega_1 := \{x \in \Omega | q(x) > 1\}.$$

We set, for each $y \in \mathbb{R}^d$,

$$\delta'(y) := \mathrm{dist}(y,\bar{\Omega}) \wedge 1;$$

then δ' is continuous (and 1-Lipschitz) and positive, and $\delta'(y) \to 0$ if y converges to a point of $\partial\Omega$; we define, for $x \in \bar{\Omega}$ and $y \in \bar{\Omega}^c$,

$$g(x,y) := \frac{\|x-y\| - \delta'(y)}{\delta'(y)};$$

we have that g is continuous in y and in x, it is non negative (by $\|x-y\| \geq \delta'(y)$), and it converges to $+\infty$ if x is fixed and $|y|$ goes to $+\infty$; moreover, if $y_n \to y_0 \in \partial\Omega\backslash\{x\}$, then $g(x,y_n) \to +\infty$.

So, if $x \in \Omega$, fixed x the function $g(x,\cdot)$ has a minimum.

Let $\partial_{ss}\Omega$ the sets of elements of $\partial\Omega$ s.t. $g(x,\cdot)$ does not have a minimum.

For every $x \in \bar{\Omega}\backslash\partial_{ss}\Omega$, $g(x,\cdot)$ has a minimum.

If $x \in \Omega$, then $\|x-y\| > \mathrm{dist}(y,\bar{\Omega})$ for every $y \in \bar{\Omega}^c$, so $g(x,y) > 0$.

DEFINITION 8.2.2. We define on $\bar{\Omega}\backslash\partial_{ss}\Omega$

$$g_1(x) := \inf_{y\in\bar{\Omega}^c} \frac{\|x-y\| - \delta'(y)}{\delta'(y)} = \min_{y\in\bar{\Omega}^c} g(x,y),$$

and, on $\mathbb{R}^d$

$$\rho(x) := \begin{cases} 1 \wedge g_1(x) & \text{if } x \in \bar{\Omega}\backslash\partial_{ss}\Omega \\ 0 & \text{otherwise} \end{cases};$$

we have that, for all $x \in \bar{\Omega}\backslash\partial_{ss}\Omega$, there is a nonempty compact set $M(x)$ of minimal points of $g(x,\cdot)$; we define on $\bar{\Omega}\backslash\partial_{ss}\Omega$ a function $\delta_1(x) := \max_{y\in M(x)} \delta'(y) > 0$; we define on $\mathbb{R}^d$

$$\delta(x) := \begin{cases} \delta_1(x) & \text{if } x \in \bar{\Omega}\backslash(\Omega_1 \cup \partial_{ss}\Omega) \\ 1 & \text{otherwise} \end{cases};$$

clearly $\delta(x) \leq 1$ everywhere.

In general, $0 \leq \rho(x) \leq 1$, for $x \in \Omega$ we have that $g_1(x) > 0$, $\rho(x) > 0$, and for $x \in \overline{\Omega}^c$ we have that $\rho(x) = 0,\ g_1(x) < 0$.

We observe that $q(x) \leq g_1(x)$ for every $x \in \Omega$: in fact, for all $y \in \bar{\Omega}^c$, we have $g(x,y) \geq \|x-y\| - \delta'(y) \geq q(x)$; therefore, $q(x) \leq \rho(x)$ on $\mathbb{R}^d\backslash(\Omega_1 \cup \partial_{ss}\Omega)$.

δ is defined everywhere, but its behaviour is interesting only in $\bar{\Omega}\backslash(\Omega_1 \cup \partial_{ss}\Omega)$.

REMARK 8.2.3. If $x \in \partial\Omega$, and if there exists a ball $B_1 \subseteq \bar{\Omega}^c$ tangent in x of radius $r \leq 1$ and center y then clearly $x \notin \partial_{ss}\Omega$ and $\rho(x) = 0$ (recalling that $g(x)$ is always non negative, so $g_1(x) = g(x,y) = 0$ and 0 is a minimum) and $\delta(x) \geq r$. If $z \in \Omega$ and the above mentioned x is the nearest point of $\partial\Omega$ to z then $\delta(z) \geq r$.

In fact, if by contradiction $\delta(z) < r$, then there exists $y \in \bar{\Omega}^c$ s.t. $\delta'(y) = \delta(z) < r$ and s.t. $g(z,y) \leq g(z,w)$ where w is the center of B_1 (hence $\delta'(w) = r$); this yields, if v is an intersection of the segment between z and y with $\partial\Omega$, then

$$\|z-y\| - \delta'(y) \leq \big(\|z-w\| - \delta'(w)\big)\frac{\delta'(y)}{\delta'(w)} \leq \|z-w\| - r$$

and

$$\|z-v\| \leq \|z-y\| - \delta'(y) \leq \|z-w\| - r = \|z-x\|,$$

(the last equality is true because $x \in \partial B_1$ and r is the radius of B_1); but x is the nearest point to z of $\partial\Omega$, contradiction.

For $x \in \Omega_1$ we have

$$g_1(x) \geq \frac{1}{\delta_1(x)} \geq 1$$

so

$$\rho_{|\Omega_1} \equiv 1; \tag{8.2.2}$$

for $x \in \Omega\backslash\Omega_1$, by the Hypothesis 8.2.1 there exists y s.t. $\delta(y) > \eta$ and $\|x-y\| \leq R+1$, so

$$g_1(x) \leq c_0 := \frac{R+1}{\eta}. \tag{8.2.3}$$

For $x \in \Omega\backslash\Omega_1$ and $y \in M(x)$ we have that $\|x-y\| - \delta'(y) \geq q(x)$ (because $\delta'(y) = \mathrm{dist}(y,\Omega)$), therefore, by definition of ρ and δ

$$\rho(x) = g_1(x) \geq \frac{q(x)}{\delta'(y)} \geq \frac{q(x)}{\delta(x)};$$

hence, for every $x \in \Omega\backslash\Omega_1$, by (8.2.3),

$$\delta(x) > \frac{q(x)}{g_1(x)} \geq c_0^{-1} q(x), \tag{8.2.4}$$

clearly (8.2.4) is true for every $x \in \Omega$, because $\delta_{|\Omega_1} \equiv 1$.

LEMMA 8.2.4. *If $x_n \to x$ in $\bar{\Omega}\backslash(\Omega_1 \cup \partial_{ss}\Omega)$, with $\delta(x_n) \to \delta_1 > 0$ for some δ_1, then $\delta(x) = \delta_1$; δ is upper semicontinuous in $\bar{\Omega}\backslash(\Omega_1 \cup \partial_{ss}\Omega)$.*

PROOF. Let $x_n \to x$, with $\delta(x_n) \to \delta_1 > 0$; it is not restrictive to suppose that there exists $c > 0$ s.t. $\delta(x_n) > c > 0$; let y_n a sequence s.t. $y_n \in M(x_n)$ and $\delta'(y_n) = \delta(x_n)$, and let $z \in M(x_n)$; we have that $g(x_n,z) \to g(x,z)$, hence there exists $C > 0$ s.t. for every $n \in \mathbb{N}$ $g(x_n,z) \leq C$ and so $g(x_n,y_n) \leq C$ (because $y_n \in M(x_n)$), so

$$\|x_n - y_n\| \leq \delta'(y)C + \delta'(y) \leq C+1.$$

So, by $x_n \to x$, there exists $R > 0$ s.t. $\mathrm{dist}(y_n,x) \leq R$, hence $\{y_n\}_{n\in\mathbb{N}}$ is contained in the compact $\overline{B(x,R)}$ and up to a subsequence we have that y_n converges to some y, and, by the continuity of δ', $\delta_1 := \delta'(y) > c > 0$ and $y \in \bar{\Omega}^c$; therefore by the continuity of g on $\mathbb{R}^d \times \bar{\Omega}^c$

$$g(x,y) = \lim_{n\to\infty} g(x,y_n) = \lim_{n\to\infty} g(x_n,y_n),$$

and we can infer $y \in M(x)$ and $\delta(x) = \delta'(y)$, because if by contradiction there exists y' s.t. $g(x,y') < g(x,y)$, then by continuity of g for some n we have that $g(x_n,y') < g(x_n,y_n)$ and $y_n \notin M(x_n)$ (contradiction).

If $\delta(x_n) \to 0$, the upper semicontinuity in $\bar{\Omega}\backslash(\Omega_1 \cup \partial_{ss}\Omega)$ is obvious, because δ is non negative, so we have concluded. $\square$

REMARK 8.2.5. Now we consider a set $O \subseteq \bar{\Omega}\backslash\partial_{ss}\Omega$ and a $c > 0$ s.t. for all $x \in O$, we have $\delta(x) \geq c$; then, ρ is c^{-1}-Lipschitz in O.

In fact, for $x_1 \in O$, $x_2 \in O$ with $g_1(x_2) = \rho(x_2) < \rho(x_1)$ (hence, we can suppose $x_2 \notin \Omega_1$ by (8.2.2)), we can fix $y_2 \in M(x_2)$, so $\delta'(y_2) = \delta(x_2) \geq c$ and we have

$$\rho(x_1) - \rho(x_2) \leq g_1(x_1) - g_1(x_2) \leq \inf_{y\in\bar{\Omega}^c} \frac{\|x_1 - y\| - \delta'(y)}{\delta'(y)} - \frac{\|x_2 - y_2\| - \delta'(y_2)}{\delta'(y_2)} \leq$$
$$\leq \frac{\|x_1 - y_2\| - \delta'(y_2)}{\delta'(y_2)} - \frac{\|x_2 - y_2\| - \delta'(y_2)}{\delta'(y_2)} \leq c^{-1}\|x_1 - x_2\|.$$

REMARK 8.2.6. For $r \leq 1$, by (8.2.4), $\delta(x) > c^{-1}r$ if $x \in \Omega_r\backslash\Omega_1$ and by (8.2.2) $\rho(x) = 1$ if $x \in \Omega_1$; so ρ is cr^{-1}-Lipschitz in Ω_r for Remark 8.2.5.

We have also $\Omega = \bigcup_{n=1}^{\infty} \Omega_{\frac{1}{n}}$; therefore, ρ is locally Lipschitz, and hence continuous, on Ω; the set

$$\Omega_r^* := \{x \in \Omega | \rho(x) > r\} \tag{8.2.5}$$

is open for all $r > 0$. Besides, $\rho \equiv 0$ on $\bar{\Omega}^c$, and so ρ is continuous everywhere except on the boundary of Ω.

DEFINITION 8.2.7. We consider

$$\partial_s'\Omega = \{x \in \partial\Omega \backslash \partial_{ss}\Omega | g_1(x) > 0\}.$$

We define

$$\partial_s\Omega := \{x \in \bar{\Omega} | \exists \{x_n\}_{n\in\mathbb{N}} \text{ sequence in } \bar{\Omega} \text{ s.t } x_n \to x, \delta(x_n) \to 0\} \bigcup \overline{\partial_s'\Omega} \cup \overline{\partial_{ss}\Omega};$$

by (8.2.4) $\partial_s\Omega \subseteq \partial\Omega$. It is obvious that $\partial_s\Omega$ is closed. We will call $\partial_s\Omega$ *singular part* of $\partial\Omega$,

REMARK 8.2.8. Let $A \subseteq \mathbb{R}^d$ be an open and $s > 0$ s.t. for every $x \in \partial\Omega \cap A$ there exists a ball $B \subset \Omega^c$ s,t, $x \in \bar{B}$; then $S \cap \partial_s\Omega = \varnothing$ by Remark 8.2.3.

REMARK 8.2.9. δ is continuous in $\bar{\Omega}\backslash(\partial_s\Omega \cup \Omega_1)$ by Lemma 8.2.4; $g_1 = 0$ in $\partial\Omega\backslash\partial_s\Omega$.

DEFINITION 8.2.10. For each $a > 0$, if we define the compact set

$$\Gamma_a := \{x \in \overline{B_{\frac{1}{a}}} | \text{dist}(x, \partial_s\Omega) \geq a\}$$

(where $B_{\frac{1}{a}}$ is the ball centered in the origin with radius a).

δ is continuous on the compact $\left(\Gamma_a \cap \bar{\Omega}\right) \backslash \Omega_1$, so δ has a minimum $c > 0$ on $\left(\Gamma_a \cap \bar{\Omega}\right) \backslash \Omega_1$ (because $\partial_s\Omega \cap \Gamma_a = \varnothing$), so δ has minimum c on Γ_a (because $\delta \equiv 1$ on $\bar{\Omega}^c$ and Ω_1).

REMARK 8.2.11. We recall that, if a function is not Lipschitz in a compact, then there is at least a point in which is not locally Lipschitz.

LEMMA 8.2.12. *Let $a > 0$; ρ is Lipschitz in Γ_a. ρ is locally Lipschitz out of $\partial_s\Omega$.*

PROOF. Let c the minimum of δ on Γ_a.
We want to prove the first point. By Remark 8.2.11, we have only to verify that ρ is locally Lipschitz in every point.
In $\bar{\Omega}^c$, we have $\rho = 0$, hence the local Lipschitzianity is verified.
If $x \in \Gamma_a \cap \Omega$, then by Remark 8.2.5, ρ is c^{-1}-locally Lipschitz in x.
If $x \in \Gamma_a \cap \partial\Omega$, then $x \notin \partial_s\Omega$, so in a convex neighbourhood B of x we have

$$\rho_{|B\cap\partial\Omega} \equiv 0;$$

by Remark 8.2.5, ρ restricted to $\Gamma_a \cap \bar{\Omega}$ is c^{-1}-Lipschitz, and restricted to Ω^c is 0, so it is c^{-1}-Lipschitz in B: given $x \in \Omega \cap B$ and $y \in \Omega^c \cap B$, the segment between them intersects $B \cap \partial\Omega$ in a point z and $\rho(z) = \rho(y) = 0$, so

$$\frac{|\rho(x) - \rho(y)|}{||x-y||} \leq \frac{|\rho(x) - \rho(z)|}{||x-z||} \leq c^{-1}.$$

Hence we concluded the first part.
The second part is an obvious consequence (because $\mathbb{R}^d\backslash\partial_s\Omega = \bigcup_{a>0}\Gamma_a$). □

8.3. Finite perimeter of subsets of $C([0,1],\mathbb{R}^d)$ through approximations of characteristic functions.

As in Section 8.2, $\Omega \subseteq \mathbb{R}^d$ is an open set which satisfies Hypothesis 8.2.1. We recall the concepts of $\rho, \delta, \partial_s\Omega, \Gamma_a$ of Section 8.2.

8.3.1. Geometric properties of functions on $C([0,1],\mathbb{R}^d)$. In this subsection, $X = C([0,1],\mathbb{R}^d)$. For each $A \subset \mathbb{R}^d$ we define

$$\Theta_A = \{\omega \in X | \exists t \in [0,1] \text{ s.t. } \omega(t) \in A\}$$

and

$$\Xi_A = \{\omega \in X | \omega(t) \in A \ \forall t \in [0,1]\}.$$

If A is open in $\mathbb{R}^d$, then Ξ_A, Θ_A are open in X; if A is closed in $\mathbb{R}^d$, then Ξ_A, Θ_A are closed in X.

In particular, $\Theta_{\partial_s\Omega}$ is closed.

DEFINITION 8.3.1. We define $\overline{\rho_1} : X \to \mathbb{R}$ as

$$\overline{\rho}(\omega) := \begin{cases} \inf_{t\in[0,1]} \rho(\omega(t)) & \text{if } \omega \notin \Theta_{\partial_s\Omega} \\ 0 & \text{if } \omega \in \Theta_{\partial_s\Omega} \end{cases}.$$

LEMMA 8.3.2. *$\bar{\rho}$ is locally Lipschitz out of $\Theta_{\partial_s\Omega}$; in particular it is continuous on $X \backslash \Theta_{\partial_s\Omega}$, and Borel measurable on X.*

PROOF. For all $\omega \in X \backslash \Theta_{\partial_s\Omega}$, we define the function

$$\tau(\omega) := \inf\{t \in [0,1] | \rho(\omega(t)) = \bar{\rho}(\omega)\},$$

and $\bar{x}(\omega) := \omega(\tau(\omega))$: it is clear that if $\omega \notin \Theta_{\partial_s\Omega}$ then $\bar{\rho}(\omega) = \rho(\bar{x}(\omega))$, because ρ is continuous on $\omega([0,1])$ and ω is continuous.

For $a > 0$, the function ρ is Lipschitz in the open set $\mathring{\Gamma}_a$ (interior of Γ_a) by Lemma 8.2.12, with a constant that we denote as δ_a^{-1}.

We consider $\Upsilon_a := \Xi_{\mathring{\Gamma}_a} \subseteq X$, we have that it is an open, and let $\omega_1, \omega_2 \in \Upsilon_r$, if $\bar{\rho}(\omega_1) > \bar{\rho}(\omega_2)$ then (recalling that $\bar{\rho}(\omega_1) \leq \bar{\rho}(\omega_1(t))$ for all t)

$$\frac{|\bar{\rho}(\omega_1) - \bar{\rho}(\omega_2)|}{\|\omega_1 - \omega_2\|_X} = \frac{\bar{\rho}(\omega_1)}{\|\omega_1 - \omega_2\|_X} - \frac{\rho(\omega_2(\tau(\omega_2)))}{\|\omega_1 - \omega_2\|_X} \leq$$

$$\leq \frac{\rho(\omega_1(\tau(\omega_2)))}{\|\omega_1 - \omega_2\|_X} - \frac{\rho(\omega_2(\tau(\omega_2)))}{\|\omega_1 - \omega_2\|_X} \leq \frac{\delta_a^{-1}\|\omega_1 - \omega_2\|_{C([0,1)]}}{\|\omega_1 - \omega_2\|_X} = \delta_a^{-1}.$$

We have that if $\omega \notin \Theta_{\partial_s\Omega}$, then $\mathrm{dist}(\omega([0,1]), \partial_s\Omega) = r' > 0$, and $\omega \in \Upsilon_a$ for some $a > 0$; so $X \backslash \Theta_{\partial_s\Omega} = \bigcup_{i=1}^{\infty} \Upsilon_{\frac{1}{i}}$, hence $\bar{\rho}$ is locally Lipschitz in $X \backslash \Theta_{\partial_s\Omega}$, and we can conclude. □

Our first issue is to give sufficient conditions to can use the Proposition 8.1.13 for the functional $\bar{\rho}$.

We can define for all $\omega \notin \Theta_{\partial_s\Omega}$ the set

$$A(\omega) := \{x \in \omega([0,1]), \rho(x) = \bar{\rho}(\omega)\}. \tag{8.3.1}$$

For $\omega \notin \Theta_{\partial_s\Omega}$ we get $\omega([0,1]) \subset \Gamma_a$ for some $a > 0$, hence, by Lemma 8.2.12, ρ is continuous on $\omega([0,1])$; this, together with the continuity of ω, yields that $A(\omega)$ is compact; δ is continuous on $\Gamma_a \backslash \Omega_1$ by Remark 8.2.9 (and it is 1 on Ω_1 and $\mathbb{R}^d \backslash \bar{\Omega}$, and less than 1 in the other points), hence it admits a minimum on $A(\omega)$.

For what we said, we can give the following definition.

DEFINITION 8.3.3. We define on $X\backslash\Theta_{\partial_s\Omega}$

$$\bar{\delta}_1(\omega) := \min_{x\in A(\omega)} \delta(x)$$

and

$$\bar{\delta}(\omega) := \begin{cases} \bar{\delta}_1(\omega) & \text{if } \omega \in \Xi_{\bar{\Omega}}\backslash\Theta_{\partial_s\Omega} \\ 1 & \text{otherwise} \end{cases}$$

(by recalling the definition of Ω_1 in the Section 8.2); clearly $0 < \bar{\delta}(\omega) \leq 1$.

LEMMA 8.3.4. *$\bar{\delta}$ is lower semicontinuous in $\Xi_{\bar{\Omega}}\backslash\Theta_{\partial_s\Omega}$; in particular, it is a Borel measurable function.*

PROOF. We have $\delta_{|\Omega_1} \equiv 1$, so $\bar{\delta}_{|\Xi_{\Omega_1}} \equiv 1$, hence in Ξ_{Ω_1} there is nothing to prove because it is open and $\bar{\delta} \leq 1$ everywhere.

Because of the lower semicontinuity, it suffices to prove that $\bar{\delta}$ is lower continuous on $\Xi_{\bar{\Omega}}\backslash(\Theta_{\partial_s\Omega}\cup \Xi_{\Omega_1})$; so, let ω_n be a sequence which uniformly converges to ω in $\Xi_{\bar{\Omega}}\backslash(\Theta_{\partial_s\Omega}\cup\Xi_{\Omega_1})$. Let $l := \liminf_{n\to\infty}\bar{\delta}(\omega_n)$, we want to prove that $l \geq \bar{\delta}(\omega)$.

If $l = 1$, there is nothing to prove.

Let $l < 1$. For every $n \in \mathbb{N}$ there exists $x_n \in A(\omega_n)$ s.t. $\delta(x_n) = \bar{\delta}(\omega_n)$, so $\delta(x_n) \to l$; there exists a sequence t_n s.t. $x_n = \omega_n(t_n)$, up to a subsequence $t_n \to t$, let $x := \omega(t)$, we get $x_n \to x$; by the continuity of $\bar{\rho}$ out of $\Theta_{\partial_s\Omega}$ and the continuity of ρ out of $\partial_s\Omega$, we have $x \in A(\omega)$; clearly $x \notin \Omega_1$ because $l < 1$, $\delta_{|\Omega_1} \equiv 1$ and Ω_1 is open; hence, by the continuity of δ in $\bar{\Omega}\backslash(\partial_s\Omega\cup\Omega_1)$ (Remark 8.2.9), we have $\delta(x) = l$, so $l \geq \bar{\delta}(\omega)$.

Hence we proved the lower semicontinuity in $\Xi_{\bar{\Omega}}\backslash\Theta_{\partial_s\Omega}$; by this and by $\Xi_{\bar{\Omega}}\backslash\Theta_{\partial_s\Omega} \in \mathfrak{B}(X)$, we get the $\mathfrak{B}(X)$-measurability (see Lemma 1.2.8). □

DEFINITION 8.3.5. For all $n \in \mathbb{N}$, $n > 1$ we define the set

$$Y_n := \left\{\omega \in X\backslash\Theta_{\partial_s\Omega} | \bar{\delta}(\omega) \leq \frac{1}{n-1}\right\},$$

$$X_n := \left\{\omega \in X\backslash\Theta_{\partial_s\Omega} | \frac{1}{n} < \bar{\delta}(\omega) \leq \frac{1}{n-1}\right\},$$

by Lemma 8.3.4, X_n and Y_n are Borel sets.

REMARK 8.3.6. The sets X_n are mutually disjoint; $\bigcup_{n=2}^{\infty} X_n = X\backslash\Theta_{\partial_s\Omega}$.

LEMMA 8.3.7. *In this setting, for every $n \in \mathbb{N}$, we have that, in each point of X_n, the Lipschitz constant of $\bar{\rho}$ (as function on X), is less than n.*

PROOF. We use the Remark 8.1.14: given $\omega \in X_n$ and $\omega_1 \in X$ we want to find a l_0 s.t. if $l < l_0$ then

$$|\bar{\rho}(\omega + l\omega_1) - \bar{\rho}(\omega)| \leq nt\,\|h\|_X\,;$$

it suffices to prove that there exists $c_\omega > 0$ s.t. for a generic ω_1 with $\|\omega_1\|_X \leq c$ we get $|\bar{\rho}(\omega + \omega_1) - \bar{\rho}(\omega)| \leq n\,\|\omega_1\|_X$.

Hereafter, $B(A(\omega), r)$ is the set of points of $\mathbb{R}^d$ at distance from $A(\omega)$ less than r.

Case 1): $\omega \in X_n \cap \Xi_{\Omega_1}$; hence the local n-Lipschitzianity it is clear because $\rho_{|\Omega_1} = 1$ and Ξ_{Ω_1} is open;

Case 2) $\omega \in X_n \cap \Xi_\Omega \backslash \Xi_{\Omega_1}$: Ξ_Ω is open , therefore there exists $c_1 > 0$ s.t. for a generic ω_1 with $\|\omega_1\|_X \le c_1$ we get $\omega + \omega_1 \in \Xi_\Omega$.

$\omega \notin \Theta_{\partial_s \Omega}$ by definition of X_n, so $\operatorname{dist}(A(\omega), \partial_s \Omega) > 0$, moreover $\operatorname{dist}(\omega([0,1]), \Omega^c) > 0$ (because $\omega \in \Xi_\Omega$ and Ω is open); by $\omega \in X_n$ we get $\delta(\omega(x)) > \frac{1}{n}$ for all $x \in A(\omega)$; we know that δ is continuous in $\bar{\Omega} \backslash \partial \Omega_s$ (Remark 8.2.9), and $\delta = 1$ in $\bar{\Omega}^c$ so there exists an $r > 0$ s.t. $\delta(x) > \frac{1}{n}$ for all $x \in B(A(\omega), r)$; hence, ρ is n-Lipschitz in $B(A(\omega), r) \cap \bar{\Omega}$ by Remark 8.2.5, therefore

$$\bar{\rho} \text{ is } n\text{-Lipschitz in } B(A(\omega), r) \tag{8.3.2}$$

because $\rho = 0$ on $\bar{\Omega}^c$.

By recalling (8.2.5), the compact $\omega([0,1]) \backslash B(A, \frac{r}{2})$ is contained into the open $\Omega^*_{\bar{\rho}(\omega)}$ by the definition of $A(\omega)$, hence $\operatorname{dist}(\omega([0,1]) \backslash B(A(\omega), \frac{r}{2}), \mathbb{R}^d \backslash \Omega^*_{\bar{\rho}(\omega)}) =: r' > 0$.

Now we consider a $\omega_1 \in X$ s.t. $\|\omega_1\|_X < \frac{r}{2} \wedge r' \wedge c_1$ (recalling that $\|\cdot\|_X$ is the L^∞ norm): let $t \in [0,1]$ s.t. $\omega(t) \in B(A(\omega), \frac{r}{2})$, we have $(\omega + \omega_1)(t) \in B(A(\omega), r) \cap \Omega$ (by $\|\omega_1\|_X < \frac{r}{2}$ and $\omega + \omega_1 \in \Xi_\Omega$), hence by (8.3.2)

$$|\rho((\omega + \omega_1)(t)) - \rho((\omega)(t))| \le n \|\omega_1\|_X$$

so

$$\rho((\omega + \omega_1)(t)) \ge \rho(\omega(t)) - n\|\omega_1\|_X \ge \bar{\rho}(\omega) - n\|\omega_1\|_X. \tag{8.3.3}$$

Moreover, by considering t_0 s.t. $\omega(t_0) \in A(\omega) \subset B(A(\omega), \frac{r}{2})$, we have also

$$\bar{\rho}(\omega) + n\|\omega_1\|_X = \rho(\omega(t_0)) + n\|\omega_1\|_X \ge \rho((\omega + \omega_1)(t_0)) \ge \bar{\rho}(\omega + \omega_1). \tag{8.3.4}$$

Let $t \in [0,1]$ s.t. $\omega(t) \notin B(A, \frac{r}{2})$ then $(\omega + \omega_1)(t) \in \Omega^*_{\bar{\rho}(\omega)}$ (by $\|\omega_1\|_X < r'$), i.e.

$$\rho((\omega + \omega_1)(t)) > \bar{\rho}(\omega),$$

so by this and (8.3.3), we have for every $t \in [0,1]$,

$$\rho(\omega + \omega_1)(t) \ge \bar{\rho}(\omega) - n\|\omega_1\|_X \tag{8.3.5}$$

hence

$$\bar{\rho}(\omega + \omega_1) \ge \bar{\rho}(\omega) - n\|\omega_1\|_X. \tag{8.3.6}$$

Now, by (8.3.4) and (8.3.6), and by the generality of ω_1 we have that $\bar{\rho}$ is Lipschitz in a neighbourhood of ω with constant n.

Case 3) $\omega \in X_n \cap \Theta_{\Omega^c} \cap \Xi_{\bar{\Omega}} \backslash \Xi_{\Omega_1}$: so, $A(\omega)$ intersects $\partial\Omega \backslash \partial_s \Omega$ and $\bar{\rho}(\omega) = 0$ (recalling that $\rho = 0$ on $\partial\Omega \backslash \partial_s \Omega$); we can repeat the arguments of Case 2), but we do not define c_1 and we impose $\|\omega_1\|_X < \frac{r}{2} \wedge r'$ (instead of $\|\omega_1\|_X < \frac{r}{2} \wedge r' \wedge c_1$): if $\omega + \omega_1 \notin \Xi_{\bar{\Omega}}$ then

$$\bar{\rho}(\omega + \omega_1) = \bar{\rho}(\omega) = 0; \tag{8.3.7}$$

otherwise, let $t \in [0,1]$ s.t. $\omega(t) \in B(A(\omega), \frac{r}{2})$, we have $(\omega + \omega_1)(t_0) \in B(A(\omega), r) \cap \bar{\Omega}$ (by $\|\omega_1\|_X < \frac{r}{2}$ and $\omega + \omega_1 \in \Xi_{\bar{\Omega}}$), hence in both cases

$$|\rho((\omega + \omega_1)(t)) - \rho(\omega)(t)| \le n\|\omega_1\|_X,$$

and we can deduce (8.3.4) and (8.3.3); (8.3.5) in this case is obvious because $\bar{\rho}(\omega) = 0$, and we can conclude in the same way of Case 2)

Case 4) $\omega \in \Theta_{\bar{\Omega}^c}$ then obviously $\bar{\rho}$ is 0 in a neighbourhood, and there is nothing to prove. $\square$

8.3.2. Stochastic properties of functions on $C([0,1],\mathbb{R}^d)$. Also in this subsection, we write $X = C([0,1],\mathbb{R}^d)$.

We recall that the Brownian motion can be described by a Markov process

$$Z = (\mathcal{A}, \mathscr{F}, \{Z_t\}_{t\in[0,+\infty]}, \{\mu_x\}_{x\in\mathbb{R}^d_\partial}, \{\mathscr{F}_t\}_{t\in[0,+\infty]})$$

(see Subsection 1.3.4).

We define $i : \mathcal{A} \to X$, $a \mapsto (t \mapsto Z_t(a))$ and, for every $x \in \mathbb{R}^d$,

$$P_x := i^{-1} \circ \mu_x$$

is a probability on X which describes the d-dimensional Brownian motion with starting point in x.

In the rest of this subsection we suppose the following Hypotheses on Ω are true.

HYPOTHESIS 8.3.8. For every $x \notin \partial_s\Omega$, we assume $P_x(\Theta_{\partial_s\Omega}) = 0$.

REMARK 8.3.9. By Hypothesis 8.3.8, $\omega([0,1] \cap \partial_s\Omega = \varnothing$ P_x-almost surely, hence in particular $\rho(\omega(\cdot))$ is continuous P_x-almost surely by Lemma 8.2.12.

We recall $\mathcal{H}^d_s$ defined in Definition 8.1.1.

HYPOTHESIS 8.3.10. For some $l > 0$, for $s < 1$, it is verified $\mathcal{H}^{d-2}_s(\partial_s\Omega) \le cs^l$ for some $c > 0$ independent of s (i.e. the set $\partial_s\Omega$ satisfies the Hypothesis 8.1.4).

HYPOTHESIS 8.3.11. There exists $c_2 > 0$ s.t. for all $r > 0$ with $r < 1$ we have $\delta(x) > c_2 r$ for all $x \notin B(\partial_s\Omega, r)$.

REMARK 8.3.12. Let

$$U := B(\partial_s\Omega, c_2^{-1}(n-1)^{-1})$$

the set of points with distance from $\partial_s\Omega$ less than $c_2^{-1}(n-1)^{-1}$. If the Hypothesis 8.3.11 is verified, then, $Y_n \subset \Theta_{\overline{U}}$ and, if $\omega \in Y_n$ then $A(\omega) \cap \overline{U} \neq \varnothing$.

In fact, if $\omega \in Y_n$, then $\bar{\delta}(\omega) \le \frac{1}{n-1}$, so there exists $x \in A(\omega)$ s.t. $\delta(x) \le \frac{1}{n-1}$ and so by the hypothesis $x \in \overline{U}$.

EXAMPLE 8.3.13. By Remark 8.2.3, if Ω satisfies an uniform outer ball condition for some radius $r > 0$, then $\partial_s\Omega = \varnothing$ (by Remark 8.2.8) and $\delta(x) > r$ everywhere in Ω (see Remark 8.2.3), in particular it satisfies Hypotheses 8.3.10 and 8.3.11.

COROLLARY 8.3.14. *Let $x \notin \partial_s\Omega$, P_x be the probability associated to the Brownian motion with starting point x. If Hypotheses 8.3.8, 8.3.10 and 8.3.11 are true, then there exists $C > 0$ and $n_1 \in \mathbb{N}$ (dependent on Ω, x) s.t. for all $n \in \mathbb{N}$, $n > n_1$ we have*

$$P_x(X_{n+1}) \le P_x(Y_{n+1}) \le P_x(\Theta_{\overline{B(\partial_s\Omega,\frac{1}{n})}}) < C(n-1)^{-l},$$

for the l in Hypothesis 8.3.10.

PROOF. We have that, by Hypothesis 8.3.11,

$$X_n \subseteq Y_n \subseteq \Theta_{\overline{B(\partial_s\Omega,\frac{1}{n})}} \subset \Theta_{B(\partial_s\Omega,\frac{1}{n-1})};$$

we suppose that $n > 1$, $n > \mathrm{dist}(x,\partial_s\Omega)^{-1}$; clearly

$$P_x(\Theta_{B(\partial_s\Omega,\frac{1}{n-1})}) \le \mu_x(\tau_{\frac{1}{n}} < \infty])$$

where $\tau_{\frac{1}{n}}$ is the hitting time of $B(\partial_s\Omega,\frac{1}{n})$ so by Hypothesis 8.3.10 we can apply Corollary 8.1.7

$$P_x(\Theta_{B(\partial_s\Omega,\frac{1}{n})})\leq C_1\mathcal{H}_s^{d-2}(\partial_s\Omega),$$

for some $C>0$ (depending on Ω, x); hence we conclude. $\square$

To prove the next result, we will argue as in the proof of [[**46**], Proposition 3.1] with $\|x-y\|-\delta'(y)$ (for $y\in M(x)$) instead of $q(x)$ and $\delta(x)<1$ instead of δ.

LEMMA 8.3.15. $\exists C_1>0$ *s.t. for all* $u\in(0,1]$, $x\in\Omega$,

$$P_x\{\omega\in X\backslash\Theta_{\partial_s\Omega}|0<\inf_{t\in[0,u]}\rho(\omega(t))\}\leq C_1g_1(x)u^{-\frac{1}{2}}.$$

PROOF. Clearly, by the continuity of ρ out of $\partial_s\Omega$,

$$\{\omega\in X\backslash\Theta_{\partial_s\Omega}|0<\inf_{t\in[0,u]}\rho(\omega(t))\}$$

is open.

Given $x\in\Omega$, let $y\in M(x)$. We recall that $\rho(z)=0$ for $z\in\Omega^c\backslash\partial_s\Omega$, and that the ball $B(y,\delta'(y))$ is in $\bar{\Omega}^c$, so

$$\{X\backslash\Theta_{\partial_s\Omega}|0<\inf_{t\in[0,u]}\rho(\omega(t))\}\subseteq\{\omega\in X|\omega([0,u])\cap\bar{\Omega}^c=\varnothing\}\subseteq$$

$$\subseteq\{\omega\in X|\omega([0,u])\cap B(y,\delta'(y))=\varnothing\}=\{\omega\in X|\inf_{t\in[0,u]}\|\omega(t)-y\|\geq\delta'(y)\}$$

(clearly they are all Borel subsets of X); now, we can apply Lemma 1.3.20 (μ corresponds to P_x), a is substituted by $\delta'(y)$, r is substituted by $\|x-y\|-\delta'(y)$: for the $c>0$ of that Lemma we have (by recalling $\delta'(y)\leq 1$, $u\leq 1$)

$$P_x\{X\backslash\Theta_{\partial_s\Omega}|0<\inf_{t\in[0,u]}\rho(\omega(t))\}\leq$$

$$\leq c\left(\frac{\|x-y\|-\delta'(y)}{\delta'(y)}+u^{-\frac{1}{2}}\left(\|x-y\|-\delta(y)\right)\right)\leq C_1g(x,y)u^{-\frac{1}{2}}=C_1g_1(x)u^{-\frac{1}{2}}$$

for some $C_1:=c+1$ independent of x,u. $\square$

Hereafter, let

$$\Lambda_r:=\Theta_{E_r}\cup\Xi_\Omega$$

where $E_r:=\{x\in\Omega\backslash\Theta_{\partial_s\Omega}|\rho_1(x)\leq r\}$ (it is closed in $\Omega\backslash\Theta_{\partial_s\Omega}$); clearly Λ_r is a Borel set.

LEMMA 8.3.16. *Let U' be a closed set s.t.*

$$U'\subseteq\partial_s\Omega\cup\{x\in\mathbb{R}^d\backslash\partial_s\Omega|\rho(x)\leq r\},$$

τ' be the hitting time associated to U', Z be the Markov process introduced above, associated to a Brownian motion. There exists $C>0$ s.t. for every $x\in\Omega$, for μ_x-almost every $a\in\mathcal{A}$, for $k,l,c\in(0,1)$

$$\mu_x\left(k<1-\tau'\leq l,\ 0<\inf_{t\in[0,cl]}\rho(Z_{\tau'+t})|\mathcal{F}_{\tau'(a)}\right)(a)\leq$$

$$\leq \mathbb{1}_{(k,l]}(1-\tau'(a))\cdot Cc^{-\frac{1}{2}}r\left(1-\tau'(a)\right)^{-\frac{1}{2}}.$$

PROOF. τ' is the hitting time associated to U, hence it corresponds to the exit time of the open U^c (and $x \in U^c$). We will use $i : \mathcal{A} \to X$; hereafter we will write also ω_a to mean $i(a)$ with $a \in \mathcal{A}$ (ω_a will be a part of the sample path of a).

We recall that $\rho \circ \omega_a$ is continuous for μ_x-almost every $a \in \mathcal{A}$ by Remark 8.3.9 hence, for the definition of τ', we have that $\rho(\omega(\tau')) \leq r < 1$ P_x-almost surely, so $\rho(\omega(\tau')) = g_1(\omega(\tau')) \leq r$; therefore, by Lemma 8.3.15

$$\mu_{\omega_a(\tau'(a))}\left(\left[0 < \inf_{t\in[0,cl]} \rho(Z_t)\right]\right) =$$

$$= P_{\omega_a(\tau'(a))}\{\omega \in X | 0 < \inf_{t\in[0,cl]} \rho(\omega(t))\} \leq C_1 c^{-\frac{1}{2}} r l^{-\frac{1}{2}}, \tag{8.3.8}$$

for μ_x-almost every $a \in \mathcal{A}$; hence, arguing as in [**46**], we have the following calculation (E_x will be the mean value of a function with respect to the measure given by μ_x): for μ_x-almost every $a \in \mathcal{A}$

$$\mu_x\left(k < 1-\tau' \leq l,\ 0 < \inf_{t\in[0,cl]} \rho(Z_{\tau'+t}) | \mathcal{F}_{\tau'(a)}\right)(a) =$$

(because $\chi_{(k,l]}(1-\tau'(\cdot))$ is $\mathcal{F}_{\tau'(a)}$ measurable)

$$= 1\!\!1_{(k,l]}(1-\tau'(a)) \cdot \mu_x\left(0 < \inf_{t\in[0,cl]} \rho(Z_{\tau'+t}) | \mathcal{F}_{\tau'(a)}\right)(a) =$$

(by the strong Markov property, and the fact that the set defined by $0 < \inf_{t\in[0,k]} \rho(\cdot)$ is Borel)

$$\leq 1\!\!1_{(k,l]}(1-\tau'(a)) \cdot \mu_{\omega_a(\tau'(a))}\left(\left[0 \leq \inf_{t\in[0,cl]} \rho(Z_t)\right]\right) \leq$$

(we know that $\rho(\omega_a(\tau')) \leq r$ for μ_x-almost every $a \in \mathcal{A}$, so by (8.3.8) there exists C s.t.)

$$\leq 1\!\!1_{(k,l]}(1-\tau'(a)) \cdot C c^{-\frac{1}{2}} r l^{-\frac{1}{2}} \leq$$

$$\leq 1\!\!1_{(k,l]}(1-\tau'(a)) \cdot C c^{-\frac{1}{2}} r \left(1-\tau'\right)^{-\frac{1}{2}}$$

and we can conclude. □

LEMMA 8.3.17. *Let $x_0 \in \Omega$. In our hypothesis there exists $C > 0$, s.t., for all $0 < r < 1$,*

$$P_{x_0}\left(\{\omega \in X | 0 < \inf_{t\in[0,1]} \rho(\omega(t)) \leq r\}\right) \leq Cr \tag{8.3.9}$$

for the l in Hypothesis 8.3.10.

PROOF. $\rho(x_0) > 0$, so it is not restrictive to suppose $r < \rho(x_0)$.

We will use the Markov process

$$Z = (\mathcal{A}, \mathscr{F}, \{Z_t\}_{t\in[0,+\infty]}, \{\mu_x\}_{x\in\mathbb{R}^d_\partial}, \{\mathscr{F}_t\}_{t\in[0,+\infty]});$$

we recall that Z is a strong Markov process, and it has the strong Markov property (see Definition 1.3.12); we want to prove for some $C > 0$

$$\mu_{x_0}(i^{-1}(\Lambda_r)) \leq Cr,$$

where $i : \mathcal{A} \to X$, $a \mapsto (t \mapsto Z_t(a))$.

We recall the set $A(\omega) = \{x \in \omega([0,1]), \rho(x) = \bar{\rho}(\omega)\}$.

We define the closed set

$$U := \partial_s\Omega \cup \{x \in \mathbb{R}^d \setminus \partial_s\Omega | \rho(x) \le r\}$$

(it is closed because ρ is continuous out of $\partial_s\Omega$); we have $x_0 \notin U$.

We define τ' the hitting time associated to U (it corresponds to the exit time of the open U^c) for the Brownian motion with starting point $x_0 \in U^c$; it is clear

$$i^{-1}(\Theta_U) = \{a \in \mathcal{A} | \tau'(a) \le 1\};$$

and

$$i^{-1}(X_{n+1} \cap \Lambda_r) \subseteq i^{-1}(\Theta_U \cap \Xi_\Omega) = \{a \in \mathcal{A} | \tau'(a) \le 1, 0 < \inf_{t\in[0,1]} \rho(Z_t(a))\} \subseteq$$

$$\subseteq \bigcup_{k=1}^{\infty} \{a \in \mathcal{A} | 2^{-k} < 1-\tau'(a) \le 2^{-k+1}, 0 < \inf_{t\in[0,1]} \rho(Z_t(a))\} \cup \{a \in \mathcal{A} | \tau'(a) = 1\} \subseteq$$

$$\subseteq \bigcup_{k=1}^{\infty} (2^{-k} < 1-\tau'(a) \le 2^{-k+1}, 0 < \inf_{t\in[0,2^{-k}]} \rho(Z_{\tau'+t}(a))) \cup (\tau'(a) = 1);$$

we have that $\mu_x(\{\tau' = 1\}) = 0$ due to Lemma 8.1.9; by Lemma 8.3.16 there exists $C_1 > 0$ s.t. for every $x \in \Omega$, $k \in \mathbb{N}$ and for μ_x-almost every $a \in \mathcal{A}$,

$$\mu_x\left(2^{-k} < 1-\tau' \le 2^{-k+1},\ 0 < \inf_{t\in[0,2^{-k}]} \rho(Z_{\tau'+t}) | \mathcal{F}_{\tau'(a)}\right)(a) =$$

$$\le 1\!\!1_{(2^{-k},2^{-k+1}]}(1-\tau'(a)) \cdot C_1 r \left(\frac{1-\tau'(a)}{2}\right)^{-\frac{1}{2}}.$$

In the following $\mathbb{E}_x$ will be the expected value respect to the probability μ_x, and $\mathbb{E}_x(\cdot,\mathcal{F})$ will be the conditional expected value with respect to the probability μ_x.

Defining $P_{x_0}^{\tau'} := \tau'^{-1} \circ \mu_x$ measure on $[0,+\infty]$ (it is the law of τ' under P_x), arguing as in [**46**], Prop. 3.3 (but by using the Lemma 8.1.9 instead of [**46**], Lem. 3.2), we have that

$$P_{x_0}\left(\left[0 < \inf_{t\in[0,1]} \rho(\omega(t)) \le r\right]\right) = P_{x_0}(i^{-1}(\Theta_U \cap \Xi_\Omega)) \le$$

$$\le \mu_{x_0}(\bigcup_{k=1}^{\infty} (2^{-k} < 1-\tau' \le 2^{-k+1}, 0 < \inf_{t\in[0,2^{-k}]} \rho(Z_{\tau'+t})) \cup (\tau' = 1)) =$$

(recalling that $\mu_x(\{\tau' = 1\}) = 0$, and by the properties of the conditional probability in Proposition 1.3.9 and of conditional expectation in Proposition 1.3.7)

$$= \mathbb{E}_{x_0}\left[\mu_{x_0}(\bigcup_{k=1}^{\infty} (2^{-k} < 1-\tau'(a) \le 2^{-k+1}, 0 < \inf_{t\in[0,2^{-k}]} \rho(Z_{\tau'+t}) | \mathcal{F}_{\tau'})\right] \le$$

$$\le \mathbb{E}_{x_0}\left[\sum_{k=1}^{\infty} \mu_{x_0}\left(\left[2^{-k} < 1-\tau' \le 2^{-k+1},\ 0 < \inf_{t\in[0,2^{-k}]} \rho(Z_{\tau'+t})\right] | \mathcal{F}_{\tau'}\right)\right] =$$

$$= \sum_{k=1}^{\infty} \mathbb{E}_{x_0}\left[\mu_{x_0}\left(\left[2^{-k} < 1-\tau' \le 2^{-k+1},\ 0 < \inf_{t\in[0,2^{-k}]} \rho(Z_{\tau'+t})\right] | \mathcal{F}_{\tau'}\right)\right] \le$$

$$\le \sum_{k=1}^{\infty} \mathbb{E}_{x_0}\left[1_{\{2^{-k} < 1-\tau' \le 2^{-k+1}\}} \cdot C_1 r \left(\frac{1-\tau'}{2}\right)^{-\frac{1}{2}}\right] =$$

$$= \mathbb{E}_{x_0}\left[1_{\{\tau'\leq 1\}}\cdot C_1 r\left(\frac{1-\tau'}{2}\right)^{-\frac{1}{2}}\right] =$$

(we can apply Lemma 8.1.9, because τ' is the exit time from an open, so $P^{\tau'}_{x_0}((0,t))$ is differentiable)

$$= C_1 r\int_0^1 \frac{d}{dt}P^{\tau'}_{x_0}((0,t))\left(\frac{1-t}{2}\right)^{-\frac{1}{2}}\, dt$$

$$\leq C_1 r\Bigg(\left(\frac{1-\frac{1}{2}}{2}\right)^{-\frac{1}{2}}\int_0^{\frac{1}{2}}\frac{d}{dt}P^{\tau'}_{x_0}((0,t))\, dt +$$

$$+\sup_{t\in[\frac{1}{2},1]}\frac{d}{dt}P^{\tau'}_{x_0}((0,t))\int_{\frac{1}{2}}^1\left(\frac{1-t}{2}\right)^{-\frac{1}{2}}\, dt\Bigg)\leq$$

(by Lemma 8.1.9, $\sup_{t\in[\frac{1}{2},1]}\frac{d}{dt}P^{\tau'}_{x_0}((0,t)) < c\sqrt{P^{\tau'}_x([0,1])}$ for some $c>0$ independent on n,r)

$$\leq C_1 r\left(P^{\tau'}_x\left(\left(0,\frac{1}{2}\right)\right)+c\sqrt{P^{\tau'}_x([0,1])}\right)\leq rC_2$$

for some $C_2>0$ and because $U\subseteq \overline{B(\partial_s\Omega,(c_2 n)^{-1})}$; clearly C_2 is independent on r, and we concluded. ☐

PROPOSITION 8.3.18. *Let $x_0\in\Omega$. In our hypothesis there exists $C>0$, $n_0\in\mathbb{N}$ s.t., for all $0<r<1$, $n\in\mathbb{N}$, $n>n_0$*

$$P_{x_0}\left(X_{n+1}\cap\{\omega\in X|0<\inf_{t\in[0,1]}\rho(\omega(t))\leq r\}\right)\leq Cr(n-1)^{\frac{-l}{2}} \tag{8.3.10}$$

for the l in Hypothesis 8.3.10.

PROOF. We have

$$X_n\cap\{\omega\in X|0<\inf_{t\in[0,1]}\rho(\omega(t))\leq r\} = X_n\cap\Lambda_r$$

because $X_n\cap\Theta_{\partial_s\Omega}=\varnothing$, hence it is a Borel set.

We will use the Markov process

$$Z=(\mathcal{A},\mathscr{F},\{Z_t\}_{t\in[0,+\infty]},\{\mu_x\}_{x\in\mathbb{R}^d_\partial},\{\mathscr{F}_t\}_{t\in[0,+\infty]});$$

we recall that Z is a strong Markov process, and it has the strong Markov property (see Definition 1.3.12); we want to prove

$$\mu_{x_0}(i^{-1}(X_n\cap\Lambda_r))\leq Crn^{\frac{-l}{2}},$$

where $i:\mathcal{A}\to X$, $a\mapsto(t\mapsto Z_t(a))$.

We recall the set $A(\omega)=\{x\in\omega([0,1]),\rho(x)=\bar{\rho}(\omega)\}$.

We suppose $n_0>\mathrm{dist}(x_0,\partial_s\Omega)^{-1}$.

Let c_2 the constant in Hypothesis 8.3.11; for $n\in\mathbb{N}$, $n>n_0$ we define the closed set

$$U:=\partial_s\Omega\cup\left(\overline{B(\partial_s\Omega,(c_2 n)^{-1})}\cap\{x\in\mathbb{R}^d\backslash\partial_s\Omega|\rho(x)\leq r\}\right)$$

(it is closed because ρ is continuous out of $\partial_s\Omega$); we have $x_0\notin U$.

By Remark 8.3.12, if $\omega\in X_{n+1}\subset Y_{n+1}$, then there exists

$$x\in A(\omega)\cap\overline{B(\partial_s\Omega,(c_2 n)^{-1})}$$

so $\rho(x)=\bar{\rho}(\omega)\leq r$, therefore

$$X_{n+1}\cap\Lambda_r\subseteq\Theta_U\cap\Xi_\Omega.$$

We define τ' the hitting time associated to U (it corresponds to the exit time of the open U^c) for the Brownian motion with starting point $x_0\in U^c$; it is clear

$$i^{-1}(\Theta_U)=\{a\in\mathcal{A}|\tau'(a)\leq 1\};$$

and

$$i^{-1}(X_{n+1}\cap\Lambda_r)\subseteq i^{-1}(\Theta_U\cap\Xi_\Omega)=\{a\in\mathcal{A}|\tau'(a)\leq 1, 0<\inf_{t\in[0,1]}\rho(Z_t(a))\}\subseteq$$

$$\subseteq\bigcup_{k=1}^{\infty}\{a\in\mathcal{A}|2^{-k}<1-\tau'(a)\leq 2^{-k+1}, 0<\inf_{t\in[0,1]}\rho(Z_t(a))\}\cup\{a\in\mathcal{A}|\tau'(a)=1\}\subseteq$$

$$\subseteq\bigcup_{k=1}^{\infty}(2^{-k}<1-\tau'(a)\leq 2^{-k+1}, 0<\inf_{t\in[0,2^{-k}]}\rho(Z_{\tau'+t}(a)))\cup(\tau'(a)=1);$$

we have that $\mu_x(\{\tau'=1\})=0$ due to Lemma 8.1.9; by Lemma 8.3.16 there exists $C_1>0$ s.t. for every $x\in\Omega$, $k\in\mathbb{N}$ and for μ_x-almost every $a\in\mathcal{A}$,

$$\mu_x\left(2^{-k}<1-\tau'\leq 2^{-k+1},\ 0<\inf_{t\in[0,2^{-k}]}\rho(Z_{\tau'+t})|\mathcal{F}_{\tau'(a)}\right)(a)=$$

$$\leq 1\!\!1_{(2^{-k},2^{-k+1}]}(1-\tau'(a))\cdot C_1 r\left(\frac{1-\tau'(a)}{2}\right)^{-\frac{1}{2}}.$$

In the following $\mathbb{E}_x$ will be the expected value respect to the probability μ_x, and $\mathbb{E}_x(\cdot,\mathcal{F})$ will be the conditional expected value with respect to the probability μ_x.

Defining $P_{x_0}^{\tau'}:=\tau'^{-1}\circ\mu_x$ measure on $[0,+\infty]$ (it is the law of τ' under P_x), arguing as in [**46**], Prop. 3.3 (but by using the Lemma 8.1.9 instead of [**46**], Lem. 3.2), we have that

$$P_{x_0}\left(X_{n+1}\cap\left[0<\inf_{t\in[0,1]}\rho(\omega(t))\leq r\right]\right)=P_{x_0}(i^{-1}(\Theta_U\cap\Xi_\Omega))\leq$$

$$\leq\mu_{x_0}(\bigcup_{k=1}^{\infty}(2^{-k}<1-\tau'\leq 2^{-k+1}, 0<\inf_{t\in[0,2^{-k}]}\rho(Z_{\tau'+t}))\cup(\tau'=1))=$$

(recalling that $\mu_x(\{\tau'=1\})=0$, and by the properties of the conditional probability in Proposition 1.3.9 and of conditional expectation in Proposition 1.3.7)

$$=\mathbb{E}_{x_0}\left[\mu_{x_0}(\bigcup_{k=1}^{\infty}(2^{-k}<1-\tau'(a)\leq 2^{-k+1}, 0<\inf_{t\in[0,2^{-k}]}\rho(Z_{\tau'+t})|\mathcal{F}_{\tau'})\right]\leq$$

$$\leq\mathbb{E}_{x_0}\left[\sum_{k=1}^{\infty}\mu_{x_0}\left(\left[2^{-k}<1-\tau'\leq 2^{-k+1},\ 0<\inf_{t\in[0,2^{-k}]}\rho(Z_{\tau'+t})\right]|\mathcal{F}_{\tau'}\right)\right]=$$

$$=\sum_{k=1}^{\infty}\mathbb{E}_{x_0}\left[\mu_{x_0}\left(\left[2^{-k}<1-\tau'\leq 2^{-k+1},\ 0<\inf_{t\in[0,2^{-k}]}\rho(Z_{\tau'+t})\right]|\mathcal{F}_{\tau'}\right)\right]\leq$$

$$\leq\sum_{k=1}^{\infty}\mathbb{E}_{x_0}\left[1_{\{2^{-k}<1-\tau'\leq 2^{-k+1}\}}\cdot C_1 r\left(\frac{1-\tau'}{2}\right)^{-\frac{1}{2}}\right]=$$

$$= \mathbb{E}_{x_0}\left[1_{\{\tau'\le 1\}}\cdot C_1 r\left(\frac{1-\tau'}{2}\right)^{-\frac{1}{2}}\right] =$$

(we can apply Lemma 8.1.9, because τ' is the exit time from an open, so $P^{\tau'}_{x_0}((0,t))$ is differentiable)

$$= C_1 r\int_0^1 \frac{d}{dt}P^{\tau'}_{x_0}((0,t))\left(\frac{1-t}{2}\right)^{-\frac{1}{2}}\,dt$$

$$\le C_1 r\Bigg(\left(\frac{1-\frac{1}{2}}{2}\right)^{-\frac{1}{2}}\int_0^{\frac{1}{2}}\frac{d}{dt}P^{\tau'}_{x_0}((0,t))\,dt+$$

$$+\sup_{t\in[\frac{1}{2},1]}\frac{d}{dt}P^{\tau'}_{x_0}((0,t))\int_{\frac{1}{2}}^1\left(\frac{1-t}{2}\right)^{-\frac{1}{2}}\,dt\Bigg)\le$$

(by Lemma 8.1.9, $\sup_{t\in[\frac{1}{2},1]}\frac{d}{dt}P^{\tau'}_{x_0}((0,t))< c\sqrt{P^{\tau'}_x([0,1])}$ for some $c>0$ independent on n,r)

$$\le C_1 r\left(P^{\tau'}_x\left(\left(0,\frac{1}{2}\right)\right)+c\sqrt{P^{\tau'}_x([0,1])}\right)\le rC_2\sqrt{P^{\tau'}_x([0,1])}\le rC_3(n-1)^{\frac{l}{2}}$$

for some $C_2,C_3>0$ and $n>n_1$ (for some n_1) by Corollary 8.3.14 and because $U\subseteq\overline{B(\partial_s\Omega,(c_2n)^{-1})}$; clearly C_3 is independent on r, $n>n_0$. Let $n_0:=\mathrm{dist}(x_0,\partial_s\Omega)^{-1}+n_1$, and $C:=C_3$ we have that the inequality (8.3.10) is verified for every $n>n_0$.

So we concluded. □

8.3.3. Finite perimeter of Ξ_Ω. In the above subsections we considered $X=C([0,1],\mathbb{R}^d)$, for every x on X it is defined the probability P_x corresponding to the Brownian motion with starting point in x.

We have that, in particular, P_0 is a probability on X, that describes the d-dimensional Brownian motion with starting point in 0 (see Section 2.6), and P_0 concentrates on $X_*:=C_*([0,1],\mathbb{R}^d)$ let H be the Cameron-Martin space. We recall that (X_*,P_0) is a Wiener space.

For a set $A\subset\mathbb{R}^d$ we define Θ^* and Ξ^* in X_* in a way similar to Θ and Ξ:

$$\Theta^*_A:=\{\omega\in X_*|\exists t\in[0,1]\text{ s.t. }\omega(t)\in A\}=\Theta^*_A\cap X_*$$

and

$$\Xi^*_A:=\{\omega\in X_*|\omega(t)\in A\ \forall t\in[0,1]\}=\Xi^*_A\cap X_*.$$

We can restrict $\bar\rho$ on X_*.

We consider an open set $\Omega\subseteq X$ which satisfies Hypotheses 8.2.1, 8.3.8, 8.3.10, 8.3.11.

We make stronger assumptions about Hypothesis 8.3.10 and Hypothesis 8.3.8.

HYPOTHESIS 8.3.19. For $s<1$, the set $\mathcal{H}^{d-2}_s(\partial_s\Omega)\le cs^5$ for some $c>0$ independent of s (in other words, Hypothesis 8.3.10 is true for some $l\ge 5$).

HYPOTHESIS 8.3.20. In addition to Hypothesis 8.3.8 ($P_x(\Theta_{\partial_s\Omega})=0$ for every $x\notin\partial_s\Omega$) we suppose that $\Theta_{\partial_s\Omega}$ has null 2-*capacity* with respect to the measure γ, i.e. $C_2(\Theta^*_{\partial_s\Omega})=0$ (see Definition 8.1.12).

THEOREM 8.3.21. *Let $0\in\Omega$, we assume Hypotheses 8.2.1, 8.3.20, 8.3.11, 8.3.19, then $1_{\Xi^*_\Omega}$ is BV (with respect to the measure P_0).*

PROOF. The first step is to prove that $\bar{\rho} \in W^{1,1}(X_*)$, by using Proposition 8.1.13.

We recall that

$$P_0(\Theta^*_{\partial_s\Omega}) = P_0(\Theta_{\partial_s\Omega}) = 0$$

(by Hypothesis 8.3.20)

We Remark that $\bar{\rho} \in W^{1,1}(X_*)$: in $X_n \cap X_*$ we have that $\bar{\rho}$ has Lipschitz constant less or equal to n (Lemma 8.3.7), hence $|\nabla_H \bar{\rho}|_H \leq n$; moreover we have by Corollary 8.3.14, by n sufficiently large

$$P_0(X_n \cap X_*) \leq c(n-2)^{\frac{-l}{2}};$$

clearly $\sum_{i=2}^{+\infty}(n-2)^{\frac{-l}{2}} < +\infty$ because $l > 4$. So, by the Remarks 8.3.6 and recalling $C_2(\Theta_{\partial_s\Omega}) = 0$ by Hypothesis 8.3.20, we have all the hypotheses of the Proposition 8.1.13 with $l_n = n$, and we can apply it (recalling the Remark 8.1.15).

We define, for $j \in \mathbb{N}$,

$$\bar{\rho}_{(j)}(\omega) := j\bar{\rho}(\omega) \wedge 1,$$

clearly, $\bar{\rho}_{(j)}$ is jn-Lipschitz a.e. in X_n, for every $n \in \mathbb{N}$; by Corollary 8.3.14 there exists $n_1 \in \mathbb{N}$ s.t. for some $c > 0$ we have (by $l > 4$)

$$\sum_{n=n_1}^{\infty} j^2 n^2 P_0(X_n) = \sum_{n=n_1}^{\infty} c j^2 (n-2)^{2-l} < \infty,$$

so $\bar{\rho}_{(j)} \in W^{1,1}(X_*)$ arguing as to prove $\bar{\rho} \in W^{1,1}(X_*)$ above (by using Proposition 8.1.13).

We recall that $P_0(\Theta_{\partial_s\Omega}) = 0$ (because it has null capacity by Hypothesis 8.3.20).

By Lemma 8.3.2, $\bar{\rho}$ is locally Lipschitz out of $\Theta^*_{\partial_s\Omega}$, so we can locally define $\nabla_H \bar{\rho}$ almost everywhere. Let $\omega \in X^* \backslash (\Xi^*_\Omega \cup \Theta^*_{\partial_s\Omega})$, we have $\bar{\rho}(\omega) = 0$ (by $\rho_{|\Omega^c \backslash \partial_s\Omega} \equiv 0$), so it is a point of minimum, hence, on each line, if the restriction of $\bar{\rho}$ is derivable then it has 0 derivative; so $\nabla_H \bar{\rho} = 0$ almost everywhere on $X_* \backslash (\Xi^*_\Omega \cup \Theta^*_{\partial_s\Omega})$.

Let $U_j := \{x \in \Omega | \rho \geq \frac{1}{j}\}$. On the set Ξ_{U_j}, $\nabla_H \bar{\rho}_{(j)} = 0$ almost everywhere in a similar way ($\bar{\rho}_{(j)} \equiv 1$, so each it is a point of maximum).

For $j \to \infty$, U_j is an increasing sequence s.t. $\bigcup_{j=1}^{+\infty} U_j = \Omega$; in particular, for n_0 in Proposition 8.3.18, we have $P_0(\bigcup_{n=2}^{n_0} X_n \backslash U_j) \to 0$.

Now, by the chain rule (see Remark 3.2.8) and Remark 8.3.6

$$\left\|\nabla_H \bar{\rho}_{(j)}\right\|_{L^1(X_*,(H,\mu))} = \left\|\nabla_H \bar{\rho}_{(j)}\right\|_{L^1(\Xi_\Omega,(H,\mu))} = \left\| j 1\!\!1_{(0<\bar{\rho}<\frac{1}{j})} |\nabla_H \bar{\rho}|_H \right\|_{L^1(\Xi_\Omega,(H,\mu))} =$$

$$= \sum_{n=2}^{\infty} j \left\| 1\!\!1_{(0<\bar{\rho}<\frac{1}{j})} \nabla_H \bar{\rho} \right\|_{L^1(X_n \cap \Xi_\Omega,(H,\mu))} =$$

(by Lemma 8.3.7, Proposition 8.3.18)

$$\leq j \sum_{n=2}^{\infty} n P_0 \left[\omega \in X_n, 0 < \inf_{t\in[0,1]} \rho(\omega(t)) < j^{-1} \right] \leq j \sum_{n=2}^{n_0} n + C j \frac{1}{j} \sum_{n=n_0}^{\infty} (n-2)^{-\frac{l}{2}+1} < \infty$$

because $\frac{l}{2} > 2$.

So 1_{Ξ_Ω} is BV due to $\bar{\rho}_{(j)} \xrightarrow{L^1} 1_{\Xi_\Omega}$ and Theorem 4.1.3: in fact, by recalling that $P_0(\Theta_{\partial_s\Omega}) = 0$, that we have $\bar{\rho}_{(j)}(\omega) = 1$ if $\inf_{t\in[0,1]} \rho(\omega(t)) \geq j^{-1}$, and $\rho_{(j)}(\omega) = 0$ in

$$X \backslash (\Xi_\Omega \cup \Theta_{\partial_s\Omega}) \subset \{\omega \in X | \inf_{t\in[0,1]} \rho(\omega(t)) = 0\},$$

and $0 \le \bar{\rho}_{(j)} \le 1$, we have that

$$\left\|\bar{\rho}_{(j)} - 1_{\Xi_\Omega}\right\|_{L^1(X)} = \int_X \left(1_{\Xi_\Omega} - \bar{\rho}_{(j)}\right)\, dP_0 \le P_0\{\omega \in X\backslash\Theta_{\partial_s\Omega} | 0 < \inf_{t\in[0,1]} \rho(\omega(t)) \le j^{-1}\} \le$$

(by Remark 8.3.6)

$$\le \sum_{n=1}^{\infty} P_0\{\omega \in X_{n+1} | 0 < \inf_{t\in[0,1]} \rho(\omega(t)) \le j^{-1}\} \le$$

$$\le \sum_{n=1}^{\infty} C j^{-1} n^{\frac{-l}{2}} = \sum_{n=2}^{n_0} C c j^{-1} n + \sum_{n=1}^{\infty} C j^{-1} n^{\frac{-l}{2}} = c_0 j^{-1} \xrightarrow{j\to\infty} 0$$

for some $c, c_0 > 0$, due to Lemma 8.3.17, Proposition 8.3.18 and because $l \ge 4$. $\square$

In other words, Ξ_Ω is a set with finite perimeter.

EXAMPLE 8.3.22. If Ω satisfies the outer ball condition, then clearly it satisfies Hypothesis 8.2.1, moreover $\partial_s\Omega = \varnothing$, and Ω satisfies also Hypotheses 8.3.20, 8.3.11, 8.3.19 (see Example 8.3.13); so, we can apply Theorem 8.3.21.

8.4. Example: complement of a cone

$X = \{\omega \in C_*([0,1], \mathbb{R}^d)\}$, P_0 as in the above Section (as we said it is a Wiener space).
For every $r > 0$, we define on $\mathbb{R}^+$

$$l_r(t) := \begin{cases} 1 & \text{if } t \in [0,r] \\ \frac{2r-t}{r} & \text{if } t \in [r,2r] \\ 0 & \text{if } t > 2r \end{cases};$$

it is r^{-1}-Lipschitz.
For a point $x_0 \in \mathbb{R}^d$ we can consider the function $f : X \to \mathbb{R}$

$$\omega \mapsto l_r(\mathrm{dist}(\omega([0,1]), x_0));$$

We have that f is r^{-1}-Lipschitz, hence it is $W^{1,2}(X)$, $\nabla_H l_{r|\left(X\backslash\Theta_{B_{2r}}\right)\cup\Theta_{B_r}} \equiv 0$ and $(|\nabla_H l_r|_H)_{|\Theta_{B_{2r}}} \le r^{-1}$; now by Lemma 1.3.21, for every $y \notin \Theta_{B_{2r}}$ we have (by recalling that, for $y \in \mathbb{R}^d$, the probability P_y correspond to μ probability associated to the Markov process with starting point y)

$$P_y(\Theta_{B_{2r}}) \le (2r)^{d-2} |x-y|^{2-d};$$

so, $|f_r|_{W_\gamma^{1,2}(X)} \xrightarrow{r\to 0} 0$ because $d > 3$.
We proved

$$P_y(\{x\}) = 0 \text{ for every } y \ne x, \quad C_2(\Theta_{\{x\}}) = 0 \tag{8.4.1}$$

where C_2 is the 2-capacity in (X, P_0).

REMARK 8.4.1. Analogously it can be proved that, if A is an affine subspace with dimension $d > 3$, then $C_2(\Theta_{\{x\}}) = 0$.

Now, let K be an open circular cone with vertex in the origin O, i.e. there exists a ball $B \subset \mathbb{R}^d$ s.t. $x \in K$ if and only if x is on a half-line starting from O and intersecting B; there exists a, a half-line starting from O and passing through the center of B; b will be the line which completes a and it is called *axis* of the cone. In our setting, K will be up, and $b \backslash a$ down.

Clearly K is convex, and it is symmetric with respect to b.

Let $\Omega := \bar{K}^c$; for such a Ω we apply the concepts of Subsection 8.2.

We have that $\partial\Omega$ is the union of half-line starting from O and tangent to B, and each of that forms with a an angle of amplitude α. It is clear, that, for each point of $z \in \partial\Omega$ except O, there is an outer ball tangent in z, and the radius of this ball is locally uniform, as in Remark 8.2.8, so for what we said in that Remark $\partial_s\Omega \subseteq \{O\}$.

Now, with a translation, we suppose that O is in the origin (only to simplify some calculation about the geometry Ω).

We will suppose that the axes of the first coordinate in $\mathbb{R}^d$, in the positive part, corresponds to a. So, each $x \in \mathbb{R}^d$ can be written as $x = (x_1, \bar{x})$ where $\bar{x} \in \mathbb{R}^{d-1}$. For each $y \in \bar{\Omega}^c$ (so $y_1 > 0$), we have that $\delta'(y) = y_1 \sin\alpha$; hereafter, we write $r := \sin\alpha$, so $\delta'(y) = ry_1$.

If $x_1 > 0$, then the point z_x of $\partial\Omega$ nearest to x is on a line l_x passing through x and orthogonal to the surface of $\partial\Omega$, so l_x intersect a in y_x with an angle $\pi/2+\alpha$ with respect to its unbounded part; we have that there is a ball in $\bar{\Omega}^c$ with center in y_x tangent to z_x and with radius $\|z_x\|\tan\alpha$ (because O, z_x and y_x form a rectangular triangle); so by Remark 8.2.3,

$$\delta(x) \geq 1 \wedge \|z_x\|\tan\alpha,$$

by $x_1 > 0$ we have also that

$$\|z_x\| \geq \|\bar{x}\|\tan\alpha \geq \|x\|\sin\alpha\tan\alpha,$$

is nearer to O than x (because l_x is orthogonal to the line through O and z_x and by the Pythagorean theorem), so

$$\delta(x) \geq 1 \wedge \|x\|\tan\alpha. \tag{8.4.2}$$

Let $x \in \Omega$ s.t. $\delta(x) < 1$ and $x_1 \leq 0$: clearly $x \notin \Omega_1$; by Remark 8.2.6, we have that $\delta(x) > 1 \wedge cq(x)$ (where $q(x)$ is the distance from the boundary) for some $c > 0$ independent on x. We have that t. $q(x) \geq \sin\alpha \mathrm{dist}\|x\|$, hence for some $c_2 > 0$

$$\delta(x) \geq 1 \wedge c_2 \|x\|. \tag{8.4.3}$$

We are ready to prove the above result.

PROPOSITION 8.4.2. *Let $d \geq 7$ the dimension, for $\Omega = \bar{K}^c$ (where K is the above described cone), and $0 \in \Omega$, we have $\Xi^*_\Omega \in BV(X)$.*

PROOF. We prove that the hypotheses of Theorem 8.3.21 are verified, so we can apply it.

It is clear that Hypothesis 8.2.1 is satisfied. Obviously $\partial_s\Omega = \{O\}$, it is a point and by (8.4.1) $C_2(\Theta_{\partial_s\Omega}) = 0$, so Hypothesis 8.1.4 is verified. Clearly Hypothesis 8.3.10 is verified for $l = 5$, in fact for some $c > 0$

$$\mathcal{H}_s^{d-2}(\partial_s\Omega) = cs^5$$

Eventually, Hypothesis 8.3.11 is verified by putting together (8.4.2),(8.4.3). □

REMARK 8.4.3. For $d \geq 7$, we can define K a spheric cone in $\mathbb{R}^d$, and then $K_1 = K \times \mathbb{R}^m \subset \mathbb{R}^{d+m}$ (a cone that is translation invariant in some directions). It can be verified that what we said can be extended to K_1, because a $d+m$-dimensional Brownian motion can be decomposed in a sum of a d-dimensional Brownian motion and a m-dimensional Brownian motion, mutually independent.

APPENDIX A

Fundamental definitions and notions

Basic notions about Lipschitzianity, Hölderianity, graphs, lower semicontinuity. If X is a metric space, a real function f is said locally Lipschitz if, for every $x \in X$, there is a ball B centered in x s.t. $f_{|B}$ is Lipschitz.

A Lipschitz function with Lipschitz constant $c > 0$ is said c-Lipschitz.

A function $f : X \to \mathbb{R}$ is said α-Hölder if there exists a constant c s.t.

$$|f(x) - f(y)|_X \leq c|x - y|^\alpha.$$

Given a function $f : X \to Y$, we define the sets $\{G = k\} := \{x \in X | G(x) = k\}$, $\{G \in A\} := \{x \in X | G(x) \in A\}$.

Given a function $f : X \to \mathbb{R}$, the *graph* of f is the set

$$\{y \in X \times \mathbb{R} | y = (x, f(x)) \text{ for some } x \in X\}.$$

Given a function $f : X \to \mathbb{R}$, the *epigraph* of f is the set

$$\{y \in X \times \mathbb{R} | y = (x, y') \text{ where } x \in X \text{ and } y' < f(x)\}.$$

Let X be a topological space. We say that a function $f : X \to \mathbb{R}$ is *lower semicontinuous* if, for every $x_n \to x$ we have

$$f(x) \leq \liminf_{n \to +\infty} f(x_n),$$

or equivalently, if $f^{-1}((r, +\infty])$ is an open set for every $r \in \mathbb{R}$.

Some geometric notions. A topological space X is said *separable* if there exists a countable basis of open set, i.e. a countable collections $\mathcal{A}$ of open set s.t. all the open sets of X can be obtained by a countable union of open sets of $\mathcal{A}$.

A set $A \subseteq \mathbb{R}^d$ satisfies an *uniform outer ball condition* if there exists $r > 0$ s.t., for every $x \in \partial A$, there exists an y s.t. $B_r(y) \cap A = \varnothing$ but $x \in \partial B_r(y)$.

If X is a normed space, a set $A \subseteq X$ is said *convex* if: if $x_1, \dots, x_m \in A$, if $\lambda_1, \dots, \lambda_m > 0$ s.t. $\sum_{i=1}^m \lambda_i = 1$, then

$$\sum_{i=1}^m \lambda_i x_i \in A.$$

If X is a Banach space and A is a convex subset, for every point $x \in \partial A$ there exists at least an hyperplane π s.t. $x \in \pi$ and one of the two halfspaces does not intersect A (a hyperplane with this property is said tangent hyperplane).

The intersection of convex sets is always convex; in particular, the intersection of a convex and a line is an interval on the line.

A convex set in $\mathbb{R}^d$, has always Lipschitz boundary.

Let Ω be a subset of a metric space X, $l > 0$, f be a l-Lipschitz function on Ω; then f admits the McShane extension which is Lipschitz on X

$$\bar{f}(x) := \sup\{f(y) - l\,\|x-y\|\,|y \in O\}$$

(clearly $\bar{f}_{|\Omega} = f$); if f is Lipschitz and bounded, we can consider a truncated McShane extension which has the same Lipschitz constant an the same sup-norm of f.

Vector spaces, Banach spaces, complexifications. If E is a real vector space, we consider a complexification $E_{\mathbb{C}}$ in this way: as set $E_{\mathbb{C}} = E \times E$, the sum is defined in the obvious way, and if $z \in \mathbb{C}$ then

$$z(x_1,x_2) := (\Re z x_1 - \Im z x_2, \Im z x_1 + \Re z x_2)$$

we will write $x_1 + \mathrm{i} x_2$ to mean (x,y), and $E_{\mathbb{C}}$ is a complex vector space.

In a real (or complex) vector space X a norm is a nonnegative function $\|\cdot\|_X$ on X s.t.: $\|rx\|_X = |r|\,\|x\|_X$ for every $r \in \mathbb{R}$ ($r \in \mathbb{C}$) and $x \in X$;$\|x\|_X = 0$ iff $x = 0$ (for $x \in X$); $\|x+y\|_X \le \|x\|_X + \|y\|_X$ for every $x,y \in X$; a space provided with a norm is said a normed space; it is a a metric space with $\mathrm{dist}(x,y) = \|x-y\|_X$ (and it has a topology).

A normed space it is complete if $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of points of x s.t., for every $\varepsilon > 0$ there exists m_ε s.t. $\|x_n - x_m\|_X < \varepsilon$ for every $n > m_\varepsilon$, then x_n converges to some $x \in X$ for $n \to \infty$.

A complete normed real vector space X is said a *(real) Banach space*.

A complete normed complex vector space X is said a *complex Banach space*

A (real or complex) *Hilbert space H* is a (real or complex) Banach space provided with an inner product $\langle\cdot,\cdot\rangle_H$ s.t. $\|x\|_H = \sqrt{\langle x,x\rangle_H}$; we denote $\|\cdot\|_H$ also as $|\cdot|_H$. Every separable Hilbert space H admits a orthonormal basis $\{h_i\}_{i\in\mathbb{N}}$, s.t. each $h \in H$ can be written in only one way as $h = \sum_{i=1}^{+\infty} a_i h_i$ where $a_i \in \mathbb{R}$ for every $i \in \mathbb{N}$ and $\displaystyle\sum_{i}^{+\infty} |a_i|^2 < +\infty$ (we will call a real Hilbert space simply Hilbert space).

Given a real Banach space X we define its *complexification* $X_{\mathbb{C}}$ in this way: $X_{\mathbb{C}}$ is the complexification of H as a vector space, and the norm is defined as

$$\|(x_1,x_2)\|_{\tilde{X}} := \sup_{-\pi\le\theta\le\pi} \|x_1\cos\theta + x_2\sin\theta\|;$$

$X_{\mathbb{C}}$ is a complex Banach space (see e.g. [**50**], Appendix).

If H is a real Hilbert space, we consider a complexification $H_{\mathbb{C}}$ in this way: $H_{\mathbb{C}}$ is the complexification of H as a vector space, and, for $x_1,x_2,x_3,x_4 \in H$,

$$\langle x_1 + \mathrm{i}x_2, x_3 + \mathrm{i}x_4\rangle_{H_{\mathbb{C}}} = \langle x_1,x_3\rangle_H + \mathrm{i}\,\langle x_2,x_3\rangle_H - \mathrm{i}\,\langle x_1,x_4\rangle_H + \langle x_2,x_4\rangle_H$$

and

$$|x_1 + \mathrm{i}x_2|_{H_{\mathbb{C}}} = \sqrt{|x_1|_H^2 + |x_2|_H^2}$$

$H_{\mathbb{C}}$ is a complex Hilbert space.

If X is a Banach space, its dual X^* is the space of bounded linear functions from X to $\mathbb{R}$, i.e. the linear functions f s.t.

$$\|f\|_{X^*} := \sup_{x\in X} \frac{f(x)}{\|x\|_X} = \sup_{x\in X,\|x\|_X=1} f(x) < +\infty;$$

X^* is a Banach space with norm $\|\cdot\|_{X^*}$.

If H is a Hilbert space, then its dual H^* is canonically isometric to H by the function

$$H \to H^*,\ h \mapsto \langle h,\cdot\rangle_H.$$

If H is a Hilbert space, and F is dense in H, then for $h \in H$, we have that $h = 0$ iff $\langle h, g \rangle_H = 0$ for every $g \in F$.

If X is a Banach space, the *weak* topology* on $X^*$is the weakest topology s.t. for each $x \in X$ the function $X^* \to \mathbb{R},\ f \mapsto f(x)$ is continuous; the *weak topology* on X is the littlest topology s.t. for each $f \in X^*$ the weakest $X \to \mathbb{R},\ x \mapsto f(x)$ is continuous; this topology always exists; if X is a Hilbert space, the weak topology and the weak* topology coincide (recalling that H is the dual of itself).

If x_n converges to x the sense of weak* topology (we also say in weak* sense), we write $x_n \rightharpoonup^* x$. If x_n converges to x the sense of weak topology (we also say in weak sense), we write $x_n \rightharpoonup x$.

In a Banach space X, a set A is said an *hyperplane* if there exists $f \in X^*$s.t. $A = f^{-1}(c)$ for some $c \in \mathbb{R}$; in this setting, we say that the hyperplane cuts X in two open *halfspaces*, $f^{-1}((c, +\infty))$ and $f^{-1}((-\infty, c))$.

A open $O \subseteq \mathbb{R}^d$ is said set with Lipschitz boundary if the boundary is locally the graph of a Lipschitz function.

For every $p \in [1, +\infty]$, and $A \subseteq \mathbb{R}^d$ open, $L^p(A, \mathscr{L}^d)$ and $L^p_{loc}(A, \mathscr{L}^d)$ (for Lebesgue measure $\mathscr{L}^d$) are defined as usual.

Let $A \subset \mathbb{R}^d$, A open. For every $p \in [1, +\infty]$ we define the *Sobolev space* $W^{1,p}(A, \mathscr{L}^d) \subset L^p(A, \mathscr{L}^d)$ (for Lebesgue measure) in this way: $f \in L^p(A, \mathscr{L}^d)$ is in $W^{1,p}(A, \mathscr{L}^d)$ if for every $i \in \{1, \ldots, d\}$ there exists $g_i \in L^p(A, \mathscr{L}^d)$ s.t. , for every $\varphi \in C^1_c(A)$,

$$\int_A f(x) \frac{\partial \varphi}{\partial x_i}(x)\, dx = - \int_A g_i(x) \varphi(x)\, dx;$$

we define the gradient $\nabla f := (g_1, \ldots, g_d)$, we will write $g_i =: \frac{\partial f}{\partial x_i}$ for every $i \in 1, \ldots, d$ and we define the norm

$$\|f\|_{W^{1,p}(A,\mathscr{L}^d)} = \|f\|_{L^p(A,\mathscr{L}^d)} + \left(\int_A \sum_{i=1}^d |\frac{\partial f}{\partial x_i}(x)|^p\, dx \right)^{\frac{1}{p}}$$

for $f \in W^{1,p}(A, \mathscr{L}^d)$. We have that, with this norm, $W^{1,p}(A, \mathscr{L}^d)$ is a Banach space (as $L^p(A, \mathscr{L}^d)$ with its norm); on $W^{1,2}(A, \mathscr{L}^d)$, it is defined this inner product: if $f_1, f_2 \in W^{1,2}(A, \mathscr{L}^d)$

$$\langle f, g \rangle_{W^{1,2}(A,\mathscr{L}^d)} = \int_A f(x) g(x)\, dx + \int_A \sum_{i=1}^d \frac{\partial f}{\partial x_i}(x) \frac{\partial g}{\partial x_i}(x)\, dx;$$

we have that $W^{1,2}(A, \mathscr{L}^d)$ with this product is a Hilbert space (as $L^2(A, \mathscr{L}^d)$ is with its norm).

We define $W^{1,p}_0(A, \mathscr{L}^d)$ as the closure in $W^{1,p}(A, \mathscr{L}^d)$ of $C^1_c(A)$; if A is a convex, $W^{1,p}_0(A, \mathscr{L}^d)$ is also the set of the restrictions to A of functions in $W^{1,p}(\mathbb{R}^d, \mathscr{L}^d)$ which are 0 a.e. out of A.

$W^{1,p}_{loc}(\mathbb{R}^d, \mathscr{L}^d)$ is the subset of of classes of measurable functions f on $\mathbb{R}^d$, s.t., for every point $x \in \mathbb{R}^d$, there exists a neighbourhood A of x s.t. $f_{|A} \in W^{1,p}_{loc}(A, \mathscr{L}^d)$.

If A has Lipschitz boundary, then each function in $W^{1,1}(A)$ can be extended to a function in $W^{1,1}(A)$; in particular, this can be done for A convex set.

Holomorphic function. If $O \subseteq \mathbb{C}$ and X is a complex Banach space, we say that a function $f : O \to X$ is holomorphic in a point $z_0 \in O$ if there exists $f' \in X$ s.t.,

$$f' = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0};$$

f is holomorphic in O if it is holomorphic in every point of O.

Convolutions. If φ,η are functions on $\mathbb{R}^d$, the *convolution* is the function given by the integral (if it is always well-defined)

$$\varphi*\eta(x)=\int\varphi(y)\eta(x-y)\,dy.$$

If for $k\in\mathbb{N}\cup\{+\infty\}$ and $\eta\in C^k(\mathbb{R}^d)$ then $\varphi*\eta\in C^k(\mathbb{R}^d)$; if $\{\eta_n\}_{n\in\mathbb{N}}\subseteq C^1_c(\mathbb{R}^d,\mathscr{L}^d)$ is a sequence s.t. $\eta_n\geq 0$, $|\eta_n|_{L^1(\mathbb{R}^d)}=1$ for every $n\in\mathbb{N}$ and $\mathrm{supp}(\eta_n)$ converges to $\{0\}$ as a set, then: if $\varphi\in L^p(\mathbb{R}^d)$ then $\varphi*\eta_n$ converges to φ in $L^p(\mathbb{R}^d,\mathscr{L}^d)$; if $\varphi\in W^{1,p}(\mathbb{R}^d,\mathscr{L}^d)$ then $\varphi*\eta_n$ converges to φ in $W^{1,p}(\mathbb{R}^d,\mathscr{L}^d)$.

So, a function in $W^{1,p}(\mathbb{R}^d,\mathscr{L}^d)$, by convolution, can be approximated by a sequence of Lipschitz functions.

If $\varphi\in L^1(\mathbb{R}^d)$ and $\eta\in L^p(\mathbb{R}^d)$ then $\varphi*\eta\in L^p(\mathbb{R}^d)$) and $|\varphi*\eta|_{L^p(\mathbb{R}^d)}\leq|\varphi|_{L^1(\mathbb{R}^d)}|\eta|_{L^p(\mathbb{R}^d)}$.

Absolute continuity. We recall some facts and definitions about absolute continuity.

DEFINITION. Let U an open subset of $\mathbb{R}$. A real function f on U is said absolutely continuous if, for every $\varepsilon>0$ there exists $\delta_\varepsilon>0$ s.t., for every finite sequence of disjoint interval $A=\{[a_1,b_i],\dots,[a_n,b_n]\}$ s.t. $[a_i,b_i]\subseteq U$ for every i and $\sum_{i=1}^n(b_i-a_i)\leq\delta_\varepsilon$, the condition

$$\sum_{i=1}^n|f(b_i)-f(a_i)|\leq\varepsilon$$

is satisfied.

A real function f on U is said locally absolutely continuous if, for every compact interval $[a,b]\subseteq\mathbb{R}$, $f_{|[a,b]}$ is absolutely continuous.

We recall the well-known facts that a absolutely continuous function in U is in $W^{1,1}(U)$ (with the Lebesgue measure), and that an element f of $W^{1,1}(U)$ always admits an absolutely continuous version $\tilde{f}$.

Riesz-Thorin interpolation theorem. For the above result and more, see e.g. [**58**], Sub. 1.3.18.

PROPOSITION. *[Riesz-Thorin theorem] Let (X,μ) be a measure space, $p,q\in[1,+\infty)$, A_1 be a contractive operator in $L^p(X,\mu)$ and A_2 be a contractive operator in $L^q(X,\mu)$, and A_1 and A_2 coincide on $L^p(X,\mu)\cap L^q(X,\mu)$; then they can be extended in a unique way to a contractive operator in $L^r(X,\mu)$ for every $r\in(p,q)$.*

Banach-Alaoglu theorem.

THEOREM. *[Banach-Alaoglu theorem] If X is a Banach space, then each bounded set X^* is compact in the weak* topology; in particular, if X is a Hilbert space, then each bounded set is compact in the weak topology.*

Hölderianity of the solution of elliptic problems. We recall that, for $\alpha>0$, $C^{k,\alpha}$ is the set of functions with k derivatives which are all α-Hölder.

DEFINITION. We will say that a set has boundary $C^{k,\alpha}$-regular if the boundary is locally a graph of a function $C^{k,\alpha}$.

REMARK A.0.1. If $O\subseteq\mathbb{R}^N$ is a set with boundary $C^{2,\alpha}$-regular for some $\alpha>0$, if L is an operator in O strictly elliptic on bounded sets (see e.g. [**43**]) with Dirichlet boundary conditions, if $y \in C^\infty(\bar{O})$ and $u := (I - \sigma L)^{-1}$ and L is an operator which is strictly elliptic on bounded sets, we have that $u \in C^2(\bar{O})$. Let's recall the proof.

In fact, for each $R' > R > 0$, we can consider two balls $B_{R'}, B_R$ centered in a point, and a smooth function θ that is 1 on B_R and 0 out of $B_{R'}$, and a bounded smooth set C s.t $C \cap B_{R'} = O \cap B_{R'}$; hence, $v := \theta u$ will be the classical solution of a Dirichlet problem

$$\begin{cases} \sigma L v - v = g & \text{in } C \\ v = 0 & \text{on } \partial C \end{cases}$$

for some g that is in $L^2(C)$ (because $u \in W^{1,2}(O)$) and L is strictly elliptic on C; therefore, $v \in W^{2,2}(O \cap B_{R'})$ (e.g. by [**43**], Thm. 9.15), hence $u \in W^{2,2}(O \cap B_R)$ (and this for all $R > 0$). Hence, by the Morrey theorem (see e.g. [**21**], Cor. 9.15) we have $W^{1,p} \subseteq L^q$ with $\frac{1}{q} = \frac{1}{p} - \frac{1}{N}$, hence u and its first derivatives are in $L^{(\frac{1}{2} - \frac{1}{N})^{-1}}$ in each bounded set. By induction, we can find that $v \in W^{2,p}(O \cap B_R)$ for a creasing sequence of $p > 1$ and $R > 0$ (at each step, by knowing that $u \in W^{2,p}(O \cap B_R)$ we can find that $g \in L^q(C)$ and hence $u \in W^{2,q}(O \cap B_R)$ for $q = (\frac{1}{p} - \frac{1}{N})^{-1}$); in particular, $u \in W^{2,p}(O \cap B_R)$ for some p s.t. $\frac{1}{p} - \frac{1}{N} < 0$; hence, $u \in C^{1,\alpha}(\bar{O} \cap B_R)$ for some $\alpha > 0$, always by the Morrey theorem. Hence, $g \in C^{0,\alpha}(\bar{O} \cap B_R)$, so θu is a classical solution and $\theta u \in C^2(\bar{O} \cap B_R)$ (by [**43**], Th. 6.14), and this for all $R > 0$. So, $u \in C^2(\bar{O})$.

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