HEAVILY SEPARABLE FUNCTORS

ALESSANDRO ARDIZZONI AND CLAUDIA MENINI

Abstract. Prompted by an example related to the tensor algebra, we introduce and investigate a stronger version of the notion of separable functor that we call heavily separable. We test this notion on several functors traditionally connected to the study of separability.

Introduction

Given a field $\Bbbk$, the functor $\mathbf{P}:\mathrm{Bialg}_{\Bbbk}\to\mathrm{Vec}_{\Bbbk}$, assigning to a $\Bbbk$-bialgebra B the $\Bbbk$-vector space of its primitive elements, admits a left adjoint $\mathbf{T}$, assigning to a vector space V the tensor algebra $\mathbf{T}V$ endowed with its canonical bialgebra structure such that the elements in V becomes primitive. By investigating the properties of the adjunction $(\mathbf{T},\mathbf{P})$, together with its unit $\boldsymbol{\eta}$ and counit $\boldsymbol{\epsilon}$, we discovered that there is a natural retraction $\boldsymbol{\gamma}:\mathbf{TP}\to\mathrm{Id}$ of $\boldsymbol{\eta}$, i.e. $\boldsymbol{\gamma}\circ\boldsymbol{\eta}=\mathrm{Id}$, fulfilling the condition $\boldsymbol{\gamma}\boldsymbol{\gamma}=\boldsymbol{\gamma}\circ\mathbf{P}\boldsymbol{\epsilon}\mathbf{T}$. The existence of a natural retraction of the unit of an adjunction is, by Rafael Theorem, equivalent to the fact that the left adjoint is a separable functor. It is then natural to wonder if the above extra condition on the retraction $\boldsymbol{\gamma}$ corresponds to a stronger notion of separability. In the present paper, we show that an affirmative answer to this question is given by what we call a heavily separable (h-separable for short) functor and we investigate this notion in case of functors usually connected to the study of separability.

Explicitly, in Section 1 we introduce the concept of h-separable functor and we recover classical results in the h-separable case such as their behaviour with respect to composition (Lemma 1.4). In Section 2, we obtain a Rafael type Theorem 2.1. As a consequence we characterize the h-separability of a left (respectively right) adjoint functor either with respect to the forgetful functor from the Eilenberg-Moore category of the associated monad (resp. comonad) in Proposition 2.3 or by the existence of an augmentation (resp. grouplike morphism) of the associated monad (resp. comonad) in Corollary 2.7. In Theorem 2.8, we prove that the induced functor attached to an A-coring is h-separable if and only if this coring has an invariant grouplike element.

Section 3 is devoted to the investigation of the h-separability of the induction functor φ^* and of the restriction of scalars functors φ_* attached to a ring homomorphism $\varphi:R\to S$. In Proposition 3.1, we prove that φ^* is h-separable if and only if there is a ring homomorphism $E:S\to R$ such that $E\circ\varphi=\mathrm{Id}$. Characterizing whether φ_* is h-separable (in this case we say that S/R is h-separable) is more laborious. In Proposition 3.4, we prove that S/R is h-separable if and only if it is endowed with what we call a h-separability idempotent, a stronger version of a separability idempotent. In Lemma 3.7 we show that the ring epimorphisms (by this we mean epimorphisms in the category of rings) provide particular examples of h-separability. In Lemma 3.11 we show that the ring of matrices is never h-separable over the base ring except in trivial cases. In the rest of the present section we investigate the particular case when S is an R-algebra i.e. $\mathrm{Im}(\varphi)\subseteq Z(S)$. In Theorem 3.13 we discover that, in this case, S/R is h-separable if and only if φ is a ring epimorphism. Moreover S becomes commutative. As a consequence, in Proposition 3.14 we show that a h-separable algebra over a field $\Bbbk$ is necessarily trivial.

Finally in Section 4 we provide a more general version of our starting example $(\mathbf{T},\mathbf{P})$ involving monoidal categories and bialgebras therein.

2010 *Mathematics Subject Classification.* Primary 16H05; Secondary 18D10.

Key words and phrases. Separable Functors, Separable extension, Monads, Monoidal Category.

This paper was written while both authors were members of the "National Group for Algebraic and Geometric Structures and their Applications" (GNSAGA-INdAM).

1. Heavily separable functors

In this section we collect general facts about heavily separable functors.

Definition 1.1. For every functor $F:\mathcal{B}\to\mathcal{A}$ we set

$$F_{X,Y}:\mathrm{Hom}_{\mathcal{B}}\left(X,Y\right)\to\mathrm{Hom}_{\mathcal{A}}\left(FX,FY\right):f\mapsto Ff$$

Recall that F is called **separable** if there is a natural transformation

$$P_{-,-}:=P_{-,-}^{F}:\mathrm{Hom}_{\mathcal{A}}\left(F-,F-\right)\to\mathrm{Hom}_{\mathcal{B}}\left(-,-\right)$$

such that $P_{X,Y}\circ F_{X,Y}=\mathrm{Id}$ for every X,Y objects in $\mathcal{B}$.

We say that F is **heavily separable** (**h-separable** for short) if it is separable and the $P_{X,Y}$'s make commutative the following diagram for every $X,Y,Z\in\mathcal{B}$.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{A}}\left(FX,FY\right)\times\mathrm{Hom}_{\mathcal{A}}\left(FY,FZ\right) \arrow[r, "P_{X,Y}\times P_{Y,Z}"] \arrow[d, "\circ"'] & \mathrm{Hom}_{\mathcal{B}}\left(X,Y\right)\times\mathrm{Hom}_{\mathcal{B}}\left(Y,Z\right) \arrow[d, "\circ"] \\
\mathrm{Hom}_{\mathcal{A}}\left(FX,FZ\right) \arrow[r, "P_{X,Z}"] & \mathrm{Hom}_{\mathcal{B}}\left(X,Z\right)
\end{tikzcd}$$

where the vertical arrows are the obvious compositions. On elements the above diagram means that $P_{X,Z}(f\circ g)=P_{Y,Z}(f)\circ P_{X,Y}(g)$.

Remark 1.2. We were tempted to use the word "strongly" at first, instead of "heavily", but a notion of "strongly separable functor" already appeared in the literature in connection with graded rings in [CGN, Definition 3.1].

Remark 1.3. The Maschke's Theorem for separable functors asserts that for a separable functor $F:\mathcal{B}\to\mathcal{A}$ a morphism $f:X\to Y$ splits (resp. cosplits) if and only if $F\left(f\right)$ does. Explicitly, if $F\left(f\right)\circ g=\mathrm{Id}$ (resp. $g\circ F\left(f\right)=\mathrm{Id}$) for some morphism g then $f\circ P_{Y,X}\left(g\right)=\mathrm{Id}$ (resp. $P_{Y,X}\left(g\right)\circ f=\mathrm{Id}$). If $F\left(f\right)\circ g=\mathrm{Id}$ and $F\left(f'\right)\circ g'=\mathrm{Id}$ for $f:X\to Y,f':Y\to Z$, then $f\circ P_{Y,X}\left(g\right)=\mathrm{Id}$ and $f'\circ P_{Z,Y}\left(g'\right)=\mathrm{Id}$ so that $f'\circ f\circ P_{Y,X}\left(g\right)\circ P_{Z,Y}\left(g'\right)=\mathrm{Id}$ so that $P_{Y,X}\left(g\right)\circ P_{Z,Y}\left(g'\right)$ is a section of $f'\circ f$. Since $F\left(f'\circ f\right)\circ g\circ g'=\mathrm{Id}$, we also have $f'\circ f\circ P_{Z,X}\left(g\circ g'\right)=\mathrm{Id}$ so that $P_{Z,X}\left(g\circ g'\right)$ is another section of $f'\circ f$. In general these two section may differ but not in case F is h-separable. Thus in some sense we get a sort of functoriality of the splitting. A similar remark holds for cosplittings. We thank J. Vercruysse for this observation.

Lemma 1.4. *Let $F:\mathcal{C}\to\mathcal{B}$ and $G:\mathcal{B}\to\mathcal{A}$ be functors.*

i) *If F and G are h-separable so is GF.*
ii) *If GF is h-separable so is F.*
iii) *If G is h-separable, then F is h-sparable if and only if so is GF.*

Proof. i). By [NVV, Lemma 1] we know that GF is separable with respect to $P_{X,Y}^{GF}:=P_{X,Y}^{F}\circ P_{FX,FY}^{G}$. As a consequence, since F and G are h-separable, the following diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{A}}\left(GFX,GFY\right)\times\mathrm{Hom}_{\mathcal{A}}\left(GFY,GFZ\right) \arrow[r, "P_{FX,FY}^{G}\times P_{FY,FZ}^{G}"] \arrow[d, "\circ"'] & \mathrm{Hom}_{\mathcal{B}}\left(FX,FY\right)\times\mathrm{Hom}_{\mathcal{B}}\left(FY,FZ\right) \arrow[r, "P_{X,Y}^{F}\times P_{Y,Z}^{F}"] \arrow[d, "\circ"] & \mathrm{Hom}_{\mathcal{C}}\left(X,Y\right)\times\mathrm{Hom}_{\mathcal{C}}\left(Y,Z\right) \arrow[d, "\circ"] \\
\mathrm{Hom}_{\mathcal{A}}\left(GFX,GFZ\right) \arrow[r, "P_{FX,FZ}^{G}"] & \mathrm{Hom}_{\mathcal{B}}\left(FX,FZ\right) \arrow[r, "P_{X,Z}^{F}"] & \mathrm{Hom}_{\mathcal{C}}\left(X,Z\right)
\end{tikzcd}$$

commutes so that GF is h-separable.

ii). By [NVV, Lemma 1] we know that $F_{X,Y}$ cosplits naturally through $P_{X,Y}^{F}:=P_{X,Y}^{GF}\circ G_{FX,FY}$. On the other hand, since G is a functor and GF is h-separable the following diagram commutes

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{B}}\left(FX,FY\right)\times\mathrm{Hom}_{\mathcal{B}}\left(FY,FZ\right) \arrow[r, "G_{FX,FY}\times G_{FY,FZ}"] \arrow[d, "\circ"'] & \mathrm{Hom}_{\mathcal{A}}\left(GFX,GFY\right)\times\mathrm{Hom}_{\mathcal{A}}\left(GFY,GFZ\right) \arrow[r, "P_{X,Y}^{GF}\times P_{Y,Z}^{GF}"] \arrow[d, "\circ"] & \mathrm{Hom}_{\mathcal{C}}\left(X,Y\right)\times\mathrm{Hom}_{\mathcal{C}}\left(Y,Z\right) \arrow[d, "\circ"] \\
\mathrm{Hom}_{\mathcal{B}}\left(FX,FZ\right) \arrow[r, "G_{FX,FZ}"] & \mathrm{Hom}_{\mathcal{A}}\left(GFX,GFZ\right) \arrow[r, "P_{X,Z}^{GF}"] & \mathrm{Hom}_{\mathcal{C}}\left(X,Z\right)
\end{tikzcd}$$

so that F is h-separable.

iii) It follows trivially from i) and ii). $\square$

Remark 1.5. The present remark was pointed out by J. Vercruysse. If the functor $F:\mathcal{B}\to\mathcal{A}$ is a split monomorphism, meaning that there is a functor $G:\mathcal{A}\to\mathcal{B}$ such that $G\circ F=\mathrm{Id}$, then F is h-separable. This follows by setting $P_{X,Y}:=G_{X,Y}$ as in Definition 1.1. It can also be proved by means of Lemma 1.4,ii).

Lemma 1.6. *A full and faithful functor is h-separable.*

Proof. If $F:\mathcal{B}\to\mathcal{A}$ is full and faithful we have that the canonical map $F_{X,Y}:\mathrm{Hom}_{\mathcal{B}}(X,Y)\to\mathrm{Hom}_{\mathcal{A}}(FX,FY)$ is invertible so that we can take $P_{X,Y}:=F_{X,Y}^{-1}$. Since F is a functor, the following diagram commutes

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{B}}(X,Y)\times\mathrm{Hom}_{\mathcal{B}}(Y,Z) \arrow[r, "F_{X,Y}\times F_{Y,Z}"] \arrow[d, "\circ"'] & \mathrm{Hom}_{\mathcal{A}}(FX,FY)\times\mathrm{Hom}_{\mathcal{A}}(FY,FZ) \arrow[d, "\circ"] \\
\mathrm{Hom}_{\mathcal{B}}(X,Z) \arrow[r, "F_{X,Z}"] & \mathrm{Hom}_{\mathcal{A}}(FX,FZ)
\end{tikzcd}$$

Reversing the horizontal arrows we obtain that F h-separable. $\square$

2. Heavily separable adjoint functors

In this section we investigate h-separable functors which are adjoint functors.

Theorem 2.1 (Rafael type Theorem). *Let (L,R,η,ϵ) be an adjunction where $L:\mathcal{B}\to\mathcal{A}$.*

(i) *L is h-separable if and only if there is a natural transformation $\gamma:RL\to\mathrm{Id}_{\mathcal{B}}$ such that $\gamma\circ\eta=\mathrm{Id}$ and*

$$\gamma\gamma=\gamma\circ R\epsilon L. \tag{1}$$

(ii) *R is h-separable if and only if there is a natural transformation $\delta:\mathrm{Id}_{\mathcal{A}}\to LR$ such that $\epsilon\circ\delta=\mathrm{Id}$ and*

$$\delta\delta=L\eta R\circ\delta. \tag{2}$$

Proof. The second part of the statement follows from the first by duality so we only have to establish (i).

First recall that, by Rafael Theorem [Ra, Theorem 1.2], the functor L is separable if and only if there is a natural transformation $\gamma:RL\to\mathrm{Id}_{\mathcal{B}}$ such that $\gamma\circ\eta=\mathrm{Id}$. Moreover, by construction

$$\gamma X=P_{RLX,X}(\epsilon LX) \tag{3}$$

so that, by naturality of $P_{-,-}$ one has

$$\gamma Y\circ Rg\circ\eta X=P_{RLY,Y}(\epsilon LY)\circ Rg\circ\eta X=P_{X,Y}(\epsilon LY\circ LRg\circ L\eta X)=P_{X,Y}(g\circ\epsilon LX\circ L\eta X)=P_{X,Y}(g).$$

Assume that (1) holds. Then, for all $f\in\mathrm{Hom}_{\mathcal{A}}(LX,LY)$ and $g\in\mathrm{Hom}_{\mathcal{A}}(LY,LZ)$, we have

$$\begin{aligned}
P_{Y,Z}(g)\circ P_{X,Y}(f) &= \gamma Z\circ Rg\circ\eta Y\circ\gamma Y\circ Rf\circ\eta X\\
&\overset{\text{nat.}\gamma}{=} \gamma Z\circ\gamma RLZ\circ RLRg\circ RL\eta Y\circ Rf\circ\eta X\\
&\overset{(1)}{=} \gamma Z\circ R\epsilon LZ\circ RLRg\circ RL\eta Y\circ Rf\circ\eta X\\
&= \gamma Z\circ Rg\circ R\epsilon LY\circ RL\eta Y\circ Rf\circ\eta X\\
&= \gamma Z\circ Rg\circ Rf\circ\eta X=P_{X,Z}(g\circ f)
\end{aligned}$$

so that $P_{Y,Z}(g)\circ P_{X,Y}(f)=P_{X,Z}(g\circ f)$ and hence L is h-separable. Conversely, if the latter condition holds for every X,Y,Z and f,g as above, we have

$$\begin{aligned}
& \gamma\gamma X=\gamma X\circ\gamma RLX\overset{(3)}{=}P_{RLX,X}(\epsilon LX)\circ P_{RLRLX,RLX}(\epsilon LRLX)\\
={}& P_{RLRLX,X}(\epsilon LX\circ\epsilon LRLX)=\gamma X\circ R(\epsilon LX\circ\epsilon LRLX)\circ\eta RLRLX\\
={}& \gamma X\circ R\epsilon LX\circ R\epsilon LRLX\circ\eta RLRLX=\gamma X\circ R\epsilon LX
\end{aligned}$$

so that (1) holds. $\square$

REMARK 2.2. Let γ be as in Theorem 2.1. Then

$$RLRL \underset{R\epsilon L}{\overset{RL\gamma}{\rightrightarrows}} RL \xrightarrow{\gamma} \mathrm{Id}$$

is a split coequalizer (in the sense of [Bo, Definition 4.4.2]) taking $u = R\epsilon L, v = RL\gamma, r = \eta RL, q = \gamma$ and $s = \eta$. In particular the coequalizer above is absolute, i.e. preserved by every functor defined on $\mathcal{B}$. One can see this as a consequence of the fact that, for every $B \in \mathcal{B}$, the pair $(B, \gamma B)$ belongs to ${}_{RL}\mathcal{B}$, i.e. the Eilenberg-Moore category of the monad $(RL, R\epsilon L, \eta)$, and hence [Bo, Lemma 4.4.3] applies to this pair. A similar remark holds for δ as in Theorem 2.1 in connection with the Eilenberg-Moore category $\mathcal{B}^{LR}$ of the comonad $(LR, L\eta R, \epsilon)$.

PROPOSITION 2.3. *Let (L, R) be an adjunction.*

1. *The functor L is h-separable if and only if the forgetful functor $U : {}_{RL}\mathcal{B} \to \mathcal{B}$ is a split epimorphism i.e. there is a functor $\Gamma : \mathcal{B} \to {}_{RL}\mathcal{B}$ such that $U \circ \Gamma = \mathrm{Id}_{\mathcal{B}}$.*
2. *The functor R is h-separable if and only if the functor forgetful $U : \mathcal{B}^{LR} \to \mathcal{B}$ is a split epimorphism i.e. there is a functor $\Gamma : \mathcal{B} \to \mathcal{B}^{LR}$ such that $U \circ \Gamma = \mathrm{Id}_{\mathcal{B}}$.*

Proof. We just prove (1) , the proof of (2) being similar. By Theorem 2.1, L is h-separable if and only if there is a natural transformation $\gamma : RL \to \mathrm{Id}_{\mathcal{B}}$ such that $\gamma \circ \eta = \mathrm{Id}$ and (1) holds. For every $B \in \mathcal{B}$, define $\Gamma B := (B, \gamma B)$. Then $\Gamma B \in {}_{RL}\mathcal{B}$ by the properties of γ. Moreover any morphism $f : B \to C$ fulfills $f \circ \gamma B = \gamma C \circ RLf$ by naturality of γ. This means that f induces a morphism $\Gamma f : \Gamma B \to \Gamma C$ such that $U\Gamma f = f$. We have so defined a functor $\Gamma : \mathcal{B} \to \mathcal{B}_{RL}$ such that $U \circ \Gamma = \mathrm{Id}_{\mathcal{B}}$.

Conversely, let $\Gamma : \mathcal{B} \to {}_{RL}\mathcal{B}$ be a functor such that $U \circ \Gamma = \mathrm{Id}_{\mathcal{B}}$. Then, for every $B \in \mathcal{B}$, we have that $\Gamma B = (B, \gamma B)$ for some morphism $\gamma B : RLB \to B$. Since $\Gamma B \in \mathcal{B}_{RL}$ we must have that $\gamma B \circ \eta B = B$ and $\gamma B \circ RL\gamma B = \gamma B \circ R\epsilon LB$. Given a morphism $f : B \to C$, we have that $\Gamma f : \Gamma B \to \Gamma C$ is a morphism in ${}_{RL}\mathcal{B}$, which means that $f \circ \gamma B = \gamma C \circ RLf$ i.e. $\gamma := (\gamma B)_{B \in \mathcal{B}}$ is a natural transformation. By the foregoing $\gamma \circ \eta = \mathrm{Id}$ and (1) holds. $\square$

COROLLARY 2.4. *Let (L, R) be an adjunction.*

1. *Assume that R is strictly monadic (i.e. the comparison functor is an isomorphism of categories). Then the functor L is h-separable if and only if R is a split epimorphism.*
2. *Assume that L is strictly comonadic (i.e. the cocomparison functor is an isomorphism of categories). Then the functor R is h-separable if and only if L is a split epimorphism.*

Proof. We just prove (1) , the proof of (2) being similar. Since the comparison functor $K : \mathcal{A} \to \mathcal{B}_{RL}$ is an isomorphism of categories and $U \circ K = R$, we get that R is a split epimorphism if and only if U is a split epimorphism. By Proposition 2.3, this is equivalent to the h-separability of L. $\square$

EXAMPLE 2.5. In Remark 4.3 we will obtain that the tensor algebra functor $T : \mathrm{Vec}_{\Bbbk} \to \mathrm{Alg}_{\Bbbk}$ is separable but not h-separable.

DEFINITION 2.6. Following [LMW, Section 4] we say that a **grouplike morphism** for a comonad $(C, \Delta : C \to CC, \epsilon : C \to \mathrm{Id})$ is a natural transformation $\delta : \mathrm{Id} \to C$ such that $\epsilon \circ \delta = \mathrm{Id}$ and $\delta\delta = \Delta \circ \delta$. Dually an **augmentation** for a monad $(M, m : MM \to M, \eta : \mathrm{Id} \to M)$ is a natural transformation $\gamma : M \to \mathrm{Id}$ such that $\gamma \circ \eta = \mathrm{Id}$ and $\gamma\gamma = \gamma \circ m$.

An immediate consequence of the previous definition and Theorem 2.1 is the following result.

COROLLARY 2.7. *Let (L, R, η, ϵ) be an adjunction.*

1. *L is h-separable if and only if the monad $(RL, R\epsilon L, \eta)$ has an augmentation.*
2. *R is h-separable if and only if the comonad $(LR, L\eta R, \epsilon)$ has a grouplike morphism.*

Consider an A-coring $\mathcal{C}$ and its set of invariant elements $\mathcal{C}^A = \{c \in \mathcal{C} \mid ac = ca, \text{ for every } a \in A\}$. In [Br, Theorem 3.3], Brzeziński proved that the induction functor $R := (-) \otimes_A \mathcal{C} : \text{Mod-}A \to \mathcal{M}^{\mathcal{C}}$ is separable if and only if there is an invariant element $e \in \mathcal{C}^A$ such that $\varepsilon_{\mathcal{C}}(e) = 1$. Next result provides a similar characterization for the h-separable case.

THEOREM 2.8. *Given an A-coring $\mathcal{C}$, the induction functor $R := (-) \otimes_A \mathcal{C} : \mathrm{Mod}\text{-}A \to \mathcal{M}^{\mathcal{C}}$ is h-separable if and only if $\mathcal{C}$ has an invariant grouplike element.*

Proof. By [Br, Lemma 3.1], the functor R is the right adjoint of the forgetful functor $L : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}_A$. Thus by Corollary 2.7, R is h-separable if and only if the comonad $(LR, L\eta R, \epsilon)$ has a grouplike morphism. A grouplike morphism for this particular comonad is equivalent to an invariant grouplike element for the coring $\mathcal{C}$ i.e. an element $e \in \mathcal{C}^A$ such that $\varepsilon_{\mathcal{C}}(e) = 1$ and $\Delta_{\mathcal{C}}(e) = e \otimes_A e$. □

REMARK 2.9. Let $\mathcal{C}$ be an A-coring. We recall that, by [Br, Lemma 5.1], if A is a right $\mathcal{C}$-comodule via $\rho_A : A \to A \otimes_A \mathcal{C}$, then $g = \rho_A(1_A)$ is a grouplike element of $\mathcal{C}$. Conversely if g is a grouplike element of $\mathcal{C}$, then A is a right $\mathcal{C}$-comodule via $\rho_A : A \to A \otimes_A \mathcal{C}$ defined by $\rho_A(a) = 1_A \otimes_A (g \cdot a)$. Moreover, if g is a grouplike element of $\mathcal{C}$, then, by [Br, page 404], g is an invariant element if and only if $A = A^{\mathrm{co}\mathcal{C}} := \{a \in A \mid ag = ga\}$

3. Heavily separable ring homomorphisms

Let $\varphi : R \to S$ be a ring homomorphism. Consider the induction functor $\varphi^* := S \otimes_R (-) : R$-Mod $\to S$-Mod.

PROPOSITION 3.1. *Let $\varphi : R \to S$ be a ring homomorphism. Then the induction functor φ^* is h-separable if and only if there is a ring homomorphism $E : S \to R$ such that $E \circ \varphi = \mathrm{Id}$.*

Proof. By [CMZ, Theorem 27], we know that φ^* is separable if and only if there is a morphism of R-bimodules $E : S \to R$ such that $E(1_S) = 1_R$. Using E one defines a natural transformation $\gamma M : S \otimes_R M \to M : s \otimes_R m \mapsto E(s)m$, for every $M \in R$-Mod, such that $\gamma \circ \eta = \mathrm{Id}$ where the unit η is defined by $\eta M : M \to S \otimes_R M : m \mapsto 1_S \otimes_R m$. All natural transformations γ such that $\gamma \circ \eta = \mathrm{Id}$ are of this form because R is a generator in R-Mod.

Finally (1) rewrites as $E(x)E(y)m = E(xy)m$ for every $x, y \in S$ and $m \in M$, for every $M \in R$-Mod. Thus it is equivalent to ask that E is multiplicative. Summing up φ^* is h-separable if and only if there is a morphism of R-bimodules $E : S \to R$ which is a ring homomorphism. This is equivalent to ask that $E : S \to R$ is a ring homomorphism such that $E \circ \varphi = \mathrm{Id}$. □

It is well-known that the restriction of scalars functor $\varphi_* : S$-Mod $\to R$-Mod is the right adjoint of the induction functor φ^*. Moreover φ_* is separable if and only if S/R is separable (see [NVV, Proposition 1.3]) if and only if it admits a separability idempotent.

We are so lead to the following definition.

DEFINITIONS 3.2. 1) S/R is **h-separable** if the functor $\varphi_* : S$-Mod $\to R$-Mod is h-separable.

2) A **heavy separability idempotent** (**h-separability idempotent** for short) of S/R is an element $\sum_i a_i \otimes_R b_i \in S \otimes_R S$ such that $\sum_i a_i \otimes_R b_i$ is a separability idempotent, i.e.

$$\sum_i a_i b_i = 1, \qquad \sum_i s a_i \otimes_R b_i = \sum_i a_i \otimes_R b_i s \quad \text{for every } s \in S, \tag{4}$$

which moreover fulfills

$$\sum_{i,j} a_i \otimes_R b_i a_j \otimes_R b_j = \sum_i a_i \otimes_R 1_S \otimes_R b_i. \tag{5}$$

REMARK 3.3. Note that a h-separability idempotent $e := \sum_i a_i \otimes_R b_i$ is exactly a grouplike element in the Sweedler's coring $\mathcal{C} := S \otimes_R S$ such that $se = es$ for every $s \in S$ i.e. which is invariant. Note that $1_S \otimes_R 1_S$ is always a grouplike element in $\mathcal{C}$ but it is not invariant in general.

PROPOSITION 3.4. *S/R is h-separable if and only if it has a h-separability idempotent.*

Proof. We observed that φ_* is the right adjoint of the induction functor $\varphi^* := S \otimes_R (-)$. Recall that S/R is separable if and only if the map $S \otimes_R S \to S$ splits as an S-bimodule map. The splitting is uniquely determined by a so-called separability idempotent i.e. an element $\sum_i a_i \otimes_R b_i \in S \otimes_R S$

such that (4) hold. Using this element we can define $\delta : \mathrm{Id} \to \varphi^* \varphi_*$ such that $\epsilon \circ \delta = \mathrm{Id}$ by $\delta M : M \to S \otimes_R M : m \mapsto \sum_i a_i \otimes_R b_i m$. This natural transformation satisfies (2) if and only if

$$(S \otimes_R \delta M) \circ \delta M = (S \otimes_R \eta M) \circ \delta M.$$

Let us compute separately the two terms of this equality on any $m \in M$,

$$(S \otimes_R \delta M)(\delta M)(m) = \sum\nolimits_i a_i \otimes_R (\delta M)(b_i m) = \sum\nolimits_{i,j} a_i \otimes_R a_j \otimes_R b_j b_i m \overset{(4)}{=} \sum\nolimits_{i,j} a_i \otimes_R b_i a_j \otimes_R b_j m,$$

$$(S \otimes_R \eta M)(\delta M)(m) = \sum\nolimits_i a_i \otimes_R (\eta M)(b_i m) = \sum\nolimits_i a_i \otimes_R 1_S \otimes_R b_i m.$$

Thus δ satisfies (2) if and only if (5) holds true. □

Remark 3.5. Let $\varphi : R \to S$ be a ring homomorphism and let $\mathcal{C} := S \otimes_R S$ be the Sweedler coring. In view of Theorem 2.8 and Remark 3.3 we obtain that S/R is h-separable (i.e. the functor $\varphi_* : S$-Mod $\to R$-Mod is h-separable) if and only if the induction functor $R := (-) \otimes_S \mathcal{C} : \text{Mod-}S \to \mathcal{M}^{\mathcal{C}}$ is h-separable. Note that here $\mathcal{M}^{\mathcal{C}}$ is isomorphic (see e.g. [BW, page 252-253.]) to the category $\mathrm{Desc}\,(S/R)$ of descent data associated to the ring homomorphism φ.

Corollary 3.6. *Let $\varphi : R \to S$ and $\psi : S \to T$ be ring homomorphisms.*

1) *If T/S and S/R are h-separable so is T/R.*
2) *If T/R is h-separable so is T/S.*
3) *If S/R is h-separable then T/S is h-separable if and only if so is T/R.*

Proof. It follows by Definition 3.2 and Lemma 1.4. □

Lemma 3.7. *Let $\varphi : R \to S$ be a ring homomorphism. The following are equivalent.*

(1) *The map φ is a ring epimorphism (i.e. an epimorphism in the category of rings);*
(2) *the multiplication $m : S \otimes_R S \to S$ is invertible;*
(3) *$1_S \otimes_R 1_S$ is a separability idempotent for S/R;*
(4) *$1_S \otimes_R 1_S$ is a h-separability idempotent for S/R.*

If these equivalent conditions hold true then S/R is h-separable.
Moreover $1_S \otimes_R 1_S$ is the unique separability idempotent for S/R.

Proof. (1) $\Leftrightarrow$ (2) follows by [St, Proposition XI.1.2 page 225].
(1) $\Leftrightarrow$ (3) follows by [St, Proposition XI.1.1 page 226].
(3) $\Leftrightarrow$ (4) depends on the fact that $1_S \otimes_R 1_S$ always fulfills (5).
By Proposition 3.4, (4) implies that S/R is h-separable.
Let us check the last part of the statement. If $\sum_i a_i \otimes_R b_i$ is another separability idempotent, we get

$$\sum\nolimits_i a_i \otimes_R b_i = \sum\nolimits_i a_i 1_S \otimes_R 1_S b_i = 1_S \otimes_R 1_S \sum\nolimits_i a_i b_i \overset{(4)}{=} 1_S \otimes_R 1_S.$$

□

Example 3.8. We now give examples of ring epimorphisms $\varphi : R \to S$.

1) Let S be a multiplicative closed subset of a commutative ring R. Then the canonical map $\varphi : R \to S^{-1}R$ is a ring epimorphism, c.f. [AMa, Proposition 3.1]. More generally we can consider a perfect right localization of R as in [St, page 229].

2) Consider the ring of matrices $\mathrm{M}_n(R)$ and the ring $\mathrm{T}_n(R)$ of $n \times n$ upper triangular matrices over a ring R. Then the inclusion $\varphi : \mathrm{T}_n(R) \to \mathrm{M}_n(R)$ is a ring epimorphism. In fact, given ring homomorphisms $\alpha, \beta : \mathrm{M}_n(R) \to B$ that coincide on $\mathrm{T}_n(R)$ then they coincide on all matrices. To see this we first check that $\alpha(E_{ij}) = \beta(E_{ij})$ for all $i > j$,

$$\begin{aligned} \alpha(E_{ij}) &= \alpha(E_{ij}E_{jj}) = \alpha(E_{ij})\alpha(E_{jj}) = \alpha(E_{ij})\beta(E_{jj}) = \alpha(E_{ij})\beta(E_{ji}E_{ij}) \\ &= \alpha(E_{ij})\beta(E_{ji})\beta(E_{ij}) = \alpha(E_{ij})\alpha(E_{ji})\beta(E_{ij}) = \alpha(E_{ij}E_{ji})\beta(E_{ij}) \\ &= \alpha(E_{ii})\beta(E_{ij}) = \beta(E_{ii})\beta(E_{ij}) = \beta(E_{ii}E_{ij}) = \beta(E_{ij}). \end{aligned}$$

Thus $\alpha(E_{ij}) = \beta(E_{ij})$ for every i, j. Now, given $r \in R$ we have

$$\alpha(rE_{ij}) = \alpha(rE_{ii}E_{ij}) = \alpha(rE_{ii})\alpha(E_{ij}) = \beta(rE_{ii})\beta(E_{ij}) = \beta(rE_{ii}E_{ij}) = \beta(rE_{ij}).$$

As a consequence $\alpha(M) = \beta(M)$ for all $M \in \mathrm{M}_n(R)$ as desired.

3) Any surjective ring homomorphism $\varphi : R \to S$ is trivially a ring epimorphism.

REMARK 3.9. A kind of dual to Lemma 3.7, establishes that a $\Bbbk$-coalgebra homomorphism $\varphi : C \to D$ is a coalgebra epimorphism if and only if the induced functor $\mathcal{M}^C \to \mathcal{M}^D$ is full, see [NT, Theorem 3.5]. Since this functor is always faithful, by Lemma 1.6 it is in this case h-separable.

PROPOSITION 3.10. *Let $\varphi : R \to S$ be a ring homomorphism. Then S/R is h-separable if and only if $S/\varphi(R)$ is h-separable.*

Proof. Write $\varphi = i \circ \overline{\varphi}$ where $i : \varphi(R) \to S$ is the canonical inclusion and $\overline{\varphi} : R \to \varphi(R)$ is the corestriction of φ to its image $\varphi(R)$. By Lemma 3.7, we have that $\varphi(R)/R$ is h-separable.

By Corollary 3.6, S/R is h-separable if and only if $S/\varphi(R)$ is h-separable. $\square$

It is well-known that the ring of matrices is separable, see e.g. [DI, Example II, page 41]. In the following result we show that however it is never h-separable.

LEMMA 3.11. *Let R be a ring and $S := \mathrm{M}_n(R)$. If S/R is h-separable, then either $n = 1$ or $R = 0$.*

Proof. By Proposition 3.4, S/R admits a h-separability idempotent $e = \sum_{i,j,s,t} r_{i,j}^{s,t} E_{i,j} \otimes_R E_{s,t}$ where $r_{i,j}^{s,t} \in R$ and $E_{i,j}$ is the canonical matrix having 1 in the entry (i,j) and zero elsewhere (e can be written in the given form since the tensor product is over R and the $E_{i,j}$'s are R-invariant). Then

$$\begin{aligned} E_{a,b}e &= \sum_{i,j,s,t} E_{a,b} r_{i,j}^{s,t} E_{i,j} \otimes_R E_{s,t} = \sum_{i,j,s,t} r_{i,j}^{s,t} E_{a,b} E_{i,j} \otimes_R E_{s,t} = \sum_{j,s,t} r_{b,j}^{s,t} E_{a,j} \otimes_R E_{s,t}, \\ eE_{a,b} &= \sum_{i,j,s,t} r_{i,j}^{s,t} E_{i,j} \otimes_R E_{s,t} E_{a,b} = \sum_{i,j,s} r_{i,j}^{s,a} E_{i,j} \otimes_R E_{s,b}. \end{aligned}$$

Note that the elements $E_{i,j} \otimes_R E_{i',j'}$'s form a basis of $S \otimes_R S$ as a free left R-module. Thus, the equalities $E_{a,b}e = eE_{a,b}$ implies $r_{b,j}^{s,t} = 0$ for $t \neq b$. Moreover if $t = b$ then $r_{b,j}^{s,t} = r_{a,j}^{s,a}$. Thus $r_{b,j}^{s,t} = \delta_{t,b} r_{a,j}^{s,a}$ for every a, b, j, s. Thus, if we set $r_j^s := r_{1,j}^{s,1}$, we obtain $r_{b,j}^{s,t} = \delta_{t,b} r_{1,j}^{s,1} = \delta_{t,b} r_j^s$. We can now rewrite

$$e = \sum_{i,j,s,t} r_{i,j}^{s,t} E_{i,j} \otimes_R E_{s,t} = \sum_{i,j,s} r_j^s E_{i,j} \otimes_R E_{s,i} = \sum_{j,s} r_j^s \sum_i E_{i,j} \otimes_R E_{s,i}.$$

Now

$$\sum_i E_{i,i} = 1 = m(e) = \sum_{j,s} r_j^s \sum_i E_{i,j} E_{s,i} = \sum_j r_j^j \sum_i E_{i,i}$$

so that

$$\sum_j r_j^j = 1_R. \tag{6}$$

From (5) and the fact that the $E_{i,j}$'s are R-invariant we deduce that

$$\sum_{j,s} \sum_{j',s'} r_j^s r_{j'}^{s'} \sum_i \sum_{i'} E_{i,j} \otimes_R E_{s,i} E_{i',j'} \otimes_R E_{s',i'} = \sum_{j,s} r_j^s \sum_i E_{i,j} \otimes_R 1 \otimes_R E_{s,i}$$

i.e.

$$\sum_{j,s} \sum_{j',s'} r_j^s r_{j'}^{s'} \sum_i E_{i,j} \otimes_R E_{s,j'} \otimes_R E_{s',i} = \sum_{j,s} r_j^s \sum_{j',i} E_{i,j} \otimes_R E_{j',j'} \otimes_R E_{s,i}.$$

Equivalently for all i, j we have

$$\sum_s \sum_{j',s'} r_j^s r_{j'}^{s'} E_{s,j'} \otimes_R E_{s',i} = \sum_s r_j^s \sum_{j'} E_{j',j'} \otimes_R E_{s,i}$$

i.e., replacing s with s' in the second term,

$$\sum_s \sum_{j',s'} r_j^s r_{j'}^{s'} E_{s,j'} \otimes_R E_{s',i} = \sum_{s'} r_j^{s'} \sum_{j'} E_{j',j'} \otimes_R E_{s',i}.$$

Thus for every s', i, j

$$\sum_s \sum_{j'} r_j^s r_{j'}^{s'} E_{s,j'} = r_j^{s'} \sum_{j'} E_{j',j'}.$$

From this equality we deduce $r_j^s r_{j'}^{s'} = \delta_{s,j'} r_j^{s'}$ for all s, j, s', j'. Now

$$r_j^{s'} = \sum\nolimits_{j'} \delta_{s,j'} r_j^{s'} = \sum\nolimits_{j'} r_j^s r_{j'}^{j'} = r_j^s \sum\nolimits_{j'} r_{j'}^{j'} \overset{(6)}{=} r_j^s \cdot 1_R = r_j^s$$

so that we can set $r_j := r_j^1$ and we get $r_j^s = r_j$ for each s, j. Hence the equality $r_j^s r_{j'}^{s'} = \delta_{s,j'} r_j^{s'}$ rewrites as $r_j r_{j'} = \delta_{s,j'} r_j$ for all s, j, j'.

If $n \geq 2$, then for every j' there is always $s \neq j'$ *so* that we obtain $r_j r_{j'} = 0$ for all j, j'. Now $0 = \sum_j \sum_{j'} r_j r_{j'} = \sum_j r_j \sum_{j'} r_{j'} = 1_R \cdot 1_R = 1_R$, a contradiction. □

3.1. **Heavily separable algebras.**

3.12. Let R be a commutative ring, let S be a ring and let $Z(S)$ be its center. We recall that a S is said to be an R-algebra, or that S is an algebra over R, if there is a unital ring homomorphism $\varphi : R \to S$ such that $\varphi(R) \subseteq Z(S)$. In this case we set

$$r \cdot s = \varphi(r) \cdot s \text{ for every } r \in R \text{ and } s \in S.$$

Since $Im(\varphi) \subseteq Z(S)$, we have $r \cdot s = s \cdot r$ for every $r \in R$ and $s \in S$ and

$$r \cdot 1_S = \varphi(r) \cdot 1_S = \varphi(r) \cdot \varphi(1_R) = \varphi(r \cdot 1_R) = \varphi(r) \text{ for every } r \in R \text{ so that } R1_S = Im(\varphi) \subseteq Z(S).$$

Theorem 3.13. *Let S be and R-algebra. Then S/R is h-separable if and only if the canonical map $\varphi : R \to S$ is a ring epimorphism. Moreover if one of these conditions holds, then S is commutative.*

Proof. $(\Rightarrow)$. Let $\sum_i a_i \otimes_R b_i$ be an h-separability idempotent. Since $\varphi(R) \subseteq Z(S)$, we get that the map $\tau : A \otimes_R A \to A \otimes_R A, \tau(a \otimes_R b) = b \otimes_R a$, is well-defined and left R-linear. Hence we can apply $A \otimes_R \tau$ on both sides of (5) to get $\sum_{i,j} a_i \otimes_R b_j \otimes_R b_i a_j = \sum_i a_i \otimes_R b_i \otimes_R 1_S$. By multiplying, we obtain $\sum_{i,j} a_i b_j \otimes_R b_i a_j = \sum_i a_i b_i \otimes_R 1_S$. By (4), we get

$$\sum\nolimits_{i,j} a_i b_j \otimes_R b_i a_j = 1_S \otimes_R 1_S. \tag{7}$$

By (4), we get that $\sum_t a_t s b_t \in Z(S)$, for all $s \in S$. Using this fact we have

$$\begin{aligned} s &= 1_S \cdot 1_S \cdot s \overset{(7)}{=} \sum\nolimits_{i,j} a_i b_j b_i a_j s = \sum\nolimits_{i,j} a_i (b_j) b_i (a_j) s (1_S) \\ &\overset{(5)}{=} \sum\nolimits_{i,j,t} a_i b_j b_i (a_t s b_t) a_j = \sum\nolimits_{i,j,t} a_i b_j b_i a_j (a_t s b_t) \overset{(7)}{=} \sum\nolimits_t a_t s b_t \in Z(S). \end{aligned}$$

We have so proved that $S \subseteq Z(S)$ and hence S is commutative. Now, we compute

$$\sum\nolimits_i a_i \otimes_R b_i \overset{(4)}{=} \sum\nolimits_{i,j} a_i a_j b_j \otimes_R b_i \overset{S=Z(S)}{=} \sum\nolimits_{i,j} a_j a_i b_j \otimes_R b_i \overset{(4)}{=} \sum\nolimits_{i,j} a_i b_j \otimes_R b_i a_j$$

so that $\sum_i a_i \otimes_R b_i = 1_S \otimes_R 1_S$ by (7). We conclude by Lemma 3.7.

$(\Leftarrow)$ It follows by Lemma 3.7. □

The following result establishes that there is no non-trivial h-separable algebra over a field $\Bbbk$.

Proposition 3.14. *Let A be a h-separable algebra over a field $\Bbbk$. Then either $A = \Bbbk$ or $A = 0$.*

Proof. By Theorem 3.13, the unit $u : \Bbbk \to A$ is a ring epimorphism. By Lemma 3.7, we have that $A \otimes_\Bbbk A \cong A$ via multiplication. Since A is h-separable over $\Bbbk$ it is in particular separable over $\Bbbk$. By [Pi, page 184], the separable $\Bbbk$-algebra A is finite-dimensional. Thus, from $A \otimes_\Bbbk A \cong A$ we deduce that A has either dimensional one or zero over $\Bbbk$. □

Example 3.15. $\mathbb{C}/\mathbb{R}$ is separable but not h-separable. In fact, by Proposition 3.14, $\mathbb{C}/\mathbb{R}$ is not h-separable. On the other hand $e = \frac{1}{2}(1 \otimes 1 - i \otimes i)$ is a separability idempotent (it is the only possible one). It is clear that e is not a h-separability idempotent.

Remark 3.16. Let $\Bbbk$ be a field. Set $A = B = \Bbbk, R = \Bbbk \times \Bbbk = S$. Then A and B are R-algebras and $S = A \times B$ is their product in the category of R-algebras. Moreover S/R is h-separable as $S = R$. Hence the product of R-algebras may be h-separable.

LEMMA 3.17. *Let A and B be R-algebras and let $S=A\times B$ be their product in the category of R-algebras. Set $e_1:=(1_A,0_B)\in S$ and $e_2:=(0_A,1_B)\in S$. The following are equivalent.*

(*i*) *S/R is h-separable.*
(*ii*) *A/R and B/R are h-separable and $e_1\otimes_R e_2=0=e_2\otimes_R e_1$.*

Proof. First, by Theorem 3.13 and Lemma 3.7, the conditions (*i*) and (*ii*) can be replaced respectively by

- $1_S\otimes_R 1_S$ is a separability idempotent of S/R
- $1_A\otimes_R 1_A$ and $1_B\otimes_R 1_B$ are separability idempotents of A/R and B/R respectively and $e_1\otimes_R e_2=0=e_2\otimes_R e_1$.

Note also that, if the first condition holds, then, for $i\neq j$ we get

$$e_i\otimes_R e_j=e_i1_S\otimes_R 1_Se_j=1_S\otimes_R 1_Se_ie_j=0$$

so that $e_1\otimes_R e_2=0=e_2\otimes_R e_1$. Thus the latter condition can be assumed to hold.

Denote by $p_A:S\to A$ and $p_B:S\to B$ the canonical projections.

Since $1_S=e_1+e_2$ and $e_1\otimes_R e_2=0=e_2\otimes_R e_1$, we get that $1_S\otimes_R 1_S=e_1\otimes_R e_1+e_2\otimes_R e_2$. For every $s\in S$ we have

$$\begin{aligned} s1_S\otimes_R 1_S &= se_1\otimes_R e_1+se_2\otimes_R e_2\\ &= s\left(1_A,0_B\right)\otimes_R\left(1_A,0_B\right)+s\left(0_A,1_B\right)\otimes_R\left(0_A,1_B\right)\\ &= \left(p_A\left(s\right)1_A,0_B\right)\otimes_R\left(1_A,0_B\right)+\left(0_A,p_B\left(s\right)1_B\right)\otimes_R\left(0_A,1_B\right)\\ &= \left(i_A\otimes_R i_A\right)\left(p_A\left(s\right)1_A\otimes_R 1_A\right)+\left(i_B\otimes_R i_B\right)\left(p_B\left(s\right)1_B\otimes_R 1_B\right)\end{aligned}$$

Similarly $1_S\otimes_R 1_Ss=\left(i_A\otimes_R i_A\right)\left(1_A\otimes_R 1_Ap_A\left(s\right)\right)+\left(i_B\otimes_R i_B\right)\left(1_B\otimes_R 1_Bp_B\left(s\right)\right).$

As a consequence, for every $s\in S$

$$s1_S\otimes_R 1_S=1_S\otimes_R 1_Ss\Longleftrightarrow$$

$$\begin{aligned} &\left(i_A\otimes_R i_A\right)\left(p_A\left(s\right)1_A\otimes_R 1_A\right)+\left(i_B\otimes_R i_B\right)\left(p_B\left(s\right)1_B\otimes_R 1_B\right)\\ =\;&\left(i_A\otimes_R i_A\right)\left(1_A\otimes_R 1_Ap_A\left(s\right)\right)+\left(i_B\otimes_R i_B\right)\left(1_B\otimes_R 1_Bp_B\left(s\right)\right)\Longleftrightarrow\end{aligned}$$

$$p_A\left(s\right)1_A\otimes_R 1_A=1_A\otimes_R 1_Ap_A\left(s\right)\qquad\text{and}\qquad p_B\left(s\right)1_B\otimes_R 1_B=1_B\otimes_R 1_Bp_B\left(s\right).$$

Since p_A and p_B are surjective, we get that to require that $s1_S\otimes_R 1_S=1_S\otimes_R 1_Ss$ for every $s\in S$ is equivalent to require that

$$a1_A\otimes_R 1_A=1_A\otimes_R 1_Aa\qquad\text{and}\qquad b1_B\otimes_R 1_B=1_B\otimes_R 1_Bb$$

for every $a\in A,b\in B$. We have so proved that $1_S\otimes_R 1_S$ is a separability idempotent of S/R if and only if $1_A\otimes_R 1_A$ and $1_B\otimes_R 1_B$ are separability idempotents of A/R and B/R under the assumption $e_1\otimes_R e_2=0=e_2\otimes_R e_1$. □

The following result is similar to [DI, Corollary 1.7 page 44].

LEMMA 3.18. *Let R be a commutative ring. Let A and B be R-algebras. Then, if B/R is h-separable, so is $\left(A\otimes_R B\right)/A$. As a consequence if both A/R and B/R are h-separable, so is $\left(A\otimes_R B\right)/R$.*

Proof. Since B/R is h-separable, by Theorem 3.13, we have that the unit $u_B:R\to B$ is a ring epimorphism. By Lemma 3.7 this means that $1_B\otimes_R 1_B$ is a separability idempotent. Thus also $\left(1_A\otimes_R 1_B\right)\otimes_A\left(1_A\otimes_R 1_B\right)$ is a separability idempotent.

As a consequence also $A\otimes_R u_B:A\otimes_R R\to A\otimes_R B$ is a ring epimorphism by the same lemma. If A/R is h-separable, then the unit $u_A:R\to A$ is a ring epimorphism too. Thus the composition

$$R\xrightarrow{u_A}A\cong A\otimes_R R\xrightarrow{A\otimes_R u_B}A\otimes_R B,$$

i.e. the unit of $A\otimes_R B$, is an epimorphism. By Theorem 3.13, $\left(A\otimes_R B\right)/R$ is h-separable. □

Remark 3.19. Let R be a ring, G be a group and consider RG, the group ring. S. Caenepeel posed the following problem: to characterize whether RG/R is h-separable. In general we do not have an answer to this question. However, if R is commutative, we can consider a maximal ideal M of R and take $\Bbbk := R/M$. By Lemma 3.18, we deduce that $(\Bbbk \otimes_R RG)/\Bbbk$ is h-separable i.e. $\Bbbk G/\Bbbk$ is h-separable. By Proposition 3.14, we conclude that $\Bbbk G = \Bbbk$ and hence $|G| = 1$.

4. Example on monoidal categories

In the present section $\mathcal{M}$ denotes a preadditive braided monoidal category such that

- $\mathcal{M}$ has equalizers and denumerable coproducts;
- the tensor products are additive and preserve equalizers and denumerable coproducts.

In view of the assumptions above, we can apply [AM1, Theorem 4.6] to obtain an adjunction $(\mathbf{T}, \mathbf{P})$ as in the following diagram

$$\begin{tikzcd} \mathrm{Bialg}(\mathcal{M}) \arrow[r, "\mho"] \arrow[d, shift left, "\mathbf{P}"] & \mathrm{Alg}(\mathcal{M}) \arrow[d, shift left, "\Omega"] \\ \mathcal{M} \arrow[r, "\mathrm{Id}"] \arrow[u, shift left, "\mathbf{T}"] & \mathcal{M} \arrow[u, shift left, "T"] \end{tikzcd}$$

Here $\mathrm{Alg}(\mathcal{M})$ denotes the category of algebras (or monoids) in $\mathcal{M}$, $\mathrm{Bialg}(\mathcal{M})$ is the category of bialgebras (or bimonoids) in $\mathcal{M}$, the functors $\mho$ and Ω are the obvious forgetful functors and, by construction of $\mathbf{T}$, we have $\mho \circ \mathbf{T} = T$.

It is noteworthy that, since Ω has a left adjoint T, then Ω is strictly monadic (the comparison functor is a category isomorphism), see [AM2, Theorem A.6].

Let $V \in \mathcal{M}$. By construction $\Omega TV = \oplus_{n\in\mathbb{N}} V^{\otimes n}$, see [AM1, Remark 1.2]. Denote by $\alpha_n V : V^{\otimes n} \to \Omega TV$ the canonical inclusion. The unit $\eta : \mathrm{Id}_{\mathcal{M}} \to \Omega T$ of the adjunction (T, Ω) is defined by $\eta V := \alpha_1 V$ while the counit $\epsilon : T\Omega \to \mathrm{Id}$ is uniquely defined by the equality

$$\Omega\epsilon\,(A, m, u) \circ \alpha_n A = m^{n-1} \text{ for every } n \in \mathbb{N} \tag{8}$$

where $m^{n-1} : A^{\otimes n} \to A$ denotes the iterated multiplication of an algebra (A, m, u) defined by $m^{-1} = u, m^0 = \mathrm{Id}_A$ and, for $n \geq 2, m^{n-1} = m \circ \left(m^{n-2} \otimes A\right)$.

Denote by $\boldsymbol{\eta}, \boldsymbol{\epsilon}$ the unit and counit of the adjunction $(\mathbf{T}, \mathbf{P})$.

Consider the natural transformation $\xi : \mathbf{P} \to \Omega\mho$ defined by

$$\mathbf{P} \xrightarrow{\eta P} \Omega T\mathbf{P} = \Omega\mho\mathbf{T}\mathbf{P} \xrightarrow{\Omega\mho\boldsymbol{\epsilon}} \Omega\mho.$$

We have $\epsilon\mho \circ T\xi = \epsilon\mho \circ T\Omega\mho\boldsymbol{\epsilon} \circ T\eta P = \mho\boldsymbol{\epsilon} \circ \epsilon T\mathbf{P} \circ T\eta\mathbf{P} = \mho\boldsymbol{\epsilon}$ i.e.

$$\epsilon\mho \circ T\xi = \mho\boldsymbol{\epsilon}. \tag{9}$$

so that ξ is exactly the natural transformation of [AM1, Theorem 4.6], whose components are the canonical inclusions of the subobject of primitives of a bialgebra B in $\mathcal{M}$ into $\Omega\mho B$ and hence they are regular monomorphisms.

Define the functor

$$(-)^+ : \mathrm{Bialg}(\mathcal{M}) \to \mathcal{M}$$

that assigns to every bialgebra A the kernel $(A^+, \zeta A : A^+ \to \Omega\mho A)$ of $\varepsilon_{\Omega\mho A}$ (i.e. the equalizer of $\varepsilon_{\Omega\mho A} : \Omega\mho A \to \mathbf{1}$ and the zero morphism) and to every morphism f the induced morphism f^+.

Since ζA is natural in A we get a natural transformation $\zeta : (-)^+ \to \Omega\mho$ which is by construction a monomorphism on components.

Lemma 4.1. *The natural transformation $\xi : \mathbf{P} \to \Omega\mho$ factors through the natural transformation $\zeta : (-)^+ \to \Omega\mho$ (i.e. there is $\widehat{\xi} : \mathbf{P} \to (-)^+$ such that $\xi = \zeta \circ \widehat{\xi}$) which is a monomorphism on components.*

Proof. Given $A \in \mathrm{Bialg}\,(\mathcal{M})$ we have that ξ and ζ are defined by the following kernels.

$$\begin{tikzcd}
\mathbf{P}A \arrow[r, "\xi A"] \arrow[d, "\widehat{\xi}"] & \Omega\mho A \arrow[rrr, "(u_{\Omega\mho A}\otimes\Omega\mho A)r^{-1}_{\Omega\mho A}+(\Omega\mho A\otimes u_{\Omega\mho A})\circ l^{-1}_{\Omega\mho A}-\Delta_{\Omega\mho A}"] \arrow[d, "\mathrm{Id}"] & & & \Omega\mho A\otimes\Omega\mho A \arrow[d, "m_{\mathbf{1}}(\varepsilon_{\Omega\mho A}\otimes\varepsilon_{\Omega\mho A})"] \\
A^{+} \arrow[r, "\zeta A"] & \Omega\mho A \arrow[rrr, "\varepsilon_{\Omega\mho A}"] & & & \mathbf{1}
\end{tikzcd}$$

Since the right square above commutes, there is a unique morphism $\widehat{\xi}A : \mathbf{P}A \to A^{+}$ such that $\zeta A \circ \widehat{\xi}A = \xi A$. The naturality of ζA and ξA in A implies the one of $\widehat{\xi}A$ so that $\zeta \circ \widehat{\xi} = \xi$. $\square$

There is a unique morphism $\omega V : \Omega T V \to V$ such that

$$\omega V \circ \alpha_n V = \delta_{n,1}\mathrm{Id}_V. \tag{10}$$

Given $f : V \to W$ a morphism in $\mathcal{M}$, we get for every $n \in \mathbb{N}$,

$$\omega W \circ \Omega T f \circ \alpha_n V = \omega W \circ \alpha_n W \circ f^{\otimes n} = \delta_{n,1} f^{\otimes n} = \delta_{n,1} f = f \circ \omega V \circ \alpha_n V$$

so that $\omega W \circ \Omega T f = f \circ \omega V$ which means that $\omega := (\omega V)_{V\in\mathcal{M}}$ is a natural transformation $\omega : \Omega T \to \mathrm{Id}_{\mathcal{M}}$.

Lemma 4.2. *The natural transformation ω fulfills $\omega \circ \eta = \mathrm{Id}$ and*

$$\omega\omega \circ \Omega T\zeta\mathbf{T} = \omega \circ \Omega\epsilon T \circ \Omega T\zeta\mathbf{T}. \tag{11}$$

Proof. We have

$$\omega V \circ \eta V = \omega V \circ \alpha_1 V \overset{(10)}{=} \mathrm{Id}_V$$

and hence $\omega \circ \eta = \mathrm{Id}$. Let us check (11). For every $V \in \mathcal{M}$ we compute

$$\begin{aligned}
& \omega\omega V \circ \Omega T\zeta\mathbf{T}V \circ \alpha_n\,(-)^{+}\,\mathbf{T}V \\
= \;& \omega V \circ \Omega T\omega V \circ \Omega T\zeta\mathbf{T}V \circ \alpha_n\,(-)^{+}\,\mathbf{T}V = \omega V \circ \alpha_n V \circ (\omega V)^{\otimes n} \circ (\zeta\mathbf{T}V)^{\otimes n} \\
= \;& \delta_{n,1}\,(\omega V)^{\otimes n} \circ (\zeta\mathbf{T}V)^{\otimes n} = \delta_{n,1}\omega V \circ \zeta\mathbf{T}V.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\omega V \circ \Omega\epsilon TV \circ \Omega T\zeta\mathbf{T}V \circ \alpha_n\,(-)^{+}\,\mathbf{T}V &= \omega V \circ \Omega\epsilon TV \circ \alpha_n\Omega\mho\mathbf{T}V \circ (\zeta\mathbf{T}V)^{\otimes n} \\
&= \omega V \circ \Omega\epsilon TV \circ \alpha_n\Omega TV \circ (\zeta\mathbf{T}V)^{\otimes n} \\
&\overset{(8)}{=} \omega V \circ m^{n-1}_{\Omega TV} \circ (\zeta\mathbf{T}V)^{\otimes n}.
\end{aligned}$$

Hence we have to check that

$$\delta_{n,1}\omega V \circ \zeta\mathbf{T}V = \omega V \circ m^{n-1}_{\Omega TV} \circ (\zeta\mathbf{T}V)^{\otimes n}.$$

For $n = 0$ we have

$$\delta_{0,1}\omega V \circ \zeta\mathbf{T}V = 0 = \omega V \circ \alpha_0 V = \omega V \circ u_{\Omega TV} = \omega V \circ m^{-1}_{\Omega TV} \circ (\zeta\mathbf{T}V)^{\otimes 0}.$$

For $n = 1$ we have

$$\delta_{1,1}\omega V \circ \zeta\mathbf{T}V = \omega V \circ \zeta\mathbf{T}V = \omega V \circ m^{0}_{\Omega TV} \circ (\zeta\mathbf{T}V)^{\otimes 1}.$$

For $n \geq 2$ we have $\delta_{n,1}\omega V \circ \zeta\mathbf{T}V = 0$. In order to prove that also $\omega \circ m^{n-1}_{\Omega TV} \circ (\zeta\mathbf{T}V)^{\otimes n} = 0$ we need first to give a different expression for $\omega V \circ m_{\Omega TV}$. To this aim we compute (we use the identifications $V \otimes \mathbf{1} \cong V \cong \mathbf{1} \otimes V$)

$$\begin{aligned}
& \omega V \circ m_{\Omega TV} \circ (\alpha_m V \otimes \alpha_n V) \\
= \;& \omega V \circ \alpha_{m+n} V = \delta_{m+n,1}\mathrm{Id}_V \\
= \;& \delta_{m,1}\delta_{n,0}\mathrm{Id}_{V\otimes\mathbf{1}} + \delta_{m,0}\delta_{n,1}\mathrm{Id}_{\mathbf{1}\otimes V} \\
= \;& r_V \circ (\delta_{m,1}\mathrm{Id}_V \otimes \delta_{n,0}\mathrm{Id}_{\mathbf{1}}) + l_V \circ (\delta_{m,0}\mathrm{Id}_{\mathbf{1}} \otimes \delta_{n,1}\mathrm{Id}_V) \\
= \;& r_V \circ (\omega V \otimes \varepsilon_{\Omega TV}) \circ (\alpha_m V \otimes \alpha_n V) + l_V \circ (\varepsilon_{\Omega TV} \otimes \omega V) \circ (\alpha_m V \otimes \alpha_n V) \\
= \;& (r_V \circ (\omega V \otimes \varepsilon_{\Omega TV}) + l_V \circ (\varepsilon_{\Omega TV} \otimes \omega V)) \circ (\alpha_m V \otimes \alpha_n V).
\end{aligned}$$

Since the tensor products preserve denumerable coproducts, the equalities above yield the identity

$$\omega V\circ m_{\Omega TV}=r_V\circ(\omega V\otimes\varepsilon_{\Omega TV})+l_V\circ(\varepsilon_{\Omega TV}\otimes\omega V).$$

Using it, we obtain

$$\begin{aligned}
&\omega V\circ m_{\Omega TV}^{n-1}\circ(\zeta\mathbf{T}V)^{\otimes n}\\
=\;&\omega V\circ m_{\Omega TV}\circ\left(m_{\Omega TV}^{n-2}\otimes\Omega TV\right)\circ(\zeta\mathbf{T}V)^{\otimes n}\\
=\;&(r_V\circ(\omega V\otimes\varepsilon_{\Omega TV})+l_V\circ(\varepsilon_{\Omega TV}\otimes\omega V))\circ\left(m_{\Omega TV}^{n-2}\otimes\Omega TV\right)\circ(\zeta\mathbf{T}V)^{\otimes n}\\
=\;&r_V\circ\left(\omega V\circ m_{\Omega TV}^{n-2}\otimes\varepsilon_{\Omega TV}\right)\circ(\zeta\mathbf{T}V)^{\otimes n}+l_V\circ\left(\varepsilon_{\Omega TV}\circ m_{\Omega TV}^{n-2}\otimes\omega V\right)\circ(\zeta\mathbf{T}V)^{\otimes n}\\
=\;&r_V\circ\left(\omega V\circ m_{\Omega TV}^{n-2}\circ(\zeta\mathbf{T}V)^{\otimes n-1}\otimes\varepsilon_{\Omega TV}\circ\zeta\mathbf{T}V\right)+l_V\circ\left((\varepsilon_{\Omega TV}\circ\zeta\mathbf{T}V)^{\otimes n-1}\otimes\omega V\circ\zeta\mathbf{T}V\right).
\end{aligned}$$

The last two summands are zero as $\varepsilon_{\Omega TV}\circ\zeta\mathbf{T}V=\varepsilon_{\Omega\mho\mathbf{T}V}\circ\zeta\mathbf{T}V=0$ by definition of ζ. □

Remark 4.3. As observed the comparison functor $K:\mathrm{Alg}(\mathcal{M})\to\mathcal{M}_{\Omega T}$ is an isomorphism of categories. By Corollary 2.4, T is h-separable if and only if $\Omega:\mathrm{Alg}(\mathcal{M})\to\mathcal{M}$ is a split epimorphism. Let us prove, by contradiction, that this is not the case. Assume that there is a functor $\Gamma:\mathcal{M}\to\mathrm{Alg}(\mathcal{M})$ such that $\Omega\Gamma=\mathrm{Id}$. Let $V\in\mathcal{M}$. Then $\Gamma V=(V,mV,uV)$ for some morphisms $mV:V\otimes V\to V$ and $uV:\mathbf{1}\to V$. Let $f:V\to V$ be the zero morphism. Then $\Gamma f:\Gamma V\to\Gamma V$ is an algebra morphism and $\Omega\Gamma f=f$. Thus f is unitary i.e. $uV=f\circ uV=0\circ uV=0$. Hence $\mathrm{Id}_V=mV\circ(V\otimes uV)\circ r_V^{-1}=0$. As a consequence any morphism $h:V\to W$ would be zero as $h=h\circ\mathrm{Id}_V$ for every $V,W\in\mathcal{M}$. Hence $\mathrm{Hom}_{\mathcal{M}}(V,W)=\{0\}$. This happens only if all objects are isomorphic to the unit object $\mathbf{1}$, i.e. if the skeleton of $\mathcal{M}$ is the trivial monoidal category $(\mathcal{T},\otimes,\mathbf{1})$, where $\mathrm{Ob}(\mathcal{T})=\{\mathbf{1}\}$, $\mathrm{Hom}_{\mathcal{T}}(\mathbf{1},\mathbf{1})=\{\mathrm{Id}_{\mathbf{1}}\}$ and the tensor product is given by $\mathbf{1}\otimes\mathbf{1}=\mathbf{1}$ and $\mathrm{Id}_{\mathbf{1}}\otimes\mathrm{Id}_{\mathbf{1}}=\mathrm{Id}_{\mathbf{1}}$. This is evidently a restrictive condition on $\mathcal{M}$. Thus, in general $T:\mathcal{M}\to\mathrm{Alg}(\mathcal{M})$ is not heavily separable. On the other hand the equality $\omega\circ\eta=\mathrm{Id}$ obtained in Lemma 4.2 means that the functor $T:\mathcal{M}\to\mathrm{Alg}(\mathcal{M})$ is separable.

As a particular case, we get that the functor $T:\mathrm{Vec}_{\Bbbk}\to\mathrm{Alg}_{\Bbbk}$ is separable but not h-separable.

Theorem 4.4. *Set $\boldsymbol{\gamma}:=\omega\circ\xi\mathbf{T}:\mathbf{PT}\to\mathrm{Id}_{\mathcal{M}}$. Then $\boldsymbol{\gamma}\circ\boldsymbol{\eta}=\mathrm{Id}$ and $\boldsymbol{\gamma}\boldsymbol{\gamma}=\boldsymbol{\gamma}\circ\mathbf{P}\boldsymbol{\epsilon}\mathbf{T}$. Hence the functor $\mathbf{T}:\mathcal{M}\to\mathrm{Bialg}(\mathcal{M})$ is h-separable.*

Proof. We compute

$$\begin{aligned}
\boldsymbol{\gamma}\circ\boldsymbol{\eta}&=\omega\circ\xi\mathbf{T}\circ\boldsymbol{\eta}\overset{\text{def. }\xi}{=}\omega\circ\Omega\mho\boldsymbol{\epsilon}\mathbf{T}\circ\eta\mathbf{PT}\circ\boldsymbol{\eta}\\
&=\omega\circ\Omega\mho\boldsymbol{\epsilon}\mathbf{T}\circ\Omega T\boldsymbol{\eta}\circ\eta=\omega\circ\Omega\mho\boldsymbol{\epsilon}\mathbf{T}\circ\Omega\mho\mathbf{T}\boldsymbol{\eta}\circ\eta=\omega\circ\eta=\mathrm{Id}.
\end{aligned}$$

Moreover

$$\begin{aligned}
\Omega\epsilon\mho\circ\xi\mathbf{T}\xi&=\Omega\epsilon\mho\circ\Omega\mho\mathbf{T}\xi\circ\xi\mathbf{TP}\\
&\overset{\text{def.}\xi}{=}\Omega\epsilon\mho\circ\Omega T\Omega\mho\boldsymbol{\epsilon}\circ\Omega T\eta P\circ\Omega\mho\boldsymbol{\epsilon}\mathbf{TP}\circ\eta P\mathbf{TP}\\
&=\Omega\mho\boldsymbol{\epsilon}\circ\Omega\epsilon\mho\mathbf{TP}\circ\Omega T\eta P\circ\Omega\mho\boldsymbol{\epsilon}\mathbf{TP}\circ\eta P\mathbf{TP}\\
&=\Omega\mho\boldsymbol{\epsilon}\circ\Omega\mho\boldsymbol{\epsilon}\mathbf{TP}\circ\eta P\mathbf{TP}\\
&=\Omega\mho\boldsymbol{\epsilon}\circ\Omega\mho\mathbf{TP}\boldsymbol{\epsilon}\circ\eta P\mathbf{TP}\\
&=\Omega\mho\boldsymbol{\epsilon}\circ\eta P\circ\mathbf{P}\boldsymbol{\epsilon}\overset{\text{def.}\xi}{=}\xi\circ\mathbf{P}\boldsymbol{\epsilon}
\end{aligned}$$

so that

$$\begin{aligned}
&\boldsymbol{\gamma}\boldsymbol{\gamma}=\omega\omega\circ\xi\mathbf{T}\xi\mathbf{T}=\omega\omega\circ\Omega\mho\mathbf{T}\zeta\mathbf{T}\circ\xi\mathbf{T}\widehat{\xi}\mathbf{T}=\omega\omega\circ\Omega T\zeta\mathbf{T}\circ\xi\mathbf{T}\widehat{\xi}\mathbf{T}\\
\overset{(11)}{=}\;&\omega\circ\Omega\epsilon T\circ\Omega T\zeta\mathbf{T}\circ\xi\mathbf{T}\widehat{\xi}\mathbf{T}=\omega\circ\Omega\epsilon\mho\mathbf{T}\circ\xi\mathbf{T}\xi\mathbf{T}=\omega\circ\xi\mathbf{T}\circ\mathbf{P}\boldsymbol{\epsilon}\mathbf{T}=\boldsymbol{\gamma}\circ\mathbf{P}\boldsymbol{\epsilon}\mathbf{T}
\end{aligned}$$

□

References

[AM1] A. Ardizzoni and C. Menini, *Adjunctions and Braided Objects*, J. Algebra Appl. **13**(06) (2014), 1450019 (47 pages). 10

[AM2] A. Ardizzoni and C. Menini, *Milnor-Moore Categories and Monadic Decomposition*, J. Algebra **448** (2016), 488-563. 10

[AMa] M. F. Atiyah, I. G. Macdonald, *Introduction to commutative algebra.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1969. 6

[Bo] F. Borceux, *Handbook of categorical algebra. 2. Categories and structures.* Encyclopedia of Mathematics and its Applications, **51**. Cambridge University Press, Cambridge, 1994. 4

[Br] T. Brzeziński, *The structure of corings: induction functors, Maschke-type theorem, and Frobenius and Galois-type properties.* Algebr. Represent. Theory **5** (2002), no. 4, 389–410. 4, 5

[BW] T. Brzezinski, R. Wisbauer, *Corings and comodules.* London Mathematical Society Lecture Note Series, 309. Cambridge University Press, Cambridge, 2003. 6

[CGN] F. Castaño Iglesias, J. Gómez Torrecillas, C. Năstăsescu, *Separable functors in graded rings.* J. Pure Appl. Algebra **127** (1998), no. 3, 219–230. 2

[CMZ] S. Caenepeel, G. Militaru, S. Zhu, *Frobenius and separable functors for generalized module categories and nonlinear equations.* Lecture Notes in Mathematics, 1787. Springer-Verlag, Berlin, 2002. 5

[DI] F. DeMeyer, E. Ingraham, *Separable algebras over commutative rings.* Lecture Notes in Mathematics, Vol. **181** Springer-Verlag, Berlin-New York 1971 7, 9

[LMW] M. Livernet, B. Mesablishvili, R. Wisbauer, *Generalised bialgebras and entwined monads and comonads.* J. Pure Appl. Algebra **219** (2015), no. 8, 3263–3278. 4

[NT] C. Năstăsescu, B. Torrecillas, *Torsion theories for coalgebras.* J. Pure Appl. Algebra **97** (1994), no. 2, 203-220. 7

[NVV] C. Năstăsescu, M. Van den Bergh, F. Van Oystaeyen, *Separable functors applied to graded rings.* J. Algebra **123** (1989), no. 2, 397–413. 2, 5

[Pi] R. S. Pierce, *Associative algebras.* Graduate Texts in Mathematics, **88**. Studies in the History of Modern Science, 9. Springer-Verlag, New York-Berlin, 1982. 8

[Ra] M. D. Rafael, *Separable Functors Revisited*, Comm. Algebra **18** (1990), 1445–1459. 3

[St] B. Stenström, *Rings of quotients* Die Grundlehren der Mathematischen Wissenschaften, Band **217**. An introduction to methods of ring theory. Springer-Verlag, New York-Heidelberg, 1975.

6

University of Turin, Department of Mathematics "G. Peano", via Carlo Alberto 10, I-10123 Torino, Italy
E-mail address: alessandro.ardizzoni@unito.it
URL: sites.google.com/site/aleardizzonihome

University of Ferrara, Department of Mathematics, Via Machiavelli 35, Ferrara, I-44121, Italy
E-mail address: men@unife.it
URL: sites.google.com/a/unife.it/claudia-menini