Università degli Studi di Ferrara

DOTTORATO DI RICERCA
in
MATEMATICA

In convenzione con
Università degli Studi di Parma
Università degli Studi di Modena e Reggio Emilia

CICLO XXXI

COORDINATORE Prof. Massimiliano Mella

Burnside and Mackey Theories for Abstract Groupoids

Settore Scientifico Disciplinare MAT/02

Dottorando
Dott. Spinosa Leonardo

Relatore
Prof. El Kaoutit Laiachi

Tutor
Prof. Menini Claudia

Anni accademici 2015-2018

Table of contents

List of diagrams

Introduction

This thesis, after introducing the definition of groupoid and its main properties, aims to discuss the extensions to the groupoid context of two important topics in group representation theory: Burnside and Mackey theories. In this introduction, after an historical overview about groupoids, Burnside theory and Mackey theory, we are going to outline the content of this thesis, chapter by chapter. After this, we are going to enumerate the main results of this thesis.

Historical overview

Groupoids

Groupoids are natural generalization of groups and they have proved to be useful in different branches of mathematics. Many surveys have been written about them: see, for example, [Bro87], [Car08] and [Wei96] (references therein included). Furthermore, the relevance of groupoids in category theory and in topology has been made clear in [Hig71] and in [Bro06], respectively. Actually, using category theory, a groupoid (without further structure, e.g., a topological one or a differential one, sometimes also called an abstract groupoid) can be defined as a small category whose every morphism is an isomorphism. This entail that a groupoid, as a category, is equivalent to a disjoint union of groups and, thus, that a groupoid can be considered as a "group with many objects". In the same way, a group can be seen as a groupoid with only one object. The problem is that, as explained in [Bro87], this forces unnatural choices of base points and obscures the overall structure of the situation. This becomes especially evident with the fundamental groupoid of a topological space.

As it will be realized from this thesis, even for abstract groupoids, the idea of treating these objects as a certain bundle of groups, using classical tools, obscures their internal structure and behaviour. For instance, when treating abstract groupoids, from the classical Burnside theory point of view by using the classical groupoid-set notions (i.e., functors to the core category of sets), we will see that this theory discriminate against subgroupoids with several objects, as this class of subgroupoids becomes absolutely absent. This makes manifest that the "poset" of subgroupoids of a given abstract groupoid needs new techniques in order for its study to be approached. We write the word *poset* between quotation marks because we think that the set of subgroupoids could have more structure than the poset one.

Moreover structured groupoids, like topological or differential groupoids (see [Mac87] and [Mac05]), in contrast with abstract groupoids, have "more structure" and are not equivalent, as as a category, to a disjoint union of topological or differential groups, respectively. Several mathematicians came to the conclusion (see, for instance, [Bro87] and [Con94, page 6-7]), in fact, that extending a certain well known result in the group context to the framework of groupoids is not a trivial research problem and has its own difficulties and challenges to overcome.

Burnside theory

The Burnside theory is a classical part of the representation theory of finite groups and its first introduction has been realized by Burnside in [Bur11]. Subsequently, further progresses have been accomplished by Solomon and Dress in [Sol67] and [Dre69], respectively. Apparently, there are two interrelated aspects of this theory. The first is the renowned Burnside Theorem, that codifies some basic combinatorial properties of the lattice of subgroups of a given finite group, providing, for instance, its table of marks. As we will see here, this lattice can be viewed as a category whose arrows are equivariant maps between cosets. An entry in the table of marks coincides with the number of morphisms between two objects in this category. This approach could be considered innovative, even in the case of groups.

The other aspect is the construction of the Burnside ring over the integers and its extension algebra over the rational numbers. Years after its discovery, the Burnside ring of a group has became a very powerful tool in different branches of pure mathematics. For instance, in certain equivariant stable homotopies (e.g., that of the sphere in dimension zero, see [Seg71]), the influence of the Burnside ring is conspicuously present so that, in particular, stable equivariant homotopy groups are modules over the Burnside ring (see [Die79] for further details).

Mackey theory

The classical Mackey formula, which appeared for the first time in [Mac51, Theorem 1], deals with linear representations of finite groups. To summarize, given a group G, let us consider two subgroups H and K of G. Roughly speaking (see [Ser77, Proposition 22]), this formula states that applying first the restriction functor with respect to H and, then, the induction functor with respect to K, we obtain a representation that is isomorphic, although in a not canonical way, to a coproduct of a particular set of representations.

As it was explained in [Ser77, Section 7.4], the Mackey formula is a key tool in proving the Mackey irreducibility criterion, which gives us necessary and sufficient conditions for the irreducibility of an induced representation, proving to be useful to study the linear representations of a semidirect product by an abelian group (see [Ser77, Proposition 25]).

Another formulation of the classical Mackey formula, using modules over groups algebras, was stated in [CR62, Theorem 44.2]. Successively, in [Mac52, Theorems 7.1 and 12.1], the Mackey formula was extended to the context of locally compact groups (with opportune hypotheses), and used to prove a generalization of the Frobenius Reciprocity Theorem (see [Mac52, Theorems 8.1, 8.2 and 13.1]). Afterwards, many variants and different formulations of the Mackey formula have been investigated. For example, in [Tay17], [Bon00] and [Bon03], Taylor and Bonnafé proved specific versions of this formula for algebraic groups. The importance of Mackey formula version in this context had already been made clear in [DM91] and previous work had been done in [DL76, Theorem 6.8], [LS79, Lemma 2.5] and [DL83, Theorem 7].

Outline of the thesis

Now we are going to summarize the content of this thesis.

Chapter 1

We'll start by giving the basic definitions about groupoids and showing their basic properties, introducing the notion of groupoid-set, that is, a set with a groupoid action.

After that, we will turn to the concept of groupoid-biset: this notion will be used to introduce the crucial notion of cosets by subgroupoids in Subsection 1.2.2.

Lastly, we'll discuss monoidal equivalence between categories of groupoid-sets with respect to different groupoids. Actually, these categories have two different monoidal structures: one given by the coproduct, that is, by the disjoint union, and the other given by the fibre product (a generalization of the cartesian product). The aforementioned equivalences, of course, have to be monoidal with respect to both monoidal structure, and we'll call "Laplaza" this kind of equivalences of categories (see Subsection 1.3.1).

The content of Section 1.3 has been published in the following preprint: [ES18a].

Chapter 2

In this chapter we will develop a theory of conjugations for subgroupoids, even with multiple objects, showing, through multiple examples and counterexamples, many particularities of this new theory.

Subsequently, we will apply all of this to study how to reproduce the classical Burnside Theorem (see [Bou10a, Thm. 2.4.5]) in the groupoid context (Theorem 2.2.7). This result, under reasonable finiteness conditions of the groupoid, provides necessary and sufficient conditions in order for two groupoid-sets to be isomorphic, and it will be useful later on, in Chapter 4, to study the Burnside ring of a given groupoid.

The content of this chapter has been published in the following preprint: [ES18a].

Chapter 3

The classical Mackey formula can be applied, in a not trivial way, in many different mathematical fields. In this direction, motivated by the study of the structure of biset functors over finite groups (see [Bou10a, Definitions 3.1.1, 3.2.2] for the pertinent definitions), Serge Bouc proved in [Bou10a, Lemma 2.3.24] a different version of the classical Mackey formula in the framework of group-bisets (a set with a left action and a right action that commute with each other). The gist is that, given two groups H and G and a field $\mathbb{F}$, an (H, G)-biset (of groups) is a left H-invariant and right G-invariant $\mathbb{F}$-basis of an $(\mathbb{F}H, \mathbb{F}G)$-bimodule. Since the classical Mackey formula on linear representations can be rephrased using bimodules, and bimodules induce bisets, the classical Mackey formula can be reformulated using an isomorphism of group-bisets (see the end of [Bou10a, Section 1.1.5]) which is further on reformulated as [Bou10a, Lemma 2.3.24].

We have to mention that, in [Bou10b], Bouc himself proved an additional version of the Mackey formula, which is expressed using bimodules and group-bisets.

In this chapter we will generalize the Mackey formula proved by Bouc in [Bou10a, Lemma 2.3.24] to the environment of groupoid-bisets. To achieve this goal we'll have first to prove a few technical results, which will be necessary to even state this new formula.

The content of this chapter has been published in [ES18b].

We notice that the content of this chapter, as well as of Chapter 5 and, probably, also of Chapter 6, could be investigated also in the context of linear representations of groupoids (see [EB18]) but this will be the object of future work.

Chapter 4

We'll start this chapter explaining how the right translation groupoid induces a functor from the category of groupoid-sets to the category of groupoids and we will describe its properties.

Next, we will examine another issue. Since groupoids are categories, even if two of them are not isomorphic, they can be equivalent as categories. After having examined this notion, we will use it to define a new concept: we will define two groupoid-sets as weakly translationally equivalent if their translation groupoids are equivalent and we will conclude describing the properties of this relation.

Chapter 5

From a categorical point of view, the Burnside ring can be constructed, with the help of the Grothendieck functor, from any skeletally small category with initial object and finite coproducts, which possesses a monoidal structure compatible with this coproduct (a structure we called a Laplaza category). The special case is when this category is a certain category of representations over a specific object: a group, a groupoid, a 2-group, a 2-groupoid, etc. The idea is to use this ring in order to analyse the structure of the handled object. This heavily depends, of course, on the choice of the category of representations and on the chosen "equivalence relation" between its objects. Precisely, one could use a kind of weak equivalences - if there is any relation of this type compatible with the tensor product - instead of the obvious isomorphism relation between objects.

The situation of groupoids corroborates this dependency. More precisely, we will show, in Chapter 5, that the use of the category of "classical" groupoid-sets, in building up the Burnside ring, does not reflects the groupoid structure. Namely, this construction treats a groupoid as if it were a bundle of groups, which is absolutely not the case.

The problem seems to have it origin in the decomposition of a given (right or left) groupoid-set into its orbit subsets. Indeed, each of these orbits is isomorphic to a set of right cosets of the groupoid by a subgroupoid with a single object, as proved in Proposition 1.2.10 and Corollary 1.2.11. Thus, in this "classical" Burnside theory, most "elements" in the lattice of subgroupoids just don't show up: specifically, those subgroupoids with more than one object.

The content of this chapter has been published in the following preprint: [ES18a].

Chapter 6

In this chapter we will develop a new version of the Burnside ring, named the *categorified Burnside ring*, using the idea of categorification. This ring, in a certain sense (see Remark 6.6.2), extends the classical Burnside ring and provides a finer invariant although, even in this case, it doesn't distinguish a bundle of groups from a groupoid.

The concept of categorification has been explained extensively in [BC04, pag. 495] and [BL04]: the idea is to replace the underlying set of an algebraic structure, like a group, with a category, with the goal of obtaining a new and more complex structure. Moreover, the old structure maps are replaced by functors. The aim is to explain and include more complex situations that cannot be explained using the tools of the classical algebraic structure. In the case of the category of groups, for example, the categorificaton process produces the notion of 2-group (see [BL04]), which has been proved to be equivalent to the concept of crossed module introduced by Whitehead in [Whi46] and [Whi49]. To performe the categorification of a structure, the notions of internal category, interal functor and internal natural transformation, introduced in [Ehr63a], [Ehr63b] and [Ehr66], are crucial and they will be used extensively in this chapter.

We note that there are, in the literature, other generalizations of the classical Burnside theory: see [OY01], [HY07], [DL09] and [GRR12].

The content of this chapter has been published in the following preprint: [ES18c]. Note that the entity that we call right categorified $\mathcal{G}$-sets in this thesis has been called a simplicial right $\mathcal{G}$-set in [ES18c]. In this thesis the terminology has been changed to prevent incomprehensions, as explained in Remark 6.1.5.

Appendixes

In Appendixes A and B we will briefly recall the concepts of "rig" (also called semiring) and of Grothendieck functor, fundamental tools to construct the classical and categorified Burnside theories in Chapters 5 and 6, respectively. We define a rig as a ring without negative elements, that is, without the inverses of the addition. The Grothendieck functor enables us to "add" the additive inverses to a rig to obtain a ring. It's exactly in this way that the ring of integers $\mathbb{Z}$ is constructed from the natural numbers $\mathbb{N}$, the quintessential example of rig.

In Appendix C, after recalling some definitions about monoidal categories and monoidal functors, we will proceed to prove a necessary result for our work (Proposition C.0.9) whose proof, albeit known, we have been able to find only briefly hinted.

Main results of the thesis

For the convenience of the reader, we will explicitly enumerate all the original results contained in this thesis.

(a) Theorem 1.3.15.

(b) Theorem 2.1.4, Proposition 2.2.3 and Theorem 2.2.7.

(c) Theorem 3.2.1.

(d) Corollary 4.4.2.

(e) Proposition 5.1.3, Theorem 5.1.8 Corollary 5.1.10 and the content of Section 5.2.

(f) Propositions 6.4.3 and 6.5.11, Theorems 6.6.5, 6.6.7 and 6.6.8, Corollary 6.6.9.

Acknowledgements

My deepest gratitude goes to my Ph.D. advisor, Prof. Laiachi El Kaoutit, without whom this thesis would have never been completed. The time we spent together in Ceuta has been essential to the writing of this thesis.

Special thanks are due to Prof. Claudia Menini, my tutor at the University of Ferrara, for all her help during these three years.

I would like to thank the two referees of the present work, Prof. Christian Lomp and Prof. Joost Vercruysse, who kindly agreed to read this thesis, and Prof. Gabriella Böhm, who accepted to be the non-italian member of the Ph.D. final exam committee.

I find it right to acknowledge the economic support of Prof. Claudia Menini, of the Italian mathematical association INdAM - GNSAGA and of Prof. Laiachi El Kaoutit for their financial support of my research stays in Turin and Ceuta.

It would be impossible not to mention my master thesis advisor at the University of L'Aquila, Prof. Giuseppe Baccella, for his guidance and for having introduced me to category theory and to the world of noncommutative rings.

I have to thank my parents, for having supported me all these years through all my school and university career, and for being always present.

Lastly but not least, for all the wonderful time together, I have to thank my friends in Ferrara: Sara, Giorgio, Serena, Francesco, Giulia, Giao and Elisa.

Ferrara, 22nd November 2018

Leonardo Spinosa

Chapter 1

Basic notions

1.1 Abstract groupoids: General notions and basic properties

The material of this section, which will be used throughout the rest of the thesis, is somehow considered folklore and most of its content can be found in [EK17] and [El 18]. However, for the sake of completeness and for the convenience of the reader, we are going to illustrate the basic notions, as well as some motivating examples, of the groupoid theory. The exposition is written in a very elementary way in order to render it accessible to every kind of reader.

1.1.1 Notations, basic notions and examples

Definition 1.1.1. We say that a category $\mathcal{C}$ is a **small category** if its class of arrows is actually a set.

Given functions $f\colon A \longrightarrow D$ and $g\colon B \longrightarrow D$, we will use the notation:

$$A\,{}_{f}\!\times_{g} B = \{\, (a,b) \in A \times B \mid f(a) = g(b) \,\}\,. \tag{1.1.1}$$

This set is well known as the **fiber product** (or **fibre product**) of f and g and it is the pullback of the maps f and g in the category of sets. This notation can be also adopted in a categorical setting replacing sets with small categories and functions with functors (see equation 6.1.1).

Definition 1.1.2. A **groupoid** is a (small) category such that all its morphisms are invertible. Given a groupoid $\mathcal{G}$, we will denote by $\mathcal{G}_0$ its set of objects and by $\mathcal{G}_1$ its set of morphisms, which are also called **arrows**. Given a morphism $f\colon a \longrightarrow b$ in $\mathcal{G}$, we will use the notations $a = \mathsf{s}(f)$ and $b = \mathsf{t}(f)$ where s stands for "**source**" and t stands for "**target**". Given an object a of $\mathcal{G}$, we will denote by ι_a the identity morphism at a. Moreover, the set $\{\, g \in \mathcal{G}_1 \mid \mathsf{t}(g) = \mathsf{s}(g) = a \,\}$ is obviously a group, is denoted by $\mathcal{G}^a$, is called the **isotropy group of $\mathcal{G}$ at** a and its elements are called **loops**.

As a consequence a groupoid is a pair of two sets $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_0)$ endowed with the following functions:

$$(\ )^{-1} \circlearrowright \mathcal{G}_1 \underset{\mathsf{t}}{\overset{\mathsf{s}}{\rightrightarrows}} \mathcal{G}_0,\quad \mathcal{G}_0 \xrightarrow{\iota} \mathcal{G}_1 \qquad \text{and} \qquad \mathcal{G}_1\,{}_{\mathsf{s}}\!\times_{\mathsf{t}} \mathcal{G}_1 \longrightarrow \mathcal{G}_1,$$

where ι assigns to each object its identity arrow, the multiplication (i.e. the second map in the previous equation) is associative and unital and $(\ )^{-1}$ associates to each arrow $g\colon a \longrightarrow b$ its inverse $g^{-1}\colon b \longrightarrow a$. Note that ι is an injective map thus $\mathcal{G}_0$ can be identified with a subset of $\mathcal{G}_1$. To summarize, a groupoid is a small category with more structure, namely the map that sends any arrow to its inverse. We implicitly identify a groupoid with its underlying category. Interchanging the source and the target will lead to the **opposite groupoid** which we denote by $\mathcal{G}^{\mathrm{op}}$.

Given a groupoid $\mathcal{G}$, consider two objects $x, y \in \mathcal{G}_0$: we denote by $\mathcal{G}(x,y)$ the set of all arrows with source x and target y (in particular we have $\mathcal{G}^x = \mathcal{G}(x,x)$).

Remark 1.1.3. Let $\mathcal{G}$ be a groupoid such that the composition is commutative that is, for each $g, h \in \mathcal{G}_1$ such that $\mathsf{s}(g) = \mathsf{t}(h)$, we have $\mathsf{s}(h) = \mathsf{t}(g)$ and $gh = hg$. In this case we obtain $\mathsf{s}(h) = \mathsf{t}(g) = \mathsf{t}(h)$ and $\mathsf{t}(g) = \mathsf{s}(h) = \mathsf{s}(g)$. Therefore every arrow of the groupoid $\mathcal{G}$ is a loop and $\mathcal{G}$ is merely a disjoint union of abelian groups.

Clearly each of the sets $\mathcal{G}(x,y)$ is, by the groupoid multiplication, a left $\mathcal{G}^y$-set and right $\mathcal{G}^x$-set. In fact, each of the $\mathcal{G}(x,y)$ sets is a $(\mathcal{G}^y, \mathcal{G}^x)$-biset, in the sense of [Bou10a]. Two objects $x, x' \in \mathcal{G}_0$ are said to be equivalent if and only if there is an arrow connecting them. This in fact defines an equivalence relation whose quotient set is the set of all **connected components** of $\mathcal{G}$, which we denote by $\pi_0(\mathcal{G}) := \mathcal{G}_0/\mathcal{G}$ and we call **orbit set of the groupoid** $\mathcal{G}$. Alternatively, this equivalence relation can be described as follows: given an object $x \in \mathcal{G}_0$, define

$$\mathscr{O}_x := \mathsf{t}\big(\mathsf{s}^{-1}(\{x\})\big) \;=\; \Big\{ y \in \mathcal{G}_0 \,|\, \exists\, g \in \mathcal{G}_1 \text{ such that } \mathsf{s}(g) = x,\, \mathsf{t}(g) = y \Big\}, \tag{1.1.2}$$

which is equal to the set $\mathsf{s}\big(\mathsf{t}^{-1}(\{x\})\big)$. This is a not empty set, since $x \in \mathscr{O}_x$. Two objects $x, x' \in \mathcal{G}_0$ are said to be equivalent if and only if $\mathscr{O}_x = \mathscr{O}_{x'}$. We will also use the notation $\mathrm{Orb}_{\mathcal{G}}(x)$ to denote $\mathscr{O}_x$ and we will call it the **orbit of** x.

Given a set I and a family of groupoids $\{\mathcal{G}^{(i)}\}_{i\in I}$, **the coproduct groupoid** is a groupoid denoted by $\mathcal{G} = \coprod_{i\in I} \mathcal{G}^{(i)}$ and defined by

$$\mathcal{G}_0 \;=\; \biguplus_{i\in I} \mathcal{G}_0^{(i)}, \qquad \mathcal{G}(x,y) \;=\; \begin{cases} \mathcal{G}^{(i)}(x,y), & \text{if } \exists\, i \in I \text{ such that } x, y \in \mathcal{G}_0^{(i)} \\ \varnothing, & \text{otherwise.} \end{cases}$$

Definition 1.1.4. A **groupoid** $\mathcal{G}$ is said **transitive** (or **connected**) if for every $(y,x) \in \mathcal{G}_0 \times \mathcal{G}_0$, there is $g \in \mathcal{G}_1$ such that $\mathsf{t}(g) = y$ and $\mathsf{s}(g) = x$. Equivalently, $\mathcal{G}$ is transitive if the map $(\mathsf{s}, \mathsf{t})\colon \mathcal{G}_1 \longrightarrow \mathcal{G}_0 \times \mathcal{G}_0$ is surjective.

Remark 1.1.5. In general, any groupoid can be seen as a coproduct of transitive groupoids: namely, its connected components. Note that, with this definition, the empty groupoid is transitive.

Definition 1.1.6. Given two groupoids $\mathcal{H}$ and $\mathcal{G}$, we say that $\mathcal{H}$ is a **subgroupoid** of $\mathcal{G}$ if $\mathcal{H}$ is a subcategory of $\mathcal{G}$ which is stable under the inverse map, that is, for every $h \in \mathcal{H}_1$, also $h^{-1} \in \mathcal{H}_1$.

For instance, any connected component of $\mathcal{G}$ is a subgroupoid. On the other hand, a subgroup H of an isotropy group $\mathcal{G}^x$, for an object $x \in \mathcal{G}_0$, can be considered as a **subgroupoid with only one object** of $\mathcal{G}$. Conversely, any subgroupoid of $\mathcal{G}$ with one object is of this form.

Definition 1.1.7. Given two groupoids $\mathcal{H}$ and $\mathcal{G}$, a **morphism of groupoids** $\varphi\colon \mathcal{G} \longrightarrow \mathcal{H}$ is a functor between the underlying categories.

Groupoids and their morphisms form a category denoted by **Grpd**.

In a more explicit way, a morphism of groupoids $\varphi\colon \mathcal{H} \longrightarrow \mathcal{G}$ can be characterized as a pair of functions

$$\Big(\varphi_1\colon \mathcal{H}_1 \longrightarrow \mathcal{G}_1, \qquad \varphi_0\colon \mathcal{H}_0 \longrightarrow \mathcal{G}_0\Big)$$

such that the following diagrams commute:

$$\begin{array}{ccc} \mathcal{H}_1 & \xrightarrow{\varphi_1} & \mathcal{G}_1 \\ {\scriptstyle \mathsf{s}}\downarrow & & \downarrow{\scriptstyle \mathsf{s}} \\ \mathcal{H}_0 & \xrightarrow{\varphi_0} & \mathcal{G}_0, \end{array} \qquad \begin{array}{ccc} \mathcal{H}_1 & \xrightarrow{\varphi_1} & \mathcal{G}_1 \\ {\scriptstyle \mathsf{t}}\downarrow & & \downarrow{\scriptstyle \mathsf{t}} \\ \mathcal{H}_0 & \xrightarrow{\varphi_0} & \mathcal{G}_0, \end{array} \qquad \begin{array}{ccc} \mathcal{H}_0 & \xrightarrow{\varphi_0} & \mathcal{G}_0 \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \iota} \\ \mathcal{H}_1 & \xrightarrow{\varphi_1} & \mathcal{G}_1 \end{array}$$

and

$$\begin{array}{ccc} \mathcal{H}_1\, {}_{\mathsf{s}}\!\times_{\mathsf{t}} \mathcal{H}_1 & \xrightarrow{\varphi_1 \times \varphi_1} & \mathcal{G}_1\, {}_{\mathsf{s}}\!\times_{\mathsf{t}} \mathcal{G}_1 \\ \downarrow & & \downarrow \\ \mathcal{H}_1 & \xrightarrow{\varphi_1} & \mathcal{G}_1. \end{array}$$

Of course, a morphism of groupoids $\varphi\colon \mathcal{H} \longrightarrow \mathcal{G}$ induces homomorphisms of groups $\varphi^a\colon \mathcal{H}^a \longrightarrow \mathcal{G}^{\varphi_0(a)}$ between the isotropy groups for every $a \in \mathcal{H}_0$. The homomorphisms of groups

$$\Big(\varphi^a\colon \mathcal{H}^a \longrightarrow \mathcal{G}^{\varphi_0(a)}\Big)_{a \in \mathcal{H}_0}$$

are refereed to as the **isotropy maps** of φ. Clearly any subgroupoid $\mathcal{H}$ of $\mathcal{G}$ induces a morphism $\tau : \mathcal{H} \hookrightarrow \mathcal{G}$ of groupoids whose both maps τ_0 and τ_1 are injective. In order to illustrate the foregoing notions, we quote here some standard examples of groupoids and their morphisms.

Example 1.1.8 (Trivial groupoid). Given a set X, the pair (X, X) is a small discrete category, that is, a category with only identities as arrows. This is known as the **trivial groupoid**. Note that, with this definition, the empty groupoid is the trivial groupoid $(\varnothing, \varnothing)$ which, by convention, is also considered as a transitive groupoid.

Example 1.1.9 (Product of groupoids). Given two groupoids $\mathcal{G}$ and $\mathcal{H}$, the **product groupoid** $\mathcal{G} \times \mathcal{H}$ is the direct product of their underlying categories. This means that $(\mathcal{G} \times \mathcal{H})_1 = \mathcal{G}_1 \times \mathcal{H}_1$ and $(\mathcal{G} \times \mathcal{H})_0 = \mathcal{G}_0 \times \mathcal{H}_0$. Moreover, the multiplication, inverse and unit arrow are canonically given as follows:

$$(g,h)\,(g',h') \;=\; (gg', hh'), \qquad (g,h)^{-1} \;=\; (g^{-1}, h^{-1}), \quad \iota_{(x,u)} \;=\; (\iota_x, \iota_u).$$

Example 1.1.10 (Action groupoid). Any group G can be considered as a groupoid by taking $G_1 = G$ and $G_0 = \{*\}$ (a set with one element). Now if X is a right G-set with action $\rho\colon X \times G \longrightarrow X$, it is possible to define the **action groupoid** $\mathcal{G}$, whose set of objects is $G_0 = X$ and whose set of arrows is $G_1 = X \times G$; the source and the target maps are, respectively, $\mathsf{s} = \rho$ and $\mathsf{t} = \mathrm{pr}_1$ and, lastly, the identity map sends x to $(x, e) = \iota_x$, where e is the identity element of G. The multiplication is given by $(x, g)(x', g') = (x, gg')$, whenever $xg = x'$, and the inverse is defined by $(x, g)^{-1} = (xg, g^{-1})$. Clearly the pair of maps $(\mathrm{pr}_2, *)\colon \mathcal{G} = (G_1, G_0) \longrightarrow (G, \{*\})$ defines a morphism of groupoids. For a given $x \in X$, the isotropy group $\mathcal{G}^x$ is obviously identified with $\mathrm{Stab}_G(x) = \{g \in G |\, gx = x\}$, the stabilizer subgroup of x in G (see subsection 1.1.3).

Example 1.1.11 (Equivalence relation groupoid). Here is a standard class of examples of groupoids, ordered by inclusion.

(1) One can associated to a given set X the so called the **groupoid of pairs** (called **fine groupoid** in [Bro87] and **simplicial groupoid** in [Hig71]) whose set of arrows is $X \times X$ and whose set of objects is X. The source and the target are $\mathsf{s} = \mathrm{pr}_2$ and $\mathsf{t} = \mathrm{pr}_1$, the second and the first projections, and the map of identity arrows ι is the diagonal map $x \mapsto (x, x)$. The multiplication and the inverse maps are given by

$$(x, x')\,(x', x'') \;=\; (x, x''), \quad \text{and} \quad (x, x')^{-1} \;=\; (x', x).$$

(2) Given a function $\nu\colon X \longrightarrow Y$, we define a groupoid with X as set of objects, the fiber product $X\,{}_{\nu}\!\times_{\nu}\, X$ as set of arrows, $\mathsf{s} = \mathrm{pr}_2$, $\mathsf{t} = \mathrm{pr}_1$ and the diagonal map as the map of identity arrows ι. The multiplication and the inverse map are defined as for the groupoid of pairs.

(3) Assume that $\mathcal{R} \subseteq X \times X$ is an equivalence relation on the set X. It is possible to construct a groupoid, as before in (2), but with set of arrows $\mathcal{R}$. This is an important class of groupoids known as the **groupoid of equivalence relation** (or **equivalence relation groupoid**). Obviously the inclusion $(\mathcal{R}, X) \hookrightarrow (X \times X, X)$ is a morphism of groupoid and $(\mathcal{R}, X)$ is a subgroupoid of $(X \times X, X)$ (see for instance [DG70, Example 1.4, page 301]).

Notice, that in all these examples each of the isotropy groups is the trivial group.

Example 1.1.12 (Induced groupoid). Let $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_0)$ be a groupoid and $\varsigma\colon X \longrightarrow \mathcal{G}_0$ a map. Consider the following pair of sets:

$$(\mathcal{G}^{\varsigma})_1 := X\,{}_{\varsigma}\!\times_{\mathsf{t}}\, \mathcal{G}_1\,{}_{\mathsf{s}}\!\times_{\varsigma}\, X = \left\{ (x, g, x') \in X \times \mathcal{G}_1 \times X \;\middle|\; \begin{array}{c} \varsigma(x) = \mathsf{t}(g) \\ \varsigma(x') = \mathsf{s}(g) \end{array} \right\}, \quad (\mathcal{G}^{\varsigma})_0 := X.$$

Then $\mathcal{G}^{\varsigma} = (\mathcal{G}^{\varsigma}{}_1, \mathcal{G}^{\varsigma}{}_0)$ is a groupoid, with source, target and identity maps as follow: $\mathsf{s} = \mathrm{pr}_3$, $\mathsf{t} = \mathrm{pr}_1$ and $\iota_x = (\varsigma(x), \iota_{\varsigma(x)}, \varsigma(x))$, for each $x \in X$. The multiplication is defined by $(x, g, y)(x', g', y') = (x, gg', y')$, whenever $y = x'$, and the inverse is given by $(x, g, y)^{-1} = (y, g^{-1}, x)$. The groupoid $\mathcal{G}^{\varsigma}$ is known as the **induced groupoid of $\mathcal{G}$ by the map** ς, (or the **pullback groupoid of $\mathcal{G}$ along** ς, see [Hig71] for dual notion). Clearly, there is a canonical morphism $\varphi^{\varsigma} := (\mathrm{pr}_2, \varsigma)\colon \mathcal{G}^{\varsigma} \longrightarrow \mathcal{G}$ of groupoids.

Remark 1.1.13. A particular instance of an induced groupoid is when $\mathcal{G} = G$ has a single object. Thus for any group G it is possible to consider the Cartesian product $X \times G \times X$ as a groupoid with set of objects X. This groupoid, denoted by $\mathcal{G}_{G,X}$, is clearly transitive with G as isotropy group type. It is noteworthy to mention that the class of groupoids given in Example 1.1.12 characterizes, in fact, transitive groupoids. More precisely, every transitive groupoid is isomorphic, although in a not canonical way, to a groupoid of the form $\mathcal{G}_{G,X}$ with admissible choices $X = \mathcal{G}_0$ and $G = \mathcal{G}^x$ for any $x \in \mathcal{G}_0$.

Furthermore, given groups G and H and sets S and T, it is easily shown that the following statements are equivalent:

(1) The groupoids $\mathcal{G}_{G,S}$ and $\mathcal{G}_{H,T}$ are isomorphic.

(2) There is a bijection $S \simeq T$ and an isomorphism of groups $G \cong H$.

1.1.2 Groupoid actions and equivariant maps

The following crucial definition, which we reproduce here from [EK17] and [El 18], is a natural generalization to the context of groupoids of the usual notion of group-set (see, for instance, [Bou10a]). This is an abstract formulation of that given in [Mac05, Definition 1.6.1] for Lie groupoids, and it's essentially the same definition based on the Sets-bundles notion given in [Ren80, Definition 1.11].

Definition 1.1.14. Given a groupoid $\mathcal{G}$, a set X and a map $\varsigma\colon X \longrightarrow \mathcal{G}_0$, we say that (X, ς) is a **right $\mathcal{G}$-set**, with a **structure map** ς, if there is a **right action** $\rho\colon X\, {}_{\varsigma}\!\times_{\mathsf{t}} \mathcal{G}_1 \longrightarrow X$, sending (x, g) to xg, and satisfying the following conditions:

(1) for each $x \in X$ and $g \in \mathcal{G}_1$ such that $\varsigma(x) = \mathsf{t}(g)$, we have $\mathsf{s}(g) = \varsigma(xg)$;

(2) for each $x \in X$, we have $x\iota_{\varsigma(x)} = x$;

(3) for each $x \in X$ and $g, h \in \mathcal{G}_1$ such that $\varsigma(x) = \mathsf{t}(g)$ and $\mathsf{s}(g) = \mathsf{t}(h)$, we have $(xg)\,h = x\,(gh)$.

In order to simplify the notation, the action map of a given right $\mathcal{G}$-set (X, ς) will be omitted and, by abuse of notation, we will simply refer to a right $\mathcal{G}$-set X without even mentioning the structure map. A left action is analogously defined by interchanging the source with the target map.

Definition 1.1.15. Given a groupoid $\mathcal{G}$, a set X and a map $\varsigma\colon X \longrightarrow \mathcal{G}_0$, we say that (X, ς) is a **left $\mathcal{G}$-set**, with a ς, if there is a **left action** $\rho\colon \mathcal{G}_1\, {}_{\mathsf{s}}\!\times_{\varsigma} X \longrightarrow X$, sending $(g, x) \longrightarrow gx$, and satisfying the following conditions:

(1) for each $x \in X$ and $g \in \mathcal{G}_1$ such that $\varsigma(x) = \mathsf{s}(g)$ we have $\mathsf{t}(g) = \varsigma(gx)$;

(2) for each $x \in X$, we have $\iota_{\varsigma(x)}x = x$;

(3) for each $x \in X$ and $h, g \in \mathcal{G}_1$ such that $\mathsf{s}(g) = \varsigma(x)$ and $\mathsf{s}(h) = \mathsf{t}(g)$ we have $h\,(gx) = (hg)\,x$.

In general a set X with a (right or left) groupoid action is just called a **groupoid-set** but we will also employ the terminology: a **set X with a left (or right) $\mathcal{G}$-action**.

Obviously, any groupoid $\mathcal{G}$ acts over itself on both sides by using the **regular action**, that is, the multiplication $\mathcal{G}_1\, {}_{\mathsf{s}}\!\times_{\mathsf{t}}\, \mathcal{G}_1 \longrightarrow \mathcal{G}_1$. This means that $(\mathcal{G}_1, \mathsf{s})$ is a right $\mathcal{G}$-set and $(\mathcal{G}_1, \mathsf{t})$ is a left $\mathcal{G}$-set with this action. It is also clear that $(\mathcal{G}_0, \mathrm{Id}_{\mathcal{G}_0})$ is a right $\mathcal{G}$-set endowed with the action given by

$$\begin{aligned} \mathcal{G}_0\, {}_{\mathrm{Id}_{\mathcal{G}_0}}\!\times_{\mathsf{t}}\, \mathcal{G}_1 &\longrightarrow \mathcal{G}_0 \\ (a, g) &\longrightarrow ag = \mathsf{s}(g) \end{aligned} \tag{1.1.3}$$

Given a groupoid $\mathcal{G}$, let (X, ς) be a right $\mathcal{G}$-set with action map ρ. Then the pair of sets

$$X \rtimes \mathcal{G} := \Big(X\, {}_{\varsigma}\!\times_{\mathsf{t}} \mathcal{G}_1, X\Big) \tag{1.1.4}$$

is a groupoid with structure maps $\mathsf{s}^{\rtimes} = \rho$, $\mathsf{t}^{\rtimes} = \mathrm{pr}_1$ and $\iota_x^{\rtimes} = (x, \iota_{\varsigma(x)})$ for each $x \in X$. The multiplication and the inverse maps are defined as follows: For each $(x, g), (y, h) \in X\, {}_{\varsigma}\!\times_{\mathsf{t}} \mathcal{G}_1$ such that $\mathsf{t}^{\rtimes}(y, h) = \mathsf{s}^{\rtimes}(x, g)$ the multiplication is given by

$$(x, g)(y, h) = (x, gh).$$

That is, for any pairs of elements in $X_{\varsigma}\times_{\mathsf{t}}\mathcal{G}_1$ as before, we have

$$\varsigma(x)=\mathsf{t}(g), \qquad \mathsf{s}^{\rtimes}(x,g)=\rho(x,g)=xg, \qquad \mathsf{t}^{\rtimes}(x,g)=\mathrm{pr}_1(x,g)=x,$$
$$\varsigma(x) \xleftarrow{\;\;g\;\;} \varsigma(xg)$$

and

$$\varsigma(y)=\mathsf{t}(h), \qquad \mathsf{s}^{\rtimes}(y,h)=\rho(y,h)=yh, \qquad \mathsf{t}^{\rtimes}(y,h)=\mathrm{pr}_1(y,h)=y,$$
$$\varsigma(y) \xleftarrow{\;\;h\;\;} \varsigma(yh).$$

Consequently, the multiplication is explicitly given by

$$y=\mathsf{t}^{\rtimes}(y,h)=\mathsf{s}^{\rtimes}(x,g)=xg,$$
$$\varsigma(x) \xleftarrow{\;\;g\;\;} \varsigma(xg)=\varsigma(y) \xleftarrow{\;\;h\;\;} \varsigma(yh)=\varsigma(xgh), \qquad \varsigma(x) \xleftarrow{\;\;gh\;\;} \varsigma(xgh)$$

and schematically can be presented by

$$x \xleftarrow{\;(x,g)\;} xg=y \xleftarrow{\;(y,h)\;} yh=xgh, \qquad x \xleftarrow{\;(x,gh)\;} xgh.$$

For each $(x,g)\in X_{\varsigma}\times_{\mathsf{t}}\mathcal{G}_1$ the inverse arrow is defined by $(x,g)^{-1}=(xg,g^{-1})$. The groupoid $X\rtimes\mathcal{G}$ is called the **right translation groupoid of X by $\mathcal{G}$**. Furthermore, there is a canonical morphism of groupoids $\sigma\colon X\rtimes\mathcal{G}\longrightarrow\mathcal{G}$, given by $\sigma_0=\varsigma\,\mathrm{pr}_1$ and $\sigma_1=\mathrm{pr}_2$.

Regarding the left version, let (Z,ϑ) be a left $\mathcal{G}$-set with structure map ϑ and action λ. Then

$$\mathcal{G}\ltimes Z := \Big(\mathcal{G}_1\,{}_{\mathsf{s}}\!\times_{\vartheta} Z, Z\Big)$$

is a groupoid with structure maps $\mathsf{s}^{\ltimes}=\lambda$, $\mathsf{t}^{\ltimes}=\mathrm{pr}_2$ and, for each $x\in X$, $\iota_x^{\ltimes}=(x,\iota_{\varsigma(x)})$. Moreover, for each $(g,x),(h,y)\in\mathcal{G}_1\,{}_{\mathsf{s}}\!\times_{\vartheta}X$ such that $\mathsf{t}^{\ltimes}(h,y)=\mathsf{s}^{\ltimes}(g,x)$ we define the multiplication in this way: $(g,x)(h,y)=(hg,x)$. For each $(g,x)\in\mathcal{G}_1\,{}_{\mathsf{s}}\!\times_{\vartheta}Z$ the inverse map is defined by $(g,x)^{-1}=(gx,g^{-1})$. The groupoid $\mathcal{G}\ltimes Z$ is called the **left translation groupoid of Z by $\mathcal{G}$**.

Definition 1.1.16. Given a groupoid $\mathcal{G}$, a **morphism of right $\mathcal{G}$-sets**, also called a **right $\mathcal{G}$-equivariant map**, $F\colon (X,\varsigma)\longrightarrow(Y,\theta)$ is a function $F\colon X\longrightarrow Y$ such that the following diagrams commute:

$$\begin{array}{ccc} X & \xrightarrow{\;F\;} & Y \\ & {}_{\varsigma}\searrow \quad \swarrow_{\theta} & \\ & \mathcal{G}_0 & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X\,{}_{\varsigma}\!\times_{\mathsf{t}}\mathcal{G}_1 & \longrightarrow & X \\ {\scriptstyle F\times\mathrm{Id}_{\mathcal{G}_1}}\downarrow & & \downarrow{\scriptstyle F} \\ Y\,{}_{\theta}\!\times_{\mathsf{t}}\mathcal{G}_1 & \longrightarrow & Y. \end{array}$$

Clearly any such a $\mathcal{G}$-equivariant map induces a morphism of groupoids $\mathsf{F}\colon X\rtimes\mathcal{G}\longrightarrow X'\rtimes\mathcal{G}$. A subset $Y\subseteq X$ of a right $\mathcal{G}$-set (X,ς), is said to be **$\mathcal{G}$-invariant subset** whenever the inclusion $Y\hookrightarrow X$ is a $\mathcal{G}$-equivariant map. Morphisms of left $\mathcal{G}$-sets can be defined in a similar way.

Definition 1.1.17. Given a groupoid $\mathcal{G}$, a **morphism of left $\mathcal{G}$-sets** (or **left $\mathcal{G}$-equivariant map**) $F\colon (X,\varsigma)\longrightarrow(Y,\theta)$ is a function $F\colon X\longrightarrow Y$ such that the following diagrams commute:

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ & {\scriptstyle\varsigma}\searrow \quad \swarrow{\scriptstyle\theta} & \\ & \mathcal{G}_0 & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathcal{G}_{1\ \mathsf{s}}\times_{\varsigma} X & \longrightarrow & X \\ {\scriptstyle \mathrm{Id}_{\mathcal{G}_1}\times F}\downarrow & & \downarrow{\scriptstyle F} \\ \mathcal{G}_{1\ \mathsf{s}}\times_{\theta} Y & \longrightarrow & Y. \end{array}$$

We denote by $\mathsf{Sets}\text{-}\mathcal{G}$ the category of right $\mathcal{G}$-sets and by $\mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}(X, X')$ the set of all $\mathcal{G}$-equivariant maps from (X, ς) to (X', ς'). The category of left $\mathcal{G}$-sets, denoted by $\mathcal{G}\text{-}\mathsf{Sets}$, is analogously defined and it is isomorphic to the category of right $\mathcal{G}$-sets, using the inverse map by switching the source with the target. It is noteworthy to mention that the definition of the category of groupoid-sets, as it has been recalled in Definition 1.1.14, can be rephrased using the core of the category of sets. To our purposes, it is advantageous to work with Definition 1.1.14, rather than this formal definition (see Remark 1.1.18 for further explanations).

Remark 1.1.18. A right $\mathcal{G}$-set can be defined also as a functor $\mathscr{X}: \mathcal{G}^{op} \longrightarrow \mathsf{Sets}$ to the core of the category of sets. Then to any functor of this kind we can attach a set $X = \biguplus_{a \in \mathcal{G}_0} \mathscr{X}(a)$ with the canonical map $\varsigma: X \longrightarrow \mathcal{G}_0$ and action

$$\begin{aligned} \varrho\colon X\,{}_{\varsigma}\!\times_{\mathsf{t}}\, \mathcal{G}_1 &\longrightarrow X \\ (x, g) &\longrightarrow xg := \mathscr{X}(g)(x). \end{aligned}$$

Notice that, if none of the fibers $\mathscr{X}(a)$ is an empty set, then the induced map ς is surjective. However, as the example expounded in subsection 1.1.4 shows, this is not always the case.

Any natural transformation between functors as above leads to a $\mathcal{G}$-equivariant map between the associated right $\mathcal{G}$-sets. This in fact establishes an equivalence of categories between the category of the functors of this form (i.e., functors from $\mathcal{G}^{op}$ to the core category of the category of sets) and the category of right $\mathcal{G}$-sets. The functor, which goes in the opposite direction, associates to any right $\mathcal{G}$-set (X, ς) the functor $\mathscr{X}: \mathcal{G}^{op} \longrightarrow \mathsf{Sets}$ that sends any object $a \in \mathcal{G}_0$ to the fibre $\varsigma^{-1}(\{a\})$, and any arrow g in $\mathcal{G}^{op}$ to the bijective map $\varsigma^{-1}(\{\mathsf{t}(g)\}) \longrightarrow \varsigma^{-1}(\{\mathsf{s}(g)\})$ that sends x to xg.

Formally there should not be a more advantageous choice between these two definitions. Nevertheless, in our opinion, for technical reasons, it is perhaps better to deal with groupoid-sets as given in Definition 1.1.14, instead of the aforementioned functorial approach. Specifically, the latter approach presents an inconvenient, since one is forced to distinguish, in certain "local" proofs, between the cases when the fiber is empty and when it is not. There is no such difficulty using Definition 1.1.14, as we will see in the sequel.

Example 1.1.19. Given a morphism of groupoids $\varphi: \mathcal{H} \longrightarrow \mathcal{G}$, let us consider the triple $(\mathcal{H}_0\,{}_{\varphi_0}\!\times_{\mathsf{t}}\, \mathcal{G}_1, \mathrm{pr}_1, \varsigma)$, where $\varsigma: \mathcal{H}_0\,{}_{\varphi_0}\!\times_{\mathsf{t}}\, \mathcal{G}_1 \longrightarrow \mathcal{G}_0$ sends $(u, g) \mapsto s(g)$ and pr_1 is the first projection. Then the following maps

$$\begin{aligned} \Big(\mathcal{H}_0\,{}_{\varphi_0}\!\times_{\mathsf{t}}\, \mathcal{G}_1\Big)\,{}_{\varsigma}\!\times_{\mathsf{t}}\, \mathcal{G}_1 &\longrightarrow \mathcal{H}_0\,{}_{\varphi_0}\!\times_{\mathsf{t}}\, \mathcal{G}_1 \\ \Big((u, g'), g\Big) &\longmapsto (u, g')\leftharpoondown g := (u, g'g) \\ \mathcal{H}_{1\ \mathsf{s}}\!\times_{\mathrm{pr}_1} \Big(\mathcal{H}_0\,{}_{\varphi_0}\!\times_{\mathsf{t}}\, \mathcal{G}_1\Big) &\longrightarrow \mathcal{H}_0\,{}_{\varphi_0}\!\times_{\mathsf{t}}\, \mathcal{G}_1 \\ \Big(h, (u, g)\Big) &\longmapsto h\rightharpoondown(u, g) := (t(h), \varphi(h)g) \end{aligned} \tag{1.1.5}$$

define structures of right $\mathcal{G}$-set and of left $\mathcal{H}$-set, respectively. Analogously, the maps

$$\begin{aligned} \Big(\mathcal{G}_{1\,\mathsf{s}}\times_{\varphi_0}\mathcal{H}_0\Big)\,{}_{\mathrm{pr}_2}\!\times_{\mathsf{t}}\mathcal{H}_1 &\longrightarrow \mathcal{G}_{1\,\mathsf{s}}\times_{\varphi_0}\mathcal{H}_0 \\ \Big((g,u),h\Big) &\longmapsto (g,u)\leftharpoondown h := (g\varphi(h), s(h)) \\ \mathcal{G}_{1\,\mathsf{s}}\times_{\vartheta}\Big(\mathcal{G}_{1\,\mathsf{s}}\times_{\varphi_0}\mathcal{H}_0\Big) &\longrightarrow \mathcal{G}_{1\,\mathsf{s}}\times_{\varphi_0}\mathcal{H}_0 \\ \Big(g,(g',u)\Big) &\longmapsto g\rightharpoondown(g',u) := (gg',u) \end{aligned} \tag{1.1.6}$$

where $\vartheta\colon \mathcal{G}_{1\,\mathsf{s}}\times_{\varphi_0}\mathcal{H}_0 \longrightarrow \mathcal{G}_0$ sends $(g,u)\mapsto t(g)$, define structures of right $\mathcal{H}$-set and of left $\mathcal{G}$-set on $\mathcal{G}_{1\,\mathsf{s}}\times_{\varphi_0}\mathcal{H}_0$, respectively. This in particular can be applied to any morphism of groupoids of the form

$$\begin{aligned} (X,X) &\longrightarrow (Y\times Y, Y) \\ (x,x') &\longrightarrow \big((f(x),f(x)),\, f(x')\big), \end{aligned}$$

where $f\colon X\longrightarrow Y$ is any map. On the other hand, if f is a G-equivariant map, for a group G acting on both X and Y, then the above construction applies, as well, to the morphism of action groupoids

$$\begin{aligned} (G\times X,X) &\longrightarrow (G\times Y, Y) \\ \big((g,x),x'\big) &\longrightarrow \big((g,f(x)),f(x')\big). \end{aligned}$$

The proofs of the following useful lemmas are immediate.

Lemma 1.1.20. *Given a groupoid $\mathcal{G}$, let (X,ς) be a right $\mathcal{G}$-set with action ρ and let be (X',ς') be a right $\mathcal{G}$-set with action ρ'. Let $F\colon (X,\varsigma)\longrightarrow(X',\varsigma')$ be a $\mathcal{G}$-equivariant map with bijective underlying map. Then $F^{-1}\colon (X',\varsigma')\longrightarrow(X,\varsigma)$ is also $\mathcal{G}$-equivariant.*

Proposition and Definition 1.1.21. *Given a groupoid $\mathcal{G}$, let (X,ς) be a right $\mathcal{G}$-set with action ρ and let be $X'\subseteq X$. We define*

$$\varsigma' = \varsigma|_{X'} : X'\longrightarrow \mathcal{G}_0 \qquad \textit{and} \qquad \rho' = \rho|_{X'{}_{\varsigma'}\times_{\mathsf{t}}\mathcal{G}_1} : X'{}_{\varsigma'}\times_{\mathsf{t}}\mathcal{G}_1 \longrightarrow X$$

*and let's suppose that for each $(a,g)\in X'{}_{\varsigma'}\times_{\mathsf{t}}\mathcal{G}_1$ we have $\rho(a,g)\in X'$. Then (X',ς') is a right $\mathcal{G}$-set with action map ρ' and we say that (X',ς') is a **right $\mathcal{G}$-subset** of (X,ς).*

1.1.3 Orbit sets and stabilizers

Next we recall the notion (see, for instance, [Jel03, page 11]) of the orbit set attached to a right groupoid-set. This notion is a generalization of the orbit set in the context of group-sets. Here we use the (right) translation groupoid to introduce this set.

Given a right $\mathcal{G}$-set (X,ς), the **orbit set** $X/\mathcal{G}$ of (X,ς) is the orbit set of the (right) translation groupoid $X\rtimes\mathcal{G}$, that is, $X/\mathcal{G}=\pi_0(X\rtimes\mathcal{G})$, the set of all connected component. For an element $x\in X$, the equivalence class of x, called the **orbit of** x, is denoted by:

$$[x]\,\mathcal{G} := \mathrm{Orb}_{X\rtimes\mathcal{G}}(x) = \left\{ y\in X \;\middle|\; \begin{array}{l} \exists\,(x,g)\in(X\rtimes\mathcal{G})_1 \text{ such} \\ \text{that } x=\mathsf{t}^{\rtimes}(x,g) \text{ and} \\ y=\mathsf{s}^{\rtimes}(x,g)=xg \end{array} \right\} = \left\{ xg\in X \;\middle|\; \mathsf{t}(g)=\varsigma(x) \right\}.$$

Let us denote by $\mathrm{rep}_{\mathcal{G}}(X)$ a **representative set** of the orbit set $X/\mathcal{G}$. For instance, if $\mathcal{G}=(X\times G,X)$ is an action groupoid as in Example 1.1.10, then obviously the orbit set of this groupoid coincides with the classical set of orbits X/G. Of course, the orbit set of an equivalence relation groupoid $(\mathcal{R}, X)$ (see Example 1.1.11) is precisely the quotient set $X/\mathcal{R}$. The left orbits sets for left groupoids sets are analogously defined by using the left translation groupoids. We will use the following notations: given a left $\mathcal{G}$-set (Z, ϑ), its orbit set will be denoted by $\mathcal{G}\backslash Z$ and the orbit of an element $z \in Z$ by $\mathcal{G}[z]$.

A right $\mathcal{G}$-set is said to be **transitive** if it has a single orbit, that is, if $X/\mathcal{G}$ is a singleton or, equivalently, if its associated right translation groupoid $X \rtimes \mathcal{G}$ is transitive.

Now let (X, ς) be a right $\mathcal{G}$-set with action $\rho\colon X\,{}_{\varsigma}\!\times_{\mathsf{t}} \mathcal{G}_1 \longrightarrow X$. The **right stabilizer** $\mathrm{Stab}_{\mathcal{G}}(x)$ of x in $\mathcal{G}$ is the groupoid with arrows

$$(\mathrm{Stab}_{\mathcal{G}}(x))_1 = \{\, g \in \mathcal{G}_1 \mid \varsigma(x) = \mathsf{t}(g) \quad \text{and} \quad xg = x \,\}$$

and objects

$$(\mathrm{Stab}_{\mathcal{G}}(x))_0 = \{\, u \in \mathcal{G}_0 \mid \exists\, g \in \mathcal{G}_1(\varsigma(x), u) : xg = x \,\} \subseteq \mathscr{O}_{\varsigma(x)}.$$

Therefore, we have that

$$(\mathrm{Stab}_{\mathcal{G}}(x))_0 = \{\, \varsigma(x) \,\}, \qquad \mathrm{Stab}_{\mathcal{G}}(x)^{\varsigma(x))} = (\mathrm{Stab}_{\mathcal{G}}(x))_1 \leqslant \mathcal{G}^{\varsigma(x)}.$$

and, as a groupoid with only one object $\varsigma(x)$, the set of arrow is:

$$(\mathrm{Stab}_{\mathcal{G}}(x))_1 = \left\{ g \in \mathcal{G}_1 \,\middle|\, \mathsf{s}(g) = \mathsf{t}(g) = \varsigma(xg) \text{ and } xg = x \right\}.$$

In other words, the stabilizer of an element $x \in X$ is the subgroup of the isotropy group $\mathcal{G}^{\varsigma(x)}$ consisting of those loops g which satisfy $xg = x$. The following lemma is then an immediate consequence of this observation.

Lemma 1.1.22. *Let (X, ς) be a right $\mathcal{G}$-set and consider its associated morphism of groupoids*

$$\sigma\colon X \rtimes \mathcal{G} \longrightarrow \mathcal{G},$$

given by $\sigma_0 = \varsigma\,\mathrm{pr}_1$ and $\sigma_1 = \mathrm{pr}_2$. Then, for each $x \in X$, the stabilizer $\mathrm{Stab}_{\mathcal{G}}(x)$ is the image by σ of the isotropy group $(X \rtimes \mathcal{G})^x$ and there is an isomorphism of groups:

$$(\mathrm{Stab}_{\mathcal{G}}(x))_1 \cong (X \rtimes \mathcal{G})^x.$$

The **left stabilizer** of an elements of a left groupoid set is similarly defined and enjoys analogue properties, as in Lemma 1.1.22.

1.1.4 A right groupoid-set with a non surjective structure map

We will give an example of a right $\mathcal{G}$-set whose structure map is not surjective, completing by this the observations made in Remark 1.1.18. Given a set S, let be $\varnothing \neq S' \subsetneq S$ and let R be an equivalence relation on S. Assume that there is $x_0 \in S \backslash S'$ such that for every $y \in S'$, $(x_0, y) \notin R$. Let's define $R_{S'} = R \cap (S' \times S')$, that is, the restriction of R to S'. It is clear that $R_{S'}$ is an equivalence relation on S'.

We have the following two groupoids of equivalence relation: $\mathcal{G} = (R, S)$ and $\mathcal{H} = (R_{S'}, S')$ with the inclusion $\tau\colon \mathcal{H} \hookrightarrow \mathcal{G}$ of groupoids. Let us define $X := \mathcal{H}_0\,{}_{\tau}\!\times_{\mathsf{s}} \mathcal{G}_1 = S'\,{}_{\tau}\!\times_{\mathsf{t}} R$. Note that we have to choose $S' \neq \varnothing$ otherwise X would be empty. The set X becomes a right $\mathcal{G}$-set with structure map

$$\begin{aligned} \varsigma\colon X &\longrightarrow \mathcal{G}_0 \\ \big(s', (a,b)\big) &\longrightarrow \mathsf{s}(a,b) = b \end{aligned}$$

and action

$$\begin{aligned} X\,{}_{\varsigma}\!\times_{\mathsf{s}} \mathcal{G}_1 &\longrightarrow X \\ \Big(\big(s', (a,b)\big), \big(b', d\big)\Big) &\longrightarrow \big(s', (a,b)\,(b',d)\big) = \big(s', (a,d)\big), \end{aligned}$$

where

$$b = \varsigma\,\big(s', (a,b)\big) = \mathsf{t}\,\big(b', d\big) = b'.$$

The axioms of right $\mathcal{G}$-set are not difficult to verify.

Now, we want to prove that ς is not surjective. By contradiction, let us assume that ς is surjective. Therefore, there is $(s', (a,b)) \in X = S'\,{}_{\tau}\!\times_{\mathsf{t}} R$ such that $x_0 = \varsigma\,(s', (a,b)) = b$: in particular $a = \mathsf{t}(a,b) = \tau_0\,(s') = s'$, hence $(s', x_0) = (a,b) \in R$, with $s' \in S'$, which contradicts our assumption. As a consequence x_0 doesn't belong to the image of ς in $\mathcal{G}_0$ and ς is not surjective. As a particular implementation of this example, we can choose $S = \mathbb{R}$, $S' = \mathbb{Q}$, $x_0 = \sqrt{2}$ and for every $s_1, s_2 \in S$ we can define $s_1 R s_2$ if and only if $s_1 - s_2 \in \mathbb{Z}$.

1.2 Groupoid-bisets, translation groupoids, orbits and cosets

In this section, we recall from [El 18] (with sufficient details) the notions of groupoid-biset, two sided translation groupoid, coset by a subgroupoid and tensor product of bisets. After that, we will discuss the decomposition of a set, with a groupoid acting over it, into disjoint orbits. Moreover, we will prove the bijective correspondence between groupoid-bisets and left sets over the product of the involved groupoids.

1.2.1 The category of bisets and two sided translation groupoids

The following definitions are abstract formulations of those given in [Jel03] for topological groupoids and in [MM05] for Lie groupoids. In this regard, see also [El 18].

Definition 1.2.1. Given a set X and two groupoids $\mathcal{H}$ and $\mathcal{G}$, let $\vartheta\colon X \longrightarrow \mathcal{H}_0$ and $\varsigma\colon X \longrightarrow \mathcal{G}_0$ be two maps. The triple $(X, \vartheta, \varsigma)$ is said to be an $(\mathcal{H}, \mathcal{G})$-**biset** if there is a left $\mathcal{H}$-action $\lambda\colon \mathcal{H}_1\,{}_{\mathsf{s}}\!\times_{\vartheta} X \longrightarrow X$ and a right $\mathcal{G}$-action $\rho\colon X\,{}_{\varsigma}\!\times_{\mathsf{t}} \mathcal{G}_1 \longrightarrow X$ such that:

(1) for each $x \in X$, $h \in \mathcal{H}_1$ and $g \in \mathcal{G}_1$ such that $\vartheta\,(x) = \mathsf{s}\,(h)$ and $\varsigma\,(x) = \mathsf{t}\,(g)$, we have

$$\vartheta\,(xg) = \vartheta\,(x) \qquad \text{and} \qquad \varsigma\,(hx) = \varsigma\,(x)\,;$$

(2) for each $x \in X$, $h \in \mathcal{H}_1$ and $g \in \mathcal{G}_1$ such that $\vartheta\,(x) = \mathsf{s}\,(h)$ and $\varsigma\,(x) = \mathsf{t}\,(g)$, we have

$$h\,(xg) = (hx)\,g.$$

The triple $(X, \vartheta, \varsigma)$ is referred to as a **groupoid-biset**, whenever the two groupoids $\mathcal{H}$ and $\mathcal{G}$ are clear.

Note that the first condition in the definition of biset is necessary for the second one to have meaning. For simplicity the actions maps of a groupoid-bisets are omitted in the notation.

Definition 1.2.2. Given two groupoids $\mathcal{H}$ and $\mathcal{G}$, let $(X, \vartheta, \varsigma)$ and (Y, α, β) be two $(\mathcal{H}, \mathcal{G})$-bisets. A **morpshism of $(\mathcal{H}, \mathcal{G})$-biset** is a function $f\colon X \longrightarrow Y$ such that $f\,(X, \vartheta) \longrightarrow (Y, \alpha)$ is a morphism of left $\mathcal{H}$-set and $f\colon\ (X, \varsigma) \longrightarrow (Y, \beta)$ is a morphism of right $\mathcal{G}$-sets. The resulting category will be denoted by $\mathcal{H}$-BiSets-$\mathcal{G}$.

The **two sided translation groupoid** associated to a given $(\mathcal{H}, \mathcal{G})$-biset $(X, \vartheta, \varsigma)$ is defined to be the groupoid $\mathcal{H} \ltimes X \rtimes \mathcal{G}$ whose set of objects is X and whose set of arrows is

$$\mathcal{H}_{1\,\mathsf{s}}\times_{\vartheta} X\,{}_{\varsigma}\times_{\mathsf{s}} \mathcal{G}_1 \;=\; \Big\{(h, x, g) \in \mathcal{H}_1 \times X \times \mathcal{G}_1 |\ \mathsf{s}(h) = \vartheta(x),\ \mathsf{s}(g) = \varsigma(x)\Big\}.$$

The structure maps are:

$$\mathsf{s}(h, x, g) = x, \quad \mathsf{t}(h, x, g) = hxg^{-1} \quad \text{and} \quad \iota_x = \big(\iota_{\vartheta(x)}, x, \iota_{\varsigma(x)}\big)\,.$$

The multiplication and the inverse map are given by

$$(h, x, g)(h', x', g') \;=\; \big(hh', x', gg'\big) \qquad \text{and} \qquad (h, x, g)^{-1} = \big(h^{-1}, hxg^{-1}, g^{-1}\big)\,,$$

respectively.

Example 1.2.3. Given two groups G and H and a group-biset U, the category $\langle U\rangle$ defined by Bouc in [Bou10b, Notation 2.1] is the translation groupoid of the $(\mathcal{H}, \mathcal{G})$-biset V, where $\mathcal{H}$, respectively $\mathcal{G}$, is the groupoid with only one object and isotropy group H, respectively G, and V is exactly U considered as a groupoid-biset.

The **orbit space of the two translation groupoid** is the quotient set $X/(\mathcal{H}, \mathcal{G})$ defined by the following equivalence relation: for each $x, x' \in X$, $x \sim x'$ if and only if there exist $h \in \mathcal{H}_1$ and $g \in \mathcal{G}_1$ with $\mathsf{s}(h) = \vartheta(x)$ and $\mathsf{t}(g) = \varsigma(x')$ such that $hx \;=\; x'g$. We will also employ the notation $\mathcal{H}\backslash X/\mathcal{G} := X/(\mathcal{H},\, \mathcal{G})$ and denote by $\mathrm{rep}_{(\mathcal{H},\mathcal{G})}(X)$ one of its **representative sets**.

Example 1.2.4. Let $\varphi\colon \mathcal{H} \longrightarrow \mathcal{G}$ be a morphism of groupoids. Consider, as in Example 1.1.19, the associated triples $(\mathcal{H}_0\,{}_{\varphi_0}\times_{\mathsf{t}} \mathcal{G}_1, \varsigma, \mathrm{pr}_1)$ and $(\mathcal{G}_{1\,\mathsf{s}}\times_{\varphi_0} \mathcal{H}_0, \mathrm{pr}_2, \vartheta)$ with actions defined as in equations (1.1.5) and (1.1.6). Then these triples are an $(\mathcal{H}, \mathcal{G})$-biset and a $(\mathcal{G}, \mathcal{H})$-biset, respectively.

Proposition 1.2.5. *Let $(X, \vartheta, \varsigma)$ be an $(\mathcal{H}, \mathcal{G})$-biset with actions*

$$\lambda\colon \mathcal{H}_{1\,\mathsf{s}}\times_{\vartheta} X \longrightarrow X \qquad \textit{and} \qquad \rho\colon X\,{}_{\varsigma}\times_{\mathsf{t}} \mathcal{G}_1 \longrightarrow X.$$

Then $\mathcal{H}\backslash X$ is a right $\mathcal{G}$-set with structure map and action as follows:

$$\begin{aligned} \widehat{\varsigma}\colon \mathcal{H}\backslash X &\longrightarrow \mathcal{G}_0 \\ \mathcal{H}[x] &\longrightarrow \widehat{\varsigma}\,(\mathcal{H}[x]) = \varsigma\,(x) \end{aligned} \qquad \textit{and} \qquad \begin{aligned} \widehat{\rho}\colon (\mathcal{H}\backslash X)\,{}_{\widehat{\varsigma}}\times_{\mathsf{t}} \mathcal{G}_1 &\longrightarrow \mathcal{H}\backslash X \\ (\mathcal{H}[x], g) &\longrightarrow \widehat{\rho}\,(\mathcal{H}[x], g) = \mathcal{H}[xg] = \mathcal{H}[\rho\,(x, g)]. \end{aligned}$$

Proof. Let be $x_1, x_2 \in X$ such that $\mathcal{H}[x_1] \;=\; \mathcal{H}[x_2]$. Then by definition of orbit there is $h \in \mathcal{H}_1$ such that $x_1 \;=\; hx_2$ and $\vartheta\,(x_2) \;=\; \mathsf{s}\,(h)$. One of the biset conditions says that $\varsigma\,(x_1) = \varsigma\,(hx_2) = \varsigma\,(x_2)$. This shows that $\widehat{\rho}$ is well defined. Now let be $g \in \mathcal{G}_1$, $x_1, x_2 \in X$ such that $\mathcal{H}[x_1] = \mathcal{H}[x_2]$, $\widehat{\varsigma}\,(\mathcal{H}[x_1]) = \mathsf{t}\,(g)$ and $\widehat{\varsigma}\,(\mathcal{H}[x_2]) = \mathsf{t}\,(g)$. We have

$$\varsigma\,(x_1) = \widehat{\varsigma}\,(\mathcal{H}[x_1]) = \mathsf{t}\,(g) = \widehat{\varsigma}\,(\mathcal{H}[x_2]) = \varsigma\,(x_2)$$

and

$$\rho\,(x_1, g) = x_1 g = (hx_2)\, g = h\,(x_2 g) = h\rho\,(x_2, g)$$

so $\mathcal{H}[x_1 g] \;=\; \mathcal{H}[x_2 g]$ which shows that $\widehat{\rho}$ is well defined.

Now we only have to check the axioms of right $\mathcal{G}$-set but this is easy and is left to the reader. As a consequence, we have proved that $\mathcal{H}\backslash X$ is a right $\mathcal{G}$-set as stated. □

The left version of Proposition 1.2.5 also holds true. Precisely, given an $(\mathcal{H},\mathcal{G})$-biset (X,ς,ϑ), since X is obviously a $(\mathcal{G}^{\mathrm{op}},\mathcal{H}^{\mathrm{op}})$-biset, applying Proposition 1.2.5 we obtain that $\mathcal{G}^{\mathrm{op}}\backslash X$ is right $\mathcal{H}^{\mathrm{op}}$-set, that is, $X/\mathcal{G}$ is a left $\mathcal{H}$-set.

1.2.2 Left (right) cosets by subgroupoids and decompositions

Now we will introduce the notion of left and right cosets associated to a morphism of groupoids. Let us assume that a morphism of groupoids $\varphi\colon \mathcal{H} \longrightarrow \mathcal{G}$ is given and denote by

$$^{\varphi}\mathcal{H}(\mathcal{G}) = \mathcal{H}_{0\,\varphi_0}\times_{\mathsf{t}}\mathcal{G}_1$$

the underlying set of the $(\mathcal{H},\mathcal{G})$-biset of Example 1.2.4. Then the left translation groupoid is given by

$$\mathcal{H} \ltimes {}^{\varphi}\mathcal{H}(\mathcal{G}) = \mathcal{H} \ltimes \left(\mathcal{H}_{0\,\varphi_0}\times_{\mathsf{t}}\mathcal{G}_1\right) = \left(\mathcal{H}_{1\,\mathsf{s}}\times_{\mathrm{pr}_1}\left(\mathcal{H}_{0\,\varphi_0}\times_{\mathsf{t}}\mathcal{G}_1\right), \mathcal{H}_{0\,\varphi_0}\times_{\mathsf{t}}\mathcal{G}_1\right)$$

where the source $\mathsf{s}^{\ltimes}$ is the action $\rightharpoonup$ described in equation (1.1.5) and the target $\mathsf{t}^{\ltimes}$ is the second projection pr_2 on X. The multiplication of two arrows (h_1,a_1,g_1) and (h_2,a_2,g_2) of $\mathcal{H} \ltimes {}^{\varphi}\mathcal{H}(\mathcal{G})$ such that $\mathsf{s}^{\ltimes}(h_1,a_1,g_1) = \mathsf{t}^{\ltimes}(h_2,a_2,g_2)$ is given as follows. Since

$$\Big(\mathsf{t}(h_1),\varphi_1(h_1)g_1\Big) = \Big(h_1\rightharpoonup(a_1,g_1)\Big) = \mathsf{s}^{\ltimes}(h_1,a_1,g_1) = \mathsf{t}^{\ltimes}(h_2,a_2,g_2) = (a_2,g_2)$$

and $\mathsf{s}(h_2) = \mathrm{pr}_1(a_2,g_2) = a_2 = \mathsf{t}(h_1)$ we can write h_2h_1. We calculate $\mathsf{t}^{\ltimes}(h_1,a_1,g_1) = (a_1,g_1)$ and

$$\begin{aligned}\mathsf{s}^{\ltimes}(h_2,a_2,g_2) &= \Big(h_2\rightharpoonup(a_2,g_2)\Big) = \Big(\mathsf{t}(h_2),\varphi_1(h_2)g_2\Big) = \Big(\mathsf{t}(h_2h_1),\varphi_1(h_2)\varphi_1(h_1)g_1\Big)\\ &= \Big(\mathsf{t}(h_2h_1),\varphi_1(h_2h_1)g_1\Big) = \Big((h_2h_1)\rightharpoonup(a_1,g_1)\Big),\end{aligned}$$

therefore the situation is as follows

$$(a_1,g_1) \xleftarrow{(h_1,a_1,g_1)} \Big(\mathsf{t}(h_1),\varphi_1(h_1)g_1\Big) \xleftarrow{(h_2,a_2,g_2)} \Big((h_2h_1)\rightharpoonup(a_1,g_1)\Big)$$

and we obtain

$$(h_1,a_1,g_1)(h_2,a_2,g_2) = (h_2h_1,a_1,g_1).$$

Definition 1.2.6. Given a morphism of groupoids $\varphi\colon \mathcal{H} \longrightarrow \mathcal{G}$, we define

$$(\mathcal{G}/\mathcal{H})^{\mathsf{R}}_{\varphi} := \pi_0\Big(\mathcal{H} \ltimes {}^{\varphi}\mathcal{H}(\mathcal{G})\Big) = \mathcal{H}\backslash{}^{\varphi}\mathcal{H}(\mathcal{G})$$

and, for each $(a,g) \in \mathcal{H}_{0\,\varphi_0}\times_{\mathsf{t}}\mathcal{G}_1$, we set

$$^{\varphi}\mathcal{H}[(a,g)] = \left\{ \Big(h\rightharpoonup(a,g)\Big) \in {}^{\varphi}\mathcal{H}(\mathcal{G}) \;\Big|\; h \in \mathcal{H}_1,\, \mathsf{s}(h) = a \right\}.$$

If $\mathcal{H}$ is a subgroupoid of $\mathcal{G}$, that is, if $\varphi := \tau\colon \mathcal{H} \hookrightarrow \mathcal{G}$ is the inclusion functor, we use the notation

$$(\mathcal{G}/\mathcal{H})^{\mathsf{R}} = \mathcal{H}\backslash\mathcal{H}(\mathcal{G}) = \mathcal{H}\backslash{}^{\tau}\mathcal{H}(\mathcal{G}) = \mathcal{H}\backslash\left(\mathcal{H}_{0\,\tau_0}\times_{\mathsf{t}}\mathcal{G}_1\right) = \left\{\, \mathcal{H}[(a,g)] \mid (a,g) \in \mathcal{H}_{0\,\tau_0}\times_{\mathsf{t}}\mathcal{G}_1, \,\right\} \tag{1.2.1}$$

and, for each $(a,g) \in \mathcal{H}_{0\,\tau_0}\times_{\mathsf{t}}\mathcal{G}_1$,

$$\mathcal{H}[(a,g)] = {}^{\tau}\mathcal{H}[(a,g)] = \left\{ \Big(h\rightharpoonup(a,g)\Big) \in \mathcal{H}(\mathcal{G}) \;\Big|\; h \in \mathcal{H}_1,\, \mathsf{s}(h) = a \right\}. \tag{1.2.2}$$

Moreover, $(\mathcal{G}/\mathcal{H})^{\mathsf{R}}$ is called the **set of right cosets of $\mathcal{G}$ by $\mathcal{H}$**.

For each arrow (h, a, g) of $\mathcal{H} \ltimes {}^{\tau}\mathcal{H}(\mathcal{G})$ we have

$$\Big(h \rightharpoonup (a, g)\Big) = (t(h), \tau_1(h)\, g) = (t(h), hg) \qquad \text{and} \qquad \mathsf{s}(h) = a = \tau_0(a) = \mathsf{t}(g)$$

thus, for each $(a, g) \in \mathcal{H}_0\, {}_{\tau_0}\!\times_{\mathsf{t}} \mathcal{G}_1$, we obtain

$$\mathcal{H}[(a, g)] = \Big\{ (\mathsf{t}(h), hg) \in \mathcal{H}(\mathcal{G}) \Big| h \in \mathcal{H}_1,\ \mathsf{s}(h) = \mathsf{t}(g) \Big\}.$$

Keeping the notation of the previous definition we can state:

Lemma 1.2.7. *Given $(a_1, g_1), (a_2, g_2) \in \mathcal{H}_{0\,\tau_0}\!\times_{\mathsf{t}} \mathcal{G}_1$, we have $\mathcal{H}[(a_1, g_1)] = \mathcal{H}[(a_2, g_2)]$ if and only if there is $h \in \mathcal{H}(a_2, a_1)$ such that $h = g_1 g_2^{-1}$.*

Proof. We have $\mathcal{H}[(a_1, g_1)] = \mathcal{H}[(a_2, g_2)]$ if and only if $(a_1, g_1) \in \mathcal{H}[(a_2, g_2)]$, if and only if there is $h \in \mathcal{H}_1$ such that $\mathsf{s}(h) = \mathsf{t}(g_2)$ and $(a_1, g_1) = (\mathsf{t}(h), hg_2)$, if and only if there is $h \in \mathcal{H}_1$ such that

$$\begin{cases} \mathsf{t}(g_1) = a_1 = \mathsf{t}(h) \\ g_1 = hg_2, \end{cases}$$

if and only if there is $h \in \mathcal{H}_1$ such that $\mathsf{s}(h) = \mathsf{t}(g_2) = a_2$, $\mathsf{t}(h) = \mathsf{t}(g_1) = a_1$, $h = g_1 g_2^{-1}$, if and only if there is $h \in \mathcal{H}(a_1, a_2)$ such that $h = g_1 g_2^{-1}$. □

The set of left coset of $\mathcal{G}$ by $\mathcal{H}$ is defined using the $(\mathcal{G}, \mathcal{H})$-biset

$$\mathcal{H}(\mathcal{G})^{\tau} := \mathcal{G}_1\, {}_{\mathsf{s}}\!\times_{\tau_0} \mathcal{H}_0$$

of Example 1.2.4 with action maps as in equation (1.1.6). If $\tau \colon \mathcal{H} \longrightarrow \mathcal{G}$ is the inclusion functor, then we use the notation

$$(\mathcal{G}/\mathcal{H})^{\mathsf{L}} = \mathcal{H}(\mathcal{G})^{\tau}/\mathcal{H} = \Big\{ [(g, u)]\mathcal{H} \Big|\ (g, u) \in \mathcal{H}(\mathcal{G})^{\tau} \Big\} \tag{1.2.3}$$

and, for each $(g, u) \in \mathcal{H}(\mathcal{G})^{\tau}$,

$$[(g, u)]\mathcal{H} = \Big\{ \Big((g, u) \leftharpoonup h\Big) = (gh, \mathsf{s}(h)) \in \mathcal{H}(\mathcal{G})^{\tau} \Big|\ h \in \mathcal{H}_1, \mathsf{s}(g) = u = \mathsf{t}(h) \Big\}.$$

Moreover, $(\mathcal{G}/\mathcal{H})^{\mathsf{L}}$ is called the **set of left cosets of $\mathcal{G}$ by $\mathcal{H}$**. The following is an analogue of Lemma 1.2.7.

Lemma 1.2.8. *If $\tau \colon \mathcal{H} \longrightarrow \mathcal{G}$ is an inclusion functor, then for each $(g_1, u_1), (g_2, u_2) \in \mathcal{G}_{1\mathsf{s}}\!\times_{\tau_0} \mathcal{H}_0$ we have $[(g_1, u_1)]\mathcal{H} = [(g_2, u_2)]\mathcal{H}$ if and only if $g_2^{-1} g_1 \in \mathcal{H}_1$.*

Proof. Is similar to that of Lemma 1.2.7. □

As a corollary of Proposition 1.2.5, we obtain:

Corollary 1.2.9. *Let $\mathcal{H}$ be a subgroupoid of $\mathcal{G}$ via the inclusion functor $\tau \colon \mathcal{H} \longrightarrow \mathcal{G}$. Then $(\mathcal{G}/\mathcal{H})^{\mathsf{R}}$ becomes a right $\mathcal{G}$-set with structure map and action given as follows:*

$$\begin{aligned} \varsigma \colon (\mathcal{G}/\mathcal{H})^{\mathsf{R}} &\longrightarrow \mathcal{G}_0 \\ \mathcal{H}[(a, g)] &\longrightarrow \mathsf{s}(g) \end{aligned} \qquad \textit{and} \qquad \begin{aligned} \rho \colon (\mathcal{G}/\mathcal{H})^{\mathsf{R}}\, {}_{\varsigma}\!\times_{\mathsf{t}} \mathcal{G}_1 &\longrightarrow (\mathcal{G}/\mathcal{H})^{\mathsf{R}} \\ (\mathcal{H}[(a, g_1)], g_2) &\longrightarrow \mathcal{H}[(a, g_1 g_2)]. \end{aligned} \tag{1.2.4}$$

In a similar way the left coset $(\mathcal{G}/\mathcal{H})^{\mathsf{L}}$ becomes a left $\mathcal{G}$-set with a structure and action maps given by:

$$\begin{aligned} \vartheta\colon (\mathcal{G}/\mathcal{H})^{\mathsf{L}} &\longrightarrow \mathcal{G}_0 \\ [(g,u)]\mathcal{H} &\longrightarrow \mathsf{t}(g) \end{aligned} \quad\text{and}\quad \begin{aligned} \lambda\colon \mathcal{G}_{1\,\mathsf{s}}\times_{\vartheta} (\mathcal{G}/\mathcal{H})^{\mathsf{L}} &\longrightarrow (\mathcal{G}/\mathcal{H})^{\mathsf{L}} \\ \big(g_1,[(g_2,u)]\mathcal{H}\big) &\longrightarrow [(g_1g_2,u)]\mathcal{H}. \end{aligned} \tag{1.2.5}$$

The following crucial proposition characterizes, as in the classical case of groups, right orbits of an element (i.e. right transitive $\mathcal{G}$-sets) using the stabilizer subgroupoid of that element.

Proposition 1.2.10. *Let (X,ς) be a right $\mathcal{G}$-set with action map $\rho\colon X\,{}_{\varsigma}\!\times_{\mathsf{t}}\mathcal{G}_1 \longrightarrow X$. Given $x\in X$, let us consider $\mathcal{H}=\mathrm{Stab}_{\mathcal{G}}(x)$ its stabilizer as a subgroupoid of $\mathcal{G}$ (see Subsection 1.1.3). Then the following map*

$$\begin{aligned} \varphi\colon\ (\mathcal{G}/\mathcal{H})^{\mathsf{R}} &\longrightarrow [x]\mathcal{G} \\ \mathcal{H}[(a,g)] &\longmapsto xg \end{aligned}$$

establishes an isomorphism of right $\mathcal{G}$-sets. Likewise, a similar statement is true for left cosets. That is, for a given left $\mathcal{G}$-set (Y,ϑ) with action λ, we have, for every $y\in Y$, an isomorphism of left $\mathcal{G}$-set

$$\begin{aligned} \psi\colon\ (\mathcal{G}/\mathcal{H})^{\mathsf{L}} &\longrightarrow \mathcal{G}[y] \\ [(g,u)]\mathcal{H} &\longmapsto gy, \end{aligned}$$

where $\mathcal{H}=\mathrm{Stab}_{\mathcal{G}}(y)$ is the stabilizer of y.

Proof. We only show the right side of the statement. Given $\mathcal{H}[(a,g)]\in(\mathcal{G}/\mathcal{H})^{\mathsf{R}}$, we have $a\in\mathcal{H}_0=\{\,\varsigma(x)\,\}$ and, since $(a,g)\in\mathcal{H}_0\,{}_{\tau_0}\!\times_{\mathsf{t}}\mathcal{G}_1$, where $\tau\colon\mathcal{H}\longrightarrow\mathcal{G}$ is the inclusion functor, we have $\mathsf{t}(g)=a=\varsigma(x)$ and we can write xg. Now consider $(a_1,g_1),(a_2,g_2)\in\mathcal{H}_0\,{}_{\tau_0}\!\times_{\mathsf{t}}\mathcal{G}_1$ (that is, $a_1=\mathsf{t}(g_1)$ and $a_2=\mathsf{t}(g_2)$) such that $\mathcal{H}[(a_1,g_1)]=\mathcal{H}[(a_2,g_2)]$. Then, by Lemma 1.2.7, there exists $h\in\mathcal{H}_1$ such that $\mathsf{s}(h)=a_2$, $\mathsf{t}(h)=a_1$ and $h=g_1g_2^{-1}$. Since $\mathcal{H}=\mathrm{Stab}_{\mathcal{G}}(x)$ we have $a_1=a_2=\varsigma(x)$ and $xh=x$ so $x=xh=xg_1g_2^{-1}$, whence $xg_2=xg_1$. This shows that φ is well defined. Now let be

$$\mathcal{H}[(a_1,g_1)],\quad \mathcal{H}[(a_2,g_2)]\in(\mathcal{G}/\mathcal{H})^{\mathsf{R}}$$

such that $\varphi\big(\mathcal{H}[(a_1,g_1)]\big)=\varphi\big(\mathcal{H}[(a_2,g_2)]\big)$. Then we have

$$xg_1=\varphi\big(\mathcal{H}[(a_1,g_1)]\big)=\varphi\big(\mathcal{H}[(a_2,g_2)]\big)=xg_2,$$

so $xg_1g_2^{-1}=x$, which means that $g_1g_2^{-1}\in\mathrm{Stab}_{\mathcal{G}}(x)=\mathcal{H}$. Therefore $\mathcal{H}[(a_1,g_1)]=\mathcal{H}[(a_2,g_2)]$ and φ is injective. Now consider an element $xg\in[x]\mathcal{G}$: by definition we have $\varphi\left(\mathcal{H}[(\varsigma(x),g)]\right)=xg$, therefore φ is a surjective map. As a consequence, φ is bijective.

By Corollary 1.2.9 and Proposition 1.1.21 it follows that $(\mathcal{G}/\mathcal{H})^{\mathsf{R}}$ and $[x]\mathcal{G}$ are right $\mathcal{G}$-sets. We denote by ς_x and ρ_x the structure and the action maps of $[x]\mathcal{G}$, respectively. To prove that φ is a morphism of right $\mathcal{G}$-set we have to prove that the following two diagrams are commutative:

$$\begin{array}{ccc} (\mathcal{G}/\mathcal{H})^{\mathsf{R}}\,{}_{\widehat{\varsigma}}\!\times_{\mathsf{t}}\mathcal{G}_1 & \xrightarrow{\ \widehat{\rho}\ } & (\mathcal{G}/\mathcal{H})^{\mathsf{R}} \\ {\scriptstyle \varphi\times\mathrm{Id}_{\mathcal{G}_1}}\downarrow & & \downarrow{\scriptstyle\varphi} \\ ([x]\mathcal{G})\,{}_{\varsigma_x}\!\times_{\mathsf{t}}\mathcal{G}_1 & \xrightarrow{\ \rho_x\ } & [x]\mathcal{G} \end{array} \quad\text{and}\quad \begin{array}{ccc} (\mathcal{G}/\mathcal{H})^{\mathsf{R}} & \xrightarrow{\ \widehat{\varsigma}\ } & \mathcal{G}_0 \\ {\scriptstyle\varphi}\downarrow & \nearrow{\scriptstyle\varsigma_x} & \\ [x]\mathcal{G}, & & \end{array}$$

where we used the notation $\widehat{\varsigma}$ and $\widehat{\rho}$ for the structural and the action maps given in (1.2.4). Let us check the commutativity of the triangle. So take $\mathcal{H}[(a,g)] \in (\mathcal{G}/\mathcal{H})^{\mathsf{R}}$, using the definition of right action, we have

$$\varsigma_x \varphi\left(\mathcal{H}[(a,g)]\right) = \varsigma_x\left(xg\right) = \varsigma\left(xg\right) = \mathsf{s}\left(g\right) = \widehat{\varsigma}\left(\mathcal{H}[(a,g)]\right)$$

since $\mathsf{t}(g) = a = \varsigma(x)$. As for the rectangle, take an arbitrary element $\Big(\mathcal{H}[(a,g)], g_1\Big) \in (\mathcal{G}/\mathcal{H})^{\mathsf{R}}\, {}_{\widehat{\varsigma}}\times_{\mathsf{t}} \mathcal{G}_1$, we have

$$\rho_x\left(\varphi \times \mathrm{Id}_{\mathcal{G}_1}\right)\Big(\mathcal{H}[(a,g)], g_1\Big) = \rho_x\left(\varphi\left(\mathcal{H}[(a,g)]\right), g_1\right)$$
$$= \rho_x\left(xg, g_1\right) = \rho\left(xg, g_1\right) = \left(xg\right)g_1 = x\left(gg_1\right) = \varphi\left(\mathcal{H}[(a, gg_1)]\right) = \varphi\widehat{\rho}\big(\mathcal{H}[(a,g)], g_1\big).$$

Therefore, φ is compatible with the action and by using Lemma 1.1.20, we conclude that φ is an isomorphism of right $\mathcal{G}$-set as desired. ☐

Corollary 1.2.11. *Let (X, ς) be a right $\mathcal{G}$-set. Then there is an isomorphism of right $\mathcal{G}$-sets:*

$$X \cong \biguplus_{x \in \mathrm{rep}_{\mathcal{G}}(X)} \left(\mathcal{G}/\mathrm{Stab}_{\mathcal{G}}\left(x\right)\right)^{\mathsf{R}}.$$

where the right hand side is the coproduct in the category of right $\mathcal{G}$-sets and with $\mathrm{rep}_{\mathcal{G}}(X)$ we indicate a set of representatives of the orbits of the right $\mathcal{G}$-set X.

Proof. Immediate from Proposition 1.2.10, considering the fact that a right $\mathcal{G}$-set is a disjoint union of its orbits. ☐

1.2.3 Groupoid-bisets versus (left) groupoid-sets

In this subsection we give the complete proof of the fact that the category of groupoid $(\mathcal{H}, \mathcal{G})$-bisets is isomorphic to the category of left groupoid $(\mathcal{H} \times \mathcal{G}^{\mathrm{op}})$-sets (equivalently right $(\mathcal{H}^{\mathrm{op}} \times \mathcal{G})$-sets). Here the groupoid structure of the (cartesian) product of two groupoids is the one given by the product of the underlying categories as described in Example 1.1.9.

Proposition 1.2.12. *Given a set X and two groupoids $\mathcal{H}$ and $\mathcal{G}$, there is a bijective correspondence between structures of $(\mathcal{H}, \mathcal{G})$-bisets on X and structures of left $(\mathcal{H} \times \mathcal{G}^{\mathrm{op}})$-sets on X.*

Proof. Let X be an $(\mathcal{H}, \mathcal{G})$-biset with actions and structures map as follows:

$$\vartheta\colon X \longrightarrow \mathcal{H}_0 \qquad \text{and} \qquad \varsigma\colon X \longrightarrow \mathcal{G}_0$$
$$\lambda\colon \mathcal{H}_1\, {}_{\mathsf{s}}\times_{\vartheta} X \longrightarrow X \qquad\qquad \rho\colon X\, {}_{\varsigma}\times_{\mathsf{t}} \mathcal{G}_1 \longrightarrow X.$$

We define the structure map and action as follows

$$\alpha\colon X \longrightarrow \left(\mathcal{H} \times \mathcal{G}^{\mathrm{op}}\right)_0 \qquad \text{and} \qquad \beta\colon \left(\mathcal{H} \times \mathcal{G}^{\mathrm{op}}\right)_1\, {}_{\mathsf{s}}\times_{\alpha} X \longrightarrow X$$
$$x \longmapsto \left(\vartheta\left(x\right), \varsigma\left(x\right)\right) \qquad\qquad \left((h,g), x\right) \longmapsto h\left(xg\right).$$

Now the verification that (X, α) is a left $(\mathcal{H} \times \mathcal{G}^{\mathrm{op}})$-set is obvious.

Conversely, let X be an $(\mathcal{H} \times \mathcal{G}^{\mathrm{op}})$-left set with

$$\alpha\colon X \longrightarrow \left(\mathcal{H} \times \mathcal{G}^{\mathrm{op}}\right)_0 \qquad \text{and} \qquad \beta\colon \left(\mathcal{H} \times \mathcal{G}^{\mathrm{op}}\right)_1\, {}_{\mathsf{s}}\times_{\alpha} X \longrightarrow X$$

as structure map and action. Let be pr_1 and pr_2 the canonical projections

$$\begin{aligned}\mathrm{pr}_1\colon (\mathcal{H}\times\mathcal{G}^{\mathrm{op}})_0 &\longrightarrow \mathcal{H}_0\\ (h,g) &\longmapsto h\end{aligned} \qquad\text{and}\qquad \begin{aligned}\mathrm{pr}_2\colon (\mathcal{H}\times\mathcal{G}^{\mathrm{op}})_0 &\longrightarrow \mathcal{G}_0\\ (h,g) &\longmapsto g.\end{aligned}$$

We define a structure of left $\mathcal{H}$-set as follows

$$\begin{aligned}\vartheta\colon X &\longrightarrow \mathcal{H}_0\\ x &\longmapsto \mathrm{pr}_1\,\alpha(x)\end{aligned} \qquad\text{and}\qquad \begin{aligned}\lambda\colon \mathcal{H}_{1\,\mathsf{s}}\!\times_{\vartheta} X &\longrightarrow X\\ (x,g) &\longmapsto \beta\left((h,\iota_{\varsigma(x)}),x\right)\end{aligned}$$

and a structure of right $\mathcal{G}$-set as follows:

$$\begin{aligned}\varsigma\colon X &\longrightarrow \mathcal{G}_0\\ x &\longmapsto \mathrm{pr}_2\,\alpha(x)\end{aligned} \qquad\text{and}\qquad \begin{aligned}\rho\colon X_{\varsigma}\!\times_{\mathsf{t}} \mathcal{G}_1 &\longrightarrow X\\ (x,g) &\longmapsto \beta\left((\iota_{\vartheta(x)},g),x\right).\end{aligned}$$

For each $(h,x)\in \mathcal{H}_{1\mathsf{s}}\!\times_{\vartheta} X$ we have

$$\alpha(x) = (\vartheta(x),\varsigma(x)) = \left(\mathsf{s}(h),\mathsf{t}\left(\iota_{\varsigma(x)}\right)\right) = \mathsf{s}\left(h,\iota_{\varsigma(x)}\right)$$

so λ is well defined. The verification that (X,ϑ) is a left $\mathcal{H}$-set is obvious. As for the right action, for each $(x,g)\in X_{\varsigma}\!\times_{\mathsf{t}}\mathcal{G}_1$, we have

$$\alpha(x) = (\vartheta(x),\varsigma(x)) = \left(\mathsf{s}\left(\iota_{\vartheta(x)}\right),\mathsf{t}(g)\right) = \mathsf{s}\left(\iota_{\vartheta(x)},g\right)$$

so ρ is well defined. The verification that (X,ς) is a right $\mathcal{G}$-set is also obvious. Now we only have to verify the properties of a biset but this is easy and it is left to the reader. As a consequence, X is an $(\mathcal{H},\mathcal{G})$-biset. Lastly, it is clear that these two constructions are mutually inverse and this completes the proof. □

A similar proof to that of Proposition 1.2.12, works to show that there is a bijective correspondence between right $(\mathcal{H}^{\mathrm{op}}\times\mathcal{G})$-set structures and $(\mathcal{H},\mathcal{G})$-set structures. Furthermore, any $(\mathcal{H},\mathcal{G})$-equivariant map (i.e., any morphism of $(\mathcal{H},\mathcal{G})$-bisets) is canonically transformed, under this correspondence, to a left $(\mathcal{H}\times\mathcal{G}^{\mathrm{op}})$-equivariant map. This allows us to state the following corollary.

Corollary 1.2.13. *Let $\mathcal{H}$ and $\mathcal{G}$ be two groupoids. Then there are canonical isomorphisms of categories between the category of $(\mathcal{H},\mathcal{G})$-bisets, the category of left $(\mathcal{H}\times\mathcal{G}^{op})$-sets and the category of right $(\mathcal{H}^{op}\times\mathcal{G})$-sets.*

1.2.4 Orbits and stabilizers of bisets and double cosets

We will use the notations of the proof of Proposition 1.2.12. Let X be an $(\mathcal{H},\mathcal{G})$-biset and let's consider $x\in X$. Thanks to Proposition 1.2.12, X becomes an $(\mathcal{H}\times\mathcal{G}^{\mathrm{op}})$-left set, therefore we can define the **bilateral stabilizer** $\mathrm{Stab}_{(\mathcal{H},\mathcal{G})}(x)$ of x in $(\mathcal{H},\mathcal{G})$ as the left stabilizer of x in $\mathcal{H}\times\mathcal{G}^{\mathrm{op}}$. As a result we have

$$\left(\mathrm{Stab}_{(\mathcal{H},\mathcal{G})}(x)\right)_0 = \left(\mathrm{Stab}_{(\mathcal{H}\times\mathcal{G}^{\mathrm{op}})}(x)\right)_0 = \{\,(\vartheta(x),\varsigma(x))\,\}$$

and

$$\begin{aligned}\left(\mathrm{Stab}_{(\mathcal{H},\mathcal{G})}(x)\right)_1 = \left(\mathrm{Stab}_{(\mathcal{H}\times\mathcal{G}^{\mathrm{op}})}(x)\right)_1 &= \left\{ (h,g)\in\mathcal{H}_1\times\mathcal{G}_1^{\mathrm{op}} \;\middle|\; \begin{array}{c}\mathsf{s}(h,g)=\mathsf{t}(h,g)=\alpha(x)\\ (h,g)\,x=x\end{array}\right\}\\ &= \left\{ (h,g)\in\mathcal{H}_1\times\mathcal{G}_1^{\mathrm{op}} \;\middle|\; \begin{array}{c}\mathsf{s}(h)=\mathsf{t}(h)=\vartheta(x)\\ \mathsf{s}(g)=\mathsf{t}(g)=\varsigma(x)\\ hxg=x\end{array}\right\}.\end{aligned}$$

Similarly, we can define the **orbit of** x, with respect to the $(\mathcal{H}, \mathcal{G})$-biset X, as the orbit set of x with respect to the left $(\mathcal{H} \times \mathcal{G}^{\mathrm{op}})$-set X and we denote it by $\mathrm{Orb}_{(\mathcal{H},\mathcal{G})}(x)$ or $\mathcal{H}[x]\mathcal{G}$. Consequently, we have

$$\mathrm{Orb}_{(\mathcal{H},\mathcal{G})}(x) = \mathcal{H}[x]\mathcal{G} = \left\{ hxg = (h, g)\, x \in X \,\middle|\, \mathsf{s}(h) = \vartheta(x),\ \varsigma(x) = \mathsf{t}(g) \right\}.$$

The following proposition is an useful precursor of Proposition 3.1.7.

Proposition 1.2.14. *Given a groupoid $\mathcal{H}$, let $\mathcal{A}$ and $\mathcal{B}$ be subgroupoid of $\mathcal{H}$. We define*

$$X = \mathcal{A}_0 \,{}_{}\!\times_{\mathsf{t}} \mathcal{H}_{\mathsf{s}}\!\times \mathcal{B}_0 = \left\{ (a, h, b) \in \mathcal{A}_0 \times \mathcal{H}_1 \times \mathcal{B}_0 \,\middle|\, a = \mathsf{t}(h), \mathsf{s}(h) = b \right\}.$$

Then X is an $(\mathcal{A}, \mathcal{B})$-biset with structure maps

$$\begin{aligned} \vartheta\colon X &\longrightarrow \mathcal{A}_0 \\ (a,h,b) &\longmapsto a \end{aligned} \qquad \textit{and} \qquad \begin{aligned} \varsigma\colon X &\longrightarrow \mathcal{B}_0 \\ (a,h,b) &\longmapsto b \end{aligned}$$

and action maps

$$\begin{aligned} \lambda\colon \mathcal{A}_{1\,\mathsf{s}}\!\times_{\vartheta} X &\longrightarrow X \\ (r,(a,h,b)) &\longmapsto (\mathsf{t}(r), rh, b) \end{aligned} \qquad \textit{and} \qquad \begin{aligned} \rho\colon X_{\varsigma}\!\times_{\mathsf{t}} \mathcal{B}_1 &\longrightarrow X \\ ((a,h,b), q) &\longmapsto (a, hq, \mathsf{s}(q)). \end{aligned}$$

Proof. We have to check the properties of a groupoid right action.

(1) For each $(a, h, b) \in X$ and $q \in \mathcal{B}_1$ such that $\varsigma(a, h, b) = \mathsf{t}(q)$ we have

$$\varsigma((a, h, b)\, d) = \varsigma(a, hq, \mathsf{s}(q)) = \mathsf{s}(q).$$

(2) For each $(a, h, b) \in X$ we have

$$(a, h, b)\, \iota_{\varsigma(a,h,b)} = (a, h, b)\, \iota_b = (a, h\iota_b, \mathsf{s}(\iota_b)) = (a, h, b).$$

(3) For each $(a, h, b) \in X$ and $q, q' \in \mathcal{B}_1$ such that $\varsigma(a, h, b) = \mathsf{t}(q)$ and $\mathsf{s}(q) = \mathsf{t}(q')$ we have

$$((a, h, b)\, q)\, q' = (a, hq, \mathsf{s}(q))\, q' = \left(a, hqq', \mathsf{s}(q')\right) = \left(a, hqq', \mathsf{s}(qq')\right) = (a, h, b)\left(qq'\right).$$

The properties of the left action are similarly proved. Now, we have to check the compatibility conditions of a biset. For each $(a, h, b) \in X$, $r \in \mathcal{A}_1$ and $q \in \mathcal{B}_1$ such that $\vartheta(a, h, b) = \mathsf{s}(a)$ and $\varsigma(a, h, b) = \mathsf{t}(q)$ we have

$$\begin{aligned} \vartheta((a, h, b)\, q) = \vartheta(a, hq, \mathsf{s}(q)) = a = \vartheta(a, h, b), \\ \varsigma(r\,(a, h, b)) = \varsigma(\mathsf{t}(r), rh, b) = b = \varsigma(a, h, b) \end{aligned}$$

and

$$r\,((a, h, b)\, q) = r\,(a, hq, \mathsf{s}(q)) = (\mathsf{t}(r), rhq, \mathsf{s}(q)) = (\mathsf{t}(r), rh, b)\, q = (r\,(a, h, b))\, q,$$

and this finishes the proof. □

Proposition 1.2.14 allows us to define the orbit set $\mathcal{A}\backslash X/\mathcal{B}$ as the **set of double cosets of $\mathcal{H}$ by $\mathcal{A}$ and $\mathcal{B}$**, thus a double coset of $\mathcal{H}$ by $\mathcal{A}$ and $\mathcal{B}$ will be in the form $\mathcal{A}(a, h, b)\mathcal{B}$ with $(a, h, b) \in X$.

1.2.5 The tensor product of groupoid-bisets

Now we are going to recall, from [MM05, page 161], [DG70, Chap. III, §4, 3.1] and [Gir71, Définition 1.3.1, page 114] the definition of the tensor product of two groupoid-bisets and we are going to show its universal property.

Given groupoids $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{K}$, let $(X, \vartheta, \varsigma)$ be a $(\mathcal{H}, \mathcal{G})$-biset and let (Y, χ, ϱ) be a $(\mathcal{G}, \mathcal{K})$-biset. Considering the triple $\big(X\, {}_{\varsigma}\!\times_{\mathsf{t}} \mathcal{G}_1\, {}_{\mathsf{s}}\!\times_{\chi} Y, \overline{\vartheta}, \overline{\varrho}\big)$, where

$$\begin{aligned} \overline{\vartheta}\colon X\, {}_{\varsigma}\!\times_{\mathsf{t}} \mathcal{G}_1\, {}_{\mathsf{s}}\!\times_{\chi} Y &\longrightarrow \mathcal{H}_0 \\ (x,g,y) &\longrightarrow \vartheta(x) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \overline{\varrho}\colon X\, {}_{\varsigma}\!\times_{\mathsf{t}} \mathcal{G}_1\, {}_{\mathsf{s}}\!\times_{\chi} Y &\longrightarrow \mathcal{K}_0 \\ (x,g,y) &\longrightarrow \varrho(y), \end{aligned}$$

we have that $\big(X\, {}_{\varsigma}\!\times_{\mathsf{t}} \mathcal{G}_1\, {}_{\mathsf{s}}\!\times_{\chi} Y, \overline{\vartheta}, \overline{\varrho}\big)$ is an $(\mathcal{H}, \mathcal{K})$-biset with actions

$$\begin{aligned} \Big(X\, {}_{\varsigma}\!\times_{\mathsf{t}} \mathcal{G}_1\, {}_{\mathsf{s}}\!\times_{\chi} Y\Big)\, {}_{\overline{\varrho}}\!\times_{\mathsf{t}} \mathcal{K}_1 &\longrightarrow X\, {}_{\varsigma}\!\times_{\mathsf{t}} \mathcal{G}_1\, {}_{\mathsf{s}}\!\times_{\chi} Y \\ \big((x,g,y),k\big) &\longrightarrow (x,g,yk) \end{aligned}$$

and

$$\begin{aligned} \mathcal{H}_1\, {}_{\mathsf{s}}\!\times_{\overline{\vartheta}} \Big(X\, {}_{\varsigma}\!\times_{\mathsf{t}} \mathcal{G}_1\, {}_{\mathsf{s}}\!\times_{\chi} Y\Big) &\longrightarrow X\, {}_{\varsigma}\!\times_{\mathsf{t}} \mathcal{G}_1\, {}_{\mathsf{s}}\!\times_{\chi} Y \\ \big(h,(x,g,y)\big) &\longrightarrow (hx,g,y). \end{aligned}$$

Then we can define a right $\mathcal{G}$-set $(X\, {}_{\varsigma}\!\times_{\chi} Y, \omega)$ with structure map and action

$$\begin{aligned} \omega\colon X\, {}_{\varsigma}\!\times_{\chi} Y &\longrightarrow \mathcal{G}_0 \\ (x,y) &\longrightarrow \varsigma(x) = \chi(y) \end{aligned} \qquad \text{and} \qquad \begin{aligned} (X\, {}_{\varsigma}\!\times_{\chi} Y)\, {}_{\omega}\!\times_{\mathsf{t}} \mathcal{G}_1 &\longrightarrow (X\, {}_{\varsigma}\!\times_{\chi} Y) \\ ((x,y),g) &\longrightarrow \big(xg, g^{-1}y\big)\,, \end{aligned}$$

respectively.

Following the notation and the terminology of [El 18, Remark 2.12], we use the notation

$$X \otimes_{\mathcal{G}} Y := (X\, {}_{\varsigma}\!\times_{\chi} Y)\,/\mathcal{G}$$

to denote the orbit set of the right $\mathcal{G}$-set $(X\, {}_{\varsigma}\!\times_{\chi} Y, \omega)$ and we refer to it as the **tensor product over $\mathcal{G}$ of X and Y**. We also use the notation $x \otimes_{\mathcal{G}} y$ to denote the equivalence class of the element $(x,y) \in X\, {}_{\varsigma}\!\times_{\chi} Y$, therefore for every $g \in \mathcal{G}_1$ such that $\varsigma(x) = \chi(y) = \mathsf{t}(g)$ we have $xg \otimes_{\mathcal{G}} y = x \otimes_{\mathcal{G}} gy$. Moreover, $X \otimes_{\mathcal{G}} Y$ admits a structure of $(\mathcal{H}, \mathcal{K})$-biset whose left and right structure maps are

$$\begin{aligned} \tilde{\vartheta}\colon X \otimes_{\mathcal{G}} Y &\longrightarrow \mathcal{H}_0 \\ x \otimes_{\mathcal{G}} y &\longrightarrow \vartheta(x) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \tilde{\varrho}\colon X \otimes_{\mathcal{G}} Y &\longrightarrow \mathcal{K}_0 \\ x \otimes_{\mathcal{G}} y &\longrightarrow \varrho(y) \end{aligned}$$

respectively, and whose left and right actions are

$$\begin{aligned} \mathcal{H}_1\, {}_{\mathsf{s}}\!\times_{\tilde{\vartheta}} (X \otimes_{\mathcal{G}} Y) &\longrightarrow X \otimes_{\mathcal{G}} Y \\ (h, x \otimes_{\mathcal{G}} y) &\longrightarrow hx \otimes_{\mathcal{G}} y \end{aligned} \qquad \text{and} \qquad \begin{aligned} (X \otimes_{\mathcal{G}} Y)\, {}_{\tilde{\varrho}}\!\times_{\mathsf{t}} \mathcal{K}_1 &\longrightarrow X \otimes_{\mathcal{G}} Y \\ (x \otimes_{\mathcal{G}} y, k) &\longrightarrow x \otimes_{\mathcal{G}} yk \end{aligned}$$

respectively.

Now we will state the universal property of the tensor product between groupoid-bisets (in this regard, see also [EK17, Remark 2.2]). We will denote with $\pi\colon X\, {}_{\varsigma}\!\times_{\chi} Y \longrightarrow X \otimes_{\mathcal{G}} Y$ the canonical projection to the quotient.

Lemma 1.2.15. *Let $\mathcal{G}, \mathcal{H}$ and $\mathcal{K}$ be three groupoids and $(X, \vartheta, \varsigma)$, (Y, χ, ϱ) the above groupoid-bisets. Then the following diagram*

$$X\,{}_{\varsigma}\!\times_{\mathsf{t}}\mathcal{G}_1\,{}_{\mathsf{s}}\!\times_{\chi} Y \underset{1_Y\times\lambda}{\overset{\rho\times 1_X}{\rightrightarrows}} X\,{}_{\varsigma}\!\times_{\chi} Y \xrightarrow{\ \pi\ } X\otimes_{\mathcal{G}} Y \tag{1.2.6}$$

is the co-equalizer, in the category of $(\mathcal{H}, \mathcal{K})$-bisets, of the pair of morphisms $\big(\rho\times 1_X, 1_Y\times\lambda\big)$. Furthermore, if $\mathfrak{f}\colon X \longrightarrow X'$ and $\mathfrak{g}\colon Y \longrightarrow Y'$ are morphism of $(\mathcal{H}, \mathcal{G})$-biset and $(\mathcal{G}, \mathcal{K})$-biset, respectively, then there is a unique map $\mathfrak{f}\otimes_{\mathcal{G}}\mathfrak{g}\colon X\otimes_{\mathcal{G}} Y \longrightarrow X'\otimes_{\mathcal{G}} Y'$ rendering commutative the following diagram:

$$\begin{array}{ccccc}
X\,{}_{\varsigma}\!\times_{\mathsf{t}}\mathcal{G}_1\,{}_{\mathsf{s}}\!\times_{\chi} Y & \underset{1_Y\times\lambda}{\overset{\rho\times 1_X}{\rightrightarrows}} & X\,{}_{\varsigma}\!\times_{\chi} Y & \xrightarrow{\ \pi\ } & X\otimes_{\mathcal{G}} Y \\
{\scriptstyle \mathfrak{f}\times \mathrm{Id}_{\mathcal{G}_1}\times\mathfrak{g}}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{f}\times\mathfrak{g}} & & \big\downarrow{\scriptstyle \mathfrak{f}\otimes_{\mathcal{G}}\mathfrak{g}} \\
X'\,{}_{\varsigma}\!\times_{\mathsf{t}}\mathcal{G}_1\,{}_{\mathsf{s}}\!\times_{\chi} Y' & \underset{1_{Y'}\times\lambda'}{\overset{\rho'\times 1_{X'}}{\rightrightarrows}} & X'\,{}_{\varsigma}\!\times_{\chi} Y' & \xrightarrow{\ \pi'\ } & X'\otimes_{\mathcal{G}} Y'.
\end{array}$$

Proof. Straightforward. □

Remark 1.2.16. It is clear from its universal construction that the tensor product establishes a functor:

$$\begin{array}{rcl}
\mathcal{H}\text{-}\mathsf{BiSets}\text{-}\mathcal{G}\times\mathcal{G}\text{-}\mathsf{BiSets}\text{-}\mathcal{K} & \longrightarrow & \mathcal{H}\text{-}\mathsf{BiSets}\text{-}\mathcal{K} \\
\big(X, Y\big) & \longmapsto & X\otimes_{\mathcal{G}} Y \\
\big(\mathfrak{f}, \mathfrak{g}\big) & \longmapsto & \mathfrak{f}\otimes_{\mathcal{G}}\mathfrak{g}
\end{array} \tag{1.2.7}$$

Although we will not employ the notion of bicategory in the context of bisets, it is noteworthy to mention that groupoids and groupoid-bisets give rise to a bicategory as follows: 0-cells are groupoids, 1-cells are groupoid-bisets and 2-cells are morphisms between groupoid-bisets. The horizontal and vertical compositions, as well as the coherence constraints, are given by the tensor product stated in Lemma 1.2.15 and equation (1.2.7).

1.3 Monoidal equivalences between groupoid-sets

The material in this section will be essential in Chapter 5 to construct the Burnside ring of a groupoid. Mainly, we will prove some isomorphisms of categories of groupoid-sets, after having described two monoidal structures on them. Before starting doing this, however, we have to establish some definitions that will help us to state the future theorems and propositions in a more succinct way.

1.3.1 Laplaza categories and their functors

Given two monoidal structures $\diamond$ and $\boxplus$ on a category $\mathcal{C}$, let's assume that one of them, let's say $\diamond$, distributes over the other, that is, for each object A, B and C of $\mathcal{C}$, there are natural isomorphisms

$$A\diamond(B\boxplus C)\cong A\diamond B\boxplus A\diamond C \qquad \text{and} \qquad (A\boxplus B)\diamond C\cong A\diamond C\boxplus B\diamond C.$$

To distinguish the two monoidal structures, we call $\diamond$ the "multiplicative" monoidal structure of $\mathcal{C}$ and $\boxplus$ the "additive" monoidal structure of $\mathcal{C}$. This situation was foreshadowed in [Lap72a] and studied more thoroughly in [Lap72b, page 29], where a complete set of coherence conditions is provided. We will call such a category, for lack of a better name, a **Laplaza category** and we will denote the category of small Laplaza categories by **LPZCat**. An example of a small Laplaza category, as we will see in subsection 1.3.2, is the category $\mathsf{Sets}\text{-}\mathcal{G}$ of the finite right $\mathcal{G}$-sets over a groupoid $\mathcal{G}$ where $\boxplus$ is the coproduct, i.e., the disjoint union $\uplus$, and $\diamond$ is the fibered product $\underset{\mathcal{G}_0}{\times}$.

In this subsection, for brevity reasons, we will omit the unity of the monoidal structures involved and we will denote a Laplaza category with the notation $(\mathcal{C}, \diamond, \boxplus)$ where the first monoidal product, $\diamond$, is the multiplicative one the second monoidal structure the additive one. For the definition of strong monoidal functor and monoidal natural transformation we remand to Appendix C.

Definition 1.3.1. Let $(\mathcal{C}_1, \diamond_1, \boxplus_1)$ and $(\mathcal{C}_2, \diamond_2, \boxplus_2)$ be Laplaza categories. A **Laplaza functor**

$$F\colon\ (\mathcal{C}_1, \diamond_1, \boxplus_1) \longrightarrow (\mathcal{C}_2, \diamond_2, \boxplus_2)$$

is simultaneously both a strong monoidal functor $F\colon\ (\mathcal{C}_1, \diamond_1) \longrightarrow (\mathcal{C}_2, \diamond_2)$ and a strong monoidal functor $F\colon\ (\mathcal{C}_1, \boxplus_1) \longrightarrow (\mathcal{C}_2, \boxplus_2)$.

Definition 1.3.2. Let $(\mathcal{C}_1, \diamond_1, \boxplus_1)$ and $(\mathcal{C}_2, \diamond_2, \boxplus_2)$ be Laplaza categories. A Laplaza functor

$$F\colon\ (\mathcal{C}_1, \diamond_1, \boxplus_1) \longrightarrow (\mathcal{C}_2, \diamond_2, \boxplus_2)$$

is said to be an **isomorphism of Laplaza categories** if it is an isomorphism of categories and the inverse functor F^{-1} is also a Laplaza functor.

Definition 1.3.3. Given two Laplaza categories $(\mathcal{C}_1, \diamond_1, \boxplus_1)$ and $(\mathcal{C}_2, \diamond_2, \boxplus_2)$, let

$$F, G\colon\ (\mathcal{C}_1, \diamond_1, \boxplus_1) \longrightarrow (\mathcal{C}_2, \diamond_2, \boxplus_2)$$

be two strong monoidal functors. A **Laplaza natural transformation** (respectively, a **Laplaza natural isomorphism**)

$$\varphi\colon F \longrightarrow G\colon\ (\mathcal{C}_1, \diamond_1, \boxplus_1) \longrightarrow (\mathcal{C}_2, \diamond_2, \boxplus_2)$$

is simultaneously both a monoidal natural transformation (respectively, a monoidal natural isomorphism)

$$\varphi\colon F \longrightarrow G\colon\ (\mathcal{C}_1, \diamond_1) \longrightarrow (\mathcal{C}_2, \diamond_2)$$

and a monoidal natural transformation (respectively, a monoidal natural isomorphism)

$$\varphi\colon F \longrightarrow G\colon\ (\mathcal{C}_1, \boxplus_1) \longrightarrow (\mathcal{C}_2, \boxplus_2)\,.$$

Definition 1.3.4. Given $(\mathcal{C}_1, \diamond_1, \boxplus_1)$ and $(\mathcal{C}_2, \diamond_2, \boxplus_2)$ two Laplaza categories, let

$$F\colon\ (\mathcal{C}_1, \diamond_1, \boxplus_1) \longrightarrow (\mathcal{C}_2, \diamond_2, \boxplus_2)$$

be a Laplaza functor. We say that F realizes a **Laplaza equivalence of categories** if there is a Laplaza functor

$$G\colon\ (\mathcal{C}_2, \diamond_2, \boxplus_2) \longrightarrow (\mathcal{C}_1, \diamond_1, \boxplus_1)$$

and Laplaza natural isomorphisms $\eta\colon\ \mathrm{Id}_{\mathcal{C}_1} \longrightarrow GF$ and $\varepsilon\colon FG \longrightarrow \mathrm{Id}_{\mathcal{C}_2}$.

1.3.2 The monoidal structures of the category of (right) $\mathcal{G}$-sets and the induction functors

We will start describing the Laplaza structure on Sets-$\mathcal{G}$ and, after that, we will explore the concept of the induction functor and of the induced natural transformation.

Given a groupoid $\mathcal{G}$ the category of right $\mathcal{G}$-sets is a symmetric monoidal category with respect to the disjoint union $\biguplus$. This structure is given as follows: given two right $\mathcal{G}$-set (X, ς) and (Y, ϑ), we set $(X, \varsigma) \uplus (Y, \vartheta) = (Z, \mu)$ where $Z = X \uplus Y$ and the map $\mu \colon Z \longrightarrow \mathcal{G}_0$ is defined by the conditions $\mu_{|X} = \varsigma$ and $\mu_{|Y} = \vartheta$. The action is defined by:

$$Z\,{}_{\mu}\!\times_{\mathsf{t}} \mathcal{G}_1 \longrightarrow Z$$
$$(z, g) \longrightarrow zg$$

where zg stands for xg if $z = x \in X$ or yg if $z = y \in Y$. The identity object of this monoidal structure is the right $\mathcal{G}$-set with an empty underlying set whose action is, by convention, the empty one.

On the other hand, the fibre product $- \underset{\mathcal{G}_0}{\times} -$ induces another symmetric monoidal structure (see, for instance, [EK17, section 2]). Explicitly, the product of (X, ς) and (Y, ϑ) is defined as follows:

$$(X, \varsigma) \underset{\mathcal{G}_0}{\times} (Y, \vartheta) \;=\; \big(X \underset{\mathcal{G}_0}{\times} Y, \varsigma\vartheta\big),$$

where $X \underset{\mathcal{G}_0}{\times} Y = X\,{}_{\varsigma}\!\times_{\theta} Y$ and $\varsigma\vartheta \colon X \underset{\mathcal{G}_0}{\times} Y \longrightarrow \mathcal{G}_0$ sends (x, y) to $\varsigma(x) = \vartheta(y)$. The action is given by $(x, y)g = (xg, yg)$ for each $g \in \mathcal{G}_1$ and $(x, y) \in X \underset{\mathcal{G}_0}{\times} Y$ such that $\varsigma\vartheta(x, y) = \mathsf{t}(g)$. The identity object is the right $\mathcal{G}$-set $(\mathcal{G}_0, \mathrm{Id}_{\mathcal{G}_0})$ with action given as in (1.1.3). Furthermore, up to isomorphisms, $(\mathcal{G}_0, \mathrm{Id}_{\mathcal{G}_0})$ is the only dualizable object with respect to this monoidal category.

The compatibility between the two monoidal structure is expressed by the subsequent lemma, while the coherence condition expressed in [Lap72b, page 29] are clear.

Lemma 1.3.5. *Given a groupoid $\mathcal{G}$, let be $((X_i, \varsigma_i))_{i \in I}$ and $((Y_j, \vartheta_j))_{j \in J}$ two families of right $\mathcal{G}$-sets. Then there is an isomorphism of right $\mathcal{G}$-sets*

$$\biguplus_{\substack{i \in I \\ j \in J}} \left((X_i, \varsigma_i) \underset{\mathcal{G}_0}{\times} (Y_j, \vartheta_j) \right) \;\cong\; \left(\biguplus_{i \in I} (X_i, \varsigma_i) \right) \underset{\mathcal{G}_0}{\times} \left(\biguplus_{j \in J} (Y_j, \vartheta_j) \right).$$

Proof. It is omitted, since it is a direct verification. □

Let $\varphi \colon \mathcal{H} \longrightarrow \mathcal{G}$ be a morphism of groupoids. We define the induced functor, referred to as the **induction functor**

$$\varphi^* \colon \mathsf{Sets}\text{-}\mathcal{G} \longrightarrow \mathsf{Sets}\text{-}\mathcal{H},$$

which sends the right $\mathcal{G}$-set (X, ς) to the right $\mathcal{H}$-set $(X\,{}_{\varsigma}\!\times_{\varphi_0} \mathcal{H}_0, \mathrm{pr}_2)$ with the following action:

$$(X\,{}_{\varsigma}\!\times_{\varphi_0} \mathcal{H}_0)\,{}_{\mathrm{pr}_2}\!\times_{\mathsf{t}} \mathcal{H}_1 \longrightarrow X\,{}_{\varsigma}\!\times_{\varphi_0} \mathcal{H}_0$$
$$\big((x, a), h\big) \longrightarrow (x\varphi_1(h), \mathsf{s}(h)).$$

Given a morphism of right $\mathcal{G}$-set $f \colon (X, \varsigma) \longrightarrow (Y, \vartheta)$, we define the morphism

$$\varphi^*(f) \colon \varphi^*(X, \varsigma) \longrightarrow \varphi^*(Y, \vartheta)$$

as the morphism

$$f \times \mathcal{H}_0 \colon (X\, {}_{\varsigma}\!\times_{\varphi_0} \mathcal{H}_0, \mathrm{pr}_2) \longrightarrow (Y\, {}_{\theta}\!\times_{\varphi_0} \mathcal{H}_0, \mathrm{pr}_2)$$
$$(x, a) \longrightarrow (f(x), a).$$

For instance, we have that $\varphi^*(\mathcal{G}_0, \mathrm{Id}_{\mathcal{G}_0}) = \Big(\mathcal{G}_0\, {}_{\mathrm{Id}_{\mathcal{G}_0}}\!\times_{\varphi_0} \mathcal{H}_0, \mathrm{pr}_2\Big)$. The following is a well known property of the induction functor (see [EK17]). However, for the sake of completeness and for the convenience of the reader, we give here an elementary proof.

Proposition 1.3.6. *The functor $\varphi^* \colon \mathsf{Sets}\text{-}\mathcal{G} \longrightarrow \mathsf{Sets}\text{-}\mathcal{H}$ is Laplaza, that is, is monoidal with respect to both the disjoint union $\uplus$ and the fibered product $- \underset{\mathcal{G}_0}{\times} -$.*

Proof. The fact that φ^* is well defined is a routine computation and we leave it to the reader. Let us check that φ^* is monoidal with respect to $\uplus$. Given right sets (X, ς) and (Y, ϑ) we have the natural isomorphisms

$$\begin{aligned}\varphi^*\left((X, \varsigma) \uplus (Y, \vartheta)\right) &= \varphi^*\left(X \uplus Y, \varsigma \uplus \vartheta\right) = \left((X \uplus Y)\, {}_{\varsigma \uplus \vartheta}\!\times_{\varphi_0} \mathcal{H}_0, \mathrm{pr}_2\right) \\ &\cong \left(X\, {}_{\varsigma}\!\times_{\varphi_0} \mathcal{H}_0, \mathrm{pr}_2\right) \uplus \left(Y\, {}_{\vartheta}\!\times_{\varphi_0} \mathcal{H}_0, \mathrm{pr}_0\right) = \varphi^*\left(X, \varsigma\right) \uplus \varphi^*\left(Y, \vartheta\right)\end{aligned}$$

and, we also have that

$$\varphi^*\left(\varnothing, \varnothing\right) = \left(\varnothing\, {}_{\varnothing}\!\times_{\varphi_0} \mathcal{H}_0, \mathrm{pr}_2\right) = \left(\varnothing, \varnothing\right).$$

Now we have to prove that φ^* is monoidal with respect to the fibered product. Given right $\mathcal{G}$-sets (X, ς) and (Y, ϑ) we have the natural isomorphisms

$$\begin{aligned}\varphi^*\left((X, \varsigma) \underset{\mathcal{G}_0}{\times} (Y, \vartheta)\right) &= \varphi^*\left(X\, {}_{\varsigma}\!\times_{\vartheta} Y, \varsigma\vartheta\right) = \left((X\, {}_{\varsigma}\!\times_{\vartheta} Y)\, {}_{\varsigma\vartheta}\!\times_{\varphi_0} \mathcal{H}_0, \mathrm{pr}_2\right) \\ &\cong \left((X\, {}_{\varsigma}\!\times_{\varphi_0} \mathcal{H}_0)\, {}_{\mathrm{pr}_2}\!\times_{\mathrm{pr}_2} (Y\, {}_{\vartheta}\!\times_{\varphi_0} \mathcal{H}_0), (\mathrm{pr}_2)(\mathrm{pr}_2)\right) \\ &= \left(X\, {}_{\varsigma}\!\times_{\varphi_0} \mathcal{H}_0, \mathrm{pr}_2\right) \underset{\mathcal{H}_0}{\times} \left(Y\, {}_{\vartheta}\!\times_{\varphi_0}, \mathrm{pr}_2\right) = \varphi^*\left(X, \varsigma\right) \underset{\mathcal{H}_0}{\times} \varphi^*\left(Y, \vartheta\right),\end{aligned}$$

because an element of $(X\, {}_{\varsigma}\!\times_{\varphi_0} \mathcal{H}_0)\, {}_{\mathrm{pr}_2}\!\times_{\mathrm{pr}_2} (Y\, {}_{\vartheta}\!\times_{\varphi_0} \mathcal{H}_0)$ is of the kind (x, a, y, a) with $x \in X$, $y \in Y$, $a \in \mathcal{H}_0$. Therefore, we have a natural isomorphism

$$\varphi^*\left((X, \varsigma) \underset{\mathcal{G}_0}{\times} (Y, \vartheta)\right) \cong \varphi^*\left(X, \varsigma\right) \underset{\mathcal{H}_0}{\times} \varphi^*\left(Y, \vartheta\right),$$

for every pair of right $\mathcal{G}$-sets (X, ς) and (Y, ϑ). On the other hand, we have that

$$\varphi^*\left(\mathcal{G}_0, \mathrm{Id}_{\mathcal{G}_0}\right) = \left(\mathcal{G}_0\, {}_{\mathrm{Id}_{\mathcal{G}_0}}\!\times_{\varphi_0} \mathcal{H}_0, \mathrm{pr}_2\right) \cong \left(\mathcal{H}_0, \mathrm{Id}_{\mathcal{H}_0}\right),$$

since the map $\mathrm{pr}_2 \colon \mathcal{G}_0\, {}_{Id}\!\times_{\varphi_0} \mathcal{H}_0 \longrightarrow \mathcal{H}_0$ establishes an isomorphism of right $\mathcal{H}$-sets. □

Proposition and Definition 1.3.7. *Given groupoid $\mathcal{H}$ and $\mathcal{G}$, let $\varphi, \psi \colon \mathcal{H} \longrightarrow \mathcal{G}$ be two morphisms of groupoids and consider a natural transformation $\alpha \colon \varphi \longrightarrow \psi$. We define an* ***induced natural transformation***

$$\alpha^* \colon \varphi^* \longrightarrow \psi^*$$

between the induced functors

$$\varphi^*,\ \psi^* \colon \mathsf{Sets}\text{-}\mathcal{G} \longrightarrow \mathsf{Sets}\text{-}\mathcal{H}$$

as follows: for each right $\mathcal{G}$-set (X,ς) and for each $(x,a) \in \varphi^*(X,\varsigma)$, we set

$$\alpha^*_{(X,\varsigma)} \colon \varphi^*(X,\varsigma) \longrightarrow \psi^*(X,\varsigma)$$
$$(x,a) \longrightarrow \alpha^*_{(X,\varsigma)}(x,a) = \big(x \cdot (\alpha a)^{-1}, a\big)\,.$$

Then α^* is a Laplaza transformation (as defined in subsection 1.3.1).

Proof. Given a right $\mathcal{G}$-set (X,ς) and $(x,a) \in \varphi^*(X,\varsigma)$, the situation is as follows:

$$\varsigma(x) = \varphi_0(a) \xrightarrow{\quad \alpha a \quad} \psi_0(a).$$

We have $\varsigma(x) = \varphi_0(a) = \mathsf{s}(\alpha a) = \mathsf{t}\left((\alpha a)^{-1}\right)$ thus we can write $x \cdot (\alpha a)^{-1}$ and we have

$$\varsigma\left(x \cdot (\alpha a)^{-1}\right) = \mathsf{s}\left((\alpha a)^{-1}\right) = \mathsf{t}(\alpha a) = \psi_0(a).$$

Therefore $\alpha^*(X,\varsigma)$ is well defined. We have to check that $\varphi^*(X,\varsigma)$ is a morphism of right $\mathcal{H}$-sets. The condition on the structure map is obviously satisfied. Regarding the condition on the actions, let be $(x,a) \in \varphi^*(X,\varsigma)$ and $h \in \mathcal{H}_1$ such that $a = \mathrm{pr}_2(x,a) = \mathsf{t}(h)$: the arrow $h \colon b \longrightarrow a$ is a morphism in $\mathcal{H}$ thus the following diagram is commutative:

$$\begin{array}{ccc} \varphi_0(b) & \xrightarrow{\quad \alpha(b) \quad} & \psi_0(b) \\ {\scriptstyle \varphi_1(h)}\downarrow & & \downarrow{\scriptstyle \psi_1(h)} \\ \varphi_0(a) & \xrightarrow{\quad \alpha(a) \quad} & \psi_0(a). \end{array}$$

As a consequence we can compute

$$\begin{aligned} \alpha^*_{(X,\varsigma)}((x,a)h) &= \alpha^*_{(X,\varsigma)}\left(x\varphi_1(h), b\right) = \big(x \cdot \varphi_1(h) \cdot (\alpha b)^{-1}, b\big) = \big(x \cdot (\alpha b)^{-1} \cdot \psi_1(h), b\big) \\ &= \big(x \cdot (\alpha b)^{-1}, a\big) \cdot h = \Big(\alpha^*_{(X,\varsigma)}(x,a)\Big) \cdot h, \end{aligned}$$

which show that $\alpha^*_{(X,\varsigma)}$ is an $\mathcal{H}$-equivariant map.

We have to check that α^* is natural that is, given a morphism of right $\mathcal{G}$-sets $f \colon (X,\varsigma) \longrightarrow (Y,\vartheta)$, that the following diagram is commutative:

$$\begin{array}{ccc} \varphi^*(X,\varsigma) = (X\,{}_{\varsigma}\!\times_{\varphi_0} \mathcal{H}_0, \mathrm{pr}_2) & \xrightarrow{\quad \alpha^*(X,\varsigma) \quad} & \psi^*(X,\varsigma) = (X\,{}_{\varsigma}\!\times_{\psi_0} \mathcal{H}_0, \mathrm{pr}_2) \\ {\scriptstyle \varphi^*(f) = f \times \mathrm{Id}_{\mathcal{H}_0}}\downarrow & & \downarrow{\scriptstyle \psi^*(g) = g \times \mathrm{Id}_{\mathcal{H}_0}} \\ \varphi^*(Y,\vartheta) = (Y\,{}_{\vartheta}\!\times_{\varphi_0} \mathcal{H}_0, \mathrm{pr}_2) & \xrightarrow{\quad \alpha^*(Y,\vartheta) \quad} & \psi^*(Y,\vartheta) = (Y\,{}_{\vartheta}\!\times_{\psi_0} \mathcal{H}_0, \mathrm{pr}_2)\,. \end{array}$$

This follows from the following computation: given $(x,a) \in \varphi^*(X,\varsigma)$, we have

$$\begin{aligned} \left(\psi^* f\right)\left(\alpha^*(X,\varsigma)\right)(x,a) &= \left(\psi^* f\right)\left(x \cdot (\alpha a)^{-1}, a\right) = \left(f\left(x \cdot (\alpha a)^{-1}\right), a\right) = \left(f(x)(\alpha a)^{-1}, a\right) \\ &= \alpha^*(Y,\vartheta)\left(f(x), a\right) = \alpha^*(Y,\vartheta)\left(\varphi^* f\right)(x,a). \end{aligned}$$

The fact that α^* is a Laplaza transformation is proved directly and it is left to the reader. □

Proposition 1.3.8. *Given groupoids $\mathcal{K}$, $\mathcal{H}$ and $\mathcal{G}$, let's consider the following homomorphism of groupoids:*

$$\mathcal{K} \xrightarrow{\quad \psi \quad} \mathcal{H} \xrightarrow{\quad \varphi \quad} \mathcal{G}\,.$$

Then the following diagrams commute up to a natural isomorphism

$$\begin{array}{ccc} \mathsf{Sets}\text{-}\mathcal{G} & \xrightarrow{(\varphi\psi)^*} & \mathsf{Sets}\text{-}\mathcal{K} \\ {\scriptstyle\varphi^*}\downarrow & \cong & \nearrow{\scriptstyle\psi^*} \\ \mathsf{Sets}\text{-}\mathcal{H} & & \end{array} \qquad \textit{and} \qquad \mathsf{Sets}\text{-}\mathcal{G} \underset{\mathrm{Id}_{\mathsf{Sets}\text{-}\mathcal{G}}}{\overset{(\mathrm{Id}_{\mathcal{G}})^*}{\rightrightarrows}} \mathsf{Sets}\text{-}\mathcal{G}\ (\cong).$$

That is, there are Laplaza natural isomorphisms

$$\gamma\colon (\varphi\psi)^* \longrightarrow \psi^*\varphi^* \qquad \textit{and} \qquad \beta\colon (\mathrm{Id}_{\mathcal{G}})^* \longrightarrow \mathrm{Id}_{\mathsf{Sets}\text{-}\mathcal{G}}\,.$$

Proof. Given a homomorphism $f\colon (X,\varsigma) \longrightarrow (Y,\theta)$ in $\mathsf{Sets}\text{-}\mathcal{G}$, we have

$$\begin{array}{rcl} (X\,{}_{\varsigma}\!\times_{\varphi\psi}\mathcal{K}_0, \mathrm{pr}_2) = (\varphi\psi)^*(X,\varsigma) & \xrightarrow{(\varphi\psi)^*(f)} & (\varphi\psi)^*(Y,\theta) = (Y\,{}_{\theta}\!\times_{\varphi\psi}\mathcal{K}_0, \mathrm{pr}_2) \\ (x,a) & \longrightarrow & (f(x),a) \end{array}$$

and

$$\begin{array}{ccc} \psi^*(X\,{}_{\varsigma}\!\times_{\varphi_0}\mathcal{H}_0, \mathrm{pr}_2) = \psi^*\varphi^*(X,\varsigma) & \xrightarrow{\psi^*\varphi^*(f)} & \psi^*\varphi^*(Y,\theta) = \psi^*(Y\,{}_{\theta}\!\times_{\varphi_0}\mathcal{H}_0, \mathrm{pr}_2) \\ \| & & \| \\ \big((X\,{}_{\varsigma}\!\times_{\varphi_0}\mathcal{H}_0)\,{}_{\mathrm{pr}_2}\!\times_{\psi_0}\mathcal{K}_0, \mathrm{pr}_3\big) & & \big((Y\,{}_{\theta}\!\times_{\varphi_0}\mathcal{H}_0)\,{}_{\mathrm{pr}_2}\!\times_{\psi_0}\mathcal{K}_0, \mathrm{pr}_3\big) \\ (x,a,b) & \longrightarrow & (\varphi^*(x,a),b) = (f(x),a,b)\,. \end{array}$$

Now, for each right $\mathcal{G}$-set (X,ς) we consider the $\mathcal{G}$-equivariant maps

$$\begin{array}{rcl} \gamma\,(X,\varsigma)\colon \psi^*\varphi^*(X,\varsigma) & \longrightarrow & (\varphi\psi)^*(X,\varsigma) \\ (x,a,b) & \longrightarrow & (x,b) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \beta\,(X,\varsigma)\colon (\mathrm{Id}_{\mathcal{G}})^*(X,\varsigma) & \longrightarrow & (X,\varsigma) \\ (x,a) & \longrightarrow & x \end{array}$$

that give us the desired natural transformations. The proof of the fact that γ and β are Laplaza isomorphisms is immediate. □

Proposition 1.3.9. *Given groupoids $\mathcal{H}$ and $\mathcal{G}$ and morphisms of groupoids $\varphi, \psi, \mu\colon \mathcal{H} \longrightarrow \mathcal{G}$, let's consider natural transformations $\alpha\colon \varphi \longrightarrow \psi$ and $\beta\colon \psi \longrightarrow \mu$. Then the diagrams*

$$\begin{array}{ccc} \varphi^* & \xrightarrow{(\beta\alpha)^*} & \mu^* \\ {\scriptstyle\alpha^*}\downarrow & & \nearrow{\scriptstyle\beta^*} \\ \psi^* & & \end{array} \qquad \textit{and} \qquad \varphi^* \underset{\mathrm{Id}_{\varphi^*}}{\overset{(\mathrm{Id}_{\varphi})^*}{\rightrightarrows}} \varphi^*$$

are commutative. Moreover, if α is a natural isomorphism, then we have $\left(\alpha^{-1}\right)^* = (\alpha^*)^{-1}$.

Proof. Straightforward. □

We finish this subsection with the following useful results.

Proposition 1.3.10. *Given a groupoid $\mathcal{G}$, let $\mathcal{A}$ be a subgroupoid of $\mathcal{G}$. Then the functor*

$$\begin{array}{rcl} F\colon \mathsf{Sets}\text{-}\mathcal{G} & \longrightarrow & \mathsf{Sets}\text{-}\mathcal{A} \\ (X,\varsigma) & \longrightarrow & \left(\varsigma^{-1}(\mathcal{A}_0)\,,\, \varsigma|_{\varsigma^{-1}(\mathcal{A}_0)}\right), \end{array}$$

opportunely defined on morphisms, is a Laplaza functor.

Proof. The fact that F is a well defined functor is obvious. We have to prove that F is monoidal with respect to the disjoint union. So, let be $(X_1, \varsigma_1), (X_2, \varsigma_2) \in \mathsf{Sets}\text{-}\mathcal{G}$: we have

$$\begin{aligned} F(X_1, \varsigma) \uplus F(X_2, \varsigma_2) &= \left(\varsigma_1^{-1}(\mathcal{A}_0), \varsigma_1|_{\varsigma_1^{-1}(\mathcal{A}_0)}\right) \uplus \left(\varsigma_2^{-1}(\mathcal{A}_0), \varsigma_2|_{\varsigma_2^{-1}(\mathcal{A}_0)}\right) \\ &= \left(\varsigma_1^{-1}(\mathcal{A}_0) \uplus \varsigma_2^{-1}(\mathcal{A}_0), \varsigma_1|_{\varsigma_1^{-1}(\mathcal{A}_0)} \uplus \varsigma_2|_{\varsigma_2^{-1}(\mathcal{A}_0)}\right) \\ &= \left((\varsigma_1 \uplus \varsigma_2)^{-1}(\mathcal{A}_0), (\varsigma_1 \uplus \varsigma_2)|_{(\varsigma_1 \uplus \varsigma_2)^{-1}(\mathcal{A}_0)}\right) \\ &= F(X_1 \uplus X_2, \varsigma_1 \uplus \varsigma_2) = F((X_1, \varsigma_1) \uplus (X_2, \varsigma_2)) \end{aligned}$$

and evidently, we have $F(\varnothing, \varnothing) = (\varnothing, \varnothing)$. Therefore, since the coherency conditions are immediate to verify, F is monoidal strict with respect to $\uplus$. It remains to check that F is monoidal with respect to the fiber product. For each $(X_1, \varsigma_1), (X_2, \varsigma_2) \in \mathsf{Sets}\text{-}\mathcal{G}$, we have, using the notations $\vartheta_i = \varsigma_i|_{\varsigma_i^{-1}(\mathcal{A}_0)}$ for $i \in \{1, 2\}$,

$$\begin{aligned} F(X_1, \varsigma_1) \underset{\mathcal{A}_0}{\times} F(X_2, \varsigma_2) &= \left(\varsigma_1^{-1}(\mathcal{A}_0), \vartheta_1\right) \underset{\mathcal{A}_0}{\times} \left(\varsigma_2^{-1}(\mathcal{A}_0), \vartheta_2\right) \\ &= \left(\varsigma_1^{-1}(\mathcal{A}_0) \; {}_{\vartheta_1}\!\times_{\vartheta_2} \varsigma_2^{-1}(\mathcal{A}_0), \vartheta_1\vartheta_2\right) \\ &= \left((\varsigma_1\varsigma_2)^{-1}(\mathcal{A}_0), \varsigma_1\varsigma_2|_{(\varsigma_1\varsigma_2)^{-1}(\mathcal{A}_0)}\right) \\ &= F(X_1 \; {}_{\varsigma_1}\!\times_{\varsigma_2} X_2, \varsigma_1\varsigma_2) = F\left((X_1, \varsigma_1) \underset{\mathcal{G}_0}{\times}, (X_2, \varsigma_2)\right) \end{aligned}$$

and, obviously, we have that $F(\mathcal{G}_0, \mathrm{Id}_{\mathcal{G}_0}) = (\mathcal{A}_0, \mathrm{Id}_{\mathcal{A}_0})$. Therefore, since the coherency conditions are immediate to verify, F is monoidal strict with respect to the fiber product and the proof is concluded. ☐

1.3.3 Monoidal equivalences and category decompositions

We will describe several equivalences of categories of groupoid-sets that will be used in the forthcoming sections. Before starting, we need to talk about the coproduct in the category of small categories and, in particular, in the category of (small) groupoids.

Proposition 1.3.11. *Let* **Cat** *be the category of small categories and let $(\mathcal{X}_j)_{j \in I}$ be a family of objects in* **Cat** *(respectively, in* **Grpd***). Let $\mathcal{X}$ be the small category (respectively, the groupoid) whose set of object is the disjoint union of the $\mathcal{X}_j$'s, that is,*

$$\mathrm{Obj}(\mathcal{X}) = \biguplus_{j \in I} \mathrm{Obj}(\mathcal{X}_j),$$

and such that, for every $y, z \in \mathrm{Obj}(\mathcal{X})$,

$$\mathrm{Hom}_{\mathcal{X}}(y, z) = \begin{cases} \mathrm{Hom}_{\mathcal{X}_j}(y, z) & \text{if } \exists\, i \in I : y, z \in \mathrm{Obj}(\mathcal{X}_i) \\ \varnothing & \text{otherwise.} \end{cases}$$

Now let $(\mathcal{X}_j)_{j \in I}$ be a family of objects in **Cat** *(respectively, in* **Grpd***) and, for each $j \in I$, let $i_j \colon \mathcal{X}_j \longrightarrow \mathcal{X}$ be the inclusion functor. Then $(i_j \colon \mathcal{X}_j \longrightarrow \mathcal{X})_{j \in I}$ is the coproduct in* **Cat** *(respectively, in* **Grpd***) of the family $(\mathcal{X}_j)_{j \in I}$. Henceforth, we will often simply write $\mathcal{X} = \coprod_{j \in I} \mathcal{X}_j$.*

Proof. Let $(\varphi_j : \mathcal{X}_j \longrightarrow \mathcal{Y})_{j\in I}$ be a family of morphisms in $\mathcal{C}$. We have to prove that there is a unique morphism $\varphi : \mathcal{X} \longrightarrow \mathcal{Y}$ such that for every $j \in I$ the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{X}_j & \xrightarrow{\varphi_j} & \mathcal{Y} \\ {\scriptstyle i_j}\downarrow & \nearrow_{\varphi} & \\ \mathcal{X} & & \end{array} .$$

Let be x be an object of $\mathcal{X}$: then there is a unique $j \in I$ such that $x \in \mathcal{X}_j$ so we define $\varphi(x) := \varphi_j(x)$ (note that j is unique because $\mathrm{Obj}\,(\mathcal{X})$ is the disjoint union of the X_j's). Next, let $g : x \longrightarrow y$ be a morphism in $\mathcal{X}$: then there is a unique $j \in I$ such that $g : x \longrightarrow y$ is a morphism in $\mathcal{X}_j$ and we can define

$$\varphi\,(g : x \longrightarrow y) := \varphi_j\,(g : x \longrightarrow y)\,.$$

Proving that φ is actually a functor, that is, a morphism in **Cat** (respectively, in **Grpd**) is immediate, therefore we just have to show that φ is unique. Let $\varphi' : X \longrightarrow Y$ be another functor such that, for each $j \in I$, the diagram

$$\begin{array}{ccc} \mathcal{X}_j & \xrightarrow{\varphi_j} & \mathcal{Y} \\ {\scriptstyle i_j}\downarrow & \nearrow_{\varphi'} & \\ \mathcal{X} & & \end{array}$$

commutes. Given $x \in \mathcal{X}$, there is $j \in i$ such that $x \in \mathcal{X}_j$ thus $\varphi'(x) = \varphi'\,(i_j(x)) = \varphi_j(x) = \varphi(x)$. Now let be $g : x \longrightarrow y$ a morphism in X: there is $j \in J$ such that g is a morphism in $\mathcal{X}_j$, thus we have $\varphi'(g) = \varphi'\,(i_j(g)) = \varphi_j(g) = \varphi(g)$ and, consequently, $\varphi = \varphi'$.

Note that this proof works both for small categories and groupoids. □

Lemma 1.3.12. *Given a family of groupoids $(\mathcal{G}_i)_{i\in I}$, let be $j \in J$ and let's consider and object $(X, \varsigma) \in \mathsf{Sets}\text{-}\mathcal{G}_j$. Let $\mathcal{G} = \coprod_{j\in J} \mathcal{G}_j$ be the coproduct of the family $(\mathcal{G}_i)_{i\in I}$, as explained in Proposition 1.3.11. We can define an object $\widehat{(X, \varsigma)} = (X, \widehat{\varsigma}) \in \mathsf{Sets}\text{-}\mathcal{G}$ as follows. The structure map $\widehat{\varsigma} : X \longrightarrow \mathcal{G}$ is such that for every $x \in X$, $\widehat{\varsigma}(x) = \varsigma(x)$. The action*

$$\widehat{\rho} : X\,{}_{\widehat{\varsigma}}\!\times_{\mathsf{t}} \mathcal{G}_1 \longrightarrow X_j$$

is such that for every $x \in X$ and $g \in \mathcal{G}_1$ such that $\widehat{\varsigma}(x) = \mathsf{t}(g)$, we have $\widehat{\rho}(x, g) = \rho(x, g)$, where $\rho : X\,{}_{\varsigma}\!\times_{\mathsf{t}} \mathcal{G}_1 \longrightarrow X$ is the action of (X, ς).

Proof. Straightforward. □

Proposition 1.3.13. *Let $\{\mathcal{G}_i\}_{i\in I}$ be a family of groupoids and let $\{i_j : \mathcal{G}_j \longrightarrow \mathcal{G}\}_{j\in I}$ be their coproduct in* **Grpd**. *Then we have a Laplaza isomorphism of categories:*

$$\mathsf{Sets}\text{-}\left(\coprod_{j\in I} \mathcal{G}_j\right) \cong \prod_{j\in I} \mathsf{Sets}\text{-}\mathcal{G}_j.$$

Proof. We define a functor

$$T : \prod_{j\in I} \mathsf{Sets}\text{-}\mathcal{G}_j \longrightarrow \mathsf{Sets}\text{-}\left(\coprod_{j\in I} \mathcal{G}_j\right) = \mathsf{Sets}\text{-}\mathcal{G}$$

in the following way. Let be $((X_j, \varsigma_j))_{j \in I}$ an object of $\prod_{j \in I} \mathsf{Sets}\text{-}\mathcal{G}_j$. Using Lemma 1.3.12, we set

$$T\left(((X_j, \varsigma_j))_{j \in I}\right) = \biguplus_{j \in I} \widehat{(X_j, \varsigma_j)}.$$

It is clear that T becomes a functor in the expected way.

In the other direction, we define

$$S\colon \mathsf{Sets}\text{-}\left(\coprod_{j \in I} \mathcal{G}_j\right) \longrightarrow \prod_{j \in I} \mathsf{Sets}\text{-}\mathcal{G}_j$$

as follows. Given $(X, \varsigma) \in \mathsf{Sets}\text{-}\mathcal{G}$, we set

$$S(X, \varsigma) = \left(\left(\varsigma^{-1}\left((\mathcal{G}_j)_0\right), \varsigma|_{\varsigma^{-1}((\mathcal{G}_j)_0)}\right)\right)_{j \in I}.$$

It is clear that S becomes a functor in the expected way and that T and S are isomorphism of categories such that $S = T^{-1}$. To conclude, thanks to Corollary C.0.14, is it enough to prove that S is a Laplaza functor, but this follows from Proposition 1.3.10. □

Proposition 1.3.14. *Let $\mathcal{H}$ and $\mathcal{G}$ be isomorphic groupoids. Then there is a Laplaza isomorphism of categories* $\mathsf{Sets}\text{-}\mathcal{G} \cong \mathsf{Sets}\text{-}\mathcal{H}$.

Proof. It is an immediate verification. □

Given a groupoid $\mathcal{G}$ and a fixed object $x \in \mathcal{G}_0$, we denote with $\mathcal{G}^{(x)}$ the one object subgroupoid with isotropy group $\mathcal{G}^x$.

Theorem 1.3.15. *Given a transitive and not empty groupoid $\mathcal{G}$, let be $a \in \mathcal{G}_0$. Then there is a Laplaza equivalence of categories*

$$\mathsf{Sets}\text{-}\mathcal{G} \simeq \mathsf{Sets}\text{-}\mathcal{G}^{(a)}.$$

Proof. Set $S = \mathcal{G}_0$. Thanks to Propositions 1.3.14 and Remark 1.1.13, it is enough to prove the theorem when $\mathcal{G} = \mathcal{G}_{G,S}$ and $\mathcal{G}^{(a)} = \mathcal{G}_{G,\{a\}}$. We set $\mathcal{A} = \mathcal{G}^{(a)}$ for brevity. Let's define the functor

$$F\colon \mathsf{Sets}\text{-}\mathcal{G} \longrightarrow \mathsf{Sets}\text{-}\mathcal{A}$$
$$(X, \varsigma) \longrightarrow \left(\varsigma^{-1}(a), \varsigma|_{\varsigma^{-1}(a)}\right)$$

where $\varsigma\colon X \longrightarrow \mathcal{G}_0$ is the structure map of X. Thanks to Proposition 1.3.10, F is a well defined Laplaza functor, opportunely defined on morphisms, of course.

Now we have to construct a functor $G\colon \mathsf{Sets}\text{-}\mathcal{A} \longrightarrow \mathsf{Sets}\text{-}\mathcal{G}$: for each $(X, \varsigma) \in \mathsf{Sets}\text{-}\mathcal{A}$ we define $G(X, \varsigma) = (Y, \vartheta) \in \mathsf{Sets}\text{-}\mathcal{G}$ such that $Y = X \times \mathcal{G}_0$ and

$$\vartheta = \mathrm{pr}_2\colon Y = X \times \mathcal{G}_0 \longrightarrow \mathcal{G}_0$$
$$(x, b) \longrightarrow b.$$

Note that $\varsigma(x) = a$ for every $x \in X$ because $\mathcal{A}_0 = \{a\}$. We also want to extend the action $X\, {}_{\varsigma}\!\times_{\mathsf{t}} \mathcal{A}_1 \longrightarrow X$ to $Y\, {}_{\vartheta}\!\times_{\mathsf{t}} \mathcal{G}_1 \longrightarrow Y$. Let be $(x, b) \in Y$ and $(b, g, d) \in \mathcal{G}_1$: we set

$(x,b)(b,g,d) = (x(a,g,a),d)$. It's easy to prove that the action axioms are satisfied. Now let be $f\colon (X_1,\varsigma_1) \longrightarrow (X_2,\varsigma_2)$ a morphism in $\mathsf{Sets}\text{-}\mathcal{A}$: we define

$$\begin{aligned} G(f) = (f \times \mathrm{Id}_S)\colon (X_1 \times S, \mathrm{pr}_2) &\longrightarrow (X_2 \times S, \mathrm{pr}_2) \\ (x,b) &\longrightarrow (f(x),b)\,. \end{aligned}$$

It is easy to see that G is well defined and respects the properties of a functor. Thanks to Corollary $C.0.14$, it is now sufficient to show that F and G establish an ordinary equivalence of categories.

For each $(X,\varsigma) \in \mathsf{Sets}\text{-}\mathcal{A}$ we calculate

$$\begin{aligned} FG(X,\varsigma) &= F(X \times \mathcal{G}_0, \vartheta\colon X \times S \longrightarrow S) = \left(\vartheta^{-1}(a), \vartheta|_{\vartheta^{-1}(a)}\right) \\ &= (X \times \{\, a \,\}, \varsigma \times \mathrm{Id}_a) \cong (X,\varsigma)\,. \end{aligned}$$

Since the behaviour on the morphisms is obvious, we obtain $FG \cong \mathrm{Id}_{\mathsf{Sets}\text{-}\mathcal{A}}$.

For each $(X,\varsigma) \in \mathsf{Sets}\text{-}\mathcal{G}$ we have

$$GF(X,\varsigma) = G\left(\varsigma^{-1}(a), \varsigma|_{\varsigma^{-1}(a)}\right) = \left(\varsigma^{-1}(a) \times S, \mathrm{pr}_2\colon \varsigma^{-1}(a) \times S \longrightarrow S\right) := (Y,\vartheta)$$

We have to define a natural isomorphism $\alpha\colon GF \longrightarrow \mathrm{Id}_{\mathsf{Sets}\text{-}\mathcal{G}}$. Thus, given $(X,\varsigma) \in \mathsf{Sets}\text{-}\mathcal{G}$, we have to define $\alpha := \alpha_{(X,\varsigma)}\colon (Y,\vartheta) \longrightarrow (X,\varsigma)$ and prove that it is a homomorphism of right $\mathcal{G}$-set. According to the above notation, for each $(x,b) \in Y$ we set $\alpha(x,b) = x\,(a,1,b)$. We left to the reader to check that this is a $\mathcal{G}$-equivariant map turning commutative the following diagram of $\mathcal{G}$-sets

$$\begin{array}{ccc} Y_1 := \left(\varsigma_1^{-1}(a) \times S, \mathrm{pr}_2\right) & \xrightarrow{\alpha_{(X_1,\varsigma_1)}} & (X_1,\varsigma_1) \\ {\scriptstyle f|_{\varsigma_1^{-1}(a)} \times \mathrm{Id}_S}\downarrow & & \downarrow{\scriptstyle f} \\ Y_2 := \left(\varsigma_2^{-1}(a) \times S, \mathrm{pr}_2\right) & \xrightarrow{\alpha_{(X_2,\varsigma_2)}} & (X_2,\varsigma_2) \end{array}$$

and the proof is completed. □

In Chapter 4 we will explain, using Lemma 4.3.4, how to obtain another proof of Theorem 1.3.15.

Chapter 2

Conjugations and Burnside Theorem

In this chapter we will explore in depth, with several example and counterexamples, the concept of conjugation of two subgroupoids of a given groupoid, illustrating instances of new phenomenons that are not present in the group context. For example, unlike the classical case of groups, there can be two subgroupoids that are conjugated without being isomorphic (see Example 2.1.11). Another peculiar situation is described in Example 2.1.7, where we show that there is a groupoid with two conjugated subgroupoids such that not each isotropy group of the first subgroupoid is conjugated to each isotropy group of the second subgroupoid. Both Examples make manifest the complexity of the study of the "poset" of all subgroupoids using the conjugacy relation.

Next we will apply these concepts to an in-depth discussion of the Burnside Theorem for groupoids: first in the general case (Proposition 2.2.3) and, then, in the finite one (Theorem 2.2.7). The proof in the finite case is based on the fact that the table of marks, or the matrix of marks, of the groupoid under consideration can be shown to be a diagonal block matrix, where each block is a lower triangular matrix, which corresponds to the matrix of marks of an isotropy group type of the groupoid (see Proposition 2.1.21).

2.1 Conjugation of subgroupoids and the fixed points functors

We give in this section our first result concerning the conjugacy criteria between subgroupoids of a given groupoid, and we expound some illustrating examples. A discussion, from a categorical point of view, about fixed point subsets is provided here, as well as the description, under certain finiteness conditions, of the table of marks of groupoids.

2.1.1 The conjugation equivalence relation

In this subsection we clarify the notion of conjugation between subgroupoids. To illustrate the concept, crucial in the sequel, of conjugacy in the groupoid context, the following notion is needed.

Definition 2.1.1. Let $\mathcal{G}$ be a groupoid and let $\mathcal{H}$ and $\mathcal{K}$ be two subgroupoids of $\mathcal{G}$ with inclusion monomorphisms $\tau_{\mathcal{K}}\colon \mathcal{K} \longrightarrow \mathcal{G}$ and $\tau_{\mathcal{H}}\colon \mathcal{H} \longrightarrow \mathcal{G}$. We say that $\mathcal{K}$ and $\mathcal{H}$ are **conjugally equivalent subgroupoids** if there is an equivalence $F\colon \mathcal{K} \longrightarrow \mathcal{H}$, between their underlying categories, and a natural transformation $\mathfrak{g}\colon \tau_{\mathcal{H}}F \longrightarrow \tau_{\mathcal{K}}$.

Remark 2.1.2. It follows from the definition that $\mathfrak{g}$ is, actually, a natural isomorphism, because for each $a \in \mathcal{K}_0$, $\mathfrak{g}(a) \in \mathcal{G}_1$, thus it is an isomorphism, and the inverses of these arrows lead to the inverse of $\mathfrak{g}$.

We note that all the groupoids under consideration are assumed to be small, therefore the subgroupoids of a given groupoid actually constitute a set and, even more, a small category.

Lemma 2.1.3. *Being conjugally equivalent subgroupoid, as defined in Definition* 2.1.1*, induces an equivalence relation on the set of subgroupoids of* $\mathcal{G}$*.*

Proof. The fact the relation is reflexive is immediate: it's sufficient to take $F = \mathrm{Id}_{\mathcal{H}}$ and $\mathfrak{g} = \mathrm{Id}_F = \mathrm{Id}_{\mathrm{Id}_{\mathcal{H}}}$. Now let $\mathcal{H}$ and $\mathcal{K}$ be conjugally equivalent subgroupoids: there is an equivalence of categories $F \colon \mathcal{K} \longrightarrow \mathcal{H}$ and a natural transformation $\mathfrak{g} \colon \tau_{\mathcal{H}} F \longrightarrow \tau_{\mathcal{K}}$. Since F is an equivalence there is a functor $G \colon \mathcal{H} \longrightarrow \mathcal{K}$ such that there are natural isomorphisms $\varepsilon \colon FG \longrightarrow \mathrm{Id}_{\mathcal{H}}$ and $\eta \colon GF \longrightarrow \mathrm{Id}_{\mathcal{K}}$. The functor G and the natural transformation

$$\tau_{\mathcal{K}} G \xrightarrow{\ \mathfrak{g}^{-1} G\ } \tau_{\mathcal{H}} F G \xrightarrow{\ \tau_{\mathcal{H}} \varepsilon\ } \tau_{\mathcal{H}}$$

render $\mathcal{K}$ and $\mathcal{H}$ conjugally equivalent therefore the relation is transitive.

Now, to prove the transitivity, we consider subgroupoids $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{A}$ of $\mathcal{G}$ such that $\mathcal{H}$ and $\mathcal{K}$ are conjugally equivalent and such that $\mathcal{K}$ and $\mathcal{A}$ are too. This means that there are equivalences of categories $F \colon \mathcal{K} \longrightarrow \mathcal{H}$ and $L \colon \mathcal{A} \longrightarrow \mathcal{K}$, and natural transformations $\mathfrak{g} \colon \tau_{\mathcal{H}} F \longrightarrow \tau_{\mathcal{K}}$ and $\mathfrak{n} \colon \tau_{\mathcal{K}} L \longrightarrow \tau_{\mathcal{A}}$. The equivalence of categories $FL \colon \mathcal{A} \longrightarrow \mathcal{H}$ and the natural transformation

$$\tau_{\mathcal{H}} F L \xrightarrow{\ \mathfrak{g} L\ } \tau_{\mathcal{K}} L \xrightarrow{\ \mathfrak{n}\ } \tau_{\mathcal{A}}$$

imply that $\mathcal{H}$ and $\mathcal{A}$ are conjugally equivalent. □

Using elementary arguments, this definition is equivalent to say that there is a functor $F \colon \mathcal{K} \longrightarrow \mathcal{H}$, which is an equivalence of categories, such that there is a family $(g_b)_{b \in \mathcal{K}_0}$ as follows. For every $b \in \mathcal{K}_0$ it has to be $g_b \in \mathcal{G}(F(b), b)$ and for every arrow $d \colon b_1 \longrightarrow b_2$ in $\mathcal{K}$ it has to be $F(d) = g_{b_2}^{-1} d g_{b_1}$, which justifies the terminology. This will be explained in the proof $(iii) \Rightarrow (iv)$ of the following Theorem 2.1.4, which is the main result of this subsection.

Theorem 2.1.4. *Let* $\mathcal{H}$ *and* $\mathcal{K}$ *be two subgroupoids of a given groupoid* $\mathcal{G}$*. Then the following conditions are equivalent:*

(i) $(\mathcal{G}/\mathcal{H})^{\mathsf{R}} \cong (\mathcal{G}/\mathcal{K})^{\mathsf{R}}$ *as right* $\mathcal{G}$*-sets;*

(ii) There are morphisms of groupoids $F : \mathcal{K} \to \mathcal{H}$ *and* $G \colon \mathcal{H} \longrightarrow \mathcal{K}$ *together with two natural transformations* $\mathfrak{g} \colon \tau_{\mathcal{H}} F \longrightarrow \tau_{\mathcal{K}}$ *and* $\mathfrak{f} \colon \tau_{\mathcal{K}} G \longrightarrow \tau_{\mathcal{H}}$*.*

(iii) The subgroupoids $\mathcal{H}$ *and* $\mathcal{K}$ *are conjugally equivalent.*

(iv) There are families $(u_b)_{b \in \mathcal{K}_0}$ *and* $(g_b)_{b \in \mathcal{K}_0}$ *with* $u_b \in \mathcal{H}_0$ *and* $g_b \in \mathcal{G}(u_b, b)$ *for every* $b \in \mathcal{K}_0$*, such that:*

(a) for each $b_1, b_2 \in \mathcal{K}_0$ *we have* $g_{b_2}^{-1} \mathcal{K}(b_1, b_2)\, g_{b_1} = \mathcal{H}(u_{b_1}, u_{b_2})$*;*

(b) for each $u \in \mathcal{H}_0$ *there is* $z \in \mathcal{K}_0$ *such that* $\mathcal{H}(u_z, u) \neq \varnothing$*.*

(v) $(\mathcal{G}/\mathcal{H})^{\mathsf{L}} \cong (\mathcal{G}/\mathcal{K})^{\mathsf{L}}$ *as left* $\mathcal{G}$*-sets.*

Proof. (*i*) ⇒ (*ii*). Let us assume that there is a $\mathcal{G}$-equivariant isomorphism

$$\mathcal{F} \colon \left(\mathcal{G}/\mathcal{K}\right)^{\mathsf{R}} \longrightarrow \left(\mathcal{G}/\mathcal{H}\right)^{\mathsf{R}}$$

and for each class of the form $\mathcal{K}[(b, \iota_b)] \in \left(\mathcal{G}/\mathcal{K}\right)^{\mathsf{R}}$, denote by $\mathcal{H}[(a_b, g_b)]$ its image by $\mathcal{F}$. Thus, for each $b \in \mathcal{K}_0$, there could be many objects $a_b \in \mathcal{H}_0$ such that $\mathcal{F}\big(\mathcal{K}[(b, \iota_b)]\big) = \mathcal{H}[(a_b, g_b)]$ and, moreover, two such objects a_b and a'_b are isomorphic. Therefore, we can make a single choice by taking a representative element, which will be denoted by $F_0(b)$, and we will have $\mathcal{F}\big(\mathcal{K}[(b, \iota_b)]\big) = \mathcal{H}[(F_0(b), g_b)]$, according to this choice. As a consequence, we have a map $F_0 \colon \mathcal{K}_0 \longrightarrow \mathcal{H}_0$, which will be the object function of the functor we are going to built. On the other hand, considering the definition of $\mathcal{F}$, we have $\mathsf{s}(g_b) = b$ and $\mathsf{t}(g_b) = F_0(b)$, thus we obtain a family of arrows

$$\left(\mathfrak{g}_b \colon F_0(b) \longrightarrow b\right)_{b \in \mathcal{K}_0}$$

with $\mathfrak{g}_b = g_b^{-1}$ for every $b \in \mathcal{K}_0$. Now, given an arrow $k \colon b \longrightarrow b'$ in $\mathcal{K}_1$, we have $\mathcal{K}[(b, \iota_b)] = \mathcal{K}[(k \rightharpoonup (b, \iota_b))] = \mathcal{K}[(b', k)]$, which implies

$$\begin{aligned} \mathcal{H}[(F_0(b), g_b)] &= \mathcal{F}\big(\mathcal{K}[(b, \iota_b)]\big) = \mathcal{F}\big(\mathcal{K}[(b', k)]\big) = \mathcal{F}\big(\mathcal{K}[(b', \iota_{b'})]k\big) = \mathcal{F}\big(\mathcal{K}[(b', \iota_{b'})]\big)k \\ &= \mathcal{H}[(F_0(b'), g_{b'})]k = \mathcal{H}[(F_0(b'), g_{b'}k)] \end{aligned}$$

Therefore, there is a unique arrow $h \in \mathcal{H}(F_0(b), F_0(b'))$ such that we have the equality $hg_b = g_{b'}k$ in $\mathcal{G}_1$. In this way we can construct a map, at the level of arrows, $F_1 \colon \mathcal{K}_1 \longrightarrow \mathcal{H}_1$ with the property that, for any $k \in \mathcal{K}(b, b')$, we have $k\mathfrak{g}_b = \mathfrak{g}_{b'}F_1(k)$ as an equality in $\mathcal{G}_1$. The properties of groupoid action show that $F \colon \mathcal{K} \longrightarrow \mathcal{H}$ is actually a functor with a natural transformation $\mathfrak{g} \colon \tau_{\mathcal{H}}F \longrightarrow \tau_{\mathcal{K}}$, as claimed in (*ii*). To complete the proof of this implication, it's enough to use the inverse $\mathcal{G}$-equivariant map of $\mathcal{F}$ to construct, in a similar way, the functor G with the required properties.

(*ii*) ⇒ (*iii*) We have the following natural transformations:

$$\tau_{\mathcal{H}}FG \xrightarrow{\quad \mathfrak{g}G \quad} \tau_{\mathcal{K}}G \xrightarrow{\quad \mathfrak{f} \quad} \tau_{\mathcal{H}} = \tau_{\mathcal{H}}\,\mathrm{Id}_{\mathcal{H}}$$

and

$$\tau_{\mathcal{K}}GF \xrightarrow{\quad \mathfrak{f}F \quad} \tau_{\mathcal{H}}F \xrightarrow{\quad \mathfrak{g} \quad} \tau_{\mathcal{K}} = \tau_{\mathcal{K}}\,\mathrm{Id}_{\mathcal{K}}\,.$$

Since $\tau_{\mathcal{H}}$ and $\tau_{\mathcal{K}}$ are just inclusion functors, it is now clear that F and G establish an equivalence of categories.

(*iii*) ⇒ (*iv*). Let $F \colon \mathcal{K} \longrightarrow \mathcal{H}$ be the given equivalence of categories and $\mathfrak{g}$ the accompanying natural transformation. For each element $b \in \mathcal{K}_0$ we set $u_b = F(b) \in \mathcal{H}_0$ and $g_b = \mathfrak{g}_b \in \mathcal{G}(u_b, b)$. Condition (*a*) follows now from the naturality of $\mathfrak{g}$, while condition (*b*) from the fact that F is an essentially surjective functor.

(*iv*) ⇒ (*v*). We define the following map:

$$\begin{aligned} \psi \colon \left(\mathcal{G}/\mathcal{K}\right)^{\mathsf{L}} &\longrightarrow \left(\mathcal{G}/\mathcal{H}\right)^{\mathsf{L}} \\ [(g, b)]\mathcal{K} &\longrightarrow [(gg_b, u_b)]\mathcal{H} \end{aligned}$$

Condition (*a*) in the statement implies that ψ is a well defined and injective $\mathcal{G}$-equivariant map. Let us check that it is also surjective. Let be $[(p, u)]\,\mathcal{H} \in (\mathcal{G}/\mathcal{H})^{\mathsf{L}}$: thanks to the condition (*b*) there is $z \in \mathcal{K}_0$ such that there is $h \in \mathcal{H}\,(u_z, u)$. The situation is as follows:

$$z \xrightarrow{\ g_z^{-1}\ } u_z \xrightarrow{\ h\ } u \xrightarrow{\ p\ } \mathsf{t}(p).$$

Computing

$$\psi\Big(\big[(phg_z^{-1}, z)\big]\,\mathcal{K}\Big) = [(ph, u_z)]\,\mathcal{H} = [(p,u) \leftharpoondown h]\,\mathcal{H} = [(p,u)]\,\mathcal{H}$$

we obtain that ψ is surjective.

$(v) \Rightarrow (i)$. It's enough to use the isomorphism of categories between right $\mathcal{G}$-sets and left $\mathcal{G}$-sets. □

Definition 2.1.5. Let $\mathcal{H}$ and $\mathcal{K}$ be two subgroupoids of a given groupoid $\mathcal{G}$. We say that $\mathcal{H}$ and $\mathcal{K}$ are **conjugated subgroupoids** if one of the equivalent conditions in Theorem 2.1.4 is fulfilled.

Definition 2.1.6. Given a groupoid $\mathcal{G}$, let be $a, b \in \mathcal{G}_0$, H a subgroup of $\mathcal{G}^a$ and K a subgroup of $\mathcal{G}^b$. We say that H and K are **conjugated isotropy subgroups** if there is $d \in \mathcal{G}(a,b)$ such that $K = dHd^{-1}$.

Example 2.1.7. There is a groupoid $\mathcal{G}$ with not empty subgroupoids $\mathcal{H}$ and $\mathcal{K}$ which are conjugally equivalent and satisfy the following property: there are $x \in \mathcal{H}_0$ and $w \in \mathcal{K}_0$ such that $\mathcal{H}^x$ and $\mathcal{K}^w$ are not conjugated. Namely, let us consider a group G, with two distinct subgroups A and B which are not conjugated, and a set S with at least four elements x, y, z and w. Set $\mathcal{G} = \mathcal{G}_{G,S}$, as in Remark 1.1.13, we construct two subgroupoids $\mathcal{H}$ and $\mathcal{K}$ of $\mathcal{G}$, with only loops as arrows, in the following way. We set $\mathcal{H}_0 = \{\, x, z \,\}$, $\mathcal{K}_0 = \{\, y, w \,\}$,

$$\mathcal{H}^x = (x, A, x), \qquad \mathcal{H}^z = (z, B, z), \qquad \mathcal{K}^y = (y, A, y), \qquad \text{and} \qquad \mathcal{K}^w = (w, B, w)$$

where we made the abuse of notation $(x, A, x) = \{\, x \,\} \times A \times \{\, x \,\}$. We want to prove the condition (iv) of Theorem 2.1.4: to this purpose we set $u_y = x$ and $u_w = z$. We obtain $\mathcal{H}(u_y, x) = \mathcal{H}^x \neq \varnothing$ and $\mathcal{H}(u_w, z) = \mathcal{H}^z \neq \varnothing$ thus (b) is proved. Since all the arrows of $\mathcal{H}$ and $\mathcal{K}$ are loops we just have to prove that there are $g_y \in \mathcal{G}(u_y, y) = \mathcal{G}(x, y)$ and $g_w \in \mathcal{G}(u_w, w) = \mathcal{G}(z, w)$ such that

$$g_y^{-1}\mathcal{K}(y,y)\,g_y = \mathcal{H}(u_y, u_y) \qquad \text{and} \qquad g_w^{-1}\mathcal{K}(w,w)\,g_y = \mathcal{H}(u_w, u_w)\,.$$

To this end we set $g_y = (y, 1, x)$ and $g_w = (w, 1, z)$ and we calculate

$$g_y^{-1}\mathcal{K}(y,y)\,g_y = g_y^{-1}\mathcal{K}^y g_y = (x,1,y)(y,A,y)(y,1,x) = (x,A,x) = \mathcal{H}^x = \mathcal{H}(u_y,u_y)$$

and

$$g_w^{-1}\mathcal{K}(w,w)\,g_w = g_w^{-1}\mathcal{K}^w g_w = (z,1,w)(w,B,w)(w,1,z) = (z,B,z) = \mathcal{H}^z = \mathcal{H}(u_w,u_w)$$

proving (a) and, thus, the claim. Now, by contradiction, let be $d\colon w \longrightarrow x$ such that $\mathcal{K}^w = d^{-1}\mathcal{H}d$. Of course, there has to be $g \in G$ such that $d = (x, g, w)$. Calculating

$$(w, B, w) = \mathcal{K}^w = d^{-1}\mathcal{H}^x d = \big(x, g^{-1}x\big)\,(x, A, X)\,(x, g, w) = \big(w, g^{-1}Ag, w\big)$$

we obtain $g^{-1}Ag = B$, which is a contradiction.

Proposition 2.1.8. *Given a groupoid $\mathcal{G}$, let's consider two conjugated subgroupoids $\mathcal{H}$ and $\mathcal{K}$. Then $\mathcal{H}$ is transitive if and only if $\mathcal{K}$ is transitive. Moreover, in this case, every isotropy group of $\mathcal{H}$ is conjugated to every isotropy group of $\mathcal{K}$.*

Proof. Let's assume $\mathcal{H}$ transitive and let's consider $b_1, b_2 \in \mathcal{K}_0$: for $i \in \{1, 2\}$ there are $u_{b_i} \in \mathcal{H}_0$ and $g_{b_i} \in \mathcal{G}(u_{b_i}, b_i)$ such that

$$g_{b_2}^{-1}\mathcal{K}(b_1, b_1)\, g_{b_1} = \mathcal{H}(u_{b_1}, u_{b_2})$$

therefore $\mathcal{K}(b_1, b_2) \neq \varnothing$. If we assume $\mathcal{K}$ to be transitive, we can obtain $\mathcal{H}$ to be transitive using (iv) of Theorem 2.1.4 with the two subgroupoids exchanged.

Now let's assume $\mathcal{H}$ and $\mathcal{K}$ to be transitive (i.e., connected) and let be $u \in \mathcal{H}_0$ and $v \in \mathcal{K}_0$. Thanks to (iv) of Theorem 2.1.4 there are $u_v \in \mathcal{H}_0$ and $g_v \in \mathcal{G}(u_v, v)$ such that $g_v^{-1}\mathcal{K}^v g_v = \mathcal{H}^{u_v}$. Since $\mathcal{H}$ is transitive there is $h \in \mathcal{H}(u_v, u)$ such that $\mathcal{H}^{u_v} = h^{-1}\mathcal{H}^u h$ therefore

$$\left(g_v h^{-1}\right)^{-1}\mathcal{K}^v\left(g_v h^{-1}\right) = h g_v^{-1}\mathcal{K}^v g_v h^{-1} = \mathcal{H}^u,$$

which shows that $\mathcal{H}^u$ and $\mathcal{K}^v$ are conjugated, and finishes the proof. □

Corollary 2.1.9. *Now let's consider $a, b \in \mathcal{G}_0$, $H \leqslant \mathcal{G}^a$ and $K \leqslant \mathcal{G}^b$. Then H and K induce subgroupoids $\mathcal{H}$ and $\mathcal{K}$ of $\mathcal{G}$ such that $\mathcal{H}_0 = \{a\}$, $\mathcal{H}_1 = \mathcal{H}^a = H$ and $\mathcal{K}_0 = \{b\}$, $\mathcal{K}_1 = \mathcal{K}^b = K$. Moreover, $\mathcal{H}$ and $\mathcal{K}$ are conjugated subgroupoids if and only if H and K are conjugated isotropy subgroups.*

Proof. It follows from Proposition 2.1.8. □

Remark 2.1.10. The conjugation in the context of groupoids is a much more complex phenomenon then in the case of groups. For instance, there is a difference between a global conjugation, between subgroupoids, and a local conjugation, between isotropy subgroups. In this regard, Example 2.1.7 shows that there is a groupoid $\mathcal{G}$, with conjugated subgroupoids $\mathcal{H}$ and $\mathcal{K}$, such that, given $x \in \mathcal{H}_0$ and $w \in \mathcal{K}_0$, $\mathcal{H}^x$ and $\mathcal{K}^w$ are not conjugated. Proposition 2.1.8 shows that this behaviour cannot happen if $\mathcal{H}$ and $\mathcal{K}$ are transitive, therefore two subgroupoids with a single object are conjugated if and only if their isotropy groups are conjugated (Corollary 2.1.9).

Another interesting phenomenon is that, in contrast with the classical case of groups, conjugated subgroupoids are not necessarily isomorphic. The issue is that conjugated subgroupoids have equivalent underlying categories, which means that they are not necessarily isomorphic as groupoids. We will show in depth the difference between the conjugacy relation and the isomorphism relation between subgroupoids in Example 2.1.11.

It is straightforward to see that the conjugation induces an equivalence relation $\sim_{\mathrm{C}}$ on the set $\mathcal{S}_{\mathcal{G}}$ of all subgroupoids of $\mathcal{G}$ with only one object. The equivalence class of a given element $\mathcal{H}$ in $\mathcal{S}_{\mathcal{G}}$ will be denoted by $[\mathcal{H}]$. Notice that any subgroup of an isotropy group of $\mathcal{G}$ can be considered as a subgroupoid with only one object (see Definition 1.1.6) and, consequently, as an element of $\mathcal{S}_{\mathcal{G}}$. The converse is, by definition, also true. We denote by $\mathrm{rep}(\mathcal{S}_{\mathcal{G}})$ a set of representative elements of $\mathcal{S}_{\mathcal{G}}$ modulo the equivalence relation $\sim_{\mathrm{C}}$. It is clear that this equivalence relation can be extended to the set of all subgroupoids of $\mathcal{G}$.

Example 2.1.11. In contrast with the classical case of groups the conjugacy relation differs form the isomorphism relation. Here we give examples of two subgroupoids which are isomorphic but not conjugated, as well as two subgroupoids which are conjugated but not isomorphic.

- Let us show that there is a groupoid $\mathcal{G}$ with two subgroupoids $\mathcal{H}$ and $\mathcal{K}$ which are isomorphic but not conjugated. Namely, given a set $J \neq \varnothing$, let's consider $A, B \subseteq J$ such that $A \neq \varnothing \neq B$ and $|A| = |B|$. Given an abelian group G, the relation of conjugacy is the same of the relation of equality, thus if we consider two distinct and isomorphic subgroups H and K of G they are not conjugated. In particular, given an abelian group U, possible choices are $G = U \times U$, $H = U \times 1$ and $K = 1 \times U$. Now let us consider the induced groupoids $\mathcal{G} = \mathcal{G}_{G,J}$, $\mathcal{H} = \mathcal{G}_{H,A}$ and $\mathcal{K} = \mathcal{G}_{K,B}$ (see Example 1.1.12 and Remark 1.1.13 for the notations). Thanks to Remark 1.1.13, we know that the groupoids $\mathcal{H}$ and $\mathcal{K}$ are isomorphic. By contradiction, let's assume that the subgroupoids $\mathcal{H}$ and $\mathcal{K}$ are conjugated. Then, by Theorem 2.1.4(iv), there are families $(a_j)_{j\in B}$ and $(g_j)_{j\in B}$ such that $a_j \in A$, $g_j \in \mathcal{G}(a_j, j)$ and $g_j^{-1}\mathcal{K}^j g_j = \mathcal{H}^{a_j}$ for each $j \in B$. By definition, for each $j \in B$ there is $\eta_j \in G$ such that $g_j = (j, \eta_j, a_j)$ thus

$$\begin{aligned}\{ a_j \} \times H \times \{ a_j \} = \mathcal{H}^{a_j} = g_j^{-1}\mathcal{K}^j g_j &= \left(a_j, \eta_j^{-1}, j\right) (\{ j \} \times K \times \{ j \}) (j, \eta_j, a_j) \\ &= \{ a_j \} \times \eta_j^{-1} K \eta_j \times \{ a_j \}.\end{aligned}$$

As a consequence we obtain $H = \eta_j^{-1} K \eta_j$, which is a contradiction with the above choices made for G, H and K.

- Let us check that there is a groupoid $\mathcal{G}$ with two subgroupoids $\mathcal{B}$ and $\mathcal{A}$ which are conjugated but not isomorphic. To this end, we consider two subsets A and B of a given set J such that $\varnothing \neq A \subseteq B \subseteq J$, and we construct the groupoids of pairs $\mathcal{G} = (J \times J, J)$, $\mathcal{B} = (B \times B, B)$ and $\mathcal{A} = (A \times A, A)$ (see Example 1.1.11 for the definition). Since $A \neq \varnothing$ we can choose $a \in A$ and, for every $b \in B$, we define the families $(u_b)_{b\in B}$ and $(g_b)_{b\in B}$ as follows:

$$u_b = \begin{cases} b, & b \in A \\ a, & b \in B\backslash A \end{cases} \qquad \text{and} \qquad g_b = \begin{cases} (b, b), & b \in A \\ (b, a), & b \in B\backslash A. \end{cases}$$

We have to check that for each $b_1, b_2 \in \mathcal{K}_0$, we have $g_{b_2}^{-1}\mathcal{K}(b_1, b_2)\, g_{b_1} = \mathcal{H}(u_{b_1}, u_{b_2})$ but this condition is trivially satisfied in a groupoid of pairs. Now for each $\alpha \in A$ we have to check that there is $b \in B$ such that $\mathcal{A}(u_b, \alpha) \neq \varnothing$ but this is obvious: it is enough to choose $b = \alpha$. Lastly, the groupoids $\mathcal{A}$ and $\mathcal{B}$ are not isomorphic if $|A| \lneq |B|$.

Lemma 2.1.12. *Given a groupoid $\mathcal{G}$, let's consider a subgroupoid $\mathcal{I}$ of $\mathcal{G}$ with a single object a, that is, $\mathcal{I}_0 = \{a\}$. Set $I = \mathcal{I}_1 \leqslant \mathcal{G}^a$, we can construct a $\mathcal{G}^a$-equivariant injective map*

$$\begin{aligned}\mathcal{G}^a/I &\longrightarrow (\mathcal{G}/\mathcal{I})^{\mathsf{R}} \\ Ip &\longmapsto \mathcal{I}[(a, p)].\end{aligned}$$

Moreover, if we assume that $\mathcal{G}^a$ and $\mathcal{G}_0$ are finite sets, then the set of right cosets $(\mathcal{G}/\mathcal{I})^{\mathsf{R}}$ must also be finite.

Proof. We have

$$(\mathcal{G}/\mathcal{I})^{\mathsf{R}} = \{ \mathcal{I}[(a, p)] \mid p \in \mathcal{G}_1,\, \mathsf{t}(p) = a \}$$

therefore, if we denote with $\mathcal{G}^{\langle a \rangle}$ the connected component of $\mathcal{G}$ containing a, using the characterization of transitive groupoids (see Remark 1.1.13), we obtain

$$\left|(\mathcal{G}/\mathcal{I})^{\mathsf{R}}\right| \leqslant \left|\left(\mathcal{G}^{\langle a \rangle}\right)_1\right| = \left|\left(\mathcal{G}^{\langle a \rangle}\right)_0\right| \times |\mathcal{G}^a| \times \left|\left(\mathcal{G}^{\langle a \rangle}\right)_0\right| \leqslant |(\mathcal{G})_0| \times |\mathcal{G}^a| \times |\mathcal{G}_0| < \infty,$$

and this proves the claim. □

The following lemma will be used in subsequent sections.

Lemma 2.1.13. *Let $\mathcal{G}$ be a groupoid and consider two elements $\mathcal{H}$ and $\mathcal{K}$ in* $\mathrm{rep}\left(\mathcal{S}_{\mathcal{G}}\right)$. *Then we have the following properties.*

(i) Let be $a, b \in \mathcal{G}_0$ such that $\mathcal{H}_0 = \{\, a \,\}$ and $\mathcal{K}_0 = \{\, b \,\}$, and let's define the set

$$T = \left\{ g \in \mathcal{G}(b,a) \;\middle|\; \mathcal{K}_1 \subseteq g^{-1}\mathcal{H}_1 g, \text{ where } \mathcal{H}_1 \leqslant \mathcal{G}^a \text{ and } \mathcal{K}_1 \leqslant \mathcal{G}^b \right\}.$$

Then there is a bijection

$$\mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}\left((\mathcal{G}/\mathcal{K}))^{\mathsf{R}}, (\mathcal{G}/\mathcal{H}))^{\mathsf{R}} \right) \longrightarrow T.$$

In particular, the set of $\mathcal{G}$-equivariant maps $\mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}\left((\mathcal{G}/\mathcal{K}))^{\mathsf{R}}, (\mathcal{G}/\mathcal{H}))^{\mathsf{R}} \right)$ *is a not empty set if and only if there is $g \in \mathcal{G}(b,a)$ such that $\mathcal{K}_1 \subseteq g^{-1}\mathcal{H}_1 g$, where $\mathcal{H}_1 \leqslant \mathcal{G}^a$ and $\mathcal{K}_1 \leqslant \mathcal{G}^b$.*

(ii) Assume that all isotropy groups of $\mathcal{G}$ are finite groups. Then the following implication

$$\mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}\left((\mathcal{G}/\mathcal{K})^{\mathsf{R}}, (\mathcal{G}/\mathcal{H})^{\mathsf{R}} \right) \neq \varnothing \implies \mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}\left((\mathcal{G}/\mathcal{H})^{\mathsf{R}}, (\mathcal{G}/\mathcal{K})^{\mathsf{R}} \right) = \varnothing$$

holds true, whenever $\mathcal{H} \neq \mathcal{K}$.

Proof. (i) Assume that we have a $\mathcal{G}$-equivariant map $F\colon (\mathcal{G}/\mathcal{K})^{\mathsf{R}} \longrightarrow (\mathcal{G}/\mathcal{H})^{\mathsf{R}}$, and let be $g \in \mathcal{G}(b,a)$ such that

$$F\big(\mathcal{K}[(b, \iota_b)]\big) = \mathcal{H}[(a,g)] \in (\mathcal{G}/\mathcal{H}))^{\mathsf{R}}.$$

For every $k \in \mathcal{K}_1$ we compute

$$\begin{aligned}\mathcal{H}[(a,g)] &= F\big(\mathcal{K}[(b,\iota_b)]\big) = F\big(\mathcal{K}[(b,k)]\big) = F\big(\mathcal{K}[(b,\iota_b)]\, k\big) = F\big(\mathcal{K}[(b,\iota_b)]\,\big)\, k \\ &= \mathcal{H}[(a,g)]\, k = \mathcal{H}[(a,gk)].\end{aligned}$$

This means that there exists $h \in \mathcal{H}_1$ such that $hg = gk$. Therefore, we obtain $\mathcal{K}_1 \subseteq g^{-1}\mathcal{H}_1 g$. The inverse function, from T to $\mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}\left((\mathcal{G}/\mathcal{K}))^{\mathsf{R}}, (\mathcal{G}/\mathcal{H}))^{\mathsf{R}} \right)$, is built in a similar way.

(ii) Assume by contradiction that we also have that $\mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}\left((\mathcal{G}/\mathcal{H}))^{\mathsf{R}}, (\mathcal{G}/\mathcal{K}))^{\mathsf{R}} \right) \neq \varnothing$. Applying the first part, we get that there exist $g_1 \in \mathcal{G}(b,a)$ and $g_2 \in \mathcal{G}(a,b)$ such that $\mathcal{K}_1 \subseteq g_1^{-1}\mathcal{H}_1 g_1$ and $\mathcal{H}_1 \subseteq g_2^{-1}\mathcal{K}_1 g_2$. Thus $\mathcal{H}_1$ and $\mathcal{K}_1$ have the same cardinality as subsets of $\mathcal{G}_1$. Since $\mathcal{G}^b$ is a finite group, we obtain $\mathcal{K}_1 = g_1^{-1}\mathcal{H}_1 g_1$. This means that $\mathcal{H}$ and $\mathcal{K}$ are conjugated, that is, they represent the same class in $\mathcal{S}_{\mathcal{G}}/\sim_{\mathrm{C}}$, which is a contradiction because, by hypothesis, $\mathcal{H} \neq \mathcal{K}$ as elements in $\mathrm{rep}\,(\mathcal{S}_{\mathcal{G}})$. □

2.1.2 Fixed points subsets of groupoid-sets and the table of marks of finite groupoids

This subsection deals with the fixed point subsets of groupoid-sets under subgroupoid actions and discusses their functorial properties. Moreover we give the analogue notion of what is known in the classical theory as the table of marks attached to a given (finite) groupoid [Bur11].

Definition 2.1.14. Given a groupoid $\mathcal{G}$, let $\mathcal{H}$ be a subgroupoid of $\mathcal{G}$ and let (X, ς) be a right $\mathcal{G}$-set. We define the **set of fixed points by $\mathcal{H}$ in X** as:

$$X^{\mathcal{H}} = \left\{ x \in X \;\middle|\; \begin{array}{l} \varsigma(x) \in \mathcal{H}_0 \\ \forall h \in \mathcal{H}_1 \text{ such that } \varsigma(x) = \mathsf{t}(h), \text{ we have that } xh = x \end{array} \right\}.$$

Notice that not any subgroupoid is allowed to stabilize the elements of a given right $\mathcal{G}$-set (X, ς). More precisely, the set of fixed point $X^{\mathcal{H}}$ can be introduced only for those subgroupoids $\mathcal{H}$ satisfying the condition $\mathcal{H}_0 \cap \varsigma(X) \neq \varnothing$ and possessing at most one object. If this is not the case, then it implicitly stands from the definition that we are setting $X^{\mathcal{H}} = \varnothing$.

Note that, if $\mathcal{H}$ and $\mathcal{H}'$ are conjugated subgroupoids of $\mathcal{G}$ with only one object (see Corollary 2.1.9), then $X^{\mathcal{H}}$ and $X^{\mathcal{H}'}$ are clearly in bijection (see Corollary 2.1.16).

If $F\colon (X, \varsigma) \longrightarrow (Y, \vartheta)$ is a $\mathcal{G}$-equivariant map, then F induces a function

$$F^{\mathcal{H}}\colon X^{\mathcal{H}} \longrightarrow Y^{\mathcal{H}}$$
$$x \longrightarrow F^{\mathcal{H}}(x) = F(x).$$

It is clear that, in this way, we obtain a functor

$$(-)^{\mathcal{H}} : \mathsf{Sets}\text{-}\mathcal{G} \longrightarrow \mathsf{Sets} \tag{2.1.1}$$

from the category of $\mathcal{G}$-sets to the category of sets. Therefore, if F is an isomorphism of $\mathcal{G}$-sets, the function $F^{\mathcal{H}}$ is a bijection. Since there is another functor

$$\mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}\Big((\mathcal{G}/\mathcal{H})^{\mathsf{R}}, -\Big) : \mathsf{Sets}\text{-}\mathcal{G} \longrightarrow \mathsf{Sets}, \tag{2.1.2}$$

we can ask ourselves if there is a natural transformation between these two functors. It turns out that the answer is affirmative, as stated in the following result, where we also state their compatibility with the coproduct.

Proposition 2.1.15. *Let $\mathcal{H}$ be a subgroupoid of $\mathcal{G}$ with only one objects. Then we have the following natural isomorphism*

$$\varepsilon\colon \mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}\Big((\mathcal{G}/\mathcal{H})^{\mathsf{R}}, -\Big) \longrightarrow (-)^{\mathcal{H}}$$

defined, for each $(X, \varsigma) \in \mathsf{Sets}\text{-}\mathcal{G}$, in the following way:

$$\varepsilon(X, \varsigma)\colon \mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}\Big((\mathcal{G}/\mathcal{H})^{\mathsf{R}}, X\Big) \longrightarrow X^{\mathcal{H}}$$
$$f \longrightarrow f\Big(\mathcal{H}\left[(a, \iota_a)\right]\Big),$$

where $a \in \mathcal{G}_0$ is such that $\mathcal{H}_0 = \{\, a \,\}$. Furthermore, the functors of formulas (2.1.1) *and* (2.1.2) *commutes with arbitrary coproducts. In particular, given a disjoint union $X = \biguplus_{i \in I} X_i$ of right $\mathcal{G}$-sets, we have a natural bijection*

$$X^{\mathcal{H}} \simeq \biguplus_{i \in I} X_i^{\mathcal{H}}. \tag{2.1.3}$$

Proof. To prove that ε is a natural transformation, we have to prove that, for each $\mathcal{G}$-equivariant map $F\colon (X, \varsigma) \longrightarrow (Y, \theta)$, the following diagram is commutative:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}\big((\mathcal{G}/\mathcal{H})^{\mathsf{R}}, X\big) & \xrightarrow{\ \varepsilon(X,\varsigma)\ } & X^{\mathcal{H}} \\
{\scriptstyle \mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}((\mathcal{G}/\mathcal{H})^{\mathsf{R}}, F)}\downarrow & & \downarrow{\scriptstyle F^{\mathcal{H}}} \\
\mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}\big((\mathcal{G}/\mathcal{H})^{\mathsf{R}}, Y\big) & \xrightarrow{\ \varepsilon(Y,\vartheta)\ } & Y^{\mathcal{H}}.
\end{array}$$

For each $f \in \mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}\big((\mathcal{G}/\mathcal{H})^{\mathsf{R}}, X\big)$ we calculate:

$$\begin{aligned} \varepsilon\,(Y,\vartheta)\,\mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}\Big((\mathcal{G}/\mathcal{H})^{\mathsf{R}}, F\Big)\,(f) &= \varepsilon\,(Y,\vartheta)\,(Ff) = Ff\,(\mathcal{H}\,[(a,\iota_a)]) \\ = F^{\mathcal{H}}\Big(f\,(\mathcal{H}\,[(a,\iota_a)])\Big) &= F^{\mathcal{H}}\varepsilon\,(X,\varsigma)\,(f). \end{aligned}$$

We want to define a natural transformation

$$\eta\colon\; (-)^{\mathcal{H}} \longrightarrow \mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}\Big((\mathcal{G}/\mathcal{H})^{\mathsf{R}}, -\Big)$$

such that, for each $(X,\varsigma) \in \mathsf{Sets}\text{-}\mathcal{G}$,

$$\begin{aligned} \eta\,(X,\varsigma)\colon X^{\mathcal{H}} &\longrightarrow \mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}\Big((\mathcal{G}/\mathcal{H})^{\mathsf{R}}, X\Big) \\ x &\longrightarrow \eta\,(X,\varsigma)\,(x) = \Big[\mathcal{H}[(a,g)] \mapsto xg\Big]. \end{aligned}$$

Note that, since $x \in X^{\mathcal{H}}$, we have $\varsigma(x) = a = \mathsf{t}(g)$ and xg is well defined. For each $f \in \mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}\big((\mathcal{G}/\mathcal{H})^{\mathsf{R}}, X\big)$ we calculate

$$\begin{aligned} \eta\varepsilon(f) = \eta f\Big(\mathcal{H}\,[(a,\iota_a)]\Big) &= \Big(\mathcal{H}\,[(a,g)] \to f\big(\mathcal{H}\,[(a,\iota_a)]\,\big)g\Big) \\ &= \Big(\mathcal{H}\,[(a,g)] \to f\big(\mathcal{H}\,[(a,g)]\,\big)\Big) = f \end{aligned}$$

and, for each $x \in X^{\mathcal{H}}$,

$$\varepsilon\eta(x) = \varepsilon\Big(\mathcal{H}[(a,g)] \mapsto xg\Big) = x\iota_a = x.$$

As a consequence ε is a natural isomorphism.

Lastly, it is clear that the functor $(-)^{\mathcal{H}}$ commutes with coproducts and, since ε is a natural isomorphism, so does $\mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}\big((\mathcal{G}/\mathcal{H})^{\mathsf{R}}, -\big)$. □

Corollary 2.1.16. *Let $\mathcal{H}$ and $\mathcal{H}'$ be two subgroupoids of $\mathcal{G}$ both with only one object. Assume that we have a $\mathcal{G}$-equivariant map $F\colon (\mathcal{G}/\mathcal{H})^{\mathsf{R}} \longrightarrow (\mathcal{G}/\mathcal{H}')^{\mathsf{R}}$. Then, for any right $\mathcal{G}$-set (X,ς), we have a commutative diagram:*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}\big((\mathcal{G}/\mathcal{H}')^{\mathsf{R}}, X\big) & \xrightarrow{\;\cong\;} & X^{\mathcal{H}'} \\ {\scriptstyle \mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}(F,X)}\downarrow & & \downarrow{\scriptstyle X^F} \\ \mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}\big((\mathcal{G}/\mathcal{H})^{\mathsf{R}}, X\big) & \xrightarrow{\;\cong\;} & X^{\mathcal{H}}. \end{array} \tag{2.1.4}$$

In particular, if $\mathcal{H}$ and $\mathcal{H}'$ are conjugated, we have a bijection $X^{\mathcal{H}} \simeq X^{\mathcal{H}'}$.

Proof. Set $\mathcal{H}_0 = \{a\}$, $\mathcal{H}'_0 = \{a'\}$ and let $g \in \mathcal{G}(a,a')$ be the arrow attached to the $\mathcal{G}$-equivariant map F, that is, g is determined by the equality $F\big(\mathcal{H}[(a,\iota_a)]\big) = \mathcal{H}'[(a',g)]$. Then the stated map $X^F\colon X^{\mathcal{H}'} \longrightarrow X^{\mathcal{H}}$ is a given by $x \mapsto xg$. The desired diagram commutativity is now clear from the the involved maps. The particular case is a direct consequence of the first claim. □

Remark 2.1.17. Let $\mathcal{G}$ be a groupoid and consider as before the set $\mathcal{S}_{\mathcal{G}}$ of all subgroupoids with only one object. One can define a category whose objects set is $\mathcal{S}_{\mathcal{G}}$ and, given two objects $\mathcal{H},\mathcal{H}' \in \mathcal{S}_{\mathcal{G}}$, the set of arrows from $\mathcal{H}'$ to $\mathcal{H}$ is the set of all $\mathcal{G}$-equivariant maps $\mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}\big((\mathcal{G}/\mathcal{H})^{\mathsf{R}}, (\mathcal{G}/\mathcal{H}')^{\mathsf{R}}\big)$. This category is also denoted by $\mathcal{S}_{\mathcal{G}}$. In this way, the set $\mathrm{rep}(\mathcal{S}_{\mathcal{G}})$ is then identified with the skeleton of the category $\mathcal{S}_{\mathcal{G}}$. On the other hand, for any right $\mathcal{G}$-set (X,ς) we obtain as, in Corollary 2.1.16, a functor $\mathcal{H} \longrightarrow X^{\mathcal{H}}$, which is naturally isomorphic to the functor $\mathcal{H} \longrightarrow \mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}\big((\mathcal{G}/\mathcal{H})^{\mathsf{R}}, X\big)$.

Next we discuss the cardinality of the fixed point subsets of right $\mathcal{G}$-cosets by subgroupoids with a single object, that is, by elements of $\mathcal{S}_{\mathcal{G}}$. First, we give the definition of the notion of finite groupoid, which we will deal with.

Definition 2.1.18. Given a **groupoid** $\mathcal{G}$, we say that $\mathcal{G}$ is **strongly finite** if its set of arrows $\mathcal{G}_1$ is finite. This obviously implies that $\mathcal{G}_0$ and $\pi_0(\mathcal{G})$ are finite sets. We say that $\mathcal{G}$ is **locally strongly finite** provided that the category $\mathcal{S}_{\mathcal{G}}$ of Remark 2.1.17 is **skeletally finite** and each of the isotropy groups of $\mathcal{G}$ is a finite set. Here skeletally finite means that $\mathcal{S}_{\mathcal{G}}/\sim_{\mathrm{C}}$ is a finite set.

Remark 2.1.19. Evidently, any strongly finite groupoid is locally strongly finite. On the other hand , it could happens that a groupoid has each of its isotropy groups finite, but $\mathrm{rep}(\mathcal{S}_{\mathcal{G}})$ is not. To see this, it suffices to look at the class of not transitive groupoids with trivial isotropy groups and with an infinite number of connected components. More precisely, one can take a groupoid of the form $\biguplus_{i \in I}\mathcal{G}_i$, where I is an infinite set and each of the groupoids $\mathcal{G}_i$ is one of those expounded in Example 1.1.11.

Let $\mathcal{H}$ be a subgroupoid of $\mathcal{G}$. From now on we will denote by $\mathcal{G}/\mathcal{H} := (\mathcal{G}/\mathcal{H})^{\mathsf{R}}$ the set of right $\mathcal{G}$-cosets.

Given a groupoid $\mathcal{G}$, we let's fix a set $\mathrm{rep}(\mathcal{G}_0)$ of representative objects modulo the regular action of $\mathcal{G}$ over itself, by using either the source or the target. In other words, $\mathrm{rep}\,(\mathcal{G}_0)$ is a set of objects representing the set of connected components $\pi_0(\mathcal{G})$ of $\mathcal{G}$.

Proposition 2.1.20. *Let $\mathcal{G}$ be a groupoid: for each $a \in \pi_0(\mathcal{G})$, we denote by $\mathcal{G}^{\langle a\rangle}$ the connected component subgroupoid of $\mathcal{G}$ containing a (this is clearly a transitive groupoid) and we consider, in a canonical way, $\mathcal{S}_{\mathcal{G}^{\langle a\rangle}}/\sim_{\mathrm{C}}$ as a subset of $\mathcal{S}_{\mathcal{G}}/\sim_{\mathrm{C}}$. Then:*

(i) We have a disjoint union

$$\mathcal{S}_{\mathcal{G}}/\sim_{\mathrm{C}} \;=\; \biguplus_{a \in \mathrm{rep}(\mathcal{G}_0)} \Big(\mathcal{S}_{\mathcal{G}^{\langle a\rangle}}/\sim_{\mathrm{C}}\Big).$$

(ii) If $\mathcal{G}$ is locally strongly finite, then $\pi_0(\mathcal{G})$ and $\mathrm{rep}\,(\mathcal{G}_0)$ are finite sets and so is each of the quotient sets $\mathcal{S}_{\mathcal{G}^{\langle a\rangle}}/\sim_{\mathrm{C}}$.

(iii) If $\mathcal{G}$ is locally strongly finite, then the set of all $\mathcal{G}$-equivariant maps $\mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}\,(\mathcal{G}/\mathcal{H}, \mathcal{G}/\mathcal{K})$ is finite, for every $\mathcal{H}, \mathcal{K} \in \mathcal{S}_{\mathcal{G}}$.

Proof. (i) and (ii). Straightforward.

(iii). By hypothesis, $\mathcal{G}$ is locally strongly finite, therefore each of the isotropy groups of $\mathcal{G}$ is a finite set. As a consequence, thanks to the characterization in Remark 1.1.13 of transitive groupoids, since every groupoid is the coproduct of its own connected (transitive) components, we obtain that $\mathcal{G}(b, a)$ is finite, or empty, for each $a, b \in \mathcal{G}_0$. Now the thesis follows from Lemma 2.1.13(i). □

Given a locally strongly finite groupoid $\mathcal{G}$, let us fix a set of representatives $\mathrm{rep}(\mathcal{S}_{\mathcal{G}})$ and a set of representatives of the quotient set $\pi_0(\mathcal{G})$, whose elements we call $a_1, \ldots, a_n \in \mathcal{G}_0$. According to Proposition 2.1.20 (i), we can write

$$\mathrm{rep}\,(\mathcal{S}_{\mathcal{G}}) \;=\; \biguplus_{i=1}^{n} \mathrm{rep}\,\big(\mathcal{S}_{\mathcal{G}^{\langle a_i\rangle}}\big)\,, \tag{2.1.5}$$

where each of the $\mathcal{G}^{\langle a_i \rangle}$ is a transitive groupoid (i.e., the connected component containing a_i). Furthermore, once fixed the choice of rep$(\mathcal{S}_{\mathcal{G}})$, we can consider the following family of not negative integers:

$$\mathsf{m}_{(\mathcal{H},\mathcal{K})} \;=\; |\mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}\left(\mathcal{G}/\mathcal{H}, \mathcal{G}/\mathcal{K}\right)|, \qquad \forall\, \mathcal{H}, \mathcal{K} \in \mathrm{rep}\left(\mathcal{S}_{\mathcal{G}}\right),$$

and by Proposition 2.1.15, we know that these entries are

$$\mathsf{m}_{(\mathcal{H},\mathcal{K})} = \left|(\mathcal{G}/\mathcal{K})^{\mathcal{H}}\right|, \quad \forall\, \mathcal{H}, \mathcal{K} \in \mathrm{rep}\left(\mathcal{S}_{\mathcal{G}}\right).$$

This, in conjunction with Lemma 2.1.13, shows that the natural numbers $\{\mathsf{m}_{(\mathcal{H},\mathcal{K})}\}_{\mathcal{H},\mathcal{K} \in \mathrm{rep}\left(\mathcal{S}_{\mathcal{G}}\right)}$ satisfy the following conditions:

$$\mathsf{m}_{(\mathcal{H},\mathcal{K})}\,\mathsf{m}_{(\mathcal{K},\mathcal{H})} \;=\; 0,\ \forall\, \mathcal{H} \neq \mathcal{K} \quad \text{ and } \quad \mathsf{m}_{(\mathcal{H},\mathcal{H})} \;\neq\; 0,\ \forall\, \mathcal{H}, \tag{2.1.6}$$

where $\mathcal{H} = \mathcal{K}$ in rep $\left(\mathcal{S}_{\mathcal{G}}\right)$ means that $\mathcal{H}$ and $\mathcal{K}$ are conjugated (or isomorphic as objects in the category $\mathcal{S}_{\mathcal{G}}$ of Remark 2.1.17). The table that we want to construct in the sequel, which will be formed by those coefficients (where $\mathcal{H}$ denotes the row position and $\mathcal{K}$ denotes the column one), is what we can call, in analogy with the classical case [Bur11, §180], the **table of marks of the groupoid** $\mathcal{G}$.

Proposition 2.1.21 (The table of marks of a finite groupoid)**.** *Let $\mathcal{G}$ be a locally strongly finite groupoid. Then the fixed set of representatives* rep$(\mathcal{S}_{\mathcal{G}})$*, can be endowed with a total order $\preceq$ satisfying the following property: for every $\mathcal{H}, \mathcal{K} \in$ rep$(\mathcal{S}_{\mathcal{G}})$, we have*

$$\mathcal{H} \preceq \mathcal{K} \Longrightarrow \begin{cases} \mathsf{m}_{(\mathcal{H},\mathcal{K})} = 0 & \text{if } \mathcal{H} \neq \mathcal{K} \\ \mathsf{m}_{(\mathcal{H},\mathcal{K})} \neq 0 & \text{if } \mathcal{H} = \mathcal{K}. \end{cases}$$

In particular, under this choice of ordering, the table (or matrix) of marks of $\mathcal{G}$ has the following form:

$$\left(\mathsf{m}_{(\mathcal{H},\mathcal{K})}\right)_{\mathcal{H},\mathcal{K} \in \mathrm{rep}\left(\mathcal{S}_{\mathcal{G}}\right)} \;=\; \left(\begin{array}{c|c|c} M_1 & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \ddots & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & M_n \end{array}\right),$$

where n is the number of connected components of $\mathcal{G}$ and each of the matrix M_i, $i = 1, \cdots, n$ is a lower triangular matrix with each diagonal entry different from zero.

Proof. To construct this total order on the finite set rep$\left(\mathcal{S}_{\mathcal{G}}\right)$, one proceeds as follows. If the handled groupoid $\mathcal{G}$ has only one object, then we are in the classical situation of a finite group and the total ordering is exactly given by comparing the cardinalities of representative subgroups of this group modulo the conjugation relation. The details are expounded in [Bur11, pages 236 and 237], and the result is one of the matrices M_i's. Regarding the case when $\mathcal{G}$ is a transitive groupoid, one can employ, for instance, Theorem 1.3.15 to reduce this case to the particular one of finite groups and proceed as in the classical case.

As for the general case, one uses equation (2.1.5) to decomposes rep$\left(\mathcal{S}_{\mathcal{G}}\right)$ into a finite disjoint union of finite sets $\left\{ \mathrm{rep}\left(\mathcal{S}_{\mathcal{G}^{\langle a_i \rangle}}\right) \right\}_{i=1,\cdots,n}$, where n is the number of connected components of $\mathcal{G}$. In this way one can extended the total ordering of each piece rep$\left(\mathcal{S}_{\mathcal{G}^{\langle a_i \rangle}}\right)$ to the whole set rep$\left(\mathcal{S}_{\mathcal{G}}\right)$, since each of the $\mathcal{G}^{\langle a_i \rangle}$'s is a transitive groupoid (following, for example, the order $1 < 2 < \cdots < n$ between the pieces). The resulting matrix (or the table of marks) of $\mathcal{G}$ will be a diagonal block-matrix whose blocks correspond to the matrix of $\mathcal{G}^{(a_i)}$ and, such that, outside of these blocks, only zeros will appear. Given two distinct elements $\mathcal{H}, \mathcal{K} \in \operatorname{rep}(\mathcal{S}_{\mathcal{G}})$ with $\mathcal{H}_0 = \{a\}$ and $\mathcal{K}_0 = \{b\}$ such that a and b are not connected, it is necessary, by Corollary 2.1.13(i), to have $\mathsf{m}_{(\mathcal{H},\mathcal{K})} = \mathsf{m}_{(\mathcal{K},\mathcal{H})} = 0$. Thus, the whole matrix will also be lower triangular with non zero entries in the diagonal as stated. □

2.2 Burnside Theorem for groupoid-sets: general and finite cases

Before introducing the ghost function for (finite) groupoids, an analogue of Burnside Theorem for right groupoid-sets will be accomplished in this section. The classical situation of groups is described as follows. Take two right G-sets X and Y and assume that their fixed point subsets under any subgroup are in bijection, that is, $X^H \simeq Y^H$, for any subgroup H of G. Under this assumption, in general X and Y are not necessarily isomorphic as right G-sets. The main objective of the Burnside Theorem (see [Bur11, Theorem I, page 238] or, for instance, [Bou10a, Theorem 2.4.5]) is to seek further conditions under which X and Y become isomorphic as right G-sets.

From a categorical point of view, one can assume, in the previous situation, a stronger hypothesis, namely, that the functors $H \longrightarrow X^H$ and $H \longrightarrow Y^H$ are naturally isomorphic (see Remark 2.1.17 for the definition of these functors). Nevertheless, this is equivalent to say that the functors $\{e\} \longrightarrow X^{\{e\}}$ and $\{e\} \longrightarrow Y^{\{e\}}$ are naturally isomorphic (here we're taking the full subcategory of the category of subgroups of G, with only one object e, the neutral element of G) which, as we will see below, is equivalent to say that X and Y are isomorphic as right G-sets. In this direction, it is not clear, at least to us, whether the condition $X^H \simeq Y^H$, for every subgroup H of G, implies that the functors $H \longrightarrow X^H$ and $H \longrightarrow Y^H$ are naturally isomorphic (it seems that, without passing through the classical Burnside's theorem, this is not known even for the finite case, that is, when G, X and Y are finite sets).

All this suggests that, in the context of groupoid-sets, one should treat separately the case when the fixed point subsets functors are naturally isomorphic.

2.2.1 The general case: two $\mathcal{G}$-sets with natural bijections between fixed points subsets

Let us first explain what is the meaning of the natural bijections, between the fixed points subsets, that was mentioned above.

Definition 2.2.1. Let (X, ς) and (Y, ϑ) be two right $\mathcal{G}$-sets. We say that (X, ς) **and** (Y, ϑ) **have naturally the same fixed points subsets**, provided there is a natural bijection $X^{\mathcal{H}} \simeq Y^{\mathcal{H}}$, for every subgroupoid $\mathcal{H}$ of $\mathcal{G}$ with only one object. This means that we have a commutative diagram

$$\begin{array}{ccc} X^{\mathcal{H}'} & \xrightarrow{X^F} & X^{\mathcal{H}} \\ \simeq\downarrow & & \downarrow\simeq \\ Y^{\mathcal{H}'} & \xrightarrow{Y^F} & Y^{\mathcal{H}} \end{array} \tag{2.2.1}$$

for any $\mathcal{G}$-equivariant map $F\colon \mathcal{G}/\mathcal{H} \longrightarrow \mathcal{G}/\mathcal{H}'$ between cosets of subgroupoids with only one object, where X^F and Y^F are the maps given as in the proof of Corollary 2.1.16.

Remark 2.2.2. In the case of groups, if we assume that two right G-sets have naturally the same fixed points subsets as in Definition 2.2.1, then this, in particular, implies that $X^{\{e\}} \simeq Y^{\{e\}}$ in a natural way (e is the neutral element of G). Thus, for any $g \in G$, the right translation map $x \mapsto xg$ from G to G gives arise to a G-equivariant map $F\colon G/\{e\} \longrightarrow G/\{e\}$ which, by the commutativity of diagram (2.2.1), shows that X and Y are isomorphic as right G-sets. Thus, in the group context, two right G-sets are isomorphic if and only if they have naturally the same fixed points subsets. The case of groupoids is a bit more elaborate, as we will see in the sequel.

Using the previous definition we can show the following result.

Proposition 2.2.3. *Let $\mathcal{G}$ be a groupoid and let's consider two right $\mathcal{G}$-sets (X, ς) and (Y, ϑ). Then the following statements are equivalent.*

(i) (X, ς) and (Y, ϑ) have naturally the same fixed points subsets under the action of each one object subgroupoid (Definition 2.2.1);

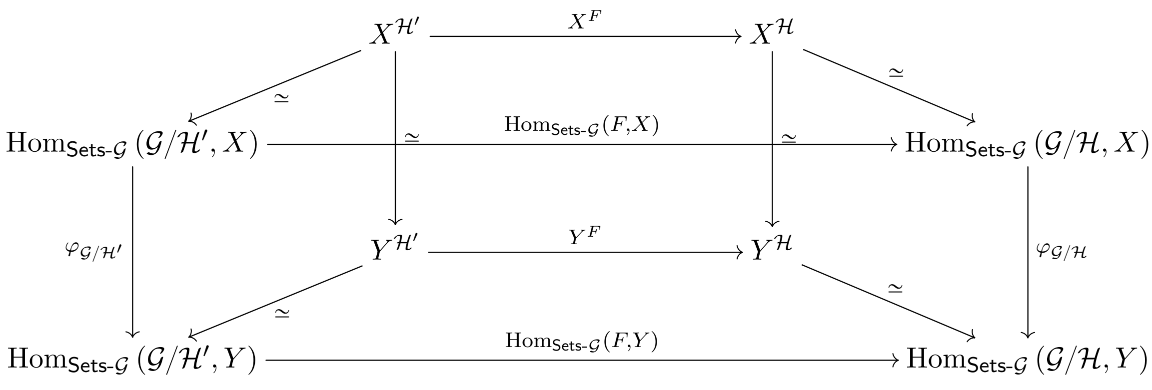
(ii) (X, ς) and (Y, ϑ) are isomorphic as $\mathcal{G}$-sets.

Proof. $(ii) \Rightarrow (i)$. It is clearly deduced from Proposition 2.1.15, Corollary 2.1.16 and Remark 2.1.17.

$(i) \Rightarrow (ii)$. Given such an $\mathcal{H}$ we have, by Proposition 2.1.15, the following a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}\left(\mathcal{G}/\mathcal{H}, X\right) & \xrightarrow{\ \cong\ } & X^{\mathcal{H}} \\
{\scriptstyle \varphi_{\mathcal{G}/\mathcal{H}}}\downarrow & & \downarrow{\scriptstyle \simeq} \\
\mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}\left(\mathcal{G}/\mathcal{H}, Y\right) & \xrightarrow{\ \cong\ } & Y^{\mathcal{H}}.
\end{array}$$

Let us check that φ_{-} establishes a natural transformation (isomorphism indeed) over the class of right $\mathcal{G}$-sets which are right cosets by one object subgroupoids. Thus, given another subgroupoid with only one object $\mathcal{H}'$ together with a $\mathcal{G}$-equivariant maps $F\colon \mathcal{G}/\mathcal{H} \longrightarrow \mathcal{G}/\mathcal{H}'$



we need to show that the front rectangle is commutative. However, this follows immediately from Corollary 2.1.16, since we already know by assumptions that the rear square commutes, and the desired natural isomorphism φ_{-} is derived.

Now, let us consider an arbitrary $\mathcal{G}$-set (Z, ζ). We know from Corollary 1.2.11 that

$$Z \cong \biguplus_{z \in \mathrm{rep}_{\mathcal{G}}(Z)} \mathcal{G}/\mathrm{Stab}_{\mathcal{G}}\left(z\right)$$

and, for each subgroupoid $\mathcal{H}$ of $\mathcal{G}$ with a single object, we have the following commutative diagram.

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}(Z, X) & \xrightarrow{\ \simeq\ } & \prod_{z \in \mathrm{rep}_{\mathcal{G}}(Z)} \mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}(\mathcal{G}/\operatorname{Stab}_{\mathcal{G}}(z), X) \\
{\scriptstyle \varphi_Z} \downarrow & & \cong \downarrow {\scriptstyle \prod_{z \in \mathrm{rep}_{\mathcal{G}}(Z)} \varphi_{\mathcal{G}/\operatorname{Stab}_{\mathcal{G}}(z)}} \\
\mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}(Z, Y) & \xrightarrow{\ \simeq\ } & \prod_{z \in \mathrm{rep}_{\mathcal{G}}(Z)} \mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}(\mathcal{G}/\operatorname{Stab}_{\mathcal{G}}(z), Y)
\end{array}
$$

This leads to a natural isomorphism

$$\mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}(Z, X) \simeq \mathrm{Hom}_{\mathsf{Sets}\text{-}\mathcal{G}}(Z, Y)$$

for each $\mathcal{G}$-set (Z, ζ). As a consequence we obtain $(X, \varsigma) \cong (Y, \vartheta)$ as right $\mathcal{G}$-sets, as claimed. $\square$

Remark 2.2.4. Combining Propositions 2.2.3 and 2.1.15, we deduce that two $\mathcal{G}$-sets are isomorphic if and only if their fixed points sets are in a natural bijection, in the sense of Definition 2.2.1. It could happen that two $\mathcal{G}$-sets have bijective fixed points subsets but not in a natural way, that is, there is no choice of a family of bijections that turns the diagrams (2.2.1) commutative (to the best of our knowledge, this is not even known for the case of groups). Since we do have neither a counterexample nor a complete proof for the fact that these diagrams are always commutative, once a bijection is given between the fixed points subsets, it is wise to consider the proof of the case when diagrams (2.2.1) do not commute. Of course, in this case, the proof of Proposition 2.2.3 does not work and the converse of the previous equivalence fails. Finiteness conditions should be imposed, in order to provide the proof of the converse implication. This seems to explain the notable difficulty of the classical Burnside theory.

2.2.2 The finite case: Two finite $\mathcal{G}$-sets with bijective fixed points subsets

Next, we will try to find sufficient conditions under which two finite $\mathcal{G}$-sets, whose fixed points subsets have the same cardinality, should be isomorphic; this will be the Burnside Theorem we are looking for.

Given groupoid $\mathcal{G}$, recall that $\mathcal{S}_{\mathcal{G}}$ denotes its set of subgroupoids with only one object and $\sim_{\mathrm{C}}$ is the equivalence relation on this set given by conjugation.

Lemma 2.2.5. *Let (X, ς) be a right $\mathcal{G}$-set and let's consider $x, x' \in X$. Then if x and x' belong to the same orbit,* $\operatorname{Stab}_{\mathcal{G}}(x)$ *and* $\operatorname{Stab}_{\mathcal{G}}(x')$ *are conjugated subgroupoids of $\mathcal{G}$. Furthermore, the canonical map $X \longrightarrow \mathcal{S}_{\mathcal{G}}$ sending $x \mapsto \operatorname{Stab}_{\mathcal{G}}(x) \leqslant \mathcal{G}^{\varsigma(x)}$, which factors through the quotient sets $X/\mathcal{G} \longrightarrow \mathcal{S}_{\mathcal{G}}/\sim_{\mathrm{C}}$, leads to a well defined map*

$$
\begin{aligned}
\wp_X \colon \mathrm{rep}_{\mathcal{G}}(X) &\longrightarrow \mathcal{S}_{\mathcal{G}}/\sim_{\mathrm{C}} \\
x &\longrightarrow [\operatorname{Stab}_{\mathcal{G}}(x)]
\end{aligned}
$$

Proof. Let be $x, x' \in X$ such that they belong to the same orbit. Thanks to Proposition 1.2.10 the right cosets $\mathcal{G}/\operatorname{Stab}_{\mathcal{G}}(x)$ and $\mathcal{G}/\operatorname{Stab}_{\mathcal{G}}(x')$ are isomorphic $\mathcal{G}$-sets therefore, using Theorem 2.1.4, we deduce that $\operatorname{Stab}_{\mathcal{G}}(x)$ and $\operatorname{Stab}_{\mathcal{G}}(x')$ are conjugated subgroupoids of $\mathcal{G}$. As a consequence the map $X/\mathcal{G} \longrightarrow \mathcal{S}_{\mathcal{G}}/\sim_{\mathrm{C}}$ that sends the orbit $[x]\,\mathcal{G}$ to $[\mathrm{Stab}_{\mathcal{G}}(x)]$ is well defined, but this clearly implies that also the map $\wp_X$ is well defined, proving the thesis. □

Using this lemma, one can construct the following map with values in the natural numbers: given a right $\mathcal{G}$-set (X, ς) (with a countable underlying set X), we define

$$\mathsf{a}_X \colon \mathcal{S}_{\mathcal{G}} \longrightarrow \mathbb{N}, \qquad \mathcal{H} \longrightarrow \left|\wp_X^{-1}([\mathcal{H}])\right|. \tag{2.2.2}$$

Of course we get that $\mathsf{a}_X(\mathcal{H}) = 0$ if no representative element x in $\mathrm{rep}_{\mathcal{G}}(X)$ has its orbit $\mathrm{Orb}_{\mathcal{G}}(x)$ isomorphic to the coset $\mathcal{G}/\mathcal{H}$, or equivalently, if its stabilizer $\mathrm{Stab}_{\mathcal{G}}(x)$ is not conjugated with the subgroupoid $\mathcal{H}$.

For any set I and Z any right $\mathcal{G}$-set we denote by $Z^{(I)}$ the disjoint union of I copies of Z, that is, the coproduct, in the category of right $\mathcal{G}$-sets, of Z with itself I-times. If I has a finite cardinal, say $n \in \mathbb{N}$, then we denote this coproduct by $n\,Z$, with the convention $0\,Z = (\varnothing, \varnothing)$.

We know, thanks to Corollary 1.2.11, that the category of right $\mathcal{G}$-sets has a cogenerator object given by the right $\mathcal{G}$-set

$$\biguplus_{\mathcal{H} \in \mathcal{S}_{\mathcal{G}}} \mathcal{G}/\mathcal{H},$$

that is, the disjoint union of all the cosets of the form $\mathcal{G}/\mathcal{H}$, where $\mathcal{H} \in \mathcal{S}_{\mathcal{G}}$. So given a right $\mathcal{G}$-set (X, ς) with a countable underlying set (or it set of representatives modulo the $\mathcal{G}$-action is countable), then we have a monomorphism of right $\mathcal{G}$-sets

$$\jmath \colon X \hookrightarrow \biguplus_{\mathcal{H} \in \mathcal{S}_{\mathcal{G}}} (\mathcal{G}/\mathcal{H})^{(I_{\mathcal{H},X})},$$

for a family of sets $(I_{\mathcal{H},X})_{\mathcal{H} \in \mathcal{S}_{\mathcal{G}}}$, whose image can be written as follows. First, we have the following isomorphism of $\mathcal{G}$-sets:

$$\biguplus_{\mathcal{H} \in \mathcal{S}_{\mathcal{G}}} (\mathcal{G}/\mathcal{H})^{(I_{\mathcal{H},X})} \;\cong\; \biguplus_{\mathcal{K} \in \mathrm{rep}\left(\mathcal{S}_{\mathcal{G}}\right)} (\mathcal{G}/\mathcal{K})^{(J_{\mathcal{K},X})},$$

where the cardinality of each of the sets $J_{\mathcal{K},X}$'s is of the form $|J_{\mathcal{K},X}| = |I_{\mathcal{K},X}|\,|[\mathcal{K}]|$, where $|[\mathcal{K}]|$ is the cardinal of the equivalence class represented by $\mathcal{K}$ in the quotient set $\mathcal{S}_{\mathcal{G}}/\sim_{\mathrm{C}}$.

Given an element $\mathcal{K} \in \mathrm{rep}\left(\mathcal{S}_{\mathcal{G}}\right)$, define the following natural numbers:

$$\mathsf{n}_{\mathcal{K}}(X) \;=\; \begin{cases} |J_{\mathcal{K},X}|, & \text{if } \jmath(X) \cap (\mathcal{G}/\mathcal{K})^{(J_{\mathcal{K},X})} \neq \varnothing \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.2.6. *Keep the above notations. Then, for every element $\mathcal{K} \in \mathrm{rep}\left(\mathcal{S}_{\mathcal{G}}\right)$, we have*

$$\mathsf{a}_X(\mathcal{K}) \leqslant \mathsf{n}_{\mathcal{K}}(X) \quad \textit{and} \quad \Big(\mathsf{a}_X(\mathcal{K}) = 0 \Leftrightarrow \mathsf{n}_{\mathcal{K}}(X) = 0\Big).$$

Furthermore, we have isomorphisms of right $\mathcal{G}$-sets:

$$X \;\cong\; \jmath(X) \;\cong\; \biguplus_{\mathcal{K} \in \mathrm{rep}\left(\mathcal{S}_{\mathcal{G}}\right)} \mathsf{a}_X(\mathcal{K})\,\mathcal{G}/\mathcal{K}, \tag{2.2.3}$$

where the map a_X is the one of equation (2.2.2).

Proof. It is immediate. □

Observe that if the underlying set X of (X, ς) is finite, then there are finitely many elements $\mathcal{K} \in \operatorname{rep}(\mathcal{S}_{\mathcal{G}})$ with the property $\mathsf{a}_X(\mathcal{K}) \neq 0$. Thus, in the finite $\mathcal{G}$-sets case, the support sets $\{ \mathcal{K} \in \operatorname{rep}(\mathcal{S}_{\mathcal{G}}) | \, \mathsf{a}_X(\mathcal{K}) \neq 0 \}$ have to be finite as well.

The subsequent theorem is the main result of this section.

Theorem 2.2.7 (Burnside Theorem). *Let $\mathcal{G}$ be a locally strongly finite groupoid (Definition 2.1.18). Consider two finite right $\mathcal{G}$-sets (X, ς) and (Y, ϑ). Then the following statements are equivalent.*

(1) The right $\mathcal{G}$-sets (X, ς) and (Y, ϑ) are isomorphic.

(2) For each subgroupoid $\mathcal{H}$ of $\mathcal{G}$ with a single object, we have that

$$\left|X^{\mathcal{H}}\right| = \left|Y^{\mathcal{H}}\right|.$$

In particular, this applies to any strongly finite groupoid.

Proof. (1) ⇒ (2). Follows from Propositions 2.1.15 or 2.2.3.

(2) ⇒ (1). Using the isomorphisms given in equation (2.2.3), we know that

$$X \cong \biguplus_{\mathcal{K} \in \operatorname{rep}(\mathcal{S}_{\mathcal{G}})} \mathsf{a}_X(\mathcal{K})\, \mathcal{G}/\mathcal{K} \qquad \text{and} \qquad Y \cong \biguplus_{\mathcal{K} \in \operatorname{rep}(\mathcal{S}_{\mathcal{G}})} \mathsf{a}_Y(\mathcal{K})\, \mathcal{G}/\mathcal{K}.$$

By hypothesis it is assumed that $\left|X^{\mathcal{H}}\right| = \left|Y^{\mathcal{H}}\right|$ for each subgroupoid $\mathcal{H}$ of $\mathcal{G}$ with a single object. Applying the bijections of equation (2.1.3) to the previous isomorphisms, we get the following equalities

$$\sum_{\mathcal{K} \in \operatorname{rep}(\mathcal{S}_{\mathcal{G}})} \mathsf{a}_X(\mathcal{K}) \left|(\mathcal{G}/\mathcal{K})^{\mathcal{H}}\right| = \left| \biguplus_{\mathcal{K} \in \operatorname{rep}(\mathcal{S}_{\mathcal{G}})} \mathsf{a}_X(\mathcal{K})\, (\mathcal{G}/\mathcal{K})^{\mathcal{H}} \right| = \left| \left(\biguplus_{\mathcal{K} \in \operatorname{rep}(\mathcal{S}_{\mathcal{G}})} \mathsf{a}_X(\mathcal{K})\, \mathcal{G}/\mathcal{K} \right)^{\mathcal{H}} \right|$$
$$= \left|X^{\mathcal{H}}\right| = \left|Y^{\mathcal{H}}\right| = \sum_{\mathcal{K} \in \operatorname{rep}(\mathcal{S}_{\mathcal{G}})} \mathsf{a}_Y(\mathcal{K}) \left|(\mathcal{G}/\mathcal{K})^{\mathcal{H}}\right|,$$

for every subgroupoid $\mathcal{H} \in \mathcal{S}_{\mathcal{G}}$. Therefore, for each $\mathcal{H} \in \operatorname{rep}(\mathcal{S}_{\mathcal{G}})$, we have the equality

$$\sum_{\mathcal{K} \in \operatorname{rep}(\mathcal{S}_{\mathcal{G}})} (\mathsf{a}_X(\mathcal{K}) - \mathsf{a}_Y(\mathcal{K})) \left|(\mathcal{G}/\mathcal{K})^{\mathcal{H}}\right| = 0. \tag{2.2.4}$$

Now by Proposition 2.1.21 the entries $\{ \mathsf{m}_{(\mathcal{H},\mathcal{K})} \}$ form a lower triangular square matrix, which is non-singular. It follows that, in the system (2.2.4), we have $\mathsf{a}_X(\mathcal{H}) = \mathsf{a}_Y(\mathcal{H})$ for each $\mathcal{H} \in \operatorname{rep}(\mathcal{S}_{\mathcal{G}})$. Therefore (X, ς) and (Y, ϑ) are isomorphic as right $\mathcal{G}$-sets. Lastly, the particular claim is clear and this finishes the proof. □

An important consequence of the Burnside Theorem is the injectivity of the ghost map of a groupoid (see Corollary 5.2.2). Moreover, it also implies a sort of cancellative property, with respect to the internal operation of disjoint union $\uplus$.

Corollary 2.2.8. *Given a locally strongly finite groupoid $\mathcal{G}$, let be (X,ς), (Y,ϑ) and (Z,ζ) finite right $\mathcal{G}$-sets such that there is an isomorphism of right $\mathcal{G}$-sets of the form*

$$(X,\varsigma) \uplus (Z,\zeta) \cong (Y,\vartheta) \uplus (Z,\zeta)\,.$$

Then we have an isomorphism $(X,\varsigma) \cong (Y,\vartheta)$ of $\mathcal{G}$-sets.

Proof. Given $\mathcal{H} \in \mathcal{S}_{\mathcal{G}}$, using the bijection of equation (2.1.3), we obtain

$$\left|(X,\varsigma)^{\mathcal{H}}\right| + \left|(Z,\zeta)^{\mathcal{H}}\right| = \left|(Y,\vartheta)^{\mathcal{H}}\right| + \left|(Z,\zeta)^{\mathcal{H}}\right|$$

therefore $\left|(X,\varsigma)^{\mathcal{H}}\right| = \left|(Y,\vartheta)^{\mathcal{H}}\right|$. As a consequence, thanks to Theorem 2.2.7, we get $(X,\varsigma) \cong (Y,\vartheta)$ as right $\mathcal{G}$-sets. □

Corollary 2.2.8 gives us a strong property on the behaviour of $\mathcal{G}$-sets that turns out to be extremely useful. For example, it will be crucial to prove an important property of the Burnside ring of a groupoid, as explained in Remark 5.1.6.

Chapter 3

Mackey formula for bisets over groupoids

The main goal of this chapter is to extend the Mackey formula obtained Bouc about group-bisets (see [Bou10a, Lemma 2.3.24]) to the context of groupoid-bisets. Since the proof of this formula is very technical and for the sake of completeness, we decided to include all the steps and most of the details of the proof. Moreover, in Subsection 3.3.2, we will provide an elementary application of this formula to groupoids of equivalence relations (see Example 1.1.11).

3.1 Preliminaries

Before proving the Mackey formula for Groupoids, we have to introduce some particular groupoid-bisets and prove their properties. We will also have to define a specific kind of product of two subgroupoids (see Definition 3.1.4).

Let $\mathcal{G}$ and $\mathcal{H}$ be two groupoids and $\mathcal{L}$ a subgroupoid of the product $\mathcal{H} \times \mathcal{G}$. Consider the set of equivalence classes $\left(\frac{\mathcal{H}\times\mathcal{G}}{\mathcal{L}}\right)^{\mathsf{L}}$ as in equation (1.2.3). An element in this set is an equivalence class of a fourfold element

$$(h, g, u, v) \in \mathcal{L}(\mathcal{H} \times \mathcal{G})^{\tau} = \Big(\mathcal{H}_1 \times \mathcal{G}_1\Big)\, {}_{\mathsf{s}}\!\times_{\tau_0} \mathcal{L}_0$$

where $\tau \colon \mathcal{L} \hookrightarrow \mathcal{H} \times \mathcal{G}$ is the inclusion functor, that is,

$$[(h, g, u, v)]\mathcal{L} = \left\{ \Big(hh_1, gg_1, \mathsf{s}(h_1), \mathsf{s}(g_1)\Big) \in \mathcal{L}(\mathcal{H} \times \mathcal{G})^{\tau} \;\middle|\; (h_1, g_1) \in \mathcal{L}_1 \text{ with } \begin{array}{l} \mathsf{t}(h_1) = \mathsf{s}(h) \\ \mathsf{t}(g_1) = \mathsf{s}(g) \end{array} \right\}.$$

Lemma 3.1.1. *Given two groupoids $\mathcal{G}$ and $\mathcal{H}$, let $\mathcal{L}$ be a subgroupoid of $\mathcal{H} \times \mathcal{G}$. Then the left $\mathcal{L}$-coset*

$$X = \left(\frac{\mathcal{H} \times \mathcal{G}}{\mathcal{L}}\right)^{\mathsf{L}}$$

is an $(\mathcal{H}, \mathcal{G})$-biset with structure maps

$$\begin{aligned} \vartheta \colon X &\longrightarrow \mathcal{H}_0 \\ [(h, g, u, v)]\mathcal{L} &\longmapsto \mathsf{t}\,(h) \end{aligned} \qquad \textit{and} \qquad \begin{aligned} \varsigma \colon X &\longrightarrow \mathcal{G}_0 \\ [(h, g, u, v)]\mathcal{L} &\longmapsto \mathsf{t}\,(g)\,, \end{aligned}$$

left action

$$\begin{aligned} \lambda \colon \mathcal{H}_1\, {}_{\mathsf{s}}\!\times_{\vartheta} X &\longrightarrow X \\ (h_1, [(h, g, u, v)]\mathcal{L}) &\longmapsto [(h_1 h, g, u, v)]\mathcal{L} \end{aligned}$$

and right action

$$\begin{aligned}\rho\colon X\,{}_{\varsigma}\!\times_{\mathsf{t}}\mathcal{G}_1 &\longrightarrow X\\ \Big([(h,g,u,v)]\mathcal{L}, g_1\Big) &\longmapsto \Big[\big(h, g_1^{-1}g, u, v\big)\Big]\mathcal{L}.\end{aligned}$$

Proof. Let us first check that ϑ and ς are well defined maps. Given two representatives of the same equivalence class $[(h,g,u,v)]\mathcal{L} = [(h',g',u',v')]\mathcal{L}$, by Lemma 1.2.8, we know that $(h',g')^{-1}(h,g) \in \mathcal{L}_1$, from which we obtain $\mathsf{t}(h') = \mathsf{t}(h)$ and $\mathsf{t}(g') = \mathsf{t}(g)$. Regarding λ and ρ, let's take $h_1 \in \mathcal{H}_1$ and $g_1 \in \mathcal{G}_1$ such that $\mathsf{s}(h_1) = \vartheta\big([(h,g,u,v)]\mathcal{L}\big) = \mathsf{t}(h)$, $\mathsf{t}(g_1) = \varsigma\big([(h,g,u,v)]\mathcal{L}\big) = \mathsf{t}(g)$ and $[(h,g,u,v)]\mathcal{L} = [(h',g',u',v')]\mathcal{L}$. We have

$$\begin{aligned}\big(h_1h',\ g_1^{-1}g'\big)^{-1}\big(h_1h,\ g_1^{-1}g\big) &= \big(h'^{-1}h_1^{-1}h_1h,\ g'^{-1}g_1g_1^{-1}g\big) = \big(h'^{-1}h,\ g'^{-1}g\big)\\ &= \big(h',g'\big)^{-1}(h,g) \in \mathcal{L}_1\end{aligned}$$

which shows that

$$\Big[\big(h_1h, g_1^{-1}g, u, v\big)\Big]\mathcal{L} = \Big[\big(h_1h', g_1^{-1}g', u', v'\big)\Big]\mathcal{L}.$$

Therefore, λ and ρ are well defined. The verification that (X,ϑ,ς) is a biset is now easy and is left to the reader. □

Lemma 3.1.2. *Given two groupoids $\mathcal{H}$ and $\mathcal{G}$, let (X,ϑ,ς) be an $(\mathcal{H},\mathcal{G})$-biset and take $x \in X$. We define*

$$(\mathcal{L}_x)_1 = \Big\{ (h,g) \in \mathcal{H}\times\mathcal{G} \Big|\ \mathsf{s}(h) = \vartheta(x),\ \mathsf{t}(g) = \varsigma(x),\ hx = xg \Big\} \tag{3.1.1}$$

and

$$(\mathcal{L}_x)_0 = \Big\{ (\vartheta(x), \varsigma(x)) \Big\}. \tag{3.1.2}$$

Then $\mathcal{L}_x$ is a subgroupoid of the groupoid $\mathcal{H}\times\mathcal{G}$.

Proof. It is immediate, since by using the first axiom of a biset, we know that, for every $(h,g) \in (\mathcal{L}_x)_1$, we have

$$\vartheta(x) = \vartheta(xg) = \vartheta(hx) = \mathsf{t}(h) \qquad \text{and} \qquad \varsigma(x) = \varsigma(hx) = \varsigma(xg) = \mathsf{s}(g).$$

Thus, $\mathcal{L}_x$ is a subgroup of the isotropy group $(\mathcal{H}\times\mathcal{G})^{(\vartheta(x),\varsigma(x))}$ and then a subgroupoid with only one object $\{(\vartheta(x), \varsigma(x))\}$. □

Proposition 3.1.3. *Given two groupoids $\mathcal{H}$ and $\mathcal{G}$, let X be an $(\mathcal{H},\mathcal{G})$-biset and take $x \in X$. We define:*

$$\begin{aligned}(\mathcal{L}_x)_0 = (\mathcal{K}_x)_0 &= \Big\{ (\vartheta(x), \varsigma(x)) \Big\},\\ (\mathcal{L}_x)_1 &= \Big\{ (h,g) \in \mathcal{H}\times\mathcal{G} \Big|\ hx = xg,\quad \vartheta(x) = \mathsf{s}(h),\quad \mathsf{t}(g) = \varsigma(x) \Big\},\\ (\mathcal{K}_x)_1 &= \Big\{ (h,g) \in \mathcal{H}\times\mathcal{G} \Big|\ hxg = x,\quad \vartheta(x) = \mathsf{s}(h),\quad \mathsf{t}(g) = \varsigma(x) \Big\},\end{aligned}$$

that is, $\mathcal{K} = \mathrm{Stab}_{(\mathcal{H},\mathcal{G})}(x)$. Then

$$\begin{aligned}\varphi_x\colon \left(\frac{\mathcal{H}\times\mathcal{G}^{op}}{\mathcal{K}_x}\right)^{\mathsf{L}} &\longrightarrow \left(\frac{\mathcal{H}\times\mathcal{G}}{\mathcal{L}_x}\right)^{\mathsf{L}}\\ \Big[\big(h,g,\vartheta(x),\varsigma(x)\big)\Big]\mathcal{K}_x &\longrightarrow \Big[\big(h,g^{-1},\vartheta(x),\varsigma(x)\big)\Big]\mathcal{L}_x\end{aligned}$$

is a well-defined isomorphism of $(\mathcal{H},\mathcal{G})$-bisets with structure given as in Lemma 3.1.1.

Proof. For each $h', h \in \mathcal{H}_1$ and $g', g \in \mathcal{G}_1$ with $\mathsf{s}(h) = \vartheta(x) = \mathsf{s}(h')$ and $\mathsf{s}(g) = \varsigma(x) = \mathsf{s}(g')$, we have

$$[(h', g', \vartheta(x), \varsigma(x))]\mathcal{K}_x = [(h, g^{-1}, \vartheta(x), \varsigma(x))]\mathcal{K}_x$$

if and only if

$$\left(h^{-1}h', g'g^{-1}\right) = \left(h^{-1}h', g^{-1} \overset{\text{op}}{\cdot} g'\right) = (h, g)^{-1}\left(h', g'\right) \in (\mathcal{K}_x)_1,$$

if only if $h^{-1}bxag^{-1} = x$, if and only if $h^{-1}h'x = xg^{-1}$, if and only if

$$\left(h, g^{-1}\right)^{-1}\left(h', g'^{-1}\right) = \left(h^{-1}h', gg'^{-1}\right) \in \mathcal{L}_x,$$

if and only if

$$[(h', g'^{-1}, \vartheta(x), \varsigma(x))]\mathcal{K}_x = [(h, g^{-1}, \vartheta(x), \varsigma(x))]\mathcal{K}_x.$$

Therefore φ_x is well defined and injective. For each $h' \in \mathcal{H}_1$ and $g' \in \mathcal{G}_1$ we have

$$\varphi_x\left([(h', g'^{-1}, \vartheta(x), \varsigma(x))]\mathcal{K}_x\right) = [(h', g', \vartheta(x), \varsigma(x))]\mathcal{L}_x$$

hence φ_x is surjective.

Now given $y = [(h', g', \vartheta(x), \varsigma(x))]\mathcal{K}_x \in \left(\frac{\mathcal{H}\times\mathcal{G}^{\text{op}}}{\mathcal{K}_x}\right)^{\mathsf{L}}$, $h \in \mathcal{H}_1$ and $g \in \mathcal{G}_1$ such that $\mathsf{s}(h) = \vartheta(y)$ and $\varsigma(y) = \mathsf{t}(g)$ we have

$$\begin{aligned}\varphi_x(hyg) &= \varphi_x\left([\left(hh', g \overset{\text{op}}{\cdot} g', \vartheta(x), \varsigma(x)\right)]\mathcal{K}_x\right) = \varphi_x\left([\left(hh', g'g, \vartheta(x), \varsigma(x)\right)]\mathcal{K}_x\right)\\ &= [(hh', g^{-1}g'^{-1}, \vartheta(x), \varsigma(x))]\mathcal{L}_x = h\left([(h', g'^{-1}, \vartheta(x), \varsigma(x))]\mathcal{L}_x\right)g = h\varphi_x(y)\,g\end{aligned}$$

thus φ_x is an isomorphism of $(\mathcal{H}, \mathcal{G})$-bisets as stated. ☐

Now we have to define a particular kind of product between two subgroupoids of a given groupoid. This product will be essential to state the Mackey formula.

Definition 3.1.4. Given groupoids $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{K}$, let $\mathcal{L}$ be a subgroupoid of $\mathcal{H} \times \mathcal{G}$ and $\mathcal{M}$ be a subgroupoid of $\mathcal{K} \times \mathcal{H}$. We define

$$(\mathcal{M} * \mathcal{L})_1 = \left\{ (k, g) \in \mathcal{K}_1 \times \mathcal{G}_1 \,\middle|\, \exists\, h \in \mathcal{H}_1 \text{ such that } (k, h) \in \mathcal{M}_1 \text{ and } (h, g) \in \mathcal{L}_1 \right\}$$

and

$$(\mathcal{M} * \mathcal{L})_0 = \left\{ (v, a) \in \mathcal{K}_0 \times \mathcal{G}_0 \,\middle|\, \exists\, u \in \mathcal{H}_0 \text{ such that } (v, u) \in \mathcal{M}_0 \text{ and } (u, a) \in \mathcal{L}_0 \right\}.$$

Notice that, if $\text{pr}_2(\mathcal{M}) \cap \text{pr}_1(\mathcal{L}) = \varnothing$, where pr_1 and pr_2 are the first and second projections, then $(\mathcal{M} * \mathcal{L})_0$ is obviously an empty set.

Lemma 3.1.5. *Given groupoids $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{K}$ such that $\mathcal{H}$ has only one object, let $\mathcal{L}$ be a subgroupoid of $\mathcal{H} \times \mathcal{G}$ and $\mathcal{M}$ be a subgroupoid of $\mathcal{K} \times \mathcal{H}$. Then $\mathcal{M} * \mathcal{L}$, as defined in Definition* 3.1.4*, is a subgroupoid of $\mathcal{K} \times \mathcal{G}$.*

Proof. Given $(k, g) \in (\mathcal{M} * \mathcal{L})_1$, then there is $h \in \mathcal{H}_1$ such that $(k, h) \in \mathcal{M}_1$ and $(h, g) \in \mathcal{L}_1$ so $\left(k^{-1}, h^{-1}\right) \in \mathcal{M}_1$ and $\left(h^{-1}, g^{-1}\right) \in \mathcal{L}_1$ thus $(k, g)^{-1} \in (\mathcal{M} * \mathcal{L})_1$. Now let be $(k_1, g_1), (k_2, g_2) \in (\mathcal{M} * \mathcal{L})_1$ such that $\mathsf{s}(k_1, g_1) = \mathsf{t}(k_2, g_2)$. There are $h_1, h_2 \in \mathcal{H}_1$ such that $(k_i, h_i) \in \mathcal{M}_1$ and $(h_i, g_i) \in \mathcal{L}_1$ for each $i \in \{1, 2\}$. Since $\mathcal{H}$ has only one object we have $\mathsf{s}(h_1) = \mathsf{t}(h_2)$ thus we can write h_1h_2 and we have

$$(k_1, h_1)(k_2, h_2) = (k_1k_2, h_1h_2) \in \mathcal{M}_1, \quad (h_1, g_1)(h_2, g_2) = (h_1h_2, g_1g_2) \in \mathcal{L}_1.$$

Therefore $(k_1, g_1)(k_2, g_2) = (k_1k_2, g_1g_2) \in (\mathcal{M} * \mathcal{L})_1$, and this completes the proof. ☐

The next example shows that there are situations where the subgroupoid given by the product $*$ is actually not empty.

Example 3.1.6. Given groupoids $\mathcal{K}$, $\mathcal{H}$ and $\mathcal{G}$ such that $\mathcal{H}$ has only one object ω, we consider subgroupoids $\mathcal{D} \leqslant \mathcal{K}$, $\mathcal{C} \leqslant \mathcal{H}$, $\mathcal{B} \leqslant \mathcal{H}$ and $\mathcal{A} \leqslant \mathcal{G}$ where $\mathcal{C}$ and $\mathcal{B}$ are not empty, that is, have exactly the object ω. Then $\mathcal{M} = \mathcal{D} \times \mathcal{C}$ is a subgroupoid of $\mathcal{K} \times \mathcal{H}$ and $\mathcal{L} = \mathcal{B} \times \mathcal{A}$ is a subgroupoid of $\mathcal{H} \times \mathcal{G}$. For each $d_0 \in \mathcal{D}_0$ and $a_0 \in \mathcal{A}_0$ we have $(d_0, \omega) \in \mathcal{M}_0$ and $(\omega, a_0) \in \mathcal{L}_0$ thus $(d_0, a_0) \in (\mathcal{M} * \mathcal{L})_0$. We have $\iota_\omega \in \mathcal{C}_1 \cap \mathcal{B}_1$ so for each $d_1 \in \mathcal{D}_1$ and $a_1 \in \mathcal{A}_1$ we have $(d_1, \iota_\omega) \in \mathcal{M}_1$ and $(\iota_\omega, a_1) \in \mathcal{L}_1$ therefore $(d_1, a_1) \in (\mathcal{M} * \mathcal{L})_1$. As a consequence we have $\mathcal{D}_i \times \mathcal{A}_i \subseteq (\mathcal{M} * \mathcal{L})_i$ for $i = 0$ and $i = 1$. For each $i \in \{0, 1\}$ and for each $(k_i, g_i) \in (\mathcal{M} * \mathcal{L})_i$ there is $h_i \in \mathcal{H}_i$ such that $(k_i, h_i) \in \mathcal{M}_i = \mathcal{D}_i \times \mathcal{C}_i$ and $(h_i, g_i) \in \mathcal{L}_i = \mathcal{B}_i \times \mathcal{A}_i$, thus $k_i \in \mathcal{D}_i$ and $g_i \in \mathcal{A}_i$, therefore $(k_i, g_i) \in \mathcal{D}_i \times \mathcal{A}_i$ and $(\mathcal{M} * \mathcal{L})_i \subseteq \mathcal{D}_i \times \mathcal{A}_i$. This shows that $\mathcal{M} * \mathcal{L} = \mathcal{D} \times \mathcal{A}$ is not an empty groupoid if both $\mathcal{D}$ and $\mathcal{A}$ are not so.

The next result, crucial to even state the Mackey formula (Theorem 3.2.1), is a sort of generalization of the double cosets construction, in the groupoid context, realized in Proposition 1.2.14.

Proposition 3.1.7. *Given groupoids $\mathcal{K}$, $\mathcal{H}$ and $\mathcal{G}$, let $\mathcal{M}$ be a subgroupoid of $\mathcal{K} \times \mathcal{H}$ and $\mathcal{L}$ be a subgroupoid of $\mathcal{H} \times \mathcal{G}$. Let be*

$$X = \left\{ (w, u, h, v, a) \in \mathcal{K}_0 \times \mathcal{H}_0 \times \mathcal{H}_1 \times \mathcal{H}_0 \times \mathcal{G}_0 \;\middle|\; \begin{array}{c} (w, u) \in \mathcal{M}_0,\ (v, a) \in \mathcal{L}_0, \\ u = \mathsf{t}(h),\ v = \mathsf{s}(h) \end{array} \right\}. \tag{3.1.3}$$

Then X is a $(\mathcal{M}, \mathcal{L})$-biset with structure maps

$$\begin{aligned} \vartheta \colon X &\longrightarrow \mathcal{M}_0 \\ (w, u, h, v, a) &\longmapsto (w, u) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \varsigma \colon X &\longrightarrow \mathcal{L}_0 \\ (w, u, h, v, a) &\longmapsto (v, a), \end{aligned}$$

left action

$$\begin{aligned} \lambda \colon \mathcal{M}_1 {}_{\mathsf{s}}\!\times_{\vartheta} X &\longrightarrow X \\ \Big((k, h'), (w, u, h, v, a)\Big) &\longmapsto \big(\mathsf{t}(k), \mathsf{t}(h'), h'h, v, a\big) \end{aligned}$$

and right action

$$\begin{aligned} \rho \colon X {}_{\varsigma}\!\times_{\mathsf{t}} \mathcal{L}_1 &\longrightarrow X \\ \Big((w, u, h, v, a), (h'', g)\Big) &\longmapsto \big(w, u, hh'', \mathsf{s}(h''), \mathsf{s}(g)\big). \end{aligned}$$

Proof. We only check the properties of a right action, since a similar proof shows the left action properties.

(1) For each $y = (w, u, h, v, a) \in X$ and $(h'', g) \in \mathcal{L}_1$ such that $\varsigma(y) = \mathsf{t}(h'', g)$ we have

$$\varsigma\big(y(h'', g)\big) = \varsigma\big(w, u, hh'', \mathsf{s}(h''), \mathsf{s}(g)\big) = \big(\mathsf{s}(h''), \mathsf{s}(g)\big) = \mathsf{s}(h'', g)$$

(2) For each $y = (w, u, h, v, a) \in X$ we have

$$y\iota_{\varsigma(y)} = y\iota_{(v,a)} = y(\iota_v, \iota_a) = (w, u, h\,\iota_v, \mathsf{s}(\iota_v), \mathsf{s}(\iota_a)) = y.$$

(3) For each $y = (w, u, h, v, a) \in X$ and $(h_1, g), (h_2, g') \in \mathcal{L}_1$ such that $\varsigma(y) = \mathsf{t}(h_1, g)$ and $\mathsf{s}(h_1, g) = \mathsf{t}(h_2, g')$ we have

$$\Big(y(h_1, g)\Big)(h_2, g') = \Big(w, u, hh_1, \mathsf{s}(h_1), \mathsf{s}(g)\Big)(h_2, g') = \Big(w, u, hh_1h_2, \mathsf{s}(h_2), \mathsf{s}(g')\Big)$$
$$= \Big(w, u, hh_1h_2, \mathsf{s}(h_1h_2), \mathsf{s}(gg')\Big) = y\Big(h_1h_2, g, g'\Big) = y\Big((h_1, g) \cdot (h_2, g')\Big).$$

Now we have to check the properties of a biset on X, that is, condition (2) in Definition 1.2.1 for the stated actions λ and ρ. For each $y = (w, u, h, v, a) \in X$, $(k, h') \in \mathcal{M}_1$ and $(h'', g) \in \mathcal{L}_1$ such that $\mathsf{s}(k, h') = \vartheta(y)$ and $\varsigma(y) = \mathsf{t}(h'', g)$ we have

$$\vartheta\left(y\left(h'', g\right)\right) = \vartheta\left(w, u, hh'', \mathsf{s}\left(h''\right), \mathsf{s}(g)\right) = (w, u) = \vartheta(y),$$
$$\varsigma\left(\left(k, h'\right) y\right) = \varsigma\left(\mathsf{t}(k), \mathsf{t}\left(h'\right), h'h, v, a\right) = (v, a) = \varsigma(y)$$

and

$$\begin{aligned}\left(k, h'\right)\left(y\left(h'', g\right)\right) &= \left(k, h'\right)\Big(w, u, hh'', \mathsf{s}\left(h''\right), \mathsf{s}(g)\Big) = \Big(\mathsf{t}(k), \mathsf{t}\left(h'\right), h'hh'', \mathsf{s}\left(h''\right), \mathsf{s}(g)\Big)\\ &= \Big(\mathsf{t}(k), \mathsf{t}\left(h'\right), h'h, v, a\Big)\left(h'', g\right) = \Big(\left(k, h'\right) y\Big)\left(h'', g\right).\end{aligned}$$

In this way we obtain the desired properties, completing the proof. □

3.2 Mackey formula: The Theorem

Let's keep the notations of Proposition 3.1.7 and let's assume we are given $w \in \mathcal{K}_0$, $u \in \mathcal{H}_0$ and $a \in \mathcal{G}_0$ such that $(w, u) \in \mathcal{M}_0$ and $(u, a) \in \mathcal{L}_0$. Under these assumptions, identifying isotropy groups with one-object subgroupoids, we can apply Lemma 3.1.5 to the groupoids $\mathcal{K}^w$, $\mathcal{H}^u$ and $\mathcal{G}^a$ by taking the subgroupoids $\mathcal{M}^{(w,u)}$ of $\mathcal{K}^w \times \mathcal{H}^u$ and $\mathcal{L}^{(u,a)}$ of $\mathcal{H}^u \times \mathcal{G}^a$. Of course, here we are identifying the isotropy groups $\mathcal{M}^{(w,u)}$ and $\mathcal{L}^{(u,a)}$ with groupoids having only one object (w, u) and (u, a), respectively. In this way, we obtain that

$$\mathcal{M}^{(w,u)} * \left({}^{(h,\iota_a)}\mathcal{L}^{(u,a)}\right) \tag{3.2.1}$$

is a subgroupoid of $\mathcal{K}^w \times \mathcal{G}^a$ for every $(h, \iota_a) \in \mathcal{L}_1$ with $\mathsf{s}(h) = \mathsf{t}(h) = u$, where we used the notation ${}^gH = gHg^{-1}$, with H a subgroup of a group G and $g \in G$. Since we know that $\mathcal{K}^{(w,w)} \times \mathcal{G}^{(a,a)}$ is a subgroupoid of $\mathcal{K} \times \mathcal{G}$, we have that $\mathcal{M}^{(w,u)} * \left({}^{(h,\iota_a)}\mathcal{L}^{(u,a)}\right)$ is a subgroupoid of $\mathcal{K} \times \mathcal{G}$. This will be used implicitly in the sequel.

The next theorem is the main result of this chapter.

Theorem 3.2.1 (Mackey formula for groupoid-bisets). *Let $\mathcal{K}$, $\mathcal{H}$, $\mathcal{G}$, $\mathcal{M}$ and $\mathcal{L}$ be as in Proposition 3.1.7. Consider the biset X defined in equation* (3.1.3) *and the subgroupoids $\mathcal{M}^{(w,u)} * \left({}^{(h,\iota_a)}\mathcal{L}^{(u,a)}\right)$ of equation* (3.2.1). *Assume that $\mathcal{M}_0 = \mathcal{K}_0 \times \mathcal{H}_0$ and $\mathcal{L}_0 = \mathcal{H}_0 \times \mathcal{G}_0$, then there is a (non canonical) isomorphism of bisets*

$$\left(\frac{\mathcal{K} \times \mathcal{H}}{\mathcal{M}}\right)^{\mathsf{L}} \otimes_{\mathcal{H}} \left(\frac{\mathcal{H} \times \mathcal{G}}{\mathcal{L}}\right)^{\mathsf{L}} \cong \biguplus_{(w,u,h,v,a) \in \mathrm{rep}_{(\mathcal{M},\mathcal{L})}(X)} \left(\frac{\mathcal{K} \times \mathcal{G}}{\mathcal{M}^{(w,u)} * \left({}^{(h,\iota_a)}\mathcal{L}^{(u,a)}\right)}\right)^{\mathsf{L}}, \tag{3.2.2}$$

where $\mathrm{rep}_{(\mathcal{M},\mathcal{L})}(X)$ is a set of representatives of the orbits of X as $(\mathcal{M}, \mathcal{L})$-biset.

Proof. Notice that under assumptions the denominator in the right hand side of Formula (3.2.2) is a well defined subgroupoid of $\mathcal{K} \times \mathcal{G}$ and thus the right hand side of this formula is well defined as well. For simplicity let us denote

$$\mathcal{V} := \left(\frac{\mathcal{K} \times \mathcal{H}}{\mathcal{M}}\right)^{\mathsf{L}} \quad \text{and} \quad \mathcal{U} := \left(\frac{\mathcal{H} \times \mathcal{G}}{\mathcal{L}}\right)^{\mathsf{L}}.$$

As expounded in Lemma 3.1.1, $\mathcal{V}$ is a $(\mathcal{K}, \mathcal{H})$-biset with structure maps

$$\begin{aligned} \Theta\colon \mathcal{V} &\longrightarrow \mathcal{K}_0 \\ [(k,h,w,u)]\mathcal{M} &\longmapsto \mathsf{t}(k) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \Xi\colon \mathcal{V} &\longrightarrow \mathcal{H}_0 \\ [(k,h,w,u)]\mathcal{M} &\longmapsto \mathsf{t}(h) \end{aligned}$$

and $\mathcal{U}$ is an $(\mathcal{H}, \mathcal{G})$-biset with structure maps

$$\begin{aligned} \Upsilon\colon \mathcal{U} &\longrightarrow \mathcal{H}_0 \\ [(h,g,v,a)]\mathcal{L} &\longmapsto \mathsf{t}(h) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \Lambda\colon \mathcal{U} &\longrightarrow \mathcal{G}_0 \\ [(h,g,v,a)]\mathcal{L} &\longmapsto \mathsf{t}(g)\,. \end{aligned}$$

Therefore, following subsection 1.2.5, the tensor product $\mathcal{V} \otimes_{\mathcal{H}} \mathcal{U}$ in the left hand side of (3.2.2) makes sense and it is a $(\mathcal{K}, \mathcal{G})$-biset by Lemma 1.2.15. The orbit of a given element

$$[(k,h,w,u)]\mathcal{M} \otimes_{\mathcal{H}} [(h',g,v,a)]\mathcal{L} \in \mathcal{V} \otimes_{\mathcal{H}} \mathcal{U}$$

will be denoted by $\mathcal{K}\Big[\Big([(k,h,w,u)]\mathcal{M} \otimes_{\mathcal{H}} [(h',g,v,a)]\mathcal{L}\Big)\Big]\mathcal{G}$. If $y \in \mathcal{K}\backslash(\mathcal{V} \otimes_{\mathcal{H}} \mathcal{U})/\mathcal{G}$ is an element in the set of orbits of $\mathcal{V} \otimes_{\mathcal{H}} \mathcal{U}$, then we will use the following notations, similar to the ones already used in Section 1.2.4 and in Proposition 3.1.3:

$$\begin{aligned} \big(\mathrm{Stab}_{(\mathcal{K},\mathcal{G})}(y)\big)_0 &= \Big((\mathcal{K},\mathcal{G})_y\Big)_0 = \Big\{ (\Theta(y), \Lambda(y)) \Big\}, \\ \Big((\mathcal{K},\mathcal{G})_y\Big)_1 &= \Big\{ (k,g) \in \mathcal{K}_1 \times \mathcal{G}_1 \Big| \; ky = yg, \quad \Theta(y) = \mathsf{s}(k), \quad \mathsf{t}(g) = \Lambda(y) \Big\}, \\ \big(\mathrm{Stab}_{(\mathcal{K},\mathcal{G})}(y)\big)_1 &= \Big\{ (k,g) \in \mathcal{K}_1 \times \mathcal{G}_1 \Big| \; kyg = y, \quad \Theta(y) = \mathsf{s}(k), \quad \mathsf{t}(g) = \Lambda(y) \Big\}. \end{aligned}$$

Since, by Lemma 1.2.11 and Proposition 1.2.12, every biset is the disjoint union of its orbits. By Proposition 3.1.3 we obtain the following isomorphisms of $(\mathcal{K}, \mathcal{G})$-bisets:

$$\mathcal{V} \otimes_{\mathcal{H}} \mathcal{U} \cong \biguplus_{y \in \mathrm{rep}_{(\mathcal{K},\mathcal{G})}(\mathcal{V} \otimes_{\mathcal{H}} \mathcal{U})} \left(\frac{\mathcal{K} \times \mathcal{G}}{\mathrm{Stab}_{(\mathcal{K},\mathcal{G})}(y)} \right)^{\mathsf{L}} \cong \biguplus_{y \in \mathrm{rep}_{(\mathcal{K},\mathcal{G})}(\mathcal{V} \otimes_{\mathcal{H}} \mathcal{U})} \left(\frac{\mathcal{K} \times \mathcal{G}}{(\mathcal{K},\mathcal{G})_y} \right)^{\mathsf{L}}.$$

Consider the map

$$\begin{aligned} \varphi\colon \mathcal{K}\backslash(\mathcal{V} \otimes_{\mathcal{H}} \mathcal{U})/\mathcal{G} &\longrightarrow \mathcal{M}\backslash X/\mathcal{L} \\ \mathcal{K}\Big[\Big([(k,h,w,u)]\mathcal{M} \otimes_{\mathcal{H}} [(h',g,v,a)]\mathcal{L}\Big)\Big]\mathcal{G} &\longmapsto \mathcal{M}\Big[(w,u,h^{-1}h',v,a)\Big]\mathcal{L}. \end{aligned}$$

We have to check that φ is well defined. Let us choose two representatives for the same orbit, that is, let us assume that we have an equality of the form

$$\begin{aligned} &\mathcal{K}\Big[[(k,h,w,u)]\mathcal{M} \otimes_{\mathcal{H}} [(h',g,v,a)]\mathcal{L}\Big]\mathcal{G} \\ &= \mathcal{K}\Big[[(l,e,r,n)]\mathcal{M} \otimes_{\mathcal{H}} [(e',f,m,b)]\mathcal{L}\Big]\mathcal{G} \; \in \; \mathcal{K}\backslash(\mathcal{V} \otimes_{\mathcal{H}} \mathcal{U})/\mathcal{G} \end{aligned}$$

in the orbit set of $\mathcal{V} \otimes_{\mathcal{H}} \mathcal{U}$, where $[(k,h,w,u)]\mathcal{M}, [(l,e,r,n]\mathcal{M} \in \mathcal{V}$ and

$$[(h',g,v,a)]\mathcal{L}, \; [(e',f,m,b)]\mathcal{L} \; \in \; \mathcal{U}.$$

By definition this equality means that there are $k_1 \in \mathcal{K}_1$ and $g_1 \in \mathcal{G}_1$ such that $\mathsf{s}(k_1) = \mathsf{t}(l)$, $\mathsf{t}(l) = \mathsf{t}(g_1)$ and

$$\begin{aligned} \big[(k,h,w,u)\big]\mathcal{M} \otimes_{\mathcal{H}} [(h',g,v,a)]\mathcal{L} &= k_1 \Big([(l,e,r,n)]\mathcal{M} \otimes_{\mathcal{H}} [(e',f,m,b)]\mathcal{L} \Big) g_1 \\ &= [(k_1 l,e,r,n)]\mathcal{M} \otimes_{\mathcal{H}} [(e',g_1^{-1}f,m,b)]\mathcal{L} \end{aligned}$$

Thus there is $h_1 \in \mathcal{H}_1$ such that $\mathsf{t}(e) = \mathsf{t}(h_1)$, $\mathsf{t}(h_1) = \mathsf{t}(e')$ and

$$\Big([(k,h,w,u)]\mathcal{M}, [\big(h',g,v,a\big)]\mathcal{L}\Big) = \Big([\big(k_1 l, h_1^{-1}e, r, n\big)]\mathcal{M}, [\big(h_1^{-1}e', g_1^{-1}f, m, b\big)]\mathcal{L}\Big) \in \mathcal{V}\times\mathcal{U}.$$

This means that

$$\begin{cases} [(k,h,w,u)]\mathcal{M} = [\big(k_1 l, h_1^{-1}e, r, n\big)]\mathcal{M} \\ [\big(h',g,v,a\big)]\mathcal{L} = [\big(h_1^{-1}e', g_1^{-1}f, m, b\big)]\mathcal{L}. \end{cases}$$

As a consequence, from one hand, there is $(k_2,h_2) \in \mathcal{M}_1$ such that $\mathsf{s}(k_2,h_2) = (w,u)$, $\mathsf{t}(k_2,h_2) = (r,n)$ and

$$(k,h,w,u) = \big(k_1 l, h_1^{-1}e, r, n\big)\,(k_2,h_2) = \big(k_1 l k_2, h_1^{-1}eh_2, \mathsf{s}(k_2), \mathsf{s}(h_2)\big). \tag{3.2.3}$$

On the other hand, there is $(h_3,g_2) \in \mathcal{L}_1$ such that $\mathsf{s}(h_3,g_2) = (v,a)$, $\mathsf{t}(h_3,g_2) = (m,b)$ and

$$\big(h',g,v,a\big) = \big(h_1^{-1}e', g_1 f, m, b\big)\,(h_3,g_2) = \big(h_1^{-1}e'h_3, g_1 f g_2, \mathsf{s}(h_3), \mathsf{s}(g_2)\big). \tag{3.2.4}$$

Therefore we obtain the following equalities from equations (3.2.3) and (3.2.4)

$$k_2 = l^{-1}k_1^{-1}k, \quad h_2 = e^{-1}h_1 h, \quad \text{and} \quad h_3 = e'^{-1}h_1 h', \quad g_2 = f^{-1}g_1^{-1}g.$$

Thus

$$\begin{aligned} (k_2,h_2)\,\big(w,u,h^{-1}h',v,a\big)\,(h_3,g_2)^{-1} &= \big(\mathsf{s}(k_2), \mathsf{t}(h_2), h_2 h^{-1}h' h_3^{-1}, \mathsf{t}(h_3), \mathsf{t}(g_2)\big) \\ &= \big(w,n,e^{-1}h_1 h h^{-1} h' h'^{-1} h_1^{-1} e', m, b\big) = \big(w,n,e^{-1}e',m,b\big), \end{aligned}$$

which shows that $\mathcal{M}[(w,u,h^{-1}h',v,a)]\mathcal{L} = \mathcal{M}[(w,n,e^{-1}e',m,b)]\mathcal{L}$ in the orbits set $\mathcal{M}\backslash X/\mathcal{L}$. Henceforth, φ is a well defined map.

In other direction, we have a well defined map given by

$$\begin{aligned} \psi\colon \mathcal{M}\backslash X/\mathcal{L} &\longrightarrow \mathcal{K}\backslash\left(\mathcal{V}\otimes_{\mathcal{H}}\mathcal{U}\right)/\mathcal{G} \\ \mathcal{M}[(w,u,h,v,a)]\mathcal{L} &\longrightarrow \mathcal{K}\Big[[(\iota_w,\iota_u,w,u)]\mathcal{M}\otimes_{\mathcal{H}}[(h,\iota_a,v,a)]\mathcal{L}\Big]\mathcal{G}. \end{aligned}$$

Let us check that φ and ψ are one the inverse of the other. So, for each orbit

$$\mathcal{K}\Big[[(k,h,w,u)]\mathcal{M}\otimes_{\mathcal{H}}[\big(h',g,v,a\big)]\mathcal{L}\Big]\mathcal{G} \in \mathcal{K}\backslash\left(\mathcal{V}\otimes_{\mathcal{H}}\mathcal{U}\right)/\mathcal{G}$$

we have

$$\begin{aligned} &\psi\circ\varphi\Big(\mathcal{K}\Big[[(k,h,w,u)]\mathcal{M}\otimes_{\mathcal{H}}[\big(h',g,v,a\big)]\mathcal{L}\Big]\mathcal{G}\Big) = \psi\left(\mathcal{M}\Big[\,\big(w,u,h^{-1}h',v,a\big)\,\Big]\mathcal{L}\right) \\ &= \mathcal{K}\Big[[(\iota_w,\iota_u,w,u)]\mathcal{M}\otimes_{H}[\big(h^{-1}h',\iota_a,v,a\big)]\mathcal{L}\Big]\mathcal{G} \\ &= \mathcal{K}\Big[\Big(k\big([(\iota_w,\iota_u,w,u)]\mathcal{M}\big)h^{-1}\otimes_{\mathcal{H}}\big([(h',\iota_a,v,a)]\mathcal{L}\big)g^{-1}\Big)\Big]\mathcal{G} \\ &= \mathcal{K}\left[[(k,h,w,u)]\mathcal{M}\otimes_{\mathcal{H}}[\big(h',g,v,a\big)]\mathcal{L}\right]\mathcal{G}, \end{aligned}$$

which shows that $\psi\circ\varphi = id$.

Conversely, for each element $\mathcal{M}[(w,u,h,v,a)]\mathcal{L} \in \mathcal{M}\backslash X/\mathcal{L}$, we have

$$\begin{aligned} &\varphi\circ\psi\left(\mathcal{M}[(w,u,h,v,a)]\mathcal{L}\right) = \varphi\Big(\mathcal{K}\Big[[(\iota_w,\iota_u,w,u)]\mathcal{M}\otimes_{\mathcal{H}}[(h,\iota_a,v,a)]\mathcal{L}\Big]\mathcal{G}\Big) \\ &= \mathcal{M}\left[\big(w,u,\iota_u^{-1}h,u,a\big)\right]\mathcal{L} = \mathcal{M}[(w,u,h,u,a)]\mathcal{L}, \end{aligned}$$

whence $\varphi \circ \psi = id$. Therefore φ is bijective with inverse ψ.

Let us check that, for every element of the form

$$y = [(\iota_w, \iota_u, w, u)]\mathcal{M} \otimes_{\mathcal{H}} [(h, \iota_a, v, a)]\mathcal{L} \in [\mathcal{K}\backslash (\mathcal{V} \otimes_{\mathcal{H}} \mathcal{U}) /\mathcal{G}],$$

there is the following equality of subgroupoids

$$(\mathcal{K}, \mathcal{G})_y = \mathcal{M}^{(w,u)} * {}^{(h,\iota_a)}\mathcal{L}^{(u,a)}.$$

So, taking $(k_3, g_3) \in \mathcal{K}_1 \times \mathcal{G}_1$ such that $\mathsf{s}(k_3) = \mathsf{t}(k_3) = w$ and $\mathsf{s}(g_3) = \mathsf{t}(g_3) = a$ we have $k_3 y = y g_3$ if and only if

$$y = [(k_3, \iota_u, w, u)]\mathcal{M} \otimes_{\mathcal{H}} [(h, g_3, u, a)]\mathcal{L},$$

if and only if there exists $h_4 \in \mathcal{H}_1$ such that $\mathsf{s}(h_4) = \mathsf{t}(h_4) = u$ and

$$\begin{aligned} &\Big([(\iota_w, \iota_u, w, u)]\mathcal{M},\ [(h, \iota_a, u, a)]\mathcal{L}\Big) \\ &= \Big(([(k_3, \iota_u, w, u)]\mathcal{M})h_4,\ h_4^{-1}([(h, g_3, u, a)]\mathcal{L})\Big) \\ &= \Big([(k_3, \iota_u h_4^{-1}, w, u)]\mathcal{M},\ [(h_4^{-1}h, g_3, u, a)]\mathcal{L}\Big) \\ &= \Big([(k_3, h_4^{-1}, w, u)]\mathcal{M},\ [(h_4^{-1}h, g_3, u, a)]\mathcal{L}\Big) \in \mathcal{V} \times \mathcal{U}. \end{aligned}$$

This holds true, if and only if, there exists $h_4 \in \mathcal{H}^{(u,u)}$ such that

$$\begin{cases} [(\iota_w, \iota_u, w, u)]\mathcal{M} = [(k_3, h_4^{-1}, w, v)]\mathcal{M} \in \mathcal{V}, \\ [(h, \iota_a, u, a)]\mathcal{L} = [(h_4^{-1}h, g_3, u, a)]\mathcal{L} \in \mathcal{U}, \end{cases}$$

if and only if there exists $h_4 \in \mathcal{H}^{(u,u)}$ such that

$$\begin{cases} (k_3, h_4^{-1}) \in \mathcal{M}^{(w,u)} \\ (h, \iota_a)^{-1}(h_4^{-1}, g_3)(h, \iota_a) = (h, \iota_a)^{-1}(h_4^{-1}h, g_3) \in \mathcal{L}^{(v,a)}, \end{cases}$$

if and only if there exists $h_4 \in \mathcal{H}^{(u,u)}$ such that

$$\begin{cases} (k_3, h_4^{-1}) \in \mathcal{M}^{(w,u)} \\ (h_4^{-1}, g_3) \in {}^{(h,\iota_a)}\mathcal{L}^{(u,a)}, \end{cases}$$

if and only if

$$(k_3, g_3) \in \mathcal{M}^{(w,u)} * {}^{(h,\iota_a)}\mathcal{L}^{(u,a)}.$$

As a consequence we get the following isomorphisms of $(\mathcal{K}, \mathcal{G})$-bisets:

$$\mathcal{V} \otimes_{\mathcal{H}} \mathcal{U} \cong \biguplus_{y \in \mathrm{rep}_{(\mathcal{K},\mathcal{G})}(\mathcal{V} \otimes_{\mathcal{H}} \mathcal{U})} \left(\frac{\mathcal{K} \times \mathcal{G}}{(\mathcal{K}, \mathcal{G})_y}\right)^{\mathsf{L}} \cong \biguplus_{(w,u,h,v,a) \in \mathrm{rep}_{(\mathcal{M},\mathcal{L})}(X)} \left(\frac{\mathcal{K} \times \mathcal{G}}{\mathcal{M}^{(w,u)} * {}^{(h,\iota_a)}\mathcal{L}^{(u,a)}}\right)^{\mathsf{L}},$$

which depends on the choice of a representatives set of the orbits of the biset X. The proof is now completed. □

As a final remark of this section we note that, as it has been done in [Bou10a, Remark 4.1.6], the Mackey formula could be used to characterize the admissible subcategories of the biset category of finite groupoids, once this has been opportunely defined. We do not go into the details because it would be very technical and it will be the object of future work.

3.3 Examples using the equivalence relation groupoid

In this section we will give a simple application of Theorem 3.2.1, using the case of groupoids of equivalence relations as subgroupoids of groupoids of pairs (see Example 1.1.11). In other words, we want to test this formula for this case. As we will see at the end of subsection 3.3.2, this result is not surprising, although not immediate to decipher.

3.3.1 Subgroupoids and equivalence relations

Given a set H, consider as in Example 1.1.11 the groupoid of pairs $\mathcal{H} = (H \times H, H)$ and let R be an equivalence relation on H. Given the groupoid of equivalence relation $\mathcal{R} = (R, H)$, we can consider the inclusion of groupoids

$$\mathcal{R} = (R, H) \hookrightarrow \mathcal{H} = (H \times H, H)\,. \tag{3.3.1}$$

Following equation (1.2.3), we know that

$$\left(\frac{\mathcal{H}}{\mathcal{R}}\right)^{\mathsf{L}} = \left\{ [(a,u)]\,\mathcal{R} \,\middle|\, (a,u) \in \mathcal{R}\,(\mathcal{H})^{\tau} = \mathcal{H}_{1\,\mathsf{s}}\times_{\tau_0} \mathcal{R}_0 \right\}$$

where $a \in \mathcal{H}_1$, $\mathsf{s}\,(a) = u \in \mathcal{R}_0 = H$ and where

$$[(a,u)]\,\mathcal{R} = \left\{ (ar, \mathsf{s}\,(r)) \in \mathcal{R}\,(\mathcal{H})^{\tau} \,\middle|\, r \in \mathcal{R}_1, \mathsf{s}\,(a) = \mathsf{t}\,(r) \right\}$$

with $r \in \mathcal{R}_1$; so there are $h_3, h_4 \in H$ (in the same equivalence class) such that $r = (h_3, h_4)$, and we also have $a = (h_1, h_2) \in \mathcal{H}_1$ with $h_1, h_2 \in H$. In particular, we have $h_2 = \mathsf{s}\,(a) = \mathsf{t}\,(r) = h_3$, $h_4 = \mathsf{s}\,(r) \in \mathcal{R}_0 = H$, $h_2 = \mathsf{s}\,(a) = u \in \mathcal{R}_0$ and

$$ar = (h_1, h_2)\,(h_3, h_4) = (h_1, h_4)\,.$$

As a consequence

$$[(a,u)]\,\mathcal{R} = [((h_1,h_2), h_2)]\,\mathcal{R} = \{\, ((h_1,h_4), h_4) \in \mathcal{H}_{1\,\mathsf{s}}\times_{\tau_0} \mathcal{R}_0 = (H \times H) \times H \,\}$$

and

$$\left(\frac{\mathcal{H}}{\mathcal{R}}\right)^{\mathsf{L}} = \left\{ [((h_1,h_2), h_2)]\,\mathcal{R} \,\middle|\, h_1, h_2 \in H \right\}.$$

Lemma 3.3.1. *Let H/R be the quotient set of H by the equivalence relation R. Then $H \times (H/R)$ becomes a left $\mathcal{H}$-set with structure map and action given by*

$$\begin{aligned} \varsigma \colon H \times \frac{H}{R} &\longrightarrow \mathcal{H}_0 = H \\ (h_1, \overline{h_2}) &\longrightarrow h_1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathcal{H}_{1\,\mathsf{s}}\times_{\varsigma} \left(H \times \frac{H}{R}\right) &\longrightarrow H \times \frac{X}{R} \\ ((h_3, h_1), (h_1, \overline{h_2})) &\longrightarrow (h_3, \overline{h_2})\,, \end{aligned}$$

where for every $h \in H$, $\overline{h}$ denotes the equivalence class of h modulo the relation R.

Proof. It is immediate. ☐

The following lemma is also straightforward.

Lemma 3.3.2. *Given $h_1, h_2, h_3, h'_1, h'_2, h'_3 \in H$ we have that $((h_1, h_2)\,h_2) \sim ((h'_1, \underline{h'_2}), \underline{h'_2)}$, as representative elements of the coset $[((h_1,h_2), h_2)]\,\mathcal{R}$, if and only if $h_1 = h'_1$ and $\overline{h_2} = \overline{h'_2}$ as equivalence classes in H/R.*

Now we are able to deduce the following isomorphism of groupoid-sets.

Proposition 3.3.3. *Using the morphism of groupoids* (3.3.1)*, we consider* $(\mathcal{H}/\mathcal{R})^{\mathsf{L}}$ *as a left* $\mathcal{H}$*-set with structure map and action given explicitly in equation* (1.2.5)*. Then, there is an isomorphism of left* $\mathcal{H}$*-set*

$$\psi\colon \left(\frac{\mathcal{H}}{\mathcal{R}}\right)^{\mathsf{L}} \longrightarrow H \times \frac{H}{R}$$
$$[((h_1, h_2), h_2)]\,\mathcal{R} \longrightarrow \left(h_1, \overline{h_2}\right).$$

Proof. The map ψ is well defined and injective thanks to Lemma 3.3.2, and the surjectivity is obvious. Therefore, we only have to check that ψ is a homomorphism of left $\mathcal{H}$-set. The condition on the structure maps is trivial so we only have to check the condition on the actions. Henceforth, take $(y_1, h_1) \in \mathcal{H}_1$ and $[((h_1, h_2), h_2)]\,\mathcal{R} \in (\mathcal{H}/\mathcal{R})^{\mathsf{L}}$: using the action of equation (1.2.5), we compute

$$(y_1, h_1) \cdot [((h_1, h_2), h_2)]\,\mathcal{R} = [((y_1, h_2), h_2)]\,\mathcal{R}$$

and we apply ψ to obtain $\left(y_1, \overline{h_2}\right)$. Now we apply again ψ to $[((h_1, h_2), h_2)]\,\mathcal{R}$, we obtain $\left(h_1, \overline{h_2}\right)$ and finish by applying the action once again to obtain $(y_1, h_1) \cdot \left(h_1, \overline{h_2}\right) = \left(y, \overline{h_2}\right)$. □

3.3.2 Equivalence relations and Mackey formula

Given sets H, K and G, let us consider the groupoids of pairs $\mathcal{H} = (H \times H, H)$, $\mathcal{K} = (K \times K, K)$ and $\mathcal{G} = (G \times G, G)$. We have the isomorphisms of groupoids (they're just a switch)

$$\mathcal{K} \times \mathcal{H} = (K \times K \times H \times H, K \times H) \cong (K \times H \times K \times H, K \times H) =: \mathcal{A}$$

and

$$\mathcal{H} \times \mathcal{G} = (H \times H \times G \times G, H \times G) \cong (H \times G \times H \times G, H \times G) =: \mathcal{B}$$

that we call

$$\gamma_1\colon \mathcal{K} \times \mathcal{H} \longrightarrow \mathcal{A} \qquad \text{and} \qquad \gamma_2\colon \mathcal{H} \times \mathcal{G} \longrightarrow \mathcal{B},$$

respectively. Note that $\mathcal{A}$ is the groupoid of pairs with respect to the set $K \times H$ and $\mathcal{B}$ is the groupoid of pairs with respect to the set $H \times G$. Let R be an equivalence relation on $K \times H$ and Q be an equivalence relation on $H \times G$: we have $R \subseteq \mathcal{A}_1$ and $Q \subseteq \mathcal{B}_1$ and we can consider the inclusion of groupoids

$$\mathcal{R} := (R, K \times H) \hookrightarrow \mathcal{A} \qquad \text{and} \qquad \mathcal{Q} = (Q, H \times G) \hookrightarrow \mathcal{B}.$$

Defined the groupoids $\mathcal{M} = \gamma_1^{-1}(\mathcal{R})$ and $\mathcal{L} = \gamma_2^{-1}(\mathcal{Q})$, we clearly have the isomorphisms

$$\left(\frac{\mathcal{K} \times \mathcal{H}}{\mathcal{M}}\right)^{\mathsf{L}} \cong \left(\frac{\mathcal{A}}{\mathcal{R}}\right)^{\mathsf{L}} \quad \text{and} \quad \left(\frac{\mathcal{H} \times \mathcal{G}}{\mathcal{L}}\right)^{\mathsf{L}} \cong \left(\frac{\mathcal{B}}{\mathcal{Q}}\right)^{\mathsf{L}}$$

of $(\mathcal{K}, \mathcal{H})$-bisets and of $(\mathcal{H}, \mathcal{G})$-bisets, respectively. In this way we get an other isomorphism of $(\mathcal{K}, \mathcal{G})$-bisets

$$\left(\frac{\mathcal{K} \times \mathcal{H}}{\mathcal{M}}\right)^{\mathsf{L}} \otimes_{\mathcal{H}} \left(\frac{\mathcal{H} \times \mathcal{G}}{\mathcal{L}}\right)^{\mathsf{L}} \cong \left(\frac{\mathcal{A}}{\mathcal{R}}\right)^{\mathsf{L}} \otimes_{\mathcal{H}} \left(\frac{\mathcal{B}}{\mathcal{Q}}\right)^{\mathsf{L}}$$

where, in the right hand term, the structure map and the action are opportunely defined.

On the other hand, we know that

$$\left(\frac{\mathcal{A}}{\mathcal{R}}\right)^{\mathsf{L}} = \Big\{ \left[\left(\left(k_1, h_1, k_2, h_2\right), \left(k_2, h_2\right)\right)\right] \mathcal{R} | h_1, h_2 \in H, k_1, k_2 \in K \Big\}$$

and

$$\left(\frac{\mathcal{B}}{\mathcal{Q}}\right)^{\mathsf{L}} = \Big\{ \left[\left(\left(h_3, g_1, h_4, g_2\right), \left(h_4, g_2\right)\right)\right] \mathcal{Q} | h_3, h_4 \in H, g_1, g_2 \in G \Big\}.$$

Lemma 3.3.4. *The following Cartesian product of sets*

$$(K \times H) \times \left(\frac{K \times H}{R}\right)$$

admits a structure of a $(\mathcal{K}, \mathcal{H})$*-biset.*

Proof. The fact that it is a left $\mathcal{K}$-set can be proved like in Lemma 3.3.1. Therefore we only have to show that it is a right $\mathcal{H}$-set. The structure map is given by

$$\begin{aligned} \vartheta \colon (K \times H) \times \left(\frac{K \times H}{R}\right) &\longrightarrow \mathcal{H} \\ \left((k_1, h_1), \overline{(k_2, h_2)}\right) &\longrightarrow h_1 \end{aligned}$$

and action is defined by

$$\begin{aligned} (K \times H) \times \left(\frac{K \times H}{R}\right) {}_{\vartheta}\!\times_{\mathsf{s}} \mathcal{H}_1 &\longrightarrow (K \times H) \times \left(\frac{K \times H}{R}\right) \\ \Big(\left((k_1, h_1), \overline{k_2, h_2}\right), (h_1, h_3) \Big) &\longrightarrow \left((k_1, h_3), \overline{(k_2, h_2)}.\right) \end{aligned}$$

Now we have to prove the action conditions. The neutral element conditions and the associativity are trivial. Regarding the other condition, let be

$$\Big(\left((k_1, h_1), \overline{(k_2, h_2)}\right), (h_1, h_3) \Big) \in \left((K \times H) \times \left(\frac{K \times H}{R}\right) \right) {}_{\vartheta}\!\times_{\mathsf{t}} \mathcal{H}_1 :$$

we have

$$\vartheta \left((k_1, h_3), \overline{(k_2, h_2)}\right) = h_3 = \mathsf{s}\,(h_1, h_3)\,.$$

Lastly, the compatibility conditions of the left and right actions are immediate to verify. □

Proposition 3.3.5. *Keeping the above notations, we have the following isomorphism of* $(\mathcal{K}, \mathcal{H})$*-bisets*

$$\begin{aligned} \varphi \colon \left(\frac{\mathcal{A}}{\mathcal{R}}\right)^{\mathsf{L}} &\longrightarrow (K \times H) \times \frac{K \times H}{R} \\ \left[\left(\left(k_1, h_1, k_2, h_2\right), \left(k_2, h_2\right)\right)\right] \mathcal{R} &\longrightarrow \left((k_1, h_1), \overline{(k_2, h_2)}\right). \end{aligned}$$

Proof. As a map of left $\mathcal{K}$-sets, φ is an isomorphism thanks to Proposition 3.3.3 applied to the set $K \times H$. So we just have to prove that it is an homomorphism of right $\mathcal{H}$-sets. The condition on the structure map is obvious. Regarding the condition on the action maps, given $x = \left[\left(\left(k_1, h_1, k_2, h_2\right), \left(k_2, h_2\right)\right)\right] \mathcal{R} \in (\mathcal{A}/\mathcal{R})^{\mathsf{L}}$ and $(h_1, h_3) \in \mathcal{H}_1$, we have

$$x \cdot (h_1, h_3) = \left[\left(\left(k_1, h_3, k_2, h_2\right), \left(k_2, h_2\right)\right)\right] \mathcal{R} \overset{\varphi}{\longmapsto} \left((k_1, h_3), \overline{(k_2, h_2)}\right).$$

On the other hand, we have

$$\varphi(x)\cdot(h_1,h_3) \;=\; \Big(\Big(k_1,h_1,\overline{(k_2,h_2)}\Big)\Big)\cdot(h_1,h_3) \;=\; \Big((k_1,h_3)\,,\overline{(k_2,h_2)}\Big),$$

and this finishes the proof. $\square$

Since a similar proposition is true also for $(\mathcal{B}/\mathcal{Q})^{\mathsf{L}}$, we have the following isomorphism of $(\mathcal{K},\mathcal{G})$-bisets:

$$\left(\frac{\mathcal{A}}{\mathcal{R}}\right)^{\mathsf{L}}\otimes_{\mathcal{H}}\left(\frac{\mathcal{B}}{\mathcal{Q}}\right)^{\mathsf{L}}\longrightarrow\Big((K\times H)\times\Big(\frac{K\times H}{R}\Big)\Big)\otimes_{\mathcal{H}}\Big((H\times G)\times\Big(\frac{H\times G}{Q}\Big)\Big). \tag{3.3.2}$$

The typical element of the right hand side of formula (3.3.2) is

$$y=\Big((k_1,h_3)\,,\overline{(k_2,h_2)}\Big)\otimes_{\mathcal{H}}\Big((h_3,g_1)\,,\overline{(h_4,g_2)}\Big) \tag{3.3.3}$$

for $k_1,k_2\in K$, $h_2,h_3,h_4\in H$ and $g_1,g_2\in G$. Notice that, by the definition of the tensor product over $\mathcal{H}$, the element h_3 should appear in both factors of the tensor product. Therefore, there is the following isomorphism of $(\mathcal{K},\mathcal{G})$-bisets given explicitly by:

$$\begin{aligned}&\Big((K\times H)\times\Big(\frac{K\times H}{R}\Big)\Big)\otimes_{\mathcal{H}}\Big((H\times G)\times\Big(\frac{H\times G}{Q}\Big)\Big)\longrightarrow K\times\frac{K\times H}{R}\times\frac{H\times G}{Q}\times G\\&\Big((k_1,h_3)\,,\overline{(k_2,h_2)}\Big)\otimes_{\mathcal{H}}\Big((h_3,g_1)\,,\overline{(h_4,g_2)}\Big)\longrightarrow\Big(k_1,\overline{(k_2,h_2)},\overline{(h_4,g_2)},g_1\Big).\end{aligned}$$

This gives us the left hand side of the Mackey formula (3.2.2) in the situation under consideration.

Let us pass to the right hand side of the formula (3.2.2). Consider the biset X given in equation (3.1.3), and fix a representative set $\mathrm{rep}_{(\mathcal{M},\mathcal{L})}(X)$. Using the notations of Proposition 3.1.7, we have

$$X=\left\{(w,u,h,v,a)\in\mathcal{K}_0\times\mathcal{H}_0\times\mathcal{H}_1\times\mathcal{H}_0\times\mathcal{G}_0\;\middle|\;\begin{array}{c}(w,u)\in\mathcal{M}_0,(v,a)\in\mathcal{L}_0,\\ u=\mathsf{t}(h)\,,v=\mathsf{s}(h)\end{array}\right\}.$$

In our case, it is clear that X is identified, as a $(\mathcal{M},\mathcal{L})$-biset with

$$Y=\{\,(k,(h_1,h_2)\,,g)\in K\times H\times H\times G\,\}=K\times\mathcal{H}_1\times G.$$

Therefore, one can choose a bijection between their representative sets $\mathrm{rep}_{(\mathcal{M},\mathcal{L})}(X)$ and $\mathrm{rep}_{(\mathcal{M},\mathcal{L})}(Y)$. As a consequence, we obtain the following isomorphism of $(\mathcal{K},\mathcal{G})$-bisets:

$$\begin{aligned}&\biguplus_{(k,h_1,(h_1,h_2),h_2,g)\,\in\,\mathrm{rep}_{(\mathcal{M},\mathcal{L})}(X)}\left(\frac{\mathcal{K}\times\mathcal{H}}{\mathcal{M}^{(k,h_1)}*\big({}^{((h_1,h_2),\iota_g)}\mathcal{L}^{(h_2,g)}\big)}\right)^{\mathsf{L}}\\ \cong\;&\biguplus_{(k,(h_1,h_2),g)\,\in\,\mathrm{rep}_{(\mathcal{M},\mathcal{L})}(Y)}\left(\frac{\mathcal{K}\times\mathcal{H}}{\mathcal{M}^{(k,h_1)}*\big({}^{((h_1,h_2),\iota_g)}\mathcal{L}^{(h_2,g)}\big)}\right)^{\mathsf{L}}.\end{aligned} \tag{3.3.4}$$

On the other hand, given $k,k'\in K$, $h_1,h_2,h_1',h_2'\in H$ and $g,g'\in G$, we have

$$(k,(h_1,h_2)\,,g)\sim\big(k',(h_1',h_2')\,,g\big)$$

as elements of the $(\mathcal{M},\mathcal{L})$-biset Y, if and only if there are $m = (k,k',h_1,h_1') \in \mathcal{M}_1$ and $l = (h_2,h_2',g,g') \in \mathcal{L}_1$ such that

$$(k,(h_1,h_2),g) = m\left(k',\left(h_1',h_2'\right),g'\right)l,$$

if and only if $(k,k',h_1,h_1') \in \mathcal{M}_1$ and $(h_2,h_2',g,g') \in \mathcal{L}_1$, if and only if we have $(k,h_1)\,R\,(k',h_1')$ and $(h_2,g)\,Q\,(h_2',g')$. Therefore, we have a bijection

$$\operatorname{rep}_{(\mathcal{M},\mathcal{L})}(Y) \simeq \frac{K\times H}{R}\times\frac{H\times G}{Q}. \tag{3.3.5}$$

Let us now decipher the denominator parts of the right hand-side of equation (3.3.4). For any element $(k,(h_1,h_2),g) \in Y$, we have

$$\mathcal{M}^{(k,h_1)} = \Big\{ (k,k,h_1,h_1) \Big\}, \qquad \mathcal{L}^{(h_2,g)} = \Big\{ (h_2,h_2,g,g) \Big\}$$

and, using the multiplication of groupoids of pairs we get

$$\begin{aligned}
{}^{((h_1,h_2),\iota_g)}\mathcal{L}^{(h_2,g)} &= {}^{((h_1,h_2),\,(g,g))}\mathcal{L}^{(h_2,g)}\\
&= \Big\{ \Big((h_1,h_2),(g,g)\Big)\Big((h_2,h_2),(g,g)\Big)\Big((h_1,h_2),(g,g)\Big)^{-1} \Big\}\\
&= \Big\{ \Big((h_1,h_2)(h_2,h_2)(h_2,h_1),(g,g)(g,g)(g,g)\Big) \Big\}\\
&= \Big\{ \Big((h_1,h_1),(g,g)\Big) \Big\} = \Big\{ (h_1,h_1,g,g) \Big\}.
\end{aligned}$$

As a consequence, we have

$$\mathcal{M}^{(k,h_1)} * \left({}^{((h_1,h_2),\iota_g)}\mathcal{L}^{(h_2,g)}\right) = \{(k,k,g,g)\} = (\mathcal{K}\times\mathcal{G})^{(k,g)},$$

which leads to the following isomorphism of $(\mathcal{K},\mathcal{G})$-biset

$$\left(\frac{\mathcal{K}\times\mathcal{H}}{\mathcal{M}^{(k,h_1)} * \left({}^{((h_1,h_2),\iota_g)}\mathcal{L}^{(h_2,g)}\right)}\right)^{\mathsf{L}} \cong \Big\{ (k_1,k,g_1,g) \in K\times K\times G\times G |\, (k_1,g_1) \in K\times G\}.$$

Combining with formula (3.3.4), we arrive to the isomorphism of $(\mathcal{K},\mathcal{G})$-bisets

$$\begin{aligned}
&\biguplus_{(k,h_1,(h_1,h_2),h_2,g)\,\in\,\operatorname{rep}_{(\mathcal{M},\mathcal{L})}(X)} \left(\frac{\mathcal{K}\times\mathcal{H}}{\mathcal{M}^{(k,h_1)} * \left({}^{((h_1,h_2),\iota_g)}\mathcal{L}^{(h_2,g)}\right)}\right)^{\mathsf{L}}\\
\cong &\biguplus_{(k,(h_1,h_2),g)\,\in\,\operatorname{rep}_{(\mathcal{M},\mathcal{L})}(Y)} (K\times G)
\end{aligned}$$

Therefore, using the bijection of equation (3.3.5), in this case the Mackey formula can be read as the following isomorphism of $(\mathcal{K},\mathcal{G})$-bisets

$$K\times\frac{K\times H}{R}\times\frac{H\times G}{Q}\times G \cong \biguplus_{\frac{K\times H}{R}\times\frac{H\times G}{Q}} (K\times G) \tag{3.3.6}$$

where the right hand side is the coproduct of $\frac{K\times H}{R}\times\frac{H\times G}{Q}$ times $K\times G$ in the category of $(\mathcal{K},\mathcal{G})$-bisets. As anticipated at the beginning of this section, formula (3.3.6) isn't surprising but the Mackey Formula of Theorem 3.2.1 enabled us to shed new light on the fact the it's actually an isomorphism of $(\mathcal{K},\mathcal{G})$-bisets, and not a simple bijection.

Chapter 4

Weak equivalent groupoids

Let $\mathcal{G}$ be a groupoid: it is possible to prove that the right translation groupoid induces a functor $- \rtimes \mathcal{G}\colon \mathsf{Sets}\text{-}\mathcal{G} \longrightarrow \mathbf{Grpd}$. We will show that this functor is faithful and cocontinuous but, in general, it is neither full nor essentially surjective.

Subsequently, we will turn to an obvious question that arises since groupoids are, in particular, categories: how to characterise groupoid that are equivalent as categories.

Lastly, using the aforementioned theory, we will explain how to introduce a particular equivalence relation on the category of finite right groupoid-sets (see Definition 4.4.1): we will say that two groupoid-sets are weakly translationally equivalent if their translation groupoids are equivalent as categories.

4.1 A few preliminary results

Let $f, h\colon (X, \varsigma) \longrightarrow (Y, \theta)$ be morphisms in $\mathsf{Sets}\text{-}\mathcal{G}$. We define a relation $\sim$ on Y in the following way: for each $y_1, y_2 \in Y$ we set $y_1 \sim y_2$ if one of the following three conditions is satisfied:

(1) there is $x \in X$ such that $y_1 = f(x)$ and $y_2 = h(x)$;

(2) there is $x \in X$ such that $y_1 = h(x)$ and $y_2 = f(x)$;

(3) $y_1 = y_2$.

The relation $\sim$ is clearly reflexive and symmetric.

We define $\approx$ as the equivalence relation generated by $\sim$, we denote with Z its quotient set $Z = Y/\approx$ and with $[y]$ the equivalence class of $y \in Y$. Moreover, we denote with π the canonical projection to the quotient

$$\begin{aligned} \pi\colon Y &\longrightarrow Z \\ y &\longrightarrow [y]\,. \end{aligned}$$

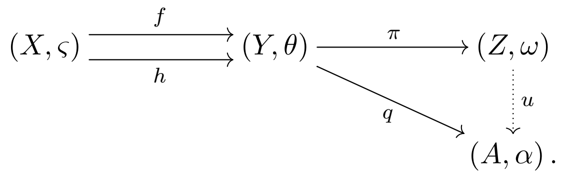
Proposition 4.1.1. *The quotient set Z is a right $\mathcal{G}$-set with structure map and action*

$$\begin{aligned} \omega\colon Z &\longrightarrow \mathcal{G}_0 \\ [y] &\longrightarrow \theta\,(y) \end{aligned} \qquad \textit{and} \qquad \begin{aligned} Z\,{}_{\omega}\!\times_{\mathsf{t}} \mathcal{G}_1 &\longrightarrow Z \\ ([y]\,, g) &\longrightarrow [y]\cdot g = [yg]\,, \end{aligned}$$

respectively. Moreover, the canonical projection π is a $\mathcal{G}$-equivariant map and a coequalizer of the morphisms f and g in the category $\mathsf{Sets}\text{-}\mathcal{G}$.

Proof. Let be $y_1, y_2 \in Y$ such that $y_1 \sim y_2$: if there is $x \in X$ such that $y_1 = f(x)$ and $y_2 = h(x)$ we obtain
$$\theta(y_1) = \theta f(x) = \varsigma(x) = \theta h(x) = \theta(y_2).$$
Otherwise, if there is $x \in X$ such that $y_1 = h(x)$ and $y_2 = f(x)$ or if $y_1 = y_2$, we obtain the same result, that is, $\theta(y_1) = \theta(y_2)$. As a consequence $y_1 \sim y_2$ implies $\theta(y_1) = \theta(y_2)$ and $y_1 \approx y_2$ implies $\theta(y_1) = \theta(y_2)$. This proves that ω is well defined. Now we only have to show that the action is well defined. Let be $g \in \mathcal{G}_1$ and $y_1, y_2 \in Y$ such that $\theta(y_1) = \theta(y_2) = \mathsf{t}(g)$ and $y_1 \sim y_2$: if there is $x \in X$ such that $y_1 = f(x)$ and $y_2 = h(x)$ we obtain $y_1 g = f(xg)$ and $y_2 g = h(xg)$, thus $y_1 g \sim y_2 g$. Otherwise, if there is $x \in X$ such that $y_1 = h(x)$ and $y_2 = f(x)$ or if $y_1 = y_2$ we obtain the same result, that is, $y_1 g \sim y_2 g$. As a consequence the action of $\mathcal{G}$ on Z is well defined and π is a $\mathcal{G}$-invariant map.

It is evident that $\pi f = \pi h$: for each $x \in X$ we have $\pi f(x) = [f(x)] = [h(x)] = \pi h(x)$. Now let $q\colon (Y, \theta) \longrightarrow (A, \alpha)$ be a morphism of right $\mathcal{G}$-sets such that $qf = qh$. We want to construct a $\mathcal{G}$-equivariant map $u\colon (Z, \omega) \longrightarrow (A, \alpha)$ such that $u\pi = q$. The situation is as follows:



Let be $y \in Y$: we define $u([y]) = q(y)$. We have to check that u is well defined. Let be $y_1, y_2 \in Y$ such that $y_1 \sim y_2$: if there is $x \in X$ such that $y_1 = f(x)$ and $y_2 = h(x)$ we obtain $q(y_1) = qf(x) = qh(x) = q(y_2)$. Otherwise, if there is $x \in X$ such that $y_1 = h(x)$ and $y_2 = f(x)$, or if $y_1 = y_2$, we obtain the same result, that is, $q(y_1) = q(y_2)$. As a consequence $y_1 \sim y_2$ implies $q(y_1) = q(y_2)$ and $y_1 \approx y_2$ implies $q(y_1) = q(y_2)$. This proves that u is well defined. For each $y \in Y$ we calculate
$$\alpha u([y]) = \alpha q(y) = \theta(y) = \omega([y])$$
and, for each $([y], g) \in Z\ {}_\omega\!\times_{\mathsf{t}} \mathcal{G}_1$, we calculate
$$u([y]\, g) = u([yg]) = q(yg) = q(y)g = u([y])\, g.$$
As a consequence we have proved that u is a $\mathcal{G}$-equivariant map. Moreover, it is clear that $u\pi = q$ from the definition of u.

Lastly, let be $v\colon (Z, \omega) \longrightarrow (A, \alpha)$ such that $q = v\pi$. For each $y \in Y$ we calculate $u([y]) = q(y) = v\pi(y) = v([y])$ therefore $u = v$ and the universal property of the coequalizer is now proved. □

Theorem 4.1.2. *The category* Sets-$\mathcal{G}$ *has (small) colimits, that is, is cocomplete.*

Proof. Thanks to the dual version of [Bor94, Thm. 2.8.1], the thesis follows from Proposition 4.1.1 and the fact that the category Sets-$\mathcal{G}$ clearly has (small) coproducts, which are given by the disjoint union. □

4.2 The right translation functor

Example 4.2.1. Let $\mathcal{G}$ be a groupoid and consider $(\mathcal{G}_0, \mathrm{Id}_{\mathcal{G}_0})$ as a right $\mathcal{G}$-set with action given by formula (1.1.3). Then the right translation groupoid $\mathcal{G}_0 \rtimes \mathcal{G}$ is isomorphic to $\mathcal{G}$: this isomorphism is given at the level of arrows by the map

$$\begin{aligned} \mathrm{pr}_2\colon \mathcal{G}_0\,{}_{\mathrm{Id}_{\mathcal{G}_0}}\!\times_{\mathsf{t}} \mathcal{G}_1 &\longrightarrow \mathcal{G}_1 \\ (\mathsf{t}(g), g) &\longrightarrow g. \end{aligned}$$

Give a morphism of right $\mathcal{G}$-sets $F\colon (X,\varsigma) \longrightarrow (Y,\theta)$ we have a morphism of groupoids

$$F \rtimes \mathcal{G}\colon X \rtimes \mathcal{G} \longrightarrow X' \rtimes \mathcal{G}$$

defined by $(F \rtimes \mathcal{G})_0 = F$ and $(F \rtimes \mathcal{G})_1 = F \times \mathrm{Id}_{\mathcal{G}_1}$, that is,

$$\begin{array}{ccc} X \rtimes \mathcal{G} & \longrightarrow & X' \rtimes \mathcal{G} \\ x & \longmapsto & F(x) \\ \left(xg \stackrel{(x,g)}{\longrightarrow} x\right) & \longmapsto & \left(F(xg) = F(x)g \stackrel{(F(x),g)}{\longrightarrow} F(x)\right). \end{array} \tag{4.2.1}$$

This implies that $- \rtimes \mathcal{G}$ establishes a functor as claimed by the following statement.

Proposition 4.2.2. *Given a groupoid $\mathcal{G}$. Then the right translation groupoids establish a functor*

$$- \rtimes \mathcal{G}\colon \mathsf{Sets}\text{-}\mathcal{G} \longrightarrow \mathbf{Grpd}$$

defined, on morphisms, as in equation (4.2.1). Furthermore, $- \rtimes \mathcal{G}$ preserves coproducts, that is, we have an isomorphism of groupoids

$$\left(\biguplus_{j\in I} X_j\right) \rtimes \mathcal{G} \cong \coprod_{j\in I} (X_j \rtimes \mathcal{G}).$$

for every family $((X_j,\varsigma_j))_{j\in I}$ of right $\mathcal{G}$-sets. Moreover, the functor $- \rtimes \mathcal{G}$ is faithful.

Proof. The proof of the first statement is immediate. The proof of the second and the third claims use routine computations and are left to the reader. □

Remark 4.2.3. Notice that the functor $- \rtimes \mathcal{G}\colon \mathsf{Sets}\text{-}\mathcal{G} \longrightarrow \mathbf{Grpd}$ is not always full. In other words, not any morphism of groupoids $X \rtimes \longrightarrow Y \rtimes \mathcal{G}$ comes from a $\mathcal{G}$-equivariant map $(X,\varsigma) \longrightarrow (Y,\vartheta)$, as shown in Example 4.4.4 and Remark 4.4.5.

Proposition 4.2.4. *The functor $- \rtimes \mathcal{G}$ is not essentially surjective.*

Proof. Given a right $\mathcal{G}$-set (X,ς), the connected component of the right translation groupoid $X \rtimes \mathcal{G}$ are of the type

$$(\mathcal{G}/\mathcal{H}) \rtimes \mathcal{G},$$

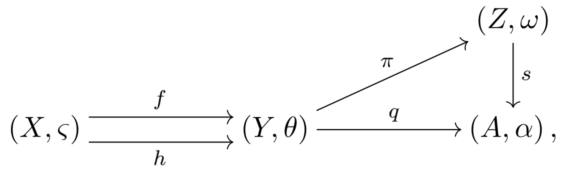
where $\mathcal{H}$ is a subgroupoid of $\mathcal{G}$ with a single object. The isotropy groups of $(\mathcal{G}/\mathcal{H}) \rtimes \mathcal{G}$ are isomorphic to $\mathcal{H}_1$. This means that the isotropy groups of $X \rtimes \mathcal{G}$ are all subgroups of the isotropy groups of $\mathcal{G}$. Now let us consider $\mathcal{A}$, a connected groupoid whose isotropy groups are not isomorphic to any subgroups of the isotropy groups of $\mathcal{G}$ (such groupoid $\mathcal{A}$ clerly exists for cardinality reasons). It is evident that there cannot be a transitive $\mathcal{G}$-set whose right translation groupoid is isomorphic to $\mathcal{A}$, which implies the thesis. □

Proposition 4.2.5. *The right translation functor $- \rtimes \mathcal{G}\colon \mathsf{Sets}\text{-}\mathcal{G} \longrightarrow \mathbf{Grpd}$ preserves the coequalizers.*

Proof. We consider a situation as in the following diagram, with q coequalizer of f and h:

$$(X,\varsigma) \underset{h}{\overset{f}{\rightrightarrows}} (Y,\theta) \xrightarrow{\quad q \quad} (A,\alpha)\,.$$

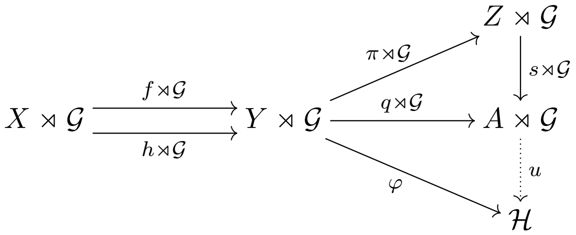
We consider another coequalizer of f and h, the morphism $\pi\colon (Y,\theta) \longrightarrow (Z,\omega)$, as defined in the proof of Proposition 4.1.1. The situation is as follows:



where s is an isomorphism of right $\mathcal{G}$-sets that exists thanks to the universal property of the coequalizers. Since $- \rtimes \mathcal{G}$ is a functor we obviously obtain

$$(q \rtimes \mathcal{G})\,(f \rtimes \mathcal{G}) = (q \rtimes \mathcal{G})\,(h \rtimes \mathcal{G}) \qquad \text{and} \qquad (\pi \rtimes \mathcal{G})\,(f \rtimes \mathcal{G}) = (\pi \rtimes \mathcal{G})\,(h \rtimes \mathcal{G})\,.$$

We have to prove that $q \rtimes \mathcal{G}$ is a coequalizer of $f \rtimes \mathcal{G}$ and $h \rtimes \mathcal{G}$. Let be $\varphi\colon Y \rtimes \mathcal{G} \longrightarrow \mathcal{H}$ such that $\varphi\,(f \rtimes \mathcal{G}) = \varphi\,(h \rtimes \mathcal{G})$. We have to construct a morphism of right $\mathcal{G}$-sets $u\colon A \rtimes \mathcal{G} \longrightarrow \mathcal{H}$ such that $u\,(q \rtimes \mathcal{G}) = \varphi$. The situation is as in the follows diagram.



Let be $a \in A = (A \rtimes \mathcal{G})_0$: there is $y \in Y$ such that $a = s\,([y]) = q(y)$ thus we can define $u_0(a) = \varphi_0(y)$. We have to check that u_0 is well defined. Let be $y_1, y_2 \in Y$ such that $s\,([y_1]) = a = s\,([y_2])$: since s is an isomorphism we obtain $[y_1] = [y_2]$ thus $y_1 \approx y_2$. This means that there are $z_0, \ldots, z_n \in Y$ such that $z_0 \sim z_1 \sim \cdots \sim z_n$. If $n = 0$ and $y_1 = y_2$ we have proved that u_0 is well defined. Otherwise, without loss of generality, we can assume $n \geqslant 1$ and $z_i \neq z_{i+1}$ for each $i = 0, \ldots, n-1$. We have

$$\varphi_0 f = \varphi_0\,(f \rtimes \mathcal{G})_0 = \varphi_0\,(h \rtimes h)_0 = \varphi_0 h.$$

Let be $i \in \{\,1, \ldots, n-1\,\}$: there is $x_i \in X$ such that either $f\,(x_i) = z_i$ and $h\,(x_i) = z_{i+1}$, or $h\,(x_i) = z_i$ and $f\,(x_i) = z_{i+1}$. In the first case we obtain $\varphi_0\,(z_i) = \varphi_0 f\,(x_i) = \varphi_0 h\,(x_i) = \varphi_0\,(z_{i+1})$ and in the second $\varphi_0\,(z_i) = \varphi_0 h\,(x_i) = \varphi_0 f\,(x_i) = \varphi_0\,(z_{i+1})$. Therefore $\varphi_0\,(y_1) = \varphi_0\,(y_2)$, u_0 is well defined and, by definition,

$$u_0\,(q \rtimes \mathcal{G})_0 = u_0 q = \varphi_0.$$

Now let be $(a, g) \in (A \rtimes \mathcal{G})_1$: there is $y \in Y$ such that $a = s\,([y]) = s\pi(y) = q(y)$ and we define $u_1(a,g) = \varphi_1(y,g)$. We have to check that u_1 is well defined. Let be $y_1, y_2 \in Y$ such that $y_1 \approx y_2$. There are $z_0, \ldots, z_n \in Y$ such that $z_0 \sim z_1 \sim \cdots \sim z_n$. If $n = 0$ and $y_1 = y_2$ we have proved that u_1 is well defined. Otherwise, as before, without loss of generality, we can assume $n \geqslant 1$ and $z_i \neq z_{i+1}$ for each $i = 0, \ldots, n-1$. We have

$$\varphi_1\,(f \rtimes \mathcal{G})_1 = \varphi_1\,(h \rtimes h)_1\,.$$

Let be $i \in \{1, \dots, n-1\}$: there is $x_i \in X$ such that either $f(x_i) = z_i$ and $h(x_i) = z_{i+1}$, or $h(x_i) = z_i$ and $f(x_i) = z_{i+1}$. In the first case we obtain

$$\varphi_1(z_i, g) = \varphi_1(f(x_i), g) = \varphi_1(h(x_i), g) = \varphi_1(z_{i+1}, g)$$

and in the second

$$\varphi_1(z_i, g) = \varphi_1(h(x_i), g) = \varphi_1(f(x_i), g) = \varphi_1(z_{i+1}, g).$$

Therefore $\varphi_1(y_1, g) = \varphi_1(y_2, g)$, u_1 is well defined and, by definition,

$$u_1 (q \rtimes \mathcal{G})_1 = u_1 (q \times \mathrm{Id}_{\mathcal{G}_1}) = \varphi_1.$$

Now we have to prove that u is a morphism of groupoids. Given $(a, g) \in (A \rtimes \mathcal{G})_1$, let be $y \in Y$ such that $a = q(y)$; we calculate

$$\begin{aligned} \mathsf{s}^{\rtimes} u_1(a, g) &= \mathsf{s}^{\rtimes} u_1(q(y), g) = \mathsf{s}^{\rtimes} \varphi_1(y, g) = \varphi_0 \mathsf{s}^{\rtimes}(y, g) = \varphi_0(yg) = u_0(q(yg)) \\ &= u_0(q(y)g) = u_0(ag) = u_0 \mathsf{s}^{\rtimes}(a, g) \end{aligned}$$

and

$$\mathsf{t}^{\rtimes} u_1(a, g) = \mathsf{t}^{\rtimes} u_1(q(y), g) = \mathsf{t}^{\rtimes} \varphi_1(y, g) = \varphi_0 \mathsf{t}^{\rtimes}(y, g) = \varphi_0(y) = u_0 q(y) = u_0(a) = u_0 \mathsf{t}^{\rtimes}(a, g).$$

Let be $(a, g), (ag, h) \in (A \rtimes \mathcal{G})_1$: then (a, g) and (ag, h) are composable and we have $(a, g)(ag, h) = (a, gh)$. Given $y \in Y$ such that $a = q(y)$, we calculate

$$\begin{aligned} u_1\Big((a, g)(ag, h)\Big) &= u_1(a, gh) = \varphi_1(y, gh) = \varphi_1\Big((y, g)(yg, h)\Big) \\ &= \varphi_1(y, g)\varphi_1(yg, h) = u_1(a, g)u_1(ag, h). \end{aligned}$$

Now, given $a \in A$, let be $y \in Y$ such that $a = q(y)$: we calculate, since $\alpha(a) = \alpha q(y) = \theta(y)$,

$$u_1\left(a, \iota_{\alpha(a)}\right) = \varphi_1\left(y, \iota_{\alpha(a)}\right) = \varphi_1\left(y, \iota_{\theta(y)}\right) = \varphi_1\left(\iota_y^{\rtimes}\right) = \iota_{\varphi_0(y)} = \iota_{u_0(a)}.$$

As a consequence we have proved that u is a morphism of groupoids.

Now we only have to prove that a groupoid morphism u such that $u(q \rtimes \mathcal{G}) = \varphi$ is unique. Therefore, let be $v\colon A \rtimes \mathcal{G} \longrightarrow \mathcal{H}$ such that $v(q \rtimes \mathcal{G}) = \varphi$. Note that, since $s\pi = q$, then q, as a function, is surjective. We calculate

$$v_0 q = v_0 (q \rtimes \mathcal{G})_0 = \varphi_0 = u_0 (q \rtimes \mathcal{G})_0 = u_0 q$$

thus, thanks to the surjectivity of q, we obtain $v_0 = u_0$. We have to prove that

$$(q \rtimes \mathcal{G}_1) : Y\,{}_{\theta}{\times}_{\mathsf{s}}\,\mathcal{G}_1 \longrightarrow A\,{}_{\alpha}{\times}_{\mathsf{s}}\,\mathcal{G}_1$$

is surjective. Let be $(a, g) \in A\,{}_{\alpha}{\times}_{\mathsf{s}}\,\mathcal{G}_1$: there is $y \in Y$ such that $a = q(y)$, thus $\theta(y) = \alpha q(y) = \alpha(a) = \mathsf{s}(g)$, therefore $(y, g) \in Y\,{}_{\theta}{\times}_{\mathsf{s}}\,\mathcal{G}_1$ and $(q \rtimes \mathcal{G}_1)(y, g) = (q(y), g) = (a, g)$. This implies that $(q \rtimes \mathcal{G}_1)_1$ is surjective and, since $v_1 (q \rtimes \mathcal{G})_1 = \varphi_1 = u_1 (q \rtimes \mathcal{G})_1$, we obtain $v_1 = u_1$.

As a consequence $u = v$ and the thesis is proved. □

Theorem 4.2.6. *The functor* $- \rtimes \mathcal{G}\colon \mathsf{Sets}\text{-}\mathcal{G} \longrightarrow \mathbf{Grpd}$ *is cocontinuous, that is, it preserves all (small) colimits.*

Proof. Thanks to the dual version of [Bor94, Prop. 2.9.2], the thesis follows from Propositions 4.1.1, 4.2.2 and 4.2.5. □

4.3 Weakly equivalent groupoids

Definition 4.3.1. We say that a morphism of groupoids $\varphi\colon \mathcal{H} \longrightarrow \mathcal{G}$ is a **weak equivalence** if φ, as a functor, is an equivalence of categories. We say that two groupoids $\mathcal{H}$ and $\mathcal{G}$ are **weakly equivalent groupoids** if there is a groupoid $\mathcal{K}$ and there are two weak equivalences φ and ψ as follows:

$$\mathcal{H} \xleftarrow{\ \varphi\ } \mathcal{K} \xrightarrow{\ \psi\ } \mathcal{G}. \tag{4.3.1}$$

Remark 4.3.2. Evidently, any identity morphism is a weak equivalence. Therefore, if there is a weak equivalence $\mathcal{H} \longrightarrow \mathcal{G}$ (or $\mathcal{G} \longrightarrow \mathcal{H}$), then $\mathcal{H}$ and $\mathcal{G}$ are obviously weakly equivalent. Conversely, if $\mathcal{H}$ and $\mathcal{G}$ are weakly equivalent, then, as was explained in [EK17, page 562], one can choose an inverse of one of the arrows in the diagram (4.3.1) and obtain, in this way, a weak equivalence. As the reader will notice in the sequel (see Proposition 4.3.9), the fact that we should express the weak equivalence "relation" in the form of a "span", as in diagram (4.3.1), is essentially due to its connection with principal-bisets and their two sided-translation groupoids (see [EK17, Lemma 2.8]).

Proposition 4.3.3. *Given groupoids $\mathcal{G}$ and $\mathcal{H}$, let $\varphi\colon \mathcal{G} \longrightarrow \mathcal{H}$ be a weak equivalence. Then the induced functor*

$$\varphi^*\colon \mathsf{Sets}\text{-}\mathcal{H} \longrightarrow \mathsf{Sets}\text{-}\mathcal{G}.$$

is a Laplaza equivalence of categories.

Proof. By hypothesis there is a morphism of groupoids $\psi\colon \mathcal{H} \longrightarrow \mathcal{G}$ such that there are natural isomorphisms

$$\varepsilon\colon \psi\varphi \longrightarrow \mathrm{Id}_{\mathcal{G}} \qquad \text{and} \qquad \eta\colon \varphi\psi \longrightarrow \mathrm{Id}_{\mathcal{H}}\,.$$

Considering the induced functors and induced natural transformation and using Proposition 1.3.9 we obtain

$$\varphi^*\psi^* \cong (\psi\varphi)^* \xrightarrow{\ \ \varepsilon^*\ \ } (\mathrm{Id}_{\mathcal{G}})^* \cong \mathrm{Id}_{\mathsf{Sets}\text{-}\mathcal{G}}, \qquad \psi^*\varphi^* \cong (\varphi\psi)^* \xrightarrow{\ \ \eta^*\ \ } (\mathrm{Id}_{\mathcal{H}})^* \cong \mathrm{Id}_{\mathsf{Sets}\text{-}\mathcal{H}}$$

To conclude, it's sufficient to apply Propositions 1.3.7 and 1.3.8. ☐

Given a groupoid $\mathcal{G}$ and a fixed object $x \in \mathcal{G}_0$, recall that by $\mathcal{G}^{(x)}$ we denote the one object subgroupoid of $\mathcal{G}$ with isotropy group $\mathcal{G}^x$ and object x.

Lemma 4.3.4. *Given a transitive groupoid $\mathcal{G}$, let be $x \in \mathcal{G}_0$. Then the inclusion functor*

$$\tau^{(x)}\colon \mathcal{G}^{(x)} \longrightarrow \mathcal{G}$$

establishes a weak equivalence of groupoids.

Proof. The functor $\tau^{(x)}$ is clearly fully faithful; moreover, it is essentially surjective because $\mathcal{G}$ is transitive. As a consequence $\tau^{(x)}$ is an equivalence of categories. ☐

Notice that Lemma 4.3.4 and Proposition 4.3.3 generate another proof of Theorem 1.3.15.

Proposition 4.3.5. *Given two transitive groupoids $\mathcal{G}$ and $\mathcal{H}$, then the following statements are equivalent.*

(1) *There is a weak equivalence* $\mathcal{G} \longrightarrow \mathcal{H}$.

(2) *For each* $u \in \mathcal{H}_0$ *and for each* $x \in \mathcal{G}_0$ *there is an isomorphism of groups* $\mathcal{G}^x \cong \mathcal{H}^u$.

(3) *There are* $u \in \mathcal{H}_0$ *and* $x \in \mathcal{G}_0$ *such that there is an isomorphism of groups* $\mathcal{G}^x \cong \mathcal{H}^u$.

Proof. Without loss of generality we can assume $\mathcal{G}_0 \neq \varnothing \neq \mathcal{H}_0$.

(1) $\Rightarrow$ (2). Given a weak equivalence $\varphi\colon \mathcal{G} \longrightarrow \mathcal{H}$ we have an isomorphism of groups $\mathcal{G}^x \cong \mathcal{H}^{\varphi_0(x)}$, for every $x \in \mathcal{G}_0$. Moreover, if $u \in \mathcal{H}_0$ is an arbitrary element, then $\mathcal{H}^u$ and $\mathcal{H}^{\varphi_0}(x)$ are conjugated subgroup, since $\mathcal{H}$ is transitive. As a consequence $\mathcal{G}^x$ and $\mathcal{H}^u$ are isomorphic groups.

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (1). Fix the pair $(x, u) \in \mathcal{G}_0 \times \mathcal{H}_0$ such that $\mathcal{G}^x \cong \mathcal{H}^{\varphi_0(x)}$ as groups. Then clearly $\mathcal{H}^{(u)}$ and $\mathcal{G}^{(x)}$ are isomorphic as groupoids. Therefore, there is a weak equivalence $\mathcal{G} \longrightarrow \mathcal{H}$, as, thanks to Lemma 4.3.4, there is a weak equivalence between $\mathcal{G}$ and $\mathcal{G}^{(x)}$ and also between $\mathcal{H}$ and $\mathcal{H}^{(u)}$. □

Definition 4.3.6. Let (X, α, β) a $(\mathcal{G}, \mathcal{H})$-biset. We say that (X, α, β) is a **left principal** $(\mathcal{G}, \mathcal{H})$**-biset** (or **left principal** $(\mathcal{G}, \mathcal{H})$**-bundle**) if the following conditions are satisfied:

(1) the structure map $\beta\colon X \longrightarrow \mathcal{H}_0$ is surjective;

(2) the canonical map

$$\begin{aligned} \nabla^l\colon \mathcal{G}_1\, {}_{\mathsf{s}}\times_{\alpha} X &\longrightarrow X\, {}_{\beta}\times_{\beta} X \\ (g, x) &\longrightarrow (gx, x) \end{aligned}$$

is bijective.

Definition 4.3.7. Let (X, α, β) a $(\mathcal{G}, \mathcal{H})$-biset. We say that (X, α, β) is a **right principal** $(\mathcal{G}, \mathcal{H})$**-biset** (or **right principal** $(\mathcal{G}, \mathcal{H})$**-bundle**) if the following conditions are satisfied:

(1) the structure map $\alpha\colon X \longrightarrow \mathcal{G}_0$ is surjective;

(2) the canonical map

$$\begin{aligned} \nabla^r\colon X\, {}_{\beta}\times_{\mathsf{t}} \mathcal{H}_1 &\longrightarrow X\, {}_{\alpha}\times_{\alpha} X \\ (x, g) &\longrightarrow (x, xg) \end{aligned}$$

is bijective.

Definition 4.3.8. We say that a biset (X, α, β) is a **principal** $(\mathcal{G}, \mathcal{H})$**-biset** (or **principal** $(\mathcal{G}, \mathcal{H})$**-bibundle**) if it is both a principal right biset and a principal left biset.

The following result, which we will need in the sequel, improves somehow the one stated in [EK17, Theorem 2.9].

Proposition 4.3.9. *Given two coproduct of groupoids* $\mathcal{G} = \coprod_{j \in J} \mathcal{G}_j$ *and* $\mathcal{H} = \coprod_{i \in I} \mathcal{H}_i$*, where* $\mathcal{G}_j$ *and* $\mathcal{H}_i$ *are not empty and transitive for each* $j \in J$ *and* $i \in I$*. Then the following statements are equivalent:*

(1) *The groupoid* $\mathcal{G}$ *and* $\mathcal{H}$ *are weakly equivalent.*

(2) *There is a weak equivalence* $\mathcal{G} \longrightarrow \mathcal{H}$.

(3) *There is a bijection* $\alpha\colon J \longrightarrow I$ *such that for each* $i \in I$ *and for each* $y \in (\mathcal{H}_i)_0$*, there is an isomorphism of groups* $\left(\mathcal{G}_{\alpha^{-1}(i)}\right)^x \cong (\mathcal{H}_i)^y$*, for each* $x \in \left(\mathcal{G}_{\alpha^{-1}(i)}\right)_0$.

(4) *There is a bijection* $\alpha\colon J \longrightarrow I$ *such that there are* $i \in I$, $y \in (\mathcal{H}_i)_0$ *and* $x \in \big(\mathcal{G}_{\alpha^{-1}(i)}\big)_0$ *such that there is an isomorphism of groups* $\big(\mathcal{G}_{\alpha^{-1}(i)}\big)^x \cong (\mathcal{H}_i)^y$.

(5) *The categories* Sets-$\mathcal{G}$ *and* Sets-$\mathcal{H}$ *are Laplaza equivalent.*

(6) *The monoidal categories* Sets-$\mathcal{G}$ *and* Sets-$\mathcal{H}$ *are equivalent with respect to the fibre product monoidal structure.*

(7) *There is a principal* $(\mathcal{H}, \mathcal{G})$*-biset.*

Proof. It follows from Remark 1.1.5 and Proposition 4.3.5, in combination with the result [EK17, Theorem 2.9]. □

Now we can state the following result.

Proposition 4.3.10. *Given a groupoid* $\mathcal{G}$*, let* X *and* Y *be two right* $\mathcal{G}$*-sets and consider their sets of orbits* $J = X/\mathcal{G}$ *and* $I = Y/\mathcal{G}$*. Then the following statements are equivalent.*

(1) *There is a weak equivalence* $X \rtimes \mathcal{G} \longrightarrow Y \rtimes \mathcal{G}$.

(2) *There is a bijection* $\alpha\colon J \longrightarrow I$ *such that for each* $l \in I$ *and each* $y \in l$*, the group* $(\mathrm{Stab}_{\mathcal{G}}(y))_1$ *is isomorphic to* $(\mathrm{Stab}_{\mathcal{G}}(x))_1$*, for every* $x \in \alpha^{-1}(l)$.

(3) *There is a bijection* $\alpha\colon J \longrightarrow I$ *such that for each* $l \in I$*, there are* $y \in l$ *and* $x \in \alpha^{-1}(l)$ *such that the groups* $(\mathrm{Stab}_{\mathcal{G}}(y))_1$ *and* $(\mathrm{Stab}_{\mathcal{G}}(x))_1$ *are isomorphic.*

(4) *There is a principal* $(X \rtimes \mathcal{G}, Y \rtimes \mathcal{G})$*-biset.*

Proof. To prove (1) $\Leftrightarrow$ (4), it is sufficient to apply Lemma 1.1.22 in conjunction with Proposition 4.3.9. □

4.4 Weakly translationally equivalent groupoid-sets

We start this section by giving the subsequent definition.

Definition 4.4.1. Given a groupoid $\mathcal{G}$ and two right $\mathcal{G}$-sets (X, ς) and (Y, ϑ), we say that (X, ς) and (Y, ϑ) are **weakly translationally equivalent $\mathcal{G}$-sets** provided that one of the equivalent conditions in Proposition 4.3.10 is satisfied. When the class of objects of the category Sets-$\mathcal{G}$ is actually a set (or when we restrict ourselves to the full subcategory of $\mathcal{G}$-sets with underlying finite sets), then the previous relation induces an equivalence relation which we denote by $\sim_{\mathsf{wt}}$. By abuse of notation we will also use this symbol between any kind of right $\mathcal{G}$-sets.

According to Proposition 4.3.5, two transitive right $\mathcal{G}$-sets are weakly equivalent if and only if there is an isomorphism of groups between two of their stabilizers. More precisely, two transitive $\mathcal{G}$-sets (X, ς) and (Y, ϑ) are weakly equivalent if and only if there is a pair $(x, y) \in X \times Y$ and an isomorphism of groupoids $\mathrm{Stab}_{\mathcal{G}}(x) \cong \mathrm{Stab}_{\mathcal{G}}(y)$. Applying these observations to the case of right cosets, we obtain the following result.

Corollary 4.4.2. *Let* $\mathcal{G}$ *be a groupoid and take two subgroupoids* $\mathcal{H}, \mathcal{K}$ *with single object, that is, elements in* $\mathcal{S}_{\mathcal{G}}$*. Then the following statements are equivalent.*

(1) $\mathcal{G}/\mathcal{H}$ *and* $\mathcal{G}/\mathcal{K}$ *are weakly translationally equivalent* $\mathcal{G}$*-sets.*

(2) *There is an isomorphism of groups* $\mathcal{H}_1 \cong \mathcal{K}_1$.

Proof. Recall first that if we set $\mathcal{H}_0 = \{ a \}$, then for any coset $\mathcal{H}[(a, g)] \in \mathcal{G}/\mathcal{H}$, we have that

$$\mathrm{Stab}_{\mathcal{G}}\left(\mathcal{H}[(a,g)]\right) = g^{-1}\mathcal{H}_1 g,$$

where $\mathcal{H}_1$ is the group of loops of $\mathcal{H}_1$, viewed as a subgroup of the isotropy group $\mathcal{G}^a$. The implication $(i) \Rightarrow (ii)$ follows then from Remark 4.3.2 and Proposition 4.3.10. Assume now that we have an isomorphism of groups $\mathcal{H}_1 \cong \mathcal{K}_1$. This means that

$$\mathrm{Stab}_{\mathcal{G}}\left(\mathcal{H}[(a,\iota_a)]\right) \cong \mathrm{Stab}_{\mathcal{G}}\left(\mathcal{K}[(b,\iota_b)]\right)$$

as groups, where $\mathcal{K}_0 = \{ b \}$. The inverse implication follows then from Proposition 4.3.10. □

Any two isomorphic $\mathcal{G}$-sets are weakly equivalent, as stated in the following proposition.

Proposition 4.4.3. *Given a groupoid $\mathcal{G}$, let (X, ς) and (Y, ϑ) be isomorphic right $\mathcal{G}$-sets. Then we obtain $(X, \varsigma) \sim_{\mathsf{wt}} (Y, \vartheta)$.*

Proof. Thanks to Proposition 4.2.2, we know that isomorphic right $\mathcal{G}$-sets have isomorphic right translation groupoids, and this shows the claim. □

Example 4.4.4. It is possible that, given a group G, there are subgroups H and K of G such that G/H and G/K are not isomorphic as right G-sets but there is nonetheless a weak equivalence between the groupoids $(G/H) \rtimes G$ and $(G/K) \rtimes G$. It is enough to take two subgroups H and K of G that are isomorphic but not conjugated, and the claims will follow from Corollary 4.4.2. For example, if G is abelian, the relation of conjugacy is the same of the relation of equality, therefore it is sufficient to take two different isomorphic subgroups of an abelian groups G. In particular, given an abelian group A, possible choices are $G = A \times A$, $H = A \times 1$ and $K = 1 \times A$.

Remark 4.4.5. In relation with Remark 4.2.3, we will use the above arguments to show that $- \rtimes \mathcal{G}$ is not always a full functor. Let us keep the notations of Example 4.4.4. By contradiction, if it were full, since by Proposition 4.2.2 it is also faithful, then the isomorphism of groupoids $(G/H) \rtimes G \cong (G/K) \rtimes G$ would imply the isomorphism of right $\mathcal{G}$-sets $G/H \cong G/K$, which is false, for instance, in the case of Example 4.4.4.

Remark 4.4.6. Unfortunately, the weakly translationally equivalence relation is compatible with the disjoint union but not with the tensor product $- \underset{\mathcal{G}_0}{\times} -$. This implies that, unlike the isomorphism relation, which enables us to construct the classical Burnside rig (see Chapter 5), using the equivalence relation $\sim_{\mathsf{wt}}$ we only obtain an additive monoid, and not a rig. As a consequence, applying the Grothendieck construction (see Appendix B), we obtain only an abelian group, and not a ring.

Chapter 5

The “classical” Burnside ring of a groupoid

In this chapter we continue the study of the Burnside theory for groupoids already begun in Chapter 2. Our main aim here is twofold. First, after defining the Burnside ring of a groupoid, we characterize it as the direct product of the Burnside rings of its isotropy groups: exactly one for each connected component. Next, we prove the existence of an opportunely defined ghost map, in the groupoid context, and we show that it is injective as in the classical group case. The idempotent of the $\mathbb{Q}$-algebra $\mathbb{Q} \otimes_{\mathbb{Z}} \mathscr{B}(\mathcal{G})$ are then computed employing the decomposition stated in Corollary 5.1.10.

5.1 Burnside functor for groupoids: coproducts and products

In this section we introduce the Burnside ring attached to a groupoid with finitely many objects, whose construction is based on the skeleton of the category of the right $\mathcal{G}$-sets with underlying finite sets. For the convenience of an inexperienced audience we recall in the Appendixes A and B, with very elementary arguments, the general notions of Grothendieck functor and rig (a ring without additive inverses, also called semiring). Both are crucial in performing the construction of the Burnside ring functor. The compatibility of this functor with coproducts and product is needed in order to establish the main result of this section, which asserts that the Burnside ring of a given (finite) groupoid is the product of the Burnside rings of its isotropy groups, where the product is taken over the set of the connected components (see Theorem 5.1.8).

We assume, in this chapter, that all handled groupoids have a finite set of objects. This condition is in fact needed to have a unit for the Burnside ring we are planing to introduce, since we will make use of the skeletally small category of finite groupoid-sets to perform this construction. We also assume that functors between groupoid-sets preserve objects with finite underlying sets, and transform an empty groupoid-set to an empty one, as the induction functors do. Given a groupoid $\mathcal{G}$, we denote by $\mathsf{sets}\text{-}\mathcal{G}$ the full subcategory of right $\mathcal{G}$-sets with finite underlying sets.

5.1.1 Burnside rig functor and coproducts

Given a groupoid $\mathcal{G}$, let (X, ς) be a finite right $\mathcal{G}$-set, that is, an object in $\mathsf{sets}\text{-}\mathcal{G}$, and denote by $[(X, \varsigma)]$ its equivalence class modulo the isomorphism relation. Consider $\mathscr{L}(\mathcal{G})$, the quotient set of all finite right $\mathcal{G}$-sets modulo the isomorphism of right $\mathcal{G}$-sets equivalence relation. This means that elements of $\mathscr{L}(\mathcal{G})$ are classes $[(X,\varsigma)]$ represented by $\mathcal{G}$-sets (X,ς) with finite underlying set X. We endow the set $\mathscr{L}(\mathcal{G})$ with an addition and a multiplication operations: for every $(X,\varsigma),(Y,\vartheta) \in \mathsf{Sets}\text{-}G$, we define

$$[(X,\varsigma)] + [(Y,\vartheta)] := [(X,\varsigma) \uplus (Y,\vartheta)] = [(X \uplus Y, \varsigma \uplus \vartheta)]$$

and

$$[(X,\varsigma)] \cdot [(Y,\vartheta)] := \left[(X,\varsigma) \underset{\mathcal{G}_0}{\times} (Y,\vartheta)\right] = [(X\,{}_{\varsigma}{\times}_{\vartheta}\,Y, \varsigma\vartheta)],$$

(see subsection 1.3.2 for the notations). It is immediate to check that these operations are well defined and that, in this way, $\mathscr{L}(\mathcal{G})$ becomes a rig (also called semiring) with additive neutral element $[(\varnothing,\varnothing)]$ and multiplicative neutral element $[(\mathcal{G}_0, \mathrm{Id}_{\mathcal{G}_0})]$ (see Definition A.0.1 for further details).

The rig construction is a functorial one, as one can prove using general monoidal category theory. We give, in our case, an elementary proof.

Lemma 5.1.1. *Given two groupoids $\mathcal{G}$ and $\mathcal{H}$, let $F\colon \mathsf{Sets}\text{-}\mathcal{G} \longrightarrow \mathsf{Sets}\text{-}\mathcal{H}$ be a strong monoidal functor with respect to both monoidal structures: $\uplus$ and the fibered product. Let us define*

$$\begin{aligned} \mathsf{h}\colon \mathscr{L}(\mathcal{G}) &\longrightarrow \mathscr{L}(\mathcal{H}) \\ [(X,\varsigma)] &\longrightarrow [F(X,\varsigma)]. \end{aligned}$$

Then h is a homomorphism of rigs.

Proof. Clearly h is a well defined map, since any functor preserves isomorphisms. Now, for every $(X,\varsigma),(Y,\vartheta) \in \mathsf{Sets}\text{-}\mathcal{G}$ we have the following isomorphisms of right $\mathcal{H}$-sets

$$F(X \uplus Y, \varsigma \uplus \vartheta) = F((X,\varsigma) \uplus (Y,\vartheta)) \cong F\Big((X,\varsigma)\Big) \uplus F\Big((Y,\vartheta)\Big) \cong F(X,\varsigma) + F(Y,\vartheta),$$

and

$$F\Big(X\,{}_{\varsigma}{\times}_{\vartheta}\,Y, \varsigma\vartheta\Big) = F\left((X,\varsigma) \underset{\mathcal{G}_0}{\times} (Y,\vartheta)\right) \cong F(X,\varsigma) \underset{\mathcal{G}_0}{\times} F(Y,\vartheta) \cong F(X,\varsigma) \cdot F(Y,\vartheta).$$

Passing to the isomorphism classes and applying h, leads to the the equalities

$$\begin{aligned} &\mathsf{h}([(X,\varsigma)] + [(Y,\vartheta)]) = \mathsf{h}([(X \uplus Y, \varsigma \uplus \vartheta)]) = [F(X \uplus Y, \varsigma \uplus \vartheta)] \\ &= [F(X,\varsigma) + F(Y,\vartheta)] = [F(X,\varsigma)] + [F(Y,\vartheta)] = \mathsf{h}([(X,\varsigma)]) + \mathsf{h}([(Y,\vartheta)]) \end{aligned}$$

and

$$\begin{aligned} &\mathsf{h}([(X,\varsigma)] \cdot [(Y,\vartheta)]) = \mathsf{h}([(X\,{}_{\varsigma}{\times}_{\vartheta}\,Y, \varsigma\vartheta)]) = [F(X\,{}_{\varsigma}{\times}_{\vartheta}\,Y, \varsigma\vartheta)] \\ &= [F(X,\varsigma) \cdot F(Y,\vartheta)] = [F(X,\varsigma)] \cdot [F(Y,\vartheta)] = \mathsf{h}([(X,\varsigma)]) \cdot \mathsf{h}([(Y,\vartheta)]). \end{aligned}$$

On the other hand, we have the isomorphisms of right $\mathcal{H}$-sets

$$F(\varnothing,\varnothing) \cong (\varnothing,\varnothing) \qquad \text{and} \qquad F(\mathcal{G}_0, \mathrm{Id}_{\mathcal{G}_0}) \cong (\mathcal{H}_0, \mathrm{Id}_{\mathcal{H}_0}).$$

We then obtain $\mathsf{h}([(\varnothing,\varnothing)]) = [F(\varnothing,\varnothing)] = [(\varnothing,\varnothing)]$ and

$$\mathsf{h}([(\mathcal{G}_0, \mathrm{Id}_{\mathcal{G}_0})]) = [F(\mathcal{G}_0, \mathrm{Id}_{\mathcal{G}_0})] = [(\mathcal{H}_0, \mathrm{Id}_{\mathcal{H}_0})].$$

As a consequence we have proved that, as desired, h is a homomorphism of rigs. ∎

Now, let $\varphi\colon \mathcal{H} \longrightarrow \mathcal{G}$ be a homomorphism of groupoids. By Proposition 1.3.6 we can consider the induced functor $\varphi^*\colon \mathsf{Sets}\text{-}\mathcal{G} \longrightarrow \mathsf{Sets}\text{-}\mathcal{H}$ and, thanks to Lemma 5.1.1, from this functor we obtain a homomorphism of rigs from $\mathscr{L}(\mathcal{G})$ to $\mathscr{L}(\mathcal{H})$, induced by φ^*, which we denote by $\mathscr{L}(\varphi)$. More precisely, we have

$$\begin{aligned} \mathscr{L}(\varphi)\colon \mathscr{L}(\mathcal{G}) &\longrightarrow \mathscr{L}(\mathcal{H}) \\ [(X,\varsigma)] &\longrightarrow [\varphi^*(X,\varsigma)]\,. \end{aligned}$$

Proposition and Definition 5.1.2. *The correspondence $\mathscr{L}$ defines a contravariant functor from the category of groupoids* **Grpd** *to the category of rigs* **Rig** *which we call, inspired by [Sch91, page* 381*], the* ***Burnside rig functor****.*

Proof. Let $\psi\colon \mathcal{K} \longrightarrow \mathcal{H}$ and $\varphi\colon \mathcal{H} \longrightarrow \mathcal{G}$ be morphisms of groupoid. Using Proposition 1.3.8, for each $[(X,\varsigma)] \in \mathscr{L}(\mathcal{G})$, we obtain

$$\begin{aligned} \mathscr{L}(\psi)\,L(\varphi)\Big(\,[(X,\varsigma)]\,\Big) &= \mathscr{L}(\psi)\Big(\,[(\varphi^*(X,\varsigma))] = [\psi^*\varphi^*(X,\varsigma)] = \big[(\varphi\psi)^*(X,\varsigma)\big] \\ &= \mathscr{L}(\varphi\psi)\Big(\,[(X,\varsigma)]\,\Big). \end{aligned}$$

Thus the following diagram is commutative

$$\begin{array}{ccc} \mathscr{L}(\mathcal{G}) & \xrightarrow{\mathscr{L}(\varphi\psi)} & \mathscr{L}(\mathcal{K}) \\ {\scriptstyle \mathscr{L}(\varphi)}\big\downarrow & \nearrow{\scriptstyle \mathscr{L}(\psi)} & \\ \mathscr{L}(\mathcal{H})\,. & & \end{array}$$

Moreover, for each groupoid $\mathcal{G}$ we calculate, thanks again to Proposition 1.3.8,

$$\mathscr{L}(\mathcal{G})\Big(\,[(X,\varsigma)]\,\Big) = \big[(\mathrm{Id}_{\mathcal{G}})^*(X,\varsigma)\big] = [(X,\varsigma)] = \mathrm{Id}_{\mathscr{L}(\mathcal{G})}\Big(\,[(X,\varsigma)]\,\Big)$$

thus $\mathscr{L}(\mathcal{G}) = \mathrm{Id}_{\mathscr{L}(\mathcal{G})}$. This shows that $\mathscr{L}$ is a well defined functor as desired. □

We finish this subsection by discussing the compatibility of the Burnside rig functor with coproduct.

Proposition 5.1.3. *The Burnside rig functor $\mathscr{L}$ sends coproduct to product. In particular, given a family of groupoids $(\mathcal{G}_j)_{j\in I}$, let $(i_j\colon \mathcal{G}_j \longrightarrow \mathcal{G})_{j\in I}$ be their coproduct in* **Grpd**. *Then*

$$\left(\mathscr{L}(i_j) : \mathscr{L}(\mathcal{G}) \longrightarrow \mathscr{L}(\mathcal{G}_j)\right)_{j\in I}$$

is the product of the family $(\mathscr{L}(\mathcal{G}_j))_{j\in I}$ in the category **Rig**.

Proof. Let $(f_j\colon A \longrightarrow \mathscr{L}(\mathcal{G}_j))_{j\in I}$ be a family of homomorphisms of rigs. We have to prove that there is a unique homomorphism $f\colon A \longrightarrow \mathscr{L}(\mathcal{G})$ of rigs such that the following diagram commutes for every $j \in I$:

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & \mathscr{L}(\mathcal{G}) \\ & {\scriptstyle f_j}\searrow & \big\downarrow{\scriptstyle \mathscr{L}(i_j)} \\ & & \mathscr{L}(\mathcal{G}_j). \end{array} \tag{5.1.1}$$

Let $a \in A$: for every $j \in I$ there is $(X_j, \varsigma_j) \in \mathsf{sets}\text{-}G_j$ such that $f_j(a) = [(X_j, \varsigma_j)] \in \mathscr{L}(\mathcal{G}_j)$. Henceforth, we define, thanks to Lemma 1.3.12,

$$f(a) = \left[\biguplus_{j \in I} \widehat{(X_j, \varsigma_j)} \right],$$

obtaining, in this way, a function $f\colon A \longrightarrow \mathscr{L}(\mathcal{G})$. Furthermore, for every $l \in I$ we have

$$\mathscr{L}(i_l)\, f(a) = \left[(i_l)^* \left(\biguplus_{j \in I} \widehat{(X_j, \varsigma_j)} \right) \right] = \left[\biguplus_{j \in I} (i_l)^* \left(\widehat{(X_j, \varsigma_j)} \right) \right]. \tag{5.1.2}$$

For every $l, j \in J$ such that $l \neq j$ we clearly have

$$(i_l)^* \left(\widehat{(X_j, \varsigma_j)} \right) = \left(X_j\, {}_{\hat{\varsigma}_j}\!\times_{(i_l)_0} (\mathcal{G}_l)_0\,, \mathrm{pr}_2 \right) = (\varnothing, \varnothing)$$

because $(\mathcal{G}_j)_0 \cap (\mathcal{G}_l)_0 = \varnothing$. Instead, for every $j \in I$ we have the following isomorphism of right $\mathcal{G}_j$-sets:

$$\begin{aligned} \mathrm{pr}_1\colon (i_j)^* \left(\widehat{(X_j, \varsigma_j)} \right) = \left(X_j\, {}_{\hat{\varsigma}_j}\!\times_{(i_j)_0} (\mathcal{G}_j)_0\,, \mathrm{pr}_2 \right) &\longrightarrow (X_j, \varsigma_j) \\ (x, c) &\longrightarrow x. \end{aligned}$$

Continuing from formula (5.1.2) we obtain that

$$\mathscr{L}(i_l)\, f(a) = [(X_l, \varsigma_l)] = f_l(a), \quad \text{for every} \;\; l \in I, \;\text{ and } a \in A.$$

This shows that the diagram (5.1.1) commutes.

The fact that f is a morphism of rigs, that is, f is compatible with the addition and the multiplication, is proved by direct computations using in part Lemma 1.3.5. Lastly, if $\gamma\colon A \longrightarrow \mathscr{L}(\mathcal{G})$ is another homomorphism of rigs which turns commutative diagrams (5.1.1), then for a given $a \in A$, let be $(X, \varsigma) \in \mathsf{Sets}\text{-}\mathcal{G}$ such that $\gamma(a) = [(X, \varsigma)]$. Setting, for every $j \in I$, $X_j = \varsigma_j^{-1}(\mathcal{G}_{l\,0})$ and $\varsigma_l = \varsigma|_{\varsigma^{-1}(\mathcal{G}_l)} : X_l \longrightarrow (\mathcal{G}_l)_0$, and restricting appropriately the action, we have

$$\gamma(a) = \left[\biguplus_{j \in I} \widehat{(X_j, \varsigma_j)} \right].$$

Thus, using properties of $(i_j)^*$ already proved in this proof, we get

$$f_i(a) = \mathscr{L}(i_j)(\gamma(a)) = \left[(i_j)^* (X, \varsigma) \right] = [(X_j, \varsigma_j)]$$

Therefore, by definition of f, we obtain that $f(a) = \gamma(a)$, for every $a \in A$, and this shows that f is unique and finishes the proof. □

5.1.2 "Classical" Burnside ring functor and product decomposition.

Now we introduce, using the Burnside rig functor, the classical Burnside ring functor, and give our main result dealing with the decomposition of the Burnside ring of a given groupoid as a product of the classical Burnside rings of its isotropy groups.

Definition 5.1.4. We define the **Burnside ring functor** $\mathscr{B}$ as the composition of the Burnside rig functor $\mathscr{L}$ with the Grothendieck functor $\mathscr{G}$, that is, $\mathscr{B} = \mathscr{G}\mathscr{L}$.

The situation is explained in the following commutative diagrams of functors:

$$\begin{array}{ccc} \mathbf{Grpd} & \xrightarrow{\mathscr{B}} & \mathbf{CRing} \\ {\scriptstyle \mathscr{L}}\downarrow & \nearrow_{\mathscr{G}} & \\ \mathbf{Rig} & & \end{array}$$

where **CRing** denotes the category of commutative rings. Of course, since $\mathscr{L}$ is contravariant functor and $\mathscr{G}$ is a covariant one, $\mathscr{B}$ is a contravariant functor.

Theorem 5.1.5. *Let be $\mathcal{G}$ and $\mathcal{A}$ be groupoids such that there is a Laplaza equivalence of categories*

$$\mathsf{Sets}\text{-}\mathcal{G} \simeq \mathsf{Sets}\text{-}\mathcal{A},$$

that is, with respect to both $\uplus$ and the fibre product. Then there is an isomorphism of commutative rings

$$\mathscr{B}(\mathcal{G}) \cong \mathscr{B}(\mathcal{A}).$$

In particular two weakly equivalent groupoids (see Definition 4.3.1*) have isomorphic classical Burnside rings.*

Proof. Let us denote by

$$F\colon \mathsf{Sets}\text{-}\mathcal{G} \longrightarrow \mathsf{Sets}\text{-}\mathcal{A} \qquad \text{and} \qquad G\colon \mathsf{Sets}\text{-}\mathcal{A} \longrightarrow \mathsf{Sets}\text{-}\mathcal{G}$$

the strong monoidal functors which give the stated equivalence. Thanks to Proposition B.0.2, it is enough to prove that there is an isomorphism of rigs $\mathscr{L}(\mathcal{G}) \cong \mathscr{L}(\mathcal{A})$. By Lemma 5.1.1, we have the following

$$\begin{aligned} \mathsf{f}\colon \mathscr{L}(\mathcal{G}) &\longrightarrow \mathscr{L}(\mathcal{A}) \\ [(X,\varsigma)] &\longrightarrow [F((X,\varsigma))] \end{aligned} \qquad\qquad \begin{aligned} \mathsf{g}\colon \mathscr{L}(\mathcal{A}) &\longrightarrow \mathscr{L}(\mathcal{G}) \\ [(Y,\vartheta)] &\longrightarrow [G((Y,\vartheta))] \end{aligned}$$

well defined homomorphism of rigs. It is left to reader to check that f and g are mutually inverse. ☐

Remark 5.1.6. Observe that, for every finite right $\mathcal{G}$-sets (X,ς), (Y,ϑ), (Z,ζ) and (W,ω), we have that

$$\Big[\, [(X,\varsigma)]\,, [(Y,\vartheta)] \,\Big] = \Big[\, [(Z,\zeta)]\,, [(W,\omega)] \,\Big],$$

as elements in $\mathscr{B}(\mathcal{G})$ (the notation is the one adopted in Appendix B), if and only if there is a finite right $\mathcal{G}$-set (U,υ) such that

$$[(X,\varsigma)] + [(W,\omega)] + [(U,\upsilon)] = [(Z,\zeta)] + [(Y,\vartheta)] + [(U,\upsilon)].$$

If the groupoid $\mathcal{G}$ is locally strongly finite, thanks to Corollary 2.2.8, this is equivalent to say that $[(X,\varsigma)] + [(W,\omega)] = [(Z,\zeta)] + [(Y,\vartheta)]$. Note that a locally strongly finite groupoid with a finite set of object is actually strongly finite.

Corollary 5.1.7. *The Burnside ring functor $\mathscr{B}$ sends coproduct to product. Specifically, given a family of groupoids $(\mathcal{G}_j)_{j\in I}$, let $(i_j\colon \mathcal{G}_j \longrightarrow \mathcal{G})_{j\in I}$ be their coproduct in* **Grpd***. Then*

$$\left(\mathscr{B}(i_j) : \mathscr{B}(\mathcal{G}) \longrightarrow \mathscr{B}(\mathcal{G}_j)\right)_{j\in I}$$

is the product of the family $(\mathscr{B}(\mathcal{G}_j))_{j\in I}$ in **CRing**.

Proof. Immediate from Proposition 5.1.3 and Proposition $B.0.3$. □

Given a groupoid $\mathcal{G}$, we let's fix a set $\mathrm{rep}(\mathcal{G}_0)$ of representative objects modulo the regular action of $\mathcal{G}$ over itself, by using either the source or the target. In other words, $\mathrm{rep}(\mathcal{G}_0)$ is a set of objects representing the set of connected components $\pi_0(\mathcal{G})$ of $\mathcal{G}$.

The next theorem is the main result of this section.

Theorem 5.1.8. *Given a groupoid $\mathcal{G}$, fix a set of representative objects $\mathrm{rep}(\mathcal{G}_0)$. For each $a \in \mathrm{rep}(\mathcal{G}_0)$, let $\mathcal{G}^{\langle a \rangle}$ be the connected component of $\mathcal{G}$ containing a, which we consider as a groupoid. Then we have the following isomorphism of rings:*

$$\mathscr{B}(\mathcal{G}) \cong \prod_{a \in \mathrm{rep}(\mathcal{G}_0)} \mathscr{B}\left(\mathcal{G}^{\langle a \rangle}\right).$$

Proof. Immediate from Corollary 5.1.7, since we already know that $\left(\mathcal{G}^{\langle a \rangle} \longrightarrow \mathcal{G}\right)_{a \in \mathrm{rep}(\mathcal{G}_0)}$ is a coproduct in the category of groupoids. □

Each connected component of a given groupoid is clearly a transitive groupoid, and the Burnside ring of a transitive groupoid is considered in the following proposition. First, notice that the classical Burnside ring of a group, as introduced in [Sol67] (see also [Die79]), is isomorphic to the Burnside ring of the groupoid, with a single object, that has the group itself as the only isotropy group.

Proposition 5.1.9. *Given a transitive groupoid $\mathcal{G}$, let $a \in \mathcal{G}_0$ and let $G = \mathcal{G}^a$ be its isotropy group. Let $\mathcal{A}$ be the subgroupoid of $\mathcal{G}$ such that $\mathcal{A}_0 = \{ a \}$ and $\mathcal{A}_1 = \mathcal{G}^a$. Then we have a chain of isomorphism of rings*

$$\mathscr{B}(\mathcal{G}) \;\cong\; \mathscr{B}(\mathcal{A}) \;\cong\; \mathscr{B}(G)$$

where $\mathscr{B}(G)$ is the classical Burnside ring of the group G introduced in [Sol67] (see also [Die79]).

Proof. It is immediately obtained by combining Theorems 1.3.15 and 5.1.5. □

The following corollary is the main conclusion of this section. It gives the desired decomposition of $\mathscr{B}(\mathcal{G})$ into a finite product of Burnside rings of groups, although, in a not canonical way. As a consequence, it shows that the Burnside functor, as defined in Definition 5.1.4, does not distinguish the arrows of a given groupoid. As one can see from this Corollary, the stated isomorphism depends on a given choice of a set of representative, that is, the decomposition is not unique. Let's keep the notations of Theorem 5.1.8 and Proposition 5.1.9.

Corollary 5.1.10. *Given a groupoid $\mathcal{G}$, we have the following isomorphism of rings:*

$$\mathscr{B}(\mathcal{G}) \cong \prod_{a \in \mathrm{rep}(\mathcal{G}_0)} \mathscr{B}(\mathcal{G}^a),$$

where the right hand side term is the product of commutative rings.

Proof. It follows from Proposition 5.1.9 and Theorem 5.1.8. □

Remark 5.1.11. It was proved in [Sol67] that the Burnside ring of a group G is isomorphic to a ring that is a free abelian group over the set of conjugacy classes $\mathcal{S}_G/\sim_{\mathrm{C}}$. Therefore, thanks to Corollary 5.1.10, the Burnside ring of a groupoid is also a free abelian group.

Examples 5.1.12. We expound examples of the Burnside ring of certain groupoids.

(1) It clear that if $G = \{\star\}$ is a trivial group, then $\mathscr{B}(G)$ is the ring of integers $\mathbb{Z}$. Therefore, the Burnside ring of any groupoid whose isotropy groups are trivial is the product ring $\mathbb{Z}^I$ for some set I. This is the case for instance of all the relation equivalence groupoids expounded in Example 1.1.11.

(2) Let G be a cyclic group of order a prime number $p \geqslant 2$. Thanks to Remark 5.1.11, we have the isomorphisms of abelian groups $\mathscr{B}(G) \cong \mathbb{Z}v \oplus \mathbb{Z}w$, where $v = [G/G] = [1]$ and $w = [G/1] = [G]$. Now we have to study the multiplicative structure of $\mathscr{B}(G)$. It is immediate to see that $v^2 = v$ and $vw = wv = w$. Since $|G \times G| = p^2$, we deduce that either $w^2 = [G \times G] = p^2 v$ or $w^2 = pw$. Considering that $G \times G$ can be decomposed into the p orbits $\left\{ \left(a^{i+j}, a^j\right) \mid j \in \{ 0, \ldots, p-1 \} \right\}$ for $i \in \{ 0, \ldots, p-1 \}$, we obtain $w^2 = pw$. Now it is easy to deduce that we have the following isomorphism of rings

$$\mathscr{B}(G) \cong \frac{\mathbb{Z}[X]}{\langle X^2 - pX \rangle}.$$

(3) Given not empty sets S_1, S_2 and S_3, we denote with G_1 the trivial group, with G_2 a cyclic group or order a prime $p \geqslant 2$ and with G_3 the alternating group A_5. We consider the groupoid $\mathcal{G}$ with the following three connected component: $\mathcal{G}_{S_1,G_1}$, $\mathcal{G}_{S_2,G_2}$ and $\mathcal{G}_{S_3,G_3}$. If follows from Corollary 5.1.10, the two previous examples and [Die79, page 10], that we have the following isomorphism of rings

$$\mathscr{B}(\mathcal{G}) \cong \mathscr{B}(G_1) \times \mathscr{B}(G_2) \times \mathscr{B}(G_3) \cong \mathbb{Z} \times \frac{\mathbb{Z}[X]}{\langle X^2 - pX \rangle} \times R,$$

where R is the Burnside ring of the group A_5 described in [Die79, page 10].

Remark 5.1.13. It was proved in [Dre69] that a group G is solvable if and only if the prime ideal spectrum of $\mathscr{B}(G)$ is connected. Since it is a known fact that the prime ideal spectrum of a direct product of commutative rings is the disjoint union of their spectrums, we deduce, thanks to Corollary 5.1.10, that the prime ideal spectrum of the Burnside ring of a groupoid $\mathcal{G}$ is connected if and only if $\mathcal{G}$ is transitive and it has a solvable isotropy group type.

Remark 5.1.14. Now let $(G_j)_{j\in J}$ be the connected components of the groupoid $\mathcal{G}$. Let be $A = \prod_{l\in J} \mathscr{L}(\mathcal{G}_l)$ and $R = \mathscr{L}(\mathcal{G})$: the families

$$\left(\mathscr{L}(i_j) : \mathscr{L}(\mathcal{G}) \longrightarrow \mathscr{L}(\mathcal{G}_j)\right)_{j\in J} \qquad \text{and} \qquad \left(\pi_j \colon \prod_{l\in J} \mathscr{L}(\mathcal{G}_l) \longrightarrow \mathscr{L}(\mathcal{G}_j)\right)_{j\in J}$$

are products in the category **Rig** therefore there are homomorphism of rigs $f\colon A \longrightarrow R$ and $h\colon R \longrightarrow A$ such that the following diagrams commute for every $j \in J$:

$$\begin{array}{ccc} A & \xrightarrow{f} & R \\ & \searrow^{\pi_j} & \downarrow{\scriptstyle L(i_j)} \\ & & \mathscr{L}(\mathcal{G}_j) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} R & \xrightarrow{h} & A \\ & \searrow^{L(i_j)} & \downarrow{\scriptstyle \pi_j} \\ & & \mathscr{L}(\mathcal{G}_j). \end{array}$$

Using the universal property of the product of rings and Lemma 1.3.12, we obtain that the following homomorphism of rigs

$$f\colon \prod_{j\in J} \mathscr{L}(\mathcal{G}_j) \longrightarrow \mathscr{L}(\mathcal{G})$$
$$\Big([(X_j, \varsigma_j)]\Big)_{j\in J} \longrightarrow \left[\biguplus_{j\in J} \widehat{(X_j, \varsigma_j)}\right]$$

and

$$h\colon \mathscr{L}(\mathcal{G}) \longrightarrow \prod_{j\in J} \mathscr{L}(\mathcal{G}_j)$$
$$[(X,\varsigma)] \longrightarrow \left(\left[\left(X\ {}_{\varsigma}\!\times_{\mathrm{Id}_{(\mathcal{G}_j)_0}} (\mathcal{G}_j)_0, \mathrm{pr}_2\right)\right]\right)_{j\in J} = \left(\left[\left(\varsigma^{-1}\left((\mathcal{G}_j)_0\right), \varsigma|_{\varsigma^{-1}\left((\mathcal{G}_j)_0\right)}\right)\right]\right)_{j\in J}$$

are isomorphism such that $h = f^{-1}$. It is now obvious that $\mathscr{G}(f)$ and $\mathscr{G}(h)$ are isomorphism of rings between $\mathscr{B}(\mathcal{G})$ and $\prod_{j\in J} \mathscr{B}(\mathcal{G}_j)$.

Remark 5.1.15. We note that Definition 4.4.1 can't be used to construct a new Burnside ring because the functor $- \rtimes \mathcal{G}$ is not monoidal.

5.2 The Burnside algebra of a groupoid and the ghost map

In this section we will continue to assume that $\mathcal{G}$ is a groupoid with a finite set of object $\mathcal{G}_0$. We define the **Burnside algebra of $\mathcal{G}$ over** $\mathbb{Q}$ as $\mathbb{Q}\mathscr{B}(\mathcal{G}) = \mathbb{Q} \otimes_{\mathbb{Z}} \mathscr{B}(\mathcal{G})$ and, given a group G, its Burnside algebra over $\mathbb{Q}$ is defined as $\mathbb{Q}\mathscr{B}(G) = \mathbb{Q} \otimes_{\mathbb{Z}} \mathscr{B}(G)$ (see [Bou10a, Page 31]). Thanks to Corollary 5.1.10 we have

$$\mathscr{B}(\mathcal{G}) \cong \prod_{a\,\in\,\mathrm{rep}(\mathcal{G}_0)} \mathscr{B}(\mathcal{G}^a)$$

and, tensoring with $\mathbb{Q}$, we obtain, since over a finite set the direct product and the direct sum of $\mathbb{Z}$-modules coincide,

$$\mathbb{Q}\mathscr{B}(\mathcal{G}) = \mathbb{Q}\otimes_{\mathbb{Z}} \mathscr{B}(\mathcal{G}) \cong \mathbb{Q}\otimes_{\mathbb{Z}} \prod_{a\,\in\,\mathrm{rep}(\mathcal{G}_0)} \mathscr{B}(\mathcal{G}^a) \cong \prod_{a\,\in\,\mathrm{rep}(\mathcal{G}_0)} (\mathbb{Q}\otimes_{\mathbb{Z}} \mathscr{B}(\mathcal{G}^a)) = \prod_{a\,\in\,\mathrm{rep}(\mathcal{G}_0)} \mathbb{Q}\mathscr{B}(\mathcal{G}^a).$$

Therefore also the Burnside algebra $\mathbb{Q}\mathscr{B}(\mathcal{G})$ is a split semi simple commutative $\mathbb{Q}$-algebra, exactly like the Burnside algebra of a group. As a consequence the idempotents of $\mathbb{Q}\mathscr{B}(\mathcal{G})$ are in a bijective correspondence with the set of elements $(x_a)_{a\,\in\,\mathrm{rep}(\mathcal{G}_0)}$, where x_a is an idempotent of $\mathbb{Q}\mathscr{B}(\mathcal{G}^a)$ for each $a \in \mathrm{rep}(\mathcal{G})$. We recall that the idempotents of the Burnside algebra $\mathbb{Q}\mathscr{B}(G)$ of a group G were completely characterized in [Bou10a, Theorem 2.5.2].

Example 5.2.1. Keep the notation of Example 5.1.12.

(1) It has been stated in Example 5.1.12 that the Burnside ring of the trivial group is $\mathbb{Z}$ therefore, of course, its Burnside algebra is $\mathbb{Q}$ whose only idempotents are 0 and 1. This implies that the Burnside algebra of a groupoid $\mathcal{G}$ with all isotropy group types trivial and a finite set of objects is $\prod_{a\,\in\,\mathrm{rep}(\mathcal{G}_0)} \mathbb{Q}$. Therefore, this can be applied to any of the groupoids given in Example 1.1.11.

(2) Let G be a cyclic group of order a prime $p \geqslant 2$. Thanks to Example 5.1.12 we know that $\mathscr{B}(G) \cong \mathbb{Z}v \oplus \mathbb{Z}w$, where

$$v = [G/G] = [1],\ w = [G/1] = [G],\ v^2 = v,\ vw = wv = w \text{ and } w^2 = pw.$$

We will use [Bou10a, Theorem 2.5.2]: since the only subgroups of G are only G itself and 1, we have that $\mu(1,1) = \mu(G,G) = 1$ and $\mu(1,G) = -1$ where μ is the Moebius function on the poset of subgroups of G. Applying the quoted theorem and computing, we obtain

$$e_1^G = \frac{1}{p}\mu(1,1)\left[\frac{G}{1}\right] = \frac{1}{p}w$$

and

$$e_G^G = \frac{1}{p}\left(\mu(1,G)\left[\frac{G}{1}\right] + p\mu(G,G)\left[\frac{G}{G}\right]\right) = \frac{-1}{p}w + v,$$

the two primitive idempotents of $\mathbb{Q}\mathscr{B}(G)$. Notice that, in the case of a cyclic group of order p, we can, by abuse of notation, avoid distinguishing a subgroup of G from its conjugacy class. Trying to rewrite e_1^G and e_G^G with the notations used in this paper we obtain

$$e_1^G = \frac{1}{p} \otimes_{\mathbb{Z}} \left[[G]\,,0\right]$$

and

$$e_G^G = \frac{-1}{p} \otimes_{\mathbb{Z}} \left[[G]\,,0\right] + 1 \otimes_{\mathbb{Z}} \left[[1]\,,0\right].$$

(3) Now let's consider a groupoid $\mathcal{G}$ with two connected components such that $\mathcal{G}_0 = \{\, a, b \,\}$, $\mathcal{G}^a$ is a trivial group and $\mathcal{G}^b$ is the cyclic group of order p. We consider the subgroupoids $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A}_0 = \{\, a \,\}$, $\mathcal{A}_1 = \{\, \iota_a \,\}$, $\mathcal{B}_0 = \{\, b \,\}$ and $\mathcal{B}_1 = \{\, \iota_b \,\}$. We denote with $\varsigma\colon \mathcal{A} \longrightarrow \mathcal{G}_0$ a structure map with image $\{\, a \,\}$ and with $\vartheta\colon \mathcal{G}/\mathcal{B} \longrightarrow \mathcal{G}_0$ and $\gamma\colon \mathcal{G}/\mathcal{B}^{(b)} \longrightarrow \mathcal{G}_0$ two structures maps with image $\{\, b \,\}$. Thanks to Remark 5.1.14 we deduce that the Burnside algebra $\mathbb{Q}\mathscr{B}(\mathcal{G})$ has the following four primitive idempotents:

$$\begin{aligned}
e_1 &= 1 \otimes_{\mathbb{Z}} [0,0] = 0,\\
e_2 &= 1 \otimes_{\mathbb{Z}} \left[[(\mathcal{A}_0, \varsigma)]\,,0\right],\\
e_3 &= \frac{1}{p} \otimes_{\mathbb{Z}} \left[\left[\left(\frac{\mathcal{G}}{\mathcal{B}}, \vartheta\right)\right],0\right],\\
e_4 &= \frac{-1}{p} \otimes_{\mathbb{Z}} \left[\left[\left(\frac{\mathcal{G}}{\mathcal{B}}, \vartheta\right)\right],0\right] + 1 \otimes_{\mathbb{Z}} \left[\left[\left(\frac{\mathcal{G}}{\mathcal{G}^{(b)}}, \gamma\right)\right],0\right].
\end{aligned}$$

In the subsequent, we are going to construct the Ghost map of the groupoid $\mathcal{G}$ and prove that is injective. Let $\mathcal{H} \in \operatorname{rep}(\mathcal{S}_{\mathcal{G}})$ be a subgroupoid of $\mathcal{G}$ with only one object a. We want to prove that the function

$$\begin{aligned}
\varphi_{\mathcal{H}}\colon \mathscr{L}(\mathcal{G}) &\longrightarrow \mathbb{N}\\
[(X,\varsigma)] &\longrightarrow \left|X^{\mathcal{H}}\right|
\end{aligned}$$

is a homomorphism of rigs. We have $\varphi_{\mathcal{H}}([\varnothing]) = 0$ and $\varphi_{\mathcal{H}}([\mathcal{G}_0]) = \left|\mathcal{G}_0^{\mathcal{H}}\right| = |\{\, a \,\}| = 1$. Given finite right $\mathcal{G}$-sets (X,ς) and (Y,ϑ), we calculate

$$\begin{aligned}
\varphi_{\mathcal{H}}([X,\varsigma] + [Y,\vartheta]) &= \varphi_{\mathcal{H}}([X \uplus Y, \varsigma \uplus \vartheta]) = \left|(X \uplus Y, \varsigma \uplus \vartheta)^{\mathcal{H}}\right|\\
&= \left|(X,\varsigma)^{\mathcal{H}}\right| + \left|(Y,\vartheta)^{\mathcal{H}}\right| = \varphi_{\mathcal{H}}([X,\varsigma]) + \varphi_{\mathcal{H}}([Y,\vartheta])
\end{aligned}$$

and

$$\begin{aligned}\varphi_{\mathcal{H}}\left([X,\varsigma]\,[Y,\vartheta]\right) &= \varphi_{\mathcal{H}}\left(\left[X \underset{\mathcal{G}_0}{\times} Y, \varsigma\vartheta\right]\right) = \left|\left(X \underset{\mathcal{G}_0}{\times} Y, \varsigma\vartheta\right)^{\mathcal{H}}\right| = \left|\left((\varsigma\vartheta)^{-1}(a)\right)^{\mathcal{H}}\right| \\ &= \left|\left(\varsigma^{-1}(a)\right)^{\mathcal{H}} \times \left(\vartheta^{-1}(a)\right)^{\mathcal{H}}\right| = \left|\left(\varsigma^{-1}(a)\right)^{\mathcal{H}}\right| \left|\left(\vartheta^{-1}(a)\right)^{\mathcal{H}}\right| \\ &= \left|(X,\varsigma)^{\mathcal{H}}\right| \left|(Y,\vartheta)^{\mathcal{H}}\right| = \varphi_{\mathcal{H}}\left([X,\varsigma]\right)\varphi_{\mathcal{H}}\left([Y,\vartheta]\right).\end{aligned}$$

Applying the Grothendieck functor $\mathscr{G}$ we obtain the homomorphism of rings

$$\begin{aligned}\mathscr{G}\left(\varphi_{\mathcal{H}}\right)\colon \mathscr{B}\left(\mathcal{G}\right) = \mathscr{G}\left(\mathscr{L}\left(\mathcal{G}\right)\right) &\longrightarrow \mathscr{G}\left(\mathbb{N}\right) = \mathbb{Z} \\ \left[[X],[Y]\right] &\longrightarrow \left|X^{\mathcal{H}}\right| - \left|Y^{\mathcal{H}}\right|.\end{aligned}$$

Using the universal property of the direct product, we are able to define the following homomorphism of rings, which is called the **ghost map of the groupoid** $\mathcal{G}$:

$$\begin{aligned}\mathfrak{g}\colon \mathscr{B}\left(\mathcal{G}\right) &\longrightarrow \prod_{\mathcal{H}\in\operatorname{rep}(\mathcal{S}_{\mathcal{G}})} \mathbb{Z} \\ \Big[[X],[Y]\Big] &\longrightarrow \left(\left|X^{\mathcal{H}}\right| - \left|Y^{\mathcal{H}}\right|\right)_{\mathcal{H}\in\operatorname{rep}(\mathcal{S}_{\mathcal{G}})}.\end{aligned} \tag{5.2.1}$$

Now let's suppose that there are $[[X],[Y]],[[A],[B]] \in B\left(\mathcal{G}\right)$ such that $\mathfrak{g}\left([[X],[Y]]\right) = \mathfrak{g}\left([[A],[B]]\right)$. Then that for each $\mathcal{H} \in \operatorname{rep}\left(\mathcal{S}_{\mathcal{G}}\right)$ we obtain

$$\left|(X \uplus B)^{\mathcal{H}}\right| = \left|X^{\mathcal{H}}\right| + \left|B^{\mathcal{H}}\right| = \left|A^{\mathcal{H}}\right| + \left|Y^{\mathcal{H}}\right| = \left|(A \uplus Y)^{\mathcal{H}}\right|.$$

which, thanks to the Burnside Theorem 2.2.7, implies $X \uplus B \cong A \uplus B$. As a consequence we have $[X] + [B] = [A] + [Y]$, therefore $\big[[X],[Y]\big] = \big[[A],[B]\big]$ and $\mathfrak{g}$ is injective. We have now proved the following result.

Corollary 5.2.2. *Given a groupoid* $\mathcal{G}$, *its ghost map* $\mathfrak{g}$, *as defined in equation* (5.2.1), *is injective.*

Chapter 6

The categorified Burnside ring

We discovered, in Chapter 5, that the Burnside contravariant functor does not distinguish between a given groupoid and its bundle of isotropy groups. Specifically, it has been realized that, under appropriate finiteness conditions, the classical Burnside ring of a given groupoid is isomorphic to the product of the Burnside rings of its isotropy group types, although, not in a canonical way.

The crux of the matter is that the isomorphism relation between finite (and not finite) groupoid-sets leads only to the consideration of (right) cosets by subgroupoids with a single object, and this, somehow, obscures the whole structure of the handled groupoid. In other words, the classical Burnside ring of a (finite) groupoid does not reflect, as in the classical case, the whole "lattice" of subgroupoids, since the subgroupoids with several objects are not relevant in this context.

In this chapter we attempt to give an alternative approach to the Burnside ring of groupoids, considering the category (2-category, actually) of internal categories inside the category of (right) groupoid-sets. The objects of this category (also called 0-cells) are named categorified groupoid-sets and, by abuse of terminology, the associated ring is called the categorified Burnside ring of the given groupoid. A point is worth mentioning: albeit, in this way, we get a commutative ring that strictly contains the classical one, we'll show that this new ring also can be decomposed, in a not canonical way, as a product of rings, which are the categorified Burnside rings of the isotropy group types of the groupoid. This makes manifest that also the idea of employing the categorification of the notion of groupoid-sets does not reflect the groupoid structural characteristics.

The main idea is to categorify the notion of groupoid action on a set to obtain a particular category, with a groupoid action on both the set of objects and of morphisms. Moreover, the source, target, identity and composition maps of this category will have to be compatible with the groupoid action. Regarding the usual right translation groupoid, it will be replaced by a right translation double category (an internal category in the category of small categories) to illustrate the new higher dimensional situation.

After this, using these concepts, we elaborate a new Burnside theory based on a particular notion of weak equivalence between these new categories endowed with a groupoid action.

6.1 Internal categories and basic definitions

In this section we recall the notion of internal categories in small categories with pullbacks, and we use this notion to introduce what we will call the category of right categorified groupoid-sets. This is a categorificaton of the usual notion of right groupoid-set object.

The fibre product of equation 1.1.1 can be generalized: given small categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$, and functors $F\colon \mathcal{A} \longrightarrow \mathcal{D}$ and $G\colon \mathcal{B} \longrightarrow \mathcal{D}$, we set, for $i = 0, 1$,

$$(\mathcal{A}\,{}_F\!\times_G \mathcal{B})_i = \{\, (a, b) \in \mathcal{A}_i \times \mathcal{B}_i \mid F(a) = G(b) \,\}. \tag{6.1.1}$$

The category $\mathcal{A}\,{}_F\!\times_G \mathcal{B}$ is called the **fiber product** (or **fibre product**) of f and g and it is the pullback of the functors F and G in the category of small categories. The following Definition 6.1.1 is taken from [BC04, pag. 495].

Definition 6.1.1. Given a category with pullbacks $\mathcal{C}$, we define an **internal category** $\mathcal{X}$ **in** $\mathcal{C}$ as a couple of objects $\mathcal{X}_0$ and $\mathcal{X}_1$ of $\mathcal{C}$ and morphisms

$$\mathcal{X}_1 \underset{\mathsf{s}_{\mathcal{X}}}{\overset{\mathsf{t}_{\mathcal{X}}}{\rightrightarrows}} \mathcal{X}_0 \xrightarrow{\ \iota_{\mathcal{X}}\ } \mathcal{X}_1 \xleftarrow{\ m_{\mathcal{D}}\ } \mathcal{X}_2 := \mathcal{X}_1\,{}_{\mathsf{s}_{\mathcal{X}}}\!\times_{\mathsf{t}_{\mathcal{X}}} \mathcal{X}_1\,,$$

where $\mathsf{s}_{\mathcal{X}}$ and $\mathsf{t}_{\mathcal{X}}$ are called the **source** and the **target morphisms**, respectively, $\iota_{\mathcal{X}}$ is called the **identity morphism** and m_X is called the **composition morphism**, or "**multiplication morphism**", such that the following diagrams are commutative (note that we will use the notation $\mathcal{X}_3 := \mathcal{X}_1\,{}_{\mathsf{s}_{\mathcal{X}}}\!\times_{\mathsf{t}_{\mathcal{X}}} \mathcal{X}_1\,{}_{\mathsf{s}_{\mathcal{X}}}\!\times_{\mathsf{t}_{\mathcal{X}}} \mathcal{X}_1$).

$$\begin{array}{ccc} \mathcal{X}_0 & \xrightarrow{\iota_{\mathcal{X}}} & \mathcal{X}_1 \\ {\scriptstyle\iota_{\mathcal{X}}}\downarrow & \searrow^{\mathrm{Id}_{\mathcal{X}_0}} & \downarrow{\scriptstyle\mathsf{s}_{\mathcal{X}}} \\ \mathcal{X}_1 & \xrightarrow{\mathsf{t}_{\mathcal{X}}} & \mathcal{X}_0 \end{array} \qquad \begin{array}{ccc} \mathcal{X}_2 & \xrightarrow{m_{\mathcal{X}}} & \mathcal{X}_1 \\ {\scriptstyle\mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle\mathsf{t}_{\mathcal{X}}} \\ \mathcal{X}_1 & \xrightarrow{\mathsf{t}_{\mathcal{X}}} & \mathcal{X}_0 \end{array} \qquad \begin{array}{ccc} \mathcal{X}_2 & \xrightarrow{m_{\mathcal{X}}} & \mathcal{X}_1 \\ {\scriptstyle\mathrm{pr}_2}\downarrow & & \downarrow{\scriptstyle\mathsf{s}_{\mathcal{X}}} \\ \mathcal{X}_1 & \xrightarrow{\mathsf{s}_{\mathcal{X}}} & \mathcal{X}_0 \end{array} \qquad \begin{array}{ccc} \mathcal{X}_3 & \xrightarrow{m_{\mathcal{X}}\times \mathrm{Id}_{\mathcal{X}_1}} & \mathcal{X}_2 \\ {\scriptstyle\mathrm{Id}_{\mathcal{X}}\times m_{\mathcal{X}_1}}\downarrow & & \downarrow{\scriptstyle m_{\mathcal{X}}} \\ \mathcal{X}_2 & \xrightarrow{m_{\mathcal{X}}} & \mathcal{X}_1 \end{array}$$

$$\begin{array}{ccccc} \mathcal{X}_0\,{}_{\mathrm{Id}_{\mathcal{X}_0}}\!\times_{\mathsf{t}_{\mathcal{X}}} \mathcal{X}_1 & \xrightarrow{\iota_{\mathcal{X}}\times \mathrm{Id}_{\mathcal{X}_1}} & \mathcal{X}_1\,{}_{\mathsf{s}_{\mathcal{X}}}\!\times_{\mathsf{t}_{\mathcal{X}}} \mathcal{X}_1 & \xleftarrow{\mathrm{Id}_{\mathcal{X}_1}\times \iota_{\mathcal{X}}} & \mathcal{X}_1\,{}_{\mathsf{s}_{\mathcal{X}}}\!\times_{\mathrm{Id}_{\mathcal{X}_0}} \mathcal{X}_0 \\ & \searrow_{\mathrm{pr}_2} & \downarrow{\scriptstyle m_{\mathcal{X}}} & \swarrow_{\mathrm{pr}_1} & \\ & & \mathcal{X}_1 & & \end{array}$$

Internal categories constitute the 0-cells of a specific 2-category (see Definition 6.1.4).

Definition 6.1.2. Given a category with pullbacks $\mathcal{C}$, let's consider two internal categories $\mathcal{X}$ and $\mathcal{Y}$ in $\mathcal{C}$. We define an **internal functor** $F\colon \mathcal{X} \longrightarrow \mathcal{Y}$ **in** $\mathcal{C}$ as a couple of morphism $F_0\colon \mathcal{X}_0 \longrightarrow \mathcal{Y}_0$ and $F_1\colon \mathcal{X}_1 \longrightarrow \mathcal{Y}_1$ such that the following diagrams are commutative, where we use the notation $F_2 = F_1 \times F_1\colon \mathcal{X}_2 \longrightarrow \mathcal{Y}_2$.

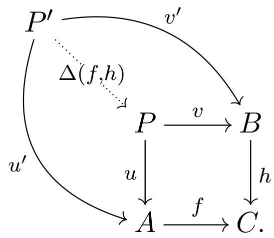
$$\begin{array}{ccc} \mathcal{X}_1 & \xrightarrow{\mathsf{s}_{\mathcal{X}}} & \mathcal{X}_0 \\ {\scriptstyle F_1}\downarrow & & \downarrow{\scriptstyle F_0} \\ \mathcal{Y}_1 & \xrightarrow{\mathsf{s}_{\mathcal{Y}}} & \mathcal{Y}_0 \end{array} \qquad \begin{array}{ccc} \mathcal{Y}_1 & \xrightarrow{\mathsf{t}_{\mathcal{X}}} & \mathcal{X}_0 \\ {\scriptstyle F_1}\downarrow & & \downarrow{\scriptstyle F_0} \\ \mathcal{Y}_1 & \xrightarrow{\mathsf{t}_{\mathcal{Y}}} & \mathcal{Y}_0 \end{array} \qquad \begin{array}{ccc} \mathcal{X}_0 & \xrightarrow{\iota_{\mathcal{X}}} & \mathcal{X}_1 \\ {\scriptstyle F_0}\downarrow & & \downarrow{\scriptstyle F_1} \\ \mathcal{Y}_0 & \xrightarrow{\iota_{\mathcal{Y}}} & \mathcal{Y}_1 \end{array} \qquad \begin{array}{ccc} \mathcal{X}_2 & \xrightarrow{m_{\mathcal{X}}} & \mathcal{X}_1 \\ {\scriptstyle F_2}\downarrow & & \downarrow{\scriptstyle F_1} \\ \mathcal{Y}_2 & \xrightarrow{m_{\mathcal{Y}}} & \mathcal{Y}_1 \end{array}$$

Before giving the next definition we have to state precisely the universal property of the pullback. Given a category $\mathcal{E}$, let us consider two morphism $f\colon A \longrightarrow C$ and $h\colon B \longrightarrow C$ in $\mathcal{E}$.

We say that a triple given by an object P and two morphisms $u\colon P \longrightarrow A$ and $v\colon P \longrightarrow B$ of $\mathcal{E}$ constitute the pullback of f and h if the following diagram is commutative

$$\begin{array}{ccc} P & \xrightarrow{v} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle h} \\ A & \xrightarrow{f} & C \end{array} \tag{6.1.2}$$

and if, given another object P' of $\mathcal{E}$ with two other morphisms $u'\colon P' \longrightarrow A$ and $v'\colon P \longrightarrow B$ in $\mathcal{E}$, there is a unique morphism $\Delta(f,h)\colon P' \longrightarrow P$ such that the following diagram is commutative:



Definition 6.1.3. Given a category with pullbacks $\mathcal{C}$, let's consider two internal categories $\mathcal{X}$ and $\mathcal{Y}$ in $\mathcal{C}$, two internal functors $F, G\colon \mathcal{X} \longrightarrow \mathcal{Y}$ in $\mathcal{C}$, we define an **internal natural transformation** $\alpha\colon F \longrightarrow G$ **in** $\mathcal{C}$ as a morphism $\alpha\colon \mathcal{X}_0 \longrightarrow \mathcal{Y}_1$ in $\mathcal{C}$ such that the following diagrams are commutative.

$$\begin{array}{ccc} \mathcal{X}_0 & \xrightarrow{\alpha} & \mathcal{Y}_1 \\ & {\scriptstyle F_0}\searrow & \downarrow{\scriptstyle \mathsf{s}_{\mathcal{X}}} \\ & & \mathcal{Y}_0 \end{array} \qquad \begin{array}{ccc} \mathcal{X}_0 & \xrightarrow{\alpha} & \mathcal{Y}_1 \\ & {\scriptstyle G_0}\searrow & \downarrow{\scriptstyle \mathsf{t}_{\mathcal{X}}} \\ & & \mathcal{Y}_0 \end{array} \qquad \begin{array}{ccc} \mathcal{X}_1 & \xrightarrow{\Delta(G_1, \alpha\mathsf{s}_{\mathcal{X}})} & \mathcal{Y}_2 \\ {\scriptstyle \Delta(\alpha\mathsf{t}_{\mathcal{X}}, F_1)}\downarrow & & \downarrow{\scriptstyle m_{\mathcal{Y}}} \\ \mathcal{Y}_2 & \xrightarrow{m_{\mathcal{Y}}} & \mathcal{Y}_1 \end{array}$$

Internal natural transformations can be composed (horizontally or vertically) in a similar way to ordinary natural transformation and we refer to [BC04, Pag. 498] for a more detailed explanation.

We note that the experienced reader will not fail to see the similarities between the theory of internal categories and enriched category theory (see [Kel05]). In this work, however, we chose to keep the internal categories approach already used by Baez in [BL04] and [BC04].

Before giving the next definition we have to precise the construction of pullbacks in $\mathsf{Sets}\text{-}\mathcal{G}$. Let $f\colon (X, \varsigma) \longrightarrow (Z, \omega)$ and $h\colon (Y, \theta) \longrightarrow (Z, \omega)$ be two $\mathcal{G}$-equivariant maps. The pullback of f and h in $\mathsf{Sets}\text{-}\mathcal{G}$ is given by the set $\mathcal{X}\ {}_f\!\times_h \mathcal{Y}$ with the structure map and action given, respectively, by

$$\begin{array}{rl} \varsigma\theta\colon \mathcal{X}\ {}_f\!\times_h \mathcal{Y} \longrightarrow \mathcal{G}_0 \\ (x,y) \longrightarrow \varsigma(x) = \theta(y) \end{array} \qquad \text{and} \qquad \begin{array}{l} (\mathcal{X}\ {}_f\!\times_h \mathcal{Y})\ {}_{\varsigma\theta}\!\times_{\mathsf{t}} \mathcal{G}_1 \\ \big((x,y), g\big) \longrightarrow (xg, yg). \end{array}$$

The maps u and v of Diagram (6.1.2) are given by pr_1 and pr_2, respectively.

Definition 6.1.4. Given a groupoid $\mathcal{G}$, we define a **right categorified** $\mathcal{G}$**-set** as an internal category in the category of right $\mathcal{G}$-sets, a **morphism of right categorified** $\mathcal{G}$**-sets** as an internal functor in the category of right $\mathcal{G}$-sets and a **2-morphism between morphisms of right categorified** $\mathcal{G}$**-sets** as an internal natural transformation in $\mathsf{Sets}\text{-}\mathcal{G}$. In this way, thanks to [Ehr63a] and [BC04, Prop. 2.4], we obtain a 2-category that we denote with $\mathsf{CSets}\text{-}\mathcal{G}$. We will also employ the terminology **right categorified groupoid-set**, whenever the groupoid under consideration isn't relevant. The category of left categorified $\mathcal{G}$-set is similarly defined, and clearly Laplaza isomorphic to the right one (see section 6.2 for the construction of the pertinent monoidal structures).

Remark 6.1.5. The category of right simplicial $\mathcal{G}$-sets is defined as the category of functors $[\mathbf{\Delta}^{op}, \mathsf{Sets}\text{-}\mathcal{G}]$ from the opposite category of $\mathbf{\Delta}$ of finite sets $\Delta_n = \{0, 1, \cdots, n\}$, with increasing maps as arrows, to the category $\mathsf{Sets}\text{-}\mathcal{G}$ of right $\mathcal{G}$-sets. Obiously, this category "contains" $\mathsf{CSets}\text{-}G$ as a subcategory. As the reader will realize, the majority of the constructions of this chapter could be extended to $[\mathbf{\Delta}^{op}, \mathsf{Sets}\text{-}\mathcal{G}]$. Notwithstanding this, we will not further pursue this line of research here since, in our opinion, the study of the whole categoy $[\mathbf{\Delta}^{op}, \mathsf{Sets}\text{-}\mathcal{G}]$ deserves a separate project.

Remark 6.1.6. We have to note that right categorified $\mathcal{G}$-sets, morphisms of right categorified $\mathcal{G}$-sets, and the relative 2-morphisms constitute, respectively, categories, functors and natural transformations in the usual sense. This means that many definitions of the usual category theory can be extended in this setting. For example, given a right $\mathcal{G}$-set $\mathcal{X}$, an element $f \in \mathcal{X}_1$ is called an **isomorphism** if there is $h \in \mathcal{X}_1$ such that $hf = \iota_{\mathcal{X}}(\mathsf{s}_{\mathcal{X}}(f))$ and $fh = \iota_{\mathcal{X}}(\mathsf{t}_{\mathcal{X}}(f))$. In this way, we obtain a forgetful functor from the 2-category of internal categories in a category $\mathcal{C}$ to the category of ordinary small categories.

Let be $(\mathcal{X}, \varsigma)$ a right categorified $\mathcal{G}$-set. As with groupoid, given $a, b \in \mathcal{X}_0$, we will use the notation

$$\mathcal{X}(a,b) = \{\, f \in \mathcal{X}_1 \mid \mathsf{s}_{\mathcal{X}}(f) = a \quad \text{and} \quad \mathsf{t}_{\mathcal{X}}(f) = b \,\}.$$

As was mentioned above, the set $\mathcal{X}_2$ admits in a canonical way a $\mathcal{G}$-action, given by $(p,q)g = (pg, qg)$, for every $(p,q) \in \mathcal{X}_2$ and $g \in \mathcal{G}_1$ such that $\varsigma_1(p) = \varsigma_1(q) = \mathsf{t}(g)$ and, moreover, we have the equality

$$m_{\mathcal{X}}(pg, qg) = \big(m_{\mathcal{X}}(p,q)\big)g,$$

which could be rewritten as $(p \circ q)g = (pg) \circ (qg)$. In this direction, we have that a morphism $p \in \mathcal{X}_1$ is an isomorphism if and only if pg is an isomorphism for some $g \in \mathcal{G}$ such that $\varsigma_1(p) = \mathsf{t}(g)$. Furthermore, for any element $a \in \mathcal{X}_0$ and $g \in \mathcal{G}_1$ such that $\varsigma_0(a) = \mathsf{t}(g)$, we have that the map

$$\begin{aligned} \mathcal{X}(a,a) &\longrightarrow \mathcal{X}(ag, ag) \\ p &\longrightarrow pg \end{aligned}$$

is a morphism of monoids, well defined because we have $\varsigma_1(p) = \varsigma_0 \mathsf{t}_{\mathcal{X}}(p) = \varsigma_0(a) = \mathsf{t}(g)$.

In the rest of the paper we will consider $\mathsf{CSets}\text{-}\mathcal{G}$ mainly as a category (the 2-category level will be used to define the concept of weak equivalence in Definition 6.3.3).

Remark 6.1.7. Let be $(\mathcal{X}, \varsigma) \in \mathsf{CSets}\text{-}\mathcal{G}$: we consider the decomposition of $(\mathcal{X}_0, \varsigma_0)$ into orbits (i.e. transitive right G-sets) $(\mathcal{X}_0, \varsigma_0) = \biguplus_{\alpha \in A} [x_\alpha]\, \mathcal{G}$ with $x_\alpha \in \mathcal{X}_0$ for each $\alpha \in A$. Since $\iota_{\mathcal{X}}$ is a morphism of right $\mathcal{G}$-sets, for each $\alpha \in A$ we obtain $\iota_{\mathcal{X}}([x_\alpha]\, \mathcal{G}) = [\iota_{\mathcal{X}}(x_\alpha)]\, \mathcal{G}$, therefore we can state that

$$(\mathcal{X}_1, \varsigma_1) = \left(\biguplus_{\alpha \in A} [\iota_{\mathcal{X}}(x_\alpha)]\, \mathcal{G} \right) \uplus \left(\biguplus_{\beta \in B} [y_\beta]\, \mathcal{G} \right)$$

with $x_\alpha \in \mathcal{X}_0$ for each $\alpha \in A$ and $y_\beta \in \mathcal{X}_1 \backslash \iota_{\mathcal{X}}(\mathcal{X}_0)$ for each $\beta \in B$, with $B \cap A = \varnothing$. As a consequence we can state that $\mathcal{X}$ is a discrete category if and only if $\mathcal{X}_1 = \biguplus_{\alpha \in A} [\iota_{\mathcal{X}}(x_\alpha)]\, \mathcal{G}$.

Remark 6.1.8. A groupoid $\mathcal{G}$ cannot be a right categorified $\mathcal{G}$-set. If this were the case, the map $\mathsf{t}\colon \mathcal{G}_1 \longrightarrow \mathcal{G}_0$ should be compatible with the right action by $\mathcal{G}$: this means that for each $a, b \in \mathcal{G}_1$ such that $\mathsf{s}(a) = \mathsf{t}(b)$, it should be $\mathsf{t}(ab) = \mathsf{t}(a)b$ but, since the situation is as follows,

$$\mathsf{t}(a) \xleftarrow{\quad a \quad} \mathsf{s}(a) = \mathsf{t}(b) \xleftarrow{\quad b \quad} \mathsf{s}(b),$$

b cannot act on $\mathsf{t}(a)$, in general. Actually, a groupoid $\mathcal{G}$ can be seen as a "twisted" and "asymmetrical" version of a right categorified $\mathcal{G}$-set: this idea will be the object of future work.

6.2 Monoidal structures

Let $\mathcal{G}$ be a groupoid. In this section we will describe the two symmetric monoidal structures on the category of right categorified $\mathcal{G}$-sets: one is given by the disjoint union, i.e., the coproduct $\biguplus$, and the other by the fibre product $\underset{\mathcal{G}_0}{\times}$. Moreover, we will show that the fibre product is distributive with respect to the disjoint union, rendering $\mathsf{CSets}\text{-}\mathcal{G}$ a Laplaza category (see subsection 1.3.1).

We define a monoidal structure $(\mathsf{CSets}\text{-}\mathcal{G}, \biguplus, \varnothing)$, based on the disjoint union, as follows:

$$\begin{aligned} \biguplus \colon \mathsf{CSets}\text{-}\mathcal{G} \times \mathsf{CSets}\text{-}\mathcal{G} &\longrightarrow \mathsf{CSets}\text{-}\mathcal{G} \\ ((\mathcal{X}, \varsigma), (\mathcal{Y}, \theta)) &\longrightarrow (\mathcal{X}, \varsigma) \biguplus (\mathcal{Y}, \theta) = (\mathcal{X} \biguplus \mathcal{Y}, \varsigma \biguplus \theta) \end{aligned}$$

where $(\mathcal{X} \biguplus \mathcal{Y}, \varsigma \biguplus \theta)$ is defined as

$$\left((\mathcal{X}_0 \biguplus \mathcal{Y}_0, \varsigma_0 \biguplus \theta_0), (\mathcal{X}_1 \biguplus \mathcal{Y}_1, \varsigma_1 \biguplus \theta_1), \mathsf{s}_{\mathcal{X} \biguplus \mathcal{Y}}, \mathsf{t}_{\mathcal{X} \biguplus \mathcal{Y}}, \iota_{\mathcal{X} \biguplus \mathcal{Y}}, m_{\mathcal{X} \biguplus \mathcal{Y}}\right).$$

The source, target, identity and composition maps are defined, in $\mathsf{Sets}\text{-}\mathcal{G}$, as the following morphisms: $\mathsf{s}_{\mathcal{X} \biguplus \mathcal{Y}} = \mathsf{s}_{\mathcal{X}} \biguplus \mathsf{s}_{\mathcal{Y}}$, $\mathsf{t}_{\mathcal{X} \biguplus \mathcal{Y}} = \mathsf{t}_{\mathcal{X}} \biguplus \mathsf{t}_{\mathcal{Y}}$, $\iota_{\mathcal{X} \biguplus \mathcal{Y}} = \iota_{\mathcal{X}} \biguplus \iota_{\mathcal{Y}}$ and

$$m_{\mathcal{X} \biguplus \mathcal{Y}} = m_{\mathcal{X}} \biguplus m_{\mathcal{Y}} \colon \mathcal{X}_2 \biguplus \mathcal{Y}_2 \longrightarrow \mathcal{X}_1 \biguplus \mathcal{Y}_1.$$

It is clear that $(\mathcal{X} \biguplus \mathcal{Y}, \varsigma \biguplus \theta)$ is an object in $\mathsf{CSets}\text{-}\mathcal{G}$, because the commutativity of the diagrams of Definition 6.1.1 to be verified derives from the commutativity of the respective diagrams of $(\mathcal{X}, \vartheta)$ and $(\mathcal{Y}, \theta)$.

Moreover, given morphisms $\varphi\colon \mathcal{X} \longrightarrow \mathcal{Y}$ and $\psi\colon \mathcal{A} \longrightarrow \mathcal{B}$ in $\mathsf{CSets}\text{-}G$, we define the morphism $\varphi \biguplus \psi\colon \mathcal{X} \biguplus \mathcal{A} \longrightarrow \mathcal{Y} \biguplus \mathcal{B}$ in $\mathsf{CSets}\text{-}\mathcal{G}$ as the couple of morphisms $(\varphi \biguplus \psi)_0 = \varphi_0 \biguplus \psi_0$ and $(\varphi \biguplus \psi)_1 = \varphi_1 \biguplus \psi_1$ in $\mathsf{Sets}\text{-}\mathcal{G}$. As before, the commutativity of the diagrams of Definition 6.1.2 to be verified derives from the commutativity of the respective diagrams of φ and ψ.

Proposition 6.2.1. *The object*

$$\varnothing = \Big((\varnothing, \varnothing), (\varnothing, \varnothing), \varnothing, \varnothing, \varnothing, \varnothing \Big),$$

is initial in $\mathsf{CSets}\text{-}G$*,* $\biguplus$ *is a coproduct in* $\mathsf{CSets}\text{-}G$ *and* $(\mathsf{CSets}\text{-}G, \biguplus, \varnothing)$ *is a strict monoidal category.*

Proof. Immediate. $\square$

Now we want to define an "inclusion" functor from the category of normal right $\mathcal{G}$-sets to the category of right categorified $\mathcal{G}$-sets:

$$\begin{aligned} \mathcal{I}\text{-}\mathcal{G}\colon \mathsf{Sets}\text{-}\mathcal{G} &\longrightarrow \mathsf{CSets}\text{-}\mathcal{G} \\ (X,\varsigma) &\longrightarrow \Big((X,\varsigma),(X,\varsigma),\mathsf{s}_X,\mathsf{t}_X,\iota_X,m_X\Big) \end{aligned} \tag{6.2.1}$$

where $\mathsf{s}_X = \mathsf{t}_X = \iota_X = \mathrm{Id}_X$ and

$$\begin{aligned} m_X = \mathrm{pr}_1\colon (X,\varsigma)_2 &\longrightarrow (X,\varsigma) \\ (a,b) &\longrightarrow a, \end{aligned}$$

with $a = \mathrm{Id}_X(a) = \mathsf{s}_{\mathcal{X}}(a) = \mathsf{t}_X(b) = b$. Regarding the morphisms, given a morphism of right $\mathcal{G}$-sets $\varphi\colon\ (X,\varsigma)\longrightarrow (Y,\theta)$, we define

$$\mathcal{I}\text{-}\mathcal{G}(\varphi)\colon \mathcal{I}\text{-}\mathcal{G}(X,\varsigma)\longrightarrow \mathcal{I}\text{-}\mathcal{G}(Y,\theta)$$

in the following way: $(\mathcal{I}\text{-}\mathcal{G}(\varphi))_0 = \varphi$ and $(\mathcal{I}\text{-}\mathcal{G}(\varphi))_1 = \varphi$. Basically, the image of $\mathcal{I}\text{-}G$ is given by discrete categories. Moreover, we will use the abuse of notation $\mathcal{G}_0 = \mathcal{I}\text{-}G\left(\mathcal{G}_0\right)$.

Now we want to construct a monoidal structure $\left(\mathsf{CSets}\text{-}\mathcal{G}, \underset{\mathcal{G}_0}{\times}, \mathcal{G}_0\right)$. Given $(\mathcal{X},\varsigma),(\mathcal{Y},\theta)\in$ $\mathsf{CSets}\text{-}\mathcal{G}$ we define

$$\left(\mathcal{X}\underset{\mathcal{G}_0}{\times}\mathcal{Y}\right)_0 = \left(\mathcal{X}_0\underset{\mathcal{G}_0}{\times}\mathcal{Y}_0,\varsigma_0\theta_0\right), \qquad \left(\mathcal{X}\underset{\mathcal{G}_0}{\times}\mathcal{Y}\right)_1 = \left(\mathcal{X}_1\underset{\mathcal{G}_0}{\times}\mathcal{Y}_1,\varsigma_1\theta_1\right)$$

and

$$(\mathcal{X},\varsigma)\underset{\mathcal{G}_0}{\times}(\mathcal{Y},\theta) = \left(\left(\mathcal{X}_0\underset{\mathcal{G}_0}{\times}\mathcal{Y}_0,\varsigma_0\theta_0\right),\left(\mathcal{X}_1\underset{\mathcal{G}_0}{\times}\mathcal{Y}_1,\varsigma_1\theta_1\right),\mathsf{s}_{\mathcal{X}\underset{\mathcal{G}_0}{\times}\mathcal{Y}},\mathsf{t}_{\mathcal{X}\underset{\mathcal{G}_0}{\times}\mathcal{Y}},\iota_{\mathcal{X}\underset{\mathcal{G}_0}{\times}\mathcal{Y}},m_{\mathcal{X}\underset{\mathcal{G}_0}{\times}\mathcal{Y}}\right),$$

where $\mathsf{t}_{\mathcal{X}\underset{\mathcal{G}_0}{\times}\mathcal{Y}} = (\mathsf{t}_{\mathcal{X}},\mathsf{t}_{\mathcal{Y}})$, $\mathsf{s}_{\mathcal{X}\underset{\mathcal{G}_0}{\times}\mathcal{Y}} = (\mathsf{s}_{\mathcal{X}},\mathsf{s}_{\mathcal{Y}})$ and $\iota_{\mathcal{X}\underset{\mathcal{G}_0}{\times}\mathcal{Y}} = (\iota_{\mathcal{X}},\iota_{\mathcal{Y}})$. Regarding the composition, we define

$$\begin{aligned} m_{\mathcal{X}\underset{\mathcal{G}_0}{\times}\mathcal{Y}}\colon \left(\mathcal{X}\underset{\mathcal{G}_0}{\times}\mathcal{Y}\right)_2 &\longrightarrow \left(\mathcal{X}\underset{\mathcal{G}_0}{\times}\mathcal{Y}\right)_1 \\ \big((x,y),(a,b)\big) &\longrightarrow \left(m_{\mathcal{X}(x,a)},m_{\mathcal{Y}(y,b)}\right). \end{aligned}$$

It is a direct verification to check that $\mathsf{s}_{\mathcal{X}\underset{\mathcal{G}_0}{\times}\mathcal{Y}}$, $\mathsf{t}_{\mathcal{X}\underset{\mathcal{G}_0}{\times}\mathcal{Y}}$, $\iota_{\mathcal{X}\underset{\mathcal{G}_0}{\times}\mathcal{Y}}$ and $m_{\mathcal{X}\underset{\mathcal{G}_0}{\times}\mathcal{Y}}$ are well defined morphisms in the category $\mathsf{Sets}\text{-}\mathcal{G}$. We have to prove that the diagrams of Definition 6.1.1 about $\mathcal{X}\underset{\mathcal{G}_0}{\times}\mathcal{Y}$ are commutative, but this is a direct verification and follows from the analogous diagrams about $\mathcal{X}$ and $\mathcal{Y}$. Now, given morphisms $\varphi\colon \mathcal{X}\longrightarrow\mathcal{Y}$ and $\psi\colon \mathcal{A}\longrightarrow\mathcal{B}$ in $\mathsf{CSets}\text{-}\mathcal{G}$, we define the morphism

$$\varphi\underset{\mathcal{G}_0}{\times}\psi\colon \mathcal{X}\underset{\mathcal{G}_0}{\times}\mathcal{A}\longrightarrow\mathcal{Y}\underset{\mathcal{G}_0}{\times}\mathcal{B}$$

in $\mathsf{CSets}\text{-}\mathcal{G}$ as the couple of morphisms

$$(\varphi\times\psi)_0 = \varphi_0\times\psi_0 \qquad \text{and} \qquad (\varphi\times\psi)_1 = \varphi_1\times\psi_1$$

in $\mathsf{Sets}\text{-}\mathcal{G}$. We have to prove that the diagrams of Definition 6.1.2 about $\varphi\times\psi$ are commutative, but this is a direct verification and follows from the analogous diagrams about φ and ψ. As a consequence we have constructed a functor

$$\left(-\underset{\mathcal{G}_0}{\times}-\right)\colon \mathsf{CSets}\text{-}\mathcal{G}\underset{\mathcal{G}_0}{\times}\mathsf{CSets}\text{-}\mathcal{G}\longrightarrow\mathsf{CSets}\text{-}\mathcal{G}.$$

Now we want to construct natural isomorphism

$$\Phi\colon \left(\mathrm{Id}_{\mathsf{CSets}\text{-}G} \underset{\mathcal{G}_0}{\times} \mathcal{G}_0\right) \longrightarrow \mathrm{Id}_{\mathsf{CSets}\text{-}G}, \qquad \Psi\colon \left(\mathcal{G}_0 \underset{\mathcal{G}_0}{\times} \mathrm{Id}_{\mathsf{CSets}\text{-}G} \longrightarrow \mathrm{Id}_{\mathsf{CSets}\text{-}G}\right)$$

and the associator on $\mathsf{CSets}\text{-}G$

$$\left(\left(- \underset{\mathcal{G}_0}{\times} -\right) \underset{\mathcal{G}_0}{\times} -\right) \longrightarrow \left(- \underset{\mathcal{G}_0}{\times} \left(- \underset{\mathcal{G}_0}{\times} -\right)\right).$$

The associator is the identity, which is clearly a natural isomorphism and satisfies the pentagonal identity. We will construct only Φ because Ψ can be construct in a similar way. Let be $(\mathcal{X}, \varsigma)$: for $i = 0, 1$ we define

$$\begin{aligned} \Phi\left(\mathcal{X}\right)_i\colon \mathcal{X}_i \underset{\mathcal{G}_0}{\times} \mathcal{G}_0 &\longrightarrow \mathcal{X}_i \\ (a, b) &\longrightarrow a \end{aligned}$$

where $\varsigma_i(a) = b$. We have to check that $\Phi\left(\mathcal{X}\right)_i$, for $i = 0, 1$, is a well defined morphism in $\mathsf{CSets}\text{-}\mathcal{G}$ and that it's a bijection: both are direct verifications. Now we just have to prove that

$$\Phi\colon \left(\mathrm{Id}_{\mathsf{CSets}\text{-}\mathcal{G}} \underset{\mathcal{G}_0}{\times} \mathcal{G}_0\right) \longrightarrow \mathrm{Id}_{\mathsf{CSets}\text{-}\mathcal{G}}$$

is a natural transformation: let $\alpha\colon \mathcal{X} \longrightarrow \mathcal{Y}$ be a morphism in $\mathsf{CSets}\text{-}\mathcal{G}$. We have to prove that the diagram

$$\begin{matrix} \mathcal{X} \underset{\mathcal{G}_0}{\times} \mathcal{G}_0 & \xrightarrow{\Phi(\mathcal{X})} & \mathcal{X} \\ {\scriptstyle \alpha \underset{\mathcal{G}_0}{\times} \mathcal{G}_0}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \mathcal{Y} \underset{\mathcal{G}_0}{\times} \mathcal{G}_0 & \xrightarrow{\Phi(\mathcal{Y})} & \mathcal{Y} \end{matrix}$$

is commutative, which is equivalent to say that the diagram

$$\begin{matrix} \mathcal{X}_i \underset{\mathcal{G}_0}{\times} \mathcal{G}_0 & \xrightarrow{\Phi(\mathcal{X})_i} & \mathcal{X}_i \\ {\scriptstyle \alpha_i \underset{\mathcal{G}_0}{\times} \mathcal{G}_0}\downarrow & & \downarrow{\scriptstyle \alpha_i} \\ \mathcal{Y}_i \underset{\mathcal{G}_0}{\times} \mathcal{G}_0 & \xrightarrow{\Phi(\mathcal{Y})_i} & \mathcal{Y}_i \end{matrix}$$

is commutative for $i = 0, 1$. Given $i = 0, 1$, for each $(x, a) \in \mathcal{X}_i \underset{\mathcal{G}_0}{\times} \mathcal{G}_0$ we have

$$\alpha_i\left(\Phi\left(\mathcal{X}\right)_i\right)(x, a) = \alpha_i(x) = \left(\Phi\left(\mathcal{Y}\right)_i\right)(\alpha_i(x), \alpha) = \left(\Phi\left(\mathcal{Y}\right)_i\right)\left(\alpha_i \underset{\mathcal{G}_0}{\times} \mathcal{G}_0\right)(x, a).$$

Lastly, it is obvious that Φ and Ψ satisfies the triangular identity; therefore we have proved the following proposition.

Proposition 6.2.2. *The triple* $\left(\mathsf{CSets}\text{-}\mathcal{G}, \underset{\mathcal{G}_0}{\times}, \mathcal{G}_0\right)$ *is a monoidal category.*

We now have two monoidal structures, $(\mathsf{CSets}\text{-}\mathcal{G}, \uplus, \varnothing)$ and $\left(\mathsf{CSets}\text{-}\mathcal{G}, \underset{\mathcal{G}_0}{\times}, \mathcal{G}_0\right)$, and it is necessary to prove the distributivity of $\underset{\mathcal{G}_0}{\times}$ over $\uplus$. Let be $(\mathcal{X}, \varsigma), (\mathcal{Y}, \theta), (\mathcal{A}, \omega) \in \mathsf{CSets}\text{-}\mathcal{G}$: we have to construct a morphism

$$\lambda\colon \left[(\mathcal{X}, \varsigma) \uplus (\mathcal{Y}, \theta)\right] \underset{\mathcal{G}_0}{\times} (\mathcal{A}, \omega) \longrightarrow \left[(\mathcal{X}, \varsigma)\right] \underset{\mathcal{G}_0}{\times} (\mathcal{A}, \omega) \uplus \left[(\mathcal{Y}, \theta)\right] \underset{\mathcal{G}_0}{\times} (\mathcal{A}, \omega)$$

in CSets-$\mathcal{G}$. We define it as the couple of morphisms in Sets-$\mathcal{G}$, for $i = 0, 1$,

$$\lambda_i\colon\ [(\mathcal{X}_i, \varsigma_i) \uplus (\mathcal{Y}_i, \theta_i)] \underset{\mathcal{G}_0}{\times} (\mathcal{A}_i, \omega_i) \longrightarrow \left[(\mathcal{X}_i, \varsigma_i) \underset{\mathcal{G}_0}{\times} (\mathcal{A}_i, \omega_i)\right] \uplus \left[(\mathcal{Y}_i, \theta_i) \underset{\mathcal{G}_0}{\times} (\mathcal{A}_i, \omega_i)\right],$$

that send (a, b) to (a, b) both if $\varsigma_i(a) = \omega_i(b)$ with $a \in \mathcal{X}_i$ and if $\theta_i(a) = \omega_i(b)$ with $a \in \mathcal{Y}_i$. It is clear that λ is a morphism in CSets-$\mathcal{G}$.

6.3 2-morphisms and weak equivalences

We introduce the notion of weak equivalences in the category of categorified groupoid-sets. Then, we show that both operations $\uplus$ and $\underset{\mathcal{G}_0}{\times}$ are compatible with 2-functor and, therefore, with the weak equivalence relation. This will be crucial to build up the weak Burnside ring functor $\mathscr{B}_{\mathsf{C}}$ in section 6.6.

Given morphisms in CSets-$\mathcal{G}$ $\varphi, \psi\colon\ (\mathcal{X}, \varsigma) \longrightarrow (\mathcal{Y}, \theta)$ and $\varepsilon, \eta\colon\ (\mathcal{A}, \lambda) \longrightarrow (\mathcal{B}, \mu)$, let's consider 2-morphisms in CSets-$\mathcal{G}$ $\alpha\colon \varphi \longrightarrow \psi$ and $\beta\colon \varepsilon \longrightarrow \eta$: we have

$$\varphi \uplus \varepsilon,\ \psi \uplus \eta\colon\ (\mathcal{X}, \varsigma) \uplus (\mathcal{A}, \lambda) \longrightarrow (\mathcal{Y}, \theta) \uplus (\mathcal{B}, \mu)\,.$$

We want to define a 2-morphism in CSets-$\mathcal{G}$ as the morphism in Sets-$\mathcal{G}$:

$$\begin{array}{rrcl} \alpha \uplus \beta\colon & \mathcal{X}_0 \uplus \mathcal{A}_0 & \longrightarrow & \mathcal{Y}_1 \uplus \mathcal{B}_1 \\ & \mathcal{X}_0 \ni x & \longrightarrow & \alpha(x) \in \mathcal{Y}_1 \\ & \mathcal{A}_0 \ni x & \longrightarrow & \beta(x) \in \mathcal{B}_1. \end{array}$$

The verification that $\alpha \uplus \beta$ renders the diagrams of Definition 6.1.3 commutative is immediate and derives from the relative diagrams regarding α and β. Now we consider $\psi'\colon \mathcal{X} \longrightarrow \mathcal{Y}$ and $\eta'\colon \mathcal{A} \longrightarrow \mathcal{B}$, morphisms in CSets-$\mathcal{G}$, and $\alpha'\colon \psi \longrightarrow \psi'$ and $\beta'\colon \eta \longrightarrow \eta'$, 2-morphism in CSets-$\mathcal{G}$.

Lemma 6.3.1. *We have*

$$\left(\alpha' \uplus \beta'\right)\left(\alpha \uplus \beta\right) = \left(\alpha'\alpha\right) \uplus \left(\beta'\beta\right) : \varphi \uplus \varepsilon \longrightarrow \psi \uplus \eta \qquad \textit{and} \qquad \mathrm{Id}_{\varphi \uplus \psi} = \mathrm{Id}_{\varphi} \uplus \mathrm{Id}_{\psi}\,.$$

Proof. Immediate. □

Let's consider

$$\varphi \underset{\mathcal{G}_0}{\times} \varepsilon,\ \psi \underset{\mathcal{G}_0}{\times} \eta\colon\ (\mathcal{X}, \varsigma) \underset{\mathcal{G}_0}{\times} (\mathcal{A}, \lambda) \longrightarrow (\mathcal{Y}, \theta) \underset{\mathcal{G}_0}{\times} (\mathcal{B}, \mu) :$$

we want to define a 2-morphism

$$\alpha \underset{\mathcal{G}_0}{\times} \beta\colon \varphi \underset{\mathcal{G}_0}{\times} \varepsilon \longrightarrow \psi \underset{\mathcal{G}_0}{\times} \eta$$

in CSets-G as the morphism

$$\begin{array}{rcl} \alpha \underset{\mathcal{G}_0}{\times} \beta\colon \mathcal{X}_0 \underset{\mathcal{G}_0}{\times} \mathcal{A}_0 & \longrightarrow & \mathcal{Y}_1 \underset{\mathcal{G}_0}{\times} \mathcal{B}_1 \\ (x, a) & \longrightarrow & (\alpha(x), \beta(a))\,. \end{array}$$

Given $(x,a) \in \mathcal{X}_0 \underset{\mathcal{G}_0}{\times} \mathcal{A}_0$ we have $\theta_1\alpha(x) = \varsigma_0(x) = \lambda_0(a) = \mu_1\beta(a)$ thus $\alpha \underset{\mathcal{G}_0}{\times} \beta$ is well defined. It is immediate to see that $\alpha \underset{\mathcal{G}_0}{\times} \beta$ is a morphism in $\mathsf{Sets}\text{-}\mathcal{G}$. Now we have to prove that $\alpha \underset{\mathcal{G}_0}{\times} \beta$ satisfies the three following diagrams of the definition of 2-morphism in $\mathsf{CSets}\text{-}\mathcal{G}$.

$$
\begin{array}{ccc}
\mathcal{X}_0 \underset{\mathcal{G}_0}{\times} \mathcal{A}_0 & \xrightarrow{\alpha \underset{\mathcal{G}_0}{\times} \beta} & \mathcal{Y}_1 \underset{\mathcal{G}_0}{\times} \mathcal{B}_1 \\
& \searrow^{\varphi_0 \underset{\mathcal{G}_0}{\times} \varepsilon_0} & \downarrow{\scriptstyle \mathsf{s}_{\mathcal{Y} \underset{\mathcal{G}_0}{\times} \mathcal{B}}} \\
& & \mathcal{Y}_0 \underset{\mathcal{G}_0}{\times} \mathcal{B}_0,
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{X}_0 \underset{\mathcal{G}_0}{\times} \mathcal{A}_0 & \xrightarrow{\alpha \underset{\mathcal{G}_0}{\times} \beta} & \mathcal{Y}_1 \underset{\mathcal{G}_0}{\times} \mathcal{B}_1 \\
& \searrow^{\psi_0 \underset{\mathcal{G}_0}{\times} \eta_0} & \downarrow{\scriptstyle \mathsf{t}_{\mathcal{Y} \underset{\mathcal{G}_0}{\times} \mathcal{B}}} \\
& & \mathcal{Y}_0 \underset{\mathcal{G}_0}{\times} \mathcal{B}_0
\end{array}
$$

and

$$
\begin{array}{ccc}
\mathcal{X}_1 \underset{\mathcal{G}_0}{\times} \mathcal{A}_1 & \xrightarrow{\Delta\left(\psi_1 \underset{\mathcal{G}_0}{\times} \eta_1, \left(\alpha \underset{\mathcal{G}_0}{\times} \beta\right) \mathsf{s}_{\mathcal{X} \underset{\mathcal{G}_0}{\times} \mathcal{A}}\right)} & \left(\mathcal{Y}_1 \underset{\mathcal{G}_0}{\times} \mathcal{B}_1\right)_2 \\
\downarrow{\scriptstyle \Delta\left(\left(\alpha \underset{\mathcal{G}_0}{\times} \beta\right) \mathsf{t}_{\mathcal{X} \uplus \mathcal{A}}, \varphi_1 \underset{\mathcal{G}_0}{\times} \varepsilon_1\right)} & & \downarrow{\scriptstyle m_{\mathcal{Y} \underset{\mathcal{G}_0}{\times} \mathcal{B}}} \\
\left(\mathcal{Y}_1 \underset{\mathcal{G}_0}{\times} \mathcal{B}_1\right)_2 & \xrightarrow{m_{\mathcal{Y} \underset{\mathcal{G}_0}{\times} \mathcal{B}}} & \mathcal{Y}_1 \underset{\mathcal{G}_0}{\times} \mathcal{B}_1.
\end{array}
$$

The commutativity of the two triangular diagrams is obvious because $\mathsf{s}_{\mathcal{Y} \underset{\mathcal{G}_0}{\times} \mathcal{B}} = \mathsf{s}_{\mathcal{Y}} \underset{\mathcal{G}_0}{\times} \mathsf{s}_{\mathcal{B}}$ and $\mathsf{t}_{\mathcal{Y} \underset{\mathcal{G}_0}{\times} \mathcal{B}} = \mathsf{t}_{\mathcal{Y}} \underset{\mathcal{G}_0}{\times} \mathsf{t}_{\mathcal{B}}$. Regarding the commutativity of the third we calculate, for each $(x,a) \in \mathcal{X}_1 \underset{\mathcal{G}_0}{\times} \mathcal{A}_1$,

$$
\begin{aligned}
& m_{\mathcal{Y} \underset{\mathcal{G}_0}{\times} \mathcal{B}} \Delta\left(\psi_1 \underset{\mathcal{G}_0}{\times} \eta_1, \left(\alpha \underset{\mathcal{G}_0}{\times} \beta\right) \mathsf{s}_{\mathcal{X} \underset{\mathcal{G}_0}{\times} \mathcal{A}}\right)(x,a) = m_{\mathcal{Y} \underset{\mathcal{G}_0}{\times} \mathcal{B}}\Big((\psi_1(x), \eta_1(a)), (\alpha \mathsf{s}_{\mathcal{X}}(x), \beta \mathsf{s}_{\mathcal{A}}(a)) \Big) \\
&= \Big(m_{\mathcal{Y}}(\psi_1(x), \alpha \mathsf{s}_{\mathcal{X}}(x)), m_{\mathcal{B}}(\eta_1(a), \beta \mathsf{s}_{\mathcal{A}}(a)) \Big) = \Big(m_{\mathcal{Y}}(\alpha \mathsf{t}_{\mathcal{X}}(x), \varphi_1(x)), m_{\mathcal{B}}(\beta \mathsf{s}_{\mathcal{A}}(a), \varepsilon_1(a)) \Big) \\
&= m_{\mathcal{Y} \underset{\mathcal{G}_0}{\times} \mathcal{B}}\Big((\alpha \mathsf{t}_{\mathcal{X}}(x), \beta \mathsf{t}_{\mathcal{A}}(a)), (\varphi_1(a), \varepsilon_1(a)) \Big) = m_{\mathcal{Y} \underset{\mathcal{G}_0}{\times} \mathcal{B}} \Delta\left(\left(\alpha \underset{\mathcal{G}_0}{\times} \beta\right) \mathsf{t}_{\mathcal{X} \underset{\mathcal{G}_0}{\times} \mathcal{A}}, \varphi_1 \underset{\mathcal{G}_0}{\times} \varepsilon_1\right)(x,a).
\end{aligned}
$$

Lemma 6.3.2. *We have*

$$
\left(\alpha' \underset{\mathcal{G}_0}{\times} \beta'\right)\left(\alpha \underset{\mathcal{G}_0}{\times} \beta\right) = (\alpha'\alpha) \underset{\mathcal{G}_0}{\times} (\beta'\beta) : \varphi \underset{\mathcal{G}_0}{\times} \varepsilon \longrightarrow \psi \underset{\mathcal{G}_0}{\times} \eta \qquad \textit{and} \qquad \mathrm{Id}_{\varphi \underset{\mathcal{G}_0}{\times} \psi} = \mathrm{Id}_{\varphi} \underset{\mathcal{G}_0}{\times} \mathrm{Id}_{\psi}.
$$

Proof. For each $(x,a) \in \mathcal{X}_0 \underset{\mathcal{G}_0}{\times} \mathcal{A}_0$ we calculate

$$
\begin{aligned}
\mathrm{Id}_{\varphi} \underset{\mathcal{G}_0}{\times} \mathrm{Id}_{\varepsilon}(x,a) &= \mathrm{Id}_{\varphi}(x) \underset{\mathcal{G}_0}{\times} \mathrm{Id}_{\varphi}(a) = \iota_{\varphi_0(x)} \underset{\mathcal{G}_0}{\times} \iota_{\varphi_0(a)} = \iota_{\varphi_0(x) \underset{\mathcal{G}_0}{\times} \varphi_0(a)} \\
&= \iota_{\varphi_0 \underset{\mathcal{G}_0}{\times} \varepsilon_0(x,a)} = \mathrm{Id}_{\varepsilon \underset{\mathcal{G}_0}{\times} \varepsilon}(x,a)
\end{aligned}
$$

and

$$\left(\alpha' \underset{\mathcal{G}_0}{\times} \beta'\right)\left(\alpha \underset{\mathcal{G}_0}{\times} \beta\right)(x,a) = m_{\mathcal{Y}\underset{\mathcal{G}_0}{\times}\mathcal{B}}\Delta\left(\alpha' \underset{\mathcal{G}_0}{\times} \beta', \alpha \underset{\mathcal{G}_0}{\times} \beta\right)(x,a)$$

$$= m_{\mathcal{Y}\underset{\mathcal{G}_0}{\times}\mathcal{B}}\left(\left(\alpha' \underset{\mathcal{G}_0}{\times} \beta'\right)(x,a), \left(\alpha \underset{\mathcal{G}_0}{\times} \beta\right)(x,a)\right) = m_{\mathcal{Y}\underset{\mathcal{G}_0}{\times}\mathcal{B}}\Big(\left(\alpha'(x), \beta'(a)\right), \left(\alpha(x), \beta(a)\right)\Big)$$

$$= \Big(m_{\mathcal{Y}}\left(\alpha'(x), \alpha(x)\right), m_{\mathcal{B}}\left(\beta'(a), \beta(a)\right)\Big)$$

$$= \Big(\left(\alpha'\alpha\right)(x), \left(\beta'\beta\right)(a)\Big) = \left(\left(\alpha'\alpha\right) \underset{\mathcal{G}_0}{\times} \left(\beta'\beta\right)\right)(x,a).$$

□

Now we are going to give the main definition of this section.

Definition 6.3.3. Let be $(\mathcal{X},\varsigma), (\mathcal{Y},\theta) \in \mathsf{CSets}\text{-}\mathcal{G}$. We say that $(\mathcal{X},\varsigma)$ and $(\mathcal{Y},\theta)$ are **weak equivalent** and we write $(\mathcal{X},\varsigma) \sim_{\mathsf{we}} (\mathcal{Y},\theta)$ if there are morphisms in $\mathsf{CSets}\text{-}\mathcal{G}$ $\varphi\colon (\mathcal{X},\varsigma) \longrightarrow (\mathcal{Y},\theta)$ and $\psi\colon (\mathcal{Y},\theta) \longrightarrow (\mathcal{X},\varsigma)$ and 2-isomorphisms in $\mathsf{CSets}\text{-}\mathcal{G}$ $\alpha\colon \psi\varphi\colon \longrightarrow \mathrm{Id}_{(\mathcal{X},\varsigma)}$ and $\beta\colon \varphi\psi \longrightarrow \mathrm{Id}_{(\mathcal{Y},\theta)}$.

The following lemma, which will be essential to define the weak Burnside rig $\mathscr{L}_{\mathsf{C}}$ in section 6.6, states that the disjoint union and the fibre product are compatible with the weak equivalence relation.

Lemma 6.3.4. *Let be $(\mathcal{X},\varsigma), (\mathcal{Y},\theta), (\mathcal{A},\lambda), (\mathcal{B},\mu) \in \mathsf{CSets}\text{-}\mathcal{G}$ such that*

$$(\mathcal{X},\varsigma) \sim_{\mathsf{we}} (\mathcal{Y},\theta) \qquad \textit{and} \qquad (\mathcal{A},\lambda) \sim_{\mathsf{we}} (\mathcal{B},\mu).$$

Then

$$\Big[(\mathcal{X},\varsigma) \uplus (\mathcal{A},\lambda)\Big] \sim_{\mathsf{we}} \Big[(\mathcal{Y},\theta) \uplus (\mathcal{B},\mu)\Big] \qquad \textit{and} \qquad \Big[(\mathcal{X},\varsigma) \underset{\mathcal{G}_0}{\times} (\mathcal{A},\lambda)\Big] \sim_{\mathsf{we}} \Big[(\mathcal{Y},\theta) \underset{\mathcal{G}_0}{\times} (\mathcal{B},\mu)\Big].$$

Proof. Let be

$$\varphi\colon (\mathcal{X},\varsigma) \longrightarrow (\mathcal{Y},\theta) \qquad \eta\colon (\mathcal{A},\lambda) \longrightarrow (\mathcal{B},\mu)$$
$$\psi\colon (\mathcal{Y},\theta) \longrightarrow (\mathcal{X},\varsigma) \qquad \varepsilon\colon (\mathcal{B},\mu) \longrightarrow (\mathcal{A},\lambda)$$

morphisms in $\mathsf{CSets}\text{-}\mathcal{G}$ such that there are 2-isomorphisms in $\mathsf{CSets}\text{-}\mathcal{G}$

$$\alpha\colon \psi\varphi \longrightarrow \mathrm{Id}_{(\mathcal{X},\varsigma)} \qquad \gamma\colon \varepsilon\eta \longrightarrow \mathrm{Id}_{(\mathcal{A},\lambda)}$$
$$\beta\colon \varphi\psi \longrightarrow \mathrm{Id}_{(\mathcal{Y},\theta)} \qquad \delta\colon \eta\varepsilon \longrightarrow \mathrm{Id}_{(\mathcal{B},\mu)}.$$

We calculate

$$\left(\alpha^{-1} \uplus \gamma^{-1}\right)\left(\alpha \uplus \gamma\right) = \left[\alpha^{-1}\alpha\right] \uplus \left(\gamma^{-1}\gamma\right) = \mathrm{Id}_{\psi\varphi} \uplus \mathrm{Id}_{\varepsilon\eta} = \mathrm{Id}_{\psi\varphi\uplus\varepsilon\eta}$$

and

$$\left(\alpha \uplus \gamma\right)\left(\alpha^{-1} \uplus \gamma^{-1}\right) = \left(\alpha\alpha^{-1}\right) \uplus \left(\gamma\gamma^{-1}\right) = \mathrm{Id}_{\mathrm{Id}_{(\mathcal{X},\varsigma)}} \uplus \mathrm{Id}_{\mathrm{Id}_{(\mathcal{A},\lambda)}}$$
$$= \mathrm{Id}_{\mathrm{Id}_{(\mathcal{X},\varsigma)} \uplus \mathrm{Id}_{(\mathcal{Y},\theta)}} = \mathrm{Id}_{\mathrm{Id}_{(\mathcal{X},\varsigma)\uplus(\mathcal{Y},\theta)}}$$

thus

$$\alpha \uplus \gamma\colon \psi\varphi \uplus \varepsilon\eta \longrightarrow \mathrm{Id}_{(\mathcal{X},\varsigma)\uplus(\mathcal{A},\lambda)}$$

is a 2-isomorphism in $\mathsf{CSets}\text{-}\mathcal{G}$. We can prove in the same way that

$$\beta \uplus \delta\colon \varphi\psi \uplus \eta\varepsilon \longrightarrow \mathrm{Id}_{(\mathcal{Y},\theta)\uplus(\mathcal{B},\mu)}$$

is a 2-isomorphism in CSets-$\mathcal{G}$. As a consequence we obtain

$$\Big[(\mathcal{X},\varsigma) \uplus (\mathcal{A},\lambda)\Big] \sim_{\mathsf{we}} \Big[(\mathcal{Y},\theta) \uplus (\mathcal{B},\mu)\Big]$$

and we can prove that

$$\Big[(\mathcal{X},\varsigma) \underset{\mathcal{G}_0}{\times} (\mathcal{A},\lambda)\Big] \sim_{\mathsf{we}} \Big[(\mathcal{Y},\theta) \underset{\mathcal{G}_0}{\times} (\mathcal{B},\mu)\Big]$$

in the same way. □

As already noted in Remark 6.1.6, many concept of ordinary category theory can be extended to right categorified $\mathcal{G}$-sets. With reference to [AHS04, Pag. 51], we will now briefly explain how to extend the concept of skeleton of a category. The proof are essentially the same thus we will omit them.

Definition 6.3.5. Let be $(\mathcal{X},\varsigma)$ and $(\mathcal{Y},\theta)$ right categorified $\mathcal{G}$-sets.

(1) We say that $(\mathcal{Y},\theta)$ is a **right categorified $\mathcal{G}$-subset** of $(\mathcal{X},\varsigma)$ if the following two conditions are satisfied:

   (a) $(\mathcal{Y}_0,\theta_0)$ and $(\mathcal{Y}_1,\theta_1)$ are right $\mathcal{G}$-subset respectively of $(\mathcal{X}_0,\varsigma_0)$ and $(\mathcal{X}_1,\varsigma_1)$;

   (b) the structure on $(\mathcal{Y},\theta)$ is appropriately induced by that on $(\mathcal{X},\varsigma)$ by restriction, in the usual sense for subcategories.

(2) We say that $(\mathcal{Y},\theta)$ is a **full right categorified $\mathcal{G}$-subset** of $(\mathcal{X},\varsigma)$ if it is a right categorified $\mathcal{G}$-subset of $(\mathcal{X},\varsigma)$ such that, for each $a,b \in \mathcal{Y}_0$, we have $\mathcal{Y}(a,b) = \mathcal{X}(a,b)$.

(3) We say that $(\mathcal{Y},\theta)$ is a **isomorphism-dense right categorified $\mathcal{G}$-subset** of $(\mathcal{X},\varsigma)$ if it is a right categorified $\mathcal{G}$-subset of $(\mathcal{X},\varsigma)$ such that, for each $a \in \mathcal{Y}_0$, there is $b \in \mathcal{X}_0$ such that there is an isomorphism $f \in \mathcal{X}_1$, with $\mathsf{s}_{\mathcal{X}}(f) = a$ and $\mathsf{t}_{\mathcal{X}}(f) = a$.

Definition 6.3.6. The **skeleton of a right categorified $\mathcal{G}$-set** is a full, isomorphism-dense right categorified $\mathcal{G}$-subset in which no two distinct objects are isomorphic.

Proposition 6.3.7. *(1) Every right categorified $\mathcal{G}$-set has a skeleton.*

(2) Two skeletons of a right categorified $\mathcal{G}$-set $(\mathcal{X},\varsigma)$ are isomorphic.

(3) Every skeleton of a right categorified $\mathcal{G}$-set $(\mathcal{X},\varsigma)$ is weak equivalent to $(\mathcal{X},\varsigma)$.

Corollary 6.3.8. *Two right categorified $\mathcal{G}$-set $(\mathcal{X},\varsigma)$ and $(\mathcal{Y},\theta)$ are weak equivalent if and only their skeletons are isomorphic as right categorified $\mathcal{G}$-sets.*

6.4 A class of examples

Let be $\mathcal{C}$ a small category, $\mathcal{G}$ a groupoid, and (X,ς) a right $\mathcal{G}$-sets. We set $\mathcal{X}_0 = \mathcal{C}_0 \times X$ and $\mathcal{X}_1 = \mathcal{C}_1 \times X$. The structure maps and the actions, for $i = 0,1$, will be

$$\begin{aligned}\varsigma_i \colon \mathcal{X}_i &\longrightarrow \mathcal{G}_0 \\ (a,x) &\longrightarrow \varsigma(x)\end{aligned} \qquad \text{and} \qquad \begin{aligned}\mathcal{X}_i\, {}_{\varsigma_i}\!\times_{\mathsf{t}} \mathcal{G} &\longrightarrow \mathcal{X}_i \\ \Big((a,x)\,,g\Big) &\longrightarrow (a,xg)\,.\end{aligned}$$

We set

$$\begin{aligned}\mathsf{s}_{\mathcal{X}} \colon \mathcal{X}_1 &\longrightarrow \mathcal{X}_0 \\ (a,x) &\longrightarrow (\mathsf{s}_{\mathcal{C}}(a),x)\,,\end{aligned} \qquad \begin{aligned}\mathsf{t}_{\mathcal{X}} \colon \mathcal{X}_1 &\longrightarrow \mathcal{X}_0 \\ (a,x) &\longrightarrow (\mathsf{t}_{\mathcal{C}}(a),x)\,,\end{aligned} \qquad \begin{aligned}\iota_{\mathcal{X}} \colon \mathcal{X}_1 &\longrightarrow \mathcal{X}_0 \\ (a,x) &\longrightarrow (\iota_{\mathcal{C}}(a),x)\end{aligned}$$

and

$$\begin{aligned} m_{\mathcal{X}}\colon \mathcal{X}_2 &\longrightarrow \mathcal{X}_1 \\ \big((a,x),(b,y)\big) &\longrightarrow (m_{\mathcal{C}}(a,b),x) \end{aligned}$$

with $(\mathsf{s}_{\mathcal{C}}(a),x) = \mathsf{s}_{\mathcal{X}}(a,x) = \mathsf{t}(b,y) = (\mathsf{t}_{\mathcal{C}}(b),y)$. It is evident that $\mathsf{s}_{\mathcal{C}}$, $\mathsf{t}_{\mathcal{C}}$, $\iota_{\mathcal{C}}$ and $m_{\mathcal{C}}$ are morphisms of right $\mathcal{G}$-sets thus we just have to prove that the diagrams of Definition 6.1.1 about $\mathcal{X}$ are commutative, but this follows immediately by the analogous diagrams about $\mathcal{C}$ (an ordinary category can be considered as an internal category in the category of sets).

Now let $\mathcal{C}$ and $\mathcal{D}$ be small categories, let (X,ς) and (Y,θ) be right $\mathcal{G}$-sets and set $\mathcal{X}_0 = \mathcal{C}_0 \times X$, $\mathcal{X}_1 = \mathcal{C}_1 \times X$, $\mathcal{Y}_0 = \mathcal{D}_0 \times Y$ and $\mathcal{Y}_1 = \mathcal{D}_1 \times Y$. Let be $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ a functor and $\varphi\colon (X,\varsigma) \longrightarrow (Y,\theta)$ a morphism in $\mathsf{Sets}\text{-}\mathcal{G}$. We want to define a morphism

$$(F,\varphi)\colon (\mathcal{X},\varsigma) \longrightarrow (\mathcal{Y},\theta)$$

in $\mathsf{CSets}\text{-}\mathcal{G}$ setting $(F,\varphi)_0 = (F_0,\varphi)$ and $(F,\varphi)_1 = (F_1,\varphi)$. It is evident that $(F,\varphi)_0$ and $(F,\varphi)_1$ are morphism of right $\mathcal{G}$-sets, thus we just have to prove that the diagrams of Definition 6.1.2 about (F,φ) are commutative, but this follows immediately by the analogous diagrams about F (an ordinary functor can be considered as an internal functor in the category of sets).

Given another functor $G\colon \mathcal{C} \longrightarrow \mathcal{D}$, we consider a natural transformation $\mu\colon F \longrightarrow G$. With the notations already introduced, we have a morphism $(G,\psi)\colon (\mathcal{X},\varsigma) \longrightarrow (\mathcal{Y},\theta)$ in $\mathsf{CSets}\text{-}\mathcal{G}$ such that $(G,\varphi)_0 = (G_0,\varphi)$ and $(G,\psi)_1 = (G_1,\varphi)$. We want to define a 2-morphism $(\mu,\varphi) : (F,\varphi) \longrightarrow (G,\varphi)$ given by a morphism of right $\mathcal{G}$-sets

$$\begin{aligned} (\mu,\varphi)\colon \mathcal{X}_0 &\longrightarrow \mathcal{Y}_1 \\ (a,x) &\longrightarrow (\mu(a),\varphi(x)) \end{aligned}$$

therefore, we have to prove that the following diagrams commute,

$$\begin{array}{ccc} \mathcal{X}_0 & \xrightarrow{(\mu,\varphi)} & \mathcal{Y}_1 \\ & {\scriptstyle (F,\varphi)_0}\searrow & \downarrow{\scriptstyle \mathsf{s}_{\mathcal{Y}}} \\ & & \mathcal{Y}_0 \end{array} \qquad \begin{array}{ccc} \mathcal{X}_0 & \xrightarrow{(\mu,\varphi)} & \mathcal{Y}_1 \\ & {\scriptstyle (G,\varphi)_0}\searrow & \downarrow{\scriptstyle \mathsf{t}_{\mathcal{Y}}} \\ & & \mathcal{Y}_0 \end{array} \qquad \begin{array}{ccc} \mathcal{X}_1 & \xrightarrow{\Delta((G,\varphi)_1,(\mu,\varphi)\mathsf{s}_{\mathcal{X}})} & \mathcal{Y}_2 \\ {\scriptstyle \Delta((\mu,\varphi)\mathsf{t}_{\mathcal{X}},(F,\varphi)_1)}\downarrow & & \downarrow{\scriptstyle m_{\mathcal{Y}}} \\ \mathcal{Y}_2 & \xrightarrow{m_{\mathcal{Y}}} & \mathcal{Y}_1 \end{array}$$

but this is just a direct verification.

Remark 6.4.1. Let be $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ small categories, $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ and $G\colon \mathcal{D} \longrightarrow \mathcal{E}$ functor, $\varphi\colon (X,\varsigma) \longrightarrow (Y,\theta)$ and $\psi\colon (Y,\theta) \longrightarrow (Z,\omega)$ morphisms of right $\mathcal{G}$-sets. Continuing to use the notation of Section 6.4 we set $\mathcal{X}_i = \mathcal{C}_i \times X$, $\mathcal{Y}_i = \mathcal{D} \times Y$ and $\mathcal{Z}_i = \mathcal{E} \times Z$ for $i = 0,1$. It is then clear that

$$\mathrm{Id}_{\mathcal{X}} = (\mathrm{Id}_{\mathcal{C}},\mathrm{Id}_X) : (\mathcal{X},\varsigma) \longrightarrow (\mathcal{X},\varsigma) \quad \text{and} \quad (GF,\psi\varphi) = (G,\psi)(F,\varphi)\colon (\mathcal{X},\varsigma) \longrightarrow (\mathcal{Z},\omega)\,.$$

Proposition 6.4.2. *Let be $\mathcal{C}$ and $\mathcal{D}$ small categories, $(X,\varsigma)\,,(Y,\theta) \in \mathsf{Sets}\text{-}\mathcal{G}$, $F,G,H\colon \mathcal{C} \longrightarrow \mathcal{D}$ functors, $\varphi\colon (X,\varsigma) \longrightarrow (Y,\theta)$ a morphism of right $\mathcal{G}$-sets, $\mu\colon F \longrightarrow G$ and $\lambda\colon G \longrightarrow H$ natural transformations. We define 2-morphisms*

$$(\mu,\varphi) : (F,\varphi) \longrightarrow (G,\varphi), \quad (\lambda,\varphi) : (G,\varphi) \longrightarrow (H,\varphi) \quad \textit{and} \quad (\lambda\mu,\varphi) : (F,\varphi) \longrightarrow (H,\varphi)$$

in $\mathsf{CSets}\text{-}\mathcal{G}$ given by morphisms of right $\mathcal{G}$-sets

$$\begin{aligned} (\mu,\varphi)\colon \mathcal{X}_0 &\longrightarrow \mathcal{Y}_1 \\ (a,x) &\longrightarrow (\mu(a),\varphi(x)), \end{aligned} \qquad \begin{aligned} (\lambda,\varphi)\colon \mathcal{X}_0 &\longrightarrow \mathcal{Y}_1 \\ (a,x) &\longrightarrow (\lambda(x),\varphi(x)) \end{aligned}$$

and

$$\begin{aligned}(\lambda\mu,\varphi)\colon (F,\varphi) &\longrightarrow (H,\varphi)\\ (a,x) &\longrightarrow ((\lambda\mu)(a),\varphi(x)).\end{aligned}$$

Then $(\lambda\mu,\varphi)=(\lambda,\varphi)\,(\mu,\varphi)$ *and* $(\mathrm{Id}_F,\varphi)=\mathrm{Id}_{(F,\varphi)}$.

Proof. We consider the 2-morphism $(\lambda,\varphi)\,(\mu,\varphi):(F,\varphi)\longrightarrow(H,\varphi)$ given by a morphism in right $\mathcal{G}$-set

$$\begin{aligned}r\colon \mathcal{X}_0 &\longrightarrow \mathcal{Y}_1\\ (a,x) &\longrightarrow r(a,x)=m_{\mathcal{Y}}\Big(\,(\lambda,\varphi)\,(a,x),(\mu,\varphi)\,(a,x)\Big).\end{aligned}$$

For each $(a,x)\in\mathcal{X}_0$ we calculate

$$\begin{aligned}r(a,x)&=m_{\mathcal{Y}}\left((\lambda,\varphi)\,(a,x),(\mu,\varphi)\,(a,x)\right)=m_{\mathcal{Y}}\left((\lambda(a),\varphi(x))\,,(\mu(a),\varphi(x))\right)\\&=\left(m_{\mathcal{C}}\left(\lambda(a),\mu(a)\right),\varphi(x)\right)=((\lambda\mu)(a),\varphi(x))=(\lambda\mu,\varphi)\,(a,x).\end{aligned}$$

The 2-morphism $\mathrm{Id}_{(F,\varphi)}$ is given by the morphism of right $\mathcal{G}$-sets

$$\begin{aligned}z\colon \mathcal{X}_0 &\longrightarrow \mathcal{Y}_1\\ (a,x) &\longrightarrow \iota_{\mathcal{Y}}\left(F(a),\varphi(x)\right)=(\iota_{\mathcal{D}}F(a),\varphi(x))\end{aligned}$$

and (Id_F,φ) is given by

$$\begin{aligned}(\mathrm{Id}_F,\varphi)\colon \mathcal{X}_0 &\longrightarrow \mathcal{Y}_1\\ (a,x) &\longrightarrow (\mathrm{Id}_F(a),\varphi(x))=(\iota_{\mathcal{D}}(F(a)),\varphi(x))\end{aligned}$$

therefore we obtain $\mathrm{Id}_{(F,\varphi)}=(\mathrm{Id}_F,\varphi)$. □

Now let $\mathcal{C}$ and $\mathcal{D}$ be two categories, $F\colon\mathcal{C}\longrightarrow\mathcal{D}$ and $G\colon\mathcal{D}\longrightarrow\mathcal{C}$ two functors, $\varepsilon\colon FG\longrightarrow\mathrm{Id}_{\mathcal{D}}$ and $\eta\colon GF\longrightarrow\mathrm{Id}_{\mathcal{C}}$ two natural isomorphisms and $\varphi\colon\,(X,\varsigma)\longrightarrow(Y,\theta)$ a morphism of right $\mathcal{G}$-sets. Continuing to use the notation of Section 6.4 we set $\mathcal{X}_i=\mathcal{C}_i\times X$ and $\mathcal{Y}_i=\mathcal{D}\times Y$ for $i=0,1$, and we consider morphisms of right categorified G-sets $(F,\varphi)\colon\,(\mathcal{X},\varsigma)\longrightarrow(\mathcal{Y},\theta)$ and $(G,\varphi)\colon\,(\mathcal{Y},\theta)\longrightarrow(\mathcal{X},\varsigma)$. We have

$$\left(G,\varphi^{-1}\right)(F,\varphi)=\left(GF,\varphi^{-1}\varphi\right)=\left(GF,\mathrm{Id}_{(X,\varsigma)}\right),\quad \mathrm{Id}_{(\mathcal{X},\varsigma)}=\left(\mathrm{Id}_{\mathcal{C}},\mathrm{Id}_{(X,\varsigma)}\right):\,(\mathcal{X},\varsigma)\longrightarrow(\mathcal{X},\varsigma)$$

and

$$(F,\varphi)\left(G,\varphi^{-1}\right)=\left(FG,\varphi\varphi^{-1}\right)=\left(FG,\mathrm{Id}_{(Y,\theta)}\right),\quad \mathrm{Id}_{(\mathcal{Y},\theta)}=\left(\mathrm{Id}_{\mathcal{D}},\mathrm{Id}_{(Y,\theta)}\right):\,(\mathcal{Y},\theta)\longrightarrow(\mathcal{Y},\theta)\,.$$

We define 2-morphisms

$$\left(\varepsilon,\mathrm{Id}_{(X,\varsigma)}\right):(G,\psi)(F,\varphi)\longrightarrow\mathrm{Id}_{(\mathcal{X},\varsigma)}\qquad\text{and}\qquad\left(\eta,\mathrm{Id}_{(Y,\theta)}\right):(F,\varphi)(G,\psi)\longrightarrow\mathrm{Id}_{(\mathcal{Y},\theta)}$$

in CSets-$\mathcal{G}$ as the morphisms of right $\mathcal{G}$-sets

$$\begin{aligned}\left(\varepsilon,\mathrm{Id}_{(X,\varsigma)}\right)\colon(\mathcal{X}_0,\varsigma_0)&\longrightarrow(\mathcal{X}_1,\varsigma_1)\\ (a,x)&\longrightarrow(\varepsilon(a),x)\end{aligned}\qquad\text{and}\qquad\begin{aligned}\left(\eta,\mathrm{Id}_{(Y,\theta)}\right)\colon(\mathcal{Y}_0,\theta_0)&\longrightarrow(\mathcal{Y}_1,\theta_1)\\ (\beta,y)&\longrightarrow(\eta(\beta),y)\end{aligned}$$

respectively. We have already proved that $\left(\varepsilon,\mathrm{Id}_{(X,\varsigma)}\right)$ and $\left(\eta,\mathrm{Id}_{(Y,\theta)}\right)$ are well defined. Considering $\varepsilon^{-1}\colon\,\mathrm{Id}_{\mathcal{D}}\longrightarrow FG$ and $\eta^{-1}\colon\,\mathrm{Id}_{\mathcal{C}}\longrightarrow GF$ we can construct the 2-morphisms

$$\left(\varepsilon^{-1},\mathrm{Id}_{(X,\varsigma)}\right):\,\mathrm{Id}_{(\mathcal{X},\varsigma)}\longrightarrow(G,\psi)(F,\varphi)\qquad\text{and}\qquad\left(\eta^{-1},\mathrm{Id}_{(Y,\theta)}\right):\,\mathrm{Id}_{(\mathcal{Y},\theta)}\longrightarrow(F,\varphi)(G,\psi)$$

in $\mathsf{CSets}\text{-}\mathcal{G}$

$$\begin{aligned}(\varepsilon^{-1}, \mathrm{Id}_{(X,\varsigma)})\colon (\mathcal{X}_0, \varsigma_0) &\longrightarrow (\mathcal{X}_1, \varsigma_1)\\ (a,x) &\longrightarrow (\varepsilon^{-1}(a), x)\end{aligned} \quad \text{and} \quad \begin{aligned}(\eta^{-1}, \mathrm{Id}_{(Y,\theta)})\colon (\mathcal{Y}_0, \theta_0) &\longrightarrow (\mathcal{Y}_1, \theta_1)\\ (\beta, y) &\longrightarrow (\eta^{-1}(\beta), y).\end{aligned}$$

We calculate

$$\mathrm{Id}_{\left(FG, \mathrm{Id}_{(Y,\theta)}\right)} = \left(\mathrm{Id}_{FG}, \mathrm{Id}_{(Y,\theta)}\right) = \left(\varepsilon^{-1}\varepsilon, \mathrm{Id}_{(Y,\theta)}\right) = \left(\varepsilon^{-1}, \mathrm{Id}_{(Y,\theta)}\right)\left(\varepsilon, \mathrm{Id}_{(Y,\theta)}\right)$$

and

$$\mathrm{Id}_{\left(\mathrm{Id}_{\mathcal{D}}, \mathrm{Id}_{(Y,\theta)}\right)} = \left(\mathrm{Id}_{\mathrm{Id}_{\mathcal{D}}}, \mathrm{Id}_{(Y,\theta)}\right) = \left(\varepsilon\varepsilon^{-1}, \mathrm{Id}_{(Y,\theta)}\right) = \left(\varepsilon, \mathrm{Id}_{(Y,\theta)}\right)\left(\varepsilon^{-1}, \mathrm{Id}_{(Y,\theta)}\right).$$

In the same way we prove that

$$\mathrm{Id}_{\left(GF, \mathrm{Id}_{(X,\varsigma)}\right)} = \left(\eta^{-1}, \mathrm{Id}_{(X,\varsigma)}\right)\left(\eta, \mathrm{Id}_{(X,\varsigma)}\right) \quad \text{and} \quad \mathrm{Id}_{\left(\mathrm{Id}_{\mathcal{C}}, \mathrm{Id}_{(X,\varsigma)}\right)} = \left(\eta, \mathrm{Id}_{(X,\varsigma)}\right)\left(\eta^{-1}, \mathrm{Id}_{(X,\varsigma)}\right)$$

therefore $(\mathcal{X}, \varsigma) \sim_{\mathsf{we}} (\mathcal{Y}, \theta)$.

Using the forgetful functor of Remark 6.1.6, we deduce, from the previous argumentations, that $\mathcal{X}$ and $\mathcal{Y}$ cannot be weakly equivalent if the categories $\mathcal{C}$ and $\mathcal{D}$ are not equivalent. Furthermore, we proved the following result.

Proposition 6.4.3. *Given a groupoid $\mathcal{G}$, we have a functor*

$$\begin{aligned}\mathbf{Cat} \times \mathsf{Sets}\text{-}\mathcal{G} &\longrightarrow \mathsf{CSets}\text{-}\mathcal{G}\\ \left(\mathcal{C}, (X,\varsigma)\right) &\longrightarrow \mathcal{C} \times (X, \varsigma)\end{aligned}$$

where **Cat** *denotes the category of small categories. Moreover, two right categorified $\mathcal{G}$-sets of the form $\mathcal{C} \times (X,\varsigma)$ and $\mathcal{D} \times (Y,\theta)$ are weakly equivalent if and only if the $\mathcal{C}$ and $\mathcal{D}$ are equivalent categories and (X,ς) and (Y,θ) are isomorphic $\mathcal{G}$-sets.*

Proof. Straightforward. □

6.5 The right double translation category

Given a groupoid $\mathcal{G}$, let us consider a right $\mathcal{G}$-set (X,ς). The right translation groupoid $X \rtimes \mathcal{G}$ (see eq. (1.1.4)) illustrates the orbits that the groupoid $\mathcal{G}$ creates acting on (X,ς). To extend this notion to the new right categorified $\mathcal{G}$-sets we need the notion of double category, introduced for the first time in [Ehr63a].

Definition 6.5.1. A **double category** is an internal category in the category of small categories.

Given a double category $\mathcal{D}$, the relevant functors are illustrated in the following diagram:

$$\mathcal{D}_1 \underset{\mathsf{S}_{\mathcal{D}}}{\overset{\mathsf{T}_{\mathcal{D}}}{\rightrightarrows}} \mathcal{D}_0 \xrightarrow{\ \mathsf{I}_{\mathcal{D}}\ } \mathcal{D}_1 \xleftarrow{\ \mathsf{M}_{\mathcal{D}}\ } \mathcal{D}_2$$

where $\mathcal{D}_0$, $\mathcal{D}_1$ and $\mathcal{D}_2 = \mathcal{D}_1\, {}_{\mathsf{S}_{\mathcal{D}}}\!\times_{\mathsf{T}_{\mathcal{D}}} \mathcal{D}_1$ are categories (we remind to the reader that the category of small categories has pullbacks).

$$
\begin{aligned}
&\mathsf{T}_{\mathcal{D}}\left(\begin{array}{ccc} x & \xleftarrow{(x,g)} & xg \\ f\big\downarrow & \overset{(f,g)}{\Longleftarrow} & \big\downarrow fg \\ y & \xleftarrow{(y,g)} & yg \end{array} \circ \begin{array}{ccc} xg & \xleftarrow{(xg,h)} & xgh \\ fg\big\downarrow & \overset{(fg,h)}{\Longleftarrow} & \big\downarrow fgh \\ yg & \xleftarrow{(yg,h)} & ygh \end{array}\right) = \mathsf{T}_{\mathcal{D}}\left(\begin{array}{ccc} x & \xleftarrow{(x,gh)} & xgh \\ f\big\downarrow & \overset{(f,gh)}{\Longleftarrow} & \big\downarrow fgh \\ y & \xleftarrow{(y,gh)} & ygh \end{array}\right) \\
&= \left(y \xleftarrow{(y,gh)} ygh \right) = \left(y \xleftarrow{(y,g)} yg \right) \circ \left(yg \xleftarrow{(y,gh)} ygh \right) \\
&= \mathsf{T}_{\mathcal{D}}\left(\begin{array}{ccc} x & \xleftarrow{(x,g)} & xgh \\ f\big\downarrow & \overset{(f,gh)}{\Longleftarrow} & \big\downarrow fg \\ y & \xleftarrow{(y,g)} & ygh \end{array}\right) \mathsf{T}_{\mathcal{D}}\left(\begin{array}{ccc} xg & \xleftarrow{(xg,h)} & xgh \\ fg\big\downarrow & \overset{(fg,h)}{\Longleftarrow} & \big\downarrow fgh \\ yg & \xleftarrow{(yg,h)} & ygh \end{array}\right)
\end{aligned}
$$

Figure 6.1: Target functor, composition condition

Definition 6.5.2. Given a double category $\mathcal{D}$, the set $(\mathcal{D}_0)_0$ is called the **set of objects**, the set $(\mathcal{D}_0)_1$ is called the **set of horizontal morphisms**, the set $(\mathcal{D}_1)_0$ is called the **set of vertical morphisms** and the set $(\mathcal{D}_1)_1$ is called the **set of squares**. Moreover, the category $\mathcal{D}_0$ is called the **category of objects** and $\mathcal{D}_1$ is called the **category of morphisms**.

The reason behind Definition 6.5.2 will be manifest in the forthcoming diagrams of this section that will also illustrate how to operate with double categories.

Given a groupoid $\mathcal{G}$ and a right categorified $\mathcal{G}$-set $(\mathcal{X}, \varsigma)$, we set $\mathcal{D}_i = \mathcal{X}_i \rtimes \mathcal{G}$ for $i = 0, 1$. We are now going to construct a structure of double category $\mathcal{D}$ starting from the categories $\mathcal{D}_0$ and $\mathcal{D}_1$. We define the **target functor** $\mathsf{T}_{\mathcal{D}} \colon \mathcal{D}_1 \longrightarrow \mathcal{D}_0$ as in the following diagrams, where $\varsigma_0(y) = \varsigma_0 \mathsf{t}_{\mathcal{D}_0}(f) = \varsigma_1(f) = \varsigma_0 \mathsf{s}_{\mathcal{D}_0}(f) = \varsigma_0(x)$:

$$
\left(\begin{array}{c} x \\ \big\downarrow f \\ y \end{array}\right) \xrightarrow{(\mathsf{T}_{\mathcal{D}})_0} y \qquad \text{and} \qquad \left(\begin{array}{ccc} x & \xleftarrow{(x,g)} & xg \\ f\big\downarrow & \overset{(f,g)}{\Longleftarrow} & \big\downarrow fg \\ y & \xleftarrow{(y,g)} & yg \end{array}\right) \xrightarrow{(\mathsf{T}_{\mathcal{D}})_1} \left(y \xleftarrow{(y,g)} yg \right).
$$

We will now check that $\mathsf{T}_{\mathcal{D}}$ is a functor. Given a vertical morphism $f \colon x \longrightarrow y$ in $(\mathcal{D}_1)_0$ we have

$$
\mathsf{T}_{\mathcal{D}}\left(\begin{array}{ccc} x & \xleftarrow{\left(x,\iota_{\varsigma_0(x)}\right)} & xg \\ f\big\downarrow & \overset{\left(f,\iota_{\varsigma_1(f)}\right)}{\Longleftarrow} & \big\downarrow fg \\ y & \xleftarrow{\left(y,\iota_{\varsigma_0(y)}\right)} & yg \end{array}\right) = \left(y \xleftarrow{\left(y,\iota_{\varsigma_0(y)}\right)} yg \right),
$$

and, regarding the composition, see figure (6.1). We define the **source functor** $\mathsf{S}_{\mathcal{D}} \colon \mathcal{D}_1 \longrightarrow \mathcal{D}_0$ as in the following diagrams:

$$
\left(\begin{array}{c} x \\ \big\downarrow f \\ y \end{array}\right) \xrightarrow{(\mathsf{S}_{\mathcal{D}})_0} x \qquad \text{and} \qquad \left(\begin{array}{ccc} x & \xleftarrow{(x,g)} & xg \\ f\big\downarrow & \overset{(f,g)}{\Longleftarrow} & \big\downarrow fg \\ y & \xleftarrow{(y,g)} & yg \end{array}\right) \xrightarrow{(\mathsf{S}_{\mathcal{D}})_1} \left(x \xleftarrow{(x,g)} xg \right).
$$

The **identity functor** $\mathsf{I}_\mathcal{D}\colon \mathcal{D}_0 \longrightarrow \mathcal{D}_1$ can be defined as in the following diagrams, where $\varsigma_0(x) = \varsigma_0 \mathsf{t}_{\mathcal{D}_0}(\mathrm{Id}_x) = \varsigma_1(\mathrm{Id}_x) = \varsigma_0 \mathsf{s}_{\mathcal{D}_0}(\mathrm{Id}_x) = \varsigma_0(x)$:

$$x \xrightarrow{(\mathsf{I}_\mathcal{D})_0} \begin{pmatrix} x \\ \big\downarrow \mathrm{Id}_x \\ x \end{pmatrix} \qquad\text{and}\qquad \left(x \xleftarrow{(x,g)} xg \right) \xrightarrow{(\mathsf{I}_\mathcal{D})_1} \begin{pmatrix} x & \xleftarrow{(x,g)} & xg \\ \mathrm{Id}_x \big\downarrow & \xLeftarrow{(\mathrm{Id}_x,g)} & \big\downarrow \mathrm{Id}_x\, g \\ x & \xleftarrow{(x,g)} & xg \end{pmatrix}.$$

The proof that $\mathsf{S}_\mathcal{D}$ and $\mathsf{I}_\mathcal{D}$ are functors is similar to the one of $\mathsf{T}_\mathcal{D}$. Regarding the **multiplication functor** (or **composition functor**) $\mathsf{M}_\mathcal{D}\colon \mathcal{D}_2 \longrightarrow \mathcal{D}_1$, we define

$$\begin{pmatrix} y & & x \\ \big\downarrow l & , & \big\downarrow f \\ z & & y \end{pmatrix} \xrightarrow{(\mathsf{M}_\mathcal{D})_0} \begin{pmatrix} x \\ \big\downarrow lf \\ z \end{pmatrix}$$

and

$$\begin{pmatrix} y & \xleftarrow{(y,g)} & yg & & x & \xleftarrow{(x,g)} & xg \\ l \big\downarrow & \xLeftarrow{(l,g)} & \big\downarrow lg & , & f \big\downarrow & \xLeftarrow{(f,g)} & \big\downarrow fg \\ z & \xleftarrow{(z,g)} & zg & & y & \xleftarrow{(y,g)} & yg \end{pmatrix} \xrightarrow{(\mathsf{M}_\mathcal{D})_1} \begin{pmatrix} x & \xleftarrow{(x,g)} & xg \\ lf \big\downarrow & \xLeftarrow{(lf,g)} & \big\downarrow (lf)g \\ z & \xleftarrow{(z,g)} & zg \end{pmatrix}.$$

Now we have to prove that $\mathsf{M}_\mathcal{D}$ is a functor: we calculate

$$\mathsf{M}_\mathcal{D} \begin{pmatrix} y & \xleftarrow{(y,\iota_{\varsigma_0(y)})} & y & & x & \xleftarrow{(x,\iota_{\varsigma_0(x)})} & x \\ l \big\downarrow & \xLeftarrow{(l,\iota_{\varsigma_1(l)})} & \big\downarrow l & , & f \big\downarrow & \xLeftarrow{(f,\iota_{\varsigma_1(f)})} & \big\downarrow f \\ z & \xleftarrow{(z,\iota_{\varsigma_0(z)})} & z & & y & \xleftarrow{(y,\iota_{\varsigma_0(y)})} & y \end{pmatrix} = \begin{pmatrix} x & \xleftarrow{(x,\iota_{\varsigma_0(x)})} & x \\ lf \big\downarrow & \xLeftarrow{(lf,\iota_{\varsigma_1(lf)})} & \big\downarrow lf \\ z & \xleftarrow{(z,\iota_{\varsigma_0(z)})} & z \end{pmatrix}$$

and, since $(lf)g = (lg)(fg)$ and $(lf)gh = (l(gh))(f(gh))$, we can prove the composition condition as in figure (6.2).

Definition 6.5.3. The double category $\mathcal{D}$ just constructed is called the **right translation double category** of the right categorified $\mathcal{G}$-set $(\mathcal{X}, \varsigma)$ and we denote it by $\mathcal{X} \rtimes \mathcal{G}$.

To clarify this concept we give an example in the context of groups.

Example 6.5.4. Given a small category $\mathcal{C}$ and a group G, let $\mathcal{X} = \mathcal{C} \times G$ be the right categorified G-set of section 6.4. We want to describe the double translation category $\mathcal{D} = \mathcal{C} \times X$. For $i = 0, 1$ we have $(\mathcal{D}_0)_0 = (\mathcal{X}_0 \rtimes G)_0 = \mathcal{X}_0$, $(\mathcal{D}_0)_1 = (\mathcal{X}_0 \times G)_1 = \mathcal{X}_0 \times G$, $(\mathcal{D}_1)_0 = (\mathcal{X}_1 \rtimes G)_0 = \mathcal{X}_1$ and $(\mathcal{D}_1)_1 = (\mathcal{X}_1 \times G)_1 = \mathcal{X}_1 \times G$. The target, source and identity functors are as follows:

$$\mathsf{T}_\mathcal{D} = (\mathsf{t}_\mathcal{C} \times \mathrm{Id}_X, \mathsf{t}_\mathcal{C}), \qquad \mathsf{S}_\mathcal{D} = (\mathsf{s}_\mathcal{C} \times \mathrm{Id}_X, \mathsf{s}_\mathcal{C}) \qquad\text{and}\qquad \mathsf{I}_\mathcal{D} = (\iota_\mathcal{C} \times \mathrm{Id}_X, \iota_\mathcal{C}),$$

Regarding the composition functor, we have:

$$\begin{aligned} &\mathsf{M}_\mathcal{D}\colon \mathcal{D}_2 \longrightarrow \mathcal{D}_1 \\ &\quad (\mathcal{D}_2)_0 \ni (h,f) \longrightarrow m_\mathcal{X}(h,f) \\ &\quad (\mathcal{D}_2)_1 \ni ((h,g),(f,g)) \longrightarrow (m_\mathcal{X}(h,f),g). \end{aligned}$$

Now we want to introduce the concept of orbit category and, to do this, we need some preparation.

$$
\mathsf{M}_{\mathcal{D}}\left(\left(\begin{array}{ccc} y & \xleftarrow{(y,g)} & yg \\ {\scriptstyle l}\big\downarrow & \xLeftarrow{(l,g)} & \big\downarrow{\scriptstyle lg} \\ z & \xleftarrow{(z,g)} & zg \end{array},\ \begin{array}{ccc} x & \xleftarrow{(x,g)} & xg \\ {\scriptstyle f}\big\downarrow & \xLeftarrow{(f,g)} & \big\downarrow{\scriptstyle fg} \\ y & \xleftarrow{(y,g)} & yg \end{array}\right)\circ\left(\begin{array}{ccc} yg & \xleftarrow{(yg,h)} & ygh \\ {\scriptstyle lg}\big\downarrow & \xLeftarrow{(lg,h)} & \big\downarrow{\scriptstyle lgh} \\ zg & \xleftarrow{(zg,h)} & zgh \end{array},\ \begin{array}{ccc} xg & \xleftarrow{(xg,h)} & xgh \\ {\scriptstyle fg}\big\downarrow & \xLeftarrow{(fg,h)} & \big\downarrow{\scriptstyle fgh} \\ yg & \xleftarrow{(yg,h)} & ygh \end{array}\right)\right)
$$

$$
= \mathsf{M}_{\mathcal{D}}\left(\begin{array}{ccc} y & \xleftarrow{(y,g)} & yg \\ {\scriptstyle l}\big\downarrow & \xLeftarrow{(l,g)} & \big\downarrow{\scriptstyle lg} \\ z & \xleftarrow{(z,g)} & zg \end{array}\circ\begin{array}{ccc} yg & \xleftarrow{(yg,h)} & ygh \\ {\scriptstyle lg}\big\downarrow & \xLeftarrow{(lg,h)} & \big\downarrow{\scriptstyle lgh} \\ zg & \xleftarrow{(zg,h)} & zgh \end{array},\ \begin{array}{ccc} x & \xleftarrow{(x,g)} & xg \\ {\scriptstyle f}\big\downarrow & \xLeftarrow{(f,g)} & \big\downarrow{\scriptstyle fg} \\ y & \xleftarrow{(y,g)} & yg \end{array}\circ\begin{array}{ccc} xg & \xleftarrow{(xg,h)} & xgh \\ {\scriptstyle fg}\big\downarrow & \xLeftarrow{(fg,h)} & \big\downarrow{\scriptstyle fgh} \\ yg & \xleftarrow{(yg,h)} & ygh \end{array}\right)
$$

$$
= \mathsf{M}_{\mathcal{D}}\left(\begin{array}{ccc} y & \xleftarrow{(y,gh)} & ygh \\ {\scriptstyle l}\big\downarrow & \xLeftarrow{(l,gh)} & \big\downarrow{\scriptstyle lgh} \\ z & \xleftarrow{(z,gh)} & zgh \end{array},\ \begin{array}{ccc} x & \xleftarrow{(x,gh)} & xgh \\ {\scriptstyle f}\big\downarrow & \xLeftarrow{(f,gh)} & \big\downarrow{\scriptstyle fgh} \\ y & \xleftarrow{(y,gh)} & ygh \end{array}\right) = \left(\begin{array}{ccc} x & \xleftarrow{(x,gh)} & xgh \\ {\scriptstyle lf}\big\downarrow & \xLeftarrow{(lf,gh)} & \big\downarrow{\scriptstyle (lf)gh} \\ z & \xleftarrow{(z,gh)} & zgh \end{array}\right)
$$

$$
= \left(\begin{array}{ccc} x & \xleftarrow{(x,g)} & xg \\ {\scriptstyle lf}\big\downarrow & \xLeftarrow{(lf,g)} & \big\downarrow{\scriptstyle (lf)g} \\ z & \xleftarrow{(z,g)} & zg \end{array}\right)\circ\left(\begin{array}{ccc} xg & \xleftarrow{(xg,h)} & xgh \\ {\scriptstyle (lf)g}\big\downarrow & \xLeftarrow{((lf)g,h)} & \big\downarrow{\scriptstyle (lf)gh} \\ zg & \xleftarrow{(zg,h)} & zgh \end{array}\right)
$$

$$
= \mathsf{M}_{\mathcal{D}}\left(\begin{array}{ccc} y & \xleftarrow{(y,g)} & yg \\ {\scriptstyle l}\big\downarrow & \xLeftarrow{(l,g)} & \big\downarrow{\scriptstyle lg} \\ z & \xleftarrow{(z,g)} & zg \end{array},\ \begin{array}{ccc} x & \xleftarrow{(x,g)} & xg \\ {\scriptstyle f}\big\downarrow & \xLeftarrow{(f,g)} & \big\downarrow{\scriptstyle fg} \\ y & \xleftarrow{(y,g)} & yg \end{array}\right)\circ\mathsf{M}_{\mathcal{D}}\left(\begin{array}{ccc} yg & \xleftarrow{(yg,h)} & ygh \\ {\scriptstyle lg}\big\downarrow & \xLeftarrow{(lg,h)} & \big\downarrow{\scriptstyle lgh} \\ zg & \xleftarrow{(zg,h)} & zgh \end{array},\ \begin{array}{ccc} xg & \xleftarrow{(xg,h)} & xgh \\ {\scriptstyle fg}\big\downarrow & \xLeftarrow{(fg,h)} & \big\downarrow{\scriptstyle fgh} \\ yg & \xleftarrow{(yg,h)} & ygh \end{array}\right).
$$

Figure 6.2: Multiplication functor, composition condition

Definition 6.5.5. Given a double category $\mathcal{D}$, let us consider a square $Q \in (\mathcal{D}_1)_1$. We define the **vertices of the square** Q as $\mathsf{s}_{\mathcal{D}_0}(\mathsf{S}_{\mathcal{D}}(Q))$, $\mathsf{t}_{\mathcal{D}_0}(\mathsf{S}_{\mathcal{D}}(Q))$, $\mathsf{s}_{\mathcal{D}_0}(\mathsf{T}_{\mathcal{D}}(Q))$ and $\mathsf{t}_{\mathcal{D}_0}(\mathsf{T}_{\mathcal{D}}(Q))$.

Example 6.5.6. Given $(\mathcal{X}, \varsigma) \in \mathsf{CSets}\text{-}\mathcal{G}$, in the following square in $\mathcal{X} \rtimes \mathcal{G}$

$$\begin{array}{ccc} x & \xleftarrow{(x,g)} & xg \\ {\scriptstyle f}\big\downarrow & \overset{(f,g)}{\Longleftarrow} & \big\downarrow{\scriptstyle fg} \\ y & \xleftarrow{(y,g)} & yg \end{array}$$

the vertices are x, xg, y and yg.

Definition 6.5.7. Let be $\mathcal{X} \in \mathsf{CSets}\text{-}\mathcal{G}$: for each $a, b \in \mathcal{X}_0$ we define $a \,\underline{\mathsf{sq}}\, b$ if and only if there is a square in $\mathcal{X} \rtimes \mathcal{G}$ with a and b amongst its vertices.

Remark 6.5.8. Given $(\mathcal{X}, \varsigma) \in \mathsf{CSets}\text{-}\mathcal{G}$, if we have a morphism $(f : a \longrightarrow b) \in \mathcal{X}_1$ then we have the diagram

$$\begin{array}{ccc} a & \xleftarrow{\left(x,\iota_{\varsigma_0(x)}\right)} & a \\ {\scriptstyle f}\big\downarrow & \overset{\left(f,\iota_{\varsigma_1(f)}\right)}{\Longleftarrow} & \big\downarrow{\scriptstyle f} \\ b & \xleftarrow{\left(b,\iota_{\varsigma_0(b)}\right)} & b, \end{array}$$

therefore $a \,\underline{\mathsf{sq}}\, b$.

Remark 6.5.9. Given $(\mathcal{X}, \varsigma) \in \mathsf{CSets}\text{-}\mathcal{G}$, for each $a \in \mathcal{X}_0$ and $g \in \mathcal{G}_1$ such that $\varsigma_0(a) = \mathsf{t}(g)$, we have the diagram

$$\begin{array}{ccc} a & \xleftarrow{(a,g)} & ag \\ {\scriptstyle \mathrm{Id}_a}\big\downarrow & \overset{(\mathrm{Id}_a,g)}{\Longleftarrow} & \big\downarrow{\scriptstyle \mathrm{Id}_a\, g = \mathrm{Id}_{ag}} \\ a & \xleftarrow{(a,g)} & ag, \end{array}$$

therefore $a \,\underline{\mathsf{sq}}\, ag$.

The relation $\underline{\mathsf{sq}}$ is reflexive and symmetric but not transitive: it's enough to consider the following example with a trivial action: $\mathcal{X}_0 = \{ a, b, c \}$ and only $f : a \longrightarrow b$ and $h : a \longrightarrow c$ as not isomorphism arrows. In this case it is clear that $a \,\underline{\mathsf{sq}}\, b$ and $a \,\underline{\mathsf{sq}}\, c$ but we don't have $b \,\underline{\mathsf{sq}}\, c$. This suggests us to give the next definition.

Definition 6.5.10. Let be $(\mathcal{X}, \varsigma) \in \mathsf{CSets}\text{-}\mathcal{G}$: for each $a, b \in \mathcal{X}_0$ we define $a \,\underline{\mathsf{Sq}}\, b$ if and only if there are $a_0, \ldots, a_n \in \mathcal{X}_0$, with $n \in \mathbb{N}^+$, such that for each $i \in \{ 0, \ldots, n-1 \}$, $a_i \,\underline{\mathsf{sq}}\, a_{i+1}$. This means that $\underline{\mathsf{Sq}}$ is the equivalence relations generated by $\underline{\mathsf{sq}}$. Given $a \in \mathcal{X}_0$, we denote with $\mathrm{Orb}_{\underline{\mathsf{Sq}}}(a)$ the full subcategory of $\mathcal{X}$ such that the set of objects of $\mathrm{Orb}_{\underline{\mathsf{Sq}}}(a)$ is the equivalence class of a with respect to $\underline{\mathsf{Sq}}$. We also set $\mathrm{Orb}_{\underline{\mathsf{Sq}}}(f) := \mathrm{Orb}_{\underline{\mathsf{Sq}}}(\mathsf{s}_{\mathcal{X}}(f)) = \mathrm{Orb}_{\underline{\mathsf{Sq}}}(\mathsf{t}_{\mathcal{X}}(f))$ for every $f \in \mathcal{X}_1$. Moreover, we denote with $\mathrm{rep}_{\underline{\mathsf{Sq}}}(\mathcal{G})$ a set of objects of $\mathcal{X}$ that acts as a set of representative elements with respect to the relation $\underline{\mathsf{Sq}}$. Note that, for every $f \in \mathcal{X}_1$ (respectively, for each $a \in \mathcal{X}_0$), $\mathrm{Orb}_{\underline{\mathsf{Sq}}}(f)$ (respectively, $\mathrm{Orb}_{\underline{\mathsf{Sq}}}(a)$) contains both the $\mathcal{G}$-orbit of f (respectively, of a) and the connected component of the category $\mathcal{X}$ (see [Mac98, pag. 88, 90]) that contains f (respectively, a). As a consequence, both $\mathrm{Orb}_{\underline{\mathsf{Sq}}}(f)$ and $\mathrm{Orb}_{\underline{\mathsf{Sq}}}(a)$ are right $\mathcal{G}$-sets, can be decomposed in $\mathcal{G}$-orbits and are called the **orbit categories** of f and a, respectively.

Proposition 6.5.11. *Let be* $(\mathcal{X}, \varsigma) \in \mathsf{CSets}\text{-}\mathcal{G}$*: we have*

$$\mathcal{X} = \biguplus_{a \in \operatorname{rep}_{\underline{\mathsf{Sq}}}(\mathcal{X})} \operatorname{Orb}_{\underline{\mathsf{Sq}}}(a)$$

and, for every $a \in \operatorname{rep}_{\underline{\mathsf{Sq}}}(\mathcal{X})$*, there cannot be two categorified* G*-sets* $\mathcal{Y}$ *and* $\mathcal{Y}'$*, both not empty, such that* $\operatorname{Orb}_{\underline{\mathsf{Sq}}}(a) = \mathcal{Y} \uplus \mathcal{Y}'$*.*

Proof. Immediate from the definitions. □

6.6 Categorified Burnside theory

We will give the main steps to build up the weak Burnside ring functor, using its category of categorified groupoid-sets, ad we will compare this new ring with the classical one, providing a natural transformation between the two contravariant functors.

Given a morphism of groupoids $\varphi\colon \mathcal{H} \longrightarrow \mathcal{G}$, in similar way to how it has been done in subsection 1.3.2, we define the **induction functor**

$$\varphi^*\colon \mathsf{CSets}\text{-}\mathcal{G} \longrightarrow \mathsf{CSets}\text{-}\mathcal{H},$$

which sends the right categorified $\mathcal{G}$-set $(\mathcal{X}, \varsigma)$ to the right categorified $\mathcal{H}$-set $\varphi^*(\mathcal{X}, \varsigma)$ such that, for $i = 0, 1$,

$$(\varphi^*(\mathcal{X}, \varsigma))_i = (\mathcal{X}_i \,{}_{\varsigma_i}\!\times_{\varphi_0} \mathcal{H}_0, \mathrm{pr}_2)$$

is the right $\mathcal{H}$-set with the following action:

$$\begin{aligned} (\mathcal{X}_i \,{}_{\varsigma_i}\!\times_{\varphi_0} \mathcal{H}_0) \,{}_{\mathrm{pr}_2}\!\times_{\mathsf{t}} \mathcal{H}_1 &\longrightarrow \mathcal{X}_i \,{}_{\varsigma_i}\!\times_{\varphi_0} \mathcal{H}_0 \\ \big((x, a), h\big) &\longrightarrow (x\varphi_1(h), \mathsf{s}(h)). \end{aligned}$$

The target, source and identity maps of $\varphi^*(\mathcal{X}, \varsigma)$ are defined as follows:

$$\mathsf{s}_{\varphi^*(\mathcal{X},\varsigma)} = \mathsf{s}_{\mathcal{X}} \times \mathrm{Id}_{\mathcal{H}_0}, \qquad \mathsf{t}_{\varphi^*(\mathcal{X},\varsigma)} = \mathsf{t}_{\mathcal{X}} \times \mathrm{Id}_{\mathcal{H}_0} \qquad \text{and} \qquad \iota_{\varphi^*(\mathcal{X},\varsigma)} = \iota_{\mathcal{X}} \times \mathrm{Id}_{\mathcal{H}_0}.$$

Regarding the composition, we set

$$\begin{aligned} m_{\varphi^*(\mathcal{X},\varsigma)}\colon (\mathcal{X}_1 \,{}_{\varsigma_i}\!\times_{\varphi_0} \mathcal{H}_0) \,{}_{\mathsf{s}_{\varphi^*(\mathcal{X},\varsigma)}}\!\times_{\mathsf{t}_{\varphi^*(\mathcal{X},\varsigma)}} (\mathcal{X}_1 \,{}_{\varsigma_i}\!\times_{\varphi_0} \mathcal{H}_0) &\longrightarrow (\mathcal{X}_1 \,{}_{\varsigma_i}\!\times_{\varphi_0} \mathcal{H}_0) \\ \Big((x, a), (y, b)\Big) &\longrightarrow (m_{\mathcal{X}}(x, y), a) \end{aligned}$$

Given a morphism of right categorified G-set $f\colon (\mathcal{X}, \varsigma) \longrightarrow (\mathcal{Y}, \theta)$, we define the morphism of categorified right $\mathcal{H}$-sets

$$\varphi^*(f)\colon \varphi^*(\mathcal{X}, \varsigma) \longrightarrow \varphi^*(\mathcal{Y}, \theta)$$

as the morphism $f \times \mathrm{Id}_{\mathcal{H}_0}$. In a similar way to how it has been done in Proposition 1.3.6 it is possible to prove that φ^* is monoidal with respect to both $\uplus$ and $\underset{\mathcal{G}_0}{\times}$.

Let now $\mathcal{G}$ be a groupoid: we will develop a Burnside theory based on categorified right $\mathcal{G}$-sets, in a similar way to how it has been done in Chapter 5. We assume, in this section, that all handled groupoids have a finite set of objects: this condition is needed for the categorified Burnside ring we are planing to introduce to have a neutral element with respect to the multiplication. We also assume that functors between categorified sets with a right action preserve objects with finite underlying sets, and transform an empty categorified groupoid-set to an empty one, as the induction functors do.

We denote with csets-$\mathcal{G}$ the 2-category of finite right categorified $\mathcal{G}$-sets; however, as with CSets-G, we will mainly consider it as a category. We define $\mathscr{L}_{\mathsf{C}}(\mathcal{G})$ as the quotient set of csets-$\mathcal{G}$ by the equivalence relation $\sim_{\mathsf{we}}$: thanks to Lemma 6.3.4, it becomes a rig. Moreover, given a morphism of groups $\varphi\colon \mathcal{H} \longrightarrow \mathcal{G}$, in a similar way to how it has been proved in Lemma 5.1.1, it is clear that we have a monomorphism of rigs

$$\begin{aligned}\mathscr{L}_{\mathsf{C}}(\varphi)\colon \mathscr{L}_{\mathsf{C}}(\mathcal{G}) &\longrightarrow \mathscr{L}_{\mathsf{C}}(\mathcal{H})\\ [(X,\varsigma)] &\longrightarrow [\varphi^*(X,\varsigma)]\,.\end{aligned}$$

To see that $\mathscr{L}_{\mathsf{C}}(\varphi)$ is well defined it's enough to consider that, given a 2-morphism $\alpha\colon f \longrightarrow h$ between two morphisms $f,h\colon (\mathcal{X},\varsigma) \longrightarrow (\mathcal{Y},\theta)$ in CSets-$\mathcal{G}$, we can define a 2-morphism

$$\alpha \times \mathrm{Id}_{\mathcal{H}_0}\colon f \times \mathrm{Id}_{\mathcal{H}_0} \longrightarrow h \times \mathrm{Id}_{\mathcal{H}_0}$$

in CSets-$\mathcal{H}$ between $f \times \mathrm{Id}_{\mathcal{H}_0}$ and $g \times \mathrm{Id}_{\mathcal{H}_0}$. As a consequence, $(\mathcal{X},\varsigma) \sim_{\mathsf{we}} (\mathcal{Y},\theta)$ in CSets-$\mathcal{G}$ implies $\varphi^*(\mathcal{X},\varsigma) \sim_{\mathsf{we}} \varphi^*(\mathcal{Y},\theta)$ in CSets-$\mathcal{H}$. In this way we obtain a contravariant functor $\mathscr{L}_{\mathsf{C}}\colon$ **Grpd** $\longrightarrow$ **Rig**, that is, from the category of groupoids to the category of rigs.

Given a groupoid $\mathcal{G}$, using the functor $\mathcal{I}$-G of Eq. (6.2.1), we can construct an injective morphism from the classical Burnside rig of $\mathcal{G}$ to the categorified Burnside rig of $\mathcal{G}$ in the following way:

$$\begin{aligned}\mathscr{L}_{\mathcal{I}}(\mathcal{G})\colon \mathscr{L}(\mathcal{G}) &\longrightarrow \mathscr{L}_{\mathsf{C}}(\mathcal{G})\\ [(X,\varsigma)] &\longrightarrow [\mathcal{I}\text{-}\mathcal{G}(X,\varsigma)]\,.\end{aligned}$$

Note that $\mathscr{L}_{\mathcal{I}}(\mathcal{G})$ cannot be surjective because, thanks to Corollary 6.3.8, categories with a not discrete skeleton cannot be weak equivalent to discrete ones. In this way we obtain a natural transformation $\mathscr{L}_{\mathcal{I}}\colon \mathscr{L} \longrightarrow \mathscr{L}_{\mathsf{C}}$, that is, given a morphism of groups $\varphi\colon \mathcal{H} \longrightarrow \mathcal{G}$, the following diagram is commutative:

$$\begin{array}{ccc}\mathscr{L}(\mathcal{G}) & \xrightarrow{\mathscr{L}_{\mathcal{I}}(\mathcal{G})} & \mathscr{L}_{\mathsf{C}}(\mathcal{G})\\ {\scriptstyle \mathscr{L}(\varphi)}\downarrow & & \downarrow{\scriptstyle \mathscr{L}_{\mathsf{C}}(\varphi)}\\ \mathscr{L}(\mathcal{H}) & \xrightarrow{\mathscr{L}_{\mathcal{I}}(\mathcal{H})} & \mathscr{L}_{\mathsf{C}}(\mathcal{H})\,.\end{array}$$

We define $\mathscr{B}_{\mathsf{C}} = \mathscr{G}\mathscr{L}_{\mathsf{C}}$ and $\mathscr{B}_{\mathcal{I}} = \mathscr{G}\mathscr{L}_{\mathcal{I}}$, where $\mathscr{G}$ is the Grothendieck functor (see Appendix B), that is, a functor that associates to each rig an opportune ring with a specific universal property. In this way we obtain a contravariant functor $\mathscr{B}_{\mathsf{C}}\colon$ **Grp** $\longrightarrow$ **CRing**, that is, from the category of groups to the category of commutative rings, and a natural transformation

$$\mathscr{B}_{\mathcal{I}} = \mathscr{G}\mathscr{L}_{\mathcal{I}}\colon \mathscr{B} = \mathscr{G}\mathscr{L} \longrightarrow \mathscr{B}_{\mathsf{C}} = \mathscr{G}\mathscr{L}_{\mathsf{C}}.$$

Definition 6.6.1. We call $\mathscr{L}_{\mathsf{C}}$ the **categorified Burnside rig functor** and $\mathscr{B}_{\mathsf{C}}$ the **categorified Burnside ring functor**. In particular, given a groupoid $\mathcal{G}$, we call $\mathscr{B}_{\mathsf{C}}(\mathcal{G})$ the **categorified Burnside ring of** $\mathcal{G}$.

Remark 6.6.2. An obvious question is if, given a groupoid $\mathcal{G}$, the morphism of rings $\mathscr{B}_{\mathcal{I}}(\mathcal{G})$ is injective (we will use the abuses of notation $\mathscr{L}_{\mathcal{I}} = \mathscr{L}_{\mathcal{I}}(\mathcal{G})$ and $\mathscr{B}_{\mathcal{I}} = \mathscr{B}_{\mathcal{I}}(\mathcal{G})$). Let be

$$\Big[\,[(\mathcal{X},\varsigma)]\,,[(\mathcal{Y},\theta)]\,\Big],\quad \Big[\,[(\mathcal{A},\lambda)]\,,[(\mathcal{B},\mu)]\,\Big] \in \mathscr{B}_{\mathsf{C}}(\mathcal{G})$$

such that

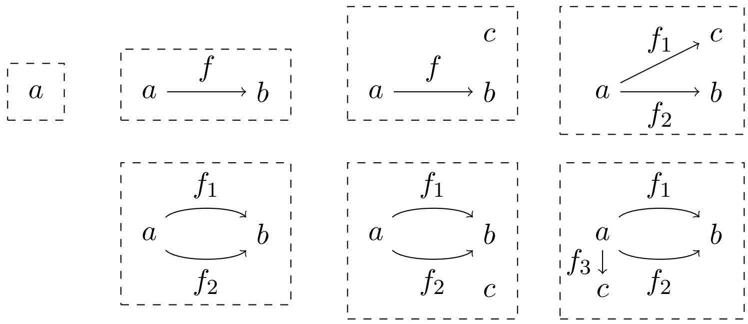
$$\begin{aligned}\Big[\mathscr{L}_{\mathcal{I}}([(\mathcal{X},\varsigma)])\,,\mathscr{L}_{\mathcal{I}}([(\mathcal{Y},\theta)])\Big] &= \mathscr{B}_{\mathcal{I}}\Big(\Big[\,[(\mathcal{X},\varsigma)]\,,[(\mathcal{Y},\theta)]\,\Big]\Big)\\ = \mathscr{B}_{\mathcal{I}}\Big(\Big[\,[(\mathcal{A},\lambda)]\,,[(\mathcal{B},\theta)]\,\Big]\Big) &= \Big[\mathscr{L}_{\mathcal{I}}([(\mathcal{A},\lambda)])\,,\mathscr{L}_{\mathcal{I}}([(\mathcal{B},\mu)])\Big]\,:\end{aligned}$$

then there is $[(\mathcal{Z},\omega)] \in \mathscr{L}_{\mathsf{C}}(G)$ such that

$$\mathscr{L}_{\mathcal{I}}\left([(\mathcal{X},\varsigma)]\right) + \mathscr{L}_{\mathcal{I}}\left([(\mathcal{B},\mu)]\right) + [(\mathcal{Z},\omega)] = \mathscr{L}_{\mathcal{I}}\left([(\mathcal{A},\lambda)]\right) + \mathscr{L}_{\mathcal{I}}\left([(\mathcal{Y},\theta)]\right) + [(\mathcal{Z},\omega)]\,.$$

If $\mathscr{L}_{\mathsf{C}}(\mathcal{G})$, as an additive monoid, satisfies the cancellative property, then, since $\mathscr{L}_{\mathcal{I}}$ is injective, we can easily deduce that $\mathscr{B}_{\mathcal{I}}$ is injective too. In the classical case this is guaranteed by the Burnside Theorem (see [Bou10a, Thm. 2.4.5]) but in this context we still don't know whether a similar theorem is true or not and, consequently, the question about the injectivity of $\mathscr{B}_{\mathcal{I}}$ remains open.

Example 6.6.3. Given a groupoid $\mathcal{G}$, the following categories, with only the identities as isomorphisms and with the actions and structure maps opportunely defined, thanks to Corollary 6.3.8,



are all examples of not weak equivalent right categorified $\mathcal{G}$-sets, thus they give rise to different elements in $\mathscr{L}_{\mathsf{C}}(\mathcal{G})$. For example, let be $\mathcal{G} = 1$, the groupoid with one object and one arrow: in this case the $\mathcal{G}$-action is trivial, thus $\mathscr{L}(\mathcal{G}) = \mathbb{N}$ (we just have finite sets) but, regarding $\mathscr{L}_{\mathsf{C}}(\mathcal{G})$, we have to consider all the classes given by all the previous not weak equivalent right categorified $\mathcal{G}$-sets. More specifically, for each $n \in \mathbb{N}^+$, in the case of $\mathscr{L}(1)$, we just have n points, but in the case of $\mathscr{L}_{\mathsf{C}}(1)$, we have to consider all the possible graphs with n vertices!

The following results are adaptations of the corresponding results from Chapter 5 to the new situation of categorified sets.

Proposition 6.6.4. *Given a groupoid $\mathcal{G}$, let $\mathcal{A}$ be a subgroupoid of $\mathcal{G}$. We define a functor*

$$F\colon \mathsf{CSets}\text{-}\mathcal{G} \longrightarrow \mathsf{CSets}\text{-}\mathcal{A}$$

in the following way: let be $(\mathcal{X},\varsigma) \in \mathsf{CSets}\text{-}\mathcal{G}$. We define $F((\mathcal{X},\varsigma))$ as the internal category in $\mathsf{Sets}\text{-}\mathcal{A}$ *with set of objects $\left(\varsigma_0^{-1}(\mathcal{A}_0), \varsigma_0|_{\varsigma_0^{-1}(\mathcal{A}_0)}\right)$ and set of morphisms $\left(\varsigma_1^{-1}(\mathcal{A}_0), \varsigma_1|_{\varsigma_1^{-1}(\mathcal{A}_0)}\right)$. The source, target, identity and composition maps of $F((\mathcal{X},\varsigma))$ are the opportune restriction to $\varsigma_0^{-1}(\mathcal{A}_0)$ and $\varsigma_1^{-1}(\mathcal{A}_0)$ of the relative maps of $(\mathcal{X},\varsigma)$. Then F, opportunely defined on morphisms, is a Laplaza functor.*

Proof. It proceeds as the proof of Proposition 1.3.10. □

Given a groupoid $\mathcal{G}$ and a fixed object $x \in \mathcal{G}_0$, we recall that by $\mathcal{G}^{(x)}$ we denote the one object subgroupoid with isotropy group $\mathcal{G}^x$.

Theorem 6.6.5. *Given a transitive and not empty groupoid $\mathcal{G}$, let be $a \in \mathcal{G}_0$. Then there is a Laplaza equivalence of categories*

$$\mathsf{CSets}\text{-}\mathcal{G} \simeq \mathsf{CSets}\text{-}\mathcal{G}^{(a)}.$$

Proof. It is similar to the proof of Theorem 1.3.15 but there are a few differences. Let's set $\mathcal{A}_0 = \{ a \}$. The functor $F\colon \mathsf{CSets}\text{-}\mathcal{G} \longrightarrow \mathsf{CSets}\text{-}\mathcal{A}$ is the one constructed in Proposition 6.6.4. Regarding $G\colon \mathsf{CSets}\text{-}A \longrightarrow \mathsf{CSets}\text{-}G$, we define $G((\mathcal{X}, \varsigma))$ as the internal category in $\mathsf{Sets}\text{-}\mathcal{G}$ with set of objects

$$\Big(\mathcal{Y}_0 = \mathcal{X}_0 \times \mathcal{G}_0, \quad \vartheta_0 = \mathrm{pr}_2\colon \mathcal{Y}_0 = \mathcal{X}_0 \times \mathcal{G}_0 \longrightarrow \mathcal{G}_0\Big)$$

and object of morphisms

$$\Big(\mathcal{Y}_1 = \mathcal{X}_1 \times \mathcal{G}_0, \quad \vartheta_1 = \mathrm{pr}_2\colon \mathcal{Y}_1 = \mathcal{X}_1 \times \mathcal{G}_0 \longrightarrow \mathcal{G}_0\Big).$$

The source, target, identity and composition maps of $G((\mathcal{X}, \varsigma))$ are defined as $\mathsf{s}_{G((\mathcal{X},\varsigma))} = \mathsf{s}_{(\mathcal{X},\varsigma)} \times \mathcal{G}_0$, $\mathsf{t}_{G((\mathcal{X},\varsigma))} = \mathsf{t}_{(\mathcal{X},\varsigma)} \times \mathcal{G}_0$, $\iota_{G((\mathcal{X},\varsigma))} = \iota_{(\mathcal{X},\varsigma)} \times \mathcal{G}_0$ and $m_{G((\mathcal{X},\varsigma))}\Big((x,a),(y,b)\Big) = \Big(m_{(\mathcal{X},\varsigma)}(x,y), a\Big)$ for each $((x,a),(y,b)) \in \mathcal{X}_2$. □

Proposition 6.6.6. *The Burnside rig functor $\mathscr{L}_{\mathsf{C}}$ sends coproduct to product. In particular, given a family of groupoids $(\mathcal{G}_j)_{j\,\in\, I}$, let $(i_j\colon \mathcal{G}_j \longrightarrow \mathcal{G})_{j\,\in\, I}$ be their coproduct in* **Grpd**. *Then*

$$\left(\mathscr{L}_{\mathsf{C}}(i_j) : \mathscr{L}_{\mathsf{C}}(\mathcal{G}) \longrightarrow \mathscr{L}_{\mathsf{C}}(\mathcal{G}_j)\right)_{j\in I}$$

is the product of the family $(\mathscr{L}_{\mathsf{C}}(\mathcal{G}_j))_{j\,\in\, I}$ in the category **Rig**.

Proof. The idea of the proof is similar to Proposition 5.1.3: the only difference to keep in mind is that we are dealing with particular small categories and not sets. □

Theorem 6.6.7. *Let be $\mathcal{G}$ and $\mathcal{A}$ be groupoids such that there is a Laplaza equivalence of categories $\mathsf{CSets}\text{-}\mathcal{G} \simeq \mathsf{CSets}\text{-}\mathcal{A}$. Then there is an isomorphism of commutative rings $\mathscr{B}_{\mathsf{C}}(\mathcal{G}) \cong \mathscr{B}_{\mathsf{C}}(\mathcal{A})$.*

Proof. It proceeds as in Theorem 5.1.5. □

Theorem 6.6.8. *Given a groupoid $\mathcal{G}$, fix a set of representative objects $\mathrm{rep}(\mathcal{G}_0)$ representing the set of connected components $\pi_0(\mathcal{G})$. For each $a \in \mathrm{rep}(\mathcal{G}_0)$, let $\mathcal{G}^{\langle a\rangle}$ be the connected component of $\mathcal{G}$ containing a, which we consider as a groupoid. Then we have the following isomorphism of rings:*

$$\mathscr{B}_{\mathsf{C}}(\mathcal{G}) \cong \prod_{a\,\in\,\mathrm{rep}(\mathcal{G}_0)} \mathscr{B}_{\mathsf{C}}\left(\mathcal{G}^{\langle a\rangle}\right).$$

Proof. It follows directly from Proposition 6.6.6. □

Corollary 6.6.9. *Given a groupoid $\mathcal{G}$, we have the following isomorphism of rings:*

$$\mathscr{B}_{\mathsf{C}}(\mathcal{G}) \cong \prod_{a\,\in\,\mathrm{rep}(\mathcal{G}_0)} \mathscr{B}_{\mathsf{C}}(\mathcal{G}^{a}),$$

where the right hand side term is the product of commutative rings.

Proof. Immediate from Theorem 6.6.8, Theorem 6.6.7, and Theorem 6.6.5. □

Appendix A

The category Rig

One of the essential notion to introduce a Burnside ring is that of rig.

Definition A.0.1. Let S be a set with two associative and commutative internal operations $\cdot$ and $+$. We call S a **rig** if the following conditions are satisfied:

(1) $+$ has a neutral element 0;

(2) $\cdot$ has a neutral element 1;

(3) $\cdot$ distributes over $+$ on the right and on the left that is, for each $a, b, c \in S$,

$$a(b+c) = ab + ac \qquad \text{and} \qquad (a+b)c = ac + ab;$$

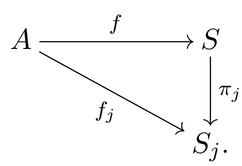
(4) S respect the absorption/annihilation laws that is, for each $a \in S$ we have $a \cdot 0 = 0 = 0 \cdot a$.

A homomorphism of rigs $f\colon S \longrightarrow T$ is a function which is a homomorphism of monoids both as $f\colon (S,+) \longrightarrow (T,+)$ and as $f\colon (S,\cdot) \longrightarrow (T,\cdot)$. The category of rigs will be denoted by **Rig**.

With this definition we choose to follow the definitions given by [Gla02, page 7], and [Sch91, page 379]. The reader should know, however, that what we called a rig is called a semiring by other authors ([Gol99, page 1]). Nevertheless, in analogy with the word semigroup that describes a monoid without a neutral element, we think that the word semiring should be reserved to a ring that lacks both the negative elements (i.e., the inverses with respect to the addition) and the additive neutral element.

Proposition A.0.2. *Given a family of rigs $(S_i)_{i\in I}$, set $S = \prod_{i\in I} S_i$ and let $\pi_i\colon S \longrightarrow S_i$ be the canonical projection. Then $(\pi_i\colon S \longrightarrow S_i)_{i\in I}$ is the product of the family $(S_i)_{i\in I}$ in the category* **Rig**.

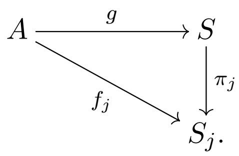
Proof. Given a rig A, let $(f_j\colon A \longrightarrow S_j)_{j\in I}$ be a family of rigs. We have to prove that there is only one morphism $f\colon A \longrightarrow S$ such that the following diagram commutes for every $j \in I$:



For each $a \in A$ we define $f(a) = (f_j(a))_{j \in I}$: obviously, f is a homomorphism of rigs because so is f_j for every $j \in I$. Regarding the commutativity of the diagram, for every $j \in I$ and for every $a \in A$ we have:

$$\pi_j\left(f(a)\right) = \pi_j\left((f_i(a))_{i \in I}\right) = f_j(a).$$

Now let $g\colon A \longrightarrow S$ be another morphism of rigs such that, for every $j \in I$, the following diagram commutes:



Then for every $j \in I$ and $a \in A$ we have

$$\pi_j\left(g(a)\right) = f_j(a) = \pi_j\left(f(a)\right)$$

thus $f(a) = g(a)$ and $f = g$. □

Appendix B

The Grothendieck functor

This topic has been treated in [Ros94, Thm. 1.1.3], even if only in the additive version.

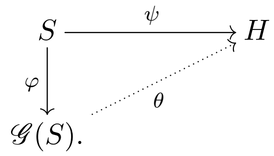
We will denote by $\mathscr{G}$ the **Grothendieck functor** which sends a rig S to the ring $\mathscr{G}(S)$ constructed as follows. We define a equivalence relation $\sim$ on $S \times S$ such that for every $(a,b),(c,d) \in S \times S$, we have $(a,b) \sim (c,d)$ if and only if there is $e \in S$ such that $a+d+e = c+b+e$. The equivalence class of the couple $(a,b) \in S \times S$ will be denoted with $[(a,b)]$, or simply by $[a,b]$ to make the notation more clear, and the quotient set of $S \times S$ with $\mathscr{G}(S)$. We will define an addition and a multiplication on $\mathscr{G}(S)$ as follows: for every $[a,b],[c,d] \in \mathscr{G}(S)$,

$$[a,b]+[c,d] = [a+c, b+d] \qquad \text{and} \qquad [a,b]\cdot[c,d] = [ac+bd, ad+bc].$$

In this way $\mathscr{G}(S)$ becomes a commutative rings with $[0,0]$ as neutral element with respect to $+$ and $[1,0]$ as neutral element with respect to $\cdot$.

Given a rig S, the ring $\mathscr{G}(S)$ has the following universal property.

Proposition B.0.1. *Given a rig S, for any ring H and for any homomorphism of rigs $\psi\colon S \longrightarrow H$, there is a unique homomorphism of rings $\theta\colon \mathscr{G}(S) \longrightarrow H$ such that $\psi = \theta\varphi$, that is, such that the following diagram is commutative:*



Using the universal property of Proposition $B.0.1$, given a homomorphism of rigs $f\colon S \longrightarrow T$ we can define

$$\begin{aligned} \mathscr{G}(f)\colon \mathscr{G}(S) &\longrightarrow \mathscr{G}(T) \\ [a,b] &\longrightarrow [f(a), f(b)]\,. \end{aligned}$$

It is possible to prove that $\mathscr{G}(f)$ is a homomorphism of rings and that, with these definitions, $\mathscr{G}$ becomes a covariant functor from the category of rigs **Rig** to the category of commutative rings **CRing**.

Proposition B.0.2. *Given an isomorphism of rigs $f\colon S \longrightarrow T$ we obtain an isomorphism of rings*

$$\mathscr{G}(f)\colon \mathscr{G}(S) \longrightarrow \mathscr{G}(T).$$

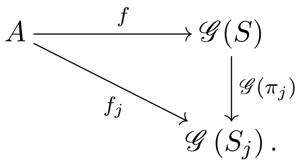
Proof. Immediate. □

Proposition B.0.3. *The Grothendieck functor $\mathscr{G}$ preserves all products. In particular, given a family of rigs $(S_j)_{j\in I}$, let be $(\pi_j \colon S \longrightarrow S_j)_{j\in I}$ their product in riG. Then*

$$\left(\mathscr{G}(\pi_j) : \mathscr{G}(S) \longrightarrow \mathscr{G}(S_j)\right)_{j\in I}$$

is the product of the family $(\mathscr{G}(S_j))_{j\in I}$ in **CRing**.

Proof. Given a ring A, let $(A \longrightarrow \mathscr{G}(S_j))_{j\in I}$ be a family of morphisms in **CRing**. We have to prove that there is a unique homomorphism of rings $f \colon A \longrightarrow \mathscr{G}(S)$ such that for every $j \in I$ the following diagram commutes:



Thanks to Proposition $A.0.2$, we will assume that $S = \prod_{j\in I} S_j$ and that $\pi_j \colon S \longrightarrow S_j$ is the canonical projection for every $j \in I$ (the categorical product is unique up to isomorphism in every category so there is no loss of generality in this choice). Let $a \in A$: for every $j \in I$ there are $x_j, y_j \in S_j$ such that $f_j(a) = [x_j, y_j]$ thus we can define

$$f(a) = \left[(x_j)_{j\in I}, (y_j)_{j\in J}\right].$$

We have to prove that this is a good definition. For every $j \in I$ let be $z_j, w_j \in S$ such that $[x_j, y_j] = [z_j, w_j]$: then there is $e_j \in S$ such that $x_j + w_j + e_j = z_j + y_j + e_j$. As a consequence we have

$$(x_j)_{j\in I} + (w_j)_{j\in I} + (e_j)_{j\in I} = (z_j)_{j\in I} + (y_j)_{j\in I} + (e_j)_{j\in I}$$

thus

$$\left[(x_j)_{j\in I}, (y_j)_{j\in I}\right] = \left[(z_j)_{j\in I}, (w_j)_{j\in I}\right]$$

and f is well defined.

Now we have to prove that f is a homomorphism of rings. Given $a, b \in A$, for every $j \in I$ let be $a_j, \alpha_j, b_j, \beta_j \in S_j$ such that $f_j(a) = [a_j, \alpha_j]$ and $f_j(b) = [b_j, \beta_j]$. We have

$$f_j(a+b) = f_j(a) + f_j(b) = [a_j, \alpha_j] + [b_j, \beta_j] = [a_j + b_j, \alpha_j + \beta_j]$$

and

$$f_j(ab) = f_j(a) f_j(b) = [a_j, \alpha_j][b_j, \beta_j] = [a_j b_j + \alpha_j\beta_j, a_j\beta_j + \alpha_j b_j]$$

thus

$$\begin{aligned} f(a) + f(b) &= \left[(a_j)_{j\in I}, (\alpha_j)_{j\in I}\right] + \left[(b_j)_{j\in I}, (\beta_j)_{j\in I}\right] = \left[(a_j)_{j\in I} + (b_j)_{j\in I}, (\alpha_j)_{j\in I} + (\beta_j)_{j\in I}\right] \\ &= \left[(a_j + b_j)_{j\in I}, (\alpha_j + \beta_j)_{j\in I}\right] = f(a+b) \end{aligned}$$

and

$$\begin{aligned} f(a)f(b) &= \left[(a_j)_{j\in I}, (\alpha_j)_{j\in I}\right]\left[(b_j)_{j\in I}, (\beta_j)_{j\in I}\right] \\ &= \left[(a_j)_{j\in I}(b_j)_{j\in I} + (\alpha_j)_{j\in I}(\beta_j)_{j\in I}, (a_j)_{j\in I}(\beta_j)_{j\in I} + (\alpha_j)_{j\in I}(b_j)_{j\in I}\right] \\ &= \left[(a_j b_j + \alpha_j\beta_j)_{j\in I}, (a_j\beta_j + \alpha_j b_j)_{j\in I}\right] = f(ab). \end{aligned}$$

Moreover, for each $j \in I$ we have $f_j(0) = [0, 0]$ and $f_j(1) = [1, 0]$ thus

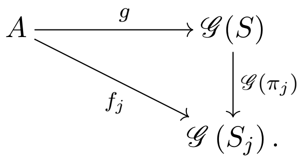
$$f(0) = \left[(0_j)_{j\in I}, (0_j)_{j\in I}\right] \qquad \text{and} \qquad f(1) = \left[(1_j)_{j\in I}, (0_j)_{j\in I}\right]$$

therefore we have proved that f is a homomorphism of rings. Regarding the commutativity of the diagrams, for every $j \in I$ and every $a \in A$ let be $x_j, y_j \in S_j$ such that $f_j(a) = [x_j, y_j]$. Then $f(a) = [(x_i)_{i\in I}, (y_i)_{i\in I}]$ thus

$$\mathscr{G}(\pi_j)(f(a)) = [\pi_j (x_i)_{i\in I}, \pi_j (y_i)_{i\in I}] = [x_j, y_j] = f_j$$

therefore $\mathscr{G}(\pi_j) f = f_j$ and the commutativity of the diagrams is proved.

Now let be $g\colon A \longrightarrow \mathscr{G}(S)$ another homomorphism of rings such that, for every $j \in I$, the following diagram commutes:



For every $j \in I$ and for every $a \in A$ we have

$$\mathscr{G}(\pi_j)(g(a)) = f_j(a) = \mathscr{G}(\pi_j)(f(a))$$

Let be $x_j, y_j \in S$ such that $f_j(a) = [x_j, y_j]$ and let be $(e_i)_{i\in I}, (f_i)_{i\in I} \in S$ such that $g(a) = [(e_i)_{i\in I}, (f_i)_{i\in i}]$. We calculate:

$$\begin{aligned}[e_j, f_j] &= [\pi_j [(e_i)_{i\in i}], \pi_j ((f_i)_{i\in I})] = \mathscr{G}(\pi_j)(g(a)) = \mathscr{G}(\pi_j)(f(a)) \\ &= \mathscr{G}(\pi_j)([(x_i)_{i\in I}, (y_i)_{i\in I}]) = [\pi_j ((x_i)_{i\in I}), \pi_j ((y_i)_{i\in I})] = [x_j, y_j]\end{aligned}$$

thus we obtain that there is $\varepsilon_j \in S_j$ such that

$$e_j + y_j + \varepsilon_j = x_j + f_j + \varepsilon_j$$

therefore

$$(e_j)_{j\in I} + (y_j)_{j\in I} + (\varepsilon_j)_{j\in I} = (x_j)_{j\in I} + (f_j)_{j\in I} + (\varepsilon_j)_{j\in I}$$

and $g(a) = [(x_i)_{i\in I}, (y_i)_{i\in I}] = f(a)$. We have now proved that $f = g$. □

Appendix C

Monoidal categories

In this appendix we will state a few definitions over which there isn't still a complete consensus (Definition $C.0.2$) and, after that, we will prove a few known results whose proofs we have been unable to find elsewhere.

We refer the reader to [TV17] for the following definitions.

Definition C.0.1. A **monoidal category** is a category $\mathcal{C}$ endowed with:

(1) a functor $\mathcal{C} \boxplus \mathcal{C} \longrightarrow \mathcal{C}$ called the **monoidal product**;

(2) an object $J \in \mathcal{C}$ called the **monoidal unit object**;

(3) a natural isomorphism

$$a\colon ((- \boxplus -) \boxplus -) \longrightarrow (- \boxplus (- \boxplus -))$$

called the **associativity constraint**;

(4) a natural isomorphism

$$r\colon (\mathrm{Id}_{\mathcal{C}} \boxplus J) \longrightarrow \mathrm{Id}_{\mathcal{C}}$$

called the **right unitality constraint**;

(5) a natural isomorphism

$$l\colon (J \boxplus \mathrm{Id}_{\mathcal{C}}) \longrightarrow \mathrm{Id}_{\mathcal{C}}$$

called the **left unitality constraint**.

Moreover, the following conditions have to be satisfied.

(1) The following diagram, called the **pentagonal identity**, has to be commutative for each quadruple (X, Y, Z, W) of objects of $\mathcal{C}$:

$$\begin{array}{ccccc}
 & & (X \boxplus Y) \boxplus (Z \boxplus W) & & \\
 & \nearrow^{a(X \otimes Y, Z, W)} & & \searrow^{a(X, Y, Z \boxplus W)} & \\
((X \boxplus Y) \boxplus Z) \boxplus W & & & & X \boxplus (Y \boxplus (Z \boxplus W)) \\
\downarrow{\scriptstyle a(X,Y,Z) \boxplus \mathrm{Id}_W} & & & & \uparrow{\scriptstyle \mathrm{Id}_X \boxplus a(Y,Z,W)} \\
(X \boxplus (Y \boxplus Z)) \boxplus W & & \xrightarrow{\quad a(X, Y \boxplus Z, W) \quad} & & X \boxplus ((Y \boxplus Z) \boxplus W).
\end{array}$$

(2) The following diagram, called the **triangular identity**, has to be commutative for each couple (X, Y) of objects of $\mathcal{C}$:

$$\begin{array}{ccc} & X \boxplus Y & \\ \nearrow^{r(X)\boxplus \mathrm{Id}_Y} & & \nwarrow^{\mathrm{Id}_X \boxplus l(Y)} \\ (X \boxplus J) \boxplus Y & \xrightarrow{a(X,J,Y)} & X \boxplus (J \boxplus Y). \end{array}$$

We will use the notation $\mathcal{C} = (\mathcal{C}, \boxplus, J, a, l, r)$. If a, r and l are identities we say that $\mathcal{C}$ is a **strict monoidal category** and we use the notation $\mathcal{C} = (\mathcal{C}, \boxplus, J)$.

Definition C.0.2. Let $\mathcal{C} = (\mathcal{C}, \boxplus, J, a, l, r)$ and $\mathcal{D} = (\mathcal{D}, \otimes, I, a', l', r')$ be two monoidal categories. A **monoidal functor** from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ endowed with a morphism $F_0\colon I \longrightarrow FJ$ and a natural transformation

$$F_2(-,-)\colon F \otimes F \longrightarrow F(\boxplus)$$

such that the following conditions are satisfied.

(1) For each triple (X, Y, Z) of objects of $\mathcal{C}$ the following diagram is commutative.

$$\begin{array}{ccc} (F(X) \otimes F(Y)) \otimes F(Z) & \xrightarrow{a'(F(X),F(Y),F(Z))} & F(X) \otimes (F(Y) \otimes F(Z)) \\ {\scriptstyle F_2(X,Y)\otimes \mathrm{Id}_{F(Z)}}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}_{F(X)} \otimes F_2(Y,Z)} \\ F(X \boxplus Y) \otimes F(Z) & & F(X) \otimes F(Y \boxplus Z) \\ {\scriptstyle F_2(X\boxplus Y,Z)}\downarrow & & \downarrow{\scriptstyle F_2(X,Y\boxplus Z)} \\ F((X \boxplus Y) \boxplus Z) & \xrightarrow{F(a(X,Y,Z))} & F(X \boxplus (Y \boxplus Z)) \end{array}$$

(2) for each object X of $\mathcal{C}$ the two following diagrams are commutative.

$$\begin{array}{ccc} I \otimes F(X) & \xrightarrow{l'(F(X))} & F(X) \\ {\scriptstyle F_0\otimes \mathrm{Id}_{F(X)}}\downarrow & & \uparrow{\scriptstyle F(l(X))} \\ F(J) \otimes F(X) & \xrightarrow{F_2(J,X)} & F(J \boxplus X) \end{array} \qquad \begin{array}{ccc} F(X) \otimes I & \xrightarrow{r'(F(X))} & F(X) \\ {\scriptstyle \mathrm{Id}_{F(X)} \otimes F_0}\downarrow & & \uparrow{\scriptstyle F(r(X))} \\ F(X) \otimes F(J) & \xrightarrow{F_2(X,J)} & F(X \boxplus J) \end{array}$$

We say that F is a **strong monoidal functor** (respectively, a **strict monoidal functor**) if F_2 and F_0 are both isomorphism (respectively, both identities).

Example C.0.3. It is clear that the identity functor is a strict monoidal functor.

Definition C.0.4. Given monoidal categories $\mathcal{C} = (\mathcal{C}, \boxplus, J, a, l, r)$ and $\mathcal{D} = (\mathcal{D}, \otimes, I, a', l', r')$, let $F, G\colon \mathcal{C} \longrightarrow \mathcal{D}$ be two monoidal functors and let's consider a natural transformation $\mu\colon F \longrightarrow G$. We say that μ is a **monoidal natural transformation** if the the following diagrams are commutative:

$$\begin{array}{ccc} I & \xrightarrow{G_0} & G(J) \\ & {\scriptstyle F_0}\searrow \quad \nearrow{\scriptstyle \mu(J)} & \\ & F(J) & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} F(X) \otimes G(X) & \xrightarrow{\mu(X)\otimes\mu(Y)} & G(X) \otimes G(Y) \\ {\scriptstyle F_2(X,Y)}\downarrow & & \downarrow{\scriptstyle G_2(X,Y)} \\ F(X \boxplus Y) & \xrightarrow{\mu(X\boxplus Y)} & G(X \boxplus Y). \end{array}$$

Definition C.0.5. Given monoidal categories $\mathcal{C} = (\mathcal{C}, \boxplus, J, a, l, r)$ and $\mathcal{D} = (\mathcal{D}, \otimes, I, a', l', r')$, let's consider a couple of monoidal functors

$$(F\colon \mathcal{C} \longrightarrow \mathcal{D}, \quad G\colon \mathcal{D} \longrightarrow \mathcal{C})$$

and monoidal natural transformations $\eta\colon \mathrm{Id}_{\mathcal{C}} \longrightarrow GF$ and $\varepsilon\colon FG \longrightarrow \mathrm{Id}_{\mathcal{D}}$. We say that:

(1) (F, G) is an **monoidal adjunction of categories** if $\varepsilon F \circ F\eta = \mathrm{Id}_F$ and $G\varepsilon \circ \eta G = \mathrm{Id}_G$;

(2) (F, G) is an **monoidal equivalence of categories** if η and ε are isomorphisms;

(3) (F, G) is an **monoidal adjoint equivalence of categories** if it is an equivalence of monoidal categories such that η and ε are isomorphisms.

Definition C.0.6. Let be $\mathcal{C} = (\mathcal{C}, \boxplus, J, a, l, r)$ and $\mathcal{D} = (\mathcal{D}, \otimes, I, a', l', r')$ monoidal categories. We say that $\mathcal{C}$ and $\mathcal{D}$ are **monoidally equivalent categories** is there is a monoidal equivalence between them.

As proved in [Mac98, Chap. 7], there is no loss of generality to restring ourselves to strict monoidal categories.

Theorem C.0.7. *Every monoidal category is monoidally equivalent to a strict one.*

Theorem C.0.7 allow us, henceforth, in this appendix, to assume that all the monoidal categories under consideration are strict.

Lemma C.0.8. *Given monoidal categories $(\mathcal{C}, \boxplus, J)$, $(\mathcal{D}, \otimes, I)$ and $(\mathcal{E}, \diamond, K)$, consider the monoidal functors*

$$F\colon (\mathcal{C}, \boxplus, J) \longrightarrow (\mathcal{D}, \otimes, I) \qquad \text{and} \qquad G\colon (\mathcal{D}, \otimes, I) \longrightarrow (\mathcal{E}, \diamond, K).$$

Then GF is a monoidal functor with

$$(GF)_0 = (GF_0)\, G_0 = \left(K \xrightarrow{\quad G_0 \quad} G(I) \xrightarrow{\quad G(F_0) \quad} GF\,(J) \right)$$

and $(GF)_2 = (GF_2)\,(G_2(F, F))$ that is, for every couple (X, Y) of objects of $\mathcal{C}$, $(GF)_2$ is defined by the following commutative diagram:

$$\begin{array}{ccc} GF(X) \diamond GF(Y) & \xrightarrow{\quad (GF)_2(X,Y) \quad} & GF(X \boxplus Y) \\ {\scriptstyle G_2(F(X),F(Y))}\downarrow & \nearrow_{GF_2(X,Y)} & \\ G\,(F(X) \otimes F(Y)) \,. & & \end{array}$$

Proof. To prove that $(GF)_2$ and $(GF)_0$ allow GF to become a monoidal functor it is enough to consider the commutative diagrams of Definition C.0.2 relative to F and G and use them to prove the respective diagrams for GF. □

Proposition C.0.9. *Let be $(\mathcal{C}, \boxplus, J)$ and $(\mathcal{D}, \otimes, I)$ be monoidal categories and let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a strong monoidal functor that realises an adjunction*

$$(F\colon \mathcal{C} \longrightarrow \mathcal{D}, \quad G\colon \mathcal{D} \longrightarrow \mathcal{C})$$

with unit $\eta\colon \mathrm{Id}_{\mathcal{C}} \longrightarrow GF$ and counit $\varepsilon\colon FG \longrightarrow \mathrm{Id}_{\mathcal{D}}$. Then we can endow G with a strong monoidal structure in such a way that ε and η become monoidal natural transformations.

Proof. We define a natural transformation

$$G_2(-,-) : G \boxplus G \longrightarrow G(\otimes)$$

as follow:

$$G_2(-,-) = G(\varepsilon \otimes \varepsilon) \circ GF_2^{-1}(G,G) \circ \eta(G \boxplus G).$$

This means that for every objects $A, B \in \mathcal{D}$ the following diagram is commutative:

$$\begin{array}{ccc} G(FGA \otimes FGB) & \xleftarrow{GF_2^{-1}(GA,GB)} & GF(GA \boxplus GB) \\ {\scriptstyle G(\varepsilon A \otimes \varepsilon B)} \downarrow & & \uparrow {\scriptstyle \eta(GA \boxplus GB)} \\ G(A \otimes B) & \xleftarrow{G_2(A,B)} & GA \boxplus GB. \end{array}$$

Moreover, we define $G_0 \colon J \longrightarrow GI$ according to the following commutative diagram

$$\begin{array}{ccc} J & \xrightarrow{G_0} & GI \\ {\scriptstyle \eta J} \downarrow & \nearrow {\scriptstyle GF_0^{-1}} & \\ GFJ & & \end{array}.$$

Note that, by hypothesis, $\varepsilon F \circ F\eta = \mathrm{Id}_F$ and $G\varepsilon \circ \eta G = \mathrm{Id}_G$. Reading the diagram in figure (C.1) we obtain that, for every $X \in \mathcal{D}$, the following diagram is commutative

$$\begin{array}{ccc} GX \boxplus J & = & GX \\ {\scriptstyle \mathrm{Id}_{GX} \boxplus G_0} \downarrow & & \| \\ GX \boxplus GI & & \| \\ {\scriptstyle G_2(X,I)} \downarrow & & \| \\ G(X \otimes I) & = & GX. \end{array}$$

The commutativity of the diagram

$$\begin{array}{ccc} J \boxplus GX & = & GX \\ {\scriptstyle G_0 \boxplus \mathrm{Id}_{GX}} \downarrow & & \| \\ GI \boxplus GX & & \| \\ {\scriptstyle G_2(I,X)} \downarrow & & \| \\ G(I \otimes X) & = & GX. \end{array}$$

can be proved in the same way or by applying what has already been demonstrated to the opposite monoidal categories of $(\mathcal{C}, \boxplus, J)$ and $(\mathcal{D}, \otimes, I)$ (here we mean opposite with respect to the monoidal products, not opposite with respect to the direction of the morphisms).

Now we have to prove the associativity coherency conditions. Reading the diagram in figures $C.2$, $C.3$ and $C.4$ we obtain that, for every $X, Y, Z \in \mathcal{D}$, the following diagram is commutative:

$$\begin{array}{ccc} GX \boxplus GY \boxplus GZ & \xrightarrow{G_2(X,Y) \boxplus GZ} & G(X \otimes Y) \boxplus GZ \\ {\scriptstyle GX \boxplus G_2(Y,Z)} \downarrow & & \downarrow {\scriptstyle G_2(X \otimes Y, Z)} \\ GX \boxplus G(Y \otimes Z) & \xrightarrow{G_2(X, Y \otimes Z)} & G(X \otimes Y \otimes Z). \end{array}$$

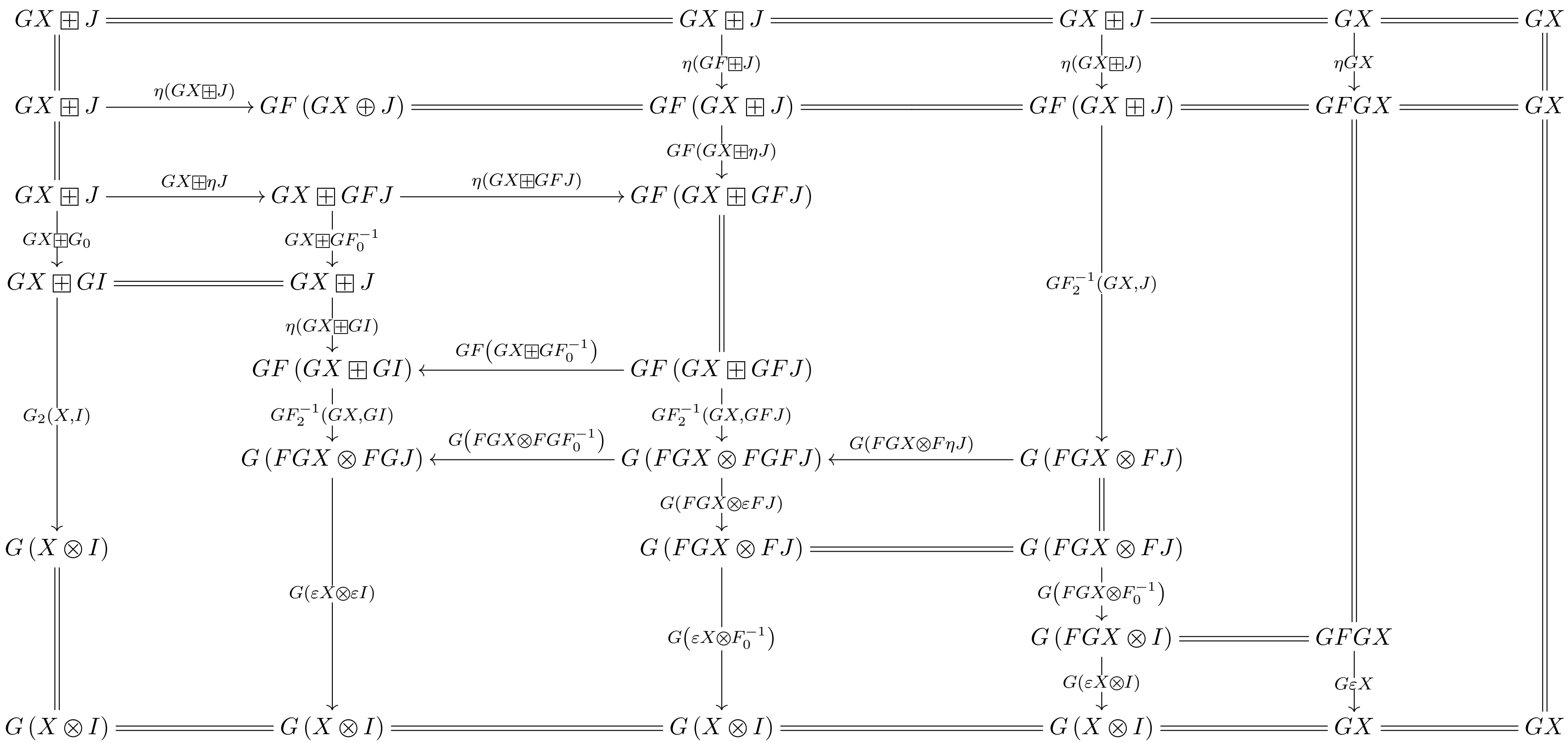


Figure C.1: Coherence diagram unity condition

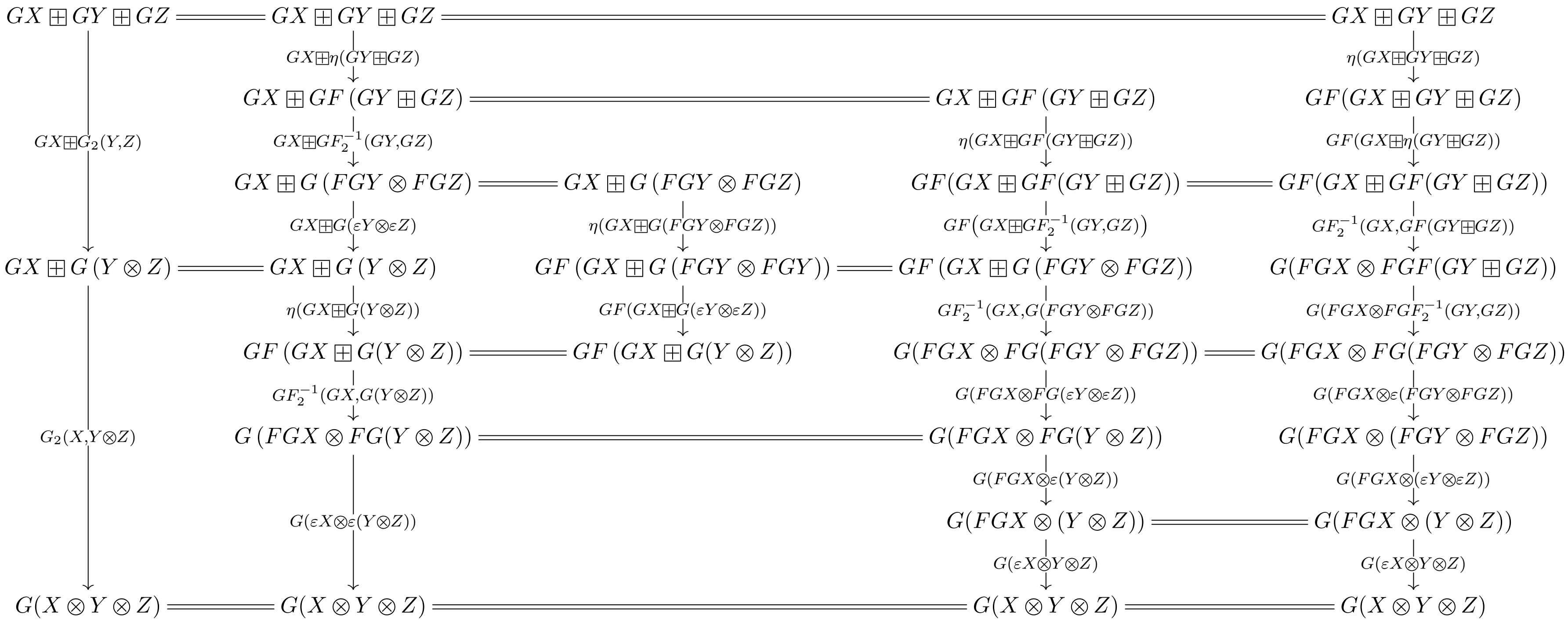


Figure C.2: Coherence diagram associativity condition - Part 1

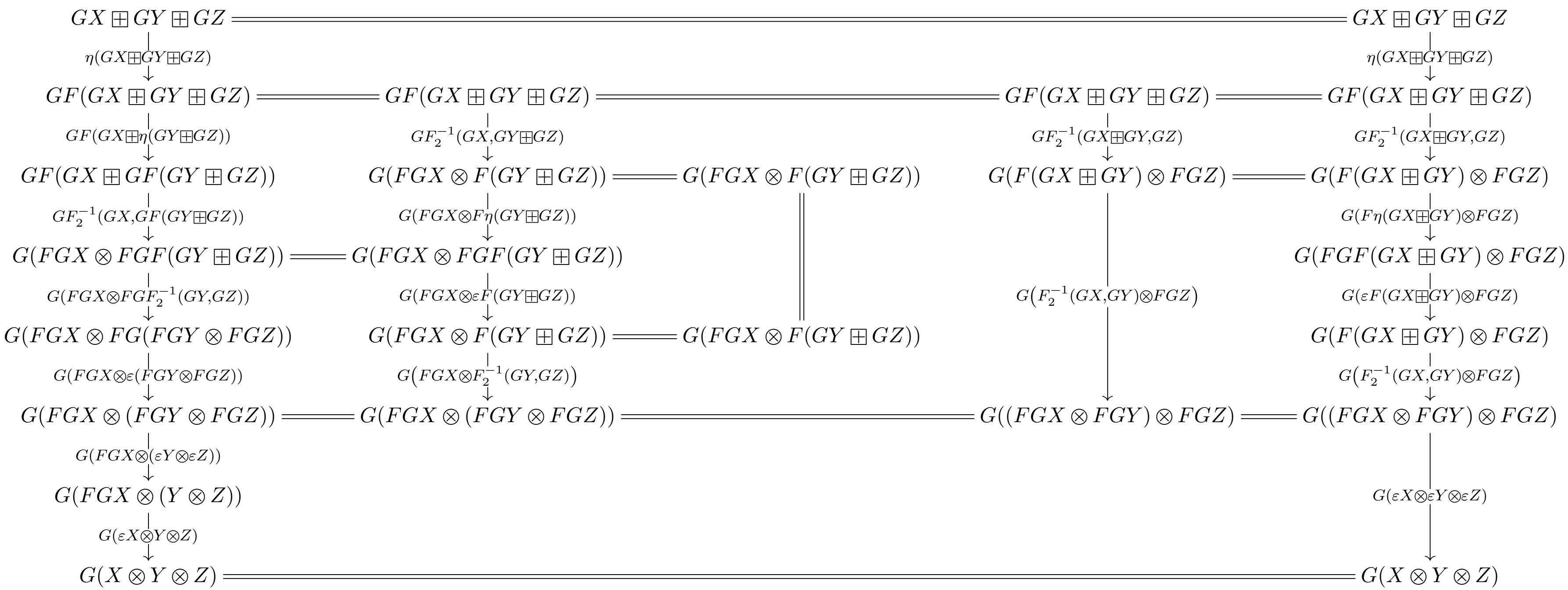


Figure C.3: Coherence diagram associativity condition - Part 2

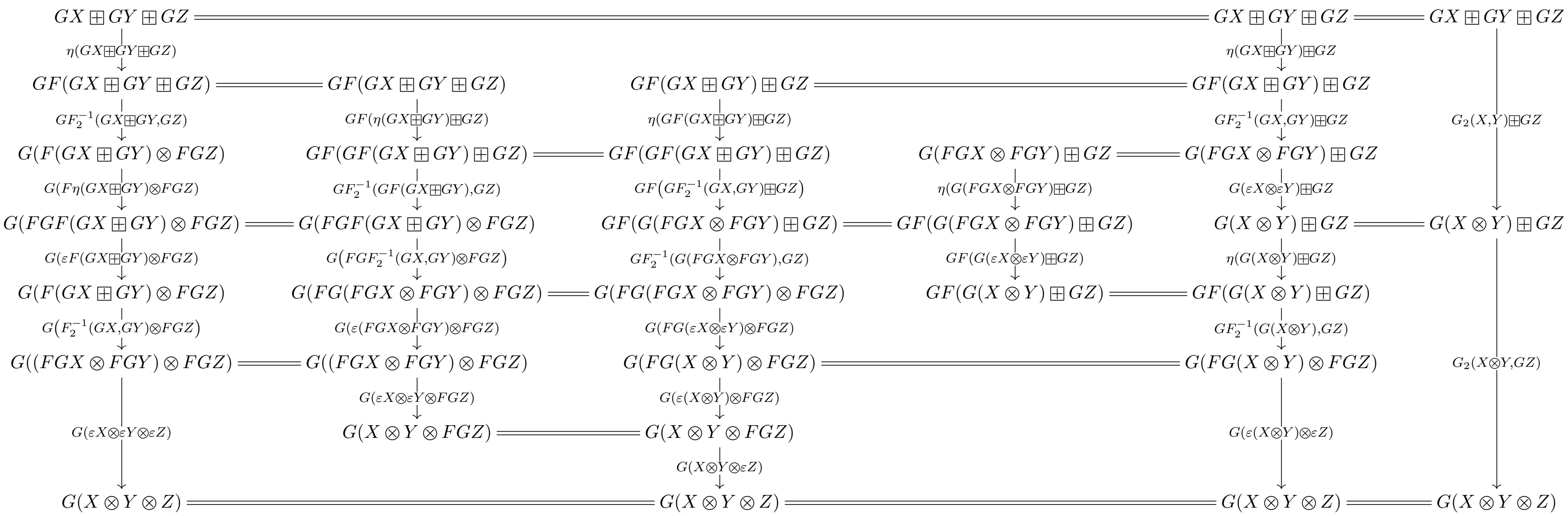


Figure C.4: Coherence diagram associativity condition - Part 3

As a consequence we have proved that G is a monoidal functor. We still have to prove that η and ε are monoidal natural transformations.

To prove that η is monoidal we have to show that the two following diagrams are commutative:

$$\begin{array}{ccc} J & \xrightarrow{(GF)_0} & GF(J) \\ {\scriptstyle \mathrm{Id}_J}\big\downarrow & \nearrow_{\eta(J)} & \\ J & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X \boxplus Y & \xrightarrow{\eta(X)\boxplus\eta(Y)} & (GF)(X) \boxplus (GF)(Y) \\ \Big\| & & \big\downarrow{\scriptstyle (GF)_2(X,Y)} \\ X \boxplus Y & \xrightarrow{\eta(X\boxplus Y)} & GF(X \boxplus Y). \end{array}$$

Using Lemma $C.0.8$ we calculate

$$(GF)_0 = (GF_0) \circ G_0 = (GF_0) \circ \left(GF_0^{-1}\right) = \eta J.$$

Now let be (X, Y) a couple of objects in $\mathcal{C}$. We calculate:

$$\begin{aligned} & (GF)_2\,(X, Y) \circ \eta X \boxplus \eta Y \\ = {} & GF_2(X, Y) \circ G_2(FX, FY) \circ \eta X \boxplus \eta Y \\ = {} & GF_2(X, Y) \circ G(\varepsilon FX \otimes \varepsilon FY) \circ GF_2^{-1}(GFX, GFY) \circ \eta(GFX \boxplus GFY) \circ \eta X \boxplus \eta Y \\ = {} & GF_2(X, Y) \circ G(\varepsilon FX \otimes \varepsilon FY) \circ GF_2^{-1}(GFX, GFY) \circ GF(\eta X \boxplus FY) \circ \eta(X \boxplus Y) \\ = {} & GF_2(X, Y) \circ G(\varepsilon FX \otimes \varepsilon FY) \circ G(F\eta X \otimes F\eta Y) \circ GF_2^{-1}(X, Y) \circ \eta(X \boxplus Y) \\ = {} & GF_2(X, Y) \circ GF_2^{-1}(X, Y) \circ \eta(X \boxplus Y) = \eta(X \boxplus Y). \end{aligned}$$

We have now proved that η is monoidal.

To prove that ε is monoidal we have to show that the two following diagrams are commutative:

$$\begin{array}{ccc} I & \xrightarrow{\mathrm{Id}_I} & I \\ {\scriptstyle (FG)_0}\big\downarrow & \nearrow_{\varepsilon(I)} & \\ FG(I) & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} (FG)(X) \otimes (FG)(Y) & \xrightarrow{\varepsilon(X)\otimes\varepsilon(Y)} & X \otimes Y \\ \big\downarrow{\scriptstyle (FG)_2(X,Y)} & & \Big\| \\ FG(X \otimes Y) & \xrightarrow{\varepsilon(X\otimes Y)} & X \otimes Y. \end{array}$$

Using Lemma $C.0.8$ we calculate

$$\varepsilon I \circ FG_0 = \varepsilon I \circ FG_0 \circ F_0 = \varepsilon I \circ FGF_0^{-1} \circ F\eta J \circ F_0 = F_0^{-1} \circ \varepsilon(FJ) \circ F\eta J \circ F_0 = \mathrm{Id}_I\,.$$

Now let be (X, Y) a couple of objects in $\mathcal{C}$. We calculate:

$$\begin{aligned} & \varepsilon(X \otimes Y) \circ (FG)_2\,(X, Y) \\ = {} & \varepsilon(X \otimes Y) \circ FG_2(X, Y) \circ F_2(GX, GY) \\ = {} & \varepsilon(X \otimes Y) \circ FG(\varepsilon X \otimes \varepsilon Y) \circ FGF_2^{-1}(GX, GY) \circ F\eta(GX \boxplus GY) \circ F_2(GX, GY) \\ = {} & \varepsilon X \otimes \varepsilon Y \circ \varepsilon(FGX \otimes FGY) \circ FGF_2^{-1}(GX, GY) \circ F\eta(GX \boxplus GY) \circ F_2(GX, GY) \\ = {} & \varepsilon X \otimes \varepsilon Y \circ F_2^{-1}(GX, GY) \circ \varepsilon F(GX \boxplus GY) \circ F\eta(GX \boxplus GY) \circ F_2(GX, GY) \\ = {} & \varepsilon X \otimes \varepsilon Y \circ F_2^{-1}(GX, GY) \circ F_2(GX, GY) = \varepsilon X \otimes \varepsilon Y. \end{aligned}$$

We have now proved that η is monoidal, completing the proof of the proposition. $\square$

Now we are going to apply the monoidal category theory developed so far to Laplaza categories. We have already introduced them and given their basic definitions in Subsection 1.3.1 but, to prove the results we need, we have to give two more definitions about them.

Definition C.0.10. Given $(\mathcal{C}_1, \diamond_1, \boxplus_1)$ and $(\mathcal{C}_2, \diamond_2, \boxplus_2)$ two Laplaza categories, we say that a **Laplaza adjunction** is a couple of Laplaza functors

$$F\colon (\mathcal{C}_1, \diamond_1, \boxplus_1) \longrightarrow (\mathcal{C}_2, \diamond_2, \boxplus_2) \qquad \text{and} \qquad G\colon (\mathcal{C}_2, \diamond_2, \boxplus_2) \longrightarrow (\mathcal{C}_1, \diamond_1, \boxplus_1)$$

such that there are Laplaza transformations

$$\eta\colon \mathrm{Id}_{\mathcal{C}_1} \longrightarrow GF \qquad \text{and} \qquad \varepsilon\colon FG \longrightarrow \mathrm{Id}_{\mathcal{C}_2}$$

such that $\varepsilon F \circ F\eta = \mathrm{Id}_F$ and $G\varepsilon \circ \eta G = \mathrm{Id}_G$.

Definition C.0.11. Given $(\mathcal{C}_1, \diamond_1, \boxplus_1)$ and $(\mathcal{C}_2, \diamond_2, \boxplus_2)$ two Laplaza categories, we say that a **Laplaza adjoint equivalence** is a couple of Laplaza functors

$$F\colon (\mathcal{C}_1, \diamond_1, \boxplus_1) \longrightarrow (\mathcal{C}_2, \diamond_2, \boxplus_2) \qquad \text{and} \qquad G\colon (\mathcal{C}_2, \diamond_2, \boxplus_2) \longrightarrow (\mathcal{C}_1, \diamond_1, \boxplus_1)$$

such that there are Laplaza natural isomorphisms

$$\eta\colon \mathrm{Id}_{\mathcal{C}_1} \longrightarrow GF \qquad \text{and} \qquad \varepsilon\colon FG \longrightarrow \mathrm{Id}_{\mathcal{C}_2}$$

such that $\varepsilon F \circ F\eta = \mathrm{Id}_F$ and $G\varepsilon \circ \eta G = G$.

Proposition C.0.12. *Let be $(\mathcal{C}_1, \diamond_1, \boxplus_1)$ and $(\mathcal{C}_2, \diamond_2, \boxplus_2)$ be two Laplaza categories and let $F\colon \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ be a Laplaza functor which realises an adjunction*

$$(F\colon \mathcal{C}_1 \longrightarrow \mathcal{C}_2, \quad G\colon \mathcal{C}_2 \longrightarrow \mathcal{C}_1)$$

with unit $\eta\colon \mathrm{Id}_{\mathcal{C}_1} \longrightarrow GF$ and counit $\varepsilon\colon FG \longrightarrow \mathrm{Id}_{\mathcal{C}_2}$. Then we can render G a Laplaza functor in such a way that ε and η become Laplaza transformations.

Proof. It is sufficient to apply Proposition C.0.9 separately fist to the strong monoidal functor $F\colon (\mathcal{C}_1, \diamond_1) \longrightarrow (\mathcal{C}_2, \diamond_2)$ and then to the strong monoidal functor $F\colon (\mathcal{C}_1, \boxplus_1) \longrightarrow (\mathcal{C}_2, \boxplus_2)$. □

The following result is a standard categorical lemma.

Lemma C.0.13. *Given categories $\mathcal{C}_1$ and $\mathcal{C}_2$, let us consider functors $F\colon \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ and $G\colon \mathcal{C}_2 \longrightarrow \mathcal{C}_1$ that realise an equivalence of categories. Then there are natural isomorphisms $\eta\colon \mathrm{Id}_{\mathcal{C}_1} \longrightarrow GF$ and $\varepsilon\colon FG \longrightarrow \mathrm{Id}_{\mathcal{C}_2}$ such that $\varepsilon F \circ F\eta = \mathrm{Id}_F$ and $G\varepsilon \circ \eta G = \mathrm{Id}_G$.*

Proof. Let be $\alpha\colon \mathrm{Id}_{\mathcal{C}_1} \longrightarrow GF$ and $\beta\colon FG \longrightarrow \mathrm{Id}_{\mathcal{C}_2}$ two natural isomorphisms that realise the equivalence (F, G) and define $\eta = \alpha$ and $\varepsilon = \beta \circ F\alpha^{-1}G \circ FG\beta^{-1}$. We calculate:

$$\begin{aligned} G\varepsilon \circ \eta G &= G\beta \circ GF\alpha^{-1}G \circ GFG\beta^{-1} \circ \alpha G \\ &= G\beta \circ GF\alpha^{-1}G \circ \alpha GFG \circ G\beta^{-1} \\ &= G\beta \circ \alpha G \circ \alpha^{-1}G \circ G\beta^{-1} = \mathrm{Id}_G\,. \end{aligned}$$

Since $FG\beta^{-1} \circ \beta^{-1} = \beta^{-1}FG \circ \beta^{-1}$ and $\alpha^{-1} \circ GF\alpha^{-1} = \alpha^{-1} \circ \alpha^{-1}GF$ we obtain $FG\beta^{-1} = \beta^{-1}FG$ and $GF\alpha^{-1} = \alpha^{-1}GF$ therefore we have

$$\begin{aligned} \varepsilon F \circ F\eta &= \beta F \circ F\alpha^{-1}GF \circ FG\beta^{-1}F \circ F\alpha \\ &= \beta F \circ FGF\alpha^{-1} \circ \beta^{-1}FGF \circ F\alpha \\ &= \beta F \circ \beta^{-1}F \circ F\alpha^{-1} \circ F\alpha = \mathrm{Id}_F\,. \end{aligned}$$

□

Now we can finally state the following result, which is the objective of this entire appendix.

Corollary C.0.14. *Let be* $(\mathcal{C}_1, \diamond_1, \boxplus_1)$ *and* $(\mathcal{C}_2, \diamond_2, \boxplus_2)$ *be two Laplaza categories and let* $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ *be a Laplaza functor which is an equivalence of categories. Then there is a Laplaza functor* $G\colon \mathcal{D} \longrightarrow \mathcal{C}$ *such that there are Laplaza natural isomorphisms*

$$\eta\colon \mathrm{Id}_{\mathcal{C}_1} \longrightarrow GF \qquad \textit{and} \qquad \varepsilon\colon FG \longrightarrow \mathrm{Id}_{\mathcal{C}_2}$$

such that (F, G) *realises an adjunction with unit* η *and counit* ε*.*

In particular, the inverse of a Laplaza functor is a Laplaza functor itself.

Proof. Thanks to Lemma $C.0.13$ there are natural isomorphisms $\eta\colon \mathrm{Id}_{\mathcal{C}_1} \longrightarrow GF$ and $\varepsilon\colon FG \longrightarrow \mathrm{Id}_{\mathcal{C}_2}$ such that $\varepsilon F \circ F\eta = \mathrm{Id}_F$ and $G\varepsilon \circ \eta G = \mathrm{Id}_G$. As a consequence we can apply Proposition $C.0.12$ obtaining the thesis. ☐

Bibliography

[AHS04] J. Adámek, H. Herrlich, and G. E. Strecker. *Abstract and Concrete Categories, The Joy of Cats*. Online edition. 2004. URL: http://katmat.math.uni-bremen.de/acc/acc.pdf.

[BC04] J. C. Baez and A. S. Crans. "Higher-Dimensional Algebra VI: Lie 2-Algebras". *Theory and Applications of Categories* 12.15 (2004), pp. 492–538. URL: http://www.tac.mta.ca/tac/volumes/12/15/12-15abs.html.

[BL04] J. C. Baez and A. D. Lauda. "Higher-dimensional algebra V: 2-Groups". *Theory and Applications of Categories* 12.14 (2004), pp. 423–491. URL: http://www.tac.mta.ca/tac/volumes/12/14/12-14abs.html.

[Bon00] C. Bonnafé. "Mackey formula in type A". *Proc. London Math. Soc.* 80.3 (May 2000), pp. 545–574. DOI: 10.1112/S0024611500012399.

[Bon03] C. Bonnafé. "Corrigenda: Mackey formula in type A". *Proc. London Math. Soc.* 86.2 (Mar. 2003), pp. 435–442. DOI: 10.1112/S0024611502014132.

[Bor94] F. Borceux. *Handbook of Categorical Algebra 1 - Basic Category Theory*. Vol. 50. Encyclopedia of Mathematics and its Applications. Cambridge Univ. Press, 1994.

[Bou10a] S. Bouc. *Biset Functors for finite Groups*. Vol. 1990. LNM. Springer-Verlag Berlin Heidelberg, 2010.

[Bou10b] S. Bouc. "Bisets as Categories and Tensor Product of Induced Bimodules". *Appl. Categ. Structures* 18.5 (Oct. 2010), pp. 517–521. DOI: 10.1007/s10485-008-9180-1.

[Bro06] R. Brown. *Topology and Groupoids*. BookSurge Publishing, Feb. 2006.

[Bro87] R. Brown. "From groups to groupoids: a brief survey". *Bull. London Math. Soc.* 19.2 (Mar. 1987), pp. 113–134. DOI: 10.1112/blms/19.2.113.

[Bur11] W. Burnside. *Theory of groups of finite order*. Second Edition. Cambridge Univ. Press, 1911.

[Car08] P. Cartier. "Groupoïdes de Lie et leurs algébroïdes". *Séminaire Bourbaki 60ème année, 2007-2008* num. 987 (Mar. 2008), pp. 165–196.

[Con94] A. Connes. *Noncommutative Geometry*. Academic Press, Inc., San Diego, CA, 1994.

[CR62] C. W. Curtis and I. Reiner. *Representation Theory of Finite Groups and Associative Algebras*. Interscience Publishers, a division of John Wiley & Sons, New York - London - Sydney, 1962.

[DG70] M. Demazure and P. Gabriel. *Groupes Algébriques - Tome 1: Géométrie Algébrique, Généralités, Groupes Commutatifs*. Masson & Cie, Éditeur, Paris. Avec un appendice *Corps de classes local* par Michiel Hazewinkel. North-Holland Publishing Co., Amsterdam, 1970.

[Die79] T. T. Dieck. *Transformation Groups and Representation Theory*. Vol. 766. Lecture Notes in Mathematics. Springer Verlag, 1979.

[DL09] A. Díaz and A. Libman. "The Burnside ring of fusion systems". *Advances in Math.* 222 (Dec. 2009), pp. 1943–1963. DOI: 10.1016/j.aim.2009.06.023.

[DL76] P. Deligne and G. Lusztig. "Representations of Reductive Groups Over Finite Fields". *Ann. of Math.* 103.1 (Jan. 1976), pp. 103–161. DOI: 10.2307/1971021.

[DL83] P. Deligne and G. Lusztig. "Duality for representations of a reductive group over a finite field, II". *J. Algebra* 81.2 (Apr. 1983), pp. 540–545. DOI: 10.1016/0021-8693(83)90202-8.

[DM91] F. Digne and J. Michel. *Representations of Finite Groups of Lie Type.* Vol. 21. London Math. Soc. Student Texts. Cambridge Univ. Press, Cambridge, 1991.

[Dre69] A. Dress. "A Characterisation of Solvable Groups". *Mathematische Zeitschrift* 110 (1969), pp. 213–217.

[EB18] L. El Kaoutit and J. J. Barbarán Sánchez. "Linear representations and Frobenius morphisms of groupoids". June 2018. URL: https://arxiv.org/abs/1806.09327.

[Ehr63a] C. Ehresmann. "Catégories structurées". *Ann. Ec. Normale Sup.* 80.4 (1963), pp. 349–426. DOI: 10.24033/asens.1125.

[Ehr63b] C. Ehresmann. "Catégories structurées III: Quintettes et applications covariantes". *Topol. et Géom. Diff. Sém. C. Ehresmann. Institut H. Poincaré, Paris.* 5 (1963), p. 21.

[Ehr66] C. Ehresmann. *Ehresmann, Introduction to the theory of structured categories.* Technical Report Univ. of Kansas at Lawrence. 1966.

[EK17] L. El Kaoutit and N. Kowalzig. "Morita theory for Hopf algebroids, principal bibundles, and weak equivalences". *Doc. Math.* 22 (2017), pp. 551–609. URL: https://www.math.uni-bielefeld.de/documenta/vol-22/17.html.

[El 18] L. El Kaoutit. "On geometrically transitive Hopf algebroids". *J. Pure Appl. Algebra* 222.11 (Nov. 2018), pp. 3483–3520. DOI: 10.1016/j.jpaa.2017.12.019.

[ES18a] L. El Kaoutit and L. Spinosa. "Burnside theory for groupoids". July 2018. URL: https://arxiv.org/abs/1807.04470.

[ES18b] L. El Kaoutit and L. Spinosa. "Mackey formula for bisets over groupoids". *J. Alg. and Its Appications* (July 2018). DOI: 10.1142/S0219498819501093.

[ES18c] L. El Kaoutit and L. Spinosa. "Simplicial sets with a groupoid action and Burnside ring". Nov. 2018. URL: https://arxiv.org/abs/1811.11966.

[Gir71] J. Giraud. *Cohomologie non abélienne.* Vol. 179. Die Grundlehren der mathematischen Wissenschaften. Springer-Verlag, Berlin, Heidelberg, New York, 1971.

[Gla02] K. Glazek. *A guide to the literature on semirings and their applications in mathematics and information sciences.* With complete bibliography. Kluwer Academic Publishers, Dordrecht, 2002.

[Gol99] J. S. Golan. *Semirings and their Applications.* Springer Netherlands, 1999. DOI: 10.1007/978-94-015-9333-5.

[GRR12] P. E. Gunnells, A. Rose, and D. Rumynin. "Generalised Burnside rings, G-categories and module categories". *J. Algebra* 358 (May 2012), pp. 33–50. DOI: 10.1016/j.jalgebra.2012.02.016.

[Hig71] P. J. Higgins. *Notes on categories and groupoids.* Vol. 32. Mathematical Studies. Van Nostrand Reinhold, 1971.

[HY07] R. Hartmann and E. Yalçın. "Generalized Burnside rings and group cohomology". *J. Algebra* 310.2 (Apr. 2007), pp. 917–944. DOI: 10.1016/j.jalgebra.2006.10.037.

[Jel03] B. Jelenc. "Serre fibrations in the Morita category of topological groupoids". *Topol. and its Appl.* 160.1 (Jan. 2003), pp. 9–23. DOI: 10.1016/j.topol.2012.09.013.

[Kel05] G. M. Kelly. *Basic Concepts of Enriched Category Theory.* Vol. 10. Theory and Applications of Categories. 2005, pp. 1–136. URL: `http://www.tac.mta.ca/tac/reprints/articles/10/tr10abs.html`.

[Lap72a] M. L. Laplaza. "Coherence for categories with associativity, commutativity and distributivity". *Bull. Amer. Math. Soc.* 78 (1972), pp. 220–222.

[Lap72b] M. L. Laplaza. "Coherence for distributivity". In: *Coherence in Categories.* Vol. 281. Lecture Notes in Math. Springer-Verlag, Berlin, Heidelberg, New York, 1972, pp. 29–65.

[LS79] G. Lusztig and N. Spaltenstein. "Induced Unipotent Classes". *J. London Math. Soc.* s2-19.1 (Feb. 1979), pp. 41–52. DOI: `10.1112/jlms/s2-19.1.41`.

[Mac05] K. C. H. Mackenzie. *General Theory of Lie Groupoids and Lie Algebroids.* Vol. 213. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2005.

[Mac51] G. W. Mackey. "On Induced Representations of Groups". *Amer. J. Math.* 73.3 (July 1951), pp. 576–592. DOI: `10.2307/2372309`.

[Mac52] G. W. Mackey. "Induced Representations of Locally Compact Groups I". *Ann. of Math., Second Series* 55.1 (Jan. 1952), pp. 101–139. DOI: `10.2307/1969423`.

[Mac87] K. C. H. Mackenzie. *Lie Groupoids and Lie Algebroids in Differential Geometry.* Vol. 124. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 1987.

[Mac98] S. Mac Lane. *Categories for the Working Mathematician.* Second Edition. Vol. 5. Graduate Texts in Mathematics. Springer-Verlag, New York, Berlin, Heidelberg, 1998. DOI: `10.1007/978-1-4757-4721-8`.

[MM05] I. Moerdijk and J. Mrčun. "Lie groupoids, sheaves and cohomology". In: *Poisson Geometry, Deformation Quantisation and Group Representations.* Ed. by S. Gutt, J. H. Rawnsley, and D. Sternheimer. Vol. 323. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2005, pp. 145–272.

[OY01] F. Oda and T. Yoshida. "Crossed Burnside Rings I. The Fundamental Theorem". *J. Algebra* 236.1 (Feb. 2001), pp. 29–79. DOI: `10.1006/jabr.2000.8341`.

[Ren80] J. Renault. *A Groupoid Approach to C^*-Algebras.* Vol. 793. Lecture Notes in Math. Springer-Verlag Berlin Heidelberg, 1980. DOI: `10.1007/BFb0091072`.

[Ros94] J. Rosenberg. *Algebraic K-Theory and its applications.* Vol. 147. Graduate Texts in Mathematics. Springer-Verlag, New York, Berlin, Heidelberg, 1994. DOI: `10.1007/978-1-4612-4314-4`.

[Sch91] S. H. Schanuel. "Negative sets have Euler characteristic and dimension". In: *Proceedings of the International Conference held in Como, Italy, July 22-28, 1990.* Vol. 1488. Lecture Notes in Math. Springer-Verlag, Berlin, Heidelberg, New York, 1991, pp. 379–385.

[Seg71] G. B. Segal. "Equivariant stable homotopy theory". In: *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2.* Ed. by Gauthier-Villars. Paris, 1971, pp. 59–63.

[Ser77] J. P. Serre. *Linear Representations of Finite Groups.* Translated from the second French edition by Leonard L. Scott. Vol. 42. Graduate Texts in Mathematics. Springer-Verlag New York, Inc., 1977.

[Sol67] L. Solomon. "The Burnside algebra of a finite group". *J. Combin. Theory* 2.4 (June 1967), pp. 603–615. DOI: `10.1016/S0021-9800(67)80064-4`.

[Tay17] J. Taylor. *On The Mackey Formula for Connected Centre Groups.* Aug. 2017. URL: `https://arxiv.org/abs/1707.04773v2`.

[TV17] V. Turaev and A. Virelizier. *Monoidal Categories and Topological Field Theory.* Vol. 322. Progress in Mathematic. Birkhäuser, Springer International Publishing AG, 2017. DOI: 10.1007/978-3-319-49834-8.

[Wei96] A. Weinstein. "Groupoids: unifying internal and external symmetry. A tour through some examples." *Notices Amer. Math. Soc.* 43.7 (July 1996), pp. 744–752.

[Whi46] J. H. C. Whitehead. "Note on a previous paper entitled 'On adding relations to homotopy groups'". *Ann. of Math.* 2.47 (1946), pp. 806–810. DOI: 10.2307/1969237.

[Whi49] J. H. C. Whitehead. "Combinatorial homotopy II". *Bull. Amer. Math. Soc.* 55 (1949), pp. 453–496. DOI: 10.1090/S0002-9904-1949-09213-3.

Index

Index of notations