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Waring decompositions via degenerations

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Contents

Introduction

The Waring problem was first stated as a number theory problem, as a generalization of the four-squares theorem. Lagrange proved in 1770 that every natural number can be expressed as a sum of at most 4 squares. Waring stated that every number can be decomposed as a sum of 9 cubes, 14 fourth powers, and so on. This question can be generalized to homogeneous polynomials, asking when a degree d form $f \in \mathbb{C}[x_0, \ldots, x_n]_d$ admits a decomposition as sum of d-th powers of linear forms, that is if there exist $l_1, \ldots, l_r \in \mathbb{C}[x_0, \ldots, x_n]_1$ such that

$$f = l_1^d + \ldots + l_r^d.$$

Many different questions arise in this context. For instance, given n and d, what is the minimum r such that all $f \in \mathbb{C}[x_0, \ldots, x_n]_d$ admit such a decomposition? And what is the minimum r such that this holds for the general f? These are called the *Little Waring Problem* and the *Big Waring Problem*, respectively. While the latter was completely solved by Alexander-Hirschowitz in [1], the former is still unsolved in its generality. Many authors have devoted their time to different versions of the problems, using a variety of approaches, algebraic, geometric and computational. A survey of such results can be found in [20, Section 7].

The set of all decompositions of a given polynomial f naturally has the structure of an algebraic variety $VSP(f)$, called the *variety of sums of powers* of f. The interest in such varieties greatly increased after Mukai, in [59], gave a description of the Fano 3-fold V_{22} as VSP of a general quartic polynomial in three variables. Since then many authors investigated this area and generalized Mukai's techniques to other polynomials. In [36], Dolgachev presents most of the strategies used and the results achieved in this direction. With a different approach, it is also possible to work out some properties of $VSP(f)$ such as rationality, unirationality, rational connectedness and so on (see for example [55]).

When the decomposition is unique (that is, when VSP is a point) the polynomial has a canonical form and it is said to be *identifiable*. Identifiability is a desirable feature whenever one wants to "infer the parameters of the model from the data", so it has applications in many areas of mathematics. Examples range from Blind Signal Separation to Phylogenetic and Algebraic Statistic, see [53] for an account. Even if we always work over the complex numbers, for some applications it is also interesting to study identifiability over $\mathbb{R}$ instead of $\mathbb{C}$, as suggested in [32] and [4].

Few generically identifiable cases are known, and finding all of them is a challenging task. A computational approach allowed us to find a new example, presented in Theorem 3.4. Despite this result comes from a computer-aided analysis, there is another interpretation of generic identifiability in terms of the secant variety of the Veronese variety $V_{n,d}$. The Waring problem can be stated for different classes of tensors as well (see for instance [24], [3] and [13]), and Segre varieties play a similar role in this case. One of the advantages of this point of view is that the uniqueness of the decomposition implies the birationality of a certain tangential projection, so in order to disprove identifiability it is enough to show that the degree of the map is greater than 1. For this reason we work with the associated linear system. This topic is widely studied, and we could use different techniques, in particular degenerations.

The study of such degenerations led us to consider flat limits of 0-dimensional subschemes of $\mathbb{P}^n$. Unlike the standard specialization approach, we find it convenient to also consider the collision of some of the fat points. This yields the new problem to fully understand and describe such limit scheme. However, once it is done we have a new possible degeneration which proves useful to pursue our goal.

This thesis is organized as follows.

Chapter 1 introduces the first definitions about 0-dimensional schemes and linear systems on $\mathbb{P}^n$. Moreover we recall some useful tools we will use to work with these systems, such as degenerations, the Castelnuovo exact sequence and few nonspeciality results. We also recall the definitions we need about secant varieties.

In Chapter 2 we focus on specializations with collapsing points, and we try to determine the flat limit scheme Z_0 of a fat points scheme $Z \subset \mathbb{P}^n$. This relies on ideas developed by Ciliberto-Miranda and Nesci for $n = 2$, and we attempt to generalize them to any dimension. The method to compute the multiplicity of the limit scheme involves the minimum degree k of a divisor containing Z. The base locus of the corresponding linear system gives information on the first order neighbourhood of Z, so it is necessary to work with linear systems on $\mathbb{P}^n$ with assigned singularities. Despite a general description of the limit scheme seems to be out of grasp, this analysis provides answers for several cases. In particular we are able to describe the limit of $n+1$ collapsing double points in $\mathbb{P}^n$, which we will be the key ingredient of the degeneration argument in Chapter 4.

Chapter 3 is a joint work with Elena Angelini, Massimiliano Mella and Giorgio Ottaviani which faces the Waring problem for forms, [5]. We give the definition of Waring decomposition of a vector of homogeneous polynomials, focusing on identifiability. After a brief review of the state of the art, we present a new identifiable case. Both the theoretical and the computational approach are discussed. We also address the simultaneous identifiability of a pair of ternary forms, and we show that such a pair of degree a and $a+1$ forms is not identifiable for $a \geq 3$.

We were also able to disprove the existence of new identifiable cases when we are dealing with only one form. This joint work with Massimiliano Mella, [41] is the content of Chapter 4. As we mentioned above, we translate the problem about uniqueness to a question about the degree of a rational map. We work with the associated linear system and we set up a degeneration involving limits of double points which allows us to use induction, and therefore to focus on the planar case.

Chapter 1

Notations and preliminaries

We work over the complex field $\mathbb{C}$. Every scheme will be projective, unless we specify it is not. For a scheme X and a subscheme $Y \subset X$, we will write $\mathcal{I}_{Y,X}$ to denote the defining ideal of Y in X. With abuse of notation, we use the same symbol to indicate the associated ideal sheaf on X. If no ambiguity is likely to arise, we will write simply $\mathcal{I}_Y$ instead of $\mathcal{I}_{Y,X}$.

We start by recalling some definitions and facts about 0-dimensional schemes.

Definition 1.1. Let X be a 0-dimensional scheme. The *degree*, or *length*, of X, denoted by $\deg X$, is the dimension of its ring of regular functions as a complex vector space.

If X is supported on a point p, we define the *multiplicity* of X, denoted by $\operatorname{mult} X$, to be the largest natural number k such that X contains the k-ple point supported on p.

Proposition 1.2. Let X be a 0-dimensional scheme.

i) The degree of X is the limit value of the Hilbert polynomial of X.

ii) Let $Y \subset X$ be a 0-dimensional subscheme. If $\deg X = \deg Y$, then $X = Y$.

1.1 Linear systems and degenerations

Since we will deal with linear systems on $\mathbb{P}^n$ with assigned singularities and tangent directions, we now introduce the notations we are going to use.

Notation 1.3. Let $p_1, \ldots, p_r \in \mathbb{P}^n$. Let $\{q_1, \ldots, q_j\} \subset \mathbb{P}(\mathbb{T}_{p_1}\mathbb{P}^n)$ be a set of tangent directions (infinitely near points) in p_1. The *linear system*

$$\mathcal{L}_{n,d}(m_1, \ldots, m_r)(p_1[\{q_1, \ldots, q_j\}], p_2, \ldots, p_r) \subset \mathbb{P}(H^0\mathcal{O}_{\mathbb{P}^n}(d))$$

is the projective space of hypersurfaces of $\mathbb{P}^n$ having multiplicities at least m_i at the point p_i and whose tangent cone at p_1 contains $\{q_1, \ldots, q_j\}$. If either the points $p_1, \ldots, p_r$ and $q_1, \ldots, q_j$ are in general position, or no confusion is likely to arise, we indicate

$$\mathcal{L}_{n,d}(m_1[j], m_2, \ldots, m_r) := \mathcal{L}_{n,d}(m_1, m_2, \ldots, m_r)(p_1[\{q_1, \ldots, q_j\}], p_2, \ldots, p_r)$$

and

$$\mathcal{L}_{n,d}(m_1,\dots,m_r) := \mathcal{L}_{n,d}(m_1[0],m_2,\dots,m_r).$$

Moreover, if $m_1 = \ldots = m_g = m$, then we indicate

$$\mathcal{L}_{n,d}(m^g,m_{g+1},\dots,m_r) := \mathcal{L}_{n,d}(m_1,\dots,m_r).$$

Again, if $\mathcal{L}$ is a linear system, sometimes with abuse of notation we will use the same symbol to indicate the associated ideal sheaf.

Definition 1.4. The *virtual dimension* of such a linear system is

$$\operatorname{vdim}\mathcal{L}_{n,d}(m_1[j],m_2,\dots,m_r) = \binom{d+n}{n} - 1 - \sum_{i=1}^{r}\binom{m_i-1+n}{n} - j.$$

The *expected dimension* is defined as

$$\operatorname{expdim}\mathcal{L}_{n,d}(m_1[j],m_2,\dots,m_r) = \max\left\{\operatorname{vdim}\mathcal{L}_{n,d}(m_1[j],m_2,\dots,m_r),-1\right\},$$

where expected dimension -1 indicates that the linear system is expected to be empty. Note that

$$\dim\mathcal{L}_{n,d}(m_1[j],m_2,\dots,m_r) \geq \operatorname{expdim}\mathcal{L}_{n,d}(m_1[j],m_2,\dots,m_r).$$

If $\dim\mathcal{L}_{n,d}(m_1[j],m_2,\dots,m_r) > \dim\mathcal{L}_{n,d}(m_1[j],m_2,\dots,m_r)$, then the linear system is said to be *special*. Otherwise it is called *nonspecial*.

The speciality of linear systems has been extensively studied, see [27] for an account, but very little is known in general. The most studied cases are linear systems with only double points and linear systems of plane curves. If all $m_i = 2$, there is the famous Alexander–Hirschowitz' theorem (see [1]).

Theorem 1.5 (Alexander-Hirschowitz)**.** The linear system $\mathcal{L}_{n,d}(2^h)$ is special if and only if (n,d,h) is one of the following:

i) $(n,2,h)$ with $2 \leq h \leq n$,

ii) $(2,4,5)$, $(3,4,9)$, $(4,3,7)$, $(4,4,14)$.

Remark 1.6. Further note that for all special linear systems in Theorem 1.5 ii) the dimension is 0, while the expected dimension is negative.

In order to compute the dimension of a linear system, degeneration is a useful tool. Since we are going to use degeneration arguments to work with such linear systems, we give here the necessary definitions.

Definition 1.7. A *degeneration* is a morphism $\pi : V \to \Delta$, where $\Delta \ni 0,1$ is a complex disk, V is a smooth variety and π is proper and flat. For any $t \in \Delta$, we denote by V_t the fiber of π over t. Let $\sigma_i : \Delta \to X$ be sections of π and let Z be a scheme with $Z_{\mathrm{red}} = \bigcup_i \sigma_i(\Delta)$. For $t \neq 0$, let $Z_t := Z_{|X_t}$, and let Z_0 be their flat limit. We say that Z_0 is a *specialization* of Z_t.

Construction 1.8 (Specialization without collisions)**.** Let X be the blow-up of $\mathbb{P}^n$ at the point p_1, with exceptional divisor E, $V := X \times \Delta$, and $\pi : V \to \Delta$ the canonical projection. Fix j disjoint sections $\tau_1, \dots, \tau_j$ such that $\tau_i(\Delta) \subset E \times \Delta$, and $r-1$ disjoint sections $\sigma_2, \dots, \sigma_r$ such that $\sigma_i(\Delta) \cap (E \times \Delta) = \varnothing$. Let

$$Z := \bigcup_{i=2}^{r} \sigma_i(\Delta)^{m_i} \cup \bigcup_{h=1}^{j} \tau_h(\Delta)$$

be the scheme supported on the sections with multiplicity m_i along $\sigma_i(\Delta)$. Let

$$\mathbb{L}_{n,d}(m_1[j], m_2, \dots, m_r)(p_1[\{\tau_1, \dots, \tau_j\}], \sigma_2, \dots, \sigma_r)$$

be the linear subsystem on V associated to degree d divisors having multiplicities at least m_i along $\sigma_i(\Delta)$, m_1 in p_1 and whose tangent cone contains $\tau_j(\Delta)$. Then, for any $t \in \Delta$, the linear system

$$\mathbb{L}_{n,d}(m_1[j], m_2, \dots, m_r)(p_1[\{\tau_1, \dots, \tau_j\}], \sigma_2, \dots, \sigma_r)_{|V_t}$$

is

$$\mathcal{L}_t := \mathcal{L}_{n,d}(m_1[j], m_2, \dots, m_r)(p_1[\{\tau_1(t), \dots, \tau_j(t)\}], \sigma_2(t), \dots, \sigma_r(t)).$$

By semicontinuity we have

$$\mathrm{h}^0(V_0, \mathcal{L}_0) \geq \mathrm{h}^0(V_t, \mathcal{L}_t).$$

Therefore to prove the nonspeciality of $\mathcal{L}_t$ it is enough to produce a specialization having $\mathcal{L}_0$ nonspecial.

Remark 1.9. Let $H \subset \mathbb{P}^n$ be a hyperplane containing the point p_1 and let

$$Z_1 := \{p_2^{m_2}, \dots, p_s^{m_s}\} \cup \{t_1, \dots, t_j\}$$

be a 0-dimensional scheme as in Construction 1.8. Furthermore, assume that $p_2, \dots, p_h \in H$ and $t_1, \dots, t_l \in \mathbb{T}_{p_1}H$. A classical way to study the speciality of a linear system is via the *Castelnuovo exact sequence*, that in this case reads

$$\begin{aligned} 0 \to \mathcal{L}_{n,d-1}((m_1-1)[j-l], m_2-1, \dots, m_h-1, m_{h+1}, \dots, m_s) \to \\ \to \mathcal{L}_{n,d}(m_1[j], m_2, \dots, m_s) \to \mathcal{L}_{n-1,d}(m_1[l], m_2, \dots, m_h). \end{aligned}$$

Therefore the nonspeciality of the linear systems

$$\mathcal{L}_{n,d-1}((m_1-1)[j-l], m_2-1, \dots, m_h-1, m_{h+1}, \dots, m_s)$$

and

$$\mathcal{L}_{n-1,d}(m_1[l], m_2, \dots, m_h)$$

implies the nonspeciality of $\mathcal{L}_{n,d}(m_1[j], m_2, \dots, m_s)$.

It is also possible to modify Construction 1.8 and to allow the specialized points to collapse.

Construction 1.10 (Specialization with h collapsing points)**.** Let $V = \mathbb{A}^n \times \Delta$ and let $\pi : V \to \Delta$ be the canonical projection. Fix a point $q \in \mathbb{A}^n \times \{0\}$ and h general sections $\sigma_1, \ldots, \sigma_h$ such that $\sigma_i(0) = q$. Let $Z := \bigcup_i \sigma_i(\Delta)^{m_i}$ and let $\nu : V \to \mathbb{A}^n$ be the projection.

Let $X \to V$ be the blow-up of V at the point q, with exceptional divisor W. Then we have natural morphisms $\nu_X : X \to \mathbb{A}^n$, a degeneration $\pi_X : X \to \Delta$, and sections $\sigma_{X,i} : \Delta \to X$. The fiber X_0 is given by $W \cup V_0$, where V_0 is $\mathbb{A}^n$ blown up at one point and $W \cong \mathbb{P}^n$. Let $R = W \cap V_0 \cong \mathbb{P}^{n-1}$ be the exceptional divisor of this blow-up. We want to stress that, since the sections σ_i's are general, $\{\sigma_{X,i}(0)\}$ is a set of general points of W.

With these notations, we say that Z_0 is the flat limit of h collapsing points of multiplicity $m_1, \ldots, m_h$. One of our problems will be to describe Z_0. This is in general quite hard and we only have partial solutions. Nonetheless, once we understand the limit, we may study the speciality of a linear system via its specializations with collapsing points, using the same technique described in Construction 1.8 and Remark 1.9.

1.2 Some nonspeciality result

Interpolation theory studies the dimension of linear systems. The problem to characterize special system has a long history, but there is still much we do not know. Later we will review some of the known results on this topic. Now we want to state some nonspeciality results about certain linear systems, which we will use in this thesis.

We start with the following well known fact.

Lemma 1.11. Let $\mathcal{L}$ be a linear system on a smooth projective variety X and let $C \subset X$ be a positive dimensional subvariety. Let $s := \operatorname{codim}_{\mathcal{L}} |\mathcal{L} \otimes \mathcal{I}_C|$. If $x_1, \ldots, x_s \in C$ are general points, then $x_1, \ldots, x_s$ impose independent conditions to $\mathcal{L}$.

We apply the above remarks to prove the nonspeciality of some linear systems we will use along the proof of Theorem 4.1. The next result is about systems with a triple point and a bunch of double points.

Proposition 1.12. Let $n \geq 3$ and $d \geq 4$. Define

$$r(n,d) := \begin{cases} \left\lceil \dfrac{\binom{n+d}{n}}{n+1} \right\rceil - n - 1 & \text{if either } n \neq 3 \text{ or } (n,d) = (3,4), \\ \\ \left\lceil \dfrac{\binom{d+3}{3}}{4} \right\rceil - 5 & \text{if } n = 3. \end{cases}$$

The linear system $\mathcal{L}_{n,d}(3, 2^a)$ is nonspecial if $a \leq r(n,d)$.

Proof. We prove the statement by induction on d. It is clear that it is enough to prove it for $a = r(n,d)$. The first step of induction is $d = 4$ and it is the content of [68, Lemma 2.4].

Assume that $d \geq 5$. Let $Z_1 := \{q^3, p_1^2, \dots, p_a^2\}$ be a 0-dimensional scheme. Fix a hyperplane H not containing q. Let Z_0 be a specialization without collisions of Z_1 with

$$h := \left\lceil \frac{\binom{d+n-1}{n-1}}{n} \right\rceil - 1$$

points on the hyperplane H. Since $q \notin H$, the Castelnuovo exact sequence reads

$$0 \to \mathcal{L}_{n,d-1}(3, 2^{a-h}, 1^h) \to \mathcal{L}_{n,d}(3, 2^a) \to \mathcal{L}_{n-1,d}(2^h),$$

where the simple base points are all on the hyperplane H. Since $d \geq 5$, the linear system on the right is nonspecial by Theorem 1.5. Therefore to conclude it is enough to prove that $\mathcal{L}_{n,d-1}(3, 2^{a-h}, 1^h)$ is nonspecial.

Claim 1.13. We have $\operatorname{expdim} \mathcal{L}_{n,d-1}(3, 2^{a-h}, 1^h) \geq 0$.

Proof. Assume first that $n > 3$. Then

$$\begin{aligned}\operatorname{vdim} \mathcal{L}_{n,d-1}(3, 2^{a-h}, 1^h) &= \binom{n+d-1}{n} - \binom{n+2}{2} - (n+1)a + nh - 1 \\ &\geq n^2 - \binom{n+2}{2} - 1 \geq 0.\end{aligned}$$

Assume that $n = 3$. Then

$$\operatorname{vdim} \mathcal{L}_{3,d-1}(3, 2^{a-h}, 1^h) = \binom{d+2}{3} - 10 - 4a + 3h - 1 \geq 16 - 14 > 0,$$

as desired. □

We start by proving that the simple base points impose independent conditions. The points are general in H, therefore, by Lemma 1.11, we have to check that

$$\dim \mathcal{L}_{n,d-2}(3, 2^{a-h}) \leq 0.$$

This is clear for $d = 4$. For $d \geq 5$ we want to apply Theorem 1.5. Checking also the special cases, we have

$$\begin{aligned}\dim \mathcal{L}_{n,d-2}(3, 2^{a-h}) &\leq \dim \mathcal{L}_{n,d-2}(2^{a-h+1}) \\ &= \binom{n+d-2}{n} - 1 - (n+1)(a-h+1). \end{aligned} \tag{1.1}$$

For $n = 3$, this reads

$$\begin{aligned}\dim \mathcal{L}_{n,d-2}(3, 2^{a-h}) &\leq \binom{d+1}{3} - 4\left(\left\lceil \frac{\binom{d+3}{3}}{4} \right\rceil - 5 - \left\lceil \frac{\binom{d+2}{2}}{3} \right\rceil + 2 \right) - 1 \\ &\leq \binom{d+1}{3} - \binom{d+3}{3} + \frac{4}{3}\binom{d+2}{2} + 15 \\ &= -\binom{d+1}{2} + \frac{1}{3}\binom{d+2}{2} + 15 \\ &= \frac{1-d^2}{3} + 15 < 0\end{aligned}$$

for every $d \geq 7$. We check the rest of the cases by a direct computation.

$(n,d)=(3,5)$ In this case $a=9$, $h=6$ and the linear system $\mathcal{L}_{3,3}(3,2^3)$ has a unique element.

$(n,d)=(3,6)$ In this case $a=16$ and $h=9$. An element of the linear system $\mathcal{L}_{3,4}(3,2^7)$ has to contain all quadrics passing through the 8 singular points. This shows that $\mathcal{L}_{3,4}(3,2^7)$ is empty.

For $n>3$, inequality (1.1) yields

$$\begin{aligned}
\dim \mathcal{L}_{n,d-2}(3,2^{a-h}) &\leq \binom{n+d-2}{n} - (n+1)\left(\left\lceil \frac{\binom{n+d}{n}}{n+1}\right\rceil - \left\lceil \frac{\binom{n+d-1}{n-1}}{n}\right\rceil - n+1\right) - 1 \\
&\leq \binom{n+d-2}{n} - \binom{n+d}{n} + \binom{n+d-1}{n-1} + \frac{\binom{n+d-1}{n-1}}{n} + n^2+n-1 \\
&= -\binom{n+d-2}{n-1} + \frac{\binom{n+d-1}{n-1}}{n} + n^2+n-1 \\
&= -\frac{(n+d-2)!}{(n-1)!(d-1)!} + \frac{(n+d-1)!}{n!d!} + n^2+n-1 \\
&= \frac{(n+d-2)!}{(n-1)!(d-1)!}\left[-1+\frac{n+d-1}{nd}\right] + n^2+n-1 \\
&= \binom{n+d-2}{n-1}\cdot\frac{(1-n)(d-1)}{nd} + n^2+n-1.
\end{aligned}$$

The latter decreases as d increases, so

$$\begin{aligned}
\dim \mathcal{L}_{n,d-2}(3,2^{a-h}) &\leq \binom{n+d-2}{n-1}\cdot\frac{(1-n)(d-1)}{nd} + n^2+n-1 \\
&\leq \frac{4}{5}\binom{n+3}{4}\cdot\frac{1-n}{n} + n^2+n-1 \\
&= \frac{-n^4-5n^3+25n^2+35n-24}{30} < 0
\end{aligned}$$

for every $n>3$.

To conclude, we prove the nonspeciality of $\mathcal{L}_{n,d-1}(3,2^{a-h})$. For this observe that

$$\begin{aligned}
a-h-r(n,d-1) &\leq \frac{\binom{n+d}{n}}{n+1} - \frac{\binom{d+n-1}{n-1}}{n} - \frac{\binom{n+d-1}{n}}{n+1} + 2 \\
&= \frac{1}{n(n+1)}\left[n\binom{n+d}{n} - (n+1)\binom{d+n-1}{n-1} - n\binom{n+d-1}{n}\right] + 2 \\
&= -\frac{\binom{n+d-1}{n-1}}{n(n+1)} + 2 < 1.
\end{aligned}$$

Therefore we apply the induction hypothesis. □

In the degeneration we will use to prove Theorem 4.1, we will deal with systems with simple base points in special position. We conclude this Section by proving a Lemma that will prove useful to handle such situation.

Lemma 1.14. Let $b, d \in \mathbb{N}$ such that $d \geq 4$ and

$$1 \leq b < \frac{\binom{d+3}{3}}{4} - 1.$$

Fix $q \in \Pi \subset \mathbb{P}^n$ a linear space of dimension 3 and b general points on it, say $x_1, \ldots, x_b$. Then the linear system

$$\mathcal{L}_{n,d}(2^a, 1^b)(q, p_1, \ldots, p_{a-1}, x_1, \ldots, x_b)$$

is nonspecial if

$$a \leq a(n, d) := \left\lfloor \frac{\binom{n+d}{n} - b - 1}{n+1} \right\rfloor - \max\{0, n-4\},$$

where the p_i are general points. For the special case $(d, b) = (4, 5)$ we prove a better estimate

$$a(n, 4) = \left\lfloor \frac{\binom{n+4}{n} - 5 - 1}{n+1} \right\rfloor - \max\{(n-7), 1\}$$

Proof. Note that, since $b \geq 1$, $d \geq 4$ and the virtual dimension is non negative, $\mathcal{L}_{n,d}(2^a)$ is nonspecial by Theorem 1.5. Therefore we have only to care about the simple points.

For $n = 3$ the statement is immediate. For $n = 4$, by Lemma 1.11, it is enough to check that $\dim \mathcal{L}_{4,d-1}(2^{a(4,d)-1}, 1) \leq 0$. By Theorem 1.5, it is enough to check that $\operatorname{vdim} \mathcal{L}_{4,d-1}(2^{a(4,d)-1}, 1) < 0$. The latter is a simple computation.

Next we prove the statement by induction on n. For $n = i + 1 \geq 5$, fix a general hyperplane $H \supset \Pi$ and consider a degeneration with $a(i, d)$ points on H. By Castelnuovo exact sequence, we only need to prove the nonspeciality of

$$\mathcal{L}_{i,d}(2^{a(i,d)}, 1^b) \quad \text{and} \quad \mathcal{L}_{i+1,d-1}(2^{a(i+1,d)-a(i,d)}, 1^{a(i,d)}).$$

The former is nonspecial by the induction step. For the latter note that

$$\begin{aligned} a(i+1, d) - a(i, d) &\geq \frac{\binom{i+1+d}{i+1} - b - 1}{i+2} - \frac{\binom{i+d}{i} - b - 1}{i+1} - 2 \\ &> \frac{\binom{i+1+d}{i+1} - b - 1}{i+2} - \frac{\binom{i+d}{i} - b - 1}{i+2} - 2 \\ &> \frac{\binom{i+d}{i+1}}{i+2} - 2 \geq \frac{\binom{i+d-1}{i+1}}{i+2}. \end{aligned}$$

For $d \geq 5$, the linear system $\mathcal{L}_{i+1,d-2}(2^{a(i+1,d)-a(i,d)})$ is empty by Theorem 1.5. For $d = 4$, it is easy to see that

$$a(i+1, 4) - a(i, 4) > i,$$

and again $\mathcal{L}_{i,2}(2^{a(i+1,4)-a(i,4)})$ is empty. Let

$$a(i, d) = \left\lfloor \frac{\binom{i+d}{i} - b - 1}{i+1} \right\rfloor - \alpha(i).$$

Then $\alpha(i+1) = \alpha(i) + 1$, and we have

$$\begin{aligned}\text{vdim}\mathcal{L}_{i+1,d-1}&(2^{a(i+1,d)-a(i,d)}, 1^{a(i,d)}) \\ &\geq (i+2)\alpha(i+1) - (i+1)(\alpha(i)+1)) - 1 > 0. \end{aligned} \tag{1.2}$$

Lemma 1.11 proves that the linear system $\mathcal{L}_{i+1,d-1}(2^{a(i+1,d)-a(i,d)}, 1^{a(i,d)})$ is non special to conclude this case.

Assume that $d = 4$ and $b = 5$. We first prove that $\mathcal{L}_{5,4}(2^{19}, 1^5)$ is nonspecial. Observe that, by degenerating 12 double points on a hyperplane, Castelnuovo exact sequence decomposes $\mathcal{L}_{5,4}(2^{19})$ into $\mathcal{L}_{4,4}(2^{12})$ and $\mathcal{L}_{5,3}(2^7, 1^{12})$. It is easy to check that these two systems are nonspecial and $\dim \mathcal{L}_{5,3}(2^7, 1^{12}) = 1$. The linear system $\mathcal{L}_{4,3}(2^{11})$ is empty, therefore there is at most a pencil of divisors in $\mathcal{L}_{5,4}(2^{19})$ that contains a given $\mathbb{P}^3$ through a double point. By hypothesis, $\text{vdim}\,\mathcal{L}_{5,4}(2^{19}, 1^5) > 1$, hence this linear system is nonspecial. To conclude the statement for $d = 4$, we then argue exactly as in the first part of the proof, checking Equation (1.2) case by case for $n \leq 8$ and then conclude as in the general case. □

1.3 The secant construction

Secant varieties play a very important role in different areas of mathematics. Here, we are interested to the fact that they naturally give a geometric meaning to the decomposition of a polynomial or a tensor as a sum of powers. We now recall the basic definitions concerning secant varieties.

Let $\mathbb{G}r_{k-1} = \mathbb{G}r(k-1, N)$ be the Grassmannian of $(k-1)$-linear spaces in $\mathbb{P}^N$. Let $X \subset \mathbb{P}^N$ be an irreducible variety of dimension n and let

$$\Gamma_k(X) \subset X \times \ldots \times X \times \mathbb{G}r_{k-1},$$

be the closure of the graph of

$$\alpha : (X \times \ldots \times X) \setminus \Delta \to \mathbb{G}r_{k-1},$$

taking $(x_1, \ldots, x_k)$ to $[\langle x_1, \ldots, x_k \rangle]$, for a k-tuple of distinct points. Observe that $\Gamma_k(X)$ is irreducible of dimension kn. Let $\pi_2 : \Gamma_k(X) \to \mathbb{G}r_{k-1}$ be the natural projection. Denote by

$$S_k(X) := \pi_2(\Gamma_k(X)) \subset \mathbb{G}r_{k-1}.$$

Again $S_k(X)$ is irreducible of dimension kn. Finally, let

$$I_k := \{(x, [\Lambda]) | x \in \Lambda\} \subset \mathbb{P}^N \times \mathbb{G}r_{k-1},$$

with natural projections p_i onto the factors. Observe that $p_2 : I_k \to \mathbb{G}r_{k-1}$ is a $\mathbb{P}^{k-1}$-bundle on $\mathbb{G}r_{k-1}$.

Definition 1.15. Let $X \subset \mathbb{P}^N$ be an irreducible variety. The *abstract k-Secant variety* is

$$\text{Sec}_k(X) := p_2^{-1}(S_k(X)) \subset I_k$$

and the *k-Secant variety* is

$$\mathbb{S}\text{ec}_k(X) := p_1(\text{Sec}_k(X)) \subset \mathbb{P}^N.$$

It is immediate that $\mathrm{Sec}_k(X)$ has dimension $kn+k-1$ and it has a $\mathbb{P}^{k-1}$-bundle structure on $S_k(X)$. One says that X is *k-defective* if

$$\dim \mathbb{S}\mathrm{ec}_k(X) < \min\{\dim \mathrm{Sec}_k(X), N\}$$

and calls *k-defect* the number

$$\delta_k = \min\{\dim \mathrm{Sec}_k(X), N\} - \dim \mathbb{S}\mathrm{ec}_k(X).$$

Remark 1.16. Let us stress that in our definition $\mathbb{S}\mathrm{ec}_1(X) = X$. A simple but useful feature of the above definition is the following. Let Λ_1 and Λ_2 be two distinct k-secant $(k-1)$-linear spaces to $X \subset \mathbb{P}^N$. Let λ_1 and λ_2 be the corresponding projective $(k-1)$-spaces in $\mathrm{Sec}_k(X)$. Then we have $\lambda_1 \cap \lambda_2 = \varnothing$.

Here is the main result we use about secant varieties.

Theorem 1.17 (Terracini Lemma, [77, 23])**.** Let $X \subset \mathbb{P}^N$ be an irreducible projective variety. If $p_1, \ldots, p_k \in X$ are general points and $z \in \langle p_1, \ldots, p_k \rangle$ is a general point, then the embedded tangent space at z is

$$\mathbb{T}_z \, \mathbb{S}\mathrm{ec}_k(X) = \langle \mathbb{T}_{p_1} X, \ldots, \mathbb{T}_{p_k} X \rangle.$$

If X is k-defective, then the general hyperplane H containing $\mathbb{T}_z \, \mathbb{S}\mathrm{ec}(X)$ is tangent to X along a variety $\Sigma(p_1, \ldots, p_k)$ of pure, positive dimension, containing $p_1, \ldots, p_k$.

Chapter 2

Limits of fat points

A standard approach to study the speciality of linear systems $\mathcal{L}_{n,d}(m_1, \dots, m_r)$ is via degeneration. This is accomplished by using a flat family in which the involved points specialize in some special configuration, for instance by sending some of the points on a hyperplane to apply induction arguments. However, sometimes it can be useful to allow points not only to be in special position, but also to collide to the same point. In other words, sometimes it is convenient to apply a degeneration as defined in Construction 1.10. Like every degeneration technique, this one is useful only if the specialized system is easier to deal with than the original one, so we want to understand what the limit scheme is. This leads to a fairly natural question, which is interesting in itself.

Question Let $n, h, m_1, \dots, m_h \in \mathbb{N}$. What is the flat limit scheme of h colliding points of multiplicities $m_1, \dots, m_h$ in $\mathbb{P}^n$?

Despite the question is easy to formulate, the answer is far from being simple. Work by Ciliberto–Miranda ([30]) and Nesci ([62]) show that there is not a definite and clean solution to this problem, even in the planar case $n = 2$.

We will use the notations of Chapter 1, in particular those of Construction 1.10. As a warm-up, we start with a very easy result that completely describes all collisions of fat points in $\mathbb{P}^1$.

Proposition 2.1. Let $m_1, \dots, m_h \in \mathbb{N}$ and let $m = m_1 + \dots + m_h$. The limit of h collapsing points of multiplicities $m_1, \dots, m_h$ is an m-ple point.

Proof. It is enough to observe that the only scheme of length m supported on a point is the m-ple point. $\square$

Now that the case $n = 1$ is settled, we assume $n \geq 2$ and we try to move to some more interesting cases in higher dimension. In order to understand what Z_0 is, the first problem to tackle is to compute its multiplicity. We will show that $\operatorname{mult} Z_0$ does not depend on the choice of the sections σ_i, and we will give a method to compute it.

The following result was proved in [62, Theorem 2.6].

Lemma 2.2. The multiplicity of the limit scheme Z_0 is at least the minimum integer j such that the linear system $\mathcal{L}_{n,j}(m_1, \dots, m_h)$ is not empty.

Proof. Set $\mu = \mathrm{mult}_q Z$. Then we have $\mathrm{mult} Z_0 \geq \mu$. Let $\mathcal{I}_{X,Z}$ be the ideal associated to Z on X. Then $\mathcal{I}_{X,Z|W} \sim \mathcal{L}_{n,\mu}(m_1, \ldots, m_h)$. Since the ideal $\mathcal{I}_{X,Z}$ is globally generated, the linear system $\mathbb{P}(\mathcal{I}_{X,Z|W})$ has to be nonempty. $\square$

We aim to show that the value predicted by Lemma 2.2 is actually achieved with equality.

Proposition 2.3. Define $k = \min\{a \in \mathbb{N} \mid H^0\mathcal{I}_{Z_1}(a) \neq 0\}$. Then $\mathrm{mult}\, Z_0 = k$. In particular, the multiplicity of the limit scheme does not depend on σ_i, as long as they are general.

Proof. Thanks to Lemma 2.2 it is enough to show that $\mathrm{mult}\, Z_0 \leq k$.

For $t \neq 0$, set $l = \dim \mathcal{I}_{Z_t}(k)$. Since the base points of Z_t are in general position, l does not depend on t, and by hypothesis we know that $l \geq 1$. Fix $p_1, \ldots, p_{l-1}$ general points on $A_t = \mathbb{A}^n$, and define $Z'_t = Z_t \cup p_1 \cup \ldots \cup p_{l-1}$. Observe $Z'_t \supset Z_t$ for every t, and there is a unique degree k divisor $D_t \subset A_t$ such that $D_t \supset Z'_t$. Let f_t be the polynomial defining D_t as a divisor in A_t. If we regard it as a polynomial in $\mathbb{C}[x_1, \ldots, x_n, t]$, f_t defines a divisor D of $\mathbb{A}^n \times \mathbb{A}^1$, which has degree k with respect to $x_1, \ldots, x_n$. Then f_0 defines a divisor D_0 of A_0 which is the flat limit of the D_t's. The degree of f_0 is at most k and $D_0 \supset Z'_0 \supset Z_0$, so $\mathrm{mult}\, Z_0 \leq k$. $\square$

In some special cases the multiplicity is enough to compute the limit scheme.

Example 2.4. Let us collapse 7 double points in $\mathbb{P}^2$. The scheme consisting of those points has length 21. Since $\mathrm{h}^0\mathcal{O}_{\mathbb{P}^2}(5) = 15$, they can not lie on a quintic curve by Theorem 1.5. On the other hand, they lie on a sestic, so the limit scheme contains a 6-ple point by Proposition 2.3. Since a 6-ple point in $\mathbb{P}^2$ has length 21, the limit scheme is a 6-ple point by Proposition 1.2.

When the scheme we are specializing does not have the degree of a multiple point, this analysis is not enough to determine the limit scheme.

In the notations of Construction 1.10, consider the limit scheme $Z_0 \subset \mathbb{A}^n$. Let

$$\Sigma_X = \bigcup_{i=1}^{h} \sigma_{X,i}(\Delta)$$

be the smooth scheme associated to strict transform Z_X of Z on X. Let $\mathcal{X} \to X$ be the blow-up of the ideal sheaf $\mathcal{I}_{\Sigma_X}$, with exceptional divisors $\mathcal{E}_1, \ldots, \mathcal{E}_h$, and let $\varphi : \mathcal{X} \to \Delta$ be the degeneration onto Δ. Note that this blow-up is an isomorphism in a neighbourhood of V_0. The central fiber is

$$\mathcal{X}_0 := \varphi^{-1}(0) = P \cup V_0,$$

where P is the blow-up of W in h general points. As before, let $R = P \cap V_0$. The linear systems we are interested in are $\mathcal{L} := \mathcal{O}_{\mathcal{X}}(-\sum_i 2\mathcal{E}_i - \mu P)$ and its restrictions $\mathcal{L}_P$, $\mathcal{L}_R$, to P and R. The linear system $\mathcal{L}$ is complete and we aim to understand when its restrictions stay complete.

Let us start with an instructive example ([62, Example 2.10]).

Example 2.5. In the case of 3 colliding double points in $\mathbb{P}^2$, the limit has multiplicity 3. On the other hand, the starting scheme Z_1 $\deg Z_1 = 9$ while a triple point has degree 6, therefore the limit is not only the triple point.

There are 3 more linear conditions the linear system has to satisfy. In order to understand them, observe that a plane cubic with 3 double points in general position is a union of 3 lines. These lines intersect the divisor R in 3 points, and the missing linear conditions are exactly the passage through those 3 points. In particular, the linear system $\mathcal{L}_R$ is not complete.

Unlike $\mathcal{L}_R$, the system $\mathcal{L}_P$ is always complete, as proved in [62, Lemma 2.11].

Lemma 2.6. The linear system $\mathcal{L}_P$ is complete.

Proof. Consider the exact sequence

$$0 \to \mathcal{L}(-P) \to \mathcal{L} \to \mathcal{L}_P \to 0.$$

To prove the claim, it is enough to show that $\mathrm{h}^1(\mathcal{L}(-P)) = 0$. To this end, note that the sheaf $\mathcal{L}(-P) \sim \mathcal{O}(\sum_i 2\mathcal{E}_i - (\mu+1)P)$ is the pull-back of the ideal sheaf $\bigcup_i \mathcal{I}^2_{\sigma_i(\Delta)} \cup \mathbf{m}_p^{\mu+1}$ on $\mathbb{A}^n \times \Delta$. Hence we have

$$H^1(\mathcal{L}(-P)) = H^1\left(\bigcup_i \mathcal{I}^2_{\sigma_i(\Delta)} \cup \mathbf{m}_p^{\mu+1}\right) = 0,$$

as desired. □

Before we move to the first results on limits, it is important to have clear in mind what kind of characterization we want. In general it will be way too complicated to determine what the limit is up to isomorphism. For instance, consider 14 collapsing simple points in $\mathbb{P}^2$. They lie on a unique quartic C, so $\operatorname{mult} Z_0 = 4$. $\operatorname{Bs}\mathcal{L}_{2,4}(1^{14}) = C$, so its restriction to the exceptional line R consists of 4 simple points. Since

$$\deg(\text{fourtuple point}) + 4 = 10 + 4 = 14 = \deg Z_1,$$

the limit is a fourtuple point with 4 infinitely near simple points. However, notice that if we change the sections $\sigma_1, \ldots, \sigma_{14}$, then we will have different tangent directions to the limit. Recall that two 4-tuples of points in $\mathbb{P}^1$ are not projectively equivalent in general, so the limits do not need to be isomorphic. Nonetheless we will be satisfied to say that the limit is a fourtuple point with 4 infinitely near simple points.

We also want to stress that our analysis works as long as we make all points collide at once. If we collide some of them to a limit scheme $\tilde{Z}_1$ and then we collide the others and $\tilde{Z}_1$, we are not guaranteed to obtain the same limit scheme as if we collide all of them at once. As an example, let Z_1 be the scheme consisting of a double point and 3 simple points in $\mathbb{P}^2$. If we make them collide, the multiplicity of the limit scheme Z_0 is 3 by Proposition 2.3. On the other hand, we could collide the 3 simple points to a double point, but the limit of 2 colliding double points has multiplicity 2.

It is time to move to the first description of the limit Z_0. The next Section is devoted to collisions of simple points.

2.1 Simple points

In this section we assume $m_1 = \ldots = m_h = 1$. When the number of colliding simple points is small compared to the dimension of the ambient space, we have enough information to understand the limit. The following Proposition gives a description of Z_0 for $h \leq n+1$.

Proposition 2.7.

1. If $h \leq n$, then the limit of h collapsing simple points in $\mathbb{P}^n$ is a simple point together with $h-1$ linear conditions on its tangent cone.
2. The limit of $n+1$ collapsing simple points in $\mathbb{P}^n$ is a double point.

Proof.

1. Since general simple points always give independent linear conditions, $\operatorname{mult} Z_0 = 1$ by Proposition 2.3. $\operatorname{Bs} \mathcal{L}_{n,1}(1^h) = \mathbb{P}^{h-1}$ consists of the linear space generated by the h points. Its intersection with R is an infinitely near $(h-2)$-dimensional linear space. This imposes $h-1$ linear conditions on the first infinitesimal neighbourhood of the limit. In order to conclude, it is enough to observe that this candidate limit scheme has length $1 + h - 1 = h = \deg Z_1$.
2. As before, the simple points give independent conditions, so $\operatorname{mult} Z_0 = 2$ by Proposition 2.3. Since a double points has length $n+1 = \deg Z_1$, Z_0 is a double point. □

When $h \geq n+2$, the situation gets more involved. As shown in [30, Proposition 3.1], the data we have on the tangent cone are not enough to define the limit.

Example 2.8. Let $Z_1 = \{p_1, p_2, p_3, p_4\} \subset \mathbb{P}^2$ be a scheme consisting of 4 simple points, and let Z_0 be the scheme obtained by colliding those 4 simple points. Then $\deg Z_1 = 4$ and $\operatorname{mult} Z_0 = 2$ by Proposition 2.3. Since a double point has length 3, there is one condition left to find. $\mathcal{L}_{2,2}(1^4)$ has no base locus beside the 4 fixed points. The general element of $\mathcal{L}_{2,2}(1^4)$ meets the exceptional divisor $R = \mathbb{P}^1$ in 2 points, but the choice of the first one identifies the other one. In other words, $\mathcal{L}_{2,2}(1^4)_{|R} \subsetneq \mathbb{P}(H^0\mathcal{O}_R(2)) = \mathbb{P}^2$. More precisely, there is no element of $\mathcal{L}_{2,2}(1^4)$ containing R, so $\mathcal{L}_{2,2}(1^4)_{|R} = \mathbb{P}^1$ is a hyperplane in $\mathbb{P}(H^0\mathcal{O}_R(2))$. If we call ι the involution sending each point $x \in R$ into the other intersection of the only conic through $x, p_1, \ldots, p_4$ with R, then the last condition defining Z_0 is that $\mathbb{P}(H^0\mathcal{I}_{Z_0}(2)) = \mathcal{L}_{2,2}(1^4)_{|R}$ is the space of quadrics of R made by two points x, y such that $y = \iota(x)$.

In higher dimension we need a more refined argument. Indeed, $\mathcal{L}_{n,2}(1^{n+1})_{|R}$ is not defined by an involution. We will define the variety parametrizing such hyperplanes, and later we will study it in low dimension.

Proposition 2.9. The limit of $n+2$ collapsing simple points in $\mathbb{P}^n$ is a double point with the additional condition that $H^0\mathcal{I}_{Z_0}(2)$ is a specific hyperplane in $H^0\mathcal{O}_{\mathbb{P}^n}(2)$.

Proof. Let $p_1, \ldots, p_{n+1} \in \mathbb{P}^n$ be the coordinate points and let

$$\mathcal{M} = \mathcal{L}_{n,2}(1^{n+1})(p_1, \ldots, p_{n+1}).$$

It is immediate that $\mathcal{M}$ is nonspecial of dimension $\binom{n+2}{2} - n - 2$. Define

$$\alpha : \mathcal{M} \to \mathbb{P}(H^0\mathcal{O}_R(2))$$

to be the restriction to R. Since there are no quadrics containing $p_1, \ldots, p_{n+1}$ and R, α is surjective. Moreover observe that $\dim \mathcal{M} = \dim \mathbb{P}(H^0\mathcal{O}_R(2))$, so α is an isomorphism. Call α^* the induced isomorphism between the duals.

For $p \in \mathbb{P}^n$, define $H_p = \mathcal{L}_{n,2}(1^{n+2})(p_1, \ldots, p_{n+1}, p)$. If p is general, then H_p has dimension $\binom{n+2}{2} - n - 3 = \binom{n+1}{2} - 2$. This means that H_p is a hyperplane in $\mathcal{M}$, and therefore $\alpha(H_p) = H_{p|R}$ is a hyperplane in $\mathbb{P}(H^0\mathcal{O}_R(2))$. Hence the association $p \to [H_{p|R}]$ defines a rational map

$$\varphi_n : \mathbb{P}^n \dashrightarrow \mathbb{P}(H^0\mathcal{O}_R(2))^*. \tag{2.1}$$

If $\varphi_{\mathcal{M}}$ is the rational map associated to $\mathcal{M}$, then the diagram

$$\begin{tikzcd}
\mathbb{P}^n \arrow[rr, dashed, "\varphi_{\mathcal{M}}"] \arrow[dr, dashed, "\varphi_n"'] & & \mathcal{M}^* \arrow[dl, "\alpha^*"] \\
& \mathbb{P}(H^0\mathcal{O}_R(2))^* &
\end{tikzcd}$$

commutes, because

$$\begin{aligned}
\alpha^*(\varphi_{\mathcal{M}}(p)) &= \alpha^*(\mathcal{L}_{n,2}(1^{n+2})(p_1, \ldots, p_{n+1}, p)) \\
&= \mathcal{L}_{n,2}(1^{n+2})(p_1, \ldots, p_{n+1}, p)_{|R} \\
&= \varphi_n(p)
\end{aligned}$$

for a general $p \in \mathbb{P}^n$.

The limit Z_0 has multiplicity 2 by Proposition 2.3, so we are studying $\mathcal{L}_{n,2}(1^{n+2})$. Up to projectivity, we can assume the $n+2$ simple points are $p_1, \ldots, p_{n+1}, x$. Since a double point has length $n+1$, there is only one condition left and such condition is the following. Once we blow-up the limit point, in the exceptional divisor R, $H^0\mathcal{I}_{Z_0}(2)_{|R} \subsetneq H^0\mathcal{O}_R(2)$ does not contain all quadrics, but only those coming from restriction to R of quadrics of $W = \mathbb{P}^n$. More precisely $H^0\mathcal{I}_{Z_0}(2)_{|R} = H_{x|R} = \alpha(H_x) = \varphi(x)$. □

It is natural to ask what are the quadrics in the ideal of the limit. This leads to studying the map φ_n, which is interesting in itself.

The map φ_n

Let Z_0 be the limit of $n+2$ collapsing simple points in $\mathbb{P}^n$. We proved that $\mathrm{h}^0\mathcal{I}_{Z_0}(2) = \binom{n+2}{2} - 1$, that is the quadrics containing Z_0 are a hyperplane of quadrics of $\mathbb{P}^n$. In order to better describe Z_0, we want to understand which hyperplanes of $\mathbb{P}(H^0\mathcal{O}_{\mathbb{P}^n}(2))$ arise as $\mathcal{I}_{Z_0}(26)$, meaning that we want to give more information about the H_x's, when $x \in \mathbb{P}^n$ is general. In other words, we want to say more about the map φ_n and its image.

Lemma 2.10. Let φ_n be the rational map (2.1) and let $Y_n = \overline{\varphi_n(\mathbb{P}^n)}$. Then

1. the indeterminacy locus of φ_n consists of the $n+1$ coordinate points,

2. $\deg Y_n = 2^n - n - 1$,

3. φ_n is birational onto Y_n, in particular Y_n is rational,

4. φ_n is dominant $\Leftrightarrow n = 2$,

5. for $n \geq 3$, the system of quadrics of $\mathbb{P}(H^0\mathcal{O}_{\mathbb{P}^n}(2))$ containing Y_n has dimension at least $2\binom{n+1}{4}$.

Proof. In Proposition 2.9 we noticed that, up to isomorphism, φ_n is defined by the linear system $\mathcal{M} = \mathcal{L}_{n,2}(1^{n+1})(p_1, \dots, p_{n+1})$. Then the indeterminacy locus of φ_n is $\operatorname{Bs}\mathcal{M} = \{p_1, \dots, p_{n+1}\}$. Moreover the degree of Y_n can be computed as the self-intersection of $\mathcal{M}$, so $\deg Y_n = 2^n - n - 1$.

In order to prove the birationality, let x be a point in the generic fiber. Since $\varphi_{\mathcal{M}}(x) = \mathcal{L}_{n,2}(1^{n+2})(p_1, \dots, p_{n+1}, x)$ has no base points except the imposed ones, there are no points in the fiber other than x.

Note that the map φ_2 is associated to the linear system $\mathcal{L}_{2,2}(1^3)$, so it is the standard quadratic Cremona map, which is dominant. On the other hand, for $n \geq 3$ we have $n < \dim \mathbb{P}(H^0\mathcal{O}_{\mathbb{P}^{n-1}}(2))^*$, so φ_n is not dominant.

For the last part, the structure exact sequence of Y_n gives

$$0 \to \mathcal{I}_{Y_n}(2) \to \mathcal{O}_{\mathbb{P}(H^0\mathcal{O}_R(2))^*}(2) \to \mathcal{O}_{Y_n}(2) \to 0,$$

so

$$\begin{aligned} \mathrm{h}^0\mathcal{I}_{Y_n}(2) &= \mathrm{h}^0\mathcal{O}_{\mathbb{P}(H^0\mathcal{O}_R(2))^*}(2) - \mathrm{h}^0\mathcal{O}_{Y_n}(2) + \mathrm{h}^1\mathcal{I}_{Y_n}(2) \\ &\geq \binom{\frac{1}{2}(n^2+n-2)+2}{2} - \dim(\mathcal{L}_{n,4}(2^{n+1})) + 1 \\ &= \binom{\frac{n^2+n+2}{2}}{2} - \left(\binom{n+4}{4} - (n+1)^2\right). \end{aligned}$$

To conclude, observe that the latter equals $2\binom{n+1}{4}$. □

A family of hyperplanes of $\mathbb{P}(H^0\mathcal{O}_R(2))$ is made by those defined by the containment of a given point. Such hyperplanes are parametrized by the Veronese variety $V_{n-1,2} \subset \mathbb{P}(H^0\mathcal{O}_R(2))^* = \mathbb{P}^{\binom{n+1}{2}-1}$, and we want to determine how Y_n and $V_{n-1,2}$ are related. The next Lemma shows that $V_{n-1,2} \subset Y_n$ and describes some other special subvarieties of Y_n.

Lemma 2.11. Assume $n \geq 3$. Let $X = \mathrm{Bl}_{p_1,\dots,p_{n+1}}(\mathbb{P}^n)$ and let

$$\begin{tikzcd} X \arrow[d, "\mu"'] \arrow[dr, "\Phi_n"] & \\ \mathbb{P}^n \arrow[r, dashed, "\varphi_n"'] & Y_n \end{tikzcd}$$

be the resolution of indeterminacy of φ_n, with exceptional divisors $E_1, \dots, E_{n+1}$. Set $l_{ij} = \langle p_i, p_j \rangle$, let $\widetilde{l_{ij}}$ be its strict transform on X and let $\Pi_j = \langle p_i \mid i \neq j \rangle$. Then

1. $\varphi_{n|R}$ is the 2-Veronese embedding of $R = \mathbb{P}^{n-1}$ in $\mathbb{P}(H^0\mathcal{O}_R(2))^*$, in particular $V_{n-1,2} = \varphi_n(R) \subset Y_n$,
2. $\varphi_{n|\Pi_j} = \varphi_{n-1}$ for every $i \in \{1, \dots, n+1\}$, in particular Y_n contains $n+1$ copies of Y_{n-1},
3. Φ_n contracts $\tilde{l_{ij}}$ to a point $y_{ij} \in V_{n-1,2} \cap \operatorname{Sing} Y_n$,
4. $\Phi_n(E_i) = \langle y_{ij} \mid j \neq i \rangle$ for every $i \in \{1, \dots, n+1\}$.

Proof. 1. It is enough to observe that $\varphi_{n|R}$ is defined by

$$\varphi_n(r) = \mathcal{L}_{n,2}(1^{n+2})(p_1, \dots, p_{n+1}, r)_{|R} = \mathcal{L}_{n-1,2}(1)(r)$$

for every $r \in R$.

2. Assume for instance that $j = n+1$. $\varphi_{n|\Pi_j}$ is associated to the linear system $\mathcal{L}_{n,2}(1^{n+1})(p_1, \dots, p_{n+1})_{|\Pi_{n+1}}$, that is $\mathcal{L}_{n-1,2}(1^n)(p_1, \dots, p_n)$, and thus it coincides with φ_{n-1}.

3. Let $x \in l_{ij} \setminus \{p_1, \dots, p_{n+1}\}$. Then $l_{ij} \subset \operatorname{Bs} \mathcal{L}_{n,2}(1^{n+2})(p_1, \dots, p_{n+1}, x)$ and therefore $\varphi_n(x)$ is the hyperplane of quadrics of R passing through $R \cap l_{ij}$. Hence Φ_n contracts $\tilde{l_{ij}}$ to a point $y_{ij} = \varphi_n(R \cap l_{ij}) \in \varphi_n(R) = V_{n-1,2}$. Notice that $\operatorname{codim} l_{ij} = n-1 > 1$, so Y_n is singular at y_{ij}.

4. Since p_i has multiplicity 1 as a base point of $\mathcal{M}$, $E_i = \mathbb{P}^{n-1}$ is embedded with degree 1, so it is a linear space of dimension $n-1$. Moreover l_{ij} contains p_i, so $\tilde{l_{ij}}$ and E_i meet. Thus $\Phi_n(E_i) \ni y_{ij}$ for every $j \neq i$. Finally observe that the p_i's are general, hence the l_{ij}'s are general and therefore the same holds for the y_{ij}'s. This means that $\{y_{ij} \mid i \neq j\}$ are n general points of $\Phi_n(E_i) = \mathbb{P}^{n-1}$ and so they span it. $\square$

Quadrics containing the limit of $n+2$ simple points in $\mathbb{P}^n$ form a hyperplane in $\mathbb{P}(H^0\mathcal{O}_{\mathbb{P}^n}(2))$, but it is interesting to point out that if $x \in \mathbb{P}^n$ is general then that hyperplane is not defined by a tangent direction. Indeed, a tangent direction is an infinitely near point, so such hyperplane is defined by the containment of a point of R. This is equivalent to require $\varphi_n(x) \in \varphi_n(R) = V_{n-1,2}$. Since $\dim Y_n > \dim V_{n-1,2}$, in general the hyperplane $\varphi_n(x)$ is not defined by a tangent direction. However, the previous Lemma shows this can happen not only for $x \in R$, but also for other points, for instance when $x \in l_{ij}$.

If we fix $p_1 = [1, 0, \dots, 0], \dots, p_{n+1} = [0, \dots, 0, 1]$ the coordinate points and $R = \mathbb{P}^{n-1}$ the hyperplane of $\mathbb{P}^n$ defined by $x_n = x_0 + \ldots + x_{n-1}$, then it is possible to give explicit equations for φ_n. Let $q = [q_0, \dots, q_n] \in \mathbb{P}^n$ be a general point. A quadric $Q \subset \mathbb{P}^n$ containing $p_1, \dots, p_{n+1}$ has equation

$$\sum_{0 \leq i < j \leq n} a_{ij} x_i x_j = 0. \tag{2.2}$$

Since q is general, we may assume $q_0, q_1 \neq 0$. By imposing $q \in Q$ we get

$$a_{01} = - \sum_{\substack{0 \leq i < j \leq n \\ (i,j) \neq (0,1)}} \frac{q_i q_j}{q_0 q_1} a_{ij}. \tag{2.3}$$

By substituting the equation of R in equation 2.2, we see that $Q_{|R}$ is defined by

$$\begin{aligned}
0 &= \sum_{0\le i<j\le n} a_{ij}x_ix_j \\
&= \sum_{0\le i<j\le n-1} a_{ij}x_ix_j + \sum_{k=0}^{n-1} a_{kn}x_kx_n \\
&= \sum_{0\le i<j\le n-1} a_{ij}x_ix_j + \sum_{k=0}^{n-1} a_{kn}x_k(x_0+\ldots+x_{n-1}) \\
&= \sum_{0\le i<j\le n-1} (a_{ij}+a_{in}+a_{jn})x_ix_j + \sum_{k=0}^{n-1} a_{kn}x_k^2.
\end{aligned}$$

Fix the canonical basis $\{x_ix_j \mid 0\le i<j\le n\}$ on $H^0\mathcal{O}_R(2)$. If we call λ_{ij} the coefficient of x_ix_j, then the hyperplane $\varphi_n(q)$ satisfies

$$\begin{cases}
\lambda_{ii} = a_{in} & \text{for every } i\in\{0,\ldots,n-1\} \\
\lambda_{01} = a_{01}+a_{0n}+a_{1n} & \\
\lambda_{ij} = a_{ij}+a_{in}+a_{jn} & \text{for every } 0\le i<j\le n-1, (i,j)\ne(0,1).
\end{cases}$$

By elimination process we get

$$\begin{cases}
a_{in} = \lambda_{ii} & \text{for every } i\in\{0,\ldots,n-1\} \\
a_{ij} = \lambda_{ij}-\lambda_{ii}-\lambda_{jj} & \text{for every } 0\le i<j\le n-1, (i,j)\ne(0,1) \\
\lambda_{01} = a_{01}+a_{0n}+a_{1n} &
\end{cases}$$

By substituting equation 2.3 in the third line, we get the defining equation of the hyperplane of $H^0\mathcal{O}_R(2)$ corresponding to $\varphi_n(q)$ as

$$\begin{aligned}
\lambda_{01} &= a_{01}+a_{0n}+a_{1n} \\
&= a_{0n}+a_{1n} - \sum_{\substack{0\le i<j\le n\\(i,j)\ne(0,1)}} \frac{q_iq_j}{q_0q_1}a_{ij} \\
&= \left(1-\frac{q_0q_n}{q_0q_1}\right)a_{0n} + \left(1-\frac{q_1q_n}{q_0q_1}\right)a_{1n} - \sum_{\substack{0\le i<j\le n\\(i,j)\ne(0,1)}} \frac{q_iq_j}{q_0q_1}a_{ij} - \sum_{i=2}^{n-1}\frac{q_iq_n}{q_0q_1}a_{in} \\
&= \left(1-\frac{q_0q_n}{q_0q_1}\right)\lambda_{00} + \left(1-\frac{q_1q_n}{q_0q_1}\right)\lambda_{11} - \sum_{\substack{0\le i<j\le n\\(i,j)\ne(0,1)}} \frac{q_iq_j}{q_0q_1}(\lambda_{ij}-\lambda_{ii}-\lambda_{jj}) - \sum_{i=2}^{n-1}\frac{q_iq_n}{q_0q_1}\lambda_{ii} \\
&= \left(1-\frac{q_0q_n}{q_0q_1}+\sum_{j=2}^{n-1}\frac{q_0q_j}{q_0q_1}\right)\lambda_{00} + \left(1-\frac{q_1q_n}{q_0q_1}+\sum_{j=2}^{n-1}\frac{q_1q_j}{q_0q_1}\right)\lambda_{11} \\
&\quad + \left(-\frac{q_2q_n}{q_0q_1}+\sum_{\substack{0\le j\le n\\j\ne 2}}\frac{q_2q_j}{q_0q_1}\right)\lambda_{22} + \ldots + \left(-\frac{q_{n-1}q_n}{q_0q_1}+\sum_{j=0}^{n-2}\frac{q_{n-1}q_j}{q_0q_1}\right)\lambda_{n-1,n-1} \\
&\quad - \sum_{\substack{0\le i<j\le n\\(i,j)\ne(0,1)}} \frac{q_iq_j}{q_0q_1}\lambda_{ij}.
\end{aligned}$$

Up to multiply by q_0q_1, the coefficient of $\varphi_n(q)$ with respect to the (i,j) element of the canonical basis of $\mathbb{P}(H^0\mathcal{O}_R(2))^*$ is

$$\varphi_n(q)_{ij} = \begin{cases} -q_iq_j & \text{for } 0 \le i < j \le n \\ q_i\left(-q_n + \sum_{k\neq i,n} q_k\right) & \text{for } 0 \le i = j \le n. \end{cases}$$

Now, for $1 \le a < b \le n+1$, we want the coordinates of the singular points y_{ab}. Since $y_{ab} = \varphi_n(\langle p_a, p_b\rangle)$, we can compute its coordinates as $\varphi_n(0,\dots,0,1,0,\dots,0,1,0,\dots,0)$, where the 1's are in the $(a-1)$-th and $(b-1)$-th slots. First assume that $b = n+1$. Then the coordinate of $y_{a,n+1}$ corresponding to the element X_{ij} of the basis is

$$(y_{a,n+1})_{ij} = \begin{cases} -1 & \text{if } i = j = a-1 \\ 0 & \text{otherwise.} \end{cases}$$

In particular $y_{a,n+1}$ is a coordinate point of $\mathbb{P}(H^0\mathcal{O}_R(2))^*$, corresponding to the element $X_{a-1,a-1}$ of the basis. Assume now that $b \le n$. Then

$$(y_{ab})_{ij} = \begin{cases} -1 & \text{if } (i,j) = (a-1,b-1) \\ 1 & \text{if } i = j = a-1 \text{ or } i = j = b-1 \\ 0 & \text{otherwise.} \end{cases}$$

Once we have the y_{ab}'s, we can get the equations of the images of the $n+1$ exceptional divisors as $\Phi_n(E_i) = \langle y_{ij} \mid j \neq i\rangle$. This is useful to study Y_n for some small values of n.

Example 2.12. Let us consider the case $n = 3$. By Lemma 2.10, $Y_3 \subset \mathbb{P}^5$ is a rational threefold of degree 4, contained in at least a pencil of quadrics, so it follows that Y_3 is the complete intersection of two quadric hypersurfaces in $\mathbb{P}^5$. We can give an explicit expression of the map.

$$\varphi_3([a,b,c,d]) = [ab+ac-ad, -ab, -ac, ab+bc-bd, -bc, ac+bc-cd].$$

We compute

$$\begin{aligned} y_{12} &= \varphi_2([1,1,0,0]) = [1,-1,0,1,0,0], \\ y_{13} &= \varphi_2([1,0,1,0]) = [1,0,-1,0,0,1], \\ y_{14} &= \varphi_2([1,0,0,1]) = [-1,0,0,0,0,0], \\ y_{23} &= \varphi_2([0,1,1,0]) = [0,0,0,1,-1,1], \\ y_{24} &= \varphi_2([0,1,0,1]) = [0,0,0,-1,0,0], \\ y_{34} &= \varphi_2([0,0,1,1]) = [0,0,0,0,0,-1]. \end{aligned}$$

Therefore the ideal of the planes $\Phi_3(E_1),\dots,\Phi_3(E_4)$ are

$$\begin{aligned} \mathcal{I}_{\Phi_3(E_1)} &= (x_4, x_1+x_3, x_2+x_5), \\ \mathcal{I}_{\Phi_3(E_2)} &= (x_2, x_0+x_1, x_4+x_5), \\ \mathcal{I}_{\Phi_3(E_3)} &= (x_1, x_0+x_2, x_3+x_4) \text{ and} \\ \mathcal{I}_{\Phi_3(E_4)} &= (x_1, x_2, x_4). \end{aligned}$$

We want to find the equations of the two quadrics generating the ideal of Y_3. It is known that

$$\mathcal{I}_{V_{2,2}} = (x_0x_3 - x_1^2, x_0x_4 - x_1x_2, x_0x_5 - x_2^2, x_1x_4 - x_2x_3, x_1x_5 - x_2x_4, x_3x_5 - x_4^2)$$

is the ideal of $V_{2,2} \subset Y_3$. The quadrics $Q_1 = (x_0x_4 - x_1x_2 + x_1x_4 - x_2x_3 = 0)$ and $Q_2 = (x_1x_4 - x_2x_3 + x_1x_5 - x_2x_4 = 0)$ contain $V_{2,2}$ and all the planes $\Phi_3(E_i)$. An explicit computation shows that $\varphi_3(p) \in Q_1 \cap Q_2$ for every $p \in \mathbb{P}^3$, hence Y_3 is the complete intersection of Q_1 and Q_2. By computing the determinant, we get $\det(\lambda Q_1 + \mu Q_2) = \lambda^2\mu^2(\lambda+\mu)^2$, so there are exactly three singular quadrics in the pencil defining Y_3, and they all have rank 4. If we set Q_3 to be the quadric defined by the sum of the two polynomials defining Q_1 and Q_2, then the three rank 4 quadrics are Q_1, singular along the line $\langle y_{12}, y_{34}\rangle$, Q_2, singular along the line $\langle y_{14}, y_{23}\rangle$, and Q_3, singular along the line $\langle y_{13}, y_{24}\rangle$.

Although Y_3 is a complete intersection, in general the same does not hold for Y_n. By Lemma 2.10, for instance, $\deg Y_4 = 2^4 - 5 = 11$ is a prime number, so Y_4 can not be a complete intersection. However, quadrics containing Y_4 play a central role in understanding its equations.

Example 2.13. Let us work out the case $n = 4$. By Lemma 2.10, $Y_4 \subset \mathbb{P}^9$ is a degree 11 rational fourfold, and it is contained in a system of quadrics of dimension at least 10. By the same argument as in Example 2.12, we can explicitly write the ideals $\mathcal{I}_{V_{3,2}}, \mathcal{I}_{\Phi_4(E_1)}, \ldots, \mathcal{I}_{\Phi_4(E_5)}$. By using the software Macaulay2 [43], we observe that the intersection ideal has exactly 10 degree two generators. The ideal of those 10 quadrics defines a variety of dimension 4 and degree 11, thus Y_4 is defined by the 10 quadrics containing $V_{3,2} \cup \Phi_4(E_1) \cup \ldots \cup \Phi_4(E_5)$.

In a similar way, $Y_5 \subset \mathbb{P}^{14}$ is a degree 26 rational 5-fold, contained in a system of quadrics of dimension at least 30. The ideal $\mathcal{I}_{V_{4,2}} \cap \mathcal{I}_{\Phi_5(E_1)} \cap \ldots \cap \mathcal{I}_{\Phi_5(E_6)}$ has 30 degree 2 generators. Those 30 quadric define a variety of dimension 5 and degree 26, so Y_5 is defined by those 30 quadrics.

Proposition 2.9 characterizes the quadrics in the limit ideal. In order to fully describe the limit Z_0 of $n+2$ collapsing simple points, it would be enough to show that it is defined by quadrics. Unfortunately this is not true in general.

Example 2.14. Consider the collision of 4 simple points in the affine plane. In the notations of Construction 1.10, we consider the sections

$$\begin{aligned}
\sigma_1(t) &= (0,0),\\
\sigma_2(t) &= (t,0),\\
\sigma_3(t) &= (0,t),\\
\sigma_4(t) &= (2t, 3t^2 + t).
\end{aligned}$$

Then an explicit computation with the software Macaulay2 shows that $H^0\mathcal{I}_{Z_0,\mathbb{A}^2}$ is not generated by quadrics.

Despite the previous Example, the ideal of Z_0 is always generated in low degree. Moreover, we can completely describe it if we make a mild assumption on the sections σ_i.

Proposition 2.15. Let V be a n-dimensional variety, and $q \in V$ a smooth point. Let $U \cong \mathbb{A}^n$ be an open affine neighbourhood of q in S. Let

$$\sigma_0, \dots, \sigma_{n+1} : \Delta \to U \times \Delta$$

be sections as in Construction 1.10, and let Z_0 be the corresponding limit of $n+2$ collapsing simple points onto q.

1. $\mathcal{I}_{Z_0}$ is generated in degree at most 3.
2. If $\sigma_{n+1}(t)$ is the barycenter of $\sigma_0(t), \dots, \sigma_n(t)$ for every $t \in \Delta \setminus \{0\}$, then $\mathcal{I}_{Z_0}$ is generated by quadric.
3. If $V = \mathbb{P}^n$, then $\mathcal{I}_{Z_0}$ is generated by quadrics. In particular, Proposition 2.9 describes the limit of $n+2$ collapsing points in $\mathbb{P}^n$.

Proof. 1. In order to prove that $\mathcal{I}_{Z_0}$ is always generated by cubics, it is enough to show that $h_{Z_0}(2) = \deg Z_0$. Let $\mu : X = \mathrm{Bl}_q A \to A$ be the blow-up of the limit point. Recall that $X_0 = W \cup V$, where $W = \mathbb{P}^n$ is the exceptional divisor and $V = \mathrm{Bl}_q A_0$. Let $[H]$ be the hyperplane of $\mathbb{P}(H^0\mathcal{O}_R(2))$ defining $\mathcal{I}_{Z_0}(2)$. A quadric $Q \subset A_0$ contains the limit Z_0 if and only if $\tilde{Q}_{|R} \in H$. Since $\mathrm{mult}\, Z_0 = 2$, Q is singular and therefore a cone. $\tilde{Q}_{|R}$ is a quadric of $R = \mathbb{P}^{n-1}$, hence it is determined by $\binom{n+1}{2} - 1$ points of R, that is by $\binom{n+1}{2} - 1$ tangent direction to the limit point. Those tangent directions are $\binom{n+1}{2} - 1$ lines of Q through q, and they cut the same number of points on a general hyperplane L of $\mathbb{A}^n$, defining a quadric D of L. So $Q = C_q(D)$ is uniquely determined by q and $\tilde{Q}_{|R}$, and $\mathrm{h}^0\mathcal{I}_{Z_0}(2) = \dim H = \mathrm{h}^0\mathcal{O}_R(2) - 1 = \binom{n+1}{2} - 1$. Thus

$$h_{Z_0}(2) = \mathrm{h}^0\mathcal{O}_{\mathbb{P}^n}(2) - \mathrm{h}^0\mathcal{I}_{Z_0}(2) = \binom{n+2}{2} - \binom{n+1}{2} + 1 = n+2 = \deg Z_0.$$

2. Up to an isomorphism of U, it is not restrictive to assume that

$$\begin{aligned} \sigma_0(t) &= (0, \dots, 0), \\ \sigma_1(t) &= (f_1(t), 0, \dots, 0), \\ &\vdots \\ \sigma_n(t) &= (0, \dots, 0, f_n(t)). \end{aligned}$$

Then $\sigma_{n+1}(t) = (f_1(t), \dots, f_n(t))$ by hypothesis. In the general fiber A_t, the ideals of the $n+2$ points are

$$\begin{aligned} \mathcal{I}_0(t) &= (x_1, \dots, x_n), \\ \mathcal{I}_1(t) &= (x_1 - f_1(t), x_2, \dots, x_n), \\ &\vdots \\ \mathcal{I}_n(t) &= (x_1, \dots, x_{n-1}, x_n - f_n(t)) \\ \mathcal{I}_{n+1}(t) &= (x_1 - f_1(t), \dots, x_n - f_n(t)). \end{aligned}$$

The set of $n+2$ points has ideal $\mathcal{I}(t) = \mathcal{I}_0(t) \cap \mathcal{I}_1(t) \cap \ldots \cap \mathcal{I}_n(t)$. Notice that $x_i(x_i - f_i(t)) \in \mathcal{I}(t)$ for every i and for every t. Then the limit ideal

$\mathcal{I}(0) = \mathcal{I}_{Z_0}$ contains $x_0^2, \dots, x_{n+1}^2$. By Proposition 2.9, $\mathcal{I}_{Z_0}(2)$ contains all quadrics of the basis but one, and we checked that no square monomial is absent. It is easy to observe that such a set of quadrics generates any cubic.

3. In $\mathbb{P}^n$ every $(n+2)$-tuple of points is equivalent, so we can assume that $\sigma_{n+1}(t)$ is the barycenter of $\sigma_0(t), \dots, \sigma_n(t)$. □

Before moving to higher multiplicity collisions, let us remark that the problem of collisions of $h \geq n+3$ simple points in $\mathbb{P}^n$ is still open. It is possible to argue as in the case of $n+2$ points, but it is much more difficult to understand the rational map and the associated linear system. Nevertheless, Ciliberto-Miranda ([30]) and Nesci ([62]) provided partial answers for the case $n = 2$.

2.2 Double points

In this section we assume that all the collapsing points have multiplicity 2. First we can easily generalize Example 2.4.

Proposition 2.16. Let $m \geq 2$, $(n, m) \notin \{(2,3), (2,5), (4,4), (4,5)\}$. Define $h = \frac{\binom{n+m-1}{n}}{n+1}$. If $h \in \mathbb{N}$, then the limit of h colliding double points in $\mathbb{P}^n$ is an m-ple point.

Proof. First we check that

$$\operatorname{vdim} \mathcal{L}_{n,m-1}(2^h) = \binom{n+m-1}{n} - 1 - (n+1)h = -1,$$

while

$$\dim \mathcal{L}_{n,m}(2^h) \geq \operatorname{vdim} \mathcal{L}_{n,m}(2^h) = \binom{n+m}{n} - 1 - (n+1)h \geq 0.$$

By our numerical assumption, together with Theorem 1.5, $\mathcal{L}_{n,m-1}(2^h)$ is non-special and therefore empty. By Proposition 2.3, Z_0 has multiplicity m. Then it is enough to check that the degree of an m-ple point is the same as $\deg Z_1$ and conclude by Proposition 1.2. □

As we already noticed, in most cases the limit is not just a point with multiplicity. As Example 2.5 shows, once we understand the minimum degree k of a divisor containing Z_1, we need informations on the base locus of such divisors.

Dealing with double points, it is convenient to work in the case $h > n$. Indeed, under this assumption we have $\operatorname{mult} Z_0 = 3$, at least for n big enough, and the base locus of cubics with assigned double points is very well understood. On the other hand, $h \leq n$ yields $\operatorname{mult} Z_0 = 2$, and $\mathcal{L}_{n,2}(2^h)$ has a nonreduced base locus and it is more difficult to describe the conditions it gives to the limit linear system.

First we need a technical result.

Lemma 2.17. Let $n \geq 2$, let $A = \{a_1, \dots, a_l\}$ be a set of l general points in $\mathbb{P}^n$ and let R be a hyperplane such that $A \cap R = \varnothing$. Let

$$B = \{p_{ij} := \langle a_i, a_j \rangle \cap R \mid 1 \leq i < j \leq l\}.$$

Assume that $l \leq n+2$. Then $\mathcal{L}_{n-1,2}(B)$ and $\mathcal{L}_{n-1,3}(B)$ are non special, that is the points of B impose independent conditions to quadrics and cubics.

Proof. It is enough to prove the claim for $l = n+1$ for quadrics and $l = n+2$ for cubics.

First assume that $l = n+1$. In this case B is a set of $\binom{n+1}{2}$ points, therefore it is enough to prove that there are no quadrics containing B. We prove the claim by induction on n. For $n = 2$ it is easy. Let $\Pi_i = \langle a_1, \dots, \hat{a_i}, \dots, a_{n+1} \rangle \cap R$. By induction, there are no quadrics in Π_i containing $\Pi_i \cap B$. Therefore any quadric containing B has to contain the hyperplanes Π_i for any i. This is enough to conclude, because $n+1 \geq 3$.

Now assume that $l = n+2$. We prove that the points of B are general for cubics by induction on n. It is easy to check that the thesis holds for $n = 2$, so we assume $n \geq 3$. Specialize $a_1, \dots, a_{n+1}$ on a general hyperplane $L = \mathbb{P}^{n-1}$. Define

$$B_1 = \{p_{ij} \mid 1 \leq i < j \leq n+1\} \text{ and } B_2 = \{p_{1,n+2}, \dots, p_{n+1,n+2}\}.$$

Observe that the points of B_2 are in general position on R, and $B = B_1 \cup B_2$. Let $H = L \cap R = \mathbb{P}^{n-2}$. Castelnuovo exact sequence reads

$$0 \to \mathcal{I}_{B_2,R}(2) \to \mathcal{I}_{B,R}(3) \to \mathcal{I}_{B_1,H}(3) \to 0.$$

Since B_2 is a set of general points of R, $\mathrm{h}^1\mathcal{I}_{B_2,R}(2) = 0$. If we set

$$A_1 = \{a_1, \dots, a_{n+1}\},$$

then A_1 is a set of general points in L and H is an hyperplane of L such that $A_1 \cap H = \varnothing$. By induction hypothesis, $\mathrm{h}^1\mathcal{I}_{B_1,H}(3) = 0$. Hence $\mathrm{h}^1\mathcal{I}_{B,R}(3) = 0$ and so B imposes independent conditions on cubics of R. $\square$

Remark 2.18. Note that even if B imposes independent conditions, the points of B are not in linear general position. Indeed there are $\binom{n+1}{t}$ linear spaces of dimension $t-2$ each containing $\binom{t}{2}$ points of B. For every choice of t points of A, their span is a $\mathbb{P}^{t-1}$, so the corresponding $\binom{t}{2}$ points of B lie on a $\mathbb{P}^{t-2}$.

The next two Propositions completely solve the cases $h = n+1$ and $h = n+2$.

Proposition 2.19. The limit of $n+1$ collapsing double points in $\mathbb{P}^n$ is a triple point with $\binom{n+1}{2}$ tangent directions. The infinitely near simple points are in the special position described by Remark 2.18.

Proof. There are no quadrics singular along $n+1$ general points of $\mathbb{P}^n$, while $\mathcal{L}_{n,3}(2^{n+1})$ has non negative dimension, so the multiplicity of the limit scheme is 3 by Proposition 2.3. Note that the degree of a triple point is $\binom{n+2}{2}$.

The base locus of cubics in $\mathbb{P}^n$ with $n+1$ general double points consists of the $\binom{n+1}{2}$ lines joining the points. Each of them cuts a simple point on $R \cong \mathbb{P}^{n-1}$.

By Lemma 2.17 these points impose independent conditions to cubics. A simple computation shows that

$$\binom{n+2}{2} + \binom{n+1}{2} = (n+1)^2 = \deg Z_0.$$

Hence the triple point, together with $\binom{n+1}{2}$ tangent directions, is the limit scheme. $\square$

Proposition 2.20. Let Z_0 be the limit of $n+2$ collapsing double points.

1. If $n = 2$, then Z_0 is a 4-ple point with the involution described by Ciliberto-Miranda in [30].
2. If $n = 3$, then Z_0 is a 4-ple point.
3. If $n \geq 4$, then Z_0 is a triple point with $\binom{n+2}{2}$ tangent directions. In this case the infinitely near simple points are in the special position described by Remark 2.18.

Proof. For $n = 2$, see [30, Proposition 3.1].

The length of Z_1 is $(n+1)(n+2)$. For $n = 3$, there are no cubics singular at 5 general points, so Z_0 contains a fourtuple point. Since $\deg Z_0 = 20$, and since a 4-ple point in $\mathbb{P}^3$ has degree 20, we conclude by Proposition 1.2.

For $n \geq 4$, the linear system $\mathcal{L}_{n,3}(2^{n+2})$ has positive dimension, hence $\operatorname{mult} Z_0 = 3$ by Proposition 2.3. It is easy to see, via reducible cubics, that the base locus of $\mathcal{L}_{n,3}(2^{n+2})$ consists of $\binom{n+2}{2}$ lines joining the points. Each line cuts a simple point on $R = \mathbb{P}^{n-1}$, and they impose independent conditions by Lemma 2.17. In order to conclude we check

$$\binom{n+2}{2} + \binom{n+2}{2} = (n+1)(n+2) = \deg Z_0.$$

Again the thesis follows by Proposition 1.2. $\square$

Despite the previous results, the limit scheme can be more complicated than a fat point with a bunch of infinitely near points. Such problems may occur even in low dimension and when all the multiplicities are 2.

Example 2.21. Exceptions of Theorem 1.5 always yield a 0-dimensional linear system. In these cases it is easy to compute the multiplicity, but we can not argue as before to describe the limit. For instance, the limit of 5 colliding double points in $\mathbb{P}^2$ is described in [30, Proposition 3.1] as a fourtuple point with a pair of infinitely near tacnodal points.

We could try to apply the argument of Propositions 2.19 and 2.20 to an higher number of colliding double points. Anyway, we can not expect the same proof to work. One of the reasons is that Lemma 2.17 does not hold for $l \geq n+3$

Example 2.22. Consider a set $A = \{a_1, \ldots, a_7\} \subset \mathbb{P}^4$ of general points. As in the setting of Lemma 2.17, let R be a hyperplane such that $A \cap R = \varnothing$ and

$$B = \{p_{ij} := \langle a_i, a_j \rangle \cap R \mid 1 \leq i < j \leq 7\}.$$

Indeed in that case $\sharp B = 21$ and $\mathrm{h}^0\mathcal{O}_R(3) = 20$. We know there is exactly one cubic C singular at $a_1, \ldots, a_7$. C contains all the lines joining pairs of points of A, so in particular $C_{|R} \supset B$. Moreover, consider the Castelnuovo exact sequence

$$0 \to \mathcal{L}_{4,2}(2^7) \to \mathcal{L}_{4,3}(2^7) \to \mathcal{I}_{B,R}(3) \to 0.$$

Since $\mathcal{L}_{4,2}(2^7)$ has no global sections, the restriction $\mathcal{L}_{4,3}(2^7) \to \mathcal{I}_{B,R}(3)$ is injective and therefore $\mathcal{L}_{4,3}(2^7) \subset \mathcal{I}_{B,R}(3)$. This means there is at least one cubic of R containing B. Since $\mathrm{h}^0\mathcal{O}_{\mathbb{P}^3}(3) = 20$, the 21 points of B impose at most 19 independent conditions on cubics of R. An easy software-aided computation shows that B actually imposes exactly 19 independent conditions.

More generally, let Z_1 be a scheme of $n+3$ double points, with $n \geq 5$. Observe that $\deg Z_1 = (n+1)(n+3)$ and $\operatorname{mult} Z_0 = 3$. It is easy to see that $\operatorname{Bs}\mathcal{L}_{n,3}(2^{n+3})$ consists of the double points and of the $\binom{n+3}{2}$ lines joining the pair of points. Then we have $\binom{n+3}{2}$ simple points infinitely near to the limit triple point. However, these simple points are not independent. Indeed, if they were, $\deg Z_0 \geq \binom{n+2}{2} + \binom{n+3}{2} = n^2 + 4n + 4 = 1 + \deg Z_1$. Hence those $\binom{n+3}{2}$ simple points impose at most $\binom{n+3}{2} - 1$ conditions on Z_0. On the other hand, at least $\binom{n+2}{2}$ of the simple points impose independent conditions by Lemma 2.17.

Remark 2.23. Let Z be an m-ple point with an infinitely near simple point, and let l be the line through Z corresponding to the infinitely near point. The restriction of Z to a general line is an m-ple point, while $Z_{|l}$ has multiplicity $m+1$. This suggest a possible description of the limit of $n+k$ collapsing double points. Assume that $\operatorname{mult} Z_0 = 3$, and let $l_1, \ldots, l_{\binom{n+k}{2}}$ be the base lines, all passing through the limit point q. Let S_i^4 be the multiplicity 4 subscheme of l_i supported at q. We know that Z_0 contains the union of the S_i^4's, and we conjecture that they coincide. Now we want to precisely formulate the problem and to provide a solution for small k.

Union of fourtuple schemes

Definition 2.24. Let $n, m \geq 2$, and let $l_1, \ldots, l_t$ be t lines in $\mathbb{A}^n$ meeting at the origin. Let S_i^m be the 0-dimensional degree m subscheme of l_i supported at the origin, and let $\mathcal{I}_{n,S_i^m}$ be the ideal defining S_i^m is $\mathbb{A}^n$. Define $Z_n(l_1, \ldots, l_t)$ to be the union scheme defined by the ideal

$$\mathcal{I}_n(l_1, \ldots, l_t) = \mathcal{I}_{n,S_1^m} \cap \ldots \cap \mathcal{I}_{n,S_t^m}.$$

If $l_1, \ldots, l_t$ are general lines and $m = 4$, then we define

$$Z_{n,t} = Z_n(l_1, \ldots, l_t) \text{ and } \mathcal{I}_{n,t} = \mathcal{I}_n(l_1, \ldots, l_t).$$

When $\operatorname{mult} Z_{n,t} = 3$ we can think of this scheme as a triple point with t infinitely near simple points, representing the directions corresponding to $l_1, \ldots, l_t$.

Remark 2.25. Consider $n+k$ colliding double points in $\mathbb{A}^n$ and assume the limit has multiplicity 3. Then the limit triple point has $\binom{n+k}{2}$ infinitely near simple points, in special position, giving possibly dependent conditions on cubic.

Nevertheless, the restriction of the limit scheme to one of the $\binom{n+k}{2}$ corresponding lines $l_1, \ldots, l_{\binom{n+k}{2}}$ has degree strictly greater than 3. In particular the limit scheme contains $Z_n\left(l_1, \ldots, l_{\binom{n+k}{2}}\right)$. So if we prove that they have the same degree, then we get an explicit description of the limit scheme.

We aim to identify the limit of a bunch of colliding double points with a scheme of the form $Z(l_1, \ldots, l_t)$. For this reason, our next task is to study such schemes. First we compute the multiplicity of the scheme $Z_n(l_1, \ldots, l_t)$.

Lemma 2.26. Let $R = \mathbb{A}^{n-1}$ be a general hyperplane in $\mathbb{A}^n$, and $p_i = l_i \cap R$. Define $P = \{p_1, \ldots, p_t\}$ and set

$$k = \min\{m \in \mathbb{N} \mid \mathcal{I}_{P,R}(m) \neq 0\}.$$

Then $\operatorname{mult} Z(l_1, \ldots, l_t) = \min(4, k)$.

Proof. First note that $\operatorname{mult} Z_n(l_1, \ldots, l_t)$ is nondecreasing with respect to t. Moreover, $\operatorname{mult} Z_n(l_1, \ldots, l_t) \leq 4$ by construction. Indeed, once multiplicity 4 is reached, the restriction to any line has degree at least 4, so by adding another S_i^4 we do not change anything. Now let $D \subset R$ be a degree m divisor containing $p_1, \ldots, p_t$. The cone C over D with vertex the origin is a degree m divisor in $\mathbb{A}^n$ containing $l_1, \ldots, l_t$, and therefore $C \supset S_1^4 \cup \ldots \cup S_t^4$. Hence the ideal of $Z_n(l_1, \ldots, l_t)$ contains a generator of degree m and so $\operatorname{mult} Z_n(l_1, \ldots, l_t) \leq m$. This implies $\operatorname{mult} Z_n(l_1, \ldots, l_t) \leq \min(4, k)$.

On the other hand, if $\operatorname{mult} Z_n(l_1, \ldots, l_t) = 4 \geq \min(4, k)$, then there is nothing else to prove. Suppose that $m := \operatorname{mult} Z_n(l_1, \ldots, l_t) \in \{1, 2, 3\}$. Then it is contained in a degree m divisor $C \subset \mathbb{A}^n$. Since it has an m-ple point, C is a cone. Moreover the restriction of $Z_n(l_1, \ldots, l_t)$ to each l_i has degree $4 > m$ so C contains each l_i, and in particular $C_{|R}$ is a degree m divisor in R containing $p_1, \ldots, p_t$. □

Corollary 2.27. Let $t \in \mathbb{N}$ and let $R = \mathbb{A}^{n-1}$ be a general hyperplane in $\mathbb{A}^n$. Set

$$k = \min\{m \in \mathbb{N} \mid \mathrm{h}^0 \mathcal{O}_R(m) > t\}.$$

If $l_1, \ldots, l_t$ are general lines, then $\operatorname{mult} Z_{n,t} = \min(4, k)$.

Proof. Apply Lemma 2.26 in the case $p_1, \ldots, p_t \in R$ are general. □

Now we want to determine the length of $Z_n(l_1, \ldots, l_t)$. The next Lemma provides a way to compute it inductively.

Lemma 2.28. Let $n \geq 2$. Then

1. $\deg Z_n(l_1) = 4$,
2. $\deg Z_n(l_1, \ldots, l_t, l_{t+1}) = \deg Z_n(l_1, \ldots, l_t) + 4 - \deg\left(Z_n(l_1, \ldots, l_t)_{|l_{t+1}}\right)$,
3. $\deg Z_{n,t+1} = \deg Z_{n,t} + 4 - \operatorname{mult} Z_{n,t}$.

Proof. 1. The length of $Z_n(l_1)$ does not depend on the immersion. Regarding $Z_n(l_1) = S_1^4$ as a divisor in $l_1 = \mathbb{P}^1$, it has degree 4 by construction.

2. Let $\mu = \deg \left(Z_n(l_1, \ldots, l_t)_{|l_{t+1}} \right)$. Of course $Z_n(l_1, \ldots, l_t) \supset S^{\mu}_{t+1}$, so

$$Z_n(l_1, \ldots, l_t) = S^4_1 \cup \ldots \cup S^4_t = S^4_1 \cup \ldots \cup S^4_t \cup S^{\mu}_{t+1}.$$

Hence the difference $\deg Z_n\left(l_1, \ldots, l_t, l_{t+1}\right) - \deg Z_n(l_1, \ldots, l_t)$ coincides with the difference $\deg S^4_{t+1} - \deg S^{\mu}_{t+1} = 4 - \mu$.

3. When $l_1, \ldots, l_t, l_{t+1}$ are general, the restriction of $Z_{n,t}$ to l_{t+1} has degree equal to $\operatorname{mult} Z_{n,t}$, so it is enough to apply (2).

□

Example 2.29. Corollary 2.27 and Lemma 2.28 allow us to compute multiplicity and degree of the scheme $Z_{n,t}$ for every n and t. As an example, here is the table for $n = 2$.

t	$\deg Z_{2,t}$	$\operatorname{mult} Z_{2,t}$
1	4	1
2	7	2
3	9	3
$t \geq 4$	10	4

Now we consider what happens when the lines are not general, in particular when they have the configuration described in Remark 2.18.

Definition 2.30. Let $\{l_{ij} \mid 1 \leq i < j \leq m\}$ be a set of $\binom{m}{2}$ lines in $\mathbb{A}^n$ meeting at the origin, such that l_{ab}, l_{bc} and l_{ac} lie on the same plane for every $\{1 \leq a < b < c \leq m\}$. Define $\tilde{Z}_{n,\binom{m}{2}} = Z_n(l_{ij} \mid 1 \leq i < j \leq m)$.

Remark 2.31. Let $n, m \geq 2$. The following simple observations will be useful.

1. $Z_{2,\binom{m}{2}} = \tilde{Z}_{2,\binom{m}{2}}$.
2. $\tilde{Z}_{n,1} = Z_{2,1}$ and $\tilde{Z}_{n,3} = Z_{2,3}$.
3. More generally, if $n \geq m$, then $\langle l_1, \ldots, l_{\binom{m}{2}} \rangle = \mathbb{A}^{m-1}$. This implies $\operatorname{mult} \tilde{Z}_{n,\binom{m}{2}} = 1$ and $\tilde{Z}_{n,\binom{m}{2}} = \tilde{Z}_{m-1,\binom{m}{2}}$.

We are now ready to compute the multiplicity and degree of $\tilde{Z}_{n,\binom{m}{2}}$. By Remark 2.31, we know the multiplicity and degree of $\tilde{Z}_{2,\binom{m}{2}}$ from Lemma 2.28. Now we tackle the cases $n = 3$ and $n = 4$.

Lemma 2.32. The next table shows the values of $\deg \tilde{Z}_{3,\binom{m}{2}}$ and $\operatorname{mult} \tilde{Z}_{3,\binom{m}{2}}$.

m	t	$\deg \tilde{Z}_{3,t}$	$\operatorname{mult} \tilde{Z}_{3,t}$
2	1	4	1
3	3	9	1
4	6	16	3
$m \geq 5$	$t \geq 10$	20	4

Degrees and multiplicities of $\tilde{Z}_{4,\binom{m}{2}}$ are presented in the following one.

m	t	$\deg \tilde{Z}_{4,t}$	$\operatorname{mult} \tilde{Z}_{4,t}$
2	1	4	1
3	3	9	1
4	6	16	1
5	10	25	3
6	15	30	3
7	21	34	3
$m \geq 8$	$t \geq 28$	35	4

Proof. We already observed that $\tilde{Z}_{n,1} = Z_{2,1}$ has multiplicity 1 and degree 4 for every n.

If $(n, m) = (3, 3)$, then we have 3 coplanar lines meeting at the origin. Let R be a general plane and let $p_{ij} = R \cap l_{ij}$. Then p_{12}, p_{13}, p_{23} are collinear, hence $\operatorname{mult} \tilde{Z}_{3,3} = 1$ by Lemma 2.26. To compute the degree, observe that $\tilde{Z}_{3,3} = Z_{2,3}$, so $\deg \tilde{Z}_{3,3} = 9$. By Remark 2.31.3, $\tilde{Z}_{4,3} = \tilde{Z}_{3,3}$, so the first two lines of both tables are filled.

Consider now $(n, m) = (3, 4)$, with 6 lines through the origin. Let R be a general plane. Then $\{p_{ij} \mid 1 \leq i < j \leq 4\}$ are 6 points in $\mathbb{P}^2$ in the special position described by Remark 2.18. By Lemma 2.17, they lie on a cubic but they do not lie on a conic, hence $\operatorname{mult} \tilde{Z}_{3,6} = 3$ by Lemma 2.26. To compute the degree, observe that $\operatorname{mult} Z_3(l_{12}, l_{13}, l_{23}) = 1$, so

$$\deg Z_3(l_{12}, l_{13}, l_{23}, l_{14}) = 3 + \deg Z_3(l_{12}, l_{13}, l_{23}) = 12$$

by Lemma 2.28. But now $\operatorname{mult} Z_3(l_{12}, l_{13}, l_{23}, l_{14}) = 2$, so

$$\begin{aligned} \deg \tilde{Z}_{3,6} &= \deg Z_3(l_{12}, l_{13}, l_{23}, l_{14}, l_{24}, l_{34}) \\ &= 2 + \deg Z_3(l_{12}, l_{13}, l_{23}, l_{14}, l_{24}) \\ &= 2 + 2 + \deg Z_3(l_{12}, l_{13}, l_{23}, l_{14}) = 16. \end{aligned}$$

In a similar way we deal with the case $m = 5$. By Lemma 2.17, the 10 points lie on a quartic but they do not lie on a cubic, hence $\operatorname{mult} \tilde{Z}_{3,10} = 4$. More precisely, $\tilde{Z}_{10}$ is the whole fourtuple point and so $\tilde{Z}_{\binom{m}{2}}$ is the fourtuple point for every $m \geq 5$. In particular, it has degree 20.

Now we consider $n = 4$. By Remark 2.31.3, $\deg \tilde{Z}_{4,6} = \deg \tilde{Z}_{3,6} = 16$ and $\operatorname{mult} \tilde{Z}_{4,3} = \operatorname{mult} \tilde{Z}_{4,6} = 1$, so the third line is filled. Moreover by Lemma 2.26 and Lemma 2.17, $\operatorname{mult} \tilde{Z}_{4,10} = \operatorname{mult} \tilde{Z}_{4,15} = 3$. The case $m = 7$ is an exception of Theorem 1.5. Notice that $\tilde{Z}_{4,21}$ is a subscheme of the limit of 7 colliding double points in $\mathbb{P}^4$, which has multiplicity 3 by Proposition 2.3. Therefore $\operatorname{mult} \tilde{Z}_{4,21} = 3$.

Note that $\tilde{Z}_{4,10}$ is obtained from $\tilde{Z}_{4,6} \subset \mathbb{A}^3 = H$ by adding $S^4_{15}, \ldots, S^4_{45}$. When we add S^4_{15}, by Lemma 2.28 the degree increases by 3 because $S^4_{15} \not\subset H$; the resulting scheme $\tilde{Z}_{4,6} \cup S^4_{15}$ has multiplicity 2. When we add S^4_{25}, the degree increases by 2 unless $\deg(\tilde{Z}_{4,6} \cup S^4_{15})_{|l_{25}} \geq 2$, i.e. unless $\tilde{Z}_{4,6} \cup S^4_{15}$ contains the infinitely near point corresponding to l_{25}. This would imply that all quadrics of R through $\{p_{ij} \mid 1 \leq i < j \leq 4\} \cup \{p_{15}\}$ contain p_{25}, and this is false by Lemma 2.17. Repeating this argument we see that all the lines increase the degree by 2 and so $\deg \tilde{Z}_{4,10} = 16 + 3 + 2 + 2 + 2 = 25$.

The same observations allow us to conclude that adding each S^4_{i6} increases the degree of $\tilde{Z}_{10}$ by 1 and then $\deg \tilde{Z}_{4,15} = 25{+}5 = 30$. Now we focus on $m = 7$.

By Example 2.22, the 21 points impose exactly 19 independent conditions. This means that each S^4_{i7} we add increases the degree by 1, except for the last two, so $\deg \tilde{Z}_{4,21} = 30 + 4 = 34$. Finally, when we add S^4_{18} the degree jumps to 35, so $\tilde{Z}_{4,28}$ is the whole fourtuple point. □

Remark 2.33. If we look at $\tilde{Z}_{3,6}$ and $\tilde{Z}_{3,10}$, we see that their multiplicities and degrees are consistent with the cases of 4 and 5 collapsing double points in $\mathbb{A}^3$.

In the same way, the numbers we found about $\tilde{Z}_{4,10}$ and $\tilde{Z}_{4,15}$ are consistent with the case of 5 and 6 colliding double points in $\mathbb{A}^4$.

We will try now to find a general statement about the degree and the multiplicity of $\tilde{Z}_{n,\binom{m}{2}}$. The situation is easy when $m \leq n$.

Proposition 2.34. Let $n \geq 2$. If $3 \leq m \leq n$, then $\operatorname{mult} \tilde{Z}_{n,\binom{m}{2}} = 1$ and $\deg \tilde{Z}_{n,\binom{m}{2}} = m^2$.

Proof. If $m \leq n$, then $\operatorname{mult} \tilde{Z}_{n,\binom{m}{2}} = 1$ by Remark 2.31.3.

We prove the statement about the degree by induction on m. We saw that $\deg \tilde{Z}_{n,3} = 9$. Let us assume $\deg \tilde{Z}_{n,\binom{m}{2}} = m^2$ and let us compute $\deg \tilde{Z}_{n,\binom{m+1}{2}}$. $\tilde{Z}_{n,\binom{m+1}{2}}$ is obtained from $\tilde{Z}_{n,\binom{m}{2}} \subset \mathbb{A}^m = H$ by adding $S^4_{1,m+1}, \dots, S^4_{m,m+1}$. Observe that $S^4_{1,m+1} \not\subset H$, so it increases the degree by 3; the resulting scheme is contained in some $P = \mathbb{A}^{m+1}$, and by adding $S^4_{1,m+1}, \dots, S^4_{m-1,m+1}$, we remain inside P. As a subscheme of P, $\tilde{Z}_{n,\binom{m+1}{2}}$ has multiplicity 2, because there are only $\binom{m}{2} + m - 1 < \mathrm{h}^0 \mathcal{O}_{\mathbb{A}^{n-1}}(2)$ lines. But we know that, even if they are in special position, they are general for quadrics, so each new addition of $S^4_{2,m+1}, \dots, S^4_{m,m+1}$ increases the degree by $4 - 2 = 2$. Hence

$$\deg \tilde{Z}_{n,\binom{m+1}{2}} = \deg \tilde{Z}_{n,\binom{m}{2}} + 3 + 2(m-1) = m^2 + 2m + 1 = (m+1)^2,$$

and therefore the thesis holds. □

Before we move to the more interesting case $m > n \geq 5$, we need some technical results. We already observed that Lemma 2.17 does not hold in the case of more than $n + 2$ points in $\mathbb{P}^n$, so our next goal is to understand what happens with larger numbers of points. In particular, we are looking for a suitable generalization of Lemma 2.17.

Lemma 2.35. For $r \in \mathbb{N}$, let $A_k = \{a_1, \dots, a_r\} \subset \mathbb{P}^r$ be a set of r general points, and let R be a hyperplane such that $A_r \cap R = \varnothing$. Let

$$B_r = \{\langle a_i, a_j \rangle \cap R\}_{i,j \in \{1,\dots,r\}}.$$

Now fix $k \in \mathbb{N}$ and define

$$n_k = \min \left\{ t \geq 2 \mid \frac{\binom{n+3}{3}}{n+1} - n > k \right\}.$$

Assume that B_{n_k+k} imposes $\binom{n_k+k}{2} - \binom{k-1}{2}$ independent conditions to cubics of R. Then B_{n+k} impose exactly $\binom{n+k}{2} - \binom{k-1}{2}$ independent conditions to cubics of R for every $n \geq n_k$.

Proof. We prove the statement by induction on $n \geq n_k$. The first step of induction is granted by hypothesis. In order to lighten the notation, throughout this proof we will write A and B instead of A_{n+k} and B_{n+k}.

Suppose that $n > n_k$. Specialize $a_1, \dots, a_{n+k-1}$ on $L = \mathbb{P}^{n-1}$. Define $B_1 = \{p_{ij} \mid 1 \leq i < j \leq n+k-1\}$ and $B_2 = \{p_{1,n+k}, \dots, p_{n+k-1,n+k}\}$. Let $H = L \cap R = \mathbb{P}^{n-2}$. When we restrict to H, Castelnuovo exact sequence reads

$$0 \to \mathcal{I}_{B_2,R}(2) \to \mathcal{I}_{B,R}(3) \to \mathcal{I}_{B_1,H}(3) \to 0.$$

First observe that the points of B_2 are general on R, so

$$\mathrm{h}^0\mathcal{I}_{B_2,R}(2) = \binom{n+1}{2} - (n+k-1) \quad \text{and} \quad \mathrm{h}^1\mathcal{I}_{B_2,R}(2) = 0.$$

Now we want to compute the dimension of the right hand side of the sequence. Note that $A_1 := \{a_1, \dots, a_{n+k-1}\}$ is a set of general points in $L = \mathbb{P}^{n-1}$, H is a hyperplane of L with $A_1 \cap H = \varnothing$ and $B_1 = \{\langle a_i, a_j\rangle \cap H \mid 1 \leq i < j \leq n+k-1\}$, so by induction hypothesis

$$\mathrm{h}^0\mathcal{I}_{B_1,H}(3) = \binom{n+1}{3} - \binom{n+k-1}{2} + \binom{k-1}{2}.$$

Therefore

$$\begin{aligned}
\mathrm{h}^0\mathcal{I}_{B,R}(3) &= \mathrm{h}^0\mathcal{I}_{B_2,R}(2) + \mathrm{h}^0\mathcal{I}_{B_1,H}(3) \\
&= \binom{n+1}{2} - (n+k-1) + \binom{n+1}{3} - \binom{n+k-1}{2} + \binom{k-1}{2} \\
&= \binom{n+2}{3} - \binom{n+k}{2} + \binom{k-1}{2}.
\end{aligned}$$

Since points of B impose $\binom{n+k}{2} - \binom{k-1}{2}$ conditions in this specialized configuration, they impose at least $\binom{n+k}{2} - \binom{k-1}{2}$ conditions in the original configuration. We already noticed they can not impose more than $\binom{n+k}{2} - \binom{k-1}{2}$ conditions. $\square$

Lemma 2.35 provides an inductive way to prove that B imposes the suitable number of conditions on cubic of R. However, in order to apply it we need the first step of induction for every k. While we are not able to prove this first step in general, we believe this is the right way to compute the number of independent conditions imposed by B.

Conjecture 2.36. Assume $k < \frac{\binom{n+3}{3}}{n+1} - n$. Let $A = \{a_1, \dots, a_{n+k}\}$ be a set of $n+k$ general points in $\mathbb{P}^n$ and R a hyperplane such that $A \cap R = \varnothing$. Let

$$B = \{\langle a_i, a_j\rangle \cap R\}_{i,j \in \{1,\dots,n+k\}}.$$

Then the points of B impose exactly $\binom{n+k}{2} - \binom{k-1}{2}$ independent conditions to cubics of R.

It is quite easy to prove that Conjecture 2.36 holds for $k \in \{0,1,2\}$, and in this way we recover some of the results of Lemma 2.17. Moreover, the software Macaulay2 allows us to prove the first step for $k \leq 4$ as well.

When Conjecture 2.36 is true, then we have a way to compute degree and multiplicity of $\tilde{Z}_{n,\binom{m}{2}}$ for every n and m.

Proposition 2.37. Let $n \geq 2$ and assume that Conjecture 2.36 holds. If $1 \leq k < \frac{\binom{n+3}{3}}{n+1} - n$ and $(n,k) \neq (4,3)$, then

$$\operatorname{mult} \tilde{Z}_{n,\binom{n+k}{2}} = 3 \quad \text{and} \quad \deg \tilde{Z}_{n,\binom{n+k}{2}} = (n+1)(n+k).$$

Proof. By our assumption we can forget about exceptions of Theorem 1.5. Observe that $\tilde{Z}_{n,\binom{n+k}{2}} \supset \tilde{Z}_{n,\binom{n+1}{2}}$ for every $k \geq 1$. By Lemma 2.26 and Lemma 2.17, $\operatorname{mult} \tilde{Z}_{n,\binom{n+1}{2}} = 3$, hence $\operatorname{mult} \tilde{Z}_{n,\binom{n+k}{2}} \geq 3$. In order to see it can not be 4, it is enough to recall that, by Remark 2.23, $\tilde{Z}_{n,\binom{n+k}{2}}$ is a subscheme of the limit of $n+k$ double points in $\mathbb{A}^n$, which has multiplicity 3 because by hypothesis $(n+k)(n+1) < \binom{n+3}{3}$.

Now that we know the multiplicity is 3, we compute the degree by induction ok k. If $k = 1$, we can argue as in Proposition 2.34. $\tilde{Z}_{n,\binom{n+1}{2}}$ is obtained from $\tilde{Z}_{n,\binom{n}{2}}$ by adding $S^4_{1,n+1}, \ldots, S^4_{n,n+1}$. By Proposition 2.34, $\tilde{Z}_{n,\binom{n}{2}}$ has multiplicity 1 and degree n^2. By adding $S^4_{1,n+1}$ the degree increases by 3 and the multiplicity becomes 2. Moreover the multiplicity stays 2 until we add $S^4_{n,n+1}$, because $\binom{n}{2} + n - 1 < \mathrm{h}^0_{\mathbb{A}^{n-1}}(2)$. So

$$\deg \tilde{Z}_{n,\binom{n+1}{2}} = n^2 + 3 + 2(n-1) = (n+1)^2.$$

Assume then $k \geq 2$. $\tilde{Z}_{n,\binom{n+k+1}{2}}$ is obtained from $\tilde{Z}_{n,\binom{n}{2}}$ by adding the schemes $S^4_{1,n+k+1}, \ldots, S^4_{n+k,n+k+1}$. By induction hypothesis $\operatorname{mult} \tilde{Z}_{n,\binom{n+k+1}{2}} = 3$, so every $S^4_{i,n+k+1}$ increases the degree by at most 1. More precisely, it increases the degree if and only if the corresponding point $p_{i,n+k+1}$ is not a base point for the cubics of R containing $\{p_{jl} \mid 1 \leq j < l \leq n+k\}$. We want to understand how many of them give their contribution. By Conjecture 2.36, the $\binom{n+k}{2}$ points $\{p_{jl} \mid 1 \leq j < l \leq n+k\}$ give $\binom{n+k}{2} - \binom{k-1}{2}$ conditions on cubic. On the other hand, all the $\binom{n+k+1}{2}$ points $\{p_{jl} \mid 1 \leq j < l \leq n+k+1\}$ give $\binom{n+k+1}{2} - \binom{k}{2}$ independent conditions. This means that exactly $\binom{k}{2} - \binom{k-1}{2} = k-1$ among $S^4_{1,n+k+1}, \ldots, S^4_{n+k,n+k+1}$ increase the degree, so

$$\deg \tilde{Z}_{n,\binom{n+k+1}{2}} = \deg \tilde{Z}_{n,\binom{n+k}{2}} + n + k - (k-1).$$

By induction hypothesis, the latter equals

$$(n+k)(n+1) + n + k - k + 1 = (n+k+1)(n+1).$$

□

Finally, we can give a description of the limit scheme of $n+k$ colliding double points in $\mathbb{P}^n$.

Corollary 2.38. Let $n \geq 2$ and let $1 \leq k < \frac{\binom{n+3}{3}}{n+1} - n$, with $(n,k) \neq (4,3)$. If Conjecture 2.36 holds, then the limit of $n+k$ collapsing double points in $\mathbb{P}^n$ is $\tilde{Z}_{n,\binom{n+k}{2}}$.

Proof. The limit scheme has degree $(n+1)(n+k)$, and by Proposition 2.37 it coincides with $\deg \tilde{Z}_{n,\binom{n+k}{2}}$. In Remark 2.23 we observed that the limit scheme contains $\tilde{Z}_{n,\binom{n+k}{2}}$, so we conclude by Proposition 1.2. □

Before we move on, we want to remark a few important facts. First, we know Corollary 2.38 holds for small values of k. In particular, it improves Propositions 2.19 and 2.20. However, this approach only works in the range

$$1 \leq k < \frac{\binom{n+3}{3}}{n+1} - n. \tag{2.4}$$

When $k \leq 0$, the limit scheme has multiplicity 2. As we already pointed out, the linear system $\mathcal{L}_{n,2}(2^{n+k})$ has nonreduced base locus, and this makes it difficult to understand the first order neighbourhood of the limit point. On the other hand, when $k+n \geq \frac{\binom{n+3}{3}}{n+1}$, the limit scheme has multiplicity at least 4 and the base locus may not give us information. It is enough to consider $(n,k) = (3,3)$ to bump into the linear system $\mathcal{L}_{3,4}(2^6)$, which has no base locus outside the imposed singularities. Our work on infinitely near points gives us no clue in this type of cases.

One could argue in a similar way with higher multiplicities, and hope to find other cases in which there are base lines. For instance, we could work with triple points, and we know that the lines joining a pair of triple points are in the base locus of quintics. Unfortunately, this strategy works only if we know the degree of the linear system we are dealing with. By Proposition 2.3, this is equivalent to compute the smallest degree of a divisor in $\mathbb{P}^n$ containing a bunch of general multiple points. This is a hard problem, and the answer is unknown in its generality even in the planar case. For $n \in \{2,3\}$, there are partial results, and we will deal with them in Section 2.3.

It is also worth mentioning that we can not produce any scheme $X \subset \mathbb{P}^n$ made by a triple point with t tangent direction as a limit of double points. Indeed, first we need that $t = \binom{n+k}{n}$ for some k in the range (2.4). Moreover, the tangent directions have to be in the special position described in Remark 2.18. It is legitimate to wonder if there are more conditions to be met in order to express X as a limit of double points. In other words, can we lift X to a bunch of double points in such a way that X is the limit of those colliding points, under the previous assumptions? We will now give an answer to this question.

Lifting problem

Remark 2.18 describes the configurations of the points in the exceptional divisor and suggests the following definition.

Definition 2.39. Let $n \geq 2, t \geq 3$. Define

$$W_{n,t} = \left\{ (x_{ij})_{1 \leq i < j \leq t} \in (\mathbb{P}^n)^{\binom{t}{2}} \mid x_{bc} \in \langle x_{ab}, x_{ac} \rangle \ \forall\ 1 \leq a < b < c \leq t \right\}.$$

Fixed a hyperplane $R = \mathbb{P}^n$ not containing any of the points x_{ij}, there is a rational map

$$\pi_{n,t} : (\mathbb{P}^{n+1})^t \dashrightarrow W_{n,t} \subset (\mathbb{P}^n)^{\binom{t}{2}}$$

defined by sending $(p_1, \ldots, p_t)$ to $(x_{ij})_{1 \leq i<j \leq t}$, where x_{ij} is the intersection of the line $\langle p_i, p_j \rangle$ with R.

For $k \leq 4$, we know that the limit of $n+k$ double points in $\mathbb{P}^n$ is a triple point with $\binom{n+k}{2}$ infinitely near simple points. The simple points form a $\binom{n+k}{2}$-tuple

$(x_{ij})_{1\leq i<j\leq n+k} \in W_{n,n+k}$. We want to understand whether all such schemes can be obtained as limits of double points. This is equivalent to ask if $\pi_{n,n+1}$ is dominant, and our next task is to give a positive answer, by proving the following result.

Theorem 2.40. $\pi_{n,t}$ is dominant for every $n \geq 2$ and every $t \geq 3$. The general fiber has dimension $n+2$.

Let us start with some simple observations.

Observation 2.41.

1. We have $\dim W_{n,t} = n(t-1)+t-2$. Indeed, one can choose freely $t-1$ general points $x_{12}, \ldots, x_{1t} \in \mathbb{P}^n$. Then, for $i \in \{3, \ldots, t\}$, it is possible to choose the $t-2$ points x_{2i} general on $\langle x_{12}, x_{1i}\rangle$. After that, for $3 \leq j < k \leq t$, the other points x_{jk} are defined by $\langle x_{1j}, x_{1k}\rangle \cap \langle x_{2j}, x_{2k}\rangle$.

2. Assume that $n \geq 3$ and let $(x_{ij})_{1\leq i<j\leq t} \in W_{n,t}$. For $1 \leq a < b < c \leq t$, let l_{abc} be the line containing x_{ab}, x_{ac}, x_{bc}. Note that l_{abc} and l_{bcd} meet at x_{bc}, so they span a plane containing l_{acd} and l_{abd} as well. This plane therefore passes through the 6 points $\{x_{ij} \mid i, j \in \{a, b, c, d\}, i < j\}$. By the same argument, for every choice of m indexes $1 \leq i_1 < \ldots < i_m \leq t$, the $\binom{m}{2}$ points $\{x_{ij} \mid 1 \leq i_1 < \ldots < i_m \leq t\}$ lie on the same $\mathbb{P}^{m-2}$.

3. In particular, if $t \leq n+1$, then $p_1, \ldots, p_t \in \mathbb{P}^{n+1}$ lie on a linear subspace $L = \mathbb{P}^{t-1}$. Hence the $\binom{t}{2}$ points $\langle p_i, p_j\rangle \cap R$ all lie on $L \cap R = \mathbb{P}^{t-2}$. Then $W_{n,t} = W_{t-2,t}$, and $\pi_{n,t}$ restricts to $\pi_{t-2,t} : L^t = (\mathbb{P}^{t-1})^t \dashrightarrow W_{t-2,t}$. For this reason, from now on we will assume $t \geq n+2$.

Next Lemma is the first step towards the proof of Theorem 2.40.

Lemma 2.42. $\pi_{n,n+2} : (\mathbb{P}^{n+1})^{n+2} \dashrightarrow W_{n,n+2}$ is dominant for every n. The general fiber has dimension $n+2$.

Proof. Let $x = (x_{ij})_{1\leq i<j\leq n+2} \in W_{n,n+2}$ be general. For $i \in \{1, \ldots, n+2\}$, let $L_i = \langle x_{jk} \mid j, k \neq i\rangle$ be the dimension $n-1$ linear subspace of $R = \mathbb{P}^n$ obtained by choosing all indexes except i. Let $\Pi_i \subset \mathbb{P}^{n+1}$ be a general hyperplane containing L_i. For $j \in \{1, \ldots, n+2\}$, define the point

$$p_j = \bigcap_{i\neq j} \Pi_i.$$

If $k, h \in \{1, \ldots, n+2\}$ and $h \neq k$, then p_k and p_h are distinct points of the line $\bigcap_{i\neq k,h} \Pi_i$, so

$$\langle p_h, p_k\rangle \cap R = \bigcap_{i\neq k,h} \Pi_i \cap R = \bigcap_{i\neq k,h} L_i,$$

which is one of the x_{ij}'s. Then, up to reorder, $(p_1, \ldots, p_{n+2})$ is a preimage of $(x_{ij})_{1\leq i<j\leq n+2}$.

To determine the dimension of the general fiber, we can either note that for each of the $n+2$ points p_i we chose a hyperplane Π_i in the pencil of those containing L_i, or we can compute the difference $\dim(\mathbb{P}^{n+1})^{n+2} - \dim W_{n,n+2}$. $\square$

It is worth to note that one could give the definition of $W_{1,t}$ and $\pi_{1,t}$ as well. However, we are computing limits under the assumption that $n = 2$. Moreover,

$W_{1,t}$ coincides with $(\mathbb{P}^1)^{\binom{t}{2}}$, so the case $n = 1$ is not very interesting for our purpose.

We are now ready to prove the result we claimed.

Proof of Theorem 2.40. As we noticed in Observation 2.41, we may assume that $t \geq n+2$. We argue by induction on t. The case $t = n+2$ is the content of Lemma 2.42, so we focus on the case $t > n+2$.

Let $(x_{ij})_{1 \leq i < j \leq t} \in W_{n,t}$ be general. By induction hypothesis exist $t-1$ general points $p_1, \ldots, p_{t-1} \in \mathbb{P}^{n+1}$ such that $\langle p_i, p_j \rangle \cap R = x_{ij}$. Define

$$p_t = \langle p_1, x_{1t} \rangle \cap \langle p_2, x_{2t} \rangle.$$

In order to conclude, we have to make sure that $\langle p_i, p_t \rangle$ meets R at x_{it} for every $i \in \{3, \ldots, t-1\}$. First observe

$$\langle x_{1i}, x_{1t} \rangle = \langle p_1, x_{1i}, x_{1t} \rangle \cap R = \langle p_1, p_i, p_t \rangle \cap R,$$

because $p_t \in \langle p_1, x_{1t} \rangle$ by construction. Hence

$$\begin{aligned}
\langle p_i, p_t \rangle \cap R &= (\langle p_1, p_i, p_t \rangle \cap \langle p_2, p_i, p_t \rangle) \cap R \\
&= (\langle p_1, p_i, p_t \rangle \cap R) \cap (\langle p_2, p_i, p_t \rangle \cap R) \\
&= \langle x_{1i}, x_{1t} \rangle \cap \langle x_{2i}, x_{2t} \rangle = x_{it}.
\end{aligned}$$

The general fiber has dimension $\dim(\mathbb{P}^{n+1})^t - \dim W_{n,t} = n+2$. $\square$

In terms of collision, this means that if $t \in \{n+1, \ldots, n+4\}$, then every scheme in $\mathbb{P}^n$ made by a triple point with $\binom{t}{2}$ infinitely near simple points x_{ij} such that $(x_{ij})_{1 \leq i < j \leq t}$ is a general point of $W_{n,t}$ can be obtained as a limit of t collapsing double points in $\mathbb{P}^{n+1}$. If Conjecture 2.36 is true, the same holds for the collision of t points, where

$$n+1 \leq t < \frac{\binom{n+3}{3}}{n+1}.$$

2.3 Homogeneous collisions in low dimension

Degenerations are widely used in interpolation theory to compute the dimension of linear systems. The most studied cases are dimension 2 and 3, where there are conjectures about the reasons why a linear system is special. For $n \in \{2, 3\}$, all known special linear systems $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \ldots, m_r)$ have a base locus containing a particular variety, with precise properties. Roughly speaking, what those conjectures state is that the only known geometric reason for a linear system to be special is the existence of such a *special effect variety* in its base locus. The precise definition about special effect varieties can be found in [11] and [12]. Some examples of special effect varieties are known, see [14] and [15], and the hard problem is to classify all of them.

In this Section we will not look into special effect varieties, but we will exploit part of the results of interpolation theory in low dimension to try to describe some limits of colliding multiple points.

Collisions in $\mathbb{P}^2$

For linear systems of plane curves there is a very precise conjecture about speciality. Since we are going to use some of its known cases, we collect here the necessary notation and statement.

Let $p_1, \dots, p_r \in \mathbb{P}^2$ be general points, and let $\tilde{\mathbb{P}}^2$ be the blow-up of $\mathbb{P}^2$ at $p_1, \dots, p_r$. Set

$$\mathcal{L} := \mathcal{L}_{2,d}(m_1, \dots, m_r)(p_1, \dots, p_r).$$

We put a tilde to indicate the strict transform of curves on $\tilde{\mathbb{P}}^2$.

Definition 2.43. A (-1)-*curve* $C \subset \mathbb{P}^2$ is a curve such that $\tilde{C}$ is a smooth rational curve with self intersection -1.

The linear system $\mathcal{L}$ on $\mathbb{P}^2$ is (-1)-*reducible* if

$$\mathcal{L} = \sum_{i=1}^{k} N_i C_i + \mathcal{M},$$

where $N_i \in \mathbb{N}$, $C_1, \dots, C_k$ are (-1)-curves, $\tilde{\mathcal{M}} \cdot \tilde{C}_i = 0$ for all $i = 1, \dots, k$, and $\operatorname{vdim}(\mathcal{M}) \geq 0$.

The system $\mathcal{L}$ is called (-1)-*special* if, in addition, there exists an index $i \in \{1, \dots, k\}$ such that $N_i > 1$.

The leading conjecture for linear systems of plane curves was first formulated by Segre in [75]. Later on, other versions of the conjecture were proposed by Gimigliano ([42]), Harbourne ([45]) and Hirschowitz ([48]). Forty years after Segre's first formulation, Ciliberto-Miranda showed in [29] that all previous versions are in fact equivalent, so now we can present the statement of the celebrated SHGH conjecture. It basically predicts that, for $n = 2$, the special effect varieties are the (-1)-curves.

Conjecture 2.44 (Segre-Harbourne-Gimigliano-Hirschowitz)**.** A linear system of plane curves $\mathcal{L}_{n,d}(m_1, \dots, m_r)$ is special if and only if it is (-1)-special.

While this conjecture was proven true in a number of cases, and most experts believe it is true, a complete proof still seems to be out of grasp. However, we will use some of the known cases to describe limits of fat points in the plane. First we need a definition.

Definition 2.45. A linear system $\mathcal{L}_{2,d}(m_1, \dots, m_r)$ is called *homogeneous* if all the multiplicities are the same, i.e. if $m_1 = \ldots = m_r$. It is called *quasi-homogeneous* if all the multiplicities but one are the same, i.e. if $m_2 = \ldots = m_r$.

We collect now some results about Conjecture 2.44.

Proposition 2.46. Let $\mathcal{L} := \mathcal{L}_{2,d}(m_1, \dots, m_r)$. Let $g_{\mathcal{L}}$ be the geometric genus of $\mathcal{L}$. Conjecture 2.44 holds in the following cases:

1. $r \leq 9$ (Castelnuovo [21], Nagata [61], Gimigliano [42], Harbourne [44]),
2. $r = h^2$ is a square (Evain [40]),
3. $r = 4^t$ is a power of four (Evain [39]),
4. $\operatorname{vdim} \mathcal{L} \geq 0$ and $g_{\mathcal{L}} \leq 4$ (Mignon [58]),

5. $m_i \leq 11$ for every $i \in \{1, \dots, r\}$ (Dumnicki-Jarnicki [38]),

6. $\mathcal{L} = \mathcal{L}_{2,d}(m^r)$ is homogeneous and $m \leq 42$ (Dumnicki [37]),

7. $\mathcal{L} = \mathcal{L}_{2,d}(m^r)$ is homogeneous and $r \geq 4m^2$ (Roè [73]),

8. $\mathcal{L} = \mathcal{L}_{2,d}(n, m^r)$ is quasi-homogeneous and $m \leq 5$ (Ciliberto-Miranda [28], Laface [50], Laface-Ugaglia [51]).

Since homogeneous (-1)-special systems have been classified (see for instance [27, Theorem 4.9]), Conjecture 2.44 has a simpler form for homogeneous systems with more than 9 points.

Conjecture 2.47. If $r \geq 10$, then $\mathcal{L}_{2,d}(m^r)$ is nonspecial.

With these tools, we can consider the scheme Z_1 made by r points of multiplicities $m_1, \dots, m_r$. Under the assumption that they satisfy one of the known cases listed in Proposition 2.46, we can compute the multiplicity d of the limit of the collision of such points via Proposition 2.3. However, in general this is not enough to completely determine the limit scheme. Usually we need information on the base locus of $\mathcal{L}_{2,d}(m_1, \dots, m_r)$, and in some cases even that is not enough. Hence this method can not give a general and unified degeneration working for every linear system, but rather it can provide concrete degenerations in many specific cases. As an example, we can easily extend Proposition 2.16 to higher multiplicity.

Proposition 2.48. Let $m, n \leq 42$, let $h = \frac{\binom{m+1}{2}}{\binom{n+1}{2}}$. If $h \in \mathbb{N}$ and $h \geq 10$, then the limit of h collapsing n-ple points in $\mathbb{P}^2$ is an m-ple point.

Proof. First observe that $\mathcal{L}_{2,m-1}(n^h)$ is a homogeneous system whose multiplicity does not exceed 42, so it satisfies Conjecture 2.44 by Proposition 2.46. Since all multiplicities are the same and $h \geq 10$, $\mathcal{L}_{2,m-1}(n^h)$ is nonspecial. Note that it has virtual dimension $\binom{m+1}{2} - 1 - h\binom{n+1}{2} < 0$, so it is empty. By Proposition 2.3, the limit has multiplicity m. To conclude we observe that the scheme made by h n-ple points has length $h\binom{n+1}{2}$, which is the same as the length of a m-tuple point. $\square$

Collisions in $\mathbb{P}^3$

Now we consider systems of surfaces in $\mathbb{P}^3$. Before we state the main conjecture about the speciality of such systems, we need a definition.

Definition 2.49. A linear system $\mathcal{L}_{3,d}(m_1, \dots, m_r)$ is *Cremona reduced* if

$$2d \geq m_{i_1} + \ldots + m_{i_4} \text{ for every } \{i_1, \dots, i_4\} \subset \{1, \dots, r\}.$$

This means the degree of the system can not be decreased by applying a Cremona transformation.

The following conjecture about the speciality of linear systems in $\mathbb{P}^3$ was stated by Laface-Ugaglia in [52, Conjecture 4.1].

Conjecture 2.50 (Laface-Ugaglia)**.** A Cremona reduced linear system

$$\mathcal{L} = \mathcal{L}_{3,d}(m_1, \ldots, m_r)$$

is special if and only if either

i) there exists a line l through two base points such that $l \cdot \mathcal{L} \leq -2$, or

ii) there exists a quadric Q such that $Q \cdot (\mathcal{L} - Q) \cdot (K_{\mathbb{P}^3} - Q) < 0$.

Like Conjecture 2.44, Laface-Ugaglia conjecture was proven true in several cases.

Proposition 2.51. Let $\mathcal{L} := \mathcal{L}_{3,d}(m_1, \ldots, m_r)$. Conjecture 2.50 holds in the following cases:

1. $m_1 = \ldots = m_r \leq 5$ (Ballico-Brambilla-Caruso-Sala [8]);
2. $r \leq 8$ (De Volder-Laface [34]);
3. $r \leq 9$ and $m \leq 8$ (Brambilla-Dumitrescu-Postinghel [15]).

By exploiting some of this known cases, we can prove a result which is the analogous of Proposition 2.48 in dimension 3.

Proposition 2.52. Let $n, m \in \mathbb{N}$ such that $n \leq 5$ and $m \geq 2n + 1$. Define $h = \frac{\binom{m+2}{3}}{\binom{n+2}{3}} = \frac{m(m+1)(m+2)}{n(n+1)(n+2)}$. If $h \in \mathbb{N}$, then the limit of h collapsing n-ple points in $\mathbb{P}^3$ is an m-ple point.

Proof. By hypothesis $\mathcal{L}_{3,m}(n^h)$ is not empty. By Proposition 2.3, in order to prove $\operatorname{mult} Z_0 = m$ we need to show that $\mathcal{L}_{3,m-1}(n^h)$ is empty. Since it has negative virtual dimension, it is enough to prove it is nonspecial. By hypothesis $m \geq 2n+1$, hence it is Cremona reduced, and since $n \leq 5$ it satisfies Conjecture 2.50. Now we will check the cases.

($n = 1$) $\mathcal{L}_{3,m-1}(1^h)$ is clearly nonspecial.

($n = 2$) We only have to observe that the only exception of Theorem 1.5 in $\mathbb{P}^3$ is $h = 9$. But there is no m such that $\frac{\binom{m+2}{3}}{4} = 9$.

($n = 3$) $h \in \mathbb{N} \Leftrightarrow m \equiv 0, 3, 4, 8, 10, 14, 15, 18, 19 \bmod 20$. Since $m \geq 7$, either $m = 8$ or $m \geq 10$, and in this range Conjecture 2.50 predicts that $\mathcal{L}_{3,m-1}(3^h)$ is nonspecial.

($n = 4$) By hypothesis $m \geq 9$. Moreover, m can not be 9, otherwise $h \notin \mathbb{N}$. So $m \geq 10$ and in this range Conjecture 2.50 predicts that $\mathcal{L}_{3,m-1}(3^h)$ is nonspecial.

($n = 5$) By hypothesis $m \geq 11$, and in this range Conjecture 2.50 predicts that $\mathcal{L}_{3,m-1}(3^h)$ is nonspecial.

Now we know $\operatorname{mult} Z_0 = m$. In order to conclude, it is enough to note that an m-ple point in $\mathbb{P}^3$ has length $\binom{m+2}{3} = \deg Z_1$. $\square$

Observe the assumption $m \geq 2n+1$ is necessary. Indeed, if we consider $n = 4$, $m = 8$ and $h = 6$, then by applying Cremona transformations we can check $\mathcal{L}_{3,7}(4^6) \cong \mathcal{L}_{3,5}(4^2, 2^4) \cong \mathcal{L}_{3,3}(2^4)$ is not empty, so $\operatorname{mult} Z_0 = 7$.

Our approach relies on a length counting, and therefore it needs a good behaviour of the numbers involved. This makes difficult to prove general results. However, there are other specific examples in which it is easy to compute the limit.

Example 2.53.

1. Let $Z_1 \subset \mathbb{P}^3$ be the scheme consisting of 4 fourtuple points $p_1, \ldots, p_4$. Then $\deg Z_1 = 80$. We want to prove that $\operatorname{mult} Z_0 = 6$. Let $H_1, \ldots, H_4$ be the planes generated by three of our four points. Since there are no plane quintics through three general fourtuple points,

   $$\mathcal{L}_{3,5}(4^4) = H_1 + \ldots + H_4 + \mathcal{L}_{3,1}(1^4)$$

   is empty. On the other hand, $\mathrm{h}^0 \mathcal{O}_{\mathbb{P}^3}(6) = 84$, so $\operatorname{mult} Z_0 = 6$. The base locus of $\mathcal{L}_{3,6}(4^4)$ consists of the 6 double lines $\langle p_i, p_j \rangle$, which cut 6 double points on R, in special positions as described in Remark 2.18. Our candidate limit scheme is a 6-ple point with these 6 infinitely near double points. It is easy to show that the latter impose independent conditions on sextics of R, for instance we can pick a line l containing three of them and use Castelnuovo exact sequence by restricting our systems to l. Then our candidate scheme has lenght $\binom{5+3}{3} + 6 \cdot 4 = \deg Z_1$. Since the degrees coincide, this is the limit scheme.

2. Let $Z_1 \subset \mathbb{P}^3$ be the scheme consisting of 5 points $p_1, \ldots, p_5$ of multiplicity 5. Then $\deg Z_1 = 175$. We want to prove that $\operatorname{mult} Z_0 = 9$. Let $H_1, \ldots, H_{10}$ be the planes generated by three of our five points. Since there are no plane 8-ics through three general points of multiplicity 5, $\mathcal{L}_{3,8}(5^4)$ should contain the 10 planes, and that is not possible. On the other hand, $\mathrm{h}^0 \mathcal{O}_{\mathbb{P}^3}(9) = 220$, so $\operatorname{mult} Z_0 = 9$. The base locus of $\mathcal{L}_{3,9}(5^4)$ consists of the 10 lines $\langle p_i, p_j \rangle$, which cut 10 simple points on R, in special positions as described in Remark 2.18. Anyway, those 10 simple points impose independent conditions on 9-ics of R by Lemma 2.17. Since

   $$\binom{8+3}{3} + \binom{5}{2} = 175 = \deg Z_1,$$

   the limit scheme is a 9-ple point with 10 infinitely near simple points.

2.4 Other collisions

Up to now we focused on homogeneous collisions, but of course there are many other cases in which we can try to determine the limit Z_0.

When one of the collapsing points has multiplicity much larger than the others, it is easy to compute the limit scheme.

Proposition 2.54. Let $m, m_1, \ldots, m_s \in \mathbb{N}$, and assume $\mathcal{L}_{n-1,m}(m_1, \ldots, m_s)$ is not empty. Then the limit scheme of $s+1$ collapsing points of multiplicity $m, m_1, \ldots, m_s$ is an m-ple point with s infinitely near points of multiplicity $m_1, \ldots, m_s$.

Proof. Clearly $\mathcal{L}_{n,m-1}(m, m_1, \dots, m_s)$ is empty. On the other hand,

$$\mathcal{L}_{n,m}(m, m_1, \dots, m_s) \cong \mathcal{L}_{n-1,m}(m_1, \dots, m_s)$$

is not empty by hypothesis, so $\operatorname{mult} Z_0 = m$ by Proposition 2.3. The base locus of $\mathcal{L}_{n,m}(m, m_1, \dots, m_s)$ contains the s lines joining the m-ple point with each of the others, counted with multiplicities $m_1, \dots, m_s$. They cut s points on R, of multiplicities $m_1, \dots, m_s$. To conclude, observe that the scheme made by an m-ple point with s infinitely near points of multiplicity $m_1, \dots, m_s$ has the same length as Z_1. $\square$

The next two Propositions will deal with the quasihomogeneous case of a fat point colliding together with a bunch of low multiplicity points.

Proposition 2.55. Let $m, n \geq 2$ and let $s = \binom{n+m-1}{n}$. Then the limit of s simple points and a point of multiplicity m colliding in $\mathbb{P}^n$ is a $(m+1)$-ple point.

Proof. By our assumption on s, $\mathcal{L}_{n,m}(m, 1^h)$ is empty while $\mathcal{L}_{n,m+1}(m, 1^h)$ is not, so $\operatorname{mult} Z_0 = m+1$ by Proposition 2.3. To conclude, observe a $(m+1)$-ple point has degree $\binom{n+m}{m} = \deg Z_1$, so Z_0 is a $(m+1)$-ple point. $\square$

Proposition 2.56. Let $m, n \geq 3$ and $(m, n) \notin \{(4,3), (3,5)\}$. Suppose $s = \frac{\binom{m+n-1}{n-1}}{n} \in \mathbb{N}$. Then the limit of s double points and a point of multiplicity m colliding in $\mathbb{P}^n$ is a $(m+1)$-ple point with s infinitely near simple points.

Proof. By hypothesis

$$\begin{aligned}
&\binom{n+m+1}{n} - \binom{n+m-1}{n} - s(n+1) \\
=&\binom{n+m+1}{n} - \binom{n+m-1}{n} - \frac{\binom{m+n-1}{n-1}}{n}(n+1) \\
=&\binom{n+m+1}{n} - \binom{n+m-1}{n} - \binom{m+n-1}{n-1} - \frac{\binom{m+n-1}{n-1}}{n} \\
=&\frac{(n+m+1)!}{n!(m+1)!} - \frac{(n+m-1)!}{n!(m-1)!} - \frac{(m+n-1)!}{(n-1)!m!} - \frac{(m+n-1)!}{m!n!} \\
=&\frac{(n+m-1)!}{(n-1)!(m-1)!}\left[\frac{(n+m+1)(n+m)}{(m+1)mn} - \frac{1}{n} - \frac{1}{m} - \frac{1}{mn}\right] \\
=&\frac{(n+m-1)!}{(n-1)!(m-1)!}\left[\frac{n^2+mn-m-1}{(m^2+m)n}\right] > 0,
\end{aligned}$$

hence $\mathcal{L}_{n,m+1}(m, 2^s)$ is not empty. On the other hand

$$\binom{n+m-1}{n-1} - sn = \binom{n+m-1}{n-1} - \binom{m+n-1}{n-1} = 0,$$

so $\mathcal{L}_{n,m}(m, 2^s) \cong \mathcal{L}_{n-1,m}(2^s)$ is expected to be empty. The latter is nonspecial by Theorem 1.5, so $\operatorname{mult} Z_0 = m+1$ by Proposition 2.3. The s lines joining the m-ple point and one of the double points are contained in the base locus of $\mathcal{L}_{n,m+1}(m, 2^s)$, and they cut s general simple points on R. The candidate limit scheme is a $(m+1)$-ple point with s infinitely near simple points, which has length $\binom{n+m-1}{n} + s = \deg Z_1$. $\square$

When we use limits to specialize a linear system, the most effective result would be a description of Z_0 as a fat point of some multiplicity. Unluckily, we saw many examples showing that this is often impossible, but the following result will be useful in such applications.

Proposition 2.57. Let $n \geq 2$, $m_1, m_2 \in \mathbb{N}$. Then exist $h, m \in \mathbb{N}$, depending on n, m_1, m_2, such that the limit of 2 points of multiplicity m_1 and m_2 and h simple points in $\mathbb{P}^n$ is an m-ple point.

Proof. Define

$$m := m(n, m_1, m_2) = m_1 + m_2 + 1 \tag{2.5}$$

and

$$h := h(n, m_1, \dots, m_s) = \binom{m+n}{n} - \binom{m_1+n-1}{n} - \binom{m_2+n-1}{n} - 1. \tag{2.6}$$

By construction $\operatorname{expdim} \mathcal{L}_{n,m}(m_1, m_2, 1^h) \geq 0$, hence $\mathcal{L}_{n,m}(m_1, m_2, 1^h)$ is not empty. Since an m-ple point has degree equal to the length of the starting scheme, it is enough to show that $\mathcal{L}_{n,m-1}(m_1, m_2, 1^h)$ is nonspecial, and therefore empty. Since the h simple points always give independent conditions, it suffices to prove that $\mathcal{L}_{n,m}(m_1, m_2)$ is nonspecial. By [14, Corollary 4.8], such system is linearly nonspecial, so we just need to observe that there are no base linear cycles, and so it is nonspecial. $\square$

2.5 First applications of limits

Proposition 2.3 shows that, in order to determine the multiplicity of the limit, we need to understand the speciality of the systems of divisors containing the starting scheme Z_1, or equivalently its Hilbert function h_{Z_1}. If we want to get the first clues about what the limit is, we must study the interpolation problem for linear systems $\mathcal{L}_{n,d}(m_1, \dots, m_h)$. Indeed, in Section 2.3 we used known results in interpolation theory to provide such clues. Therefore it is just fair to try to return the favour, using the limits we constructed as tools to specialize linear systems in order to prove their nonspeciality or nonemptiness.

In Proposition 2.46 we recalled some of the known cases of Conjecture 2.44. Now we aim to provide further examples in which it holds. We can exploit Proposition 2.48 to prove the following result.

Proposition 2.58. Let $m, n_1, \dots, n_s \in \mathbb{N}$. For $i \in \{1, \dots, s\}$, set $h_i = \frac{m(m+1)}{n_i(n_i+1)}$. Assume $h_i \in \mathbb{N}$ and $h_i \geq 10$ for every $i \in \{1, \dots, s\}$. If $m, n_1, \dots, n_s \leq 42$, then $\mathcal{L}_{2,d}(m^k, n_1^{t_1 h_1}, \dots, n_s^{t_s h_s})$ is nonspecial.

Proof. By Proposition 2.48, we can collapse h_1 of the n_1-ple points into an m-ple point, thereby degenerating $\mathcal{L}_{2,d}(m^k, n_1^{t_1 h_1}, \dots, n_s^{t_s h_s})$ to

$$\mathcal{L}_{2,d}(m^{k+1}, n_1^{(t_1-1)h_1}, \dots, n_s^{t_s h_s}).$$

By performing t_1 of these collisions, we obtain the system

$$\mathcal{L}_{2,d}(m^{k+t_1}, n_2^{t_2 h_2}, \dots, n_s^{t_s h_s}).$$

Then we apply Proposition 2.48 again to collapse h_2 of the n_2-ple points into an m-ple point. By performing t_2 of these collisions, we specialize the system to

$$\mathcal{L}_{2,d}(m^{k+t_1+t_2}, n_3^{t_3h_3}, \dots, n_s^{t_sh_s}).$$

We iterate the argument till the s-th step. At the end we are dealing with the specialized system $\mathcal{L}_{2,d}(m^{k+t_1+\dots+t_s})$. The latter is nonspecial by Proposition 2.46, and this implies $\mathcal{L}_{2,d}(m^k, n_1^{t_1h_1}, \dots, n_s^{t_sh_s})$ is nonspecial. $\square$

The next results shows how collisions can prove nonspeciality when the system has a very large expected dimension.

Proposition 2.59. Let $d, m, k \in \mathbb{N}$ be such that $k > m$ and

$$\binom{d+3}{3} \geq \binom{(m+1)(a-8)+k+4}{3} + 8\binom{m+2}{3}.$$

Then $\mathcal{L} := \mathcal{L}_{3,d}(k, m^a)$ is nonspecial.

Proof. If $a \leq 7$, then Conjecture 2.50 is known to be true for $\mathcal{L}$. First observe that $\mathcal{L}$ is Cremona reduced. Since there are only 8 imposed singularities, there can not be a special effect quadric. Moreover our assumption implies $d > \max\{2m-1, k\}$ so there are no special effect lines. Hence $\mathcal{L}$ is nonspecial.

If $a = 8$, then $\mathcal{L}$ is nonspecial by [15, Theorem 3.1], because $d \geq 2m$ and $k > m$.

Assume then $a \geq 9$. Now we can no longer rely on Conjecture 2.50, and we will exploit the large expected dimension of $\mathcal{L}$. First observe that, for every $h \in \mathbb{N}$, it suffies to show that $\mathcal{L} := \mathcal{L}_{3,d}(k, m^a, 1^h)$ is nonspecial. Set $\mathcal{L}_0 = \mathcal{L}$, $k_0 = k$ and $k_1 = k + m + 1$. By Proposition 2.57, exists

$$h_0 := h(m, k_0) = \binom{k_1+3}{3} - \binom{m+2}{3} - \binom{k+2}{3} - 1$$

such that the limit of an m-ple point, a k-ple point and h_0 simple points is a k_1-ple point. We need $\operatorname{expdim} \mathcal{L}_{3,d}(k, m^a) \geq h_0$ to guarantee $\mathcal{L}_0' := \mathcal{L}_{3,d}(k, m^a, 1^{h_0})$ is not empty, that is

$$\binom{d+3}{3} \geq \binom{k_0+m+4}{3} + (a-1)\binom{m+2}{3}.$$

If so, by Proposition 2.57 we can degenerate $\mathcal{L}_0'$ to $\mathcal{L}_1 := \mathcal{L}_{3,d}(k_1, m^{a-1})$. Then we apply the same argument to $\mathcal{L}_1$. Set $k_2 = k_1 + m + 1$, we know there exists

$$h_1 := h_1(m, k_1) = \binom{k_2+3}{3} - \binom{m+2}{3} - \binom{k_1+2}{3} - 1$$

such that we can make a k_1-ple point, an m-ple point and h_1 simple points onto a k_2-ple point. This time we need

$$\binom{d+3}{3} \geq \binom{k_1+m+4}{3} + (a-2)\binom{m+2}{3}$$

to guarantee $\mathcal{L}'_1 := \mathcal{L}_{3,d}(k, m^a, 1^{h_0})$ is not empty. If so, by Proposition 2.57 we can degenerate $\mathcal{L}'_1$ to $\mathcal{L}_2 := \mathcal{L}_{3,d}(k_2, m^{a-2})$. We keep iterating until we are left with less than 9 points. At the i-th step we require

$$\binom{d+3}{3} \geq \binom{k_{i-1}+m+4}{3} + (a-i)\binom{m+2}{3}. \tag{2.7}$$

The most restrictive among all the requirements (2.7) is the last one, that is

$$\begin{aligned}\binom{d+3}{3} &\geq \binom{k_{a-9}+m+4}{3} + 8\binom{m+2}{3} \\ &= \binom{(a-8)(m+1)+k+4}{3} + 8\binom{m+2}{3}.\end{aligned}$$

After $a-8$ steps we obtain the specialized linear system $\mathcal{L}_{a-8} = \mathcal{L}_{3,d}(k_{a-8}, m^8)$ which again is nonspecial by [15, Theorem 3.1]. $\square$

The bound provided by Proposition 2.59 is far from being sharp. Anyway, combinations of the results in Section 2.4 may prove more effective.

Chapter 3

On the number of decompositions of generic polynomials

The content of this Chapter comes from a paper written in collaboration with Elena Angelini, Massimiliano Mella and Giorgio Ottaviani, now published as [5].

We will address the Waring problem for polynomials. After giving the main definitions about identifiability and recalling some of the known results in this topic, we will present the geometric interpretation of general identifiability. Next we move to what we call the Nonabelian Apolarity lemma, which will allow us to prove the existence of a new identifiable case. Then we consider the software-aided computational approach, and we show how it was useful in the search of identifiable polynomials. Finally, we work out the decompositions of a general pair of ternary forms of degrees a and b, and we prove that there can not be identifiability if $b = a + 1$, unless $(a, b) = (2, 3)$.

Let f_1, f_2 be two general quadratic forms in $n + 1$ variables over $\mathbb{C}$. A well known theorem, which goes back to Jacobi and Weierstrass, says that f_1, f_2 can be simultaneously diagonalized. More precisely there exist linear forms $l_0, \ldots, l_n$ and scalars $\lambda_0, \ldots, \lambda_n$ such that

$$\begin{cases} f_1 = \sum_{i=0}^{n} l_i^2 \\ f_2 = \sum_{i=0}^{n} \lambda_i l_i^2 \end{cases} \tag{3.1}$$

An important feature is that the forms l_i are unique (up to order) and their equivalence class, up to multiplication by scalars, depends only on the pencil $\langle f_1, f_2 \rangle$, hence also the scalars λ_i are uniquely determined after f_1, f_2 have been chosen in this order. The canonical form (3.1) allows us to write easily the basic invariants of the pencil, like the discriminant which takes the form $\prod_{i<j}(\lambda_i - \lambda_j)^2$. We call (3.1) a *(simultaneous) Waring decomposition* of the pair (f_1, f_2). The pencil (f_1, f_2) has a unique Waring decomposition with $n+1$ summands if and only if its discriminant does not vanish. In the tensor terminology, (f_1, f_2) is *generically identifiable.*

We generalize now the decomposition (3.1) to r general forms, even allowing different degrees. For symmetry reasons, it is convenient not to distinguish f_1 from the other f_j's, so we will allow scalars λ_i^j in the decomposition of each f_j, including f_1. Throughout this Section, sometimes it will be convenient to identify the vector space of degree d homogeneous polynomials

$$\mathbb{C}[x_0 \ldots, x_n]_d$$

with the d-th symmetric power of $\mathbb{C}$, that is, with the space

$$Sym^d\mathbb{C}^{n+1}.$$

To be precise, let $f = (f_1, \ldots, f_r)$ be a vector of general homogeneous forms of degree $a_1, \ldots, a_r$ in $n+1$ variables over the complex field $\mathbb{C}$, i.e. $f_i \in \mathrm{S}ym^{a_i}\mathbb{C}^{n+1}$ for all $i \in \{1, \ldots, r\}$. Let us assume that $2 \leq a_1 \leq \ldots \leq a_r$.

Definition 3.1. A *Waring decomposition* of $f = (f_1, \ldots, f_r)$ is given by linear forms $l_1, \ldots, l_k \in \mathbb{P}(\mathbb{C}[x_0, \ldots, x_n]_1)$ and scalars $(\lambda_1^j, \ldots, \lambda_k^j) \in \mathbb{C}^k \setminus \{\underline{0}\}$, with $j \in \{1, \ldots, r\}$, such that

$$f_j = \lambda_1^j l_1^{a_j} + \ldots + \lambda_k^j l_k^{a_j} \tag{3.2}$$

for all $j \in \{1, \ldots, r\}$ or, in vector notation,

$$f = \sum_{i=1}^{k} \left(\lambda_i^1 l_i^{a_1}, \ldots, \lambda_i^r l_i^{a_r}\right). \tag{3.3}$$

The geometric argument in Section 3.1 shows that every f has a Waring decomposition. We consider two Waring decompositions of f as in (3.3) being equal if they differ just by the order of the k summands. The *rank* of f is the minimum number k of summands appearing in (3.3). This definition coincides with the classical one in the case $r = 1$ (the vector f given by a single polynomial).

Due to the presence of the scalars λ_i^j, each form l_i depends essentially only on n conditions. So the decomposition (3.2) may be thought of as a nonlinear system with $\sum_{i=1}^r \binom{a_i+n}{n}$ data (given by f_j) and $k(r+n)$ unknowns (given by kr scalars λ_i^j and k forms l_i). This is a very classical subject, see for example [70, 54, 72, 74, 78], although in most of classical papers the degrees a_i were assumed equal, with the notable exception of [72].

Definition 3.2. The space $\mathrm{S}ym^{a_1}\mathbb{C}^{n+1} \oplus \ldots \oplus \mathrm{S}ym^{a_r}\mathbb{C}^{n+1}$ is called *perfect* if there exists $k \in \mathbb{N}$ such that

$$\sum_{i=1}^{r} \binom{a_i + n}{n} = k(r+n), \tag{3.4}$$

i.e. when (3.2) corresponds to a square polynomial system.

The arithmetic condition (3.4) means that $r + n$ divides $\sum_{i=1}^r \binom{a_i+n}{n}$. In particular, the number of summands k in the system (3.2) is uniquely determined.

The case with two quadratic forms described in (3.1) corresponds to

$$r = a_1 = a_2 = 2 \text{ and } k = n + 1,$$

and it is perfect. The perfect cases are important because, by the above dimension count, we expect finitely many Waring decompositions for the generic polynomial vector in a perfect space $\mathrm{S}ym^{a_1}\mathbb{C}^{n+1} \oplus \ldots \oplus \mathrm{S}ym^{a_r}\mathbb{C}^{n+1}$.

It may happen that general elements in perfect spaces have no decompositions with the expected number k of summands. The first example, besides the one of a single plane conic, was found by Clebsch in the XIX century and regards ternary quartics, where $r = 1$, $a_1 = 4$ and $n = 2$. Equation (3.4) gives $k = 5$ but in this case the system (3.2) has no solutions and indeed 6 summands are needed to find a Waring decomposition of the general ternary quartic. It is well known that all the perfect cases with $r = 1$ when the system (3.2) has no solutions have been determined by Alexander and Hirschowitz, while more cases for $r \geq 2$ have been found in [19], where a collection of classical and modern interesting examples is listed.

Still, perfectness is a necessary condition to have finitely many Waring decompositions. So two natural questions, of increasing difficulty, arise.

Question 1 Are there other perfect cases for $a_1, \ldots, a_r, n$, beyond (3.1), where a unique Waring decomposition (3.3) exists for generic f, namely where we have generic identifiability?

Question 2 What is the number of Waring decompositions (up to order of summands) for a generic f in any perfect case?

The above two questions are probably quite difficult, but we feel it is worthwhile to state them as guiding problems. While Question 2 is open even in the case $r = 1$ of a single polynomial, limits of fat points will allow us to provide an answer to Question 1 for $r = 1$, see Chapter 4. This improves previous results in [56, 57]. The birational technique used in these papers has been generalized to our setting in Section 3.4. In the case $r = 1$, some numbers of decompositions for small a_1 and n have been computed (with high probability) in [46] by homotopy continuation techniques, with the numerical software Bertini [9].

Before stating our contributions, we need to review other known results on this topic.

In the case $n = 1$ (binary forms) there is a result by Ciliberto and Russo [31] which completely answers our Question 1.

Theorem 3.3 (Ciliberto-Russo)**.** Let $n = 1$. In all the perfect cases there is a unique Waring decomposition for generic $f \in \mathrm{S}ym^{a_1}\mathbb{C}^2 \oplus \ldots \oplus \mathrm{S}ym^{a_r}\mathbb{C}^2$ if and only if

$$a_1 + 1 \geq \frac{\sum_{i=1}^{r}(a_i + 1)}{r + 1}.$$

Note that the fraction $\frac{\sum_{i=1}^{r}(a_i+1)}{r+1}$ equals the number k of summands.

We will provide an alternative proof of Theorem 3.3 by using Apolarity, see Theorem 3.12.

As widely expected, for $n > 1$ generic identifiability is quite a rare phenomenon. It has been extensively investigated in the XIX century and at the beginning of the XX century, and the following are the only discovered cases that we are aware of:

$$\begin{cases} (i) & (Sym^2\mathbb{C}^n)^{\oplus 2}, \text{rank } n, \text{Weierstrass [79], as in (3.1)}, \\ (ii) & Sym^5\mathbb{C}^3, \text{rank } 7, \text{Hilbert [47], see also [71] and [66]}, \\ (iii) & Sym^3\mathbb{C}^4, \text{rank } 5, \text{Sylvester Pentahedral Theorem [76]}, \\ (iv) & (Sym^2\mathbb{C}^3)^{\oplus 4}, \text{rank } 4, \\ (v) & Sym^2\mathbb{C}^3 \oplus Sym^3\mathbb{C}^3, \text{rank } 4, \text{Roberts [72]}. \end{cases} \tag{3.5}$$

The interest in Waring decompositions was revived by Mukai's work on 3-folds, [59, 60]. Since then, many authors have devoted their energy to understand, interpret and expand the theory. Cases (ii) and (iii) in (3.5) were explained by Ranestad and Schreyer in [69] by using syzygies, see also [55] for an approach via projective geometry and [63] for a vector bundle approach (called in this paper "Nonabelian Apolarity", see Section 3.2). Case (v) was reviewed in [65] in the setting of Lüroth quartics. (iv) is a classical and "easy" result, there is a unique Waring decomposition of a general 4-tuple of ternary quadrics. There is a very nice geometric interpretation for this latter case. Four points in $\mathbb{P}^5$ define a $\mathbb{P}^3$ that cuts the Veronese surface in 4 points giving the required unique decomposition. See Remark 3.8 for a generalization to arbitrary (d, n).

Our main contribution with respect to unique decompositions is the following new case.

Theorem 3.4. A general $f \in Sym^3\mathbb{C}^3 \oplus Sym^3\mathbb{C}^3 \oplus Sym^4\mathbb{C}^3$ has a unique Waring decomposition of rank 7, namely it is identifiable.

The Theorem will be proved in the general setting of Theorem 3.12. Besides the new example, we think it is important to stress the way it arose. We adapted the methods in [46] to our setting, by using the software Bertini [9] and also the package *Numerical Algebraic Geometry* [49] in Macaulay2 [43], with the generous help of Jon Hauenstein and Anton Leykin, who assisted us in writing our first scripts. The computational analysis of perfect cases of forms on $\mathbb{C}^3$ suggested that the Waring decomposition is unique for $Sym^3\mathbb{C}^3 \oplus Sym^3\mathbb{C}^3 \oplus Sym^4\mathbb{C}^3$. Then we proved it via Nonabelian Apolarity, with the choice of a vector bundle. This kind of technique has many advantages. For instance, we can give a unified proof of almost all cases with a unique Waring decomposition via Nonabelian Apolarity with the choice of a vector bundle E, see Theorem 3.12.

Also, we borrow a construction from [55] to prove, see Theorem 3.15, that whenever we have uniqueness for rank k, the variety parametrizing Waring decompositions of higher rank is unirational.

For $r = 2$ and $n = 2$, the space $Sym^a\mathbb{C}^3 \oplus Sym^{a+1}\mathbb{C}^3$ is perfect if and only if $a = 2t$ is even. All the numerical computations we did suggested that identifiability holds only for $a = 2$ (by Robert's Theorem, see (3.5) (v)). Once again this pushed us to prove the non-uniqueness for these pencils of plane curves. Our main contribution to Question 2 regards this case and it is the following.

Theorem 3.5. A general $f \in Sym^a\mathbb{C}^3 \oplus Sym^{a+1}\mathbb{C}^3$ is identifiable if and only if $a = 2$, corresponding to (v) in the list (3.5). Moreover f has finitely many Waring decompositions if and only if $a = 2t$ and in this case the number of decompositions is at least

$$\frac{(3t-2)(t-1)}{2} + 1.$$

We know by equation (3.5)(v) that the bound is sharp for $t = 1$ and we verified with high probability, using [9], that it is attained also for $t = 2$. On the other hand we do not expect it to be sharp in general. Theorem 3.5 is proved in section Section 3.4. The main idea, borrowed from [56], is to bound the number of decompositions with the degree of a tangential projection, see Theorem 3.19. To bound the latter we use a degeneration argument, see Lemma 3.21, that reduces the computation needed to an intersection calculation on the plane.

3.1 Secants to a projective bundle

We show a geometric interpretation of the decomposition (3.2) by considering the k-secant variety to the projective bundle

$$X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^n}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^n}(a_r)) \subset \mathbb{P}\left(H^0\left(\oplus_i \mathcal{O}_{\mathbb{P}^n}(a_i)\right)\right) = \mathbb{P}^N,$$

where $N = \sum_{i=1}^r \binom{a_i+n}{n} - 1$. We denote by $\pi \colon X \to \mathbb{P}^n$ the bundle projection. Note that $\dim X = r + n - 1$ and the immersion in $\mathbb{P}^N$ corresponds to the canonical invertible sheaf $\mathcal{O}_X(1)$ constructed on X ([45, II, Section 7]).
Indeed X is parametrized by

$$\left(\lambda^{(1)} l^{a_1}, \ldots, \lambda^{(r)} l^{a_r}\right) \in \bigoplus_{i=1}^r H^0\left(\mathcal{O}_{\mathbb{P}^n}(a_i)\right),$$

where $\lambda^{(i)}$ are scalars. X coincides with polynomial vectors of rank 1, recall Definition 3. It follows that the k-secant variety to X is parametrized by

$$\sum_{i=1}^k \left(\lambda_i^1 l_i^{a_1}, \ldots, \lambda_i^r l_i^{a_r}\right),$$

where λ_i^j are scalars and $l_i \in \mathbb{C}[x_0, \ldots, x_n]_1$. This construction appears already in [33] in the case $a_i = i$ for $i = 1, \ldots, d$. Since X is not contained in a hyperplane, it follows that any polynomial vector has a Waring decomposition as in (3.3). Thus, the number of decompositions by means of k linear forms of $f_1, \ldots, f_r$ is equal to the *k-secant degree* of X.

If $a_i = a$ for all $i \in \{1, \ldots, r\}$, then we deal with $\mathbb{P}^{r-1} \times \mathbb{P}^n$ embedded through the Segre-Veronese map with $\mathcal{O}(1, a)$, as we can see in Proposition 1.3 of [35] or in [7]. Moreover, we remark that perfectness in the sense of Definition 3.2 is equivalent to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^n}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^n}(a_r))$ being a *perfect* variety, i.e. $(n + r)|N$.

With this language, Theorem 3.3 has the following reformulation (compare with Claim 5.3 and Proposition 1.14 of [31]):

Corollary 3.6. If (3.4) and $a_1 + 1 \geq k$ hold, then $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r))$ is k-identifiable, i.e. its k-secant degree is equal to 1.

Remark 3.7. A formula for the dimension of the k-secant variety of the rational normal scroll X for $n = 1$ has been given in [22, pag. 359] with a sign mistake, corrected in [31, Proposition 1.14].

Remark 3.8. Consider the Veronese variety $V_{d,n} \subset \mathbb{P}^{\binom{d+n}{n}-1}$, and let

$$s - 1 = \operatorname{codim} V_{d,n}.$$

Then s general points determine a unique $\mathbb{P}^{s-1}$ that intersects $V_{d,n}$ in d^n points. The d^n points are linearly independent only if $d^n = s$, that is, either $n = 1$ or $d = n = 2$. This shows that a general vector $f = (f_1, \ldots, f_s)$ of forms of degree d admits $\binom{d^n}{s}$ decompositions, see the table at the end of Section 3.3 for some numerical examples. On the other hand, from a different perspective, dropping the requirement that the linear forms giving the decompositions are linearly independent, this shows that there is a unique set of d^n linear forms that decompose the general vector f and span a linear space of dimension $s-1$. Note that this time only the forms and not the coefficients are uniquely determined. We will not dwell on this point of view here and leave it for future work.

3.2 Nonabelian Apolarity and Identifiability

Let V be a complex vector space of dimension $n+1$ and let $f \in \mathrm{S}ym^d V$. For any $e \in \mathbb{Z}$, Sylvester constructed the catalecticant map

$$C_f \colon \mathrm{S}ym^e V^* \to \mathrm{S}ym^{d-e} V,$$

which is the contraction by f. Its main property is the inequality $\operatorname{rk} C_f \leq \operatorname{rk} f$, where the rank on the left-hand side is the rank of a linear map, while the rank on the right-hand side has been defined in the Introduction. In particular the $(k+1)$-minors of C_f vanish on the variety of polynomials with rank bounded by k, which is $\mathbb{S}\mathrm{ec}_k(V_{d,n})$.

The catalecticant map behaves well with polynomial vectors. If

$$f \in \bigoplus_{i=1}^{r} \mathrm{S}ym^{a_i} V,$$

then for any $e \in \mathbb{Z}$ we define the catalecticant map

$$C_f \colon \mathrm{S}ym^e V^* \to \bigoplus_{i=1}^{r} \mathrm{S}ym^{a_i - e} V$$

which is again the contraction by f. If f has rank one, this means that there exists $l \in V$ and scalars $\lambda^{(i)}$ such that $f = \left(\lambda^{(1)} l^{a_1}, \ldots, \lambda^{(r)} l^{a_r}\right)$. It follows that rk $C_f \leq 1$, since the image of C_f is generated by $\left(\lambda^{(1)} l^{a_1 - e}, \ldots, \lambda^{(r)} l^{a_r - e}\right)$, which is zero if and only if $a_r < e$. Linearity implies the basic inequality

$$\operatorname{rk} C_f \leq \operatorname{rk} f.$$

Again the $(k+1)$-minors of C_f vanish on the variety of polynomial vectors with rank bounded by k, which is $\mathbb{S}\mathrm{ec}_k(X)$, where X is the projective bundle defined in Section 3.1.

A classical example is the following. Assume $V = \mathbb{C}^3$. London showed in [54] (see also [74]) that a pencil of ternary cubics $f = (f_1, f_2) \in \mathrm{S}ym^3 V \oplus \mathrm{S}ym^3 V$ has *border rank* ≤ 5 (the border rank of f is the smallest number k such that f is in the Zariski closure of the set of polynomial vectors in $\mathrm{S}ym^3 V \oplus \mathrm{S}ym^3 V$ of rank k) if and only if $\det C_f = 0$ where $C_f \colon \mathrm{S}ym^2 V^* \to V \oplus V$ (see [19, Remark 4.2] for a modern reference). Indeed $\det C_f$ is the equation of $\mathbb{S}\mathrm{ec}_5(X)$, where X is the Segre-Veronese variety $\left(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}_X(1,3)\right)$. Note that X is 5-defective according to Definition 1.15 and this phenomenon is pretty similar to the case of Clebsch quartics recalled in the introduction.

The following result goes back to Sylvester.

Proposition 3.9 (Classical Apolarity)**.** Let

$$f = l_1^d + \dots l_k^d \in Sym^d V$$

and let $Z = \{l_1, \dots, l_k\} \subset V$. Let $C_f \colon \mathrm{Sym}^e V^* \to \mathrm{Sym}^{d-e} V$ be the contraction by f. Assume the rank of C_f equals k. Then

$$\mathrm{Bs}\,(\ker(C_f)) \supseteq Z.$$

Proof. The Apolarity Lemma (see [69]) says that $I_Z \subset f^{\perp}$, which reads in degree e as $H^0(I_Z(e)) \subset \ker(C_f)$. Look at the subspaces in this inclusion as subspaces of $H^0(\mathbb{P}^n, \mathcal{O}(e))$. The assumption on the rank implies that

$$\operatorname{codim} H^0(I_Z(e)) \leq k = \operatorname{rk} C_f = \operatorname{codim}(\ker(C_f)),$$

hence we have the equality $H^0(I_Z(e)) = \ker(C_f)$. It follows that

$$\mathrm{Bs}(\ker\,(C_f)) = \mathrm{Bs}\, H^0(I_Z(e)) \supseteq Z.$$

□

Classical Apolarity is a powerful tool to recover Z from f, hence it is a powerful tool to write down a minimal Waring decomposition of f.

The following Proposition (compare with [63, Proposition 4.3]) is a further generalization and it reduces to classical apolarity when $(X, L) = (\mathbb{P}(V), \mathcal{O}(d))$ and $E = \mathcal{O}(e)$ is a line bundle. The vector bundle E may have larger rank which explains the name of Nonabelian Apolarity.

We recall that the natural map $H^0(E) \otimes H^0(E^* \otimes L) \to H^0(L)$ induces the linear map $H^0(E) \otimes H^0(L)^* \to H^0(E^* \otimes L)^*$, then for any $f \in H^0(L)^*$ we have the contraction map $A_f \colon H^0(E) \to H^0(E^* \otimes L)^*$.

Proposition 3.10 (Nonabelian Apolarity)**.** Let X be a variety, let $L \in \mathrm{Pic}(X)$ be a very ample line bundle. If we set $W = H^0(X, L)^*$, then L gives the embedding $X \subset \mathbb{P}(W)$. Let E be a vector bundle on X. Let $f = \sum_{i=1}^k w_i \in W$ with $z_i = [w_i] \in X$, let $Z = \{z_1, \dots, z_k\} \subset \mathbb{P}(W)$ and let

$$A_f \colon H^0(E) \to H^0(E^* \otimes L)^*$$

be the induced map. Assume that $\operatorname{rk} A_f = k \cdot \operatorname{rk} E$. Then $\mathrm{Bs}(\ker(A_f)) \supseteq Z$.

In all the cases that we apply the Proposition, we will compute separately $\mathrm{rk} A_f$.

Nonabelian Apolarity enhances the power of Classical Apolarity and may detect a minimal Waring decomposition of a polynomial in some cases when Classical Apolarity fails, see Proposition 3.11. Our main examples start with the quotient bundle Q on $\mathbb{P}^n = \mathbb{P}(V)$. It has rank n and it is defined by the Euler exact sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \otimes V^* \to Q \to 0.$$

Let $L = \mathcal{O}(d)$ and $E = Q(e)$. Any $f \in Sym^d V$ induces the contraction map

$$A_f \colon H^0(Q(e)) \to H^0(Q^*(d-e))^* \cong H^0(Q(d-e-1))^*. \tag{3.6}$$

The following was the argument used in [63] to prove cases *(ii)* and *(iii)* of (3.5).

Proposition 3.11. Let X be a variety, let $L \in \mathrm{Pic}(X)$ be a very ample line bundle and let E be a vector bundle on X with $\mathrm{rk}\, E = \dim X$ and $c_{\mathrm{rk}\, E}(E) = k$. Let $[f]$ be a point in $\mathbb{P}(H^0(L)^*)$, $k = \frac{h^0(X,L)}{\dim X+1}$ and

$$A_f \colon H^0(E) \to H^0(E^* \otimes L)^*$$

the contraction map. Assume that for general f, $\mathrm{rk} A_f = k \cdot \mathrm{rk} E$, and there is some f such that the base locus of $\ker A_f$ is given by k points. Then the k-secant map

$$\pi_k : \mathrm{Sec}_k(X) \to \mathbb{P}(H^0(L)^*)$$

is birational. The assumptions are verified in the following cases, corresponding to (ii) and (iii) of (3.5).

(X, L)	$H^0(L)$	rank	E
$(\mathbb{P}^2, \mathcal{O}(5))$	$Sym^5\mathbb{C}^3$	7	$Q_{\mathbb{P}^2}(2)$
$(\mathbb{P}^3, \mathcal{O}(3))$	$Sym^3\mathbb{C}^4$	5	$Q^*_{\mathbb{P}^3}(2)$

Specific f's in the statement may be found as random polynomials in [43]. In order to prove also cases *(iv)* and *(v)* of (3.5), and moreover our Theorem 3.4, we need to extend this result as follows.

Theorem 3.12. Let $X \xrightarrow{\pi} Y$ be a projective bundle, $L = \mathcal{O}_X(1)$ as in Section 3.1 which we assume to be very ample and embeds the fibers of π as linear spaces. Let F be a vector bundle on Y and let $E = \pi^* F$. Let $[f]$ be a point in $\mathbb{P}(H^0(L)^*)$, $k = \frac{h^0(X,L)}{\dim X+1}$ and

$$A_f \colon H^0(E) \to H^0(E^* \otimes L)^*$$

the contraction map. Let $a = \frac{\dim Y}{\mathrm{rk} F}$ be an integer and $(c_{\mathrm{rk} F} F)^a = k$. Assume that X is not k-defective and that $\mathrm{rk} A_f = k \cdot \mathrm{rk} E$ for general f. Furthermore assume that there is some f such that the base locus of $\ker A_f$ is given by k fibers of π. Then the k-secant map

$$\pi_k : \mathrm{Sec}_k(X) \to \mathbb{P}(H^0(L)^*)$$

is birational. The assumptions are verified in the following cases.

X	$H^0(L)$	rank $= k$	F	a
$\mathbb{P}(\mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(2))$	$\left(Sym^2\mathbb{C}^{n+1}\right)^{\oplus 2}$	$n+1$	$Q_{\mathbb{P}^n}(1)$	1
$\begin{cases} \mathbb{P}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i)) \\ \text{if } k \leq a_1 + 1 \end{cases}$	$\oplus_{i=1}^r Sym^{a_i}\mathbb{C}^2$	$\frac{\sum_{i=1}^r (a_i+1)}{r+1}$	$\mathcal{O}_{\mathbb{P}^1}(k)$	1
$\mathbb{P}\left(\mathcal{O}_{\mathbb{P}^2}(2)^4\right)$	$(Sym^2\mathbb{C}^3)^{\oplus 4}$	4	$\mathcal{O}_{\mathbb{P}^2}(2)$	2
$\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(3))$	$Sym^2\mathbb{C}^3 \oplus Sym^3\mathbb{C}^3$	4	$\mathcal{O}_{\mathbb{P}^2}(2)$	2
$\mathbb{P}\left(\mathcal{O}_{\mathbb{P}^2}(3)^2 \oplus \mathcal{O}_{\mathbb{P}^2}(4)\right)$	$\left(Sym^3\mathbb{C}^3\right)^{\oplus 2} \oplus Sym^4\mathbb{C}^3$	7	$Q_{\mathbb{P}^2}(2)$	1

Proof. Let Z be as in Proposition 3.10. We get $Z \subset \mathrm{Bs}(\ker A_f)$, where the base locus can be found by the common zero locus of some sections $s_1, \dots, s_a$ of E which span $\ker A_f$. Since $E = \pi^* F$ and $H^0(X, E)$ is naturally isomorphic to $H^0(Y, F)$, the zero locus of each section of E corresponds to the pullback through π of the zero locus of the corresponding section of F. By the assumption on the top Chern class of F, we expect that the base locus of $\ker A_f$ contains $k = \deg Z$ fibers of the projective bundle X. The hypothesis guarantees that this expectation is realized for a specific polynomial vector f. By semicontinuity, it is realized for the generic f. This determines the forms l_i in (3.3) for a generic polynomial vector f. It follows that f is in the linear span of the fibers $\pi^{-1}(\pi(l_i))$, where $Z = \{l_1, \dots, l_k\}$. Fix representatives for the forms l_i for $i = 1, \dots, k$. Now the scalars λ_i^j in (3.3) are found by solving a linear system. By assumption we have that X is not k-defective (note that this assumption is satisfied in the setting of Proposition 3.11, since otherwise the base locus of $\ker A_f$ should be positive dimensional). In particular, the tangent spaces at points in Z, which are general, are independent by the Terracini Lemma. Since each π-fiber is contained in the corresponding tangent space, it follows that the fibers $\pi^{-1}(l_i)$ corresponding to $l_i \in Z$ are independent. Therefore that the scalars λ_i^j in (3.3) are uniquely determined and we have generic identifiability. We checked that the assumptions are verified in the cases listed with random polynomials, with the aid of Macaulay2 package [43]. $\square$

Remark 3.13. In all the cases listed in Theorem 3.12, by the projection formula, we have the natural isomorphism $H^0(X, E^* \otimes L) \cong H^0(Y, F \otimes \pi_* L)$.

Note that the second case in the list of Theorem 3.12 corresponds to Theorem 3.3 of Ciliberto-Russo. In this case $H^0(E) = \mathrm{S}ym^k\mathbb{C}^2$ has dimension $k+1$, $H^0(E^* \otimes L) = \mathrm{S}ym^{a_1-k}\mathbb{C}^2 \oplus \ldots \oplus \mathrm{S}ym^{a_r-k}\mathbb{C}^2$ has dimension $\sum_{i=1}^r (a_i - k + 1) = k$ (if $k \leq a_1 + 1$) and the contraction map A_f has rank k, with one-dimensional kernel.

The last case in the list of Theorem 3.12 corresponds to Theorem 3.4. A general vector $f \in (\mathrm{S}ym^3\mathbb{C}^3)^{\oplus 2} \oplus \mathrm{S}ym^4\mathbb{C}^3$ induces the contraction

$$A_f \colon H^0(Q(2)) \to H^0(Q) \oplus H^0(Q) \oplus H^0(Q(1))$$

with one-dimensional kernel. Each element in the kernel vanishes on 7 points which give the seven Waring summands of f.

Moreover, observe that $\left(\mathbb{P}\left(\mathcal{O}_{\mathbb{P}^2}(2)^4\right), \mathcal{O}_X(1)\right)$ coincides with the Segre-Veronese variety $(\mathbb{P}^3 \times \mathbb{P}^2, \mathcal{O}(1,2))$.

Remark 3.14. The assumption $a_1 + 1 \geq k$ in 3.3 is equivalent to

$$\frac{1}{r+1}\sum_{i=1}^{r}(a_i + 1) \leq a_1 + 1,$$

which means that the a_i are "balanced".

We conclude this section by showing how the existence of a unique decomposition determines the birational geometry of the varieties parametrizing higher rank decompositions. The following is just a slight generalization of [55, Theorem 4.4].

Theorem 3.15. Let $X \subset \mathbb{P}^N$ be a unirational variety such that the k-secant map $\pi_k : \mathrm{Sec}_k(X) \to \mathbb{P}^N$ is birational. If $\operatorname{codim} X \geq h \geq k$, then the variety $\pi_h^{-1}(p)$ is unirational for a general point $p \in \mathbb{P}^N$. In particular, it is irreducible.

Proof. Let $p \in \mathbb{P}^N$ be a general point. For $h > k$, we have

$$\dim \pi_h^{-1}(p) = h \cdot \dim X + h - 1 - N = (h-k)(\dim X + 1).$$

Note that, for $q \in \mathbb{P}^N$ general, a general point $x \in \pi_h^{-1}(q)$ is uniquely associated to a set of h points $\{x_1, \ldots, x_h\} \subset X$ and a h-tuple $(\lambda_1, \ldots, \lambda_h) \in \mathbb{C}^h$ with the requirement that

$$q = \sum \lambda_i x_i.$$

Thus the birationality of π_k allows us to associate, to a general point $q \in \mathbb{P}^N$, its unique decomposition in sum of k factors. That is, $\pi_k^{-1}(q) = (q, [\Lambda_k(q)])$ for a general point $q \in \mathbb{P}^N$, where $[\Lambda_k(q)] \subset \mathbb{G}r_{k-1}$ is such that $q \in \Lambda_k(q)$ (see Section 1.3). Via this identification we may define a map

$$\psi_h : (X \times \mathbb{P}^1)^{h-k} \dashrightarrow \pi_h^{-1}(p)$$

given by

$$(x_1, \lambda_1, \ldots, x_{h-k}, \lambda_{h-k}) \mapsto (p, [\langle x_1, \ldots, x_{h-k}, \Lambda_k(p - \lambda_1 x_1 - \ldots - \lambda_{h-k} x_{h-k})\rangle]).$$

When $\operatorname{codim} X > h$, $\Lambda_h(q)$ intersects X in exactly h points by the Trisecant Lemma, see for instance [23, Proposition 2.6]. Hence the map ψ_h is generically finite, of degree $\binom{h}{k}$, and dominant. In a similar way, if $\operatorname{codim} X = h$, then ψ_h is generically finite of degree $\binom{\deg X}{k}$. This is sufficient to show the claim. $\square$

Theorem 3.15 applies to all decompositions that admit a unique form.

Corollary 3.16. Let $f = (f_1, \ldots, f_r)$ be a vector of general homogeneous forms. Assume f has a unique Waring decomposition of rank k. If

$$\binom{n+a_1}{n} + \ldots + \binom{n+a_r}{n} - r - n \geq h > k,$$

then the set of rank h decompositions of f is parametrized by a unirational variety.

Remark 3.17. Let's go back to our starting example (3.1) and specialize

$$f_1 = \sum_{i=0}^{n} x_i^2$$

to the euclidean quadratic form. Then any minimal Waring decomposition of f_1 consists of $n+1$ orthogonal summands, with respect to the euclidean form. It follows that the decomposition (3.1) is equivalent to the diagonalization of f_2 with orthogonal summands. Over the reals, this is possible for any f_2 by the Spectral Theorem.

Also Robert's Theorem, see (v) of (3.5), has a similar interpretation. If $f_1 = x_0^2 + x_1^2 + x_2^2$ and $f_2 \in \mathrm{S}ym^3\mathbb{C}^3$ is general, the unique Waring decomposition of the pair (f_1, f_2) consists of four representatives of lines $\{l_1, \ldots, l_4\}$ and scalars $\lambda_1, \ldots, \lambda_4$ such that

$$\begin{cases} f_1 & = & \sum_{i=1}^4 l_i^2 \\ \\ f_2 & = & \sum_{i=1}^4 \lambda_i l_i^3 \end{cases} \tag{3.7}$$

Denote by L the 3×4 matrix whose i-th column is given by the coefficients of l_i. Then the first condition in (3.7) is equivalent to the equation

$$LL^t = I. \tag{3.8}$$

This equation generalizes orthonormal bases and the columns of L make a *Parseval* frame, according to [18] Section 2.1. So Robert's Theorem states that the general ternary cubic has a unique decomposition consisting of a Parseval frame.

In general a Parseval frame for a field $\mathbb{F}$ is given by $\{l_1, \ldots, l_n\} \subset \mathbb{F}^d$ such that the corresponding $d \times n$ matrix L satisfies the condition $LL^t = I$. This is equivalent to the equation

$$\sum_{i=1}^n \left(\sum_{j=1}^d l_{ji} x_j \right)^2 = \sum_{i=1}^d x_i^2,$$

and again this corresponds to a Waring decomposition of the euclidean form in $\mathbb{F}^d$ with n summands. This makes a connection between this Chapter and [64], which studies frames in the setting of secant varieties and tensor decomposition. For example equation (7) in [64] defines a solution to (3.8) with the additional condition that the four columns have unit norm. Note that equation (8) in [64] defines a Waring decomposition of the pair (f_1, T). Unfortunately, the additional condition about unitary norm does not allow the results of [64] to be directly transferred to our setting, but we believe this connection deserves to be pushed further.

It is interesting to notice that the decompositions of moments M_2 and M_3 in [2, Section 3] are (simultaneous) Waring decompositions of the quadric M_2 and the cubic M_3.

3.3 Computational approach

In this section we describe how we can face Question 1 and Question 2 from the point of view of computational analysis.

With the aid of Bertini [9, 10] and Macaulay2 [43] software systems, we can construct algorithms, based on homotopy continuation techniques and monodromy loops, that, in the spirit of [46], yield the number of Waring decompositions of a generic polynomial vector

$$f = (f_1, \ldots, f_r) \in \mathrm{Sym}^{a_1}\mathbb{C}^{n+1} \oplus \ldots \oplus \mathrm{Sym}^{a_r}\mathbb{C}^{n+1}$$

with high probability. Precisely, given $n, r, a_1, \ldots, a_r, k \in \mathbb{N}$ satisfying (3.4) and coordinates $x_0, \ldots, x_n$, we focus on the polynomial system

$$\begin{cases} f_1 = \lambda_1^1 l_1^{a_1} + \ldots + \lambda_k^1 l_k^{a_1} \\ \quad \vdots \\ f_r = \lambda_1^r l_1^{a_r} + \ldots + \lambda_k^r l_k^{a_r} \end{cases} \tag{3.9}$$

where $f_j \in \mathrm{S}ym^{a_j}\mathbb{C}^{n+1}$ is a fixed general element, while

$$l_i = x_0 + \sum_{h=1}^{n} l_h^i x_h \in \mathbb{P}(\mathbb{C}[x_0, \dots, x_n]_1) \text{ and } \lambda_i^j \in \mathbb{C}$$

are unknown. By expanding the expressions on the right hand side of (3.9) and by applying the identity principle for polynomials, the j-th equation of (3.9) splits into $\binom{a_j+n}{n}$ conditions. Our aim is to compute the number of solutions of

$$F_{(f_1,\dots,f_r)}\left([l_1^1, \dots, l_n^1, \lambda_1^1, \dots, \lambda_1^r], \dots, [l_1^k, \dots, l_n^k, \lambda_k^1, \dots, \lambda_k^r]\right),$$

the square non linear system of order $k(r+n)$, arising from the equivalent version of (3.9) in which in each equation all the terms are on one side of the equal sign. In practice, to work with general f_j's, we assign random complex values $\overline{l}_h^i$, $\overline{\lambda}_i^j$ to l_h^i, λ_i^j and, by means of $F_{(f_1,\dots,f_r)}$, we compute the corresponding $\overline{f}_1, \dots, \overline{f}_r$, the coefficients of which are so called *start parameters*. In this way, we know a solution

$$\left(\left[\overline{l}_1^1, \dots, \overline{l}_n^1, \overline{\lambda}_1^1, \dots, \overline{\lambda}_1^r\right], \dots, \left[\overline{\lambda}_1^k, \dots, \overline{l}_n^k, \overline{\lambda}_k^1, \dots, \overline{\lambda}_k^r\right]\right) \in \mathbb{C}^{k(r+n)}$$

of $F_{(\overline{f}_1,\dots,\overline{f}_r)}$, i.e. a Waring decomposition of $\overline{f} = (\overline{f}_1, \dots, \overline{f}_r)$, which is called a *startpoint*. Then we consider F_1 and F_2, two square polynomial systems of order $k(n+r)$ obtained from $F_{(\overline{f}_1,\dots,\overline{f}_r)}$ by replacing the constant terms with random complex values. We therefore construct 3 segment homotopies

$$H_i : \mathbb{C}^{k(r+n)} \times [0,1] \to \mathbb{C}^{k(r+n)}$$

for $i \in \{0,1,2\}$: H_0 between $F_{(\overline{f}_1,\dots,\overline{f}_r)}$ and F_1, H_1 between F_1 and F_2, H_2 between F_2 and $F_{(\overline{f}_1,\dots,\overline{f}_r)}$. Through H_0, we get a *path* connecting the startpoint to a solution of F_1, called *endpoint*, which therefore becomes a startpoint for the second step given by H_1, and so on. At the end of this loop, we compare the output Waring decomposition with the starting one. If they are equal, this procedure suggests that the case under investigation is identifiable, otherwise we iterate this technique with these two *startingpoints*, and so on. If at a certain point, the number of solutions of $F_{(\overline{f}_1,\dots,\overline{f}_r)}$ stabilizes, then, with high probability, we know the number of Waring decompositions of a generic polynomial vector in $\mathrm{S}ym^{a_1}\mathbb{C}^{n+1} \oplus \dots \oplus \mathrm{S}ym^{a_r}\mathbb{C}^{n+1}$.
We have implemented the homotopy continuation technique both in the software Bertini [9], in conjunction with Matlab, and in the software Macaulay2, with the aid of the package *Numerical Algebraic Geometry* [49].

Before starting with this computational analysis, we need to check that the variety $\mathbb{P}(\mathcal{O}_{\mathbb{P}^n}(a_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^n}(a_r))$, introduced in Section 1.3, is not k-defective, in which case (3.9) has no solutions. In order to do that, by using Macaulay2, we can construct a probabilistic algorithm based on Theorem 1.17, that computes the dimension of the span of the affine tangent spaces to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^n}(a_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^n}(a_r))$ at k random points and then we can apply semicontinuity properties.

In the following table we summarize the results we are able to obtain by combining numerical and theoretical approaches. Our technique is as follows. We first apply the probabilistic algorithm, checking k-defectivity, described above.

If this suggests positive k-defect δ_k, we do not pursue the computational approach. When δ_k is zero, we apply the homotopy continuation technique. If the number of decompositions (up to order of summands) stabilizes to a number, $\#_k$, we indicate it. If homotopy technique does not stabilize to a fixed number, we apply degeneration techniques like in Section 3.4 to get a lower bound. If everything fails, we put a question mark. Bold degrees are the ones obtained via theoretical arguments.

r	n	$(a_1,\dots,a_r)$	k	δ_k	$\#_k$
2	2	$(4,5)$	9	0	3
2	2	$(6,6)$	14	0	≥ 2
2	2	$(6,7)$	16	0	≥ 8
2	3	$(2,4)$	9	2	
3	2	$(2,2,6)$	8	4	
3	2	$(3,3,4)$	7	0	**1**
3	2	$(3,4,4)$	8	0	4
3	2	$(5,5,6)$	14	0	205
3	3	$(3,3,3)$	10	0	56
4	2	$(2,2,4,4)$	7	2	
4	2	$(2,3,3,3)$	6	0	2
4	2	$(4,\dots,4)$	10	0	?
5	2	$(5,\dots,5,6)$	16	0	?
6	2	$(2,\dots,2,3)$	5	3	
6	4	$(2.\dots,2)$	9	0	45
7	3	$(2,\dots,2)$	7	0	**8**
8	2	$(3,\dots,3)$	8	0	**9**
8	2	$(2,\dots,2,6)$	7	7	
11	4	$(2,\dots,2)$	11	0	**4368**
13	2	$(4,\dots,4)$	13	0	**560**
15	2	$(4,\dots,4,6)$	14	6	
17	3	$(3,\dots,3)$	17	0	**8436285**
19	2	$(5,\dots,5)$	19	0	**177100**
26	2	$(6,\dots,6)$	26	0	**254186856**

3.4 Identifiability of pairs of ternary forms

In this Section we aim to study the identifiability of pairs of ternary forms. In particular, we study the special case of two forms of degree a and $a+1$, focusing on

$$X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(a) \oplus \mathcal{O}_{\mathbb{P}^2}(a+1)).$$

Note that X can also be seen as a special linear section of

$$\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}(a, a+1)).$$

Our main result is the following.

Theorem 3.18. Let $a \geq 2$ be an integer. A general pair of ternary forms of degree a and $a+1$ is identifiable if and only if $a = 2$. Moreover there are finitely many decompositions if and only if $a = 2t$ is even, and in this case the number of decompositions is at least

$$\frac{(3t-2)(t-1)}{2} + 1.$$

The Theorem has two directions: on one hand we need to prove that $a = 2$ is identifiable, on the other we need to show that for $a > 2$ a general pair is never identifiable. The former is a classical result we already recalled in (*iii*) of (3.5) and in Theorem 3.12. For the latter, observe that $\dim \mathrm{Sec}_k(X) = 4k-1$, therefore if $4k - 1 \neq N$, then the general pair is never identifiable. We are left to consider the perfect case $N = 4k-1$. In this case we may assume that X is not k-defective (we will prove that this is always the case in Remark 3.27), otherwise the non identifiability is immediate. Hence the core of the question is to study generically finite maps

$$\pi_k : \mathrm{Sec}_k(X) \to \mathbb{P}^N,$$

with $4k = (a+2)^2$. This yields our last numerical constraint, namely, that $a = 2t$ needs to be even.

The first step is borrowed from [56, 57], and it is a slight generalization of [56, Theorem 2.1], see also [31].

Theorem 3.19. Let $X \subset \mathbb{P}^N$ be an irreducible variety of dimension n. Assume that the natural map

$$\pi_1 : \mathrm{Sec}_k(X) \to \mathbb{P}^N$$

is dominant and generically finite of degree d. Let $z \in \mathbb{S}\mathrm{ec}_{k-1}(X)$ be a general point. Consider the projection $\varphi : \mathbb{P}^N \dashrightarrow \mathbb{P}^n$ from the embedded tangent space $\mathbb{T}_z \mathbb{S}\mathrm{ec}_{k-1}(X)$. Then

$$\varphi_{|X} : X \dashrightarrow \mathbb{P}^n$$

is dominant and generically finite of degree at most d.

Proof. Choose a general point z on a general $(k-1)$-secant linear space spanned by $\langle p_1, \ldots, p_{k-1}\rangle$. Let $f : Y \to \mathbb{P}^N$ be the blow-up of $\mathbb{S}\mathrm{ec}_{k-1}(X)$ with exceptional divisor E, and fiber $F_z = f^{-1}(z)$. Let $y \in F_z$ be a general point. This point uniquely determines a linear space Π of dimension $(k-1)(n+1)$ that contains $\mathbb{T}_z \mathbb{S}\mathrm{ec}_{k-1}(X)$. Then the projection $\varphi_{|X} : X \dashrightarrow \mathbb{P}^n$ is generically finite of degree d if and only if $(\Pi \setminus \mathbb{T}_z \mathbb{S}\mathrm{ec}_{k-1}(X)) \cap X$ consists of just d points.

Assume that $\{x_1, \ldots, x_a\} \subset (\Pi \setminus \mathbb{T}_z \mathbb{S}\mathrm{ec}_{k-1}(X)) \cap X$. By the Terracini Lemma, Theorem 1.17,

$$\mathbb{T}_z \mathbb{S}\mathrm{ec}_{k-1}(X) = \langle \mathbb{T}_{p_1} X, \ldots, \mathbb{T}_{p_{k-1}} X\rangle.$$

Consider the linear spaces $\Lambda_i = \langle x_i, p_1, \ldots, p_{k-1}\rangle$. The Trisecant Lemma yields $\Lambda_i \neq \Lambda_j$, for $i \neq j$. Let Λ_i^Y and Π^Y be the strict transforms on Y. Since $z \in \langle p_1, \ldots, p_{k-1}\rangle$ and $y = \Pi^Y \cap F_z$, Λ_i^Y contains the point $y \in F_z$. In particular we have

$$\Lambda_i^Y \cap \Lambda_j^Y \neq \varnothing.$$

Let $\pi_1 : \mathrm{Sec}_k(X) \to \mathbb{P}^N$ be the morphism from the abstract secant variety, and $\mu : U \to Y$ the induced morphism, where $U = \mathrm{Sec}_k(X) \times_{\mathbb{P}^N} Y$. Then there exists a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\mu"] \arrow[d, "p"'] & Y \arrow[d, "f"] \\ \mathrm{Sec}_k(X) \arrow[r, "\pi_1"] & \mathbb{P}^N \end{tikzcd}$$

Let λ_i and Λ_i^U be the strict transforms of Λ_i in $\mathrm{Sec}_k(X)$ and U respectively. By Remark 1.16 $\lambda_i \cap \lambda_j = \varnothing$, so that

$$\Lambda_i^U \cap \Lambda_j^U = \varnothing.$$

This proves that $\sharp\mu^{-1}(y) \geq a$. But y is a general point of a divisor in the normal variety Y. Therefore $\deg \mu$, and henceforth $\deg \pi_1$, is at least a. $\square$

To apply Theorem 3.19 we need to better understand X and its tangential projections. Recall that a divisor D is a *monoid* if it is irreducible and it is singular in a point with multiplicity $\deg D - 1$. By definition we have

$$X \cong \mathbb{P}((\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}) \otimes \mathcal{O}_{\mathbb{P}^2}(a+1).$$

Then $X \subset \mathbb{P}^N$ can be seen as the embedding of $\mathbb{P}^3$ blown up in one point q embedded by monoids of degree $a+1$ with vertex q. In other words, if we let

$$\mathcal{L} = \mathbb{P}\left(\mathcal{I}_{q^a}(a+1)\right) \quad \text{and} \quad Y = \mathrm{Bl}_q \mathbb{P}^3,$$

then we have

$$X = \varphi_{\mathcal{L}}(Y) \subset \mathbb{P}^N.$$

It is now easy, via the Terracini Lemma (Theorem 1.17), to realize that the restriction of the tangential projection $\varphi_{|X} : X \dashrightarrow \mathbb{P}^3$ is given by the linear system

$$\mathcal{H} = \mathbb{P}\left(\mathcal{I}_{q^a \cup p_1^2 \cup \ldots \cup p_{k-1}^2}(a+1)\right)$$

on $\mathbb{P}^3$. We already assumed that X is not k-defective, that is, we work under the condition

$$\dim \mathcal{H} = 3. \tag{3.10}$$

Remark 3.20. It is interesting to note that for $a = 2$ the map $\varphi_{|X}$ is the standard Cremona transformation of $\mathbb{P}^3$, given by $(x_0, \ldots, x_3) \mapsto (1/x_0, \ldots, 1/x_3)$.

We need to degenerate the linear system $\mathcal{H}$ in the sense of Construction 1.8, and so we want to understand what happens to the degree of the associated rational map under specialization.

Lemma 3.21. Let Δ be a complex disk around the origin, X a variety and $\mathcal{O}_X(1)$ a base point free line bundle. Consider the product $V = X \times \Delta$, with the natural projections, π_1 and π_2. Let $V_t = X \times \{t\}$ and $\mathcal{O}_V(d) = \pi_1^*(\mathcal{O}_X(d))$. Fix a configuration $p_1, \ldots, p_l$ of l points on V_0 and let $\sigma_i : \Delta \to V$ be sections such that $\sigma_i(0) = p_i$ and $\{\sigma_i(t)\}_{i=1,\ldots,l}$ are general points of V_t for $t \neq 0$. Let

$$P = \bigcup_{i=1}^{l} \sigma_i(\Delta) \quad \text{and} \quad P_t = P \cap V_t.$$

Consider the linear system $\mathcal{H} = \mathbb{P}(H^0(\mathcal{O}_V(d) \otimes \mathcal{I}_{P^2}))$ on V, with $\mathcal{H}_t := \mathcal{H}_{|V_t}$, and assume that

$$\dim \mathcal{H}_0 = \dim \mathcal{H}_t = \dim X \text{ for every } t \in \Delta.$$

Let $d(t)$ be the degree of the map induced by $\mathcal{H}_t$. Then $d(0) \leq d(t)$.

Proof. If $\varphi_{\mathcal{H}_t}$ is not dominant for $t \neq 0$, then there is nothing else to prove. Let $t \neq 0$ and assume that $\varphi_{\mathcal{H}_t}$ is dominant. Then $\varphi_{\mathcal{H}_t}$ is generically finite and $\deg \varphi_{\mathcal{H}_t}(X) = 1$. Let

$$\mu : Z \times \Delta \to V$$

be a resolution of the base locus, and let $V_{Z_t} = \mu^* V_t$ and $\mathcal{H}_Z = \mu_*^{-1}\mathcal{H}$ be the strict transform linear systems on Z. Then V_{Z_t} is a blow-up of $V_t = X$ and $V_{Z_0} = \mu_*^{-1} V_0 + R$, for some effective (possibly trivial) residual divisor R. By hypothesis $\mathcal{H}_0$ is the flat limit of $\mathcal{H}_t$, for $t \neq 0$. Hence flatness forces

$$d(t) = \mathcal{H}_Z^{\dim X} \cdot V_{Zt} = \mathcal{H}_Z^{\dim X} \cdot (\mu_*^{-1} V_0 + R) \geq \mathcal{H}_Z^{\dim X} \cdot \mu_*^{-1} V_0 = d(0).$$

□

Lemma 3.21 guarantee that when we specialize, the degree can not increase. Therefore it allows us to work on a degenerate configuration to study the degree of the map induced by the linear system

$$\mathbb{P}(\mathcal{I}_{q^a \cup p_1^2 \cup \ldots \cup p_{k-1}^2}(a+1)) \subset \mathbb{P}(H^0 \mathcal{O}_{\mathbb{P}^3}(a+1)).$$

Lemma 3.22. Fix the numbers

$$b := \frac{t(t+3)}{2} \quad \text{and} \quad c := \frac{t(t+1)}{2}.$$

Let $H \subset \mathbb{P}^3 \setminus \{q\}$ be a plane, $B := \{p_1, \ldots, p_b\} \subset H$ a set of b general points, and $C := \{x_1, \ldots, x_c\} \subset \mathbb{P}^3 \setminus (\{q\} \cup H)$ a set of c general points. Let $a = 2t$ and let

$$\mathcal{H} := \mathbb{P}(\mathcal{I}_{q^a \cup C^2 \cup B^2}(a+1))$$

be the linear system of degree $a+1$ monoids of $\mathbb{P}^3$ with vertex q and double points along $B \cup C$. Let $\varphi_{\mathcal{H}}$ be the associated map. Then $\dim \mathcal{H} = 3$ and

$$\deg \varphi_{\mathcal{H}} > \frac{(3t-2)(t-1)}{2}.$$

Proof. Note that the lines $\langle q, p_i \rangle$ and $\langle q, x_i \rangle$ are contained in the base locus of $\mathcal{H}$ by construction. Let us start computing $\dim \mathcal{H}$. First we prove an intermediate result.

Claim 3.23. There is a unique element in $\mathcal{H}$ containing the plane H.

Proof of Claim 3.23. Let $D \in \mathcal{H}$ such that $D \supset H$. Then $D = H + R$, with $\deg R = a$. It follows that R is a cone with vertex q over a plane curve $\Gamma \subset H$. Moreover R is singular along C and has to contain B. This forces Γ to contain B and to be singular at $\langle q, x_1 \rangle \cap H$. In other words Γ is a plane curve of degree $2t$ with c general double points and passing through b general points. Note that

$$\binom{2t+2}{2} - 3c - b = 1.$$

It is well known, see for instance [1], that the c points impose independent conditions on plane curves of degree $2t$. Clearly the latter b simple points do the same, therefore there is a unique plane curve Γ satisfying the requirements. This shows that R is unique and the claim is proved. $\square$

We are ready to compute the dimension of $\mathcal{H}$.

Claim 3.24. $\dim \mathcal{H} = 3$.

Proof of Claim 3.24. The expected dimension of $\mathcal{H}$ is 3. Consider Castelnuovo sequence

$$0 \to \mathcal{H} - H \to \mathcal{H} \to \mathcal{H}_{|H} \to 0.$$

By Claim 3.23, it is enough to show that $\dim \mathcal{H}_{|H} = 2$. Observe that $\mathcal{H}_{|H}$ is a linear system of plane curves of degree $2t+1$ with b general double points and c simple general points. As in the proof of Claim 3.23, we compute the expected dimension

$$\binom{2t+3}{2} - 3b - c = 3,$$

and conclude by [1]. $\square$

Next we want to determine the base locus scheme of $\mathcal{H}_{|H}$. Let $\epsilon : S \to H$ be the blow-up of B and $\langle q, x_i \rangle \cap H$, with $\mathcal{H}_S$ the strict transform linear system. We will first prove the following.

Claim 3.25. The scheme base locus of $\mathcal{I}_{B^2,H}(2t+1)$ is B^2.

Proof. Let $\mathcal{L}_{ij} := \mathbb{P}(\mathcal{I}_{B\setminus\{p_i,p_j\},H}(t))$. Then

$$\dim \mathcal{L}_{ij} = \binom{t+2}{2} - b - 2 - 1 = 2.$$

By the Trisecant Lemma we conclude that

$$\operatorname{Bs} \mathcal{L}_{ij} = B \setminus \{p_i, p_j\}.$$

Let $\Gamma_i, \Gamma_j \in \mathcal{L}_{ij}$ such that $\Gamma_i \ni p_i$ and $\Gamma_j \ni p_j$. By construction, we have

$$D_{ij} := \Gamma_i + \Gamma_j + \langle p_i, p_j \rangle \in \mathcal{H}.$$

Let D_{ijS}, $\mathcal{L}_{ijS}$ be the strict transforms on S. Note that Γ_h belongs to a pencil of curves in $\mathcal{L}_{hk}$ for any k. These pencils do not have common base locus outside of B since $\mathcal{L}_{ijS}$ is base point free and $\dim \mathcal{L}_{ij} = 2$. Therefore the D_{ijS} have no common base locus. $\square$

Claim 3.26. $\mathcal{H}_S$ is base point free.

Proof. To prove the Claim it is enough to prove that the simple base points associated to C impose independent conditions. Since $C \subset \mathbb{P}^3$ is general this is again implied by the Trisecant Lemma. $\square$

Then we have

$$\deg \varphi_{\mathcal{H}_S} = \mathcal{H}_S^2 = (2t+1)^2 - 4b - c = \frac{(3t-2)(t-1)}{2}.$$

To conclude, observe that, by the same argument as the claims, we can prove that $\varphi_{\mathcal{H}|R}$ is generically finite. Therefore

$$\deg \varphi_{\mathcal{H}} > \deg \varphi_{\mathcal{H}|H} = \deg \varphi_{\mathcal{H}_S} = (2t+1)^2 - 4b - c = \frac{(3t-2)(t-1)}{2}.$$

□

Remark 3.27. Lemma 3.22 proves that $\deg \varphi_{\mathcal{H}}$ is finite. As a byproduct, we get that condition (3.10) is always satisfied in our range. That is X is not k-defective for $a = 2t$.

Proof of Theorem 3.18. We already know that the number of decompositions is finite only if $a = 2t$. By Remark 3.27, we conclude that the number is finite when $a = 2t$. Let d be the number of decompositions for a general pair. By Theorem 3.19, we know that $d \geq \deg \varphi$ where $\varphi : X \dashrightarrow \mathbb{P}^3$ is the tangential projection. The required bound is obtained combining Lemma 3.21 and Lemma 3.22. □

Chapter 4

The Waring problem for general polynomials via limits

The content of this Chapter is a joint work with Massimiliano Mella, now published as [41].

As a further application of the techniques presented in Chapter 2, we exploit limits of fat points to give new results about Waring decompositions. Namely we will prove that the only generically identifiable cases for $n > 1$ are (ii) and (iii) in list 3.5. The precise statement is the following.

Theorem 4.1. Let f be a general homogeneous form of degree d in $n+1$ variables. Then f has a unique Waring decomposition with r summands if and only if

$$(n, d, r) \in \{(1, 2k-1, k), (3, 3, 5), (2, 5, 7)\}.$$

Like in Section 3.4, the starting point is [56, Theorem 2.1], where it is proved that identifiability forces a particular tangential projection of the Veronese variety to be birational. Since this projection is associated to a linear system with imposed singularities, our main result is a consequence of the following statement about Cremona modifications of $\mathbb{P}^n$, which is of interest in itself.

Theorem 4.2. Let $\mathcal{L}_{n,d}(2^h)$ be the linear system of degree d divisors of $\mathbb{P}^n$ with h double points in general position, and $\varphi_{n,d,h}$ the associated rational map. Then $\varphi_{n,d,h}$ is a Cremona transformation, i.e. $\dim \mathcal{L}_{n,d}(2^h) = n$ and $\varphi_{n,d,h} : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is birational, if and only if

- $n = 1$, $d = 2k+1$, and $h = k$,
- $n = 2$, $d = 5$ and $h = 6$,
- $n = 3$, $d = 3$ and $h = 4$.

The main difficulty in proving Theorem 4.2 is to control the singularities and the base locus of the linear system $\mathcal{L}_{n,d}(2^h)$. The first task is accomplished in [56, Corollary 4.5]. It is proved that, for $d \geq 4$, the singularities of $\mathcal{L}_{n,d}(2^h)$ are only the double points imposed. The degree 3 case was recently completed in [25]. This allowed to conclude in the mentioned range. Unfortunately, if $d \leq n$, then it is necessary to control not only the singularities but also the base locus of these linear systems to bound the degree of the map. In [57] some special cases were proved assuming a divisibility condition on the degree. Here we approach the problem from a different perspective. Instead of trying to bound directly the degree of the map associated to $\mathcal{L}_{n,d}(2^h)$, we produce a degeneration of the imposed singularities in such a way that the limit linear system admits a hyperplane on which the restricted map is still expected to be non birational. We then proceed by induction trying to bound the sectional genus of the linear systems we are considering.

This reminds of the techniques of interpolation. Indeed the linear systems we are interested in were studied for the interpolation problem and we profit both of Alexander–Hirschowitz' paper [1] and of more recent approaches due to Postinghel [68] and Brambilla–Ottaviani [16]. The proof of Theorem 4.2 is done by induction on n. The induction step is done via a careful choice of numbers. This numerology is the core of the (differentiable) Horace method in [1], where a double induction on both degree and dimension is played. We are not able to control the sectional genus along these specialization, therefore we have to develop a different approach based only on dimension induction. We let some double points collapse into a triple point with tangent directions. The latter allows us to make induction work and to study the restriction of the linear system to a hyperplane.

This leads us to study the standard interpolation problems for linear systems with one triple point, with tangent directions in general position, and a bunch of double points. The first step of induction is the study of planar linear systems. Here we benefit from the theory developed around Conjecture 2.44. In particular, we use the results Proposition 2.46.8 about quasi-homogeneous multiplicity. As usual in interpolation problems, for low degrees $d \leq 5$ and in particular for cubics, we need special arguments. Once we work out the case with tangent direction in general position, we extend it to the set of tangent direction arising in the flat limit described in Proposition 2.19, thereby concluding the proof.

First we need a definition.

Definition 4.3. The *sectional genus* of a linear system is the geometric genus of a general curve section.

We will prove that the linear systems we are interested in have positive sectional genus, and this will imply the associated linear map can not be birational. For this purpose the following remark is extremely useful.

Remark 4.4. The genus of any curve of an algebraic system of algebraic curves is not greater than the genus of the generic curve of the system, [26]. Therefore, in order to prove that a linear system $\mathcal{L}$ has positive sectional genus, it is enough to exhibit a curve of positive genus in some algebraic family of curves whose general member is a curve section of $\mathcal{L}$.

We now reduce Theorem 4.1 to Theorem 4.2 following [56]. Let n, d be integers. Then a general polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]_d$ admits a unique decomposition

$$f = l_1^d + \ldots + l_s^d,$$

with $l_i \in \mathbb{C}[x_0, \dots, x_n]_1$ for $i \in \{1, \dots, s\}$, if and only if the s-secant map

$$\pi_s : \mathrm{Sec}_s(V_{d,n}) \to \mathbb{P}^N$$

of the Veronese variety is dominant and birational, where $N = \binom{n+d}{n} - 1$. In Section 1.3 we noted that $\dim \mathrm{Sec}_s(V_{d,n}) = s(n+1) - 1$, hence, in order for π_s to be birational, the number

$$k(n,d) := \frac{\binom{d+n}{n}}{n+1} \tag{4.1}$$

has to be an integer. Now let $\mathbb{S}\mathrm{ec}_s(V_{n,d})$ be the embedded s-secant variety. For a general point $z \in \mathbb{S}\mathrm{ec}_{k(n,d)-1}(V_{n,d})$, let $\varphi : \mathbb{P}^N \dashrightarrow \mathbb{P}^n$ be the projection from the embedded tangent space $\mathbb{T}_z \mathbb{S}\mathrm{ec}_{k(n,d)-1}(V_{n,d})$. As we did in Section 3.4, we can apply Theorem 3.19 to conclude that $\varphi_{|V_{n,d}}$ is birational whenever $\pi_{k(n,d)}$ is. Again, by Theorem 1.17, this map is associated to the linear system $\mathcal{L}_{n,d}(2^{k(n,d)-1})$. Therefore the morphism $\pi_{k(n,d)}$ is birational only if the map associated to $\mathcal{L}_{n,d}(2^{k(n,d)-1})$ is birational. It follows that Theorem 4.1 is a consequence of Theorem 4.2.

4.1 The induction step

In this section we develop the induction argument we need to prove Theorem 4.2. Thanks to [56] we may assume that $n \geq d$. Since linear systems of cubics with a triple point and double points are always special, a different strategy is needed for $d = 3$, so we also assume that $d \geq 4$.

Fix a point $q \in \mathbb{P}^n$ and a general linear space $\Pi \ni q$, of dimension 3. Let Z_n be a scheme having:

- multiplicity 3 in q together with $\binom{n+1}{2} - s(3,d,n)$ general tangent directions in q and $s(3,d,n)$ tangent directions on Π,
- $k(n,d) - n - 2 - h(3,d,n)$ general double points, $h(3,d,n)$ general double points on Π.

We will define the integers $h(3,d,n)$ and $s(3,d,n)$ later on.

At the linear system level let

$$\mathcal{L}_n(d) := \mathbb{P}(\mathcal{I}_{Z_n,\mathbb{P}^n}(d)).$$

We aim to prove the nonspeciality of $\mathcal{L}_n(d)$. For this purpose, we degenerate the scheme Z_n as follows. Fix a general hyperplane $H \subset \mathbb{P}^n$ containing Π. Let Z_n^H be a specialization of Z_n such that H contains $h(n-1,d,n)$ double points and $s(n-1,d,n)$ tangent directions, at the point q. Let $\mathcal{L}_n^H(d)$ be the specialized linear system.

By Construction 1.8, the nonspeciality of $\mathcal{L}_n(d)$ is implied by the nonspeciality of $\mathcal{L}_n^H(d)$. For the latter we can use the Castelnuovo exact sequence. Hence we are left to prove that the restricted linear system

$$\mathcal{L}_{n-1,d}(3[s(n-1,d,n)], 2^{h(n-1,d,n)})$$

and the kernel of the restriction map

$$\mathcal{L}_{n,d-1}(2[s(n,d,n)-s(n-1,d,n)],2^{k(n,d)-n-2-h(n-1,d,n)},1^{h(n-1,d,n)})$$

are nonspecial, where the base points are in the described special position. To prove the nonspeciality of these linear systems, we set up an induction argument choosing a sequence of integer in such a way that the first linear system has a unique divisor, and then use induction for the latter. Let us introduce the following notation.

Definition 4.5. Let $s(i,d,n)$ and $h(i,d,n)$ be non-negative integers. Assume $n \geq d \geq 4$. Fix a general flag of linear spaces

$$H_2 \subset H_3 \subset H_4 \subset \ldots \subset H_{n-1} \subset H_n = \mathbb{P}^n,$$

with $H_i \cong \mathbb{P}^i$, $H_3 = \Pi$ and $q \in H_2$. For $i \in \{2,\ldots,n\}$, consider a 0-dimensional scheme $Z_i \subset H_i$ such that:

- Z_{i-1} is a flat limit of $Z_{i|H_{i-1}}$,
- Z_i has multiplicity 3 in q together with $s(i,d,n)$ infinitely near points, $s(i-1,d,n)$ of which are tangent directions supported in H_{i-1},
- for every $i \in \{3,\ldots,n\}$, Z_i has $h(i,d,n)$ double points, $h(i-1,d,n)$ of which are supported in H_{i-1},
- Z_2 has $h(2,d,n)$ double points.

Finally, set

$$\mathcal{L}_i^H(d) = \mathbb{P}(\mathcal{I}_{Z_i,H_i}(d)).$$

In order to define $s(i,d,n)$ and $h(i,d,n)$ in such a way we can argue by induction on n, we will need a few technical results. The first step is the following.

Lemma 4.6. Assume that for any $n \geq i \geq 2$ there are integers

$$h(i,d,n) \geq h(i-1,d,n) \ \text{ and } \ s(i,d,n) \geq s(i-1,d,n)$$

such that

i) $\operatorname{expdim} \mathcal{L}_i^H(d) = i$,

ii) $\dim \mathcal{L}_{i-1}^H(d) = i-1$,

iii) $\operatorname{expdim} \mathcal{L}_i^H(d) - H_{i-1} = 0$,

iv) $\dim \mathcal{L}_i^H(d) - 2H_{i-1} \leq 0$,

v) there is at most one divisor $D \in \mathcal{L}_i^H(d)$ with $\mathrm{mult}_q D > 3$,

vi) $\mathcal{L}_{i-1}^H(d)_{|\Pi} = \mathcal{L}_{3,d}(3[s(3,d,n)],2^{h(3,d,n)})$ for every $i > 3$,

vii)
- $h(i-1,d,n)-h(3,d,n)+s(i,d,n)-s(i-1,d,n) > (i-4)(i+1)$ for $i \geq 5$ and $d \geq 5$,
- $h(i-1,4,n)-h(3,4,n)+s(i,4,n)-s(i-1,4,n) > (i+1)$ for $5 \leq i \leq 8$,

- $h(i-1,4,n) - h(3,4,n) + s(i,4,n) - s(i-1,4,n) > (i-7)(i+1)$ for $i \geq 9$.

Then

1. $\dim \mathcal{L}_i^H(d) - H_{i-1} = 0$,
2. the divisor in $\mathcal{L}_i^H(d) - H_{i-1}$ does not contain Π,
3. $\mathcal{L}_i^H(d)$ is nonspecial,
4. $\mathcal{L}_i^H(d)_{|\Pi} = \mathcal{L}_{3,d}(3[s(3,d,n)], 2^{h(3,d,n)})$ for every $i \geq 3$.

Proof. By assumption i), $h(3,d,n) < \frac{\binom{d+3}{3}}{4} - 1$, the linear system $\mathcal{L}_i^H(d) - H_{i-1}$ is

$$\mathcal{L}_{i,d-1}(2[s(i,d,n) - s(i-1,d,n)], 2^{h(i,d,n)-h(i-1,d,n)}, 1^{h(i-1,d,n)}),$$

and it has non negative expected dimension by iii). For $i = 4$, by using Lemma 1.11, it is easy to check that the simple points on Π impose independent conditions and the general element in $\mathcal{L}_4^H(d) - \Pi$ does not contain Π. By vii) and Lemma 1.14, for $i > 4$, the simple base points on Π impose independent conditions and the general element in $\mathcal{L}_i^H(d) - H_{i-1}$ does not contain Π. This shows (2).

By assumption iv) and Lemma 1.11, the other simple base points on H_{i-1} impose independent conditions to $\mathcal{L}_{i,d-1}(2^{h(i,d,n)-h(i-1,d,n)+1})$.

Similarly assumption v) and Lemma 1.11 ensure that the tangential directions in q impose independent conditions to $\mathcal{L}_{i,d-1}(2^{h(i,d,n)-h(i-1,d,n)+1})$. Therefore we are left to prove that $\mathcal{L}_{i,d-1}(2^{h(i,d,n)-h(i-1,d,n)+1})$ is nonspecial. The latter is a linear system with only double points and positive expected dimension. Then by Theorem 1.5 we know it is nonspecial, keep also in mind Remark 1.6. This proves (1).

Then from the Castelnuovo exact sequence and assumption ii) the linear system $\mathcal{L}_i^H(d)$ is non special. To conclude observe that the nonspeciality of $\mathcal{L}_i^H(d)$, (1) and vi) yield (4). $\square$

To apply Lemma 4.6, we have first to produce the sequences of integers $h(i,d,n)$ and $s(i,d,n)$.

Proposition 4.7. Fix integers $n \geq d \geq 4$, and let $i \in \{3,\dots,n\}$. Assume that the number $k(n,d)$ defined in formula 4.1 is an integer. Then there are sequences $\{h(i,d,n)\}_{i\in\{2,\dots,n\}}$ and $\{s(i,d,n)\}_{i\in\{2,\dots,n\}}$ such that

1. $\operatorname{expdim} \mathcal{L}_{i,d-1}(2[s(i,d,n)-s(i-1,d,n)], 2^{h(i,d,n)-h(i-1,d,n)}, 1^{h(i-1,d,n)}) = 0$
2. $\operatorname{expdim} \mathcal{L}_{i-1,d}(3[s(i-1,d,n)], 2^{h(i-1,d,n)}) = i-1$

and the following properties hold:

i) we have

$$h(3,d,n) < \binom{d+2}{3} - 4,$$
$$h(n,d,n) = k(n,d) - 1 - n - 1,$$
$$h(n,d,n) - h(n-1,d,n) \geq \frac{d-1}{(n+2)(n+1)}\binom{n+d}{n} - 3 > 0,$$
$$h(i+1,d,n) - h(i,d,n) \geq \frac{d-1}{(i+2)(i+1)}\binom{i+d}{i} - 2 > 0$$

for every $i \in \{2, \dots, n-2\}$,

ii) $s(n,d,n) = \binom{n+1}{2}$ and

$$s(i-1,d,n) \in \left\{\frac{i^2-3i-2}{2}, \dots, \frac{i^2-i-4}{2}\right\},$$

iii) $s(2,d,n) \geq 0$,

iv) $s(i,d,n) \geq s(i-1,d,n)$ for every $i \in \{4, \dots, n-1\}$,

v) $s(i,d,n) - s(i-1,d,n) < \binom{i+1}{2}$ for every $i \in \{3, \dots, n\}$,

vi)
- $h(i-1,d,n) - h(3,d,n) + s(i,d,n) - s(i-1,d,n) > (i-4)(i+1)$ for $i \geq 5$ and $d \geq 5$,
- $h(i-1,4,n) - h(3,4,n) + s(i,4,n) - s(i-1,4,n) > (i+1)$ for $5 \leq i \leq 8$,
- $h(i-1,4,n) - h(3,4,n) + s(i,4,n) - s(i-1,4,n) > (i-7)(i+1)$ for $i \geq 9$.

Proof. In order to simplify the notation, we set for the moment $k := k(n,d)$, $s_i := s(i,d,n)$ and $h_i := h(i,d,n)$. Set $s_n = \binom{n+1}{2}$ and $h_n = k-1-n-1$. For $i \in \{3, \dots, n\}$, define

$$a_i =: a(i,d) := \binom{i+d-1}{i-1} - \frac{3i}{2} - \frac{i^2}{2}.$$

The linear system $\mathcal{L}^H_{i,d-1}(2[s_i - s_{i-1}], 2^{h_i - h_{i-1}}, 1^{h_{i-1}})$ has expected dimension

$$\exp_i := \binom{d-1+i}{i} - 1 - (i+1)(h_i - h_{i-1} + 1) - (s_i - s_{i-1}) - h_{i-1}.$$

Then assumption (1) reads $\exp_n = 0$ and yields

$$\binom{d-1+n}{n} - 1 - (n+1)(k-n-2-h_{n-1}+1) - h_{n-1} - \left(\binom{n+1}{2} - s_{n-1}\right) = 0.$$

This implies

$$\begin{aligned} nh_{n-1} + s_{n-1} &= -\binom{d-1+n}{n} + 2 + (n+1)k - (n+1)(n+2) + n + \binom{n+1}{2} \\ &= -\binom{d-1+n}{n} + 2 + \binom{n+d}{n} - (n+1)(n+2) + n + \binom{n+1}{2} \\ &= \binom{n+d-1}{n-1} + 2 - (n+1)(n+2) + n + \binom{n+1}{2} \\ &= a_n. \end{aligned}$$

Therefore h_{n-1} and s_{n-1} satisfy the equation

$$nh_{n-1} + s_{n-1} = a_n.$$

Note that

$$\frac{n^2 - n - 4}{2} - \frac{n^2 - 3n - 2}{2} = n - 1,$$

therefore there is a unique $t \in \left\{\frac{n^2-3n-2}{2}, \dots, \frac{n^2-n-4}{2}\right\}$ such that $a_n - t$ is a multiple of n. Call s_{n-1} that number and define

$$h_{n-1} = \frac{a_n - s_{n-1}}{n}.$$

This also settles (2), for $i = n$, since by construction

$$\operatorname{expdim} \mathcal{L}_{n,d}(3[s(n,d,n)], 2^{h(n,d,n)}) = n.$$

Hence

$$n = \exp_n + \operatorname{expdim} \mathcal{L}_{n-1,d}(3[s(n-1,d,n)], 2^{h(n-1,d,n)}) + 1.$$

In a similar fashion $\exp_{n-1} = 0$ gives

$$(n-1)h_{n-2} + s_{n-2} = \binom{n+d-2}{n-2} - \frac{n^2}{2} - \frac{n}{2} + 1 = a_{n-1}.$$

As before

$$\frac{n^2 - 3n - 2}{2} - \frac{n^2 - 5n + 2}{2} = n - 2,$$

and there is a unique $t \in \left\{\frac{n^2-5n+2}{2}, \dots, \frac{n^2-3n-2}{2}\right\}$ such that $a_{n-1} - t$ is a multiple of $n - 1$ Call s_{n-2} that number and define

$$h_{n-2} = \frac{a_{n-1} - s_{n-2}}{n-1}.$$

We iterate the argument, defining s_{i-1} to be the only natural number in $\left\{\frac{i^2-3i-2}{2}, \dots, \frac{i^2-i-4}{2}\right\}$ such that $i \mid a_i - s_{i-1}$. Hence we have

$$ih_{i-1} + s_{i-1} = a_i \tag{4.2}$$

and conditions ii) and iv) are satisfied.

Next we check $s(2,d,n) \geq 0$. By definition, $s(2,d,n) \geq -1$. Assume by contradiction that $s(2,d,n) = -1$. Then

$$a(3,d) = \frac{(d+1)(d+2)}{2} - 9 \equiv -1 \pmod 3$$

so $(d+1)(d+2) = 6t + 4 \equiv 1 \pmod 3$, and this is impossible because 1 is irreducible in $\mathbb{Z}_3$. Therefore condition iii) holds.

Let us check condition v). Assume first that $i < n$. Then

$$\begin{aligned} 2(s(i,d,n) - s(i-1,d,n)) &\leq (i+1)^2 - i - 5 - i^2 + 3i + 2 \\ &= 4i - 2 < (i+1)i \end{aligned}$$

for $i \geq 3$. Assume that $i = n$. Then

$$\begin{aligned} 2(s(n,d,n) - s(n-1,d,n)) &\leq 2\binom{n+1}{2} - n^2 + 3n + 2 \\ &= 4n + 2 < n(n+1) \end{aligned}$$

for $n \geq 4$.

Next we focus on condition i).

Claim 4.8. Set $i \geq 2$. Then

$$h_n - h_{n-1} \geq \frac{d-1}{(n+2)(n+1)}\binom{n+d}{n} - 3 > 0$$

and

$$h_{i+1} - h_i \geq \frac{d-1}{(i+2)(i+1)}\binom{i+d}{i} - 2 > 0 \text{ for every } i \leq n-2.$$

Proof of Claim 4.8. First assume that $i = n-1$. Then we have, by Equation (4.2),

$$\begin{aligned} h_n - h_{n-1} &= \frac{\binom{n+d}{n}}{n+1} - n - 3 - \frac{a_n - s_{n-1}}{n} \\ &\geq \frac{\binom{n+d}{n}}{n+1} - \frac{\binom{n+d-1}{n-1}}{n} - 3 \\ &= \frac{1}{n(n+1)}\left[n\binom{n+d}{n} - (n+1)\binom{n+d-1}{n-1}\right] - 3 \\ &= \frac{d-1}{(n+1)n}\binom{n+d-1}{n-1} - 3. \end{aligned}$$

The case $i < n-1$ is similar but a bit more painful. Keeping in mind Equation (4.2), we have

$$\begin{aligned}
h_{i+1} - h_i &= \frac{a_{i+2} - s_{i+1}}{i+2} - \frac{a_{i+1} - s_i}{i+1} \\
&\geq \frac{(i+1)a_{i+2} - (i+1) \cdot \frac{(i+2)^2 - i - 6}{2} - (i+2)a_{i+1} + (i+2) \cdot \frac{(i+1)^2 - 3(i+1) - 2}{2}}{(i+2)(i+1)} \\
&= \frac{(i+1)a_{i+2} - (i+2)a_{i+1}}{(i+2)(i+1)} - \frac{3i^2 + 7i + 6}{2(i+2)(i+1)} \\
&= \frac{1}{(i+2)(i+1)} [(i+1)a_{i+2} - (i+2)a_{i+1}] - \frac{(i+1)(3i+4)}{2(i+2)(i+1)} - 1 \\
&= \frac{1}{(i+2)(i+1)} \left[(i+1)\binom{i+d+1}{i+1} - (i+2)\binom{i+d}{i}\right] - \frac{1}{2} - \frac{3i+4}{2(i+2)} - 1 \\
&= \frac{1}{(i+2)(i+1)} \left[(i+1)\binom{i+d}{i+1} - \binom{i+d}{i}\right] - \frac{2i+3}{i+2} - 1 \\
&\geq \frac{1}{(i+2)(i+1)} \left[i\frac{(i+d)!}{(i+1)!(d-1)!} + \frac{(i+d)!}{(i+1)!(d-1)!} - \frac{(i+d)!}{i!d!}\right] - \frac{2i+4}{i+2} - 1 \\
&= \frac{(i+d)!}{i!(i+2)(i+1)(d-1)!} \left[\frac{i}{i+1} + \frac{1}{i+1} - \frac{1}{d}\right] - 3 \\
&= \frac{(i+d)!(d-1)}{i!d!(i+2)(i+1)} - 3 \\
&= \frac{d-1}{(i+2)(i+1)} \binom{i+d}{i} - 3
\end{aligned}$$

for every $i \geq 2$. Note that $h_{i+1} - h_i$ increases as d does. For $d = 4$, we have

$$h(i+1, 4, n) - h(i, 4, n) \geq \frac{3}{(i+2)(i+1)} \binom{i+4}{4} - 3 = \frac{i^2 + 7i - 12}{8} > 0$$

for every $i \geq 2$. □

Claim 4.8 proves i), so we are left with vi). Assume first that $i \geq 5$ and $d \geq 5$. Then

$$\begin{aligned}
&h(i-1, d, n) - h(3, d, n) + s(i, d, n) - s(i-1, d, n) - (i-4)(i+1) \\
&\quad \geq h(i-1, d, n) - h(3, d, n) - (i-4)(i+1) \\
&\quad = \frac{a(i, d) - s(i-1, d, n)}{i} - \frac{a(4, d) - s(3, d, n)}{4} - (i-4)(i+1) \\
&\quad \geq \frac{\binom{i+d-1}{i-1} - \frac{3i}{2} - \frac{i^2}{2} - \frac{i^2 - i - 4}{2}}{i} - \frac{\binom{d+3}{3} - 14 - 1}{4} - (i-4)(i+1).
\end{aligned}$$

The latter increases as d does, so

$$\begin{aligned}
&\frac{\binom{i+d-1}{i-1} - \frac{3i}{2} - \frac{i^2}{2} - \frac{i^2 - i - 4}{2}}{i} - \frac{\binom{d+3}{3} - 14 - 1}{4} - (i-4)(i+1) \\
\geq &\frac{\binom{i+4}{5} - \frac{3i}{2} - \frac{i^2}{2} - \frac{i^2 - i - 4}{2}}{i} - \frac{\binom{8}{3} - 14 - 1}{4} - (i-4)(i+1) \\
= &\frac{i^5 + 10i^4 - 85i^3 + 290i^2 - 846i + 240}{120i} > 0
\end{aligned}$$

for every $i \geq 5$. Now assume that $d = 4$. For $6 \leq i \leq 8$ we have

$$\begin{aligned}
h(i-1,4,n) - h(3,4,n) + s(i,4,n) - s(i-1,4,n) - i - 1 \\
&= h(i-1,4,n) + s(i,4,n) - s(i-1,4,n) - i - 6 \\
&\geq h(i-1,4,n) - i - 6 \\
&= \frac{a(i,4) - s(i-1,4,n)}{i} - i - 6 \\
&\geq \frac{\binom{i+3}{4} - \frac{3i}{2} - \frac{i^2}{2} - \frac{i^2-i-4}{2}}{i} - i - 6 \\
&= \frac{i^4 + 6i^3 - 37i^2 - 162i + 48}{24i} > 0.
\end{aligned}$$

For $i = 5$ we have

$$h(4,4,n) - h(3,4,n) + s(5,4,n) - s(4,4,n) - 6 = 9 - 5 + 9 - 5 - 6 > 0.$$

Finally suppose that $i \geq 9$. Then

$$\begin{aligned}
h(i-1,4,n) - h(3,4,n) + s(i,4,n) - s(i-1,4,n) - (i+1)(i-7) \\
&\geq \frac{a(i,4) - s(i-1,4,n)}{i} - 5 - (i+1)(i-7) \\
&\geq \frac{\binom{i+3}{4} - \frac{3i}{2} - \frac{i^2}{2} - \frac{i^2-i-4}{2}}{i} - 5 - (i+1)(i-7) \\
&= \frac{i^4 - 18i^3 + 131i^2 + 30i + 48}{24i} > 0. \qquad \square
\end{aligned}$$

Remark 4.9. There is an interesting consequence of Proposition 4.7. The sequences $h(i,d,n)$ and $s(i,d,n)$ do not vary with n, as long as $i < n$. This is crucial for all the computations we are going to do and opens also interesting generalizations of our arguments that we will explore in the future.

Example 4.10. Here we present the computation in a specific case. Assume that $n = 5$ and $d = 4$. By definition $k(5,4) = 21 \in \mathbb{N}$. Following the proof of Proposition 4.7, we set $s(5,4,5) = \binom{5}{2} = 10$ and $h(5,4,5) = 21 - 2 - 5 = 14$. Next we compute

$$\begin{aligned}
a(5,4) &= \binom{8}{4} - \frac{15}{2} - \frac{25}{2} = 50, \\
a(4,4) &= \binom{7}{3} - 6 - 8 = 21, \\
a(3,4) &= \binom{6}{2} - \frac{9}{2} - \frac{9}{2} = 6.
\end{aligned}$$

Now $s(4,5,4)$ is the only natural number $t \in \{4,\ldots,8\}$ such that $5 \mid 50-t$. This means that $s(4,5,4) = 5$ and therefore we have $h(4,5,4) = \frac{50-5}{5} = 9$. In the same way $s(3,5,4)$ is the only number $t \in \{1,\ldots,4\}$ such that $4 \mid 21-t$. Again this implies that $s(3,5,4) = 1$ and $h(3,5,4) = \frac{21-1}{4} = 5$. Finally, $s(2,5,4)$ is the only number $t \in \{0,1\}$ such that $3 \mid 6-t$, so we conclude that $s(2,5,4) = 0$ and $h(2,5,4) = 3$.

From now on we fix the sequences of integers $h(i,d,n)$ and $s(i,d,n)$ of Proposition 4.7. The following proves that assumption iv) of Lemma 4.6 is satisfied for this choice of integers.

Lemma 4.11. Assume that $n \geq d \geq 4$ and $(n,d) \neq (4,4)$. Then $\mathcal{L}_i^H(d) - 2H_{i-1}$ is empty for every $i \in \{3,\ldots,n\}$.

Proof. By definition, $\mathcal{L}_i^H(d) - 2H_{i-1}$ is

$$\mathcal{L}_{i,d-2}(1[s(i,d,n) - s(i-1,d,n)], 2^{h(i,d,n)-h(i-1,d,n)}).$$

First assume that $d = 4$ and $n \geq 5$. Consider $i = 3$. A direct computation shows that $h(3,4,n) = 5$ and $h(2,4,4) = 2$, hence $h(3,4,n) - h(2,4,n) \geq 3 = i$. Consider $i \in \{4,\ldots,n-1\}$. By i) in Proposition 4.7, we have

$$h(i,4,n) - h(i-1,4,n) \geq \frac{3}{(i+1)i}\binom{i+3}{4} - 2 = \frac{i^2+5i-10}{8} \geq i.$$

Consider $i = n > d = 4$. By i) in Proposition 4.7 we have

$$h(n,4,n) - h(n-1,4,n) \geq \frac{3}{(n+1)n}\binom{n+3}{4} - 3 = \frac{n^2+5n-18}{8} \geq n$$

for $n \geq 6$. If $n = 5$, we compute $h(5,4,5) - h(4,4,5) = 14 - 9 = 5 = i$. In all these cases the linear system we are interested in is contained in $\mathcal{L}_{n,2}(1,2^n)$, which is empty. Next we consider exceptions in Theorem 1.5 with $d \geq 4$. In our notation these are the cases $(i,d) \in \{(3,6),(4,6),(4,5)\}$. A direct computation shows that $h(i,d,n) - h(i-1,d,n)$ is greater than the exceptional value of Theorem 1.5.

Assume that $n \geq d > 4$. By hypothesis, $\mathcal{L}_{i,d-2}(2^{h(i,d,n)-h(i-1,d,n)})$ is non-special. Then, by i) in Proposition 4.7, for $i < n$ we have

$$\begin{aligned}
&\operatorname{vdim} \mathcal{L}_{i,d-2}(1[s(i,d,n) - s(i-1,d,n)], 2^{h(i,d,n)-h(i-1,d,n)}) \\
&\qquad < \binom{d+i-2}{i} - (i+1)(h(i,d,n) - h(i-1,d,n)) \\
&\qquad \leq \binom{d+i-2}{i} - (i+1)\left[\frac{d-1}{(i+1)i}\binom{i+d-1}{i-1} - 2\right] \\
&\qquad = \frac{(d+i-2)!}{i!(d-2)!} - \frac{d-1}{i}\cdot\frac{(i+d-1)!}{(i-1)!d!} + 2(i+1) \\
&\qquad = \frac{(i+d-2)!}{i!(d-2)!} - (d-1)\cdot\frac{(i+d-1)!}{i!d!} + 2(i+1) \\
&\qquad = \frac{(i+d-2)!}{i!(d-2)!}\left[1 - (d-1)\cdot\frac{i+d-1}{d(d-1)}\right] + 2(i+1) \\
&\qquad = \binom{i+d-2}{d-2}\cdot\frac{1-i}{d} + 2(i+1).
\end{aligned}$$

Note that the latter is decreasing with respect to d. For $d = 5$ we get

$$\frac{1-i}{5}\binom{i+3}{3} + 2(i+1) \leq \frac{-i^4 - 5i^3 - 5i^2 + 65i + 66}{30} \leq 0$$

for every $i \geq 3$.

To conclude, assume that $i = n$. Then the virtual dimension is bounded as follows

$$\begin{aligned}&\mathrm{vdim}\mathcal{L}_{n,d-2}(1[s(n,d,n)-s(n-1,d,n)], 2^{h(n,d,n)-h(n-1,d,n)})\\ &\quad < \frac{1-n}{d}\binom{n+d-2}{d-2} + 3(n+1).\end{aligned}$$

As before it decreases as d increases. For $d = 5$, we have

$$\frac{1-n}{5}\binom{n+3}{3} + 3(n+1) = \frac{-n^4 - 5n^3 - 5n^2 + 95n + 96}{30} < 0$$

for every $n \geq 5$. □

We are in a position to state and prove the induction step we described.

Proposition 4.12. Assume that $n \geq d \geq 4$ and $(n,d) \neq (4,4)$. Let $i \in \{3, \dots, n\}$, and suppose that

a) $\mathcal{L}^H_{i-1}(d)$ is nonspecial,

b) there is at most one divisor $D \in \mathcal{L}^H_{i-1}(d)$ with $\mathrm{mult}_q D > 3$.

Then $\mathcal{L}^H_i(d)$ is nonspecial and there is at most one divisor $D \in \mathcal{L}^H_i(d)$ with $\mathrm{mult}_q D > 3$.

Proof. Recall that $q \in H_{i-1}$. First we check that the conditions i), ii), iii), iv), vii) in Lemma 4.6 are satisfied. Point i) is (2) in Proposition 4.7, ii) is a) and (2) in Proposition 4.7, iii) is (1) in Proposition 4.7, iv) is Lemma 4.11, and vii) is vi) in Proposition 4.7.

We are left to prove that there is at most one divisor $D \in \mathcal{L}^H_i(d)$ with $\mathrm{mult}_q D > 3$. All divisors $D \in \mathcal{L}^H_i(d)$ with $\mathrm{mult}_q D > 3$ either contain H_{i-1} or restrict to divisors $D_{|H_{i-1}} \in \mathcal{L}^H_{i-1}(d)$ with $\mathrm{mult}_q D_{|H_{i-1}} > 3$. On the other hand, by assumption b), if there is a pencil of these divisors, then the unique divisor in $\mathcal{L}^H_i(d) - H_{i-1}$ has multiplicity at least 4 in q. Therefore to conclude it is enough to prove that $\mathrm{mult}_q D = 3$ for the divisor D with $D \supset H_{i-1}$. The divisor $D - H_{i-1}$ is in

$$\mathcal{L}_{i,d-1}(2[s(i,d,n)-s(i-1,d,n)], 2^{h(i,d,n)-h(i-1,d,n)}, 1^{h(i-1,d,n)}).$$

A straightforward computation shows that

$$h(i,d,n) - h(i-1,d,n) < \left\lceil \frac{\binom{i+d-1}{i}}{i+1} \right\rceil - i - 1,$$

so by Proposition 1.12 the system $\mathcal{L}_{i,d-1}(3, 2^{h(i,d,n)-h(i-1,d,n)})$ is nonspecial. By Lemma 4.11,

$$\mathcal{L}_{i,d-1}(2[s(i,d,n)-s(i-1,d,n)], 2^{h(i,d,n)-h(i-1,d,n)}) - H_{i-1}$$

is empty and so $\mathcal{L}_{i,d-1}(3, 2^{h(i,d,n)-h(i-1,d,n)}) - H_{i-1}$ is empty as well. Hence, arguing as in Proposition 4.6, we use Lemma 1.14 and Lemma 1.11 to ensure that the simple points on the hyperplane impose independent conditions and the linear system $\mathcal{L}_{i,d-1}(3, 2^{h(i,d,n)-h(i-1,d,n)}, 1^{h(i-1,d,n)})$ is nonspecial. Point (1) in Proposition 4.7 gives

$$\operatorname{expdim} \mathcal{L}_{i,d-1}(2[s(i,d,n) - s(i-1,d,n)], 2^{h(i,d,n)-h(i-1,d,n)}, 1^{h(i-1,d,n)}) = 0.$$

Point v) in Proposition 4.7 gives $s(i,d,n) - s(i-1,d,n) < \binom{i+1}{2}$. Then

$$\begin{aligned} &\operatorname{vdim} \mathcal{L}_{i,d-1}(3, 2^{h(i,d,n)-h(i-1,d,n)}, 1^{h(i-1,d,n)}) < \\ &< \operatorname{vdim} \mathcal{L}_{i,d-1}(2[s(i,d,n) - s(i-1,d,n)], 2^{h(i,d,n)-h(i-1,d,n)}, 1^{h(i-1,d,n)}) = 0. \end{aligned}$$

Since $\mathcal{L}_{i,d-1}(3, 2^{h(i,d,n)-h(i-1,d,n)}, 1^{h(i-1,d,n)})$ is nonspecial, it is empty and any divisor in $\mathcal{L}_{i,d-1}(2[s(i,d,n) - s(i-1,d,n)], 2^{h(i,d,n)-h(i-1,d,n)}, 1^{h(i-1,d,n)})$ has a double point in q. □

The next proposition proves the first step of our induction argument.

Proposition 4.13. Assume that $n \geq d \geq 4$ and $(n,d) \neq (4,4)$. Then the linear system $\mathcal{L}_{2,d}(3[s(2,d,n)], 2^{h(2,d,n)})$ is nonspecial and there is at most one divisor $D \in \mathcal{L}_{2,d}(3[s(2,d,n)], 2^{h(2,d,n)})$ with $\operatorname{mult}_q D > 3$.

Proof. A simple check of the list in [28, Lemma 7.1] shows that the linear systems $\mathcal{L}_{2,d}(3, 2^{h(2,d,n)})$ and $\mathcal{L}_{2,d}(4, 2^{h(2,d,n)})$ are nonspecial for $d \geq 5$, for the former one can also check Proposition 1.12, while a direct computation shows that $\mathcal{L}_{2,4}(3, 2^{h(2,4,n)})$ is nonspecial and $\dim \mathcal{L}_{2,4}(4, 2^{h(2,4,n)}) = 0$. In particular $\mathcal{L}_{2,d}(4, 2^{h(2,d,n)})$ is empty for $d \geq 5$ and has dimension 0 for $d = 4$.

We are left to study the $s(2,d,n)$ tangent direction. If $s(2,d,n) = 0$, we are done. Suppose now that $s(2,d,n) = 1$. This is possible only for $d \geq 5$. Since $\mathcal{L}_{2,d}(4, 2^{h(2,d,n)})$ is empty and $\mathcal{L}_{2,d}(3, 2^{h(2,d,n)})$ has positive dimension, we conclude that $\mathcal{L}_{2,d}(3[s(2,d,n)], 2^{h(2,d,n)})$ is nonspecial by Lemma 1.11. Moreover, for $d \geq 5$ any divisor in $\mathcal{L}_{2,d}(3, 2^{h(2,d,n)})$ has multiplicity 3 in q, while there is a unique divisor with multiplicity 4 in $\mathcal{L}_{2,4}(3, 2^{h(2,4,n)})$. □

We conclude this Section with the nonspeciality result we were looking for.

Proposition 4.14. If $n \geq d \geq 4$ and $(n,d) \neq (4,4)$, then

i) the linear system $\mathcal{L}^H_n(d)$ and $\mathcal{L}_n(d)$ are nonspecial,

ii) there is at most one divisor $D \in \mathcal{L}^H_n(d)$ with $\operatorname{mult}_q D > 3$.

Proof. By Proposition 4.13, $\mathcal{L}_{2,d}(3[s(2,d,n)], 2^{h(2,d,n)})$ satisfies i) and ii). Then, by Proposition 4.12, $\mathcal{L}^H_n(d)$ satisfies i) and ii). We already observed that $\mathcal{L}_n(d)$ is nonspecial if $\mathcal{L}^H_n(d)$ is nonspecial. □

4.2 The genus bound

The aim of this Section is to bound from below the sectional genus of the linear systems $\mathcal{L}^H_n(d)$. We start from $\mathcal{L}_{2,d}(3[s(2,d,n)], 2^{h(2,d,n)})$.

Proposition 4.15. The sectional genus of $\mathcal{L}_{2,d}(3[s(2,d,n)], 2^{h(2,d,n)})$ is 0 for $d = 4, 5$ and it is positive for $d \geq 6$.

Proof. The general element in $\mathcal{L}_{2,d}(3[s(2,d,n)], 2^{h(2,d,n)})$ has a triple point at q by Proposition 4.13, and double points at the remaining assigned points by [28, Theorem 8.1].

Claim 4.16. If $d \geq 8$, then the general element $D \in \mathcal{L}_{2,d}(3[s(2,d,n)], 2^{h(2,d,n)})$ is irreducible.

Proof of Claim 4.16. Assume that the general element

$$D \in \mathcal{L}_{2,d}(3[s(2,d,n)], 2^{h(2,d,n)})$$

is reducible. Then $D = D_1 + D_2$, set

$$b_i = \deg D_i \in \{1, \dots, d-1\}$$

and, by monodromy, $m_i := \text{mult}_{p_j} D_i$. Hence $m_1 + m_2 = 2$ so, up to order, there are two possibilities: either $m_1 = 0$ and $m_2 = 2$, or $m_1 = m_2 = 1$.

Assume first that $m_2 = 2$, that is, we are assuming that $\mathcal{L}_{2,b_2}(2^{h(2,d,n)})$ is not empty. We work under the hypothesis $d \geq 8$, so $h(2,d,n) \geq 12$. Therefore this linear system is nonspecial by Theorem 1.5, and its virtual dimension is

$$\binom{b_2+2}{2} - 3h(2,d,n) - 1 = \binom{b_2+2}{2} - a(3,d) + s(2,d,n) - 1.$$

A straightforward computation shows that this is non negative only for $d < 8$.

Assume now that $m_1 = m_2 = 1$. Then there are plane curves D_i of degree b_i through $p_1, \dots, p_{h(2,d,n)}$, that is $D_i \in \mathcal{L}_{2,b_i}(1^{h(2,d,n)})$. We may assume that $b_1 \leq b_2$. The virtual dimension of $\mathcal{L}_{2,b_i}(1^{h(2,d,n)})$ is

$$\binom{b_i+2}{2} - h(2,d,n) - 1,$$

and this is non-negative only if

$$3b_i^2 + 9b_i \geq d^2 + 3d - 22.$$

Therefore

$$\begin{aligned} 3b_1^2 + 9b_1 &\geq b_1^2 + b_2^2 + 2b_1b_2 + 3b_1 + 3b_2 - 22 \\ 2b_1^2 + 6b_1 &\geq b_2^2 + 2b_1b_2 + 3b_2 - 22 \geq b_1^2 + 2b_1^2 + 3b_1 - 22 \\ b_1^2 - 3b_1 - 22 &\leq 0, \end{aligned} \tag{4.3}$$

so $b_1 \leq 6$. But then by (4.3) we have

$$\begin{aligned} 2b_1^2 + 6b_1 &\geq b_2^2 + 2b_1b_2 + 3b_2 - 22 \geq b_2^2 + 2b_1^2 + 3b_2 - 22 \\ 6b_1 &\geq b_2^2 + 3b_2 - 22 \geq b_2^2 + 3b_1 - 22 \\ b_2^2 &\leq 3b_1 + 22 \leq 18 + 22 = 40 \end{aligned}$$

and we conclude $b_2 \leq 6$.

Since $h(2,8,n) = 12$, we have $b_i \geq 4$. Therefore the only possibilities are $b_1 = 4$ and $10 \geq d \geq 8$, $b_1 = 5$ and $11 \geq d \geq 10$ or $b_1 = 6$ and $d = 12$. A simple computation gives: $h(2,8,n) = 12$, $h(2,9,n) = 15$, $h(2,10,n) = 19$, $h(2,11,n) = 23$ and $h(2,12,n) = 27$. The divisor D has a triple point at q. Therefore the divisor D_1 has to belong to one of the following linear systems:

($d=8$) $\mathcal{L}_{2,4}(2,1^{12})$,

($d=9$) $\mathcal{L}_{2,4}(1^{15})$,

($d=10$) either $\mathcal{L}_{2,4}(1^{19})$ or $\mathcal{L}_{2,5}(2,1^{19})$,

($d=11$) $\mathcal{L}_{2,5}(1^{23})$,

($d=12$) $\mathcal{L}_{2,6}(2,1^{27})$.

An easy check shows that they are all empty. $\square$

Set $d \geq 8$. Since the general element is irreducible, the map $\varphi_{2,d}$ is dominant. Assume by contradiction that the geometric genus of D is 0 and, beside the singularities we imposed, there are l singular points of multiplicity m_i and t simple base points. Then we get

$$0 = \frac{(d-1)(d-2)}{2} - 3 - h(2,d,n) - \sum_{i=1}^{l} \frac{m_i(m_i-1)}{2}$$

and

$$1 \leq d^2 - 9 - 4h(2,d,n) - s(2,d,n) - t - \sum_{i=1}^{l} m_i^2.$$

This yields

$$\begin{aligned} 0 \leq & d^2 - 10 - 4h(2,d,n) - s(2,d,n) - t - \sum_{i=1}^{l} m_i^2 \\ & - \left(\frac{(d-1)(d-2)}{2} - 3 - h(2,d,n) - \sum_{i=1}^{l} \frac{m_i(m_i-1)}{2} \right) \end{aligned}$$

and, recall Equation (4.2),

$$\begin{aligned} 0 &\leq \frac{d^2+3d}{2} - 8 - 3h(2,d,n) - s(2,d,n) - t - \sum_{i=1}^{l} \frac{m_i(m_i+1)}{2} \\ &= \frac{d^2+3d}{2} - 8 - a(2,d) - t - \sum_{i=1}^{l} \frac{m_i(m_i+1)}{2} \\ &= -t - \sum_{i=1}^{l} \frac{m_i(m_i+1)}{2}, \end{aligned}$$

therefore $l = 0$ and $t = 0$. To conclude we compute the genus.

$$\begin{aligned} 0 = 2g(D) &= (d-1)(d-2) - 6 - 2h(2,d,n) = d^2 - 3d - 4 - 2h(2,d,n) \\ &\geq d^2 - 3d - 4 - \frac{2}{3}a(2,d) = \frac{2}{3}(d^2 - 6d + 2). \end{aligned}$$

This contradicts the assumption $d \geq 8$.

Assume that $d = 7$. Then $h(2,7,n) = 9$ and $s(2,7,n) = 0$. Consider a cubic $C_1 \in \mathcal{L}_{2,3}(1,1^8)$ through q and $\{p_1, \ldots, p_8\}$, and let $C_2 \in \mathcal{L}_{2,4}(2,1^7,2)$ be a quartic singular at q and p_9 and passing through $\{p_1, \dots, p_8\}$. The reducible curve $D = C_1 + C_2$ is an element of positive genus in $\mathcal{L}_{2,7}(3, 2^9)$, so we conclude by Remark 4.4.

Assume that $d = 6$. In this case $h(2, 6, n) = 6$ and $s(2, 6, n) = 1$. Consider two curves $C_1 \in \mathcal{L}_{2,3}(2, 1^6)$ and $C_2 \in \mathcal{L}_{2,3}(1[1], 1^6)$. Then $D = C_1 + C_2$ is a reducible element in $\mathcal{L}_{2,6}(3[1], 2^6)$. The curve C_2 is not rational, therefore the sectional genus of $\mathcal{L}_{2,6}(3[1], 2^6)$ is positive by Remark 4.4.

For $d = 4$ and $d = 5$ it is an easy computation to see that the movable part of $\mathcal{L}_{2,d}(3[s(2, d, n)], 2^{h(2,d,n)})$ is given respectively by $\mathcal{L}_{2,2}(1^3)$ and $\mathcal{L}_{2,3}(2, 1^4)$, which have genus 0. $\square$

For $d \in \{4, 5\}$, we bound the genus of $\mathcal{L}_{3,d}(3[s(3, d, n)], 2^{h(3,d,n)})$.

Proposition 4.17. If $n \geq 4$, then $\mathcal{L}_{3,5}(3[s(3, 5, n)], 2^{h(3,5,n)})$has positive sectional genus.

Proof. A direct computation gives $s(3, 5, n) = 2$, $h(3, 5, n) = 10$, $s(2, 5, n) = 0$ and $h(2, 5, n) = 4$. Let $H + S \in \mathcal{L}_3^H(5)$ be the unique divisor containing H. By Remark 4.4, it is enough to prove that $\mathcal{L}_{3,5}(3[s(3, 5, n)], 2^{h(3,5,n)})_{|S}$ has positive sectional genus.

The surface

$$S \in \mathcal{L}_{3,4}(2[2], 2^6, 1^4)(q[t_1, t_2], p_1, \dots, p_6, z_1, \dots, z_4)$$

is a quartic in $\mathbb{P}^3$, the points p_i are general and the points z_i are general on H. By [56, Theorem 4.1], $\mathcal{L}_{3,4}(2^7)$ is nonspecial and the general element has 7 ordinary double points as unique singularities. The points z_i are in general position on H, therefore the general element in

$$\mathcal{L}_{3,4}(2^7, 1^4)(q, p_1, \dots, p_6, z_1, \dots, z_4)$$

has only 7 ordinary double points. The linear system $\mathcal{L}_{3,4}(3, 2^5)(q, p_1, \dots, p_5)$ is nonspecial of dimension 4 by Proposition 1.12 and $\mathcal{L}_{3,3}(2, 2^5)$ is empty, therefore

$$\mathcal{L}_{3,4}(3, 2^5, 1^5)(q, p_1, \dots, p_5, p_6, z_1, \dots, z_4)$$

and henceforth $\mathcal{L}_{3,4}(3, 2^6, 1^4)$ are empty. This shows that for a general choice of 2 tangent directions the surface

$$S \in \mathcal{L}_{3,4}(2[2], 2^6, 1^4)(q[t_1, t_2], p_1, \dots, p_6, z_1, \dots, z_4)$$

has 7 ordinary double points as unique singularities. In particular S is a (singular) K3 surface and it is not uniruled. Therefore $\mathcal{L}_{3,5}(3[s(3, 5, n)], 2^{h(3,5,n)})_{|S}$ has positive sectional genus. $\square$

Proposition 4.18. If $n \geq 4$, then $\mathcal{L}_{3,4}(3[s(3, 4, n)], 2^{h(3,4,n)})$ has positive sectional genus.

Proof. Our choice of integers is $s(3, 4, n) = 1$, $h(3, 4, n) = 5$, $s(2, 4, n) = 0$ and $h(2, 4, n) = 2$. Let $S + H$ be the unique element in $\mathcal{L}_3^H(4)$ containing the hyperplane H. Then

$$S \in \mathcal{L}_{3,3}(2[1], 2^3, 1^2)(q[t], p_1, p_2, p_3, p_4, p_5)$$

is a cubic surface with 4 double points. It is easy to prove, with reducible elements, that the scheme base locus of $\mathcal{L}_3^H(4)$ is given by the assigned singularities and the lines spanned by q and $\{p_1, p_2, p_3\}$. Hence the fixed component of $\mathcal{L}_3^H(4)_{|S}$ is given by the 3 lines spanned by q and p_1, p_2, and p_3. Then the general element in the movable part of $\mathcal{L}_3^H(4)_{|S}$ has a triple point in p_1 and a double point in p_4, and therefore positive genus. This is enough to conclude by Remark 4.4. □

We conclude the section by collecting all the results we need.

Proposition 4.19. The sectional genus of $\mathcal{L}_{3,d}(3[s(3,d,n)], 2^{h(3,d,n)})$ is positive for every $n, d \geq 4$.

Proof. For $d = 4, 5$ this is the content of Propositions 4.18 and 4.17. For $d \geq 6$, observe that by construction a general curve section of $\mathcal{L}_{2,d}(3[s(2,d,n)], 2^{h(2,d,n)})$ is a curve section of $\mathcal{L}_3^H(d)$, hence the sectional genus of $\mathcal{L}_{3,d}(3[s(3,d,n)], 2^{h(3,d,n)})$ is bounded by the sectional genus of $\mathcal{L}_{2,d}(3[s(2,d,n)], 2^{h(2,d,n)})$. Thus we conclude by Proposition 4.15. □

4.3 Cubics and proof of Theorem 4.2

Fix a pair (n, d), with $n \geq d \geq 4$, a linear space $\Pi \cong \mathbb{P}^3$ and a point $q \in \Pi$. Let Z_0 be the 0-dimensional scheme obtained as a limit of $k(n,d) - 1$ double points, $n + 1$ of which collapse to the point q with $s(3,d,n)$ tangent directions, and $h(3,d,n)$ double points on Π. Such a degeneration always exists since $s(3,d,n) \leq 4$. Let

$$\mathcal{L}_0(n,d) := \mathbb{P}(\mathcal{I}_{Z_0,\mathbb{P}^n}(d))$$

be the associated linear system.

Lemma 4.20. If $n \geq d \geq 4$ and $(n,d) \neq (4,4)$, then the linear system $\mathcal{L}_0(n,d)$ is non special and its sectional genus is positive.

Proof. Thanks to Proposition 4.14, for the nonspeciality we have only to worry about the tangent directions. Let T be the set of $\binom{n+1}{2}$ tangent directions. Set $T_i := \{y_1, \ldots, y_i\} \subset T$, a subset of i tangent directions. Assume that $\mathcal{L}_{n,d}(3[i], 2^h)$ is nonspecial and $\mathcal{L}_{n,d}(3[i+1], 2^h)$ is special for a general choice of h double points. Let φ be the map associated to the linear system $\mathcal{L}_{n,d}(3[i], 2^{h-1})$.

Since the set T imposes independent conditions on cubics, we may assume that $y_j \notin \operatorname{Bs} \mathcal{L}_{n,d}(3[i], 2^{h-1})$ for $j > i$. The speciality of $\mathcal{L}_{n,d}(3[i+1], 2^h)$ forces $\varphi(y_j)$ to be a vertex of $\varphi(\mathbb{P}^n)$ for $j > i$. Hence the general $D \in \mathcal{L}_{n,d}(3[i+1], 2^h)$ is singular at y_j for $j > i$. By a monodromy argument, the general divisor in $\mathcal{L}_{n,d}(3[\binom{n+1}{2}], 2^h)$ is singular along T. Since $\mathcal{L}_{n-1,3}(2^{\binom{n+1}{2}})(T)$ is empty, this yields

$$\mathcal{L}_{n,d}\left(3\left[\binom{n+1}{2}\right], 2^h\right) \subseteq \mathcal{L}_{n,d}(4, 2^h)$$

and contradicts Proposition 4.14 ii) for $h \leq k(n,d) - n - 2$.

We are left to bound the sectional genus of $\mathcal{L}_0(n,d)$. Let $\tilde{T}_i$ a set of $\binom{n+1}{2} - i$ general tangent directions. Let

$$\mathcal{L}(T_i) := \mathcal{L}_{n,d}\left(3\left[\binom{n+1}{2}\right], 2^{k(n,d)-n-2}\right)\left(q\left[T_i \cup \tilde{T}_i\right], p_1, \ldots, p_{k(n,d)-n-2}\right).$$

By definition, we have $\mathcal{L}(T_1) = \mathcal{L}_n^H(d)$. Fix $D_1, \ldots, D_{n-3} \in \mathcal{L}_n^H(d)$ general divisors containing Π and $Y_1, Y_2 \in \mathcal{L}_n^H(d)$ general divisors. By Lemma 4.6 (2), Π is an irreducible component of $D_1 \cdot \ldots \cdot D_{n-3}$ and, by Lemma 4.6 (4), an irreducible component of $Y_1 \cdot Y_2 \cdot D_1 \cdot \ldots \cdot D_{n-3}$ is a curve section of $\mathcal{L}_{3,d}(3[s(3,d,n)], 2^{h(3,d,n)})$. Hence, by Proposition 4.19 and Remark 4.4, the claim is true for $i = 1$.

To conclude, we increase i recursively. Fix $D_1, \ldots, D_{n-3} \in \mathcal{L}(T_{i+1})$ general divisors containing Π and $Y_1, Y_2 \in \mathcal{L}(T_{i+1})$ general divisors. By construction, we have $s(3,n,d) > 0$ and therefore we may assume that $\mathcal{L}(T_{i+1})$ is a specialization of $\mathcal{L}(T_i)$ moving a tangent direction in Π. In this degenerations all sections in $\mathcal{L}(T_i)$ containing Π are also sections of $\mathcal{L}(T_{i+1})$. This shows that Π is an irreducible component of $D_1 \cdot \ldots \cdot D_{n-3}$. Next we may consider $\mathcal{L}(T_{i+1})$ as a specialization of $\mathcal{L}(T_i)$ moving a point outside Π. Via this degeneration we prove that $\mathcal{L}(T_{i+1})_{|\Pi} = \mathcal{L}(T_i)_{|\Pi}$ and therefore an irreducible component of $Y_1 \cdot Y_2 \cdot D_1 \cdot \ldots \cdot D_{n-3}$ is a curve section of $\mathcal{L}_{3,d}(3[s(3,d,n)], 2^{h(3,d,n)})$. Then Proposition 4.19 and Remark 4.4 allow us to conclude. $\square$

The case $d = 3$

The argument we used for forms of degree $d \geq 4$ does not work for cubics. Linear systems of cubics with a triple point and at least a double point are always special. This forces us to apply a different strategy to study the degree of the map associated to $\mathcal{L}_{n,3}(2^k)$. This is inspired by the proof of Alexander–Hirschowitz' Theorem in [16] and [68]. Note that we are interested in integers n such that

$$k(n) := k(n,3) = \frac{\binom{n+3}{3}}{n+1}$$

is an integer. This is equivalent to say that $n \equiv 0, 1 \pmod 3$. This property is preserved by codimension 3 linear spaces. This simple observation suggests the following induction procedure.

Assume that $k(i)$ is an integer. Let $Z_1 \subset \mathbb{P}^i$ be a 0-dimensional scheme of $k(i) - 1$ general double points. Fix a general codimension 3 linear space $\Pi \subset \mathbb{P}^i$ and let Z_0 be a specialization of Z_1 with $k(i-3) - 1$ double points on Π. Therefore the linear system $\mathcal{L}_{i,3}(2^{k(i)-1})$ specializes to a linear system $\mathcal{L}_0$ and we may split $\mathcal{L}_0$ as a direct sum of

$$\tilde{\mathcal{L}} \quad \text{and} \quad \mathcal{L}_{i-3,3}(2^{k(i-3)-1}),$$

where $\tilde{\mathcal{L}}$ is the system of cubics containing Π and singular in $i+1 = k(i) - k(i-3)$ general points of $\mathbb{P}^i$ and in $k(i-3)$ general points of Π. The linear system $\tilde{\mathcal{L}}$ is known to be nonspecial by [16, Proposition 5.4], see also [68, subsection 5.2], and $\mathcal{L}_{i-3,3}(2^{k(i-3)-1})$ is nonspecial by Theorem 1.5. Let g_i be the sectional genus of $\mathcal{L}_{i,3}(2^{k(i)-1})$, for $i \equiv 0, 1 \pmod 3$. Let D_1, D_2, D_3 be three general cubics containing Π. Considering the spaces $\mathbb{P}^3$ spanned by 4 double points, it is easy to check that Π is an irreducible component of $D_1 \cdot D_2 \cdot D_3$, hence

$$g_i \geq g_{i-3} \tag{4.4}$$

for $i \geq 6$.

We are left to prove that cubics in $\mathbb{P}^6$ and $\mathbb{P}^7$ do not define a birational map. In particular, we will show that g_6 and g_7 are positive.

Lemma 4.21. We have $g_6 > 0$.

Proof. The number $k(6)$ is 12. Let $Z_0 = \{p_1, \ldots, p_8, z_1, z_2, z_3\} \subset \mathbb{P}^6$ be a specialization with the points p_i on a hyperplane H and the points z_j general. Then $\mathcal{L}_{6,3}(2^{11})$ specializes to a linear system $\mathcal{L}^H_{6,3} = \mathcal{L} + \mathcal{L}_{5,3}(2^8, 1^3)$, with $\dim \mathcal{L}^H_{6,3} = 6$. It is easy to see that $\dim \mathcal{L} = 1$ and $\mathcal{L} = H + \Lambda$, with Λ a pencil of quadrics of rank 4 with vertex $\langle z_1, z_2, z_3 \rangle$. Then $M := \operatorname{Bs} \Lambda$ is a cone over a normal elliptic curve in $\mathbb{P}^3$. In particular M is not rationally connected and therefore $\mathcal{L}^H_{6,3}$ has positive sectional genus. Then, by Remark 4.4, we conclude that $g_6 > 0$. □

Lemma 4.22. We have $g_7 > 0$.

Proof. The number $k(7)$ is 15. Let $Z_0 = \{p_1, \ldots, p_{11}, z_1, z_2, z_3\} \subset \mathbb{P}^7$ be a specialization with the points p_i on a hyperplane H and the points z_j general. Then $\mathcal{L}_{7,3}(2^{14})$ specializes to a linear system $\mathcal{L}^H_{7,3} = \mathcal{L} + \mathcal{L}_{6,3}(2^{11}, 1^3)$, with $\dim \mathcal{L}^H_{7,3} = 7$. It is easy to see that $\dim \mathcal{L} = 3$ and $\mathcal{L} = H + \Lambda$, with Λ a linear system of quadrics of rank 5. Then $M := \operatorname{Bs} \Lambda$ is a union of 16 $\mathbb{P}^3$ meeting in $\langle z_1, z_2, z_3 \rangle$. Let $\Pi_i = \langle z_1, z_2, z_3, p_i \rangle$. Then $\Pi_i \cap \Pi_j = \langle z_1, z_2, z_3 \rangle$ and $\Pi_i \subset M$. By construction, we have $\mathcal{L}^H_{7,3|\Pi_i} \subset \mathcal{L}_{3,3}(2^4)$. On the other hand, specializations can only increase the dimension of a linear systems, therefore

$$\dim(\mathcal{L}^H_{7,3})_{|\Pi_i} \geq \dim \mathcal{L}_{7,3}(2^{14})_{|\Pi_i} = 4,$$

where the last equality is proved in [68, Section 5.4]. Let $D_1, D_2 \in \mathcal{L}^H_{7,3}$ be two general elements. Then $(D_1 \cdot D_2)_{|\Pi_i}$ contains a twisted normal curve passing through $\{z_1, z_2, z_3, p_i\}$. Let $Q_1, \ldots, Q_4 \in \Lambda$ be general elements. Then the 1-cycle $Q_1 \cap \ldots \cap Q_4 \cap D_1 \cap D_2$ contains rational curves intersecting in $\{z_1, z_2, z_3\}$ and it has positive genus. This, by Remark 4.4, shows that $g_7 > 0$. □

We collected all the results we need to prove Theorem 4.2.

Proof of Theorem 4.2. By [56, Theorem 4.3, Proposition 2.4] and [6, Theorem 3.2], we may assume that $d \leq n$ and $n \geq 4$. If $d = 2$ and $\dim \mathcal{L}_{n,2}(2^h) = n$, then the map associated to $\mathcal{L}_{n,d}(2^h)$ is always of fiber type.

If $d = 3$, $n \geq 6$ and $n \equiv 0 \pmod 3$, then Theorem 1.5 forces $h = k(n) - 1$. Then, by Equation (4.4) and Lemma 4.21, the sectional genus of $\mathcal{L}_{n,3}(2^h)$ is positive. If $d = 3$, $n \geq 7$ and $n \equiv 1 \pmod 3$, then we conclude as before via equation (4.4) and Lemma 4.22 that the sectional genus is positive. It is known that $\mathcal{L}_{4,3}(2^6)$ induces a fiber type map that contracts the rational normal curves through the 6 points. This analysis proves the theorem for $d \leq 3$.

Assume that $n \geq d \geq 4$. By Theorem 1.5, $h = k(n, d) - 1$. If $n = d = 4$, then $h = 13$. There is a pencil of quadrics in $\mathbb{P}^4$ through 13 general points, so $\mathcal{L}_{4,4}(2^{13})$ admits a linear subsystem of reducible divisors with base locus in codimension 2, hence the associated map can not be birational. Suppose then that $(n, d) \neq (4, 4)$. By Lemma 4.20, $\mathcal{L}_0(n, d)$ is a specialization of $\mathcal{L}_{n,d}(2^h)$ and it has positive sectional genus. This shows that $\mathcal{L}_{n,d}(2^h)$ does not define a Cremona transformation. □

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