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BV Functions in Metric Measure Spaces: Traces and Integration by Parts Formulæ

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A Paola

Introduction

The main scope of this work is to give new insights into the theory of functions of bounded variation (BV) in the non-smooth context of metric measure spaces. To this aim, in order to make the dissertation as self-contained as possible and to motivate the results we established during our research, we attempted to give an exhaustive - albeit far from being complete or detailed - survey on the field, discussing different notions of Sobolev and BV spaces, with particular emphasis on the connections and equivalences among them.

The Thesis is organized as follows:

- In Chapter 1 we introduce the basic notions of our analysis, namely the notion of p-summable functions $L^p(\mathcal{X})$ on metric spaces, the fundamental concept of a metric measure space $(\mathcal{X}, d, \mu)$ along with its most notable properties, and doubling measures.

- Chapter 2 can be considered as the actual starting point of our discussion. Following the classical approach of [HKST], we introduce the concept of *modulus* of families of curves on metric measure spaces, which eventually leads to the definition and properties of *(weak) upper gradients.* This tool allows for the characterization of the first-order Newton-Sobolev spaces $N^{1,p}(\mathcal{X})$, $1 \le p < \infty$, which are proven to be Banach spaces ([Sh1], [Sh2]). The chapter culminates in the discussion of Poincaré Inequalities and their consequences, the most notable - assuming additionally $(\mathcal{X}, d, \mu)$ to be doubling - being the density of Lipschitz functions inside Sobolev spaces.

- In Chapter 3 our attention shifts to other definitions of Sobolev spaces in metric measure spaces. We first consider the *test-plan* approach ([AGS2], [AGS3]), which is inherited from the theory of Optimal Transportation ([Vi]) and, making use of probability measures defined on the space of curves, allows for a characterization of weak gradients in duality with the speed of p-absolutely continuous curves. The discussion does not rely on particular assumptions on the underlying metric measure space, like the validity of a Poincaré inequality or the doubling property for the reference measure. However, Lipschitz functions are again proven to be dense inside Sobolev spaces ([AGS4], [Gi1]) and the respective notion of weak gradient turns out to be equivalent with the Newtonian one ([AGS4]). Next, we focus on the *derivations* approach described in [Di1]; this tool is reminiscent of the work [We] and makes possible to define properly a notion of *divergence* by simply imposing an integration by parts formula, from which we obtain a further definition of weak gradients and the corresponding Sobolev spaces. As for test-plans, no structural hypothesis is assumed about the ambient space but, again, a further equivalence among the various definitions of Sobolev spaces is established.

- Chapter 4 deals with three different notions of BV functions in metric measure spaces, in the same spirit of Chapters 2 and 3. We start with the well known relaxation procedure over sequences of weak upper gradients of Lipschitz - or, Sobolev - functions ([Mi], [Sh3]) under the hypothesis that $(\mathcal{X}, d, \mu)$ is doubling and supports a Poincaré inequality. We recall the notions of *total variation* of a function and of *perimeter* of a set; then, we state the Coarea Formula for BV functions along with the Isoperimetric Inequality ([Mi]) and illustrate the concentration and absolute continuity properties of the perimeter measure ([Am2]). Next, dropping the structural assumptions on the metric measure space, we briefly survey the characterization of BV functions via ∞-test plans ([AD]) and then we follow again [Di1] to describe the BV space via derivations. Similarly to Chapter 3, the analysis culminates in the equivalence between the three definitions of BV functions.

- Chapter 5 surveys the differentiable structure anticipated in [Gi1] and then developed and discussed in [Gi2]. The key-tool for this machinery is the notion of $L^p(\mu)$*-normed module*, which arises as a generalization of the $L^\infty(\mu)$-modules already introduced in [We]. While in [Gi2] the author focuses on the L^2 theory, here we preferred to re-adapt and to broaden the analysis to any exponent $p \in [1, \infty]$. Thus, we define the *cotangent module* $L^p(T^*\mathcal{X})$ via the *differentials* of Sobolev functions. Since the notion of differential is tailored to the weak gradients arising from test-plans, which are in turn defined in duality with the speed of curves, this tool yields a sort of cotangent object, whence the terminology. As expectable, the elements of the cotangent module $L^p(T^*\mathcal{X})$ play the role of *differential forms* and, by duality with $L^p(T^*\mathcal{X})$, the *tangent module* $L^q(T\mathcal{X})$, $\frac{1}{p} + \frac{1}{q} = 1$, is defined as the $L^q(\mu)$-normed module having *vector fields* as its elements. All this formalism allows for a well posed notion of *divergence* of a vector field and, moreover, it yields a characterization of the *gradient* of a given function as a vector field. Eventually, the hypothesis of *infinitesimal Hilbertianity* of the space - namely, requiring $W^{1,2}(\mathcal{X})$ to be a Hilbert space - is introduced and discussed; the most notable byproducts of this assumption are the uniqueness of the gradient for every Sobolev function, and the possibility to express the Laplace operator as the *divergence of the gradient*.

In the concluding two chapters we illustrate the results which arose from our studies:

- Chapter 6 starts by reviewing *en passant* the definition of $\mathsf{RCD}(K, \infty)$ spaces, the *heat flow* h_t along with the related *Bakry-Émery contraction estimate* involving the curvature of the space, and the notions of *test functions* and of *test vector fields* ([Gi2]). We then give a new notion of BV functions which, quite similarly to the classical one, features the suprema over divergences of suitable vector fields. We show that the $\mathsf{RCD}(K, \infty)$ structure allows for a definition of BV via a relaxation procedure over test functions which is equivalent with the more classical of [Mi] and [Sh3]. If moreover $(\mathcal{X}, d, \mu)$ is also doubling and supports a Poincaré inequality, then the equivalence involves all the characterizations we have encountered so far ([Mi], [Sh3], [AD], [Di1]), including also our definition. We eventually discuss the possibility of generalizing our notion - and thus, of an integration by parts formula - to any domain $\Omega \subset \mathcal{X}$; this, however, is still an open problem since it is not quite clear to us how to properly choose the class of vector fields to work with.

- Chapter 7 is devoted to the issue of Gauss-Green formulæ and of traces of BV functions in metric measure spaces. Inspired by [MMS], we introduce the class of *regular domains*, which proves to be the appropriate class of domains where an extended Gauss-Green formula holds for every *divergence-measure* vector field $F \in \mathcal{DM}^\infty(\mathcal{X})$, namely those vector fields whose divergence is a measure with bounded total variation in $\mathcal{X}$. The issue is then discussed in the specific context of an $\mathsf{RCD}(K, \infty)$ space and tailored to a suitable sub-class of test vector fields. Next, we step back to the more classical setting of a doubling metric measure space supporting a Poincaré inequality, where we reformulate the theory of *rough traces* of BV functions ([Ma]). We establish a further Gauss-Green type formula, this time involving explicitly the rough trace of a BV function. The last part of Chapter 7 deals with the comparison between the rough trace and the trace defined via Lebesgue points. This study was carried on during our stay at the University of Cincinnati (Ohio, USA) in collaboration with Prof. N. Shanmugalingam. Starting from the paper [LS], where the authors determine the conditions for a linear trace operator to exist on $BV(\Omega)$ - with $\Omega \subset \mathcal{X}$ bounded open set supporting a $(1,1)$-Poincaré inequality and such that $\mu\lfloor\Omega$ is doubling - and find that, for $u \in BV(\Omega)$, this trace $\mathrm{T}u(x)$ coincides with the approximate limits of its zero-extension almost everywhere on $\partial\Omega$, we first established that every bounded BV function on a domain admits a zero-extension which is of bounded variation on the whole metric space, and then that the rough trace coincides with $\mathrm{T}u(x)$ for almost every $x \in \partial\Omega$. A very interesting consequence of this equality is that the rough trace of a BV function defines *a fortiori* a linear operator.

The contents of Chapters 6 and 7 will be included in two joint works ([BM1], [BM2]) with Prof. M. Miranda, advisor for this Ph.D. Thesis.

1 Preliminary notions

In this introductory chapter, we shall survey the basic tools and notions which will recur thoroughly in our work.

We assume the reader to be familiar with the notions of (outer) measure, σ-finiteness, completeness (in the sense of measure theory), measurability (of sets and functions), Borel sets, measures and functions, Radon measures, Haudorff measures (classical, spherical) and dimension.

The main reference for the following sections is Chapter 3 of [HKST]. Due to the introductory nature of the material presented below, we shall limit ourselves to the statement of results only.

1.1 Lebesgue Spaces $L^p(\mathcal{X}, \mu; \mathcal{Y})$ and Metric Measure Spaces

In view of the theory of Newton-Sobolev functions which will be surveyed in Chapter 2, we focus here on the spaces of Banach space-valued L^p functions and discuss the concept of a metric measure space, a tool of fundamental importance. The choice of an arbitrary Banach space instead of the field of real numbers $\mathbb{R}$ is just for the sake of generality; the topics we shall discuss apply in a straightforward way to the real case.

Let $(\mathcal{X}, \mu)$ and $\mathcal{Y}$ denote a σ-finite measure space and a Banach space, respectively.

1.1.1 Definition (Bochner Integrability). Consider a simple function $f : \mathcal{X} \to \mathcal{Y}$, that is,

$$f = \sum_{i=1}^{n} v_i \cdot \mathbb{1}_{E_i},$$

where $v_i \in \mathcal{Y}$, the E_i's are measurable sets forming a partition of $\mathcal{X}$ and $\mathbb{1}_{E_i}$ denotes the characteristic function of E_i. Recall that any function $g : \mathcal{X} \to \mathcal{Y}$ is called *measurable* whenever it is a pointwise almost-everywhere limit of a sequence of simple functions. Assume $v_i = 0$ for all the indices i such that $\mu(E_i) = \infty$. Then we define the *integral* of f over $\mathcal{X}$ with respect to the measure μ as

$$\int_{\mathcal{X}} f \mathrm{d}\mu := \sum_{i=1}^{n} \mu(E_i) \, v_i.$$

The above expression gives a well defined element of $\mathcal{Y}$; being $|f|$ measurable for every simple function, see [HKST, Remark 3.1.1], we have

$$\left| \int_{\mathcal{X}} f \mathrm{d}\mu \right| \leq \int_{\mathcal{X}} |f| \mathrm{d}\mu = \sum_{i=1}^{n} \mu(E_i) \, |v_i| < \infty. \tag{1.1}$$

Then, such a simple function f will be said *integrable*.

A measurable function $f : \mathcal{X} \to \mathcal{Y}$ is called *Bochner integrable* if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of integrable simple functions such that

$$\lim_{n\to\infty} \int_{\mathcal{X}} |f - f_n| \, \mathrm{d}\mu = 0.$$

Thus, the *Bochner integral* of f is given by

$$\int_{\mathcal{X}} f \mathrm{d}\mu := \lim_{n\to\infty} \int_{\mathcal{X}} f_n \mathrm{d}\mu.$$

By (1.1), the above definition is well posed in the sense that the integral of f - as an element of $\mathcal{Y}$ - does not depend on the particular sequence $(f_n)_{n\in\mathbb{N}}$ chosen.

If $E \subset \mathcal{X}$ is any measurable subset and $f : E \to \mathcal{Y}$ is a function, we shall say that f is *integrable over* E if $f \cdot \mathbb{1}_E : \mathcal{X} \to \mathcal{Y}$ is integrable; so, we set

$$\int_E f \mathrm{d}\mu := \int_{\mathcal{X}} f \cdot \mathbb{1}_E \mathrm{d}\mu.$$

1.1.2 Proposition [HKST, Proposition 3.2.4]. If

$$\mathcal{X} = \bigcup_{k=1}^{\infty} E_k$$

where the E_k's are pairwise disjoint measurable sets and $f : \mathcal{X} \to \mathcal{Y}$ is an integrable function over each E_k with

$$\sum_{k=1}^{\infty} \int_{E_k} |f| \mathrm{d}\mu < \infty,$$

then f is integrable over $\mathcal{X}$ and, moreover,

$$\int_{\mathcal{X}} f \mathrm{d}\mu = \sum_{k=1}^{\infty} \int_{E_k} f \mathrm{d}\mu.$$

□

1.1.3 Proposition [HKST, Proposition 3.2.7]. Bochner-integrable functions coincide with measurable functions f such that $|f|$ is integrable.

□

1.1.4 Definition ($L^p(\mathcal{X}, \mu; \mathcal{Y})$ Spaces). Let $p \in [1, \infty[$. Given a measurable function $f : \mathcal{X} \to \mathcal{Y}$ we set

$$\|f\|_{\tilde{L}^p(\mathcal{X},\mu;\mathcal{Y})} := \left(\int_{\mathcal{X}} |f|^p \mathrm{d}\mu \right)^{\frac{1}{p}}. \tag{1.2}$$

This defines a semi-norm on the space

$$\tilde{L}^p(\mathcal{X},\mu;\mathcal{Y}) := \left\{ f : \mathcal{X} \to \mathcal{Y} \text{ measurable; } \|f\|_{\tilde{L}^p(\mathcal{X},\mu;\mathcal{Y})} < \infty \right\};$$

it is not a norm, since for every function which vanishes μ-almost everywhere the formula in (1.2) gives zero.

If we introduce an equivalence relation $\sim$ on $\tilde{L}^p(\mathcal{X},\mu;\mathcal{Y})$ by declaring $f \sim g$ whenever $f - g = 0$ μ-almost everywhere, then for the resulting equivalence classes $[f]$ one can define $\|[f]\|_{L^p(\mathcal{X},\mu;\mathcal{Y})}$ with no ambiguity using (1.2) on any representative of $[f]$; thus, we define the *Lebesgue Spaces* $L^p(\mathcal{X},\mu;\mathcal{Y})$ as the spaces of equivalence classes $[f]$ with $\|[f]\|_{L^p(\mathcal{X},\mu)} < \infty$, namely

$$L^p(\mathcal{X},\mu;\mathcal{Y}) := \tilde{L}^p(\mathcal{X},\mu;\mathcal{Y}) \Big/ \left\{ f \in \tilde{L}^p(\mathcal{X},\mu;\mathcal{Y});\ \|f\|_{\tilde{L}^p(\mathcal{X},\mu;\mathcal{Y})} = 0 \right\}.$$

Of course, the equivalence classes shall be dropped from the notation, so one simply writes f and $\|f\|_{L^p(\mathcal{X},\mu;\mathcal{Y})}$ instead of $[f]$ and $\|[f]\|_{L^p(\mathcal{X},\mu;\mathcal{Y})}$.

Endowed with the above norm, $L^p(\mathcal{X},\mu;\mathcal{Y})$ is a Banach space; elements of L^p spaces shall be also addressed to as *p-integrable functions*.

1.1.5 Proposition [HKST, Proposition 3.2.13]. Given $p \in [1,\infty[$, a measurable function $f : \mathcal{X} \to \mathcal{Y}$ is in $L^p(\mathcal{X},\mu;\mathcal{Y})$ if and only if there exists a sequence of simple functions $(f_n)_{n\in\mathbb{N}} \subset L^p(\mathcal{X},\mu;\mathcal{Y})$ such that

$$\lim_{n\to\infty} \int_{\mathcal{X}} |f - f_n|^p \,\mathrm{d}\mu = 0.$$

$\square$

1.1.6 Definition. For $p = \infty$ the space of *essentially bounded* measurable functions, $L^\infty(\mathcal{X},\mu;\mathcal{Y})$, is the class of functions with bounded *supremum norm*,

$$\|f\|_{L^\infty(\mathcal{X},\mu;\mathcal{Y})} := \sup\{\lambda \in \mathbb{R};\ \mu(\{x \in \mathcal{X} :\ |f(x)| > \lambda\}) \neq 0\} = \operatorname*{ess-sup}_{x\in\mathcal{X}} |f(x)|.$$

$L^\infty(\mathcal{X},\mu;\mathcal{Y})$ is a Banach space as well.

The notion of local Lebesgue spaces L^p_{loc} is given in the usual way.

1.1.7 Remark. $L^q(\mathcal{X},\mu;\mathcal{Y}^*)$ embeds isometrically in $L^p(\mathcal{X},\mu;\mathcal{Y})^*$, where $p \in [1,\infty[$ and q is its conjugate exponent, namely $\frac{1}{p} + \frac{1}{q} = 1$.

In the more usual case where $\mathcal{Y} = \mathbb{R}$, when $p \in]1,\infty[$ the dual of $L^p(\mathcal{X},\mu)$ is exactly $L^q(\mathcal{X},\mu)$ and L^p-spaces are reflexive. Under the hypothesis of σ-finiteness, the dual of $L^1(\mathcal{X},\mu)$ is (canonically isomorphic to) $L^\infty(\mathcal{X},\mu)$; the dual of the latter can be described as a space of finitely additive signed measures on $\mathcal{X}$.

Notice that, under general assumptions, L^1 and L^∞ are not reflexive.

Since they will be of practical use in the later chapters of our work, we recall here the notions of "restriction" and of "extension" of a given measure:

1.1.8 Definition. A measure μ on a set Z determines a measure on every subset $W \subset Z$ by simply restricting μ - as a set function - to the subsets of W. The resulting measure, denoted by μ_W, will be called the *restriction* of μ to W.
Alternatively, given a measure μ on a set Z, one may define its restriction to $W \subset Z$ by setting

$$\mu \lfloor W(E) \coloneqq \mu(E \cap W),$$

for $E \subset Z$.
If instead μ is a measure on a subset $W \subset Z$, we define its *extension* to Z as

$$\bar{\mu}(E) \coloneqq \mu(E \cap W),$$

for $E \subset Z$. In this way, we have $\mu \lfloor W = \overline{\mu_W}$.

Another recurring and useful notion will be the "push-forward" of a measure:

1.1.9 Definition. Let W and Z be two sets. If $f : W \to Z$ is any function and μ is a measure on W, we define the *push-forward measure* $f_{\#}\mu$ on Z by setting

$$f_{\#}\mu(E) \coloneqq \mu\left(f^{-1}(E)\right)$$

for $E \subset Z$, whenever the above formula makes sense.
When f is a Borel function and μ is a Borel measure, then $f_{\#}\mu$ is a Borel measure as well.

We now turn our attention to metric measure spaces, namely the key-tool for all the forthcoming analysis and discussions.
Let us start with a characterization of balls in metric spaces:

1.1.10 Definition. By a *ball* in a metric space $(\mathcal{X}, d)$ we mean a set of the form

$$B_\rho(x) = \{y \in \mathcal{X};\ d(x,y) < \rho\},$$

where $x \in \mathcal{X}$ is the *center* and $0 < \rho < \infty$ is the *radius*. The family of all balls in $\mathcal{X}$ will be denoted by $\mathscr{B}(\mathcal{X})$.
It is worth to underline that a ball, as a set, does not in general uniquely determine a center and a radius, which then make sense with respect to the notation $B_\rho(x)$. The *closed ball* $\overline{B}_\rho(x)$ is given by

$$\overline{B}_\rho(x) = \{y \in \mathcal{X};\ d(x,y) \leq \rho\}.$$

As an alternative notation, we may sometimes choose to write λB, $\lambda > 0$, to denote the "inflated" ball whose radius is λ times the radius of B.

1.1.11 Definition. A *metric measure space* is a triple $(\mathcal{X}, d, \mu)$ where $(\mathcal{X}, d)$ is a separable metric space and μ is a non-trivial - that is, $\mu(\mathcal{X}) > 0$ - locally finite Borel measure on $\mathcal{X}$. In this context, by "locally finite" we mean that for every $x \in \mathcal{X}$ there exists $\rho > 0$ such that $\mu\left(B_\rho(x)\right) < \infty$.

1.1.12 Remark. By the Lindelöf property of separable metric spaces - [HKST, Lemma 3.3.27] - every metric measure space can be written as a countable union of balls with finite measure; thus, in particular, every metric measure space is σ-finite - see [HKST, Lemma 3.3.28].

Notice that we are not assuming a metric measure space to be complete or even locally complete as a metric space.

The restriction of the measure μ to a subset $Z \subset \mathcal{X}$ determines a metric measure space $(Z, d, \mu\lfloor Z)$: in other words, subsets of metric measure spaces can be regarded as metric measure spaces on their own. As a consequence, every metric measure space admits a countable covering of pairwise disjoint subsets, each of them constituting itself a metric measure space.

1.1.13 Proposition [HKST, Proposition 3.3.44]. If $(\mathcal{X}, d, \mu)$ is a metric measure space such that $(\mathcal{X}, d)$ is complete, then μ is a Radon measure. In particular, $\mathcal{X}$ can be expressed as a countable union of compact sets plus a set of measure zero.

$\square$

Taking into account Proposition 1.1.13 above, since in the following chapters and sections we shall always consider complete metric measure spaces, from now on μ will be assumed to be a Radon measure.

Let us now see some properties of Lebesgue Spaces in the context of metric measure spaces.

1.1.14 Proposition [HKST, Proposition 3.3.49]. If $(\mathcal{X}, d, \mu)$ is a metric measure space and $p \in [1, \infty[$, then for every $f \in L^p(\mathcal{X}, \mu; \mathcal{Y})$ and for every $\varepsilon > 0$ there exists $g \in C(\mathcal{X}, \mathcal{Y})$ such that $\|f - g\|_{L^p(\mathcal{X}, \mu; \mathcal{Y})} < \varepsilon$.

If in particular μ is Radon, then the same holds with g being just a compactly supported function from $\mathcal{X}$ to $\mathcal{Y}$.

$\square$

Since uniform limits of sequences of continuos functions are also continuous, a combination of [HKST, Proposition 3.2.15] with Proposition 1.1.14 and [HKST, Remark 3.2.16] implies the following:

1.1.15 Corollary [HKST, Corollary 3.3.51]. Under the same hypotheses of Proposition 1.1.14, for every $f \in L^p(\mathcal{X}, \mu; \mathcal{Y})$ and for every $\varepsilon > 0$ there exists an open set $A \subset \mathcal{X}$ with $\mu(A) < \varepsilon$ such that $f|_{\mathcal{X} \setminus A}$ is continuous.

$\square$

1.1.16 Proposition [HKST, Proposition 3.3.52]. Let $(\mathcal{X}, d, \mu)$ be a metric measure space with $(\mathcal{X}, d)$ locally compact and let $p \in [1, \infty[$. Then, for every $f \in L^p(\mathcal{X}, \mu; \mathcal{Y})$ and for every $\varepsilon > 0$ there exists $g \in C_c(\mathcal{X}, \mathcal{Y})$ such that

$$\|f - g\|_{L^p(\mathcal{X},\mu;\mathcal{Y})} < \varepsilon.$$

$\square$

1.1.17 Proposition [HKST, Proposition 3.3.55]. Let $(\mathcal{X}, d, \mu)$ be a metric measure space and let $p \in [1, \infty[$. Then, $L^p(\mathcal{X}, \mu; \mathcal{Y})$ is separable - as a Banach space - if and only if $\mathcal{Y}$ is separable.

$\square$

We conclude this section by recalling quickly the notion of Lipschitz functions between metric spaces and some of their notable properties.

1.1.18 Definition. Let $(\mathcal{X}, d_\mathcal{X})$ and $(\mathcal{Y}, d_\mathcal{Y})$ be two metric spaces. A function $f : \mathcal{X} \to \mathcal{Y}$ is called *L-Lipschitz* if there exists a constant $L \geq 0$ such that

$$d_\mathcal{Y} (f(w), f(z)) \leq L d_\mathcal{X} (w, z) \tag{1.3}$$

for every pair of points $w, z \in \mathcal{X}$. The smallest value of L for which (1.3) holds is the *Lipschitz constant* of f. The class of Lipschitz functions $f : \mathcal{X} \to \mathcal{Y}$ will be usually denoted by $\mathrm{Lip}(\mathcal{X}, \mathcal{Y})$.
A *bi-Lipschitz* function is a bijective Lipschitz function whose inverse is Lipschitz as well.

Lipschitz functions are a dense class in the Lebesgue Spaces $L^p(\mathcal{X}, \mu; \mathcal{Y})$:

1.1.19 Theorem [HKST, Theorem 4.2.4]. If $(\mathcal{X}, d, \mu)$ is a metric measure space, $\mathcal{Y}$ is a Banach space and $p \in [1, \infty[$, then $\mathrm{Lip}(\mathcal{X}, \mathcal{Y})$ is dense in $L^p(\mathcal{X}, \mu; \mathcal{Y})$. If moreover $(\mathcal{X}, d)$ is locally compact, then $\mathrm{Lip}_c(\mathcal{X}, \mathcal{Y})$ is dense in $L^p(\mathcal{X}, \mu; \mathcal{Y})$.

$\square$

1.2 Doubling measures

Doubling measures constitute a notable tool in the field of nonsmooth analysis and will be of great use in our work as well. Let us see the main definitions and properties.

1.2.1 Definition. A Borel measure μ on a metric space $(\mathcal{X}, d)$ will be called *doubling* whenever for every ball $B_\rho(x) \subset \mathcal{X}$ one has $0 < \mu(B_\rho(x)) < \infty$ and there exists a constant $c \geq 1$ such that

$$\mu(B_{2\rho}(x)) \leq c\mu(B_\rho(x)) \tag{1.4}$$

for all $x \in \mathcal{X}$ and $\rho > 0$; in particular, μ is *asymptotically doubling* if

$$\lim_{\rho \to 0^+} \frac{\mu(B_{2\rho}(x))}{\mu(B_\rho(x))} < \infty$$

for all $x \in \mathcal{X}$ and $\rho > 0$. The smallest constant c fulfilling the condition in (1.4) will be said *doubling constant*; it will be denoted by C_{D},

$$C_{\mathrm{D}} = \sup_{B \in \mathscr{B}(\mathcal{X})} \frac{\mu(2B)}{\mu(B)}.$$

$(\mathcal{X}, d)$ is separable as a metric space and $(\mathcal{X}, d, \mu)$ is a metric measure space by Remark 1.1.12 and [HKST, Lemma 3.3.30] respectively.

When μ is doubling on $(\mathcal{X}, d)$, then $(\mathcal{X}, d, \mu)$ is called a *doubling metric measure space*.

1.2.2 Remark. A metric space $(\mathcal{X}, d)$ is called "metrically doubling" if there exists a constant $c \geq 1$ such that every set of diameter δ in $\mathcal{X}$ can be covered by at most c subsets whose diameter is not larger than $\delta/2$ ([He, 10.13]); in terms of balls, this is equivalent to say that any ball $B_\rho(x) \subset \mathcal{X}$ can be covered by at most c balls with radii $\rho/2$ ([BB, Section 3.1] or [HKST, Section 4.1]).

A metric measure space $(\mathcal{X}, d, \mu)$ equipped with a doubling measure μ is always metrically doubling ([Ha, Lemma 4.3]); if moreover $(\mathcal{X}, d)$ is complete, then the two conditions are equvalent ([Ha, Theorem 4.5]).

Observe that if one iterates the doubling condition (1.4) then

$$\mu(B_{\lambda\rho}(x)) \leq C_{\mathrm{D}} \lambda^{\log_2(c_{\mathrm{D}})} \mu(B_\rho(x))$$

for every $x \in \mathcal{X}$, $\lambda \geq 1$ and $\rho > 0$; the quantity $\log_2(C_{\mathrm{D}})$ can be thought as a sort of "dimension" of a doubling metric measure space $(\mathcal{X}, d, \mu)$. Indeed:

1.2.3 Proposition [BB, Lemma 3.3]. Assume $(\mathcal{X}, d, \mu)$ is a doubling metric measure space. Then, for every $x, y \in \mathcal{X}$ and for every $0 < \rho \leq r < \infty$ one has

$$\frac{\mu(B_\rho(x))}{\mu(B_r(y))} \geq \frac{1}{C_{\mathrm{D}}^2} \left(\frac{\rho}{r}\right)^s$$

with $s = \log_2(c_{\mathrm{D}})$.

$\square$

2 Upper gradients and Newton-Sobolev Spaces $N^{1,p}(\mathcal{X})$

This chapter is devoted to first-order Sobolev spaces defined by means of upper gradients. This approach relies heavily on the theory of modulus of curves in metric measure spaces and gives rise to the so-called "Newton-Sobolev" Spaces $N^{1,p}$, a term due to the fact that the definition recalls the Fundamental Theorem of Calculus.

The concept of upper gradient made its first appearances in the works of J. Heinonen and P. Koskela [HK1] and [HK2] under the name of "very weak gradients", but soon, for example in [KM], the denomination "upper gradient" was already preferred.

A systematic study of the theory of Sobolev Spaces via upper gradients was later carried on by J. Cheeger in [Ch], making use of a weak differentiable structure, and soon after in terms of p-modulus of curves by N. Shanmugalingam in her Ph.D. Thesis [Sh1] and in [Sh2], where the author also proved this latter characterization to be equivalent to Cheeger's.

In order to make our discussion coherent with Chapter 1, here we shall closely follow [HKST] again, in particular Chapters 5-8; another notable reference to be mentioned is the monograph [BB], from which we shall take some additional definitions and remarks.

2.1 Modulus of families of curves in metric measure spaces

We start with the basic facts regarding curves in metric spaces in order to introduce and discuss the concept of p-modulus of a family of curves.

2.1.1 Definition. A *curve* (or, *path*) on a metric space $(\mathcal{X}, d)$ is a continuous map $\gamma : I \to \mathcal{X}$, where I is some real interval; we shall say that γ is *compact*, *open* or *half-open* if so is I.

The interval I may also consist of one single point: in this case γ will be called a *constant curve*; in general, every curve whose image consists of only one element, shall be said constant.

Given a compact curve $\gamma : I = [a, b] \to \mathcal{X}$, its *length* is defined as

$$\ell(\gamma) \coloneqq \sup_{a=t_0<t_1<\ldots<t_k=b} \sum_{i=1}^{k} d\left(\gamma\left(t_i\right), \gamma\left(t_{i-1}\right)\right), \tag{2.1}$$

the numbers t_i being a finite decomposition of the interval $[a, b]$; when γ is non-compact, its length will be set to be the supremum of the lengths of its compact subcurves (i.e., of the restrictions of γ to compact sub-intervals of its domain).

When $\ell(\gamma)$ is finite, we shall call the curve *rectifiable*; *locally rectifiable* if each of its compact subcurves is rectifiable.

2.1.2 Remark. Given an $L-$Lipschitz map $f : \mathcal{X} \to \mathcal{Y}$ between metric spaces and a rectifiable curve $\gamma : I \to \mathcal{X}$, the composition $f \circ \gamma$ is rectifiable as well and $\ell\left(f \circ \gamma\right) \leq L\ell(\gamma)$.

Moreover, it is possible to prove that the length of any rectifiable curve is bounded from below by the 1-dimensional Hausdorff measure of its image $\mathcal{H}^1(\gamma)$, see [AT, Section 4.1] .

2.1.3 Definition. To every rectifiable curve $\gamma : [a,b] \to \mathcal{X}$ we associate a map, namely the *length function* $s_\gamma : [a,b] \to [0,\ell(\gamma)]$ given by $s_\gamma(t) := \ell\left(\gamma|_{[a,t]}\right)$.
Clearly, s_γ satisfies

$$d\left(\gamma\left(t_2\right), \gamma\left(t_1\right)\right) \leq \ell\left(\gamma|_{[t_1,t_2]}\right) = s_\gamma\left(t_2\right) - s_\gamma\left(t_1\right) \tag{2.2}$$

for $a \leq t_1 \leq t_2 \leq b$.
It can be shown that the length function is increasing and continuous, see [HKST, Lemma 5.1.4].

2.1.4 Definition. Given a rectifiable curve $\gamma : [a,b] \to \mathcal{X}$, its *arc length parametrization* is defined as the curve $\gamma_s : [0,\ell(\gamma)] \to \mathcal{X}$,

$$\gamma_s(t) := \gamma\left(s_\gamma^{-1}(t)\right).$$

By the continuity of the length function one finds that

$$s_\gamma^{-1}(t) := \sup\left\{\tau,\ s_\gamma(\tau) = t\right\} = \max\left\{\tau,\ s_\gamma(\tau) = t\right\}$$

is the one-sided inverse of s_γ; moreover, s_γ^{-1} is strictly increasing and right-continuous: this allows us to say that γ_s is the only curve in $\mathcal{X}$ such that $\gamma(t) = \gamma_s\left(s_\gamma(t)\right)$.

Via the arc length parametrization, we can also give a notion of the "length" of a curve γ in any set $E \subset \mathcal{X}$: namely, denoting by $\mathscr{L}^1$ the one-dimensional Lebesgue measure, $\ell(\gamma \cap E) := \mathscr{L}^1\left(\gamma_s^{-1}(E)\right)$. Of course, when E is the whole of $\mathcal{X}$, this definition gives just $\ell(\gamma)$.
We remark that by the properties of Hausdorff measures, one always has $\mathcal{H}^1(\gamma \cap E) \leq \ell(\gamma \cap E)$.

2.1.5 Definition. If $\gamma : [a,b] \to \mathcal{X}$ is a curve, we say that it is *absolutely continuous* provided there exists a function $f \in L^1([a,b])$ such that

$$d\left(\gamma(s), \gamma(t)\right) \leq \int_s^t f(r)\mathrm{d}r \tag{2.3}$$

for every $s,t \in]a,b[$ with $s \leq t$. The space of absolutely continuous curves $\gamma : [a,b] \to \mathcal{X}$ will be denoted by $AC([a,b],\mathcal{X})$.
If the above holds for $f \in L^p([a,b])$, $p \in [1,\infty]$, then γ is called *p-absolutely continuous* and the corresponding space of curves will be denoted by $AC^p([a,b],\mathcal{X})$.

To every p-absolutely continuous curve it is possible to associate a "derivative", namely the metric derivative, by means of an incremental ratio:

2.1.6 Theorem [AGS1, Theorem 1.1.2] Let $\gamma \in AC^p([a,b],\mathcal{X})$ with $p \in [1,\infty]$. Then the limit

$$|\dot{\gamma}|\,(t) := \lim_{s\to t}\frac{d(\gamma(s),\gamma(t))}{|s-t|}$$

exists for $\mathscr{L}^1$-almost every $t\in]a,b[$ and defines a function in $L^p(]a,b[)$. Moreover, $|\dot{\gamma}|\,(t)$ is an admissible integrand for (2.3) and it is minimal in the sense that $|\dot{\gamma}|\,(t)\le f(t)$ for $\mathscr{L}^1$-almost every $t\in]a,b[$ for all f satisfying (2.3).
We shall call $|\dot{\gamma}|\,(t)$ the *metric derivative* or the *speed* of $\gamma(t)$.

□

The metric derivative was already defined for general curves in [AT, Section 4.1].

Before discussing the concept of "modulus" of a family of curves, we need a further tool, namely line integration:

2.1.7 Definition. If $\gamma : [a,b] \to \mathcal{X}$ is a rectifiable curve and $\rho : \mathcal{X} \to [0,\infty]$ is a non-negative Borel function, we define the *line integral* of ρ over γ as the quantity

$$\int_\gamma \rho \mathrm{d}s := \int_0^{\ell(\gamma)} \rho\left(\gamma_s(t)\right)\mathrm{d}t. \tag{2.4}$$

Note that $\rho\circ\gamma_s$ is a non-negative Borel function on $[0,\ell(\gamma)]$, so the integral exists and attains its values on $[0,\infty]$.
When γ is locally rectifiable, the definition is adapted taking the supremum of the integrals of ρ over all compact subcurves of γ.
Observe that, by the properties of the length function and by the fact that $|\dot{\gamma}(t)| = \dot{s}_\gamma(t)$, when γ is absolutely continuous we can equivalently replace the right-hand side of the previous definition by

$$\int_a^b \rho\left(\gamma(t)\right)|\dot{\gamma}(t)|\,\mathrm{d}t.$$

In other words, the line integral of a non-negative Borel function makes sense over locally rectifiable curves, as no line integral is defined on curves which are not locally rectifiable; moreover, as the above equivalent formulation suggests, line integrals over constant curves are always zero.

Let us suppose that $(\mathcal{X},d,\mu)$ is a metric measure space such that $(\mathcal{X},d)$ is separable and μ is a locally finite Borel regular measure. In accordance with the notation of Chapter 1, we recall that by "measure" we mean an outer measure.

2.1.8 Definition. Given a family of curves Γ in $\mathcal{X}$, we define the *p-Modulus* of Γ, $1\le p<\infty$ as

$$\operatorname{Mod}_p(\Gamma) := \inf_{\mathcal{A}(\Gamma)}\int_{\mathcal{X}}\rho^p\mathrm{d}\mu, \tag{2.5}$$

where $\mathcal{A}(\Gamma)$ denotes the class of *admissible functions* (or *densities*) for the family Γ,

$$\mathcal{A}(\Gamma) := \left\{\rho : \mathcal{X} \to [0,\infty] \text{ Borel} : \int_\gamma \rho \mathrm{d}s \geq 1 \, \forall \gamma \in \Gamma \text{ locally rectifiable}\right\}.$$

The p-modulus can take values in $[0,\infty]$; if Γ is the family of curves in $\mathcal{X}$ which are not locally rectifiable, then $\mathrm{Mod}_p(\Gamma) = 0$, while the modulus of every family containing a constant curve is infinite.

2.1.9 Remark (properties of the p-modulus). First of all, we notice that $\mathrm{Mod}_p(\emptyset) = 0$ since the function $\rho \equiv 0$ is admissible, in this case.

If Γ_1 and Γ_2 are two families of curves such that $\Gamma_1 \subset \Gamma_2$, then $\mathrm{Mod}_p(\Gamma_1) \leq \mathrm{Mod}_p(\Gamma_2)$ because $\mathcal{A}(\Gamma_2) \subset \mathcal{A}(\Gamma_1)$; in particular, if Γ_0 and Γ are two families such that every curve $\gamma \in \Gamma$ has a subcurve $\gamma_0 \in \Gamma_0$, one has $\mathrm{Mod}_p(\Gamma) \leq \mathrm{Mod}_p(\Gamma_0)$: this happens by the simple fact that every ρ which is admissible for Γ_0 is also admissible for Γ.

The p-modulus is subadditive: in other words,

$$\mathrm{Mod}_p\left(\bigcup_{i=1}^{\infty} \Gamma_i\right) \leq \sum_{i=1}^{\infty} \mathrm{Mod}_p(\Gamma_i).$$

Indeed, let us assume - without loss of generality - that the right hand side above is finite; for $\varepsilon > 0$ fixed and for every $i \geq 1$ take $\rho_i \in \mathcal{A}(\Gamma_i)$ such that

$$\int_{\mathcal{X}} \rho_i^p \mathrm{d}\mu \leq \mathrm{Mod}_p(\Gamma_i) + 2^{-i}\varepsilon.$$

If we set

$$\rho(x) := \left(\sum_{i=1}^{\infty} \rho_i(x)^p\right)^{\frac{1}{p}},$$

then ρ is Borel measurable and $\rho \in \mathcal{A}(\Gamma_i)$ for all i because $\rho \geq \rho_i$. Thus, setting

$$\Gamma := \bigcup_{i=1}^{\infty} \Gamma_i,$$

one has the following estimate:

$$\mathrm{Mod}_p(\Gamma) \leq \int_{\mathcal{X}} \rho^p \mathrm{d}\mu \leq \sum_{i=1}^{\infty} \mathrm{Mod}_p(\Gamma_i) + \varepsilon,$$

and letting $\varepsilon \to 0$ gives the assertion.

In other words, the set function $\Gamma \mapsto \mathrm{Mod}_p(\Gamma)$ defines an outer measure on the families of curves in $\mathcal{X}$.

2.1.10 Definition. We shall call every family of curves Γ in $\mathcal{X}$ such that $\mathrm{Mod}_p(\Gamma) = 0$ *p-exceptional*; consequently, every property which fails to hold on a p-exceptional collection of curves will be said to hold for *p-almost every curve*.

Of course, by the subadditivity and by the monotonicity of the modulus it follows that $\mathrm{Mod}_p(\Gamma) = \mathrm{Mod}_p(\Gamma \setminus \Gamma')$ whenever Γ' is a subfamily of Γ with zero modulus.

The following result provides an alternative characterization of p-exceptionality:

2.1.11 Lemma. A family Γ of locally rectifiable curves in $\mathcal{X}$ is p-exceptional if and only if there exist a p-integrable Borel function $\rho : \mathcal{X} \to [0, \infty]$ satisfying

$$\int_\gamma \rho \mathrm{d}s = \infty \tag{2.6}$$

for all $\gamma \in \Gamma$.

Proof. We start with necessity. Since $\mathrm{Mod}_p(\Gamma) = 0$, for each $i \in \mathbb{N}$ we can find an admissible function ρ_i such that

$$\int_{\mathcal{X}} \rho_i^p \mathrm{d}\mu \leq 2^{-ip}.$$

The function

$$\rho(x) := \sum_{i=1}^{\infty} \rho_i(x),$$

is non-negative, p-integrable and Borel measurable; moreover, it satisfies (2.6) for every path $\gamma \in \Gamma$ as required.

Now assume that $\rho : \mathcal{X} \to [0, \infty]$ is a p-integrable Borel function satisfying (2.6) for all the curves $\gamma \in \Gamma$; thus, for very $\varepsilon > 0$, $\varepsilon\rho$ turns to be admissible, implying $\mathrm{Mod}_p(\Gamma) = 0$.

□

2.1.12 Definition. Any set $E \subset \mathcal{X}$ is said to be *p-exceptional* when the family of all non-constant curves which meet E is p-exceptional.

By the previous Lemma it is clear that E is p-exceptional if and only if the collection of all non-constant, compact curves passing through E is p-exceptional.

2.1.13 Lemma. A countable union of p-exceptional sets is p-exceptional. Moreover, if $E \subset \mathrm{M}$ and every $x \in \mathrm{M}$ has a neighborhood U_x such that $E \cap U_x$ is p-exceptional, then E is p-exceptional as well.

Proof. By the subadditivity of the p-modulus we immediately have the first assertion. For the second, we refer to [HKST, Lemma 3.3.27].

□

It is a well known fact from the classical Measure Theory that any convergent sequence of functions in L^p has a pointwise almost everywhere convergent subsequence; the following Lemma, proven by B. Fuglede in [Fu], shows that an analogous property holds as well in the present setting, involving p-exceptional families of curves. This result will be of great importance in the development of first-order Sobolev space theory in metric measure spaces.

2.1.14 Lemma (Fuglede) If $(g_i)_{i\in\mathbb{N}}$ is a sequence of Borel functions converging in $L^p(\mathcal{X},\mu)$, $p \in [1,+\infty[$, then there exists a subsequence $(g_{i_k})_{k\in\mathbb{N}}$ such that if g is any Borel representative of the L^p-limit of $(g_i)_{i\in\mathbb{N}}$, one has

$$\lim_{k\to\infty} \int_\gamma |g_{i_k} - g| \,\mathrm{d}s = 0$$

for p-almost every curve γ in $\mathcal{X}$.

PROOF. Take g as in the statement of the Lemma, and choose a subsequence $(g_{i_k})_{k\in\mathbb{N}}$ of $(g_i)_{i\in\mathbb{N}}$ such that

$$\int_{\mathcal{X}} |g_{i_k} - g| \,\mathrm{d}\mu \leq 2^{-k(p+1)}.$$

This subsequence is independent of the particular representative we have chosen.
Define

$$\rho_k \coloneqq |g_{i_k} - g|$$

and let Γ be the family of locally rectifiable curves γ in $\mathcal{X}$ such that

$$\limsup_{k\to\infty} \int_\gamma \rho_k \mathrm{d}s > 0$$

Moreover, let Γ_k be the family of locally rectifiable curves in $\mathcal{X}$ satisfying

$$\int_\gamma \rho_k \mathrm{d}s > 2^{-k}.$$

Thus,

$$\Gamma = \bigcap_{j\in\mathbb{N}} \bigcup_{k\geq j} \Gamma_k \subset \bigcup_{k=j}^{\infty} \Gamma_k$$

for all $j \geq 1$; but $2^k \rho_k$ is admissible for Γ_k for every k, and then

$$\mathrm{Mod}_p\left(\Gamma_k\right) \leq 2^{pk} \int_{\mathcal{X}} \rho_k^p \mathrm{d}\mu \leq 2^{-k}.$$

Consequently, the subadditivity of the modulus gives

$$\mathrm{Mod}_p(\Gamma) \leq \sum_{k=j}^{\infty} \mathrm{Mod}_p\left(\Gamma_k\right) \leq 2^{-j+1}$$

for every $j \geq 1$; this means $\mathrm{Mod}_p(\Gamma) = 0$ as required.

□

2.1.15 Proposition. Let $\Gamma_1 \subset \Gamma_2 \subset ...$ be an increasing sequence of path families in $\mathcal{X}$. Then, setting $\Gamma = \bigcup_{i=1}^{\infty} \Gamma_i$, for all $p > 1$ it holds

$$\lim_{i\to\infty} \mathrm{Mod}_p\,(\Gamma_i) = \mathrm{Mod}_p(\Gamma).$$

Proof. By the monotonicity of the p-modulus, one has that the quantities $\mathrm{Mod}_p\,(\Gamma_i)$ form an increasing sequence and, moreover, $\mathrm{Mod}_p\,(\Gamma_i) \le \mathrm{Mod}_p(\Gamma)$ for every i. So, assuming the above limit is finite, we need to prove the opposite inequality. For every i, consider an admissible function ρ_i for Γ_i such that

$$\int_{\mathcal{X}} \rho_i^p \mathrm{d}\mu < M + \frac{1}{i},$$

where $M \coloneqq \lim\limits_{i\to\infty} \mathrm{Mod}_p\,(\Gamma_i)$. From this we deduce that $(\rho_i)_{i\in\mathbb{N}}$ is a bounded sequence in $L^p(\mathcal{X},\mu)$ satisfying

$$\lim_{i\to\infty} \|\rho_i\|^p_{L^p(\mathcal{X},\mu)} = M.$$

Since $p > 1$, by the reflexivity of $L^p(\mathcal{X},\mu)$ we get that the sequence converges weakly to some $\rho \in L^p(\mathcal{X},\mu)$, and the lower semicontinuity of norms gives $\|\rho\|^p_{L^p(\mathcal{X},\mu)} \le M$. Now, an application of Mazur's Lemma - see for example Section 2.3 in [HKST] or the classical monograph [Yo] - allows us to consider a convex combination of the ρ_i's, say $(\tilde{\rho}_j)_{j\in\mathbb{N}}$, such that $\tilde{\rho}_j \to \rho$ in $L^p(\mathcal{X},\mu)$ as well. Taking into account that $(\rho_i)_{i\in\mathbb{N}}$ is increasing and that the admissibility condition is unaltered by convex combinations, we can assume the $\tilde{\rho}_j$'s to be admissible as well for the families Γ_j for every j. Then,

$$M \le \lim_{j\to\infty} \|\tilde{\rho}_j\|^p_{L^p(\mathcal{X},\mu)} = \|\rho\|^p_{L^p(\mathcal{X},\mu)} \le M.$$

By [HKST, Proposition 3.3.23] and by Fuglede's Lemma, we may assume ρ to be a Borel map such that

$$\int_\gamma \rho \mathrm{d}s \ge 1$$

for p-almost every curve $\gamma \in \Gamma$. At this point, invoking again the subadditivity of the p-modulus yields

$$\mathrm{Mod}_p(\Gamma) \le \int_{\mathcal{X}} \rho^p \mathrm{d}\mu = M,$$

thus concluding the proof.

□

2.1.16 Lemma [HKST, Lemma 5.2.15]. If $E \subset \mathcal{X}$ has measure zero, then for p-almost every curve γ in $\mathcal{X}$ one has $\ell(\gamma \cap E) = 0$; in particular, this means that $\mathcal{H}^1(\gamma \cap E) = 0$.

□

2.1.17 Lemma. Suppose that g and h are two non-negative Borel functions on $\mathcal{X}$ such that $g \leq h$ almost everywhere. Then,

$$\int_\gamma g \mathrm{d}s \leq \int_\gamma h \mathrm{d}s$$

for p-almost every curve γ in $\mathcal{X}$ and if the two functions agree almost everywhere, then the above inequality becomes an equality for p-almost every γ as well.

PROOF. Clearly, the set $E = \{x \in \mathcal{X},\ g(x) > h(x)\}$ is μ-negligible. By Lemma 1.1.14 above, this gives $\mathcal{H}^1\left(\gamma^{-1}\left(E\right)\right) = 0$ for p-almost every curve γ. By the definition of line integrals, it follows

$$\int_\gamma (h-g)\, \mathrm{d}s \geq 0.$$

The linearity of line integrals allows us to conclude.

□

We explicitly remark that, unfortunately, it is impossible to give a precise value for the modulus of a family of curves, except in a few cases; in general it can be also quite difficult even to give good estimates. In what follows we collect, without proof, a few examples of particular exceptions. We refer to Section 5.3 of [HKST] for the details and for other notable examples.

2.1.18 Proposition [HKST, Lemma 5.3.1]. Let Γ be a family of curves in a Borel set $A \subset \mathcal{X}$ such that for every $\gamma \in \Gamma$ one has $\ell(\gamma) \geq L > 0$. Then, $\mathrm{Mod}_p(\Gamma) \leq \mu(A)L^{-p}$.

□

2.1.19 Proposition [HKST, Lemma 5.3.2]. Given a family of curves Γ in $\mathcal{X}$ and a sequence of Borel subsets of $\mathcal{X}$, $(B_i)_{i\in\mathbb{N}}$, if every curve in Γ has a non-rectifiable subcurve in some B_i then $\mathrm{Mod}_p(\Gamma) = 0$.

□

In other words, the p-modulus is not affected by "local" non-rectifiability of non-rectifiable paths. In particular, non-rectifiable curves are p-exceptional whenever the volume growth of $\mathcal{X}$ is at most polynomial of order p:

2.1.20 Proposition [HKST, Proposition 5.3.3]. If $p > 1$ and there exists $x \in \mathcal{X}$ such that

$$\limsup_{r\to\infty} \frac{\mu\left(B_r(x)\right)}{r^p} < \infty,$$

then the p-modulus of the family of all non-rectifiable curves in $\mathcal{X}$ is zero.

□

The following Proposition shows that lower semicontinuous functions can be chosen as admissible densities:

2.1.21 Proposition [HKST, Proposition 5.3.13]. For every family of paths Γ in $\mathcal{X}$ it holds

$$\mathrm{Mod}_p(\Gamma) = \inf\left\{\int_{\mathcal{X}} \rho^p \mathrm{d}\mu;\ \rho\colon \mathcal{X} \to [0,\infty] \text{ lower semicontinuous}:\ \rho \in \mathcal{A}(\Gamma)\right\}.$$

□

We end this section with a result regarding the behaviour of the modulus with respect to the exponent p:

2.1.22 Proposition [HKST, Proposition 5.3.14]. Let Γ be a family of curves in a Borel set $A \subset \mathcal{X}$ of finite measure. Then, for $1 \leq q < p$,

$$\mathrm{Mod}_q(\Gamma)^p \leq \mu(A)^{p-q}\mathrm{Mod}_p(\Gamma)^q.$$

In particular, if Γ is exceptional for some $p > 1$ then it is q-exceptional for all $1 \leq q \leq p$.

□

2.2 Upper gradients

Throughout this section, $(\mathcal{X}, d_{\mathcal{X}}, \mu)$ will be a metric measure space with $(\mathcal{X}, d_{\mathcal{X}})$ separable and μ a locally finite Borel regular measure on $\mathcal{X}$, $(\mathcal{Y}, d_{\mathcal{Y}})$ will be a metric space and $p \in [1, \infty[$.

2.2.1 Definition. A Borel function $g : \mathcal{X} \to [0,\infty]$ is called an *upper gradient* of $u : \mathcal{X} \to \mathcal{Y}$ provided

$$d_{\mathcal{Y}}\left(u(\gamma(a)), u(\gamma(b))\right) \leq \int_\gamma g \mathrm{d}s \tag{2.7}$$

for every rectifiable path $\gamma : [a, b] \to \mathcal{X}$.

If the above inequality holds for p-almost every path γ, then g will be said a *p-weak upper gradient* of u.

The definition naturally restricts to any subset $A \subset \mathcal{X}$ if one considers A as a metric measure space itself.

Even though the concepts of upper gradient and p-weak upper gradient are very different, the former being purely metric and defined on an arbitrary metric space, the latter being dependent both on p and on the metric structure, it can be shown that they not "move too far" from one another:

2.2.2 Lemma. If $g \geq 0$ is a Borel function on $\mathcal{X}$ such that it is finite-valued μ-almost everywhere and if $u : \mathcal{X} \to \mathcal{Y}$ admits g as a p-weak upper gradient in $\mathcal{X}$, then there exists a sequence $(g_k)_{k\in\mathbb{N}}$ of upper gradients of u such that $g \leq g_{k+1} \leq g_k$ for every k and $\|g - g_k\|_{L^p} \to 0$ as $k \to \infty$.

Proof. Denote by Γ the family of all non-constant compact rectifiable curves in $\mathcal{X}$ which fail to satisfy the upper gradient condition (2.7) for the pair (u, g). Thus, $\mathrm{Mod}_p(\Gamma) = 0$ and by Lemma 2.1.11 we can find a Borel map $\rho : \mathcal{X} \to [0, \infty]$ such that

$$\int_\gamma \rho \mathrm{d}s = \infty$$

for every $\gamma \in \Gamma$ and $\rho \in L^p(\mathcal{X})$. Now, setting $g_k := g + 2^{-k}\rho$ we obtain a sequence of upper gradients of u which satisfies our claim.

□

We explicitly remark that any given function actually has infinitely many (p-weak) upper gradients.

2.2.3 Example. Let us collect a few basic examples of upper gradients:

- The function $g \equiv \infty$ is an upper gradient of every function.
- If there exist no non-constant rectifiable curves on M, then $g \equiv 0$ is an upper gradient of every function.
- If u is L-Lipschitz, then the constant function $g \equiv L$ is an upper gradient of u.
- If $f : \mathcal{Y} \to \mathcal{Z}$ is an L-Lipschitz map between metric spaces, then Lg is an upper gradient of $f \circ u : \mathcal{X} \to \mathcal{Z}$ provided g is an upper gradient of $u : \mathcal{X} \to \mathcal{Y}$.
- If $u : \mathcal{X} \to \mathcal{Y}$, $A \subset \mathcal{X}$ and g is an upper gradient of u, then $g|_A$ is an upper gradient of $u|_A$.
- The collection of upper gradients of a given function is a convex set: indeed, if g and h are upper gradients of some u and $\lambda \in [0, 1]$, then the combination $(1 - \lambda)g + \lambda h$ is an upper gradient of u.

2.2.4 Definition. Given a function $u : \mathcal{X} \to \mathcal{Y}$, we define its *pointwise lower Lipschitz constant* as

$$\operatorname{lip} u(x) = \liminf_{r\to 0} \sup_{y\in B_r(x)} \frac{d_{\mathcal{Y}}\left(u(x), u(y)\right)}{r} \tag{2.8}$$

and, similarly, its *pointwise upper Lipschitz constant*,

$$\operatorname{Lip} u(x) \coloneqq \limsup_{r\to 0} \sup_{y\in B_r(x)} \frac{d_{\mathcal{Y}}\left(u(x),u(y)\right)}{r}. \tag{2.9}$$

We observe that if u is continuous, then both $\operatorname{lip} u(x)$ and $\operatorname{Lip} u(x)$ are Borel regular functions; see [HKST, Lemma 6.2.5] for a detailed discussion.

2.2.5 Lemma. If $u : \mathcal{X} \to \mathcal{Y}$ is a locally Lipschitz map, then $\operatorname{lip} u(x)$ is an upper gradient of u.

PROOF. Suppose $\gamma : [0, \ell(\gamma)] \to \mathcal{X}$ is a non-constant rectifiable curve parametrized by arc-length. Since $u \in \operatorname{Lip}_{\mathrm{loc}}(\mathcal{X})$, then the composition $u \circ \gamma : [0, \ell(\gamma)] \to \mathcal{Y}$ is absolutely continuos and then [HKST, Proposition 4.4.25] grants

$$d_{\mathcal{Y}}\left(u(\gamma(a)),u(\gamma(b))\right) \le \int_0^{\ell(\gamma)} |(u\circ\gamma)'(t)|\,\mathrm{d}t. \tag{2.10}$$

Moreover, if we take $t \in]0, \ell(\gamma)[$ and $h \in \mathbb{R}$ with $|h|$ small enough, we find

$$\frac{d_{\mathcal{Y}}\left((u\circ\gamma)(t),(u\circ\gamma)(t+h)\right)}{|h|} \le \sup_{y\in B_{|h|}(\gamma(t))} \frac{d_{\mathcal{Y}}\left(u(\gamma(t)),u(y)\right)}{|h|}.$$

Since the left hand side goes to $|\left(u\circ\gamma\right)'(t)|$ when $|h| \to 0$ by [HKST, Theorem 4.4.8] (previously established by L. Ambrosio in [Am1]), one may conclude that $|(u\circ\gamma)'(t)| \le \operatorname{lip} u(\gamma(t))$ for almost every $t \in]0, \ell(\gamma)[$. Combining this fact with inequality (1.9), we conclude.

□

The following result shows that any non-negative Borel function which agrees almost-everywhere with a p-weak upper gradient of some function u is an upper gradient as well:

2.2.6 Lemma [HKST, Lemma 6.2.8]. If g is a p-weak upper gradient of $u : \mathcal{X} \to \mathcal{Y}$ and $h : \mathcal{X} \to [0, \infty]$ is a Borel function such that $g = h$ μ-almost everywhere in $\mathcal{X}$, then h is a p-weak upper gradient of u.
Moreover, if $E \subset \mathcal{X}$ is a Borel set of measure zero and g is as above, then $g \cdot \mathbb{1}_{\mathcal{X}\setminus E}$ is a p-weak upper gradient of u.

□

We now pass to the study of maps with p-integrable upper gradients. To this aim, we start with a generalization of the notion of absolute continuity on curves to the metric setting:

2.2.7 Definition. We shall say that a function $u : \mathcal{X} \to \mathcal{Y}$ is *absolutely continuous on a curve* γ in $\mathcal{X}$ if γ is rectifiable and, moreover, the composition $u \circ \gamma_s : [0, \ell(\gamma)] \to \mathcal{Y}$ is absolutely continuous.

2.2.8 Remark. The definitions and the absolute continuity of line integrals entail that if $u : \mathcal{X} \to \mathcal{Y}$, γ is a rectifiable compact path in $\mathcal{X}$, $g : \mathcal{X} \to [0, \infty]$ is a Borel function such that g is integrable over γ and the pair (u, g) satisfies the upper gradient condition (2.7) on γ as well as on each compact subcurve of it, then u is absolutely continuous on γ.

2.2.9 Proposition. If g is a p-integrable p-weak upper gradient of some map $u : \mathcal{X} \to \mathcal{Y}$, then p-almost every compact rectifiable path γ in $\mathcal{X}$ is such that g is integrable over γ and the pair (u, g) satisfies the upper gradient condition on γ as well as on each of its compact subcurves.

In particular, every $u : \mathcal{X} \to \mathcal{Y}$ which has a p-integrable p-weak upper gradient is absolutely continuous on p-almost every curve in $\mathcal{X}$.

Proof. If we denote by Γ_0 the family of those compact rectifiable curves γ such that either (2.7) fails to hold or g is not integrable on γ, then $\mathrm{Mod}_p(\Gamma_0) = 0$ by definition of weak upper gradients and by the subadditivity of modulus. Now, if Γ is the family of all the compact curves in $\mathcal{X}$ which have a subcurve in Γ_0, by Remark 2.1.9 we find $\mathrm{Mod}_p(\Gamma) = 0$ as well.

To prove the second assertion of the Proposition, it suffices to apply the first along with Remark 2.2.8.

□

2.2.10 Proposition. Let $u : \mathcal{X} \to \mathcal{Y}$ be any map and let $\gamma : [0, \ell(\gamma)] \to \mathcal{X}$ be a rectifiable curve parametrized by arc length. Suppose $g : \mathcal{X} \to [0, \infty]$ is a Borel function which is integrable over γ and that the pair (u, g) satisfies the upper gradient condition on γ as well as on each of its subpaths. Then, u is absolutely continuous on γ and

$$\left|(u \circ \gamma)'(t)\right| \leq (g \circ \gamma)(t) \tag{2.11}$$

for almost every $t \in [0, \ell(\gamma)]$.

In particular, if g is a p-integrable p-weak upper gradient of $u : \mathcal{X} \to \mathcal{Y}$, then the above inequality holds for p-almost every path $\gamma : [0, \ell(\gamma)] \to \mathcal{X}$ parametrized by arch length. Moreover, if u has a p-integrable p-weak upper gradient in $\mathcal{X}$ and $g : \mathcal{X} \to [0, \infty]$ is a Borel p-integrable map satisfying (2.11) for p-almost every absolutely continuous rectifiable curve γ in $\mathcal{X}$, then g is a p-weak upper gradient of u.

Proof. The absolute continuity of u on γ is a consequence of Remark 2.2.8.

Observe that, by hypothesis,

$$\frac{d_{\mathcal{Y}}\left((u \circ \gamma)(t), (u \circ \gamma)(t+h)\right)}{h} \leq \frac{1}{h} \int_t^{t+h} (g \circ \gamma)(s) \mathrm{d}s$$

for every $t \in [0, \ell(\gamma)[$ and for every $h \in]0, \ell(\gamma) - t[$. For almost every t, as $h \to 0$ the left-hand side of the above inequality tends to $|(u \circ \gamma)'(t)|$ again by [HKST, Theorem 4.4.8] and [Am1], and the right-hand side becomes $(g \circ \gamma)(t)$ by Lebesgue's Differentiation Theorem; this allows us to conclude that (2.11) holds.

The second assertion follows from the first one and by applying Proposition 2.2.9; let us then discuss the last statement. Recalling that if u has a p-integrable p-weak upper gradient on $\mathcal{X}$ then it is absolutely continuous on p-almost every curve, the proof follows by an application of [HKST, Proposition 4.4.25].

$\square$

2.2.11 Lemma. If $u : \mathcal{X} \to \mathcal{Y}$ is absolutely continuous on p-almost every curve γ in $\mathcal{X}$, and if there exists $c \in \mathcal{Y}$ such that $u \equiv c$ μ-almost everywhere in$\mathcal{X}$, then the family Γ_E of all non-constant curves in $\mathcal{X}$ which meet the set

$$E \coloneqq \{x \in \mathcal{X},\ u(x) \neq c\}$$

has p-modulus zero. In particular, the constant null function is a p-weak upper gradient of u.

As a consequence, if u has a p-integrable p-weak upper gradient and there exists $c \in \mathcal{Y}$ such that $u \equiv c$ μ-almost everywhere in $\mathcal{X}$, then $g \equiv 0$ is a p-weak upper gradient of u.

Proof. For p-almost every non-constant rectifiable curve γ in $\mathcal{X}$ we have that u is absolutely continuous on γ and that the length of γ in E is zero. So, by our hypotheses, $u \circ \gamma \equiv c$ and therefore γ cannot pass through E. However, as every curve in Γ_E meets E, by Remark 2.1.9 we find $\mathrm{Mod}_p\,(\Gamma_E) = 0$. At this point, it is obvious that $g \equiv 0$ is a p-weak upper gradient of u.

The latter assertion is a consequence of the previous ones together with Proposition 2.2.9.

$\square$

An interesting byproduct of the above result is that p-integrable p-weak upper gradients are actually a good substitute for the notion of derivative in the metric setting, since every locally constant map has the null function as p-weak upper gradient.

We stress the hypothesis of absolutely continuity of u along p-almost every curve, since if it fails to hold, then the statement of Lemma 2.2.11 is not true anymore.

It is possible to “build” new upper gradients from given ones:

2.2.12 Lemma [HKST, Lemma 6.3.8]. Let σ and τ be two p-integrable p-weak upper gradients of a map $u : \mathcal{X} \to \mathcal{Y}$; if $E \subset \mathcal{X}$ is a Borel set, then

$$g \coloneqq \sigma \cdot \mathbb{1}_E + \tau \cdot \mathbb{1}_{\mathcal{X}\setminus E}$$

is a p-weak upper gradient of u.

$\square$

Combining this result with Fuglede’s Lemma we find out the important *lattice* property of p-weak upper gradients:

2.2.13 Corollary [HKST, Corollary 6.3.12]. The collection $\mathcal{D}_p(u)$ of all p-integrable p-weak upper gradients of a function $u : \mathcal{X} \to \mathcal{Y}$ is closed under the (pointwise) operations of maximum and minimum; in other words, $\mathcal{D}_p(u)$ is a lattice in $L^p(\mathcal{X}, \mu)$.

$\square$

2.2.14 Lemma [HKST, Lemma 6.3.14]. Assume that $u : \mathcal{X} \to \mathcal{Y}$ is absolutely continuous on p-almost every compact rectifiable curve in $\mathcal{X}$. Let $E \subset \mathcal{X}$ be a Borel set and suppose there exists two maps $v, w : \mathcal{X} \to \mathcal{Y}$ such that $u = v$ μ-almost everywhere in E and $u = w$ μ-almost everywhere in $\mathcal{X} \backslash E$. Then, if v and w possess p-integrable p-weak upper gradients σ and τ respectively,

$$g \coloneqq \sigma \cdot \mathbb{1}_E + \tau \cdot \mathbb{1}_{\mathcal{X} \backslash E}$$

is a p-integrable p-weak upper gradient of u.

$\square$

2.2.15 Corollary [HKST, Corollary 6.3.16]. Let g be a p-integrable p-weak upper gradient of $u : \mathcal{X} \to \mathcal{Y}$ and take $c \in \mathcal{Y}$. Then, $g \cdot \mathbb{1}_{\mathcal{X} \backslash E}$ is a p-integrable p-weak upper gradient of u in every set $E \subset \{x \in \mathcal{X},\ u(x) = c\}$.

$\square$

2.2.16 Definition. A p-integrable p-weak upper gradient g of a map $u : \mathcal{X} \to \mathcal{Y}$ such that $g \leq h$ almost everywhere in $\mathcal{X}$ for every p-integrable p-weak upper gradient h of u will be said a *minimal p-weak upper gradient*.

By Lemma 2.2.6 it is clear that if a minimal p-weak upper gradient exists, then it shall be unique up to a set of measure zero; moreover, it has the smallest L^p-norm among all the p-weak upper gradients.

The minimal p-weak upper gradient of a function u will by denoted by g_u. The existence of a minimal p-weak upper gradient will be discussed in Theorem 2.2.18 below.

2.2.17 Remark. It is quite easy to check that the minimal p-weak upper gradient - for functions valued in some normed space - enjoys the following simple properties:

- $g_u = g_{-u}$,
- $g_{u+v} \leq g_u + g_v$,
- $g_{|u|} \leq g_u$,
- if $\lambda \in \mathbb{R}$, then $g_{\lambda u} = |\lambda| g_u$,
- if $f : \mathcal{Y} \to \mathcal{Z}$ is an L-Lipschitz map between metric spaces, then $g_{f \circ u} \leq L g_u$.

The notion of p-weak upper gradient depends on the exponent p: by the properties of modulus, for any $q \leq p$ one has $g_{u,q} \leq g_{u,p}$ μ-almost everywhere. However, as it is customary, this dependence will be omitted from the notation. See Remark 2.4.9 for additional comments.

2.2.18 Theorem. For every $p \in [1, \infty[$, the set of all p-integrable p-weak upper gradients of a map $u : \mathcal{X} \to \mathcal{Y}$ is a closed convex lattice in $L^p(\mathcal{X})$ and, if non-empty, it has a unique element of smallest L^p-norm. In particular, if a function has a p-integrable p-weak upper gradient, then it possesses a minimal one.

PROOF. By Corollary 2.2.13, we already know that the collection $\mathcal{D}_p(u)$ of all p-integrable p-weak upper gradients of u is a closed convex lattice in $L^p(\mathcal{X},\mu)$.

It is a known fact that, in general, any such set has an element of minimum L^p-norm. Indeed, consider a sequence $(g_i)_{i\in\mathbb{N}} \subset \mathcal{D}_p(u)$ such that

$$\lim_{i\to\infty} \|g_i\|_{L^p(\mathcal{X},\mu)} = \inf_{\rho\in\mathcal{D}_p(u)} \|g\|_{L^p(\mathcal{X},\mu)} .$$

If we take into account the lattice property and replace g_i by $\min_{1\le j\le i} g_j$, we can then assume that the chosen sequence decreases pointwise. As the limit function $g_u := \lim_{i\to\infty} g_i$ is Borel regular, using Lebesgue's Monotone Convergence Theorem we find that $g_i \to g_u$ in $L^p(\mathcal{X},\mu)$; moreover, Fuglede's Lemma grants that $g_u \in \mathcal{D}_p(u)$.

Clearly, we can conclude that g_u is minimal as required.

□

2.2.19 Proposition. Consider two maps $u, v : \mathcal{X} \to \mathcal{Y}$ with minimal p-weak upper gradients g_u and g_v respectively. If u and v agree almost everywhere in some Borel set E, then $g_u = g_v$ almost everywhere in E.

PROOF. The existence of p-integrable p-weak upper gradients allows us to infer that both u and v are absolutely continuous on p-almost every compact curve, thanks to Proposition 2.2.9. Now, by Lemma 2.2.14, $g_u \cdot \mathbb{1}_E + g_v \cdot \mathbb{1}_{\mathcal{X}\setminus E}$ is a p-integrable p-weak upper gradient of v. Since g_v is minimal, then $g_v \le g_u$ almost everywhere in E. A symmetric argument gives the proof.

□

Upper gradients behave well under truncation by real functions; in other words,

2.2.20 Proposition [HKST, Proposition 6.3.23]. Let $u_1, u_2 : \mathcal{X} \to \mathbb{R}$ be two measurable functions with minimal p-weak upper gradients $g_{u_1}$and g_{u_2} respectively. Then,

$$g_{\min\{u_1,u_2\}} = g_{u_1} \cdot \mathbb{1}_{\{u_1\le u_2\}} + g_{u_2} \cdot \mathbb{1}_{\{u_2<u_1\}}, \tag{2.12}$$

$$g_{\max\{u_1,u_2\}} = g_{u_1} \cdot \mathbb{1}_{\{u_1> u_2\}} + g_{u_2} \cdot \mathbb{1}_{\{u_2\ge u_1\}} \tag{2.13}$$

pointwise almost everywhere in $\mathcal{X}$.

□

Combining this Proposition with the fact that $g_u = g_{-u}$ and with the definition of modulus, $|u| := \max\{u, -u\}$, we have that for $u : \mathcal{X} \to \mathbb{R}$ with a p-integrable p-weak upper gradient, $g_u = g_{|u|}$. It is worth to remark that Proposition 2.2.20 does not assert that the functions on the right hand sides of (2.12) and (2.13) are the minimal upper gradients of $\min\{u_1, u_2\}$ and $\max\{u_1, u_2\}$ respectively; the statement actually holds only for Borel representatives of these functions.

2.2.21 Proposition [HKST, Proposition 6.3.28]. Given a Banach space $\mathcal{V}$, assume that $u : \mathcal{X} \to \mathcal{V}$ and $m : \mathcal{X} \to \mathbb{R}$ are measurable functions that are absolutely continuous on p-almost every compact rectifiable curve in $\mathcal{X}$. Assume also that g and h are p-weak upper gradients of u and m respectively. Then, every Borel representative of $|m|g + |u|h$ is a p-weak upper gradient of $mu : \mathcal{X} \to \mathcal{V}$ and, in particular, if $|m|g + |u|h \in L^p(\mathcal{X}, \mu)$, one has

$$g_{mu} \leq |m|g + |u|h$$

almost everywhere in $\mathcal{X}$.

□

2.2.22 Proposition [HKST, Proposition 6.3.29]. If $u : \mathcal{X} \to \mathcal{Y}$ is absolutely continuous on p-almost every rectifiable curve in $\mathcal{X}$ and if there exists $c \in \mathcal{Y}$ such that $u \equiv c$ almost everywhere in $\mathcal{X}$, then the set $\{x \in \mathcal{X},\ u(x) \neq c\}$ is p-exceptional. The same holds, in particular, if $u : \mathcal{X} \to \mathcal{Y}$ has a p-integrable p-weak upper gradient.

□

2.2.23 Proposition. Let $(u_i)_{i\in\mathbb{N}}$ be a sequence of functions, $u_i : \mathcal{X} \to \mathcal{Y}$, with a corresponding sequence of p-integrable p-weak upper gradients $(g_i)_{i\in\mathbb{N}}$. Suppose there exist $u : \mathcal{X} \to \mathcal{Y}$ and a p-exceptional set $E \subset \mathcal{X}$ such that $\lim_{i\to\infty} u_i(x) = u(x)$ for every $x \in \mathcal{X}\backslash E$; suppose also there exists a Borel function $g : \mathcal{X} \to [0, \infty]$ such that $g_i \to g$ in $L^p(\mathcal{X})$ as $i \to \infty$.
Then, g is ap-weak upper gradient of u.

Proof. Combining the hypotheses with Fuglede's Lemma, we find that p-almost every compact rectifiable curve γ in $\mathcal{X}$ is such that:

i) each pair (u_i, g_i) satisfies the upper gradient condition (2.7) on γ;

ii) γ does not pass through E;

iii) $\lim\limits_{i\to\infty} \int_\gamma g_i \mathrm{d}s = \int_\gamma g \mathrm{d}s$.

Given such a curve $\gamma : [a, b] \to \mathcal{X}$, one has

$$d_{\mathcal{Y}}(u(\gamma(a)), u(\gamma(b)) = \lim_{i\to\infty} d_{\mathcal{Y}} \left(u_i(\gamma(a)), u_i(\gamma(b))\right) \leq \lim_{i\to\infty} \int_\gamma g_i \mathrm{d}s.$$

Applying *iii)*, our claim follows.

□

2.3 Newton-Sobolev Spaces $N^{1,p}(\mathcal{X})$

Here we suppose that $(\mathcal{X}, d, \mu)$ is a metric measure space, $\mathcal{Y}$ is a Banach space and $p \in [1, \infty[$, unless otherwise stated.

In the classic, Euclidean theory, the *Sobolev-Dirichlet class* $D^{1,p}(\Omega)$, $\Omega \subset \mathbb{R}^n$ open set, is defined as the class of locally integrable functions in Ω with distributional - or, weak - derivatives in $L^p(\Omega)$, equipped with the semi-norm

$$\|u\|_{D^{1,p}(\Omega)} \coloneqq \left(\int_\Omega |\nabla u|^p \, \mathrm{d}x \right)^{\frac{1}{p}} .$$

$\|\cdot\|_{D^{1,p}}$ is just a semi-norm and not a norm because it vanishes on (locally) constant functions which may not be identically zero.

In the metric setting, the notion of upper gradient allows us to extend the definition of Dirichlet space in a straightforward way:

2.3.1 Definition. The *Sobolev-Dirichlet class* $D^{1,p}(\mathcal{X}, \mathcal{Y})$ consists of all the measurable functions $u : \mathcal{X} \to \mathcal{Y}$ with a p-integrable p-weak upper gradient in $\mathcal{X}$.

$D^{1,p}(\mathcal{X}, \mathcal{Y})$ is a vector space and we shall endow it with the semi-norm

$$\|u\|_{D^{1,p}(\mathcal{X},\mathcal{Y})} \coloneqq \|g_u\|_{L^p(\mathcal{X})} , \tag{2.14}$$

where g_u is the minimal p-weak upper gradient of u. When $\mathcal{Y} = \mathbb{R}$, we shall omit it from the notation.

2.3.2 Remark. The above characterization coincides with the classical one when u is defined on any open set $\Omega \subset \mathbb{R}^n$; we refer to [HKST, Proposition 7.1.2] for a proof of this statement.

2.3.3 Lemma. If $u \in D^{1,p}(\mathcal{X}, \mathcal{Y})$ and $v : \mathcal{X} \to \mathcal{Y}$ is such that $u = v$ outside a p-exceptional set, then v is in $D^{1,p}(\mathcal{X}, \mathcal{Y})$ as well. Conversely, given two functions in $D^{1,p}(\mathcal{X}, \mathcal{Y})$ which agree almost everywhere, then they agree outside a p-exceptional set.

Proof. By assumption, the set $E \coloneqq \{x \in \mathcal{X},\ u(x) \neq v(x)\}$ is p-exceptional; moreover, every p-integrable p-weak upper gradient of u is such for v as well, thus the first claim follows.

Now assume $u, v \in D^{1,p}(\mathcal{X}, \mathcal{Y})$ are equal almost everywhere, so that E has measure zero. Since $u - v$ has a p-integrable p-weak upper gradient, it follows from Proposition 2.2.22 that E is p-exceptional.

$\square$

The next result is just a simple application of the remarks given after Proposition 2.2.20:

2.3.4 Lemma [HKST, Lemma 7.1.7]. If $u \in D^{1,p}(\mathcal{X}, \mathcal{Y})$, then $|u| \in D^{1,p}(\mathcal{X})$ as well and

$$\||u|\|_{D^{1,p}(\mathcal{X})} = \|u\|_{D^{1,p}(\mathcal{X},\mathcal{Y})} .$$

$\square$

2.3.5 Proposition [HKST, Proposition 7.1.8]. Suppose $u_1, u_2 \in D^{1,p}(\mathcal{X})$ and let g_{u_1}, g_{u_2} be their respective minimal p-weak upper gradients. Then, the following hold pointwise almost everywhere in $\mathcal{X}$:

$$g_{\min\{u_1,u_2\}} = g_{u_1} \cdot \mathbb{1}_{\{u_1 \leq u_2\}} + g_{u_2} \cdot \mathbb{1}_{\{u_2 < u_1\}}, \tag{2.15}$$

$$g_{\max\{u_1,u_2\}} = g_{u_1} \cdot \mathbb{1}_{\{u_1 > u_2\}} + g_{u_2} \cdot \mathbb{1}_{\{u_2 \geq u_1\}}. \tag{2.16}$$

In particular, for $u \in D^{1,p}(\mathcal{X})$ and $t \in \mathbb{R}$ one has $g_{u_t} \leq g_u$, where $u_t \in D^{1,p}(\mathcal{X})$ is either $\min\{u, t\}$ or $\max\{u, t\}$.

$\square$

As for Proposition 2.2.20, the right hand sides of (2.15) and (2.16) cannot be considered as the minimal p-weak upper gradients of $\min\{u_1, u_2\}$ and $\max\{u_1, u_2\}$ respectively, since they may not be Borel regular; however, it is possible to modify them on sets of zero measure, finding Borel representatives for which the above identity is satisfied.

We shall not discuss further Sobolev-Dirichlet classes here, as what we already said is enough to introduce first-order Newton-Sobolev spaces in the present setting. An alternative characterization of Sobolev-Dirichlet classes will be given later in Chapter 3 in terms of curves defined in the space of probability measures, following the approach of the works by L. Ambrosio, N. Gigli and G. Savaré ([AGS2], [AGS3], [AGS4]).

Given $(\mathcal{X}, d, \mu)$, $\mathcal{Y}$ and p as above, let us denote by $\tilde{N}^{1,p}(\mathcal{X}, \mathcal{Y})$ the collection of all p-integrable functions u with an upper gradient in $L^p(\mathcal{X})$; in other words,

$$\tilde{N}^{1,p}(\mathcal{X}, \mathcal{Y}) \coloneqq D^{1,p}(\mathcal{X}, \mathcal{Y}) \cap L^p(\mathcal{X}, \mathcal{Y}).$$

The above definition has to be intended in terms of functions, not just equivalence classes. $\tilde{N}^{1,p}(\mathcal{X}, \mathcal{Y})$ is a vector space and it will be endowed with the semi-norm

$$\|u\|_{\tilde{N}^{1,p}(\mathcal{X},\mathcal{Y})} \coloneqq \|u\|_{L^p(\mathcal{X},\mathcal{Y})} + \|g_u\|_{L^p(\mathcal{X})} , \tag{2.17}$$

g_u denoting the minimal p-weak upper gradient of u. Given that a function u has an upper gradient in $L^p(\mathcal{X})$ if and only if it has a p-weak upper gradient in $L^p(\mathcal{X})$, by Lemma 2.2.2 the above semi-norm can be equivalently rewritten as

$$\|u\|_{\tilde{N}^{1,p}(\mathcal{X},\mathcal{Y})} \coloneqq \|u\|_{L^p(\mathcal{X},\mathcal{Y})} + \inf \|g\|_{L^p(\mathcal{X})} , \tag{2.18}$$

where the infimum is taken over all upper gradients of u.

It is not difficult to see that $\|\cdot\|_{\tilde{N}^{1,p}(\mathcal{X},\mathcal{Y})}$ is not a norm, in general: if $E \subset \mathcal{X}$ is a non-empty p-exceptional set of measure zero, and if $0 \neq c \in \mathcal{Y}$, then $u := c \cdot \mathbb{1}_E$ is a non-zero function from $\mathcal{X}$ to $\mathcal{Y}$ but $\|u\|_{\tilde{N}^{1,p}(\mathcal{X},\mathcal{Y})} = 0$.

2.3.6 Definition. The *Newton-Sobolev space* $N^{1,p}(\mathcal{X},\mathcal{Y})$ is the normed space of equivalence classes of functions in $\tilde{N}^{1,p}(\mathcal{X},\mathcal{Y})$, where $u_1 \sim u_2$ if and only if $\|u_1 - u_2\|_{\tilde{N}^{1,p}(\mathcal{X},\mathcal{Y})} = 0$. In other words,

$$N^{1,p}(\mathcal{X},\mathcal{Y}) := \tilde{N}^{1,p}(\mathcal{X},\mathcal{Y}) \Big/ \left\{ u \in \tilde{N}^{1,p}(\mathcal{X},\mathcal{Y}),\ \|u\|_{\tilde{N}^{1,p}(\mathcal{X},\mathcal{Y})} = 0 \right\}. \tag{2.19}$$

We shall write $\|u\|_{N^{1,p}(\mathcal{X},\mathcal{Y})}$ for the quotient norm. As for Sobolev-Dirichlet classes, when $\mathcal{Y} = \mathbb{R}$ we shall omit it from the notation.

2.3.7 Remark. For any subset $E \subset \mathcal{X}$, the restricted measure μ_E gives us the corresponding metric measure space (E, d, μ_E); it follows from the definitions that the restriction operator $u \mapsto u|_E$ yields a bounded operator from $\tilde{N}^{1,p}(\mathcal{X},\mathcal{Y})$ to $\tilde{N}^{1,p}(E,\mathcal{Y})$ such that

$$\|u|_E\|_{\tilde{N}^{1,p}(E,\mathcal{Y})} \leq \|u\|_{\tilde{N}^{1,p}(\mathcal{X},\mathcal{Y})}. \tag{2.20}$$

Passing to the quotient norm, we also have

$$\|u|_E\|_{N^{1,p}(E,\mathcal{Y})} \leq \|u\|_{N^{1,p}(\mathcal{X},\mathcal{Y})}, \tag{2.21}$$

the inequality being of course intended in terms of equivalence classes.

2.3.8 Definition. We introduce $\tilde{N}^{1,p}_{\mathrm{loc}}(\mathcal{X},\mathcal{Y})$ as the vector space of functions $u : \mathcal{X} \to \mathcal{Y}$ with the property that for every $x \in \mathcal{X}$ there exists a neighborhood U_x in $\mathcal{X}$ such that $u \in N^{1,p}(U_x,\mathcal{Y})$. Any two functions $u_1, u_2 \in \tilde{N}^{1,p}_{\mathrm{loc}}(\mathcal{X},\mathcal{Y})$ will be said equivalent if every $x \in \mathcal{X}$ has a neighborhood U_x (which, by the above remark, can be assumed to be open) such that the restrictions $u_1|_{U_x}$ and $u_2|_{U_x}$ determine the same element in $N^{1,p}(U_x,\mathcal{Y})$.
The *local Newton-Sobolev space* $N^{1,p}_{\mathrm{loc}}(\mathcal{X},\mathcal{Y})$ is the vector space of equivalence classes of functions in $\tilde{N}^{1,p}_{\mathrm{loc}}(\mathcal{X},\mathcal{Y})$, under the same equivalence relation as in Definition 2.3.6.

2.3.9 Lemma. Two functions $u_1, u_2 \in \tilde{N}^{1,p}_{\mathrm{loc}}(\mathcal{X},\mathcal{Y})$ determine the same element in $N^{1,p}_{\mathrm{loc}}(\mathcal{X},\mathcal{Y})$ if and only if $u_1 - u_2 = 0$ in $N^{1,p}(\mathcal{X},\mathcal{Y})$.

Proof. We just need to show that a function $u : \mathcal{X} \to \mathcal{Y}$ determines the zero element in $N^{1,p}_{\mathrm{loc}}(\mathcal{X},\mathcal{Y})$ if and only if $\|u\|_{\tilde{N}^{1,p}(\mathcal{X},\mathcal{Y})} = 0$.
If $\|u\|_{\tilde{N}^{1,p}(\mathcal{X},\mathcal{Y})} = 0$, by (2.20) u determines the zero element in $N^{1,p}_{\mathrm{loc}}(\mathcal{X},\mathcal{Y})$, so we pass directly to the converse implication assuming that $u = 0$ in $N^{1,p}_{\mathrm{loc}}(\mathcal{X},\mathcal{Y})$. Then, the set $E = \{x \in \mathcal{X},\ u(x) \neq 0\}$ has measure zero by [HKST, Lemma 3.3.31]; we wish to show that E is also p-exceptional.
Fix $x \in \mathcal{X}$ and denote by U_x a neighborhood of x such that $u|_{U_x} = 0$ in $N^{1,p}(U_x,\mathcal{Y})$; by Lemma 2.3.3, $E \cap U_x$ is p-exceptional and this implies, by Lemma 2.1.13, that E itself is p-exceptional. Consequently, $g \equiv 0$ is a p-weak upper gradient of u and then $\|u\|_{\tilde{N}^{1,p}(\mathcal{X},\mathcal{Y})} = 0$ as required.

□

The following Proposition summarizes what we have discussed so far and its proof is an easy byproduct of the previous Definitions and Lemmata.

2.3.10 Proposition [HKST, Proposition 7.1.31]. Given $u \in \tilde{N}^{1,p}_{\text{loc}}(\mathcal{X}, \mathcal{Y})$ and a function $v : \mathcal{X} \to \mathcal{Y}$ which agrees with u outside a μ-negligible p-exceptional set, one has that $v \in \tilde{N}^{1,p}_{\text{loc}}(\mathcal{X}, \mathcal{Y})$ and the two functions determine the same element in $N^{1,p}_{\text{loc}}(\mathcal{X}, \mathcal{Y})$. If, moreover, $u \in \tilde{N}^{1,p}(\mathcal{X}, \mathcal{Y})$, then $v \in \tilde{N}^{1,p}(\mathcal{X}, \mathcal{Y})$ as well and the two functions determine the same element in $N^{1,p}(\mathcal{X}, \mathcal{Y})$.

Conversely, if any two functions in $\tilde{N}^{1,p}_{\text{loc}}(\mathcal{X}, \mathcal{Y})$ agree almost everywhere, then they coincide outside a p-exceptional set. In particular, if two μ-representative of a function in an equivalence class of $N^{1,p}_{\text{loc}}(\mathcal{X}, \mathcal{Y})$ are both in $\tilde{N}^{1,p}_{\text{loc}}(\mathcal{X}, \mathcal{Y})$, then they differ only in a μ-negligible p-exceptional set.

□

2.3.11 Remark. Every function u in $N^{1,p}(\mathcal{X}, \mathcal{Y})$ is also in $L^p(\mathcal{X}, \mathcal{Y})$ by construction, and the inclusion $N^{1,p}(\mathcal{X}, \mathcal{Y}) \subset L^p(\mathcal{X}, \mathcal{Y})$ is a bounded embedding (and indeed an injection, by the previous Proposition).

In some cases, namely those of metric spaces without non-constant rectifiable curves (for example, totally disconnected and “snowflake” spaces), the Sobolev space $N^{1,p}$ reduces to the Lebesgue space L^p; more generally, this happens if the p-modulus of the collection of all non-constant curves in $\mathcal{X}$ is zero. The converse holds as well:

2.3.12 Proposition & Definition. The inclusion $N^{1,p}(\mathcal{X}, \mathcal{Y}) \subset L^p(\mathcal{X}, \mathcal{Y})$ is strict if and only if the p-modulus of the collection of all non-constant curves in $\mathcal{X}$ is positive.

In this case, we shall say that the Newton-Sobolev space $N^{1,p}(\mathcal{X}, \mathcal{Y})$ is *non-trivial*.

Proof. The necessary implication comes directly from the definitions; let us then pass directly to sufficiency.

Using the subadditivity of the modulus and the fact that $\mathcal{X}$ can be covered by countably many open balls $B_q(x_i)$, with the x_i's forming a countable dense subset of $\mathcal{X}$ and $q_i \in \mathbb{Q}^+$ being fixed, we conclude that there exists an open ball B in $\mathcal{X}$ for which the p-modulus of the family of curves in $\mathcal{X}$ with end points in B and $\mathcal{X} \backslash \bar{B}$ respectively is positive. We claim that the L^p function $c \cdot \mathbb{1}_B$, $0 \neq c \in \mathcal{Y}$, cannot have a representative in $N^{1,p}(\mathcal{X}, \mathcal{Y})$. Arguing by contradiction, suppose there exists such a representative u; then there exists a μ-negligible Borel set $E \subset \mathcal{X}$ such that $u|_{B\backslash E} \equiv c$ and $u|_{X\backslash(B\cup E)} \equiv 0$. Then, we can find a path γ in $\mathcal{X}$, parametrized by arc length, which passes through $B\backslash E$ and $\mathcal{X}\backslash(B \cup E)$; moreover, u is absolutely continuous on γ and γ intersects E in a set of zero length. However, this cannot be the case since on the dense set $\gamma \backslash E$ the function u takes only the two values 0 and c, and it takes both values on sets of positive length, thus violating absolute continuity. The Proposition is proven.

□

Next, we introduce the notion of Sobolev p-capacity. This tool is of fundamental importance in the present context since it allows to express p-exceptionality and $\tilde{N}^{1,p}$-equivalence in terms of p-capacity, and it will make possible to show, in the manner of [HKST, Section 7.3] and [Sh2, Section 3], that the spaces $N^{1,p}$ are Banach spaces. As before, $(\mathcal{X}, d, \mu)$ is a metric measure space and $1 \leq p < \infty$.

2.3.13 Definition (Sobolev p-capacity). Given a set $E \subset \mathcal{X}$, we define its *p-capacity* as the (possibly infinite) quantity

$$\mathrm{Cap}_p(E) := \inf \left(\int_{\mathcal{X}} |u|^p \mathrm{d}\mu + \int_{\mathcal{X}} g_u^p \mathrm{d}\mu \right), \tag{2.22}$$

the infimum being taken over all the functions $u \in N^{1,p}(\mathcal{X})$ such that $u|_E \geq 1$ outside a p-exceptional set of measure zero; such functions will be also said *p-admissible* or simply *admissible*.

2.3.14 Remark. Of course, the above definition is well-posed since functions in $N^{1,p}(\mathcal{X})$ are defined up to μ-negligible p-exceptional sets; then, any two equivalent functions in $N^{1,p}(\mathcal{X})$ are simultaneously admissible.

Without loss of generality, one may assume that admissible functions satisfy the condition $u \geq 1$ *everywhere* on E. Note also that, in the definition of p-capacity, the infimum can be equivalently taken over the functions u in $\tilde{N}^{1,p}(\mathcal{X})$ such that $u \geq 1$ on E.

When no admissible function exists for E, we set $\mathrm{Cap}_p(E) = \infty$. Every μ-negligible p-exceptional set has zero p-capacity: in fact, the characteristic function of such a set is an admissible function.

The p-capacity satisfies $\mathrm{Cap}_p(\emptyset) = 0$ and a monotonicity property, that is, $\mathrm{Cap}_p(E_1) \leq \mathrm{Cap}_p(E_2)$ whenever $E_1 \subset E_2$; moreover, it turns out to be an outer measure:

2.3.15 Lemma. The p-capacity is a countably sub-additive set function, hence an outer measure in $\mathcal{X}$. In other words,

$$\mathrm{Cap}_p \left(\bigcup_{i=1}^{\infty} E_i \right) \leq \sum_{i=1}^{\infty} \mathrm{Cap}_p(E_i) \tag{2.23}$$

for any sequence of sets $(E_i)_{i \in \mathbb{N}}$ in $\mathcal{X}$.

Proof. To begin, we observe that the p-capacity can be equivalently defined as

$$\mathrm{Cap}_p(E) := \inf \left(\int_{\mathcal{X}} |u|^p \mathrm{d}\mu + \int_{\mathcal{X}} g_u^p \mathrm{d}\mu \right),$$

where the infimum is taken over the functions $u \in N^{1,p}(\mathcal{X})$ such that $0 \leq u \leq 1$ in $\mathcal{X}$ and $u|_E = 1$. In fact, if $u \in N^{1,p}(\mathcal{X})$ is such that $u|_E \geq 1$, taking into account Proposition 2.3.5 we deduce that $\breve{u} := \max\{0, \min\{1, u\}\} \in N^{1,p}(\mathcal{X})$ is admissible and $\|\breve{u}\|_{N^{1,p}(\mathcal{X})} \leq \|u\|_{N^{1,p}(\mathcal{X})}$.

Let us now prove the Lemma. We assume that the right hand side of (2.23) is finite, the opposite case being obvious.

Fix $\varepsilon > 0$ and, for every $i \in \mathbb{N}$, pick $u_i \in N^{1,p}(\mathcal{X})$ - pointwise defined - such that $0 \leq u_i \leq 1$ on $\mathcal{X}$, $u_i|_E = 1$ and

$$\int_{\mathcal{X}} u_i^p \mathrm{d}\mu + \int_{\mathcal{X}} g_{u_i}^p \mathrm{d}\mu \leq \mathrm{Cap}_p(E_i) + 2^{-i}\varepsilon.$$

Proposition 2.3.5 grants that the functions $v_j := \max\{u_i,\, 1 \leq i \leq j\}$ are in $N^{1,p}(\mathcal{X})$ and $v(x) := \lim_{j\to\infty} v_j(x)$ is well defined for every $x \in \mathcal{X}$. Moreover, $v = 1$ on the union of the E_i's.

Now, denote by $g_i = g_{u_i}$ the minimal p-weak upper gradient of u_i; again by Proposition 2.3.5, $\sigma_j = \max\{g_i,\, 1 \leq i \leq j\}$ is a p-weak upper gradient of v_j. Moreover, since

$$0 \leq v_j^p \leq \sum_{i=1}^{j} u_i^p$$

and

$$0 \leq \sigma_j^p \leq \sum_{i=1}^{j} g_i^p,$$

one has

$$\begin{aligned} \|v_j\|_{L^p(\mathcal{X})}^p + \left\|g_{v_j}\right\|_{L^p(\mathcal{X})}^p &\leq \sum_{i=1}^{\infty} \left(\|u_i\|_{L^p(\mathcal{X})}^p + \|g_i\|_{L^p(\mathcal{X})}^p\right) \\ &\leq \sum_{i=1}^{\infty} \mathrm{Cap}_p(E_i) + \varepsilon \end{aligned} \tag{2.24}$$

for every j. The limit $\sigma(x) := \lim_{j\to\infty} \sigma_j(x)$ is a Borel p-integrable function and $\sigma_j \to \sigma$ in $L^p(\mathcal{X})$ by the Monotone Convergence Theorem. Thus, by Proposition 2.2.23 $v \in N^{1,p}(\mathcal{X})$ and σ is a p-weak upper gradient for it. Moreover, v is an admissible function for $\bigcup_{i=1}^{\infty} E_i$. Now, as $v_j \to v$ in $L^p(\mathcal{X})$, the estimates (2.24) imply that

$$\mathrm{Cap}_p\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \|v\|_{L^p(\mathcal{X})}^p + \|\sigma\|_{L^p(\mathcal{X})}^P \leq \sum_{i=1}^{\infty} \mathrm{Cap}_p(E_i) + \varepsilon.$$

Letting $\varepsilon \to 0$ gives the result. The Lemma is proven.

□

2.3.16 Proposition. A subset $E \subset \mathcal{X}$ satisfies $\mathrm{Cap}_p(E) = 0$ if and only if it is p-exceptional and $\mu(E) = 0$.

Proof. The definitions imply that a μ-negligible p-exceptional set has zero p-capacity. Indeed, the characteristic function $\mathbb{1}_E$ is p-integrable and admits $g \equiv 0$ as a p-weak upper gradient; in other words, $\mathbb{1}_E \in N^{1,p}(\mathcal{X})$ and $\|\mathbb{1}_E\|_{N^{1,p}(\mathcal{X})} = 0$.

Now assume that $\mathrm{Cap}_p(E) = 0$. For every $i \in \mathbb{N}$ there exists $u_i \in N^{1,p}(\mathcal{X})$ such that $0 \leq u_i \leq 1$ and $u_i|_E = 1$, satisfying

$$\|u_i\|^p_{L^p(\mathcal{X})} + \|g_{u_i}\|^p_{L^p(\mathcal{X})} \le 2^{-i}. \tag{2.25}$$

Arguing as in the proof of Lemma 2.3.15, we find that for positive integers $j \ge j_0$ the functions $v_j := \min\{u_i,\ j_0 \le i \le j\}$ are in $N^{1,p}(\mathcal{X})$ with p-weak upper gradients $\sigma_j = \max\{g_i,\ j_0 \le i \le j\}$ by Proposition 2.3.5. Moreover, one has $w_{j_0}(x) := \lim_{j\to\infty} v_j(x) \in N^{1,p}(\mathcal{X})$ with p-weak upper gradient $g_{j_0} := \lim_{j\to\infty} \sigma_j$, see Proposition 2.2.23. In particular, (2.25) implies $\|w_{j_0}\|_{N^{1,p}(\mathcal{X})} \le 2^{-j_0/p}$. Applying the Monotone Convergence Theorem we find a limit function $v \in N^{1,p}(\mathcal{X})$ such that $\|v\|_{N^{1,p}(\mathcal{X})} = 0$. Noting that the sequences $(w_j)_{j\in\mathbb{N}}$ and $(g_j)_{j\in\mathbb{N}}$ are monotone increasing and decreasing respectively, by Proposition 2.3.10 we conclude that E is both μ-negligible and p-exceptional.

□

2.3.17 Corollary [HKST, Corollary 7.2.10]. Two functions in $\tilde{N}^{1,p}(\mathcal{X},\mathcal{Y})$ determine the same element in $N^{1,p}(\mathcal{X},\mathcal{Y})$ if and only if they agree outside a set of zero p-capacity. Moreover, if two functions in $\tilde{N}^{1,p}(\mathcal{X},\mathcal{Y})$ agree almost everywhere, then they agree outside a set of zero p-capacity.

□

2.3.18 Proposition [HKST, Proposition 7.2.12]. If $\mathcal{X}$ is a proper metric space (i.e. closed and bounded subsets are compact), then

$$\inf_{\substack{U\supset E\\ U\subset\mathcal{X}\text{ open}}} \operatorname{Cap}_p(U) = 0$$

for every set $E \subset \mathcal{X}$ such that $\operatorname{Cap}_p(E) = 0$.

□

We are now almost ready to show that the Newton-Sobolev space $N^{1,p}(\mathcal{X},\mathcal{Y})$ is a Banach space; before doing so, we need to prove that every Cauchy sequence of $N^{1,p}(\mathcal{X},\mathcal{Y})$ functions admits a subsequence which converges in some sense.

2.3.19 Definition. A sequence of functions $u_i : \mathcal{X} \to \mathcal{Y}$ is said to converge *p-quasiuniformly* to $u : \mathcal{X} \to \mathcal{Y}$ if for every $\varepsilon > 0$ there exists a set $F_\varepsilon \subset \mathcal{X}$ such that $\operatorname{Cap}_p(F_\varepsilon) < \varepsilon$ and $u_i \to u$ uniformly on F_ε^c.

It is clear that any p-quasiuniformly convergent sequence of functions converges pointwise outside a set of zero p-capacity or, equivalently, outside a μ-negligible p-exceptional set.

2.3.20 Proposition. Every Cauchy sequence of functions in $N^{1,p}(\mathcal{X},\mathcal{Y})$ contains a p-quasiuniformly convergent subsequence. Moreover, the pointwise limit function is in $N^{1,p}(\mathcal{X},\mathcal{Y})$ as well and it does not depend upon the chosen subsequence.

Proof. Note that any Cauchy sequence in $N^{1,p}(\mathcal{X},\mathcal{Y})$ is also a Cauchy sequence in $L^p(\mathcal{X},\mathcal{Y})$; this means that any two limit functions agree almost everywhere, and combining this fact with Proposition 2.3.10 we obtain the required independence.

Given a Cauchy sequence in $N^{1,p}(\mathcal{X},\mathcal{Y})$, choose a subsequence $(u_i)_{i\in\mathbb{N}}$ converging pointwise almost everywhere to its L^p-limit $\tilde{u}$ and such that

$$\|u_i - u_{i+1}\|^p_{L^p(\mathcal{X},\mathcal{Y})} + \|g_{i+1,i}\|^p_{L^p(\mathcal{X})} \leq 2^{-i(p+1)}, \tag{2.26}$$

where $g_{i,j}$ denotes the minimal p-weak upper gradient of $u_i - u_j$. We find out that, in general,

$$g_i \coloneqq g_1 + \sum_{k=1}^{i-1} g_{k+1,k}$$

is a p-weak upper gradient of

$$u_i = u_1 + \sum_{k=1}^{i-1} (u_{k+1} - u_k) .$$

Moreover,

$$\|g_j - g_{j+i}\|_{L^p(\mathcal{X})} \leq \sum_{k=j}^{j+i-1} \|g_{k+1,k}\|_{L^p(\mathcal{X})} \leq \sum_{k=j}^{\infty} 2^{-k} \underset{j\to\infty}{\longrightarrow} 0.$$

Thus, the sequence of upper gradients $(g_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\mathcal{X},\mu)$ and then it converges in $L^p(\mathcal{X},\mu)$ to a non-negative Borel function g. Set

$$u(x) = \lim_{i\to\infty} u_i(x); \tag{2.27}$$

the limit exists because $u_i \to \tilde{u}$ almost everywhere and one has $u(x) = \tilde{u}(x)$ for almost every x and, in particular, $u \in L^p(\mathcal{X},\mathcal{Y})$.

Now we consider the sets where the sequence $(u_i)_{i\in\mathbb{N}}$ does not have the Cauchy property, namely

$$E_i \coloneqq \left\{x \in \mathcal{X},\ |u_i(x) - u_{i+1}(x)| > 2^{-i}\right\}$$

and define

$$F_j \coloneqq \bigcup_{i=j}^{\infty} E_i.$$

Outside F_j, one clearly has $|u_i(x) - u_{i+1}(x)| \leq 2^{-i}$ for every $i \geq j$; so $(u_i(x))_{i\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{Y}$ and therefore it has a limit, which is $u(x)$ as shown above. Moreover, on the one hand

$$|u(x) - u_i(x)| \leq 2^{1-i} \tag{2.28}$$

for all $i \geq j$ and $x \notin F_j$; that is, $u_i \to u$ uniformly in F_j^c. On the other hand, $2^i\,|u_i - u_{i+1}| \in N^{1,p}(\mathcal{X})$ by Lemma 2.3.4 and this quantity is greater than 1 on each of the E_i's by construction. So by the estimate (2.26) it follows

$$\mathrm{Cap}_p\,(E_i) \leq 2^{ip}\,\|u_i - u_{i+1}\|^p_{L^p(\mathcal{X})} + 2^{ip}\,\|g_{i+1,i}\|^p_{L^p(\mathcal{X})} \leq 2^{-i}.$$

The sub-additivity of capacity implies

$$\operatorname{Cap}_p(F_j) \leq \sum_{i=j}^{\infty} 2^{-i} = 2^{1-j}$$

for every $j \in \mathbb{N}$. This, together with (2.28), completes the proof.

$\square$

2.3.21 Theorem. $N^{1,p}(\mathcal{X}, \mathcal{Y})$ is a Banach space.

PROOF. Take a Cauchy sequence $(u_i)_{i\in\mathbb{N}} \subset N^{1,p}(\mathcal{X}, \mathcal{Y})$; passing to a subsequence if necessary, we may assume that $(u_i)_{i\in\mathbb{N}}$ satisfies Proposition 2.3.20 and the condition (2.26) as well. In particular, the u_i's converge pointwise to a function $u \in N^{1,p}(\mathcal{X}, \mathcal{Y})$ outside a μ-negligible p-exceptional set E. Since

$$u(x) - u_i(x) = \sum_{k=i}^{\infty} (u_{k+1}(x) - u_k(x))$$

for $x \in E^c$, and

$$\sum_{k=i}^{n} g_{k+1,k} \longrightarrow \sum_{k=i}^{\infty} g_{k+1,k}$$

in $L^p(\mathcal{X}, \mu)$ by (2.26), by Proposition 2.2.23 we have that

$$\sum_{k=i}^{\infty} g_{k+1,k}$$

is a p-weak upper gradient of $u - u_i$. Moreover, again by (2.26)

$$\begin{aligned}
\|u - u_i\|_{N^{1,p}(\mathcal{X},\mathcal{Y})} &\leq \sum_{k=i}^{\infty} \Big(\|u_k - u_{k+1}\|_{L^p(\mathcal{X},\mathcal{Y})} + \|g_{k+1,k}\|_{L^P(\mathcal{X})} \Big) \\
&\leq 2 \sum_{k=i}^{\infty} \Big(\|u_k - u_{k+1}\|^p_{L^p(\mathcal{X},\mathcal{Y})} + \|g_{k+1,k}\|^p_{L^P(\mathcal{X})} \Big)^{\frac{1}{p}} \\
&\leq 2 \sum_{k=i}^{\infty} 2^{k\left(1+\frac{1}{p}\right)} \leq 4 \cdot 2^{-i}.
\end{aligned}$$

In other words, $u_i \to u$ in $N^{1,p}(\mathcal{X}, \mathcal{Y})$. The Theorem is proven.

$\square$

2.4 Poincaré Inequalities and density of Lipschitz functions

Having the notion of upper gradient at our disposal, we can introduce Poincaré Inequalities in the metric setting and discuss some of their consequences, the most important being a density result for Lipschitz functions in the spaces $N^{1,p}(\mathcal{X})$.

In the following, $(\mathcal{X}, d, \mu)$ is a metric measure space, $\mathcal{Y}$ is a Banach space and $p \in [1, \infty[$.

2.4.1 Definition. We say that $(\mathcal{X}, d, \mu)$ supports a $(1,p)$-*Poincaré Inequality* if every ball in $\mathcal{X}$ has positive finite measure and if there exist constants $c = C_{\mathrm{P}} > 0$ and $\lambda \geq 1$ such that, for every open ball $B \in \mathscr{B}(\mathcal{X})$, for every $u : \mathcal{X} \to \mathbb{R}$ such that $u \in L^1(B, \mu; \mathbb{R})$ and every upper gradient g of u one has

$$\fint_B |u - u_B| \, \mathrm{d}\mu \leq C_{\mathrm{P}} \mathrm{diam}(B) \left(\fint_{\lambda B} g^p \mathrm{d}\mu \right)^{\frac{1}{p}},$$

u_B denoting the integral average of u over B.

The above definition extends with no modifications to Banach-space valued functions, apart from u being in $L^1(B, \mu; \mathcal{Y})$ instead of $L^1(B, \mu; \mathbb{R})$.

2.4.2 Remark. If $(\mathcal{X}, d, \mu)$ is doubling and supports a $(1,p)$-Poincaré inequality for functions attaining their values in some Banach space, then it supports a $(1,p)$-Poincaré inequality for functions valued in every Banach space, see [HKST, Theorem 8.1.42].

If a space supports a $(1,p)$-Poincaré inequality for some $p \in [1, \infty[$, then an immediate application of Hölder's Inequality yields the validity of a $(1,q)$-Poincaré inequality for all $q \geq p$.

We refer to Section 8.1 of [HKST] for a detailed discussion about Poincaré Inequalities and their consequences; here we shall focus only on the density of Lipschitz functions in the Newton-Sobolev spaces and on some geometric facts related with "generalized" (q,p)-Poincaré Inequalities.

The following result is a re-adaptation of [HKST, Theorem 8.2.1]:

2.4.3 Theorem. If $(\mathcal{X}, d, \mu)$ is a doubling metric measure space supporting a $(1,p)$-Poincaré inequality, $p \in [1, \infty[$, then $\mathrm{Lip}(\mathcal{X}, \mathcal{Y})$ is dense in $N^{1,p}(\mathcal{X}, \mathcal{Y})$.

Proof. Assume $u \in N^{1,p}(\mathcal{X}, \mathcal{Y})$. Since Newton-Sobolev functions with bounded support are dense in $N^{1,p}(\mathcal{X}, \mathcal{Y})$ by [HKST, Proposition 7.1.35], we may take u to be zero outside some given ball. Also, by passing to a representative we may assume u to be pointwise defined everywhere.

Let Mf denote the *Hardy-Littlewood maximal function* of $f : \mathcal{X} \to \mathcal{Y}$,

$$Mf(x) := \sup_{\rho > 0} \fint_{B_\rho(x)} |f(x)| \mathrm{d}\mu.$$

By [HKST, Theorem 8.1.49] there exists a μ-negiglible set $E \subset \mathcal{X}$ such that

$$|u(x) - u(y)| \leq C d(x,y) \left(M g_u^p(x) + M g_u^p(y) \right)^{\frac{1}{p}}, \tag{2.29}$$

where g_u is the minimal p-weak upper gradient of u and $x, y \in E^c$. Now, for $t > 0$ consider the p-superlevel sets of $M g_u^p(x)$, namely

$$E_t \coloneqq \{x \in \mathcal{X};\ M g_u^p(x) > t^p\}.$$

The estimate (2.29) entails that the restriction of u to $(E_t \cup E)^c$ is $c_1 t$-Lipschitz for some constant $c_1 > 0$ independent of t. By the Lipschitz extension property in [HKST, Theorem 4.1.21] there exists a $c_2 t$-Lipschitz function $u_t : \mathcal{X} \to \mathcal{Y}$ such that $u_t(x) = u(x)$ everywhere on $(E_t \cup E)^c$, with c_2 independent of t as well.

Consider a ball $B_0 \in \mathscr{B}(\mathcal{X})$ such that $u(x) = 0$ outside of it; then, it must be $E_t \subset 2B_0$ for $t > 0$ large enough. Indeed, if $x \in E_t \cap (2B_0)^c$ and we consider a ball B centered at x and such that

$$t^p < \fint_B g_u^p \mathrm{d}\mu,$$

then since $g_u = 0$ in $(B_0)^c$ by Proposition 2.2.19, we have $B \cap B_0 \neq \emptyset$. So $B_0 \subset 3B$ and then

$$t^p < c \fint_{3B} g_u^p \mathrm{d}\mu \leq \frac{c}{\mu(B_0)} \int_{\mathcal{X}} g_u^p \mathrm{d}\mu \coloneqq t_0^p,$$

which gives the claim. This implies $u_t = u = 0$ μ-almost everywhere in $(2B_0)^c$ for $t \geq t_0$. Since u_2 is $c_2 t$-Lipschitz on $\mathcal{X}$, then $|u| \leq c_3 t$ for $t \geq t_0$ with $c_3 > 0$ independent of t. Thus

$$\begin{aligned}\int_{\mathcal{X}} |u - u_t|^p \, \mathrm{d}\mu &= \int_{\{u \neq u_t\}} |u - u_t|^p \, \mathrm{d}\mu \\ &\leq c \int_{\{u \neq u_t\}} |u|^p \mathrm{d}\mu + c t^p \mu(\{u \neq u_t\}),\end{aligned}$$

where $c > 0$ does not depend on $t \geq t_0$. As $\mu(\{u \neq u_t\}) \leq \mu(E_t)$, the above inequality and the properties of maximal functions in [HKST, Proposition 3.5.15] yield $u_t \to u$ in $L^p(\mathcal{X}, \mu)$ as $t \to \infty$.

Next, in order to conclude we need to prove the same convergence for the minimal p-weak upper gradients of u and u_t. Let $F \supset E_t \cup E$ be a Borel set such that $\mu(F) = \mu(E_t \cup E)$; then $u - u_t = 0$ in $(F)^c$, so applying Remark 2.2.17 and Proposition 2.2.19 one gets

$$g_{u-u_t}(x) \leq (g_u(x) + c_2 t) \cdot \mathbb{1}_F(x)$$

for μ-almost every $x \in \mathcal{X}$. Moreover, integrating over F gives

$$\int_{\mathcal{X}} g_{u-u_t}^p \mathrm{d}\mu \leq c \int_F g_u^p \mathrm{d}\mu + c t^p \mu(E_t)$$

with $c > 0$ independent of $t \geq t_0$. This conclusion allows us to state that $g_{u-u_t} \to 0$ in $L^p(\mathcal{X}, \mu)$ as $t \to \infty$.

In other words, we have constructed a sequence of Lipschitz functions $(u_t)_{t\geq 0}$ such that

$$\lim_{t\to\infty} \|u-u_t\|_{N^{1,p}(\mathcal{X},\mathcal{Y})} = 0.$$

The Theorem is proven.

□

A result similar to Theorem 2.4.3 above had already been shown in [Ch], where under the same hypotheses the author established the density of locally Lipschitz functions inside Sobolev spaces.

The issue will be considered again at the end of Section 3.1 in the context of "test plans" (namely, probability measures on the spaces of absolutely continuous curves), a tool introduced by in [AGS2] and [AGS3], and which later in [AGS4] allowed the authors to prove a similar result without any structural assumptions like doubling measures and Poincaré inequalities.

We conclude this chapter with some comments about the aforementioned "generalized" Poincaré Inequalities. Our main reference will be Section 4.4 of [BB], which we suggest for a more detailed discussion as well as for the proofs of the results stated below.

2.4.4 Definition. Given $q\geq 1$, we say that $(\mathcal{X},d,\mu)$ supports a (q,p)-*Poincaré Inequality* if for every ball $B\in\mathscr{B}(\mathcal{X})$, every integrable function $u$in $\mathcal{X}$ and all of upper gradients g of u, there exist constants $\lambda\geq 1$ and $C_{\mathrm{P}}>0$ such that

$$\left(\fint_B |u-u_B|^q \,\mathrm{d}\mu\right)^{\frac{1}{q}} \leq C_{\mathrm{P}}\mathrm{diam}(B)\left(\fint_{\lambda B} g^p \mathrm{d}\mu\right)^{\frac{1}{p}}.$$

The next result states that, under the hypotheses of doubling measure and (q,p)-Poincaré inequality, balls satisfy a growth rate estimate with esponent s related to the "dimension" s which already appeared in Proposition 1.2.3:

2.4.5 Proposition [BB, Proposition 4.20]. If $(\mathcal{X},d,\mu)$ is a doubling metric measure space supporting a (p,q)-Poincaré inequality for some $q>p$, then for all balls $B=B_\rho(x), B'=B_{\rho'}(x')$ in $\mathcal{X}$ with $\rho'\leq\rho$ one has

$$\frac{\mu(B')}{\mu(B)} \geq C\left(\frac{\rho'}{\rho}\right)^s \tag{2.30}$$

for some constant $C>0$ and $s=qp/(q-p)$.

□

2.4.6 Theorem [BB, Theorem 4.21]. If $(\mathcal{X},d,\mu)$ supports a $(1,p)$-Poincaré inequality with dilation factor λ and the measure μ satisfies the condition (2.30) in Proposition 2.4.5 above for some $s>p$, then it supports a (p^*,p)-Poincaré inequality with $p^*=sp/(s-p)$ and dilation factor 2λ.

□

2.4.7 Corollary [BB, Corollary 4.22]. Assume $(\mathcal{X}, d, \mu)$ supports a $(1, p)$-Poincaré inequality with dilation factor λ and that (2.30) holds with $s > p$. If $u \in N^{1,p}(2\lambda B)$ then $u \in L^{p^*}(B)$ with p^* as in Theorem 2.4.6.

□

The two results above yield the "local" embedding of $N^{1,p}$ into L^{p^*}:

2.4.8 Corollary [BB, Corollary 4.22]. Assume $(\mathcal{X}, d, \mu)$ supports a $(1, p)$-Poincaré inequality and that (2.30) holds for some s. Let $\Omega \subset \mathcal{X}$ be an open set. If $s > p$ then $N^{1,p}_{\mathrm{loc}}(\Omega) \hookrightarrow L^{p^*}_{\mathrm{loc}}(\Omega)$ with p^* as above, while $s \leq p$ implies $N^{1,p}_{\mathrm{loc}}(\Omega) \hookrightarrow L^{q}_{\mathrm{loc}}(\Omega)$ for all $q \in [1, \infty[$.

□

We point out that assuming $s < p$ allows for $q = \infty$ in the above embedding; moreover, as a further byproduct of Theorem 2.4.6, one has a (q, p)-Poincaré inequality with $q \geq p$ whenever $(\mathcal{X}, d, \mu)$ is doubling and supports a $(1, p)$-Poincaré inequality (with dilation factors 2λ and λ respectively).

2.4.9 Remark. At the end of Remark 2.2.17 we made clear that, in general, the p-weak upper gradient of a function depends on the exponent p; it was proven by J. Cheeger in [Ch] that this dependence is cancelled under the hypotheses of a doubling measure and a $(1, p)$-Poincaré inequality.

As proven in [GH], the dependence on p is not an issue as well in the context of $\mathsf{RCD}(K, \infty)$ Spaces, which will be the natural setting of Chapter 6.

Another question we didn't consider in the previous sections - since a detailed discussion would go beyond the scopes of our work - is that of the reflexivity of Newton-Sobolev spaces, so we shall give some brief comments here. Again in [Ch], it was proven that if $(\mathcal{X}, d, \mu)$ is doubling and supports a $(1, p)$-Poincaré inequality with $p \in]1, \infty[$, then $N^{1,p}(\mathcal{X})$ is always reflexive. The approach presented there is slightly different from the one giving rise to $N^{1,p}(\mathcal{X})$, but this is not an issue thanks to the aforementioned equivalence shown by N. Shanmugalingam in [Sh1] and [Sh2].

Reflexivity of first-order Sobolev spaces was also proven in [ACD] under the hypothesis that $(\mathcal{X}, d, \mu)$ is such that $(\mathrm{supp}(\mu), d)$ is doubling with the measure μ being finite on bounded sets.

3 Other notions of Sobolev Spaces

In this chapter we consider two other characterizations of Sobolev Spaces in the nonsmooth setting, making use of tools such as "test plans" and "derivations" respectively. The discussion will eventually lead to notable results such as the density of Lipschitz functions inside Sobolev Spaces, and the Equivalence Theorem 3.2.9.

In both Sections 3.1 and 3.2, the underlying metric measure space $(\mathcal{X}, d, \mu)$ will not be supposed to satisfy any particular structural assumption, like a doubling measure or a $(1,p)$-Poincaré Inequality.

The following survey is based on the works of L. Ambrosio, S. Di Marino, N. Gigli and G. Savaré; the precise references will be given inside each section.

3.1 Sobolev Spaces via Test Plans

Basically speaking, test plans are probability measures over the spaces of absolutely continuous curves. This tool was first introduced in [AGS2] and [AGS3] to provide an alternative characterization of weak upper gradients - and thus, of Sobolev spaces - in the more abstract context of "extended" metric spaces, namely those where the distance between two points may also not be finite. There the authors also proved, for $p = 2$, the equivalence between the resulting notion of Sobolev spaces and the "newtonian" ones, as well as the density in energy of Lipschitz functions. The approach was then improved in [AGS4] to be extended for all exponents $p \in [1, \infty[$.

We preferred to follow N. Gigli's work [Gi1] which discusses the characterization of Sobolev-Dirichlet classes via test-plans in a more concise - yet exhaustive and self-contained - manner; we shall also refer to [Gi2] for additional comments.

Let $(\mathcal{X}, d, \mu)$ be a complete and separable metric measure space endowed with a non-negative, locally finite Radon measure μ.

We shall consider $C([0,1], \mathcal{X})$, namely the space of continuous curves equipped with the supremum norm; since the underlying metric space is complete and separable, $C([0,1], \mathcal{X})$ will be complete and separable as well.

3.1.1 Definition. We define the *evaluation map* $\mathrm{e}_t : C([0,1], \mathcal{X}) \to \mathcal{X}$, $t \in [0,1]$ as

$$\mathrm{e}_t(\gamma) \coloneqq \gamma_t = \gamma(t) \quad \forall \gamma \in C([0,1], \mathcal{X}).$$

Recall that by Definition 2.1.5, a curve $\gamma \in C([0,1], \mathcal{X})$ is called *p-absolutely continuous*, $p \in [1, \infty]$, if there exists $f \in L^p(]0,1[)$ such that

$$d(\gamma_t, \gamma_s) \le \int_t^s f(r)\mathrm{d}r, \qquad \forall\, t, s \in [0,1] \text{ with } t < s,$$

and in this case we write $\gamma \in AC^p([0,1], \mathcal{X})$.

Recall also that by Theorem 2.1.6, to every p-absolutely continuous curve we associate the *metric derivative* (or, the *speed*) $t \mapsto |\dot{\gamma}_t| \in L^p\left(]0,1[\right)$ defined as the essential infimum of all the $f \in L^p\left(]0,1[\right)$ satisfying the above condition, and which is representable in terms of an incremental ratio for almost every $t \in [0,1]$:

$$|\dot{\gamma}|\,(t) := \lim_{h\to 0} \frac{d\left(\gamma_{t+h}, \gamma_t\right)}{h}.$$

Below, $\mathscr{P}\left(C([0,1],\mathcal{X})\right)$ denotes the space of probability measures along continuous curves.

3.1.2 Definition. Let $\boldsymbol{\pi} \in \mathscr{P}\left(C([0,1],\mathcal{X})\right)$. We say that $\boldsymbol{\pi}$ has *bounded compression* whenever there exists a constant $c = c(\boldsymbol{\pi}) > 0$ such that

$$(e_t)_{\#}\,\boldsymbol{\pi} \le c(\boldsymbol{\pi})\mu \quad \forall\, t \in [0,1].$$

Let $p \in [1,\infty[$; if $\boldsymbol{\pi}$ has bounded compression, it is concentrated on $AC^p([0,1],\mathcal{X})$ and

$$\int\!\!\int_0^1 |\dot{\gamma}_t|^p \,\mathrm{d}t\mathrm{d}\boldsymbol{\pi}(\gamma) < \infty,$$

then it will called a *p-test plan*.

With the notion of p-test plans, we are entitled to define the Sobolev-Dirichlet classes $D^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})$:

3.1.3 Definition. Let $p \in [1,\infty[$ and let q be its conjugate exponent, $\frac{1}{p} + \frac{1}{q} = 1$. The *Sobolev-Dirichlet class* $D^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})$ consists of all Borel functions $f : \mathcal{X} \to \mathbb{R}$ for which there exists $g \in L^p(\mathcal{X},\mu)$ satisfying

$$\int |f\left(\gamma_1\right) - f\left(\gamma_0\right)| \,\mathrm{d}\boldsymbol{\pi}(\gamma) \le \int\!\!\int_0^1 g\left(\gamma_s\right)|\dot{\gamma}_s| \,\mathrm{d}s\mathrm{d}\boldsymbol{\pi}(\gamma) \tag{3.1}$$

for every q-test plan $\boldsymbol{\pi}$.

Following the usual tradition, we shall say that g is a *p-weak upper gradient* of f.

3.1.4 Remark. Observe that Remarks 2.2.17 and 2.4.9 apply also in this case: we are not assuming $(\mathcal{X}, d, \mu)$ to satisfy any structural properties, so p-weak upper gradients depend on p. In particular, since the class of q-test plans contains the one of q'-test plans for every $q \le q'$, then we have the inclusion $D^{1,p}_{\boldsymbol{\pi}}(\mathcal{X}) \subset D^{1,p'}_{\boldsymbol{\pi}}(\mathcal{X})$ for $p \ge p'$; in particular if $f \in D^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})$ and g is a p-weak upper gradient, then it is also a p'-weak upper gradient.

The discussions in [AGS3, Section 5.2] and [AGS4, Section 4.5] entail that for every f in the Sobolev-Dirichlet class $D^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})$ there exists a minimal $g \ge 0$ in $L^p(\mathcal{X},\mu)$ satisfying (3.1) ; such g will be obviously called *minimal p-weak upper gradient* of f and we shall denote it by $|Df|$. The choice of this notation is motivated, like in [Gi1], by the fact that the notion is given in terms of duality with speed of curves and thus it is closer to the (dual) norm of the differential, rather than of the gradient; in other words, this definition yields some kind of “cotangent” object.

We continue our discussion giving some important properties of the classes $D^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})$.

3.1.5 Remark. $D^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})$ is a vector space. Moreover,

$$|D(\alpha f + \beta g)| \leq |\alpha||Df| + |\beta||Dg|$$

μ-almost everywhere for all $f, g \in D^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})$ and $\alpha, \beta \in \mathbb{R}$. In particular, $D^{1,p}_{\boldsymbol{\pi}}(\mathcal{X}) \cap L^{\infty}(\mathcal{X}, \mu)$ is an algebra and a weak Leibniz rule holds, namely

$$|D(fg)| \leq |f||Dg| + |g||Df|$$

μ-almost everywhere for all $f, g \in D^{1,p}_{\boldsymbol{\pi}} \cap L^{\infty}(\mathcal{X}, \mu)$.
The minimal p-weak upper gradient is also a local object, in the sense that $|Df| = |Dg|$ μ-almost everywhere on $\{f = g\}$ for all $f, g \in D^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})$.
Another interesting - albeit expected - property of Sobolev-Dirichlet classes is the chain rule: if $f \in D^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function, then $\varphi \circ f \in D^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})$ and

$$|D(\varphi \circ f)| = |\varphi' \circ f||Df|$$

μ-almost everywhere.

3.1.6 Definition. On $D^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})$ we define the semi-norm

$$\|f\|_{D^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})} \coloneqq \||Df|\|_{L^p(\mathcal{X},\mu)} \,.$$

Clearly, the above does not give a norm since it vanishes on (locally) constant non-zero functions; thus, we introduce the *Sobolev space* $W^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})$ as

$$W^{1,p}_{\boldsymbol{\pi}}(\mathcal{X}) \coloneqq D^{1,p}_{\boldsymbol{\pi}} \cap L^p(\mathcal{X}, \mu)$$

endowed with the norm

$$\|f\|^p_{W^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})} \coloneqq \|f\|_{L^p(\mathcal{X},\mu)} + \|f\|_{D^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})} \,.$$

With this definition, $W^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})$ is always a Banach space; completeness follows from [Gi1, Proposition 2.7].

As anticipated in the comments right after Theorem 2.4.3 and at the beginning of the present section, test-plans allow to prove the density (in energy) of Lipschitz functions in $W^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})$:

3.1.7 Theorem. Suppose $(\mathcal{X}, d, \mu)$ is such that μ is finite on bounded sets and let $p \in]1, \infty[$. Then, for every $f \in W^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})$ there exists a sequence of Lipschitz functions $(f_n)_{n\in\mathbb{N}} \subset W^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})$ such that

$$\lim_{n\to\infty} \|f_n - f\|_{L^p(\mathcal{X},\mu)} = 0$$

and

$$\lim_{n\to\infty} \|f_n\|_{D^{1,p}_{\pi}(\mathcal{X})} = \lim_{n\to\infty} \||Df_n|\|_{L^p(\mathcal{X},\mu)} = \lim_{n\to\infty} \left\|\overline{|Df_n|}\right\|_{L^p(\mathcal{X},\mu)} = \|f\|_{D^{1,p}_{\pi}(\mathcal{X})},$$

where for any $g : \mathcal{X} \to \mathbb{R}$ the function $\overline{|Dg|} : \mathcal{X} \to [0, \infty]$ is defined to be zero on isolated points, and

$$\overline{|Dg|}(x) := \inf_{\rho>0} \sup_{y_1 \neq y_2 \in B_\rho(x)} \frac{|g(y_1) - g(y_2)|}{d(y_1, y_2)}$$

elsewhere.

Proof. Let $(\psi)_{n\in\mathbb{N}} \subset \mathrm{Lip}_b(\mathcal{X})$ be a sequence of 1-Lipschitz functions with bounded support such that $0 \leq \psi_n \leq 1$ for every n and $\psi_n \equiv 1$ on $B_n(\bar{x})$ for some $\bar{x} \in \mathcal{X}$ fixed.

If we now fix $f \in W^{1,p}_{\pi}(\mathcal{X})$, by the Dominated Convergence Theorem we get $f\psi_n \to f$ in $L^p(\mathcal{X}, \mu)$ as $n \to \infty$. Moreover, the definition of the ψ_n's and Remark 3.1.5 yield $|D(f\psi_n)| = |Df|$ μ-almost everywhere on $B_n(\bar{x})$ by the locality of the minimal p-weak upper gradient, and by the weak Leibniz rule $|D(f\psi_n)| \leq |Df| + |f|$ so that $\|f\psi_n\|_{D^{1,p}_{\pi}(\mathcal{X})} \to \|f\|_{D^{1,p}_{\pi}(\mathcal{X})}$ as $n \to \infty$ (actually, in Remark 3.1.5 we assumed the function to be essentially bounded, but the present case works as well since the ψ_n's are bounded and Lipschitz).

Thus, $(f\psi_n)$ converges in energy to f in $W^{1,p}_{\pi}(\mathcal{X})$ as $n \to \infty$. As $f\psi_n \in W^{1,p}_{\pi}(\mathcal{X})$ and has bounded support, applying [Gi1, Proposition 2.6], the finiteness of μ on bounded sets and a diagonalization argument along with the results in [AGS4] allows us to conclude.

□

3.2 Sobolev Spaces via Derivations. The Equivalence Theorem

In this section we shall follow S. Di Marino's Ph.D. Thesis [Di1, Chapter 7] and his work [Di2], where the author introduced an alternative characterization of Sobolev (and also, BV - see Section 4.3) functions in terms of "derivations", which allow a definition via an integration by parts formula.

The concept of derivation is basically taken from N. Weaver's work [We], where the a more general class of objects is considered, namely the "metric derivations", defined as bounded weak-* continuous linear maps from the space $\mathrm{Lip}(\mathcal{X})$ to some W^*-module over $L^\infty(\mathcal{X}, \mu)$ and satisfying the Leibniz rule.

Derivations will appear again - actually with a slightly different characterization - in Section 5.5, in the context of N. Gigli's differential structure.

As in Section 3.1, $(\mathcal{X}, d, \mu)$ will be a complete and separable metric measure space endowed with a non-negative, locally finite Radon measure μ; moreover, we shall suppose no further structural assumptions to hold, like the doubling condition or the $(1, p)$-Poincaré Inequality.

3.2.1 Definition. Let $L^0(\mu)$ denote the class of measurable functions. By a *derivation* we mean a linear map $\mathfrak{d} : \mathrm{Lip}_0(\mathcal{X}, d) \to L^0(\mu)$ satisfying the Leibniz rule and a weak locality condition, namely

i) for every $f, g \in \mathrm{Lip}_0(\mathcal{X}, d)$, $\mathfrak{d}(fg) = \mathfrak{d}(f)g + f\mathfrak{d}(g)$;

ii) there exists $g \in L^0(\mu)$ such that $|\mathfrak{d}(f)|(x) \leq g(x)\mathrm{lip}_a f(x)$ for μ-almost every $x \in \mathcal{X}$ and for every $f \in \mathrm{Lip}_0(\mathcal{X}, d)$.

Above, $\mathrm{lip}_a f(x)$ denotes the asymptotic Lipschitz constant of f at x,

$$\mathrm{lip}_a f(x) := \inf_{\rho>0} \mathrm{Lip}\,(f, B_\rho(x)) = \lim_{\rho\to 0^+} \mathrm{Lip}\,(f, B_\rho(x)) \,.$$

The smallest function g satisfying *ii)* will be denoted by $|\mathfrak{d}|$. By $\mathrm{D{\small ER}}(\mathcal{X})$ we shall denote the set of all derivations; $\mathfrak{d} \in L^p(\mu)$ has to be intended as $|\mathfrak{d}| \in L^p(\mu)$.

If we impose the integration by parts formula to hold, we are allowed to give the definition of divergence as follows:

3.2.2 Definition. Given a derivation $\mathfrak{d} \in L^1_{\mathrm{loc}}(\mu)$, we define its *divergence* $\mathrm{div}(\mathfrak{d})$ as the operator $\mathrm{Lip}_0(\mathcal{X}) \to \mathbb{R}$ given by

$$\mathrm{Lip}_0(\mathcal{X}, d) \ni f \longmapsto -\int_{\mathcal{X}} \mathfrak{d}(f)\mathrm{d}\mu = \int_{\mathcal{X}} f\mathrm{div}(\mathfrak{d})\mathrm{d}\mu,$$

whenever the above formula makes sense. We write $\mathrm{div}(\mathfrak{d}) \in L^p(\mu)$ to signify that the operator admits an integral representation via an $L^p(\mu)$ function h,

$$-\int_{\mathcal{X}} \mathfrak{d}(f)\mathrm{d}\mu = \int_{\mathcal{X}} hf\mathrm{d}\mu = \int_{\mathcal{X}} f\mathrm{div}(\mathfrak{d})\mathrm{d}\mu$$

for every $f \in \mathrm{Lip}_0(\mathcal{X}, d)$, and in this case it we shall write, with a slight abuse of notation, $h = \mathrm{div}(\mathfrak{d})$.
When $\mathrm{div}(\mathfrak{d}) \in L^p(\mu)$, then it is unique. Besides $\mathrm{D{\small ER}}(\mathcal{X})$ we shall also consider the spaces

$$\mathrm{D{\small ER}}^p(\mathcal{X}) := \{\mathfrak{d} \in \mathrm{D{\small ER}}(\mathcal{X});\ \mathfrak{d} \in L^p(\mathcal{X}, \mu)\}$$

and

$$\mathrm{D{\small ER}}^{p,q}(\mathcal{X}) := \{\mathfrak{d} \in \mathrm{D{\small ER}}(\mathcal{X});\ \mathfrak{d} \in L^p(\mathcal{X}, \mu),\ \mathrm{div}(\mathfrak{d}) \in L^q(\mathcal{X}, \mu)\} \,.$$

By definition, $\mathrm{D{\small ER}}(\mathcal{X})$, $\mathrm{D{\small ER}}^p(\mathcal{X})$ and $\mathrm{D{\small ER}}^{p,q}(\mathcal{X})$ are real vector spaces; in particular, the last two are also Banach spaces if we endow them with the norms

$$\|\mathfrak{d}\|_{\mathrm{DER}^p(\mathcal{X})} \coloneqq \||\mathfrak{d}|\|_{L^p(\mu)}$$
$$\|\mathfrak{d}\|_{\mathrm{DER}^{p,q}(\mathcal{X})} \coloneqq \||\mathfrak{d}|\|_{L^p(\mu)} + \|\mathrm{div}(\mathfrak{d})\|_{L^q(\mu)}$$

respectively. When both p and q are ∞, we shall write DER_b ("bounded" derivations) instead of $\mathrm{DER}^{\infty,\infty}$.
The domain of the divergence is the space

$$D(\mathrm{div}) = \left\{\mathfrak{d} \in \mathrm{DER}(\mathcal{X});\ |\mathfrak{d}|, \mathrm{div}(\mathfrak{d}) \in L^1_{\mathrm{loc}}(\mu)\right\}.$$

Notice that for every $p, q \in [1, \infty]$, one has the inclusion $\mathrm{DER}^{p,q}(\mathcal{X}) \subset D(\mathrm{div})$.

3.2.3 Definition. We define the multiplication of a derivation by a scalar function as follows: if $\mathfrak{d} \in \mathrm{DER}(\mathcal{X})$ and $u \in L^0(\mu)$, then $u\mathfrak{d}(f)(x) \coloneqq u(x) \cdot \mathfrak{d}(f)(x)$ for μ-almost every $x \in \mathcal{X}$ and for every $f \in \mathrm{Lip}_0(\mathcal{X}, d)$.

The set of derivations is closed under multiplication by scalar functions; moreover,

3.2.4 Lemma [Di1, Lemma 7.1.2]. If $\mathfrak{d} \in \mathrm{DER}(\mathcal{X})$ and $u \in L^0(\mu)$, then $|u\mathfrak{d}| = |\mathfrak{d}| \cdot |u|$. In particular, if $\mathfrak{d} \in \mathrm{DER}^{p,q}(\mathcal{X})$ and $u \in \mathrm{Lip}_b(\mathcal{X}, d)$, then $u\mathfrak{d}$ is a derivation such that

$$\mathrm{div}(u\mathfrak{d}) = u\mathrm{div}(\mathfrak{d}) + \mathfrak{d}(u)$$

and $u\mathfrak{d} \in \mathrm{DER}^{p,r}(\mathcal{X})$, with $r = \max\{p, q\}$.

□

Derivations behave well with respect to locality in $D(\mathrm{div})$; indeed, we have the following result:

3.2.5 Lemma [Di1, Lemma 7.1.3]. If $\mathfrak{d} \in D(\mathrm{div})$, then for all $f, g \in \mathrm{Lip}(\mathcal{X}, d)$ one has

i) $\mathfrak{d}(f) = \mathfrak{d}(g)$ μ-almost everywhere in $\{f = g\}$;

ii) $\mathfrak{d}(f) \leq |\mathfrak{d}| \cdot \mathrm{lip}_a(f|_C)$ μ-almost everywhere for every closed set $C \subset \mathcal{X}$.

□

With these premises done, we can now introduce Sobolev spaces in terms of derivations:

3.2.6 Definition. Let $f \in L^p(\mu)$, $p \in [1, \infty[$. Then $f \in W^{1,p}_{\mathfrak{d}}(\mathcal{X})$ if, denoting by q the conjugate exponent of p, there exists a linear continuous operator $L_f : \mathrm{DER}^{q,q}(\mathcal{X}) \to L^1(\mathcal{X}, \mu)$ such that

$$\int_{\mathcal{X}} L_f(\mathfrak{d})\mathrm{d}\mu = -\int_{\mathcal{X}} f\mathrm{div}(\mathfrak{d})\mathrm{d}\mu \tag{3.2}$$

for all $\mathfrak{d} \in \mathrm{Der}^{q,q}(\mathcal{X})$ and $L_f(h\mathfrak{d}) = hL_f(\mathfrak{d})$ for every $h \in \mathrm{Lip}_b(\mathcal{X}, d)$, $\mathfrak{d}$ as above.
When $p = 1$, we need to ask also that L_f can be extended to an L^∞-linear map in $\mathrm{Der}_b^\infty := L^\infty \cdot \mathrm{Der}_b$.

We check that Definition 3.2.6 is well posed showing that the map L_f is uniquely defined.

3.2.7 Remark. Fix $\mathfrak{d} \in \mathrm{Der}^{q,q}(\mathcal{X})$ and $f \in W_{\mathfrak{d}}^{1,p}(\mathcal{X})$. If L_f and $\tilde{L}_f$ are two different linear maps as in the definition of $W_{\mathfrak{d}}^{1,p}(\mathcal{X})$ and we take $h \in \mathrm{Lip}_b(\mathcal{X}, d)$, by Lemma 3.2.4 $h\mathfrak{d} \in \mathrm{Der}^{q,q}(\mathcal{X})$ and by applying (3.2) and the L^∞-linearity we get

$$\int_{\mathcal{X}} hL_f(\mathfrak{d})\mathrm{d}\mu = \int_{\mathcal{X}} L_f(h\mathfrak{d}) = -\int_{\mathcal{X}} f\mathrm{div}(\mathfrak{d})\mathrm{d}\mu,$$

and the same holds for $\tilde{L}_f$ as well. Moreover, as the last term does not depend on L_f,

$$\int_{\mathcal{X}} hL_f(\mathfrak{d})\mathrm{d}\mu = \int_{\mathcal{X}} h\tilde{L}_f(\mathfrak{d})\mathrm{d}\mu,$$

so the arbitrariness of $h \in \mathrm{Lip}_b(\mathcal{X}, d)$ is enough to conclude that $L_f(\mathfrak{d}) = \tilde{L}_f(\mathfrak{d})$ μ-almost everywhere. This common value will be denoted by $\mathfrak{d}(f)$.
Of course, the arguments apply also to the case $p = 1$ when $\mathfrak{d} \in \mathrm{Der}_b^\infty$.

As one may expect, the present framework allows for a consistent notion of p-weak upper gradient. So, we are given the following statement:

3.2.8 Theorem & Definition [Di1, Theorem 7.1.6]. If $f \in W_{\mathfrak{d}}^{1,p}(\mathcal{X})$, then there exists $g_f \in L^p(\mathcal{X}, \mu)$ such that

$$|\mathfrak{d}(f)| \leq g_f \cdot \mathfrak{d} \tag{3.3}$$

μ-almost everywhere for all $\mathfrak{d} \in \mathrm{Der}^{q,q}(\mathcal{X})$.
The smallest function g_f satisfying (3.3) - in the μ-almost everywhere sense - will be called the *p-weak upper gradient* of f.

□

We conclude this section with some brief comments about the equivalence between the notions of Sobolev Spaces presented here and in Chapter 2.

As we anticipated in the introductory remarks to Section 3.1, in [AGS4] the authors - using technical tools from the theories of optimal transportation and of gradient flows - established that the notion of Newton-Sobolev spaces via weak upper gradients proposed in [HKST] is analogous to the test-plan approach which had already been carried on in the previous papers [AGS2] and [AGS3], with the corresponding gradients being essentially the same objects (in the sense that they coincide μ-almost everywhere). Later, [Di1, Section 7.2] combined the equivalences shown in [AGS4] to prove that the use of derivations leads to a further equivalent definition. Namely, we have the following:

3.2.9 Theorem [Di1, Theorem 7.2.5]. If $(\mathcal{X}, d, \mu)$ is a complete and separable metric measure space endowed with a locally finite measure μ, then

$$N^{1,p}(\mathcal{X}) = W^{1,p}_{\boldsymbol{\pi}}(\mathcal{X}) = W^{1,p}_{\mathfrak{d}}(\mathcal{X}).$$

In particular, the respective notions of weak gradients coincide.

$\square$

4 Functions of Bounded Variation

Functions of bounded variation constitute the central object of our dissertation.

In accordance to Chapters 2 and 3, here we survey the theory of BV functions in the metric setting making use of the tools already encountered: upper gradients (via the well known relaxation procedure), test plans and derivations. Again, the discussion will culminate in the equivalence of the definitions arising from the three approaches, established in Theorem 4.3.5 below.

BV functions will appear again in Chapter 6, where the definition will be given in the context of $\mathsf{RCD}(K,\infty)$ spaces via suitable vector fields arising from the differential structure discussed by N. Gigli in [Gi2].

The precise references will be given inside each section of the chapter.

4.1 BV functions in the "relaxed" sense

In this section, we introduce a definition of functions of bounded variation via upper gradients. This characterization first appeared in [Mi], where the author used a relaxation procedure over sequences of locally Lipschitz functions on a metric measure space supporting a $(1,1)$-Poincaré inequality and proved that classical results like the Coarea Formula and the Isoperimetric Inequality have an analogous counterpart in the metric setting.

Here we shall follow [Sh3], where the relaxation procedure is performed over sequences of $N^{1,1}(\mathcal{X})$ functions instead of Lipschitz functions; of course, thanks to Theorem 2.4.3, this approach is equivalent by density - assuming the space is doubling and that a Poincaré inequality holds - to that of [Mi].

Let $(\mathcal{X}, d, \mu)$ be a metric measure space such that μ is a non-negative Radon measure supported in $\mathcal{X}$, with the property that $0 < \mu(B) < \infty$ for every ball B in $\mathcal{X}$.

4.1.1 Definition. A function $u \in L^1(\mathcal{X})$ will be said of *bounded variation* in $\mathcal{X}$, $u \in BV(\mathcal{X})$ if its *total variation* in $\mathcal{X}$,

$$\|Du\|(\mathcal{X}) \coloneqq \inf\left\{\liminf_{k\to\infty} \int_{\mathcal{X}} g_{u_k} \mathrm{d}\mu;\ (u_k)_{k\in\mathbb{N}} \subset N^{1,1}(\mathcal{X}),\ u_k \to u \text{ in } L^1(\mathcal{X})\right\}, \tag{4.1}$$

is finite. The same characterization applies to any open subset of the ambient metric space; that is, if $\Omega \subset \mathcal{X}$ is open, then $u \in L^1(\Omega)$ is in $BV(\Omega)$ if

$$\|Du\|(\Omega) \coloneqq \inf\left\{\liminf_{k\to\infty} \int_{\Omega} g_{u_k} \mathrm{d}\mu;\ (u_k)_{k\in\mathbb{N}} \subset N^{1,1}(\Omega),\ u_k \to u \text{ in } L^1(\Omega)\right\} < \infty. \tag{4.2}$$

Sometimes, the total variation may be also referred to as *BV-energy*.

In line with the classical, Euclidean theory, the space $BV(\mathcal{X})$ will be endowed with the norm

$$\|u\|_{BV(\mathcal{X})} \coloneqq \|u\|_{L^1(\mathcal{X},\mu)} + \|Du\|(\mathcal{X}),$$

with the obvious modifications when we consider a domain instead of the whole of $\mathcal{X}$.

We wish to prove that the total variation defines a measure; in order to do so, we start by proving that it is an outer measure (Lemma 4.1.2 below) and then make use of the well known Theorem by E. De Giorgi and G. Letta ([DL, Theorem 5.1]) within the proof of Theorem 4.1.4.

4.1.2 Lemma. The total variation satisfies the following properties:

i) $\|Du\|(\emptyset) = 0$;

ii) $\|Du\|(U) \leq \|Du\|(V)$ for any open sets U, V in $\mathcal{X}$ with $U \subseteq V$;

iii) $\|Du\|(\bigcup_i V_i) = \sum_i \|Du\|(V_i)$ whenever $\{V_i\}_i$ is a pairwise disjoint family of open subsets of $\mathcal{X}$.

PROOF. Since *i)* and *ii)* are direct consequences of the definition, we limit ourselves to the discussion of *iii)*. Recall that any Newton-Sobolev function $u \in N^{1,1}(\bigcup_i V_i)$ has restrictions $u_i \coloneqq u|_{V_i}$ in $N^{1,1}(V_i)$, and

$$\int_{\bigcup_i V_i} g_u \mathrm{d}\mu = \sum_i \int_{V_i} g_{u_i} \mathrm{d}\mu$$

for every index i, so it follows that

$$\|Du\|\left(\bigcup_i V_i\right) \geq \sum_i \|Du\|(V_i).$$

The above inequality relies on the fact that, as u_i gets closer to u in $L^1(\bigcup_i V_i)$, then u_i gets closer to u in $L^1(V_i)$. Let us prove the opposite inequality.
For every $\varepsilon > 0$, we can take $u_i \in N^{1,1}(V_i)$ such that, for every i, one has

$$\int_{V_i} |u - u_i| \,\mathrm{d}\mu < 2^{-i-2}\varepsilon \qquad \text{and} \qquad \int_{V_i} g_{u_i} \mathrm{d}\mu \leq \|Du\|(V_i) + 2^{-i-2}\varepsilon.$$

Setting

$$u_\varepsilon \coloneqq \sum_i u_i \cdot \mathbb{1}_{V_i}$$

we obtain a function in $N^{1,1}(\bigcup_i V_i)$, because there are no rectifiable curves in $\bigcup_i V_i$ with endpoints in two different V_i's (the partition being pairwise disjoint). Thus

$$\int_{\bigcup_i V_i} |u - u_\varepsilon| \,\mathrm{d}\mu \leq \sum_i \int_{V_i} |u - u_i| \,\mathrm{d}\mu \leq \frac{\varepsilon}{2}$$

and

$$\int_{\bigcup_i V_i} g_{u_\varepsilon} \mathrm{d}\mu \leq \sum_i \int_{V_i} g_{u_i} \mathrm{d}\mu \leq \frac{\varepsilon}{2} + \sum_i \|Du\| (V_i) .$$

The first inequality implies that $u_\varepsilon \to u$ in $L^1 \left(\bigcup_i V_i\right)$ as $\varepsilon \to 0$; then,

$$\|Du\| \left(\bigcup_i V_i\right) \leq \liminf_{\varepsilon \to 0} \left(\frac{\varepsilon}{2} + \sum_i \|Du\| (V_i)\right) = \sum_i \|Du\| (V_i)$$

as claimed.

□

4.1.3 Definition. Given $A \subset \mathcal{X}$, we define

$$\|Du\|^* (A) := \inf \{\|Du\| (B);\ B \subset \mathcal{X} \text{ open},\ B \subset A\} .$$

By *ii)* in Lemma 4.1.2, when $A \subset \mathcal{X}$ is open we have $\|Du\|^* (A) = \|Du\| (A)$. This observation, along with Lemma 4.1.5 below, allows us to drop the * in the notation even when A is any Borel set.

4.1.4 Theorem. If $u \in BV(\mathcal{X})$, then $\|Du\|$ is a Radon measure on $\mathcal{X}$.

Proof. Let us start by recalling the aforementioned result by De Giorgi and Letta:

4.1.5 Lemma [DL, Theorem 5.1]. If ν is a non-negative function on the class of all open subsets of $\mathcal{X}$ such that

i) $\nu(\emptyset) = 0$;

ii) if U_1 and U_2 are open sets such that $U_1 \subseteq U_2$, then $\nu (U_1) \leq \nu (U_2)$;

iii) $\nu (U_1 \cup U_2) \leq \nu (U_1) + \nu (U_2)$ for every $U_1, U_2 \subset \mathcal{X}$ open;

iv) $\nu (U_1 \cup U_2) = \nu (U_1) + \nu (U_2)$ for every disjoint open sets $U_1, U_2 \subset \mathcal{X}$;

v) given $U \subset \mathcal{X}$ open,

$$\nu(U) = \sup_{\substack{V \Subset U \\ V \text{ open}}} \nu(V),$$

then the Carathéodory extension - see for example [EG, Theorem 1.9] - of ν to all Borel subsets of $\mathcal{X}$ gives a Borel regular outer measure on $\mathcal{X}$.
Note that by $A \Subset B$ we shall always intend $A \subset B$ and $\operatorname{dist}(A, B^c) > 0$.

To prove the statement of the Theorem, we shall verify that ν satisfies the five conditions of Lemma 4.1.5. By Lemma 4.1.2 we already know that *i)*, *ii)* and *iv)* hold, so we prove *iii)* and *v)*; we start with *v)* and then use the arguments in its proof to show the validity of *iii)*.

Condition *v)* will be proven for bounded open sets; the proof of the general case may be found in [Mi, Theorem 3.4].
By the monotonicity property *ii)*, we only need to show that

$$\|Du\|(\Omega) \leq \sup_{\substack{V \Subset \Omega \\ V \text{ open}}} \|Du\|(V)$$

for any open set $\Omega \subset \mathcal{X}$. Take $\delta > 0$ and set

$$\Omega_\delta := \{x \in \Omega,\ d(x, \Omega^c) > \delta\}.$$

Given $0 < \delta_1 < \delta_2 < \mathrm{diam}(\Omega)/2$, let $V = \Omega_{\delta_1}$ and $W = \Omega \backslash \bar{\Omega}_{\delta_2}$. Note that V and W are open subsets of Ω, and $V \Subset \Omega$; moreover, $\Omega = V \cup W$ and $\partial V \cap \partial W = \emptyset$. Then, we consider a Lipschitz function η on Ω such that $0 \leq \eta \leq 1$ on Ω, $\eta = 1$ on $V \backslash W = \bar{\Omega}_{\delta_2}$, $\eta = 0$ on $W \backslash V = \Omega \backslash \Omega_{\delta_1}$, and whose Lipschitz constant is such that

$$\mathrm{Lip}(\eta) \leq \frac{2}{\delta_2 - \delta_1} \cdot \mathbb{1}_{V \cap W}.$$

Now consider two functions $v \in N^{1,1}(V)$ and $w \in N^{1,1}(W)$ and set $f = \eta v + (1-\eta) w$; by [BB, Lemma 2.18] we deduce that $f \in N^{1,1}(\Omega)$ and

$$\int_\Omega g_f \mathrm{d}\mu \leq \int_V g_v \mathrm{d}\mu + \int_W g_w \mathrm{d}\mu + \frac{2}{\delta_2 - \delta_1} \int_{V \cap W} |v - w| \mathrm{d}\mu. \tag{4.3}$$

Moreover, for any $h \in L^1(\Omega)$, using the identity $h = \eta h + (1-\eta) h$ we find

$$\int_\Omega |f - h| \mathrm{d}\mu \leq \int_V |v - h| \mathrm{d}\mu + \int_W |w - h| \mathrm{d}\mu. \tag{4.4}$$

Take a sequence $(v_k)_{k \in \mathbb{N}} \subset N^{1,1}(V)$ such that $v_k \to u$ in $L^1(V)$ and $\int_V g_{v_k} \mathrm{d}\mu \to \|Du\|(V)$ as $k \to \infty$; take a sequence $(w_k)_{k \in \mathbb{N}} \subset N^{1,1}(W)$ with the same behaviour as $(v_k)_{k \in \mathbb{N}}$ in W. Now, merge the two sequences into a new sequence $(u_k)_{k \in \mathbb{N}}$ with the same procedure as in the above definition of f. Using (4.4) with $h = u$ we get

$$\int_\Omega |u - u_k| \mathrm{d}\mu \leq \int_V |v_k - u| \mathrm{d}\mu + \int_W |w_k - u| \mathrm{d}\mu \longrightarrow 0$$

as $k \to \infty$. Now, (4.3) implies

$$\|Du\|(\Omega) \leq \liminf_{k \to \infty} \int_\Omega g_{u_k} \mathrm{d}\mu \leq \|Du\|(V) + \|Du\|(W).$$

Since $V \Subset \Omega$, we find that $\|Du\|(\Omega)$ satisfies also the following inequality,

$$\|Du\|(\Omega) \leq \sup_{\substack{V \Subset \Omega \\ V \text{ open}}} \|Du\|(V) + \|Du\|(W).$$

By the above, it suffices to show that

$$\lim_{\delta\to 0^+} \|Du\| \left(\Omega\backslash\bar{\Omega}_\delta\right) = 0. \tag{4.5}$$

First of all, we observe that the limit in (4.5) exists as, of course, $\|Du\| \left(\Omega\backslash\bar{\Omega}_\delta\right)$ acts as a decreasing function of δ. Now we fix a decreasing sequence of positive real numbers $(\delta_k)_{k\in\mathbb{N}}$ such that $\delta_k \to 0$ and for $k \geq 2$ we set $V_k := \Omega_{\delta_{2k-3}}\backslash\bar{\Omega}_{\delta_{2k}}$. Doing so, we obtain two families, namely $\{V_{2k}\}_{k\in\mathbb{N}}$ and $\{V_{2k+1}\}_{k\in\mathbb{N}}$, of pairwise disjoint open subsets of $\mathcal{X}$; invoking Lemma 4.1.2, we know that

$$\|Du\| \left(\bigcup_{k=1}^{\infty} V_{2k}\right) = \sum_{k=1}^{\infty} \|Du\| (V_{2k}) \leq \|Du\| (\Omega) < \infty$$

and

$$\|Du\| \left(\bigcup_{k=1}^{\infty} V_{2k+1}\right) = \sum_{k=1}^{\infty} \|Du\| (V_{2k+1}) \leq \|Du\| (\Omega) < \infty.$$

Then, for $\varepsilon > 0$ we may find an integer $k_\varepsilon \geq 2$ for which

$$\sum_{k=k_\varepsilon}^{\infty} \left[\|Du\| (V_{2k}) + \|Du\| (V_{2k+1})\right] < \varepsilon.$$

Now we repeat the above argument (which may be referred to as a "stitching argument", since the idea is to "stitch" Sobolev functions on an open set to Sobolev functions on another open set in order to obtain Sobolev functions on the union of such sets) for every k, taking a Lipschitz function η_k (a "stitching function") on $\bigcup_{j=k_\varepsilon}^{k+1} V_j$ such that $0 \leq \eta_k \leq 1$, $\eta_k = 1$ on $V_k\backslash V_{k-1}$, $\eta_k = 0$ on $\bigcup_{j=k_\varepsilon}^{k-1} V_j\backslash V_k$ and with upper gradient $g_{\eta_k} \leq C_k \cdot \mathbb{1}_{V_k\cap V_{k-1}}$. For each k we may find $v_{k,j} \in N^{1,1}(V_k)$ such that

$$\int_{V_k} |v_{k,j} - u| \,\mathrm{d}\mu \leq \frac{2^{-(j+k)}}{3(1+C_k)}$$

and

$$\int_{V_k} g_{v_{k,j}} \mathrm{d}\mu \leq \|Du\| (V_k) + 2^{-(j+k)}.$$

We proceed inductively, stitching the functions together. First of all, fix $i \in \mathbb{N}$; starting with $k = k_\varepsilon$, one stitches $f_{k,i}$ to $f_{k+1,i}$ using as stitch function $\eta_{k+1} = \eta_{k_\varepsilon+1}$ in order to obtain $w_{i,k} \in N^{1,1}(V_{k_\varepsilon} \cup V_{k_\varepsilon+1})$. Then,

$$\int_{V_{k_\varepsilon}\cup V_{k_\varepsilon+1}} |w_{i,k} - u| \,\mathrm{d}\mu \leq \frac{2^{-(i+k_\varepsilon)}}{1 + C_{k_\varepsilon+1}}$$

and

$$\int_{V_{k_\varepsilon}\cup V_{k_\varepsilon+1}} g_{w_{i,k}} \mathrm{d}\mu \leq \sum_{j=k_\varepsilon}^{k_\varepsilon+1} \|Du\| (V_j) + 2^{1-(i+k_\varepsilon)}.$$

Now, suppose that for some $k \in \mathbb{N}$ with $k \geq k_\varepsilon + 1$ we have constructed a Sobolev map $w_{i,k} \in N^{1,1}\left(\bigcup_{j=k_\varepsilon}^k V_j\right)$ such that

$$\int_{\bigcup_{j=k_\varepsilon}^k V_j} |w_{i,k} - u| \, \mathrm{d}\mu \leq \sum_{j=k_\varepsilon}^k \frac{2^{-(i+j)}}{1 + C_j}$$

and

$$\int_{\bigcup_{j=k_\varepsilon}^k V_j} g_{w_{i,k}} \mathrm{d}\mu \leq \sum_{j=k_\varepsilon}^k \left[\|Du\|(V_j) + 2^{1-(i+j)}\right].$$

Now we stitch $f_{k+1,i}$ to $w_{i,k}$ via η_{k+1}; we obtain $w_{i,k+1}$ which satisfies analogous inequalities as above. Since $w_{i,k+1} = w_{i,k-1}$ on V_{k-1} when $k \geq k_\varepsilon + 2$, letting $k \to \infty$ we obtain a limit function $w_i = \lim_{k\to\infty} w_{i,k} \in N^{1,1}\left(\bigcup_{k=k_\varepsilon}^\infty V_k\right)$ such that

$$\int_{\bigcup_{j=k_\varepsilon}^\infty V_j} |w_i - u| \, \mathrm{d}\mu \leq \sum_{j=k_\varepsilon}^\infty \frac{2^{-(i+j)}}{1 + C_j} \tag{4.6}$$

and

$$\int_{\bigcup_{j=k_\varepsilon}^\infty V_j} g_{w_i} \mathrm{d}\mu \leq \sum_{j=k_\varepsilon}^\infty \|Du\|(V_j) + 2^{2-i} < \varepsilon + 2^{2-i}. \tag{4.7}$$

From inequality (4.6) we get that $w_i \to u$ in $L^1\left(\bigcup_{j=k_\varepsilon}^\infty V_j\right)$ as $i \to \infty$, while using (3.7) we deduce

$$\|Du\|\left(\bigcup_{j=k_\varepsilon}^\infty V_j\right) = \|Du\|\left(\Omega \backslash \bar{\Omega}_{\delta_{k_\varepsilon}}\right) \leq \liminf_{i\to\infty} \int_{\bigcup_{j=k_\varepsilon}^\infty V_j} g_{w_i} \mathrm{d}\mu \leq \varepsilon.$$

In other words, we have proven our claim:

$$\lim_{\delta\to 0^+} \|Du\|\left(\Omega \backslash \bar{\Omega}_\delta\right) = 0,$$

which completes the proof of *v)*.

Let us now concentrate on *iii)*. By *v)*, for every $\varepsilon > 0$ there exist relatively compact open subsets $\Omega_1' \Subset \Omega_1$ and $\Omega_2' \Subset \Omega_2$ such that $\|Du\|(\Omega_1 \cup \Omega_2) \leq \|Du\|(\Omega_1' \cup \Omega_2') + \varepsilon$. Using again a stitching argument, consider a Lipschitz function η on $\mathcal{X}$ such that $0 \leq \eta \leq 1$ on $\mathcal{X}$, $\eta = 1$ on Ω_1', $\eta = 0$ on Ω_1^c and

$$g_\eta \leq \frac{1}{C_{\Omega_1,\Omega_1'}} \cdot \mathbb{1}_{\Omega_1 \backslash \Omega_1'}.$$

If we take $u_1 \in N^{1,1}(\Omega_1)$ and $u_2 \in N^{1,1}(\Omega_2)$, we obtain the stitched function $w = \eta u_1 + (1-\eta)u_2 \in N^{1,1}(\Omega_1' \cup \Omega_2')$; note that, in general, $w \notin N^{1,1}(\Omega_1 \cup \Omega_2)$ and w is not defined in $\Omega_1 \backslash (\Omega_1' \cup \Omega_2)$ because $1 - \eta$ does not vanish there and u_2 is not defined there. Thus

$$\int_{\Omega_1' \cup \Omega_2'} g_w \mathrm{d}\mu \leq \int_{\Omega_1} g_{u_1} \mathrm{d}\mu + \int_{\Omega_2} g_{u_2} \mathrm{d}\mu + \frac{1}{C_{\Omega_1,\Omega_1'}} \int_{\Omega_1 \cap \Omega_2} |u_1 - u_2| \, \mathrm{d}\mu, \tag{4.8}$$

$C_{\Omega_1,\Omega_1'} > 0$ depending on ε, and

$$\int_{\Omega_1'\cup\Omega_2'} |w-u|\,\mathrm{d}\mu \le \int_{\Omega_1} |u_1-u|\,\mathrm{d}\mu + \int_{\Omega_2} |u_2-u|\,\mathrm{d}\mu. \tag{4.9}$$

Choosing $(u_{1,k})_{k\in\mathbb{N}}$ and $(u_{2,k})_{k\in\mathbb{N}}$ as the optimal approximating sequences for u on Ω_1 and Ω_2 respectively, (4.8) tells us that the resulting sequence $(w_k)_{k\in\mathbb{N}}$ approximates u in $\Omega_1'\cup\Omega_2'$. Therefore,

$$\begin{aligned}\|Du\|(\Omega_1\cup\Omega_2) &\le \varepsilon + \|Du\|(\Omega_1'\cup\Omega_2')\\ &\le \varepsilon + \liminf_{k\to\infty}\int_{\Omega_1\cup\Omega_2} g_{w_k}\mathrm{d}\mu \le \|Du\|(\Omega_1) + \|Du\|(\Omega_2) + \varepsilon.\end{aligned}$$

Letting $\varepsilon\to 0$, *iii)* follows immediately.

□

4.1.6 Definition. Given a measurable set $E\subset\mathcal{X}$, we define its *perimeter* in $\mathcal{X}$ as the total variation of its characteristic function in $\mathcal{X}$, $P(E,\mathcal{X}) \coloneqq \|D\mathbb{1}_E\|(\mathcal{X})$. According to this, E will be said of *finite perimeter* in $\mathcal{X}$ whenever $\mathbb{1}_E\in BV(\mathcal{X})$.
More generally, for any $F\subset\mathcal{X}$ we set $P(E,F)\coloneqq\|D\mathbb{1}_E\|(F)$. Following a well established tradition, sets of finite perimeter shall be also said *Caccioppoli sets*.
Of course, by Lemma 4.1.2 and Theorem 4.1.4, the perimeter defines a measure which we shall obviously call the *perimeter measure*.
We will always prefer the notations $\|D\mathbb{1}_E\|$ and $\|D\mathbb{1}_E\|(F)$ to denote the perimeter measure and the perimeter of E in F respectively.

Below, we give the statement of Coarea Formula, a classical result which illustrates the deep connection between functions of bounded variation and sets of finite perimeter.

4.1.7 Theorem [Mi, Proposition 4.2]. If $\Omega\subset\mathcal{X}$ is an open set, for any $u\in L^1_{\mathrm{loc}}(\Omega)$ it holds

$$\int_{\mathbb{R}} \|D\mathbb{1}_{E_t}\|(A)\mathrm{d}t = \|Du\|(A), \tag{4.10}$$

for every $A\subset\Omega$ open, where $E_t\coloneqq\{x\in\Omega,\ u(x)>t\}$, $t\in\mathbb{R}$, denotes the super-level sets of u. In particular, if $u\in BV(\mathcal{X})$, then for almost every $t\in\mathbb{R}$ the sets E_t have finite perimeter and (4.10) holds for every measurable set $A\subset\mathcal{X}$.

□

4.1.8 Remark. For $u\in BV(\mathcal{X})$, we have the following generalization of Coarea Formula: given any measurable function $v:\mathcal{X}\to\mathbb{R}$ and any measurable set $A\subset\mathcal{X}$,

$$\int_{\mathbb{R}}\Big(\int_A v(x)\mathrm{d}\,\|D\mathbb{1}_{E_t}\|(x)\Big)\,\mathrm{d}t = \int_A v(x)\mathrm{d}\,\|Du\|(x).$$

We continue this survey discussing some other notable properties of BV functions; for the remaining part of this section, $(\mathcal{X}, d, \mu)$ will be assumed to be a doubling metric measure space supporting a $(1,1)$-Poincaré inequality.

4.1.9 Remark. A weaker version of Poincaré inequality holds for BV functions. Namely, if $B \in \mathscr{B}(\mathcal{X})$ is any ball in $\mathcal{X}$, with the same notation as in Definition 2.4.1 one has

$$\int_B |u - u_B| \,\mathrm{d}\mu \leq C_{\mathrm{P}} \mathrm{diam}(B) \|Du\| (\lambda B),$$

for every $u \in BV(\mathcal{X})$. In particular, if the homogeneous dimension s - arising from Propositions 1.2.3 and 2.4.5 - of $\mathcal{X}$ is greater than 1, one finds

$$\left(\fint_B |u - u_B|^{\frac{s}{s-1}} \,\mathrm{d}\mu \right)^{\frac{s-1}{s}} \leq c \mathrm{diam}(B) \frac{\|Du\| (\lambda B)}{\mu(B)},$$

where $c > 0$ is a constant depending on s and C_{D}. See [Mi], [Am2] and the references therein for more comments and details.

4.1.10 Theorem [Mi, Theorem 4.5]. If $E \subset \mathcal{X}$ is a set of finite perimeter and $B = B_\rho(x) \in \mathscr{B}(\mathcal{X})$, then there exists a constant $C_{\mathrm{I}} > 0$ such that the following *relative isoperimetric inequality* holds:

$$\min \{ \mu (E \cap B_\rho(x)), \mu (E^c \cap B_\rho(x)) \} \leq C_{\mathrm{I}} \left(\frac{\rho^s}{\mu (B_\rho(x))} \right)^{\frac{1}{s-1}} (\|D\mathbb{1}_E\| (B_{2\lambda\rho}(x)))^{\frac{s}{s-1}},$$

where $s > 1$ is as in Remark 4.1.9 and $\lambda \geq 1$ is one of the parameters in the Poincaré inequality. C_{I} will be called *isoperimetric constant*.

$\square$

A remarkable property of the perimeter measure, which we shall use extensively later, is its absolute continuity with respect to the spherical Hausdorff measure $\mathcal{S}^h$, built by applying the Carathéodory construction - see for example [Fe, Section 2.10] - to the function $h : \mathscr{B}(\mathcal{X}) \to \mathbb{R}$ defined by

$$h (B_\rho(x)) := \frac{\mu (B_\rho(x))}{\rho}.$$

Observe that $\mathcal{S}^h$ is doubling since μ is doubling as well.

Before stating the result, we recall the notion of "essential boundary" of a set, which in turn arises from the concept of "density":

4.1.11 Definition. Given a subset $E \subset \mathcal{X}$, we define its *lower density* and *upper density* at $x \in \mathcal{X}$ as

$$\Theta_{*,\mu}(E,x) := \liminf_{\rho\to 0^+}\frac{\mu\left(E\cap B_\rho(x)\right)}{\mu\left(B_\rho(x)\right)} \quad \text{and} \quad \Theta^*_\mu(E,x) := \limsup_{\rho\to 0^+}\frac{\mu\left(E\cap B_\rho(x)\right)}{\mu\left(B_\rho(x)\right)}$$

respectively.

When the two densities coincide, the common value will be denoted by $\Theta_\mu(E,x)$.

The *essential* or *measure-theoretic boundary* of E is the set of points where both E and its complement have positive upper density:

$$\partial^* E := \left\{x\in\mathcal{X};\ \Theta^*_\mu(E,x)>0,\ \Theta^*_\mu\left(E^c,x\right)>0\right\}.$$

Equivalently, the essential boundary can be characterized as the complementary set of those points where the density of E is either 0 or 1, namely $\partial^* E = \mathcal{X}\backslash\left(E^{(0)}\cup E^{(1)}\right)$.

4.1.12 Theorem [Am2, Theorem 5.4] The perimeter measure $\|D\mathbb{1}_E\|$ associated with a set of finite perimeter $E\subset\mathcal{X}$ is concentrated on the set

$$\Sigma_\gamma := \left\{x\in\mathcal{X};\ \limsup_{\rho\to 0^+}\min\left\{\frac{\mu\left(E\cap B_\rho(x)\right)}{\mu\left(B_\rho(x)\right)},\frac{\mu\left(E^c\cap B_\rho(x)\right)}{\mu\left(B_\rho(x)\right)}\right\}\geq\gamma\right\}\subset\partial^* E,$$

where $\gamma=\gamma\left(C_{\mathrm{D}},\lambda,C_{\mathrm{I}}\right)>0$. Moreover, $\mathcal{S}^h\left(\partial^* E\backslash\Sigma_\gamma\right)=0$, $\mathcal{S}^h\left(\partial^* E\right)<\infty$ and there exist $\alpha=\alpha\left(C_{\mathrm{D}},\lambda,C_{\mathrm{I}}\right)>0$ and a Borel function $\theta_E:\mathcal{X}\to[\alpha,\infty[$ such that

$$\|D\mathbb{1}_E\|\left(B\right)=\int_{\partial^* E\cap B}\theta_E(x)\mathrm{d}\mathcal{S}^h(x)$$

for every Borel set $B\subset\mathcal{X}$. Finally, the perimeter measure is asymptotically doubling; in other words, for $\|D\mathbb{1}_E\|$-almost every $x\in\mathcal{X}$ we have

$$\limsup_{\rho\to 0^+}\frac{\|D\mathbb{1}_E\|\left(B_{2\rho}(x)\right)}{\|D\mathbb{1}_E\|\left(B_\rho(x)\right)}<\infty.$$

$\square$

Actually, in the above result the function θ_E can be taken to be bounded by the doubling constant, $\theta_E\leq C_{\mathrm{D}}$; see [AMP, Proposition 4.5 and Theorem 4.6] for a detailed discussion.

4.1.13 Definition. Following [AMP], the metric measure space $(\mathcal{X},d,\mu)$ will be called *local* if, given two Caccioppoli sets E and Ω with $E\subset\Omega$, one has $\theta_E=\theta_\Omega$ $\mathcal{S}^h$-almost everywhere on $\partial^*\Omega\cap\partial^* E$.

4.2 "Weak" BV functions - a description via Test-Plans

In the concluding remarks of [AGS4], the authors suggested that, as a limiting case for $q = \infty$, functions of bounded variations could be described in terms of ∞-test plans. The approach was then reprised and developed in [AD], where the equivalence with the relaxation procedure was also shown.

The present section is taken from [AD]. The notation will be the same as in Section 3.1.

4.2.1 Definition. A measure $\boldsymbol{\pi} \in \mathscr{P}\left(C([0,1],\mathcal{X}\right)$ will be called an ∞-*test plan* whenever the following two properties hold:

i) $\boldsymbol{\pi}$ is concentrated on $AC^{\infty}([0,1],\mathcal{X})$ and $\mathrm{Lip}(\gamma) \in L^{\infty}\left((C([0,1],\mathcal{X}),\boldsymbol{\pi}\right)$;

ii) there exists a non-negative constant $c = c(\boldsymbol{\pi})$ such that $(\mathrm{e}_t)_{\#}\,\boldsymbol{\pi} \leq c\mu$ for every $t \in [0,1]$.

A Borel set $\Gamma \subset C([0,1],\mathcal{X})$ is said 1-*negligible* if $\boldsymbol{\pi}(\Gamma) = 0$ for every ∞-test plan $\boldsymbol{\pi}$; accordingly, a property of (absolutely) continuous curves which fails outside a 1-negligible set, will be said to hold 1-*almost everywhere.*

Let us denote by $\mathbf{M}^{\pm}(\mathcal{X})$ the spaces of signed Radon measure on $\mathcal{X}$. With the definition of ∞-test plans, we can characterize BV functions as follows:

4.2.2 Definition. A function $f \in L^1(\mathcal{X},\mu)$ will be said of *bounded total variation*, $f \in BV_{\boldsymbol{\pi}}(\mathcal{X})$, if:

i) for 1-almost every curve $\gamma \in C([0,1],\mathcal{X})$ one has $f \circ \gamma \in BV(]0,1[)$ and

$$|f(\gamma_1) - f(\gamma_0)| \leq \|D(f \circ \gamma)\|\,(]0,1[),$$

where $|D(f\circ\gamma)| \in \mathbf{M}^{-}(]0,1[)$ is the total variation measure associated with $f \circ \gamma : [0,1] \to \mathbb{R}$;

ii) there exists $\mathfrak{m} \in \mathbf{M}^{+}(\mathcal{X})$ such that for every Borel set $B \subset \mathcal{X}$

$$\int \gamma_{\#}\,\|D(f\circ\gamma)\|\,(B)\mathrm{d}\boldsymbol{\pi}(\gamma) \leq c(\boldsymbol{\pi})\,\|\mathrm{Lip}(\gamma)\|_{L^{\infty}(\boldsymbol{\pi})}\,\mathfrak{m}(B).$$

The least measure $\mathfrak{m}$ satisfing the above inequality in *ii)* will be called the *total variation measure* associated to f; it will be denoted by $\|Df\|_{\boldsymbol{\pi}}$.

In an equivalent way, $\|Df\|_{\boldsymbol{\pi}}$ can be seen as the supremum of the family of measures

$$\frac{1}{c(\boldsymbol{\pi})\,\|\mathrm{Lip}(\gamma)\|_{L^{\infty}(\boldsymbol{\pi})}} \int \gamma_{\#}\,\|D(f\circ\gamma)\|\,\mathrm{d}\boldsymbol{\pi}(\gamma)$$

as $\boldsymbol{\pi}$ runs over ∞-test plans.

4.3 BV functions via Derivations and an Equivalence Theorem

The notion of "derivation" has already been introduced in Section 3.2 to characterize Sobolev spaces $W^{1,p}_{\mathfrak{d}}(\mathcal{X},\mu)$; following again [Di1], the same tool is used to give an alternative notion of functions of bounded variation via an integration by parts formula.

4.3.1 Definition. Given $f \in L^1(\mathcal{X},\mu)$, we say that $f \in BV_{\mathfrak{d}}(\mathcal{X})$ if there exists a linear and continuous operator $L_f : \mathrm{D{\scriptsize ER}}_b(\mathcal{X}) \to \mathbf{M}(\mathcal{X})$ such that

$$\int_{\mathcal{X}} \mathrm{d}L_f(\mathfrak{d}) = -\int_{\mathcal{X}} f \mathrm{div}(\mathfrak{d})\mathrm{d}\mu \tag{4.11}$$

and $L_f(h\mathfrak{d}) = hL_f(\mathfrak{d})$ for every $\mathfrak{d} \in \mathrm{D{\scriptsize ER}}_b(\mathcal{X})$ and $h \in \mathrm{Lip}_b(\mathcal{X},d)$.

It is readily checked that the definition is well posed:

4.3.2 Remark. Take $f \in BV_{\mathfrak{d}}(\mathcal{X})$, $\mathfrak{d} \in \mathrm{D{\scriptsize ER}}_b(\mathcal{X})$ and suppose L_f, $\tilde{L}_f$ are two linear continuous map as in the above definition. If $h \in \mathrm{Lip}_b(\mathcal{X},d)$, by Lemma 2.2.4 one has $h\mathfrak{d} \in \mathrm{D{\scriptsize ER}}_b(\mathcal{X})$ and then, using linearity and (4.11),

$$\int_{\mathcal{X}} h\mathrm{d}L_f(\mathfrak{d}) = \int_{\mathcal{X}} \mathrm{d}L_f(h\mathfrak{d}) = -\int_{\mathcal{X}} f\mathrm{div}(h\mathfrak{d})\mathrm{d}\mu.$$

Since the same holds by definition with $\tilde{L}_f$ in place of L_f, the arbitrariness of $h \in \mathrm{Lip}_b(\mathcal{X},d)$ gives $L_f(\mathfrak{d} = \tilde{L}_f(\mathfrak{d})$; this common value will be denoted by $Df(\mathfrak{d})$.

We shall now see that the above quantity $Df(\mathfrak{d})$ induces a measure $|Df|_{\mathfrak{d}}$, namely the total variation of f, which has the usual representation formula in terms of suprema of divergences of derivations.

4.3.3 Theorem & Definition [Di1, Theorem 7.3.3]. If $f \in BV_{\mathfrak{d}}(\mathcal{X})$, then there exists a finite measure $\nu \in \mathbf{M}^+(\mathcal{X})$ such that for every Borel set $B \subset \mathcal{X}$ one has

$$\int_B \mathrm{d}Df(\mathfrak{d}) \le \int_B |\mathfrak{d}|^* \mathrm{d}\nu$$

for every $\mathfrak{d} \in \mathrm{D{\scriptsize ER}}_b(\mathcal{X})$, where $|\mathfrak{d}|^*$ denotes the upper semicontinuous envelope of $|\mathfrak{d}|$.
The least measure which fulfills the above inequality will be called the *total variation* of f, and we shall denote it by $\|Df\|_{\mathfrak{d}}$.
Moreover,

$$\|Df\|_{\mathfrak{d}}(\mathcal{X}) = \sup\left\{|Df(\mathfrak{d})|(\mathcal{X});\ \mathfrak{d} \in \mathrm{D{\scriptsize ER}}_b(\mathcal{X}),\ |\mathfrak{d}| \le 1\right\}.$$

□

4.3.4 Theorem [Di1, Theorem 7.3.4]. If $f \in BV_{\mathfrak{d}}(\mathcal{X})$, then we have the classical representation formula for $\|Df\|_{\mathfrak{d}}$; namely, for every open set $\Omega \subset \mathcal{X}$,

$$\|Df\|_{\mathfrak{d}}(\Omega) = \sup\left\{\int_{\Omega} f\mathrm{div}(\mathfrak{d})\mathrm{d}\mu;\ \mathfrak{d} \in \mathrm{D{\scriptsize ER}}_b(\mathcal{X}),\ |b| \leq 1,\ \mathrm{supp}(\mathfrak{d}) \Subset \Omega\right\}.$$

□

As in Chapter 3, we conclude this brief survey on BV functions in the metric setting with some comments on the equivalence of the definitions we presented so far. In particular, we state the following Theorem, which combines the results contained in [AD] and [Di1] to show that the relaxation procedure, the "weak" characterization via test plans and derivations actually yield to the same notion of the space of functions of bounded variation:

4.3.5 Theorem [Di1, Theorem 7.3.7]. If $(\mathcal{X}, d, \mu)$ is a complete and separable metric measure space endowed with a locally finite measure μ and $f \in L^1(\mathcal{X}, \mu)$, then

$$BV(\mathcal{X}) = BV_{\pi}(\mathcal{X}) = BV_{\mathfrak{d}}(\mathcal{X}).$$

In particular, the respective notions of total variation coincide.

□

5 The Differential Structure

Throughout this chapter, we shall follow closely the work of N. Gigli [Gi2] and occasionally the previous paper [Gi1], which contains an anticipation of the theory we are going to illustrate.

Starting from the concept of $L^\infty(\mu)$-module already introduced by N. Weaver in [We], in [Gi2] the author generalizes the discussion in order to define $L^p(\mu)$-normed modules with a well posed notion of (local) dimension. Then, focusing on the L^2 theory, this tool will serve to construct the "cotangent module" - a metric counterpart to the cotangent bundle in the smooth setting - $L^2(T^*\mathcal{X})$ of a metric measure space $(\mathcal{X}, d, \mu)$ by means of minimal weak upper gradients of Sobolev-Dirichlet functions in $D^{1,2}_{\pi}(\mathcal{X})$ (which, as pointed out in Remark 3.1.4, constitute a sort of "cotangent" object, being expressed in duality with the speed of curves), and thus to give a definition of the "differential" of a Sobolev function.

By duality with the cotangent module, the tangent module $L^2(T\mathcal{X})$ is given as a space of "vector fields". This machinery leads eventually to a well posed notion of the divergence and of the gradient and, assuming the space to be "infinitesimally Hilbertian" (or, equivalently, "infinitesimally strictly convex"), will allow to characterize the Laplacian as a linear operator expressed as the *divergence of the gradient*.

In the following, we have preferred to generalize the approach of [Gi2] to the L^p theory, with an arbitrary exponent $p \in [1, \infty]$; in regards to this - somehow delicate - choice, we refer the reader to the important Remark 5.5.22 at the end of the chapter.

5.1 L^p-normed modules

Consider a σ-finite measure space $(\mathcal{X}, \mathfrak{A}, \mu)$ where $\mathcal{X}$ is a set, $\mathfrak{A}$ a σ-algebra and μ a non-negative, locally finite Radon measure on it, with the property that there exists a countable family $(E_i)_{i\in\mathbb{N}} \subset \mathfrak{A}$ which covers $\mathcal{X}$, such that $\mu(E_i) < +\infty$ for every $i \in \mathbb{N}$.

Any two sets $A, B \in \mathfrak{A}$ shall be declared equivalent whenever $\mu((A\backslash B) \cup (B\backslash A)) = 0$; $\mathfrak{B}(\mathcal{X})$ will denote the set of all equivalence classes.

Observe that by σ-finiteness, any collection $\mathscr{C} \subset \mathfrak{B}(\mathcal{X})$ which is stable under countable union admits a unique maximal set with respect to the inclusion relation.

5.1.1 Definition. Let $(\mathscr{M}, \|\cdot\|_{\mathscr{M}})$ be a Banach space endowed with a bilinear map $L^\infty(\mathcal{X}, \mu) \times \mathscr{M} \ni (f, v) \mapsto f \cdot v \in \mathscr{M}$ such that:

$$\begin{aligned} (f\cdot g)\cdot v &= f\cdot(g\cdot v), \\ \mathbf{1}\cdot v &= v, \\ \|f\cdot v\|_{\mathscr{M}} &\leq \|f\|_{L^\infty(\mu)}\|v\|_{\mathscr{M}}, \end{aligned} \tag{5.1}$$

where $v \in \mathscr{M}$, $f, g \in L^\infty(\mu)$ and $\mathbf{1} \in L^\infty(\mu)$ denotes the function which is identically equal to 1; under these conditions, $(\mathscr{M}, \|\cdot\|_{\mathscr{M}})$ will be said an $L^\infty(\mu)$-*premodule*.

An $L^\infty(\mu)$-premodule becomes an $L^\infty(\mu)$-*module* if the following additional properties hold (with $\mathbb{1}_E$ denoting the characteristic function of any set E):

- *Locality*: for every $v \in \mathscr{M}$ and $A_n \in \mathfrak{B}(\mathcal{X})$, $n \in \mathbb{N}$, then $\mathbb{1}_{A_n} \cdot v = 0$ for each n implies $\mathbb{1}_{\cup A_n} \cdot v = 0$;
- *Gluing*: for every sequence $(v_n) \subset \mathscr{M}$ and sequence $(A_n) \subset \mathfrak{B}(\mathcal{X})$ such that

$$\mathbb{1}_{A_i \cap A_j} \cdot v_i = \mathbb{1}_{A_i \cap A_j} \cdot v_j \quad \forall\, i, j \in \mathbb{N} \qquad \text{and} \qquad \limsup_{n\to\infty} \left\| \sum_{i=1}^{n} \mathbb{1}_{A_i} \cdot v_i \right\|_{\mathscr{M}} < \infty$$

there exists $v \in \mathscr{M}$ such that

$$\mathbb{1}_{A_i} \cdot v = \mathbb{1}_{A_i} \cdot v_i \quad \forall\, i \in \mathbb{N} \qquad \text{and} \qquad \|v\|_{\mathscr{M}} \le \liminf_{n\to\infty} \left\| \sum_{i=1}^{n} \mathbb{1}_{A_i} \cdot v_i \right\|_{\mathscr{M}} .$$

A map between two $L^\infty(\mu)$-modules $\mathscr{M}_1$ and $\mathscr{M}_2$, $T : \mathscr{M}_1 \to \mathscr{M}_2$, will be said a *module morphism* if it is a bounded linear map from $\mathscr{M}_1$ to $\mathscr{M}_2$ - viewed as Banach spaces - and, moreover, if it satisfies the "locality condition" $T(f \cdot v) = f \cdot T(v)$ for every $v \in \mathscr{M}_1$, $f \in L^\infty(\mu)$.

By $\textsc{Hom}\,(\mathscr{M}_1, \mathscr{M}_2)$ we shall denote the set of all module morphisms mapping $\mathscr{M}_1$ to $\mathscr{M}_2$. In particular, any two morphisms $T \in \textsc{Hom}\,(\mathscr{M}_1, \mathscr{M}_2)$ and $S \in \textsc{Hom}\,(\mathscr{M}_2, \mathscr{M}_1)$ will be called *module isomorphisms* whenever $T \circ S = \mathrm{Id}_{\mathscr{M}_2}$ and $S \circ T = \mathrm{Id}_{\mathscr{M}_1}$. If isomorphisms exist, the involved modules will be said *isomorphic*.

An isomorphism is an *isometry* provided it is norm-preserving.

5.1.2 Example. The $L^p(\mu)$ spaces of p-summable functions, and the space of $L^p(\mathrm{vol})$ vector fields on a smooth Riemannian manifold are among the basic examples of L^∞ modules.

5.1.3 Definition (the set $\{v = 0\}$). By the locality property of $L^\infty(\mu)$-modules, we can define the set $\{v = 0\} \in \mathfrak{B}(\mathcal{X})$ for a generic element v of an $L^\infty(\mu)$-module $\mathscr{M}$. We claim that $v = 0$ μ-almost everywhere on $A \in \mathfrak{B}(\mathcal{X})$ provided $\mathbb{1}_A \cdot v = 0$; in this case, we shall say that v is *concentrated* on A^c.

Of course, given an arbitrary $L^\infty(\mu)$-premodule, if $v = 0$ μ-almost everywhere on $\{A_i\}_{i=1}^n \subset \mathfrak{B}(\mathcal{X})$, then it is zero on their union as well. Indeed, being

$$\mathbb{1}_{\cup_{n=1}^N A_n} = f \sum_{n=1}^{N} \mathbb{1}_{A_n}$$

for some $f \in L^\infty(\mu)$ and for any fixed $N \in \mathbb{N}$, one has

$$\begin{aligned} \left\| \mathbb{1}_{\cup_{n=1}^N A_n} \cdot v \right\|_{\mathscr{M}} = \left\| \left(f \sum_{n=1}^{N} \mathbb{1}_{A_n} \right) v \right\|_{\mathscr{M}} &\le \|f\|_{L^\infty(\mu)} \left\| \sum_{n=1}^{N} \mathbb{1}_{A_n} \cdot v \right\|_{\mathscr{M}} \\ &\le \|f\|_{L^\infty(\mu)} \sum_{n=1}^{N} \|\mathbb{1}_{A_n} \cdot v\|_{\mathscr{M}} , \end{aligned}$$

which is zero by hypothesis. By locality, this property can be extended to countable unions and then the respective condition in Definition 5.1.1 can be reformulated as follows: if $v = 0$ μ-almost everywhere on A_n for every $n \in \mathbb{N}$, then $v = 0$ μ-almost everywhere on $\bigcup_{n\in\mathbb{N}} A_n$.

Thus, the maximality with respect to inclusion ensures the existence of a maximal set in $\mathfrak{B}(\mathscr{X})$ where $v = 0$: this set will be denoted by $\{v = 0\}$ and its complement by $\{v \neq 0\}$.

5.1.4 Definition. Given an $L^\infty(\mu)$-module $\mathscr{M}$, a closed subspace $\mathscr{N} \subset \mathscr{M}$ which is stable with respect to multiplication by $L^\infty(\mu)$ functions is surely an $L^\infty(\mu)$-premodule with the locality property; if it is also closed with respect to the gluing operation, then it becomes also an $L^\infty(\mu)$-module and we shall call it a *submodule* of $\mathscr{M}$.

Basic examples of submodules are the kernel of a module morphism, and the module $\mathscr{M}|_E$ obtained via elements which are zero μ-almost everywhere outside of $E \in \mathfrak{B}(\mathscr{X})$.

5.1.5 Definition. With the notion of submodule $\mathscr{N} \subset \mathscr{M}$ of an $L^\infty(\mu)$-module, it comes natural to define the quotient space $\mathscr{M}/\mathscr{N}$ in the usual way.

It is easy to see that $\mathscr{M}/\mathscr{N}$ is an $L^\infty(\mu)$-premodule itself with the locality property.

5.1.6 Proposition. If $\mathscr{M}$ and $\mathscr{N}$ are two $L^\infty(\mu)$-modules, then $\textsc{Hom}(\mathscr{M}, \mathscr{N})$ inherits a canonical structure of $L^\infty(\mu)$-module.

Proof. Viewing $\mathscr{M}$ and $\mathscr{N}$ as Banach spaces, $\textsc{Hom}(\mathscr{M}, \mathscr{N})$ is a Banach space as well if we endow it with the operator norm

$$\|T\| := \sup_{v \in \mathscr{M}, \|v\|_{\mathscr{M}} \leq 1} \|T(v)\|_{\mathscr{N}} .$$

Keeping in mind that $L^\infty(\mu)$ is a commutative ring, then for $T \in \textsc{Hom}(\mathscr{M}, \mathscr{N})$ and $f \in L^\infty(\mu)$ the operator $fT : \mathscr{M} \to \mathscr{N}$ defined by $(fT)(v) := f(T(v))$ for every $v \in \mathscr{M}$ gives a module morphism and thus $\textsc{Hom}(\mathscr{M}, \mathscr{N})$ is an $L^\infty(\mu)$-premodule.

We need to prove the locality and gluing properties. Take $T \in \textsc{Hom}(\mathscr{M}, \mathscr{N})$ and a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathfrak{B}(\mathscr{X})$ such that $\mathbb{1}_{A_n} \cdot T = 0$ for all $n \in \mathbb{N}$. Thus, for a given $v \in \mathscr{M}$ one has $(\mathbb{1}_{A_n} \cdot T)(v) = 0 = \mathbb{1}_{A_n} \cdot (T(v))$; the locality property in $\mathscr{N}$ implies that $\mathbb{1}_{\cup_n A_n} \cdot T(v) = 0$ and hence the arbitrariness of v gives $\mathbb{1}_{\cup_n A_n} \cdot T = 0$. This proves locality.

To conclude, consider two sequences, $(T_n)_{n\in\mathbb{N}} \subset \textsc{Hom}(\mathscr{M}, \mathscr{N})$ and $(A_n)_{n\in\mathbb{N}} \subset \mathfrak{B}(\mathscr{X})$ respectively, such that $\mathbb{1}_{A_i \cap A_j} \cdot T_i = \mathbb{1}_{A_i \cap A_j} \cdot T_j$ for every $i, j \in \mathbb{N}$ and

$$\left\| \sum_{i=1}^n \mathbb{1}_{A_i} \cdot T_i(v) \right\|_{\textsc{Hom}(\mathscr{M},\mathscr{N})} \leq C$$

for all $n \in \mathbb{N}$ and for some $C > 0$. Now let $v \in \mathscr{M}$; using the gluing property in $\mathscr{N}$ for the sequences $(T_n(v))_{n\in\mathbb{N}}$ and $(A_n)_{n\in\mathbb{N}}$, since

$$\left\| \sum_{i=1}^n \mathbb{1}_{A_i} \cdot T_i(v) \right\|_{\mathscr{N}} \leq \left\| \sum_{i=1}^n \mathbb{1}_{A_i} \cdot T_i \right\|_{\textsc{Hom}(\mathscr{M},\mathscr{N})} \|v\|_{\mathscr{M}} \leq C \|v\|_{\mathscr{M}}$$

for every $n \in \mathbb{N}$, then there exists $w \in \mathscr{N}$ with $\|w\|_{\mathscr{N}} \leq C \|v\|_{\mathscr{M}}$ and $\mathbb{1}_{A_i} \cdot w = \mathbb{1}_{A_i} \cdot T_i(v)$ for every $i \in \mathbb{N}$. We then define $T(v) := \mathbb{1}_{\cup_n A_n} \cdot w$; notice that $\|T(v)\|_{\mathscr{N}} \leq \|w\|_{\mathscr{N}} \leq C \|v\|_{\mathscr{M}}$ and that by the locality of $\mathscr{N}$, $T(v)$ is well defined, which means it does not depend on w. Thus, the map $v \mapsto T(v)$ is a module morphism with norm bounded by C and such that $\mathbb{1}_{A_n} \cdot T = \mathbb{1}_{A_n} \cdot T_n$ for every $n \in \mathbb{N}$.

$\square$

The above Proposition, along with the fact that $L^1(\mu)$ is naturally an $L^\infty(\mu)$-module, motivates the definition of "dual" of a module:

5.1.7 Definition. The *dual module* of an $L^\infty(\mu)$-module $\mathscr{M}$ is the set

$$\mathscr{M}^* := \mathrm{Hom}\left(\mathscr{M}, L^1(\mu)\right).$$

The following important Remark shows that the dual of $L^\infty(\mu)$, viewed as an $L^\infty(\mu)$-module, is exactly $L^1(\mu)$, in contrast with the usual Banach space characterization; indeed $L^\infty(\mu)$, as a Banach space, is in general non-separable and infinite-dimensional, while as an $L^\infty(\mu)$-module it has dimension one because every function can be written as a multiple - always in the sense of modules - of the constant function $\mathbf{1}$.

5.1.8 Remark. Let $p \in [1, \infty]$. Then, the dual module of $L^p(\mu)$ can be identified with the space $L^q(\mu)$, $\frac{1}{p} + \frac{1}{q} = 1$. In other words, for every $g \in L^q(\mu)$ the map $T : L^p(\mu) \to L^1(\mu)$ given by $T(f) = fg$ is a module morphism and, vice-versa, for every $T \in \mathrm{Hom}\left(L^p(\mu), L^1(\mu)\right)$ we can find $g \in L^q(\mu)$ such that $T(f) = fg$ for every $f \in L^p(\mu)$. To see this, let us first assume that $p < \infty$ and $T \in \mathrm{Hom}\left(L^p(\mu), L^1(\mu)\right)$. Thus,

$$L^p(\mu) \ni f \mapsto \int_{\mathcal{X}} T(f)\mathrm{d}\mu \in \mathbb{R}$$

is linear and the classical L^p-L^q duality yields the existence of one - and only one - $g \in L^q(\mu)$ such that

$$\int_{\mathcal{X}} T(f)\mathrm{d}\mu = \int_{\mathcal{X}} fg\,\mathrm{d}\mu.$$

So, we claim that $T(f) = g$ μ-almost everywhere: as both the functions are in $L^1(\mu)$, it is enough to show that the above equality holds on every set $A \in \mathfrak{B}(\mathcal{X})$. In fact:

$$\int_A T(f)\mathrm{d}\mu = \int_{\mathcal{X}} \mathbb{1}_A \cdot T(f)\mathrm{d}\mu = \int_{\mathcal{X}} T\left(\mathbb{1}_A \cdot f\right)\mathrm{d}\mu = \int_{\mathcal{X}} \mathbb{1}_A \cdot fg\mathrm{d}\mu = \int_A fg\,\mathrm{d}\mu.$$

Now assume $p = \infty$. Set $g := T(\mathbf{1}) \in L^1(\mu)$; then it suffices to observe that $T(f) = T(f \cdot \mathbf{1}) = fT(\mathbf{1})$ for every $f \in L^\infty(\mu)$.

We refer to [Gi2, Example 1.2.8] for an explanation of the fact that $\mathrm{Hom}\left(L^p(\mu), L^\infty(\mu)\right) = \{0\}$.

It is interesting to characterize the duality of $L^\infty(\mu)$-module from the point of view of Banach spaces.

5.1.9 Remark. Consider an $L^\infty(\mu)$-module $\mathscr{M}$ as a Banach space and denote by $\mathscr{M}'$ its dual Banach space; by integration, we are given a map $\mathrm{Int}_{\mathscr{M}^*} : \mathscr{M}^* \to \mathscr{M}'$ defined by, for a given $T \in \mathscr{M}^*$and for every $v \in \mathscr{M}$,

$$\mathrm{I\scriptstyle NT}_{\mathscr{M}^*}(T)(v) := \int_{\mathcal{X}} T(v)\mathrm{d}\mu.$$

Of course, $\mathrm{I\scriptstyle NT}_{\mathscr{M}^*}(T)$ satisfies $\|\mathrm{I\scriptstyle NT}_{\mathscr{M}^*}(T)\|_{\mathscr{M}'} \leq \|T\|_{\mathscr{M}^*}$; it is also norm-preserving: indeed, given $T \in \mathscr{M}^*$ and $v \in \mathscr{M}$ such that $\|v\|_{\mathscr{M}} \leq 1$, let $f := \operatorname{sgn}(T(v)) \in L^\infty(\mu)$, $\tilde{v} := fv$. Thus, from $\|f\|_{L^\infty(\mu)} \leq 1$ we get $\|\tilde{v}\|_{\mathscr{M}} \leq \|v\|_{\mathscr{M}} \leq 1$ and then

$$\begin{aligned} \|T(v)\|_{L^1(\mu)} &= \int_{\mathcal{X}} |T(v)|\, \mathrm{d}\mu = \int_{\mathcal{X}} fT(v)\mathrm{d}\mu \\ &= \int_{\mathcal{X}} T(\tilde{v})\, \mathrm{d}\mu = \mathrm{I\scriptstyle NT}_{\mathscr{M}^*}(T)(\tilde{v}) \leq \|\mathrm{I\scriptstyle NT}_{\mathscr{M}^*}(T)\|_{\mathscr{M}'}. \end{aligned}$$

At this point, the claim follows from the definition of $\|T\|_{\mathscr{M}^*}$. Hence $\mathscr{M}^* \hookrightarrow \mathscr{M}'$, the embedding being actually an isometry.

Notice that, in general, such embedding is not surjective.

5.1.10 Definition. When the map $\mathrm{I\scriptstyle NT}_{\mathscr{M}^*}$ is surjective, we shall say that the module $\mathscr{M}$ has *full-dual*.

5.1.11 Definition. An *$L^p(\mu)$-normed premodule*, $p \in [1, \infty]$, is an $L^\infty(\mu)$-premodule $\mathscr{M}$ with a non-negative map $|\cdot| : \mathscr{M} \to L^p(\mu)$ such that

$$\begin{aligned} \||v|\|_{L^p(\mu)} &= \|v\|_{\mathscr{M}}, \\ |fv| &= |f|\,|v|, \end{aligned}$$

μ-almost everywhere for every $v \in \mathscr{M}$ and $f \in L^\infty(\mu)$.

The map $|\cdot|$ will be said *pointwise $L^p(\mu)$-norm*. When such an $\mathscr{M}$ is also an $L^\infty(\mu)$-module, we shall call it an *$L^p(\mu)$-normed module*.

The pointwise norm is continuous from $\mathscr{M}$ to $L^p(\mu)$:

$$\||v| - |w|\|_{L^p(\mu)} \leq \||v - w|\|_{L^p(\mu)} = \|v - w\|_{\mathscr{M}}$$

for every $v, w \in \mathscr{M}$.

With the following Proposition we summarize the fundamental properties of $L^p(\mu)$-normed modules:

5.1.12 Proposition [Gi2, Proposition 1.2.12]. Let $\mathscr{M}$ be an $L^p(\mu)$-normed premodule with $p \in [1, \infty]$. Then,

i) For $v \in \mathscr{M}$ and $E \in \mathfrak{B}(\mathcal{X})$, $v = 0$ μ-a.e. on E if and only if $|v| = 0$ μ-a.e. on E.

ii) $\mathscr{M}$ has the locality property; moreover, the pointwise $L^p(\mu)$-norm is unique and fulfills the (pointwise) triangle inequality:

$$|v + w| \leq |v| + |w| \quad \mu\text{-a.e.} \quad \forall\, v, w \in \mathscr{M}.$$

iii) If there exist $v \in \mathscr{M}$ and $E \in \mathfrak{B}(\mathcal{X})$ such that $0 \neq \chi_E v \neq v$, then $\mathscr{M}$ is not an $L^q(\mu)$-normed premodule for $q \neq p$.

iv) If $p < \infty$, then the gluing property holds and $\mathscr{M}$ becomes an $L^p(\mu)$-normed module.

v) If $\mathscr{M}_1$ and $\mathscr{M}_2$ are respectively an $L^{p_1}(\mu)$ and an $L^{p_2}(\mu)$-normed modules with $p_1, p_2 \in [1, \infty]$, $p_1 \geq p_2$, and $T : \mathscr{M}_1 \to \mathscr{M}_2$ is a linear map, then $T \in \textsc{Hom}(\mathscr{M}_1, \mathscr{M}_2)$ if and only if there exists $g \in L^q(\mu)$ with $\frac{1}{p_2} - \frac{1}{p_1} = \frac{1}{q}$ such that

$$|T(v)| \leq g\,|v| \quad \mu\text{-a.e.}, \quad \forall\, v \in \mathscr{M}_1.$$

In this case, the operator norm $\|T\|$ is given by the least of $\|g\|_{L^q(\mu)}$ among all the g's as above.

□

5.1.13 Definition & Remark. We say that an element v of any $L^p(\mu)$-normed module $\mathscr{M}$ is *bounded* whenever its pointwise norm is an $L^\infty(\mu)$ function.

Observe that a separable $L^p(\mu)$-normed module admits a countable dense subset of bounded elements. Let us show this statement.

When $p = \infty$ there is nothing to prove; for $p < \infty$, notice that for every $v \in \mathscr{M}$, the sequence $n \mapsto \mathrm{T}_n(v) := \mathbb{1}_{\{|v|<n\}} \cdot v$, $n \in \mathbb{N}$, converges to v and then if $D \subset \mathscr{M}$ is a countable and dense set, $\{\mathrm{T}_n(v);\ v \in D,\ n \in \mathbb{N}\}$ is countable and dense as well. So the claim follows.

When $p < \infty$ the following additional property holds:

$$\forall\, v \in \mathscr{M} \text{ and } (A_i)_{i \in \mathbb{N}} \in \mathfrak{B}(\mathcal{X}) \text{ disjoint,} \quad \lim_{n \to \infty} \left\| \mathbb{1}_{\cup_{i=1}^n A_i} v - \mathbb{1}_{\cup_{i=1}^\infty A_i} v \right\| = 0.$$

In fact,

$$\left\| \mathbb{1}_{\cup_{i=1}^\infty A_i} v - \mathbb{1}_{\cup_{i=1}^n A_i} v \right\|_{\mathscr{M}}^p = \left\| \mathbb{1}_{\cup_{i=1}^\infty A_i} v \right\|_{\mathscr{M}}^p = \int_{\cup_{i=1}^\infty A_i} |v|^p \,\mathrm{d}\mu,$$

and by Dominated Convergence the last term goes to zero when $n \to \infty$.

5.1.14 Proposition. If $\mathscr{M}$ is an $L^p(\mu)$-normed module, $p < \infty$, then it has full-dual.

Proof. Take $h \in \mathscr{M}'$ and for $v \in \mathscr{M}$ consider the map $\mathfrak{A} \ni \bar{A} \mapsto h(\mathbb{1}_A \cdot v)$, $A \in \mathfrak{B}(\mathcal{X})$ being the equivalence class of $\bar{A}$. This map is real-valued and additive; moreover, the previous remark yields that is is σ-additive as well. So, $h(\mathbb{1}_A \cdot v)$ defines a measure, absolutely continuous with respect to μ by construction, and then by the Radon-Nikodym Theorem there exists a unique $L(v) \in L^1(\mu)$ such that

$$h(\mathbb{1}_A \cdot v) = \int_A L(v) \mathrm{d}\mu$$

for every $A \in \mathfrak{B}(\mathcal{X})$. The map $\mathscr{M} \ni v \mapsto L(v) \in L^1(\mu)$ is linear by construction; moreover, $L(\mathbb{1}_A \cdot v) = \mathbb{1}_A \cdot L(v)$ for every $v \in \mathscr{M}$ and $A \in \mathfrak{B}(\mathcal{X})$ and

$$\int_{\mathcal{X}} L(v)\mathrm{d}\mu = h(v) \le \|h\|_{\mathscr{M}'} \|v\|_{\mathscr{M}}$$

for every $v \in \mathscr{M}$.

Now, given $v \in \mathscr{M}$ we set $f := \operatorname{sgn}(L(v)) \in L^\infty(\mu)$, $\tilde{v} := fv$; notice that $\|\tilde{v}\|_{\mathscr{M}} \le \|f\|_{L^\infty(\mu)} \|v\|_{\mathscr{M}} \le \|v\|_{\mathscr{M}}$ and that by the definition of f and by (#above), $|L(v)| = L(\tilde{v})$, so

$$\|L(v)\|_{L^1(\mu)} = \int_{\mathcal{X}} |L(v)|\,\mathrm{d}\mu = \int_{\mathcal{X}} |L(\tilde{v})|\,\mathrm{d}\mu \le \|h\|_{\mathscr{M}'} \|\tilde{v}\|_{\mathscr{M}} \le \|h\|_{\mathscr{M}'} \|v\|_{\mathscr{M}},$$

which implies the continuity of $\mathscr{M} \ni v \mapsto L(v) \in L^1(\mu)$.

To conclude, using the approximation of $L^\infty(\mu)$ functions via simple functions and the estimates above, we see that $L(fv) = fL(v)$ for every $v \in \mathscr{M}$ and $f \in L^\infty(\mu)$.

□

Let us now see how the pointwise norm behaves when passing to the dual and to the quotient modules of a given $L^p(\mu)$-normed module:

5.1.15 Proposition. Let $\mathscr{M}$ be an $L^p(\mu)$-normed module with $p \in [1,\infty]$. The following hold true:

i) If $q \in [1,\infty]$ is such that $\frac{1}{p} + \frac{1}{q} = 1$, then $\mathscr{M}^*$ is an $L^q(\mu)$-normed module with pointwise norm given by

$$|L|_* := \operatorname*{ess-sup}_{v\in\mathscr{M},\,|v|\le 1\ \mu\text{-a.e.}} |L(v)|\,.$$

ii) Given a submodule $\mathscr{N} \subset \mathscr{M}$, the quotient $\mathscr{M}/\mathscr{N}$ turns out to be an $L^p(\mu)$-normed module with pointwise norm given by

$$|[v]| := \operatorname*{ess-inf}_{w\in\mathscr{N}} |v+w|\,.$$

Proof. We limit ourselves to the proof of *i)* only; the second part is of no direct interest in our work and it may be found in [Gi2, Proposition 1.2.14].

Using point *v)* of Proposition 5.1.12 with $p_2 = 1$, one has $\|L\| = \||L|_*\|_{L^q(\mu)}$ so with the above definition of pointwise norm it follows $|fL|_* = |f|\,|L|_*$ μ-almost everywhere for all $f \in L^\infty(\mu)$ and $L \in \mathscr{M}^*$.

□

We now discuss briefly the reflexivity of modules; in what follows, $\mathscr{M}^{**}$ shall denote the bi-dual module of any module $\mathscr{M}$.

Let $\mathscr{M}$ be an $L^\infty(\mu)$-module; there is a canonical map $\mathcal{J}_{\mathscr{M}} : \mathscr{M} \to \mathscr{M}^{**}$ such that to every $v \in \mathscr{M}$ it is associated the morphism $\mathcal{J}_{\mathscr{M}}(v) : \mathscr{M}^* \to L^1(\mu)$ given by

$$\mathcal{J}_{\mathscr{M}}(v)(L) \coloneqq L(v), \qquad \forall\, L \in \mathscr{M}^*.$$

5.1.16 Remark. Since

$$\int_{\mathcal{X}} |\mathcal{J}_{\mathscr{M}}(v)(L)| \,\mathrm{d}\mu = \int_{\mathcal{X}} |L(v)| \,\mathrm{d}\mu \leq \|L\|_{\mathscr{M}^*} \|v\|_{\mathscr{M}}$$

and $\mathcal{J}_{\mathscr{M}}(v)(fL) = fL(v)$, one has $\mathcal{J}_{\mathscr{M}}(v) \in \mathscr{M}^{**}$ for every $v \in \mathscr{M}$ such that $\|\mathcal{J}_{\mathscr{M}}(v)\|_{\mathscr{M}^{**}} \leq \|v\|_{\mathscr{M}}$.

In general, one cannot assert that $\mathcal{J}_{\mathscr{M}}$ is an isometry, this being true for modules with full-dual only:

5.1.17 Proposition. Suppose $\mathscr{M}$ is a module with full-dual; then, $\mathcal{J}_{\mathscr{M}}$ is an isometry.

Proof. Take $v \in \mathscr{M}$; by the Hahn-Banach Extension Theorem, there exists a functional $h \in \mathscr{M}'$ with $\|h\|_{\mathscr{M}'} = 1$ and $h(v) = \|v\|_{\mathscr{M}}$. By hypothesis, there exists $L \in \mathscr{M}^*$ such that $h(w) = \int_{\mathcal{X}} L(w)\mathrm{d}\mu$ for all $w \in \mathscr{M}$; thus, $\|L\|_{\mathscr{M}^*} = \|h\|_{\mathscr{M}'} = 1$ and

$$\|v\|_{\mathscr{M}} = h(v) = \int_{\mathcal{X}} L(v)\mathrm{d}\mu = \int_{\mathcal{X}} \mathcal{J}_{\mathscr{M}}(v)(L)\mathrm{d}\mu \leq \|\mathcal{J}_{\mathscr{M}}(v)\|_{\mathscr{M}^{**}} \|L\|_{\mathscr{M}^*} = \|\mathcal{J}_{\mathscr{M}}(v)\|_{\mathscr{M}^{**}}.$$

□

5.1.18 Corollary. If $\mathscr{M}$ is an $L^p(\mu)$-module, $p < \infty$, and $v \in \mathscr{M}$, then there exists $L \in \mathscr{M}^*$ such that

$$|L|_*^q = |v|^p = L(v)$$

μ-almost everywhere with $\frac{1}{q} + \frac{1}{p} = 1$. When $p = 1$, the first equality should be read as $|L|_* = \mathbb{1}_{\{v \neq 0\}}$.

Proof. Consider the construction of $L \in \mathscr{M}^*$as in the proof of Proposition 5.1.17 and define $\tilde{L} \coloneqq \|v\|_{\mathscr{M}}^{p-1} L$. Then,

$$\|v\|_{\mathscr{M}}^p = \int_{\mathcal{X}} \tilde{L}(v)\mathrm{d}\mu \leq \int_{\mathcal{X}} \left|\tilde{L}\right|_* |v| \,\mathrm{d}\mu \leq \||v|\|_{L^p(\mu)} \||L|_*\|_{L^q(\mu)} = \|v\|_{\mathscr{M}} \left\|\tilde{L}\right\|_{\mathscr{M}^*} = \|v\|_{\mathscr{M}}^p.$$

At this point, when $p > 1$ it suffices to apply Hölder's Inequality to conclude; if $p = 1$ instead, replacing $\tilde{L}$ with $\mathbb{1}_{\{v \neq 0\}} \cdot \tilde{L}$ gives the assertion.

□

If $\mathscr{M}$ is an $L^p(\mu)$-module with $p < \infty$, then there exists $L \in \mathscr{M}^*$ such that $|L|_*^q = |v|^p = L(v)$ μ-almost everywhere, where q as usual denotes the conjugate exponent of p. In general, when $p \in [1, \infty]$ the following duality property holds:

$$|v| = \operatorname*{ess-sup}_{L \in \mathscr{M}^*,\, |L|_* \leq 1\ \mu\text{-a.e.}} |L(v)| \,.$$

5.1.19 Definition. An $L^\infty(\mu)$-module with full-dual will be said *reflexive* when the map $\mathcal{J}_{\mathscr{M}} : \mathscr{M} \to \mathscr{M}^{**}$ is surjective.

The reflexivity of a module does not imply, in general, its reflexivity as a Banach space; the converse instead always holds true:

5.1.20 Proposition. Suppose $\mathscr{M}$ is a module with full-dual which is reflexive as a Banach space; then, it is reflexive also as a module.

Proof. The hypotheses imply that the map $\mathrm{INT}_{\mathscr{M}^*} : \mathscr{M}^* \to \mathscr{M}'$ is an isomorphism of Banach spaces, thus its "transpose" map $\mathrm{INT}^t_{\mathscr{M}^*}$ is an isomorphism from the the bi-dual $\mathscr{M}''$ of $\mathscr{M}$ to the Banach dual $(\mathscr{M}^*)'$ of $\mathscr{M}^*$; in particular, it is then surjective.
Denoting by $\mathcal{I}_{\mathscr{M}} : \mathscr{M} \to \mathscr{M}''$ the canonical isometry between $\mathscr{M}$ and its Banach bi-dual, one obtains the following commutative diagram:

$$\begin{tikzcd} \mathscr{M} \arrow[r, "\mathcal{J}_{\mathscr{M}}"] \arrow[d, "\mathcal{I}_{\mathscr{M}}"'] & \mathscr{M}^{**} \arrow[d, "\mathrm{INT}_{\mathscr{M}^{**}}"] \\ \mathscr{M}'' \arrow[r, "\mathrm{INT}^t_{\mathscr{M}^*}"] & (\mathscr{M}^*)' \end{tikzcd} \tag{5.2}$$

Since $\mathrm{INT}_{\mathscr{M}^{**}}$ is injective and, moreover, $\mathcal{I}_{\mathscr{M}}$ and $\mathrm{INT}^t_{\mathscr{M}^*}$ are surjective, we obtain that $\mathcal{J}_{\mathscr{M}}$ is itself surjective.

□

5.1.21 Corollary. Any $L^p(\mu)$-normed module, $p \in]1, \infty[$, is reflexive if and only it is reflexive as a Banach space

Proof. The necessary part is given directly from the Proposition 5.1.20 and by the fact that $\mathscr{M}$ has full-dual, being an $L^p(\mu)$-normed module, see Proposition 5.1.14.
For the converse implication, consider the diagram in (5.2); we already know that both $\mathscr{M}$ and $\mathscr{M}^*$ have full dual, and this yields that $\mathrm{INT}_{\mathscr{M}^{**}}$ is surjective and that $\mathrm{INT}^t_{\mathscr{M}^*}$ is injective. Hence, if $\mathcal{J}_{\mathscr{M}}$ is surjective, so must be $\mathcal{I}_{\mathscr{M}}$. The assertion follows.

□

All the above justifies the introduction of "Hilbert modules", which we survey *en passant* for the sake of completeness:

5.1.22 Definition. An $L^\infty(\mu)$-module $\mathscr{M}$ will be called a *Hilbert module* whenever - viewed as a Banach space - it is indeed a Hilbert space.

It is possible to show that, as one may expect, Hilbert modules are always $L^2(\mu)$-normed:

5.1.23 Proposition [Gi2, Proposition 1.2.21]. Suppose $\mathscr{H}$ is an $L^\infty(\mu)$-premodule satisfying the locality property and such that, viewed as a Banach space, it is also an Hilbert space. Then, $\mathscr{H}$ is $L^2(\mu)$-normed.
Moreover, for the pointwise norm it holds

$$|v+w|^2 + |v-w|^2 = 2\,|v|^2 + 2\,|w|^2$$

μ-a.e. for every $v, w \in \mathscr{H}$.
As a consequence, all Hilbert modules are reflexive.

□

5.1.24 Definition & Remark. Given a Hilbert module $\mathscr{H}$, we define the *pointwise inner product* $\mathscr{H} \times \mathscr{H} \to L^1(\mu)$,

$$(v,w) \mapsto \langle v, w \rangle := \frac{1}{2}\left(|v+w|^2 - |v|^2 - |w|^2\right).$$

A polarization argument applied to the above map entails the following:

$$\begin{aligned}
\langle f_1 v_1 + f_2 v_2, w\rangle &= f_1 \langle v_1, w\rangle + f_2 \langle v_2, w\rangle\,, \\
|\langle v, w\rangle| &\leq |v|\,|w|\,, \\
\langle v, w\rangle &= \langle w, v\rangle, \\
\langle v, v\rangle &= |v|^2\,,
\end{aligned}$$

μ-a.e. for every $f_1, f_2 \in L^1(\mu)$ and every $v_1, v_2, v, w \in \mathscr{H}$; see [G] for more remarks.

If we now fix $v \in \mathscr{H}$ and define a map $L_v : \mathscr{H} \to L^1(\mu)$ by setting $L_v(w) := \langle v, w\rangle$ for every $w \in \mathscr{H}$, then it is clear from the above formulae that L_v is a module morphism, i.e. $L_v \in \mathscr{H}^*$ and, moreover, $|L_v|_* = |v|$ μ-almost everywhere; in other words, we have a Riesz-type property:

5.1.25 Theorem (Riesz Theorem for Hilbert modules). If $\mathscr{H}$ is a Hilbert module, then the map L_v is a morphism of modules, bijective and an isometry. In particular, for every $l \in \mathscr{H}'$ there exists a unique $v \in \mathscr{H}$ such that $l = \mathrm{INT}_{\mathscr{H}^*} L_v$.

PROOF. The map $v \mapsto L_v$ is linear and norm-preserving - even in the pointwise sense - as the previous discussion shows. Of course, it is also a module morphism since

$$L_{fv}(w) = \langle fv, w\rangle = f\langle v, w\rangle = (fL_v)(w)$$

μ-almost everywhere for every $v, w \in \mathscr{H}$ and $f \in L^\infty(\mu)$.
Now let $L \in \mathscr{H}^*$ and consider the linear functional $\mathrm{INT}_{\mathscr{H}^*} L \in \mathscr{H}'$; an application of the classical Riesz Theorem for Hilbert spaces yields the existence of an element $v \in \mathscr{H}$ such that

$$\int_{\mathcal{X}} L(w)\mathrm{d}\mu = \mathrm{INT}_{\mathscr{H}^*} L(w) = \langle v, w\rangle_{\mathscr{H}}$$

for every $w \in \mathscr{H}$. By a polarization argument applied to the first condition in the definition of pointwise norm for $L^p(\mu)$-normed modules, one has

$$\int_{\mathcal{X}} \langle v, w\rangle \mathrm{d}\mu = \langle v, w\rangle_{\mathscr{H}}$$

for every $v, w \in \mathscr{H}$ and then

$$\begin{aligned}\int_A L(w)\mathrm{d}\mu &= \int_{\mathcal{X}} \mathbb{1}_A \cdot L(w)\mathrm{d}\mu = \int_{\mathcal{X}} L\left(\mathbb{1}_A \cdot w\right)\mathrm{d}\mu \\ &= \langle v, \mathbb{1}_A \cdot w\rangle_{\mathscr{H}} = \int_{\mathcal{X}} \langle v, \mathbb{1}_A \cdot w\rangle \,\mathrm{d}\mu = \int_A \langle v, w\rangle \mathrm{d}\mu\end{aligned}$$

holds for every set $A \in \mathfrak{B}(\mathcal{X})$. Thus, $L(w) = \langle v, w\rangle$ μ-almost everywhere, forcing $L = L_v$. Uniqueness follows as well via the same argument.

□

5.1.26 Remark. The requirement of $L^p(\mu)$-integrability on the elements of $L^p(\mu)$-normed modules may be too restrictive in application where one may need to handle objects with a different order of integrability, or just elements of a bigger space endowed with a pointwise norm.

Any discussion on this topic would go far beyond the scopes of our work, thus we address the reader to Section 1.3 of [Gi2], where the issue is treated via the introduction of $L^0(\mu)$-modules (the notation being reminiscent of the fact that these objects arise from measurable functions).

5.2 Local dimension

In this brief section we discuss the notion of local dimension for $L^\infty(\mu)$-modules; we shall see that the definition of dimension is well-posed and that the underlying metric space admits a "dimensional decomposition" via sets for which a given module has finite dimension, a fact which is in turn linked to the reflexivity of $L^p(\mu)$-normed modules.

5.2.1 Definition. Given an $L^\infty(\mu)$-module $\mathscr{M}$ and $A \in \mathfrak{B}(\mathcal{X})$ with $\mu(A) > 0$, we shall say that a finite family $v_1, ..., v_n \in \mathscr{M}$ is *independent* on A if the identity

$$\sum_{i=1}^n f_i v_i = 0$$

holds μ-almost everywhere on A only when $f_i = 0$ μ-almost everywhere on A for every $i = 1, ..., n$.

5.2.2 Definition. Let $\mathscr{M}$ be an $L^\infty(\mu)$-module, $V \subset \mathscr{M}$ any subset and $A \in \mathfrak{B}(\mathcal{X})$. We denote by $\mathrm{Span}_A(V)$ the span of V on A, namely the subset of $\mathscr{M}$ consisting of vectors v concentrated on A with the following property: there is a disjoint sequence $(A_n)_{n\in\mathbb{N}} \subset \mathfrak{B}(\mathcal{X})$ such that $\bigcup_{n\in\mathbb{N}} A_n = A$ and, for every n elements $v_{1,n}, ..., v_{m_n,n} \in \mathscr{M}$ and every functions $f_{1,n}, ..., f_{m_n,n} \in L^\infty(\mu)$, one has

$$\mathbb{1}_{A_n} \cdot v = \sum_{i=1}^{m_n} f_{i,n} v_{i,n}.$$

Thus $\mathrm{Span}_V(A)$ is the *space spanned by* V *on* A, or simply the space spanned by V when $A = \mathcal{X}$; its closure $\overline{\mathrm{Span}_V(A)}$ is the *space generated by* V (on A).

$\mathscr{M}$ is *finitely generated* if there is a finite family $v_1, ..., v_n$ spanning $\mathscr{M}$ on the whole metric space $\mathcal{X}$; *locally finitely generated* if there exists a partition $(E_i)_{i\in\mathbb{N}}$ of $\mathcal{X}$ such that the localization $\mathscr{M}|_{E_i}$ is finitely generated for every $i \in \mathbb{N}$.

5.2.3 Remark. The above definitions are invariant under inclusion and under countable unions. Moreover, they are also invariant under isomorphism: if $v_1, ..., v_n \in \mathscr{M}_1$ are independent on A and $T : \mathscr{M}_1 \to \mathscr{M}_2$ is a module isomorphism, then $T(v_1), ..., T(v_n)$ are independent on A as well, and the same holds for local generators as well.

5.2.4 Definition. A finite family $\{v_i\}_{i=1}^n$ is a *basis* on $A \in \mathfrak{B}(\mathcal{X})$ if it is independent on A and $\mathrm{Span}_A \{v_i\}_{i=1}^n = \mathscr{M}|_A$.

If $\mathscr{M}$ has a basis of cardinality n on A, we shall say that $\mathscr{M}$ has *dimension* n on A, or that its *local dimension* on A is n. Otherwise - that is, when $\mathscr{M}$ has not dimension n for each $n \in \mathbb{N}$ - it has infinite dimension.

The former is actually a good definition of dimension:

5.2.5 Proposition. Suppose that $\mathscr{M}$ is an $L^\infty(\mu)$-module and $A \in \mathfrak{B}(\mathcal{X})$. If $\{v_i\}_{i=1}^n$ generates $\mathscr{M}$ on A and $\{w_j\}_{j=1}^m$ is independent on A, then $n \geq m$. In particular if both the v_i's and the w_j's are bases of $\mathscr{M}$ on A, then $n = m$.

Proof. By hypothesis, $\{v_i\}_{i=1}^n$ generate $\mathscr{M}$ on A; then, there exist sets $A_i \in \mathfrak{B}(\mathcal{X})$, $i \in \mathbb{N}$, such that $A = \bigcup_{i\in\mathbb{N}} A_i$ and functions $f_{i,j} \in L^\infty(\mu)$ such that

$$\mathbb{1}_{A_i} \cdot w_1 = \sum_{j=1}^{n} f_{i,j} v_j. \tag{5.3}$$

Take $i \in \mathbb{N}$ such that $\mu(A_i) > 0$; since $\{w_j\}_{j=1}^m$ is independent on A, then $w_1 \neq 0$ μ-almost everywhere on A (the opposite yielding to a contradiction). By the above identity, for some $j \in \{1, ..., n\}$ and some $\tilde{A}_i \subset A_i$ with $\mu(A_i) > 0$ one has $f_{i,j} \neq 0$ μ-almost everywhere on $\tilde{A}_i$.

Up to permutations of the v_i's, we can assume j to be 1. Hence for some $B_1 \subset \tilde{A}_i$ with positive measure and for some $c > 0$ we find $|f_{i,j}| \geq c$ μ-almost everywhere on B_1, and then $g_1 := \mathbb{1}_{B_1} \cdot \frac{1}{f_{i,1}} \in L^\infty(\mu)$.

Now, (5.3) implies

$$\mathbb{1}_{B_1} \cdot v_1 = (\mathbb{1}_{B_1} \cdot g_1)\, w_1 - \sum_{j=2}^{n} (\mathbb{1}_{B_1} \cdot g_1 f_{i,j})\, v_j.$$

This identity, along with the hypothesis on $\{v_i\}_{i=1}^n$, means that $\{w_1, v_2, ..., v_n\}$ also generates $\mathscr{M}$ on B_1.

We can now proceed by induction; let then $k < n$ and assume we already proved the existence of $B_k \in \mathfrak{B}(\mathcal{X})$ such that $\{w_1, ..., w_k, v_{k+1}, ..., v_n\}$ generates $\mathscr{M}$ on B_k. Thus, by the above argument we find $B_k' \subset B_k$ with $\mu(B_k') > 0$ and $f_1, ..., f_n \in L^\infty(\mu)$ satisfying

$$\mathbb{1}_{B_k'} \cdot w_{k+1} = \sum_{j=1}^{k} f_j w_j + \sum_{j=k+1}^{n} f_j v_j.$$

Observe that f_j cannot be zero μ-almost everywhere on B_k' for every $j = k+1, ..., n$; in fact, if this were the case we would obtain

$$\mathbb{1}_{B_k'} \cdot w_{k+1} = \sum_{j=1}^{k} f_j w_j$$

μ-almost everywhere on B_k', contradicting the hypothesis that the w_i's are independent on $A \supset B_k'$. In particular, one has $k < n$ and there exist $B_{k+1} \subset B_k'$ with $\mu(B_{k+1}) > 0$, $c > 0$ such that for some $j \in \{k+1, ..., n\}$ one has $|f_j| \geq c$ μ-almost everywhere on B_{k+1}. Relabeling the indices if necessary, we assume $j = k+1$ and arguing as above we find out that $\{w_1, ..., w_{k+1}, v_{k+2}, ..., v_n\}$ generates $\mathscr{M}$ on B_{k+1}.

Iterating this procedure up to $k = m$, the proof follows.

□

Every $L^\infty(\mu)$-module $\mathscr{M}$ admits a "dimensional decomposition", namely a partition of the underlying metric space into sets where the module has given dimension:

5.2.6 Proposition [Gi2, Proposition 1.4.5]. If $\mathscr{M}$ is an $L^\infty(\mu)$-module, then there exists a unique partition $\{E_i\}_{i\in\mathbb{N}\cup\{\infty\}}$ of $\mathcal{X}$ such that:

i) For every $i \in \mathbb{N}$ such that $\mu(E_i) > 0$, $\mathscr{M}$ has dimension i on E_i.

ii) For every $E \subset E_\infty$ with $\mu(E) > 0$, $\mathscr{M}$ has infinite dimension.

□

5.2.7 Remark & Definition. The notion of basis obviously allows us to write locally any element of a given module via *coordinates*: consider a local basis $\{v_i\}_{i=1}^n$ of $\mathscr{M}$ on A, $v \in \mathscr{M}$ and $A_i, \tilde{A}_i \in \mathfrak{B}(\mathcal{X})$, $i \in \mathbb{N}$, such that

$$A = \bigcup_{i\in\mathbb{N}} A_i = \bigcup_{i\in\mathbb{N}} \tilde{A}_i.$$

Consider also $f_{i,j}, \tilde{f}_{i,j} \in L^\infty(\mu)$ such that for every $i \in \mathbb{N}$ one has

$$\mathbb{1}_{A_i} \cdot v = \sum_{j=1}^{n} f_{i,j} v_j \quad \text{and} \quad \mathbb{1}_{\tilde{A}_i} \cdot v = \sum_{j=1}^{n} \tilde{f}_{i,j} v_j.$$

Then, $f_{i,j} = \tilde{f}_{i',j}$ μ-almost everywhere on $A_i \cap \tilde{A}_{i'}$ for every $i, i' \in \mathbb{N}$; this comes easily from the fact that

$$\sum_{j=1}^{n} \left(f_{i,j} - \tilde{f}_{i',j} \right) v_j = 0$$

μ-almost everywhere on $A_i \cap \tilde{A}_{i'}$ and from the definition of local independence.

In other words, the functions $f_j : \mathcal{X} \to \mathbb{R}$, defined for $j = 1, ..., n$ by $f_j := f_{i,j}$ μ-almost everywhere on A_i for every $i \in \mathbb{N}$, and set equal to zero outside of A, are well defined since they depend only on the local basis $\{v_i\}_{i=1}^n$ and on the vector v. Thus, we shall call the f_j's the *coordinates* of v on A with respect to the local basis $\{v_i\}_{i=1}^n$.

5.2.8 Proposition [Gi2, Proposition 1.4.6]. Let $\mathscr{M}$ be an $L^p(\mu)$-normed module, $p \in [1, \infty]$, $v_1, ..., v_n \in \mathscr{M}$ and $A \in \mathfrak{B}(\mathcal{X})$. Then, $\mathrm{Span}_A \{v_i\}_{i=1}^n$ is closed and, in particular, it is a submodule which coincides with the intersection of all the submodules of $\mathscr{M}$ containing $\chi_A v_1, ..., \chi_A v_n$.

□

Local dimension is preserved under duality:

5.2.9 Theorem. If $\mathscr{M}$ is an $L^p(\mu)$-normed module with $p < \infty$, $A \in \mathfrak{B}(\mathcal{X})$ and the local dimension of $\mathscr{M}$ on A is $n \in \mathbb{N}$, then the local dimension of the dual module $\mathscr{M}^*$ is also n. In particular, any locally finitely generated $L^p(\mu)$-normed module with $p < \infty$ is reflexive.

Proof. Take a local basis $\{v_i\}_{i=1}^n$ of $\mathscr{M}$ on A and set $\mathscr{M}_i := \mathrm{Span}_A \{v_1, ..., v_{i-1}, v_{i+1}, ..., v_n\}$ for $i = 1, ..., n$. By Proposition 5.2.8, the $\mathscr{M}_i$'s are submodules of $\mathscr{M}$ and then by Proposition 5.1.15 the quotient $\mathscr{M}/\mathscr{M}_i$, $i = 1, ..., n$, is an $L^p(\mu)$-normed module as well; thus, denoting by $\pi_i : \mathscr{M} \to \mathscr{M}/\mathscr{M}_i$ the natural projection, the fact that $\{v_i\}_{i=1}^n$ is a local basis on A gives $\pi_i(v_i) \neq 0$ μ-almost everywhere on A.

Now, Corollary 5.1.18 applied to $\mathscr{M}/\mathscr{M}_i$ and to $\pi_i(v_i)$ gives the existence of $\tilde{L}_i \in (\mathscr{M}/\mathscr{M}_i)^*$ such that $\tilde{L}_i(\pi_i(v_i)) \neq 0$ μ-almost everywhere on A. Define $L_i \in \mathscr{M}^*$ as $L_i := \tilde{L}_i \circ \pi_i$ so that $L_i(v_j) = 0$ μ-almost everywhere on A for $i \neq j$ and $L_i(v_i) \neq 0$ μ-almost everywhere on A.

Claim: $\{L_i\}_{i=1}^n$ is a basis of $\mathscr{M}^*$ on A.

Take $L \in \mathscr{M}^*$ and set $f_i := L(v_i) \in L^1(\mu)$. Thus if we write a generic $v \in \mathscr{M}$ concentrated on A using its coordinates with respect to the local basis $\{v_i\}_{i=1}^n$, one has

$$\mathbb{1}_A \cdot L = \mathbb{1}_A \cdot \sum_{i=1}^{n} f_i L_i,$$

the identity being understood in $(\mathscr{M}^*)^0$ - see [Gi2, section 1.3] for the definition of the $L^0(\mu)$-modules $\mathscr{M}^0$. In other words,

$$\mathrm{Span}_A \{L_i\}_{i=1}^n = \mathscr{M}^*|_A.$$

To prove linear independence, assume $\sum_i f_i L_i = 0$ μ-almost everywhere on A for some $f_i \in L^\infty(\mu)$ and compute this against v_j in order to obtain $f_j L_j(v_j) = 0$ μ-almost everywhere on A; but since $L_j(v_j) \neq 0$, a fortiori $f_j = 0$ μ-almost everywhere on A. The claim is proven.

The second assertion is a direct consequence of the previous arguments and of the fact that $\mathcal{J}_{\mathscr{M}} : \mathscr{M} \to \mathscr{M}^{**}$ is an isomorphism of $\mathscr{M}$ with its image.

□

The next result gives an interesting criterion to characterize the elements in the dual module via their action on a generating subspace of a given $L^p(\mu)$-normed module:

5.2.10 Proposition. Let $\mathscr{M}$ be an $L^p(\mu)$-normed module with $p < \infty$. Consider a linear subspace $V \subset \mathscr{M}$ which generates $\mathscr{M}$ and a linear map $L : V \to L^1(\mu)$ such that $|L(v)| \leq l|v|$ μ-almost everywhere for every $v \in V$ and for some $l \in L^q(\mu)$, $\frac{1}{p} + \frac{1}{q} = 1$.
Then, L can be uniquely extended to a module morphism $\tilde{L} : \mathscr{M} \to L^1(\mu)$ - that is, to an element of $\mathscr{M}^*$ - such that $\left|\tilde{L}\right| \leq l$ μ-almost everywhere.

Proof. Denote by $\tilde{V} \subset \mathscr{M}$ the set of elements of the form

$$v := \sum_{i=1}^{n} \mathbb{1}_{A_i} \cdot v_i,$$

for some $n \in \mathbb{N}$, $A_i \in \mathfrak{B}(\mathcal{X})$ and $v_i \in V$ for $i = 1, ..., n$; of course, $\tilde{V}$ is a vector space. Consider the map $\tilde{L} : \tilde{V} \to L^1(\mu)$ given by

$$\tilde{L}\left(\sum_{i=1}^{n} \mathbb{1}_{A_i} \cdot v_i\right) := \sum_{i=1}^{n} \mathbb{1}_{A_i} \cdot L(v_i).$$

The bound on $|L(v)|$ grants that the definition of $\tilde{L}$ is well posed; in other words, the right hand side above depends only on v and not on the particular way we represent the element. Hence, $\tilde{L}$ is linear.
In the definition of v, we may assume the A_i's to be disjoint; thus

$$\left\|\tilde{L}(v)\right\|_{L^1(\mu)} = \sum_{i=1}^{n} \int_{A_i} |L(v_i)| \, \mathrm{d}\mu \leq \sum_{i=1}^{n} \int_{A_i} l\,|v_i| \, \mathrm{d}\mu = \int_{\mathcal{X}} l|v| \mathrm{d}\mu \leq \|l\|_{L^q(\mu)} \|v\|_{\mathscr{M}}.$$

This implies the continuity of $\tilde{L}$, which is then extendable to a continuous linear map - still denoted by $\tilde{L}$ - from the closure of $\tilde{V}$ to $L^1(\mu)$. Taking into account the definition of $\mathrm{Span}_{\mathcal{X}}(V)$ and Remark 5.1.13, one finds that the closure of $\tilde{V}$ contains $\mathrm{Span}_{\mathcal{X}}(V)$ and then also its closure, which is the whole of $\mathcal{X}$.
The bound $\left|\tilde{L}(v)\right| \leq l|v|$ μ-almost everywhere for every $v \in \mathscr{M}$ is a direct consequence of the construction; invoking *v)* of Proposition 5.1.12 we obtain that $\tilde{L}$ is a module morphism, which in turn gives its uniqueness.

□

We conclude the section with two further interesting results about generating subsets of $L^p(\mu)$-normed modules.

5.2.11 Proposition [Gi2, Proposition 1.4.9]. Suppose $\mathscr{M}$ is an $L^p(\mu)$-normed module with $p \in]1,\infty[$ and $V \subset \mathscr{M}$ a linear subspace generating $\mathscr{M}$. Then, for every $L \in \mathscr{M}^*$ one has

$$\frac{1}{q}\,|L|_*^q = \underset{v\in V}{\text{ess-sup}}\left(L(v) - \frac{1}{p}\,|v|^p\right) \qquad \mu - \text{almost everywhere.}$$

□

5.2.12 Proposition [Gi2, Proposition 1.4.10]. Let $\mathscr{M}$ be an $L^p(\mu)$-normed module with $p < \infty$ and choose a generating set $V \subset \mathscr{M}$. If V, endowed with the induced topology, is separable, then $\mathscr{M}$ is separable as well.

□

5.3 Pullback

One of the main tools in the costruction of a differential structure in differential geometry is that of pullback bundle, which has an analogous counterpart in the non-smooth setting of $L^p(\mu)$-normed modules. We shall focus on the case $p < \infty$.

The construction of the "pullback module" $\varphi^*\mathscr{M}$ of an $L^p(\mu)$-normed module $\mathscr{M}$ we shall give below, taken from [Gi2, Section 1.6], is actually of no direct interest for our discussion, but we chose to survey this topic since it is useful in comparison with the "cotangent module" $L^p\left(T^*\mathcal{X}\right)$ which will be the central object of Section 5.4, and whose characterization will be strongly reminiscent of that of $\varphi^*\mathscr{M}$.

5.3.1 Definition & Remarks. Suppose $(\mathcal{X}_1, \mathcal{A}_1, \mu_1)$ and $(\mathcal{X}_2, \mathcal{A}_2, \mu_2)$ are two σ-finite measure spaces. A *map of bounded compression* $\varphi : \mathcal{X}_2 \to \mathcal{X}_1$ is a measurable map - more precisely, the equivalence class with respect to μ_2-almost everywhere equality - such that $\varphi_*\mu_2 \le C\mu_1$ for some $C \ge 0$.

Given two σ-finite measured spaces as above, φ of bounded compression and an $L^1(\mu_1)$-normed module $\mathscr{M}$ with $p \in [1,\infty[$, we define the *pre-pullback* as the set

$$\textsc{Ppb} := \Bigg\{ \{(v_i, A_i)\}_{i\in\mathbb{N}}\,;\,(A_i)_{i\in\mathbb{N}} \text{ is a disjoint partition of } \mathcal{X}_2, \\ v_i \in \mathscr{M}\ \forall\, i \in \mathbb{N},\ \text{and}\ \left\| \sum_{i\in\mathbb{N}} \chi_{A_i}\,|v_i| \circ \varphi \right\|_{L^p(\mu_2)} < \infty \Bigg\}.$$

On $\textsc{Ppb}$ we introduce an equivalence relation by declaring $\{(v_i, A_i)\}_{i\in\mathbb{N}} \sim \{(w_j, B_j)\}_{j\in\mathbb{N}}$ whenever $|v_i - w_j| \circ \varphi = 0$ μ_2-almost everywhere on $\{A_i \cap B_j\}$ for every $i, j \in \mathbb{N}$.

Denoting the equivalence classes by $[(v_i, A_i)_i]$, we define the sum and the multiplication by scalars on $\textsc{Ppb}/\sim$ as follows:

$$\begin{aligned}
[(v_i, A_i)_i] + \left[\,(w_j, B_j)_j\,\right] &:= \left[\,(v_i + w_j, A_i \cap B_j)_{i,j}\,\right], \\
\lambda\,[(v_i, A_i)_i] &:= [(\lambda v_i, A_i)_i]\,, \quad \lambda \in \mathbb{R}.
\end{aligned}$$

Of course, the two operations are well defined and give $\textsc{Ppb}/\sim$ the structure of a vector space.

Let $\mathrm{Sf}(\mu) \subset L^\infty(\mu)$ denote the set of simple functions already introduced in Definition 1.1.1, namely those which attain only a finite number of values; that is, functions of the form

$$g = \sum_j a_j \mathbb{1}_{B_j},$$

$(B_j)_{j\in\mathbb{N}}$ being a finite partition of $\mathscr{X}_2$. We define the multiplication by $g \in \mathrm{Sf}\,(\mu_2) \subset L^\infty\,(\mu_2)$ in the following way:

$$g\,[(v_i, A_i)_i] := \left[(a_j v_i, A_i \cap B_j)_{i,j}\right] \in \textsc{Ppb}/\sim .$$

The resulting operation is a bilinear map from $\mathrm{Sf}\,(\mu_2) \times \textsc{Ppb}/\sim$ into $\textsc{Ppb}/\sim$. Finally, let us consider the map $|\cdot| : \textsc{Ppb}/\sim \to L^p\,(\mu_2)$ given by

$$|[(v_i, A_i)_i]| := \sum_{i\in\mathbb{N}} \mathbb{1}_{A_i}\,|v_i| \circ \varphi.$$

It is easy to check that $|\cdot|$ satisfies the following conditions:

$$\begin{aligned}
|[(v_i + w_j, A_i \cap B_j)_{i,j}]| &\le |[(v_i, A_i)_i]| + |[(w_j, B_j)_j]|\,, \\
|\lambda\,[(v_i, A_i)_i]| &= |\lambda|\,|[(v_i, A_i)_i]|\,, \\
|g\,[(v_i, A_i)_i]| &= |g|\,|[(v_i, A_i)_i]|\,,
\end{aligned} \tag{5.4}$$

which hold true μ_2-almost everywhere for every $[(v_i, A_i)_i]\,, [(w_j, B_j)_j] \in \textsc{Ppb}/\sim$, $\lambda \in \mathbb{R}$ and $g \in \mathrm{Sf}\,(\mu_2)$. In this way, we are entitled to define a norm on $\textsc{Ppb}/\sim$ by setting

$$\|[(v_i, A_i)_i]\| := \||[(v_i, A_i)_i]|\|_{L^p(\mu_2)} = \left\|\sum_{i\in\mathbb{N}} \mathbb{1}_{A_i}\,|v_i| \circ \varphi\right\|_{L^p(\mu_2)}. \tag{5.5}$$

5.3.2 Definition. The completion of $\textsc{Ppb}/\sim$ with respect to the above norm $\|\cdot\|$, namely $\varphi^*\mathscr{M} := \overline{\{\textsc{Ppb}/\sim\}}^{\|\cdot\|}$, will be called the *pullback module*.

5.3.3 Proposition. For $p < \infty$, $\varphi^*\mathscr{M}$ has a canonical structure of $L^p\,(\mu_2)$-normed module.

Proof. From the third condition in (5.4) and from the definition of the norm (5.5), one has

$$\|g\,[(v_i, A_i)_i]\| \leq \|g\|_{L^\infty(\mu_2)}\,\|[(v_i, A_i)_i]\|$$

for every $g \in \mathrm{Sf}(\mu_2)$ and $[(v_i, A_i)_i] \in \mathrm{PPB}/\sim$. By the density of simple functions in L^∞, the multiplication can be uniquely extended to a bilinear, continuous map from $L^\infty(\mu_2) \times \varphi^*\mathscr{M}$ to $\varphi^*\mathscr{M}$, giving rise to an $L^\infty(\mu_2)$-premodule structure. Now, the first condition in (4.3) ensures that for a Cauchy sequence $\left([(v_{n,i}, A_{n,i})_i]\right)_{n\in\mathbb{N}} \subset \mathrm{PPB}/\sim$ the sequence $\left(|[(v_{n,i}, A_{n,i})_i]|\right)_{n\in\mathbb{N}}$ is $L^p(\mu_2)$-Cauchy. Passing to the limit as $n \to \infty$ we are given a map $|\cdot| : \varphi^*\mathscr{M} \to L^p(\mu_2)$ which, in comparison with (5.4.) and (5.5) defines in turn a pointwise norm.

At this point, by *iv)* of Proposition 5.1.12 we obtain our claim.

□

5.3.4 Definition. The *pullback map* $\varphi^* : \mathscr{M} \to \varphi^*\mathscr{M}$ is given by $\varphi^* v := [(v, \mathcal{X}_2)]$; here, $(v, \mathcal{X}_2) \in \mathrm{PPB}$ has to be intended as $(v_i, A_i)_{i\in\mathbb{N}}$ with $v_0 = v$, $A_0 = \mathcal{X}_2$ and $v_i = 0$, $A_i = \emptyset$ for all $i > 0$.

5.3.5 Remark. $\varphi^*\mathscr{M}$ is generated - in the sense of modules - by the vector space $\{\varphi^* v;\ v \in \mathscr{M}\}$; this is a direct consequence of the definition itself.

Moreover,

$$\varphi^*(fv) = f \circ \varphi \cdot \varphi^* v,$$
$$|\varphi^* v| = |v| \circ \varphi$$

μ_2-almost everywhere for every $v \in \mathscr{M}$ and $f \in L^\infty(\mu_1)$. The first comes on the one hand from the definition of pullback map, and on the other hand from the equivalence relation on PPB if $f = \mathbb{1}_A$, $A \in \mathfrak{B}(\mathcal{X}_1)$; then, by linearity it holds for every simple function and an approximation argument yields also the general case. The second condition is just given by the definition of pointwise norm on $\varphi^*\mathscr{M}$.

5.4 The Cotangent Module

From now on, $(\mathcal{X}, d, \mu)$ will denote a separable, complete metric space $(\mathcal{X}, d)$ equipped with a non-negative Radon measure μ.

The following construction is technically similar to the one of the pullback module, the main and important difference being the lack of any module as a starting point.

5.4.1 Definition. We shall call *pre-cotangent module* the set

$$
\mathrm{P\textsc{cm}} := \left\{ \{(f_i, A_i)\}_{i\in\mathbb{N}}\,;\ (A_i)_{i\in\mathbb{N}} \subset \mathfrak{B}(\mathcal{X}),\ f_i \in D^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})\ \forall\, i\in\mathbb{N},\ \sum_{i\in\mathbb{N}}\int_{A_i} |Df_i|^p \,\mathrm{d}\mu < \infty \right\},
$$

where the A_i's form a disjoint partition of $\mathcal{X}$ and $D^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})$, $1 \le p < \infty$, denotes the Sobolev-Dirichlet class of order p as in Definition 3.1.3.
By Theorem 3.2.9, $D^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})$ may be replaced with no ambiguity by $D^{1,p}(\mathcal{X})$, the two classes being equivalent; however, since the aim is to construct a metric counterpart to a cotangent bundle, in accordance with Remark 3.1.4 we prefer to adapt our notation in order to be coherent with the notion given in [Gi1] and [Gi2].

We introduce an equivalence relation $\sim$ on $\mathrm{P\textsc{cm}}$ by stating $\{(f_i, A_i)\}_{i\in\mathbb{N}} \sim \{(g_j, B_j)\}_{j\in\mathbb{N}}$ whenever $|D\,(f_i - g_j)| = 0$ μ-almost everywhere on $A_i \cap B_j$ for all $i, j \in \mathbb{N}$.
$\mathrm{P\textsc{cm}}/\sim$ turns into a vector space if we define the sum and the multiplication by scalars as

$$
\begin{gathered}
[(f_i, A_i)_i] + \left[\,(g_j, B_j)_j\right] = \left[\,(f_i + g_j, A_i \cap B_j)_{i,j}\,\right],\\
\lambda\,[(f_i, A_i)_i] = [(\lambda f_i, A_i)_i]\,.
\end{gathered}
$$

We can also define a multiplication by simple functions: if $h = \sum_j \mathbb{1}_{B_j}\cdot a_j \in \mathrm{Sf}(\mu)$ with B_j partition of $\mathcal{X}$ and $[(f_i, A_i)_i] \in \mathrm{P\textsc{cm}}/\sim$, then we set

$$
h\,[(f_i, A_i)_i] := \left[\,(a_j f_i, A_i \cap B_j)_{i,j}\,\right].
$$

This operation gives a bilinear map $\mathrm{Sf}(\mu) \times \mathrm{P\textsc{cm}}/\sim \to \mathrm{P\textsc{cm}}/\sim$ such that $\mathbf{1}\,[(f_i, A_i)_i] = [(f_i, A_i)_i]$.

5.4.2 Definition. Consider the map $|\cdot|_* : \mathrm{P\textsc{cm}}/\sim \to L^p(\mu)$ given by

$$
|[(f_i, A_i)_i]|_* := |Df_i|
$$

μ-almost everywhere on A_i for all $i \in \mathbb{N}$; this map, namely the *pointwise norm* on $\mathrm{P\textsc{cm}}/\sim$, is well defined thanks to the above definition of the equivalence relation on $\mathrm{P\textsc{cm}}$.
Since $D^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})$ is a vector space, one has the following inequalities for $|\cdot|_*$:

$$
\begin{gathered}
\left|\left[\,(f_i + g_j, A_i \cap B_j)_{i,j}\,\right]\right|_* \le |[(f_i, A_i)_i]|_* + \left|\left[\,(g_j, B_j)_j\,\right]\right|_*,\\
|\lambda\,[(f_i, A_i)_i]|_* = |\lambda|\,|[(f_i, A_i)_i]|\,,\\
|h\,[(f_i, A_i)_i]|_* = |h|\,|[(f_i, A_i)_i]|_*\,,
\end{gathered}
\tag{5.6}
$$

valid μ-almost everywhere for every $[(f_i, A_i)_i]\,,\left[(g_j, B_j)_j\right] \in \mathrm{P\textsc{cm}}/\sim$, $h \in \mathrm{Sf}(\mu)$ and $\lambda \in \mathbb{R}$.

5.4.3 Definition. The above arguments, in particular (5.6), allow us to define a norm $\|\cdot\|_{L^p(T^*\mathcal{X})} : \mathrm{P\textsc{cm}}/\sim \to [0, \infty[$ on $\mathrm{P\textsc{cm}}/\sim$ by setting

$$
\|[(f_i, A_i)_i]\|^p_{L^p(T^*\mathcal{X})} := \int_{\mathcal{X}} |[(f_i, A_i)_i]|^p \,\mathrm{d}\mu = \sum_{i\in\mathbb{N}}\int_{A_i} |Df_i|^p \,\mathrm{d}\mu.
$$

The completion of $\mathrm{P{\scriptsize CM}}/\sim$ with respect to the norm $\|\cdot\|_{L^p(T^*\mathcal{X})}$ will be called *cotangent module* and denoted by $L^p\left(T^*\mathcal{X}\right)$. Consequently, its elements will be called *cotangent vector fields* or, more traditionally, *1-forms*.

$L^p\left(T^*\mathcal{X}\right)$ is a Banach space and, moreover, it has a structure of $L^p(\mu)$-normed module: if we concentrate on the third condition in (5.6), then we see that the bilinear map $(\cdot\,,\cdot):\mathrm{Sf}(\mu)\times \mathrm{P{\scriptsize CM}}/\sim\to \mathrm{P{\scriptsize CM}}/\sim$, $(h,[(f_i,A_i)_i])\mapsto h\,[(f_i,A_i)_i]$, can be uniquely extended to a bilinear map $(g,\omega)\mapsto g\omega$ from $L^\infty(\mu)\times L^p\left(T^*\mathcal{X}\right)$ to $L^p\left(T^*\mathcal{X}\right)$ such that $|h\omega|_* = |h|\,|\omega|_*$ μ-almost everywhere for all $h\in L^\infty(\mu)$ and $\omega\in L^p\left(T^*\mathcal{X}\right)$; this gives $L^p\left(T^*\mathcal{X}\right)$ the structure of an $L^p(\mu)$-normed premodule and thus, by *iv)* of Proposition 5.1.12, it is actually an $L^p(\mu)$-normed module.

5.4.4 Remark. The choice of denoting the cotangent module by $L^p\left(T^*\mathcal{X}\right)$ is just formal: we are not given, in fact, any definition of a cotangent bundle to the metric space $\mathcal{X}$. The notation is motivated by the fact that, if $\mathcal{X}$ is a smooth manifold, the above construction gives a structure which we may canonically identify with the L^p sections of the cotangent bundle; see [Gi2, Section 2.2] for more comments.

5.4.5 Definition. Given a function $f\in D^{1,p}_\pi(\mathcal{X})$ we define its *differential* $\mathrm{d}f\in L^p\left(T^*\mathcal{X}\right)$ as

$$\mathrm{d}f := [(f,\mathcal{X})]\in \mathrm{P{\scriptsize CM}}/\sim\subset L^p\left(T^*\mathcal{X}\right).$$

Here, as in Definition 5.3.4, $(f,\mathcal{X})$ stands for $(f_i,A_i)_{i\in\mathbb{N}}$ with $f_0=f$, $A_0=\mathcal{X}$ and $f_i=0$, $A_i=\emptyset$ for every $i>0$.

By construction, the differential is linear; moreover, by the definition of the pointwisenorm on the cotangent module, one has

$$|\mathrm{d}f|_* = |Df|_* \tag{5.7}$$

μ-almost everywhere for every $f\in D^{1,p}_\pi(\mathcal{X})$.

As the next result shows, the differential is a local operator:

5.4.6 Theorem. For every $f,g\in D^{1,p}_\pi(\mathcal{X})$, one has $\mathrm{d}f=\mathrm{d}g$ μ-almost everywhere on the set $\{f=g\}$.

Proof. By the linearity of the differential, the assertion can be equivalently stated as $\mathrm{d}\,(f-g)=0$ μ-almost everywhere on $\{f-g=0\}$; this, by *i)* of Proposition 5.1.12, is the same as claiming $|\mathrm{d}\,(f-g)|_*=0$ μ-almost everywhere on $\{f-g=0\}$. Thus, (5.7) and the locality of the minimal weak upper gradient with $f-g$ instead of f and 0 in place of g allow us to conclude.

□

Another interesting byproduct of the construction is that the cotangent module is generated by differentials:

5.4.7 Proposition. $L^p(T^*\mathcal{X})$ is generated - in the sense of modules, see Definition 5.2.2 - by the space Df $:= \left\{ \mathrm{d}f;\ f \in W^{1,p}_{\pi}(\mathcal{X}) \right\}$.

In particular, if $W^{1,p}_{\pi}(\mathcal{X})$ is separable, then $L^p(T^*\mathcal{X})$ is separable as well.

PROOF. Let us first prove that $L^p(T^*\mathcal{X})$ is generated by $\left\{ \mathrm{d}f;\ f \in D^{1,p}_{\pi}(\mathcal{X}) \right\}$. Indeed, by definition, the differential of a Sobolev function, we have $\mathrm{d}f = [(f, \mathcal{X})]$ and, by the operations defined on PCM$/\sim$,

$$\sum_{i=1}^{n} \mathbb{1}_{A_i} \cdot \mathrm{d}f_i = [(f_i, A_i)_i]$$

for any finite partition $(A_i)_{i=1}^n$ of $\mathcal{X}$ and $f_i \in D^{1,p}_{\pi}(\mathcal{X})$. Now recall the definition of norm on PCM$/\sim$. We can pass to the limit, thus extending the previous property to generic elements $[(f_i, A_i)_i] \in$ PCM$/\sim$; the claim then holds by the density of PCM$/\sim$ in $L^p(T^*\mathcal{X})$.

If we take $f \in D^{1,p}_{\pi}(\mathcal{X})$ and consider the truncations $f_n := \min\{\max\{f, -n\}, n\}$, $n \in \mathbb{N}$, an application of the chain rule for Sobolev functions yields $|D(f - f_n)| = 0$ μ-almost everywhere on the set $\{|f| \leq n\}$, which implies $\mathrm{d}f_n \to \mathrm{d}f$ in $L^p(T^*\mathcal{X})$ as $n \to \infty$. Thus, by approximation one has that the cotangent module is generated by $\left\{ \mathrm{d}f;\ f \in L^\infty \cap D^{1,p}_{\pi}(\mathcal{X}, \mu) \right\}$.

Next, assume $f \in L^\infty \cap D^{1,p}_{\pi}(\mathcal{X}, \mu)$. Take $x \in \mathcal{X}$ and find a corresponding radius $\rho = \rho_x > 0$ for which $\mu(B_{2\rho}(x)) < \infty$. Setting

$$\eta_x(y) := \max\left\{ 1 - \frac{d(y, B_\rho(x))}{\rho}, 0 \right\},$$

we have that η_x is a bounded, Lipschitz function in $W^{1,p}_{\pi}(\mathcal{X})$ and by the weak Leibniz rule for weak upper gradients, see Remark 3.1.5, one has $\eta_x f \in W^{1,p}_{\pi}(\mathcal{X})$; moreover, $f = \eta_x f$ μ-almost everywhere on $B_\rho(x)$ and by Theorem 5.4.6 this means

$$\mathbb{1}_{B_\rho(x)} \cdot \mathrm{d}f = \mathbb{1}_{B_\rho(x)} \cdot \mathrm{d}(\eta_x f). \tag{5.8}$$

The Lindelöf property of $\mathcal{X}$ already mentioned in Remark 1.1.12 ensures the existence of a countable set $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{X}$ such that

$$\bigcup_{n\in\mathbb{N}} B_{\rho_n}(x_n) = \mathcal{X},$$

$\rho_n = \rho_n(x_n)$, and then, applying (5.8) we find that for all $n \in \mathbb{N}$ the 1-form

$$\omega_n := \sum_{i=1}^{n} \mathbb{1}_{B_{\rho_n}(x_n)} \cdot \mathrm{d}f$$

belongs to the submodule generated by Df. Since $|\mathrm{d}f - \omega_n|_* \leq |\mathrm{d}f|_* \in L^p(\mathcal{X}, \mu)$ and $|\mathrm{d}f - \omega_n|_* \to 0$ μ-almost everywhere as $n \to \infty$, by dominated convergence one has $\omega_n \to \mathrm{d}f$ in $L^p(T^*\mathcal{X})$ as $n \to \infty$: in other words, $\mathrm{d}f$ is in the submodule generated by Df, which turns to be the whole of $L^p(T^*\mathcal{X})$.

The second assertion of the Proposition comes from the fact that $\|\mathrm{d}f\|_{L^p(T^*\mathcal{X})} \leq \|f\|_{W^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})}$ implies the separability of Df $\subset L^p(T^*\mathcal{X})$. At this point, it suffices to apply Proposition 5.2.12 to conclude the proof.

□

Taking into account [Gi2, Theorem 2.2.6], the identity (5.8) above and the locality of the differential already proven in Theorem 5.4.6, one recovers the Leibniz and Chain rules for the differential:

5.4.8 Proposition [Gi2, Corollary 2.2.8]. For every $f, g \in D^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})$, $\mathcal{N} \subset \mathbb{R}$ Borel $\mathscr{L}^1$-negligible set and $\varphi : \mathbb{R} \to \mathbb{R}$ Lipschitz, the following hold μ-almost everywhere:

$$\begin{aligned} \mathrm{d}(fg) &= g\mathrm{d}f + f\mathrm{d}g, \\ \mathrm{d}f &= 0 \text{ on } f^{-1}(\mathcal{N}), \\ \mathrm{d}(\varphi \circ f) &= \varphi' \circ f\, \mathrm{d}f. \end{aligned}$$

□

The differential is a closed operator:

5.4.9 Theorem. Consider a sequence $(f_n)_{n\in\mathbb{N}} \subset D^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})$ converging μ-almost everywhere to $f \in L^0(\mu)$ and such that $(\mathrm{d}f_n)_{n\in\mathbb{N}}$ converges to $\omega \in L^p(T^*\mathcal{X})$ in the $L^p(T^*\mathcal{X})$ norm. Then, $f \in D^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})$ and $\mathrm{d}f = \omega$.
In particular, if $(f_n)_{n\in\mathbb{N}} \subset W^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})$ satisfies $f_n \rightharpoonup f$ and $\mathrm{d}f_n \rightharpoonup \omega$ for some $f \in L^p(\mu)$ and $\omega \in L^p(T^*\mathcal{X})$ in the weak topologies of $L^p(\mu)$ and $L^p(T^*\mathcal{X})$ respectively, then $f \in W^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})$ and $\mathrm{d}f = \omega$.

Proof. The hypotheses yield $|\mathrm{d}f_n| \to |\omega|$ in $L^p(\mu)$; then, by (5.7) and by the lower semicontinuity of minimal weak upper gradients - see (2.1.3) in [Gi2] - one has $f \in W^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})$. Since

$$\|\mathrm{d}f_n - \mathrm{d}f_m\|^p_{L^p(T^*\mathcal{X})} = \int_{\mathcal{X}} |D(f_n - f_m)|^p \, \mathrm{d}\mu,$$

again by the aforementioned property applied to the sequence $(f_n - f_m)_{n\in\mathbb{N}}$ with $m \in \mathbb{N}$ fixed gives

$$\|\mathrm{d}f - \mathrm{d}f_m\|_{L^p(T^*\mathcal{X})} \leq \liminf_{n\to\infty} \|\mathrm{d}f_m - \mathrm{d}f_n\|_{L^p(T^*\mathcal{X})}.$$

Now, take the limit supremum as $m \to \infty$ and use the Cauchy behaviour of $(\mathrm{d}f_n)_{n\in\mathbb{N}}$ in $L^p(T^*\mathcal{X})$; hence, $\mathrm{d}f_n \to \mathrm{d}f$ in $L^p(T^*\mathcal{X})$ forcing $\mathrm{d}f = \omega$.
For the second claim, just apply Mazur's Lemma. The Theorem is proven.

□

5.4.10 Remark. In [Gi2, Proposition 2.2.10], the author showed that, in the case $p = 2$, the reflexivity of $W^{1,2}_{\boldsymbol{\pi}}(\mathcal{X})$ is equivalent to the weak compactness of the differential. In other words, $W^{1,2}_{\boldsymbol{\pi}}(\mathcal{X})$ is reflexive if and only if, for every bounded sequence $(f_n)_{n\in\mathbb{N}} \subset W^{1,2}_{\boldsymbol{\pi}}(\mathcal{X})$ there exist a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ and $f \in W^{1,2}_{\boldsymbol{\pi}}(\mathcal{X})$ and such that $f_{n_k} \rightharpoonup f$ and $\mathrm{d}f_{n_k} \rightharpoonup \mathrm{d}f$ in the weak topologies of $L^2(\mu)$ and $L^2(T^*\mathcal{X})$ respectively.

Reflexivity of $W^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})$ for general p has been established in [Ke] using slightly different techniques; see the suggested reference for a detailed discussion.

5.5 The Tangent Module. Gradients, Divergence and Laplacian

In this section, $p, q \in [1, \infty]$ will always denote two conjugate exponents, $\frac{1}{p} + \frac{1}{q} = 1$. By duality with the cotangent module, we introduce the tangent module as follows.

5.5.1 Definition. The *tangent module* $L^q(T\mathcal{X})$ is the dual module of $L^p(T^*\mathcal{X})$. The elements of $L^q(T\mathcal{X})$ will be called *vector fields*.

5.5.2 Remark. By *i)* of Proposition 5.1.15, $L^q(T\mathcal{X})$ is an $L^q(\mu)$-normed module. Following the well established tradition of the smooth setting, we shall keep the notation $|\cdot|$ for the pointwise norm in the tangent module as well; accordingly, the duality between $\omega \in L^p(T^*\mathcal{X})$ and $X \in L^q(T\mathcal{X})$ will be denoted by $\omega(X) \in L^1(\mu)$.

Equivalently, $L^q(T\mathcal{X})$ can be characterized in terms of "derivations"; following [Gi2], we shall see that the two approaches coincide.

The definition we are going to give below is slightly different from S. Di Marino's given in Definition 3.2.1 and, in general, it turns out to be a particular case of the tool described by N. Weaver in [We]; see Remark 5.5.22 for more comments.

5.5.3 Definition. A linear map $L : D^{1,p}_{\boldsymbol{\pi}}(\mathcal{X}) \to L^1(\mu)$ such that

$$|L(f)| \le l\,|Df| \tag{5.9}$$

μ-almost everywhere for every $f \in W^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})$ and for some $l \in L^q(\mu)$, will be called a *derivation*.

Of course, the Chain and Leibniz rules hold for derivations, since

$$|L\,(f-g)| \le l\,|D\,(f-g)| = 0$$

μ-almost everywhere on $\{f = g\}$, whence

$$L(f) = L(g)\ \mu\text{-almost everywhere on}\ \{f = g\}\,. \tag{5.10}$$

Moreover, we clearly have

$$\|L(f)\|_{L^1(\mu)} \leq \|l\|_{L^q(\mu)} \||Df|\|_{L^p(\mu)}$$

for every $f \in D^{1,p}_{\pi}(\mathcal{X})$, so that the locality property (5.10) and [Gi2, Theorem 2.2.6] applied to the module $\mathscr{M} = L^1(\mu)$ yield the desired calculus rules (Leibniz, chain).

The following result shows that vector fields and derivations actually describe the same concept from different points of view:

5.5.4 Theorem. For every vector field $X \in L^q(T\mathcal{X})$, the composition $X \circ d : D^{1,p}_{\pi}(\mathcal{X}) \to L^1(\mu)$ is a derivation.
Conversely, given a derivation L there exists a unique vector field $X \in L^q(T\mathcal{X})$ such that the following diagram

$$\begin{tikzcd} D^{1,p}_{\pi}(\mathcal{X}) \arrow[r, "\mathrm{d}"] \arrow[dr, "L"'] & L^p\left(T^*\mathcal{X}\right) \arrow[d, "X"] \\ & L^1(\mu) \end{tikzcd}$$

is commutative.

Proof. $X \circ \mathrm{d}$ is a linear map satisfying

$$\left|\left(X \circ \mathrm{d}\right)(f)\right| = |\mathrm{d}f(X)| \leq |X||\mathrm{d}f|_* = |X||Df|$$

μ-almost everywhere for all $f \in D^{1,p}_{\pi}(\mathcal{X})$. As $|X| \in L^q(\mu)$, we get that $X \circ \mathrm{d}$ fulfills (5.9) and then it is a derivation.
Now let L be a derivation. Consider the linear map $\tilde{L} : \left\{\mathrm{d}f;\ f \in D^{1,p}_{\pi}(\mathcal{X})\right\} \to L^1(\mathcal{X}, \mu)$ given by $\mathrm{d}f \mapsto \tilde{L}\left(\mathrm{d}f\right) := L(f)$ and notice that (5.9), together with the identity $|\mathrm{d}f|_* = |Df|$, ensures that $\tilde{L}$ depends only on the differential and not on the function itself. Moreover,

$$\left|\tilde{L}\left(\mathrm{d}f\right)\right| \leq l\left|\mathrm{d}f\right|_*,$$

and then by invoking Propositions 5.2.10 and 5.4.7 we conclude.

□

5.5.5 Definition. Given $f \in D^{1,p}_{\pi}(\mathcal{X})$, we say that $X \in L^q(T\mathcal{X})$ is a *gradient* of f provided

$$\mathrm{d}f(X) = |X|^q = |\mathrm{d}f|_*^p$$

μ-almost everywhere. Corollary 5.1.18 grants that the set of the gradients of f, $\mathrm{Grad}(f)$, is not empty for every $f \in D^{1,p}_{\pi}(\mathcal{X})$; in general, however, uniqueness fails.

5.5.6 Remark. Since for every $X \in L^q(T\mathcal{X})$ and every $f \in D^{1,p}_{\pi}(\mathcal{X})$ it holds

$$\mathrm{d}f(X) \leq |\mathrm{d}f|_* |X| \leq \frac{1}{p} |\mathrm{d}f|_*^p + \frac{1}{q}|X|^q,$$

then a necessary and sufficient condition for $X \in \mathrm{Grad}(f)$ is that

$$\int_{\mathcal{X}} \mathrm{d}f(X)\mathrm{d}\mu \geq \int_{\mathcal{X}} \left(\frac{1}{p} |\mathrm{d}f|_*^p + \frac{1}{q}|X|^q\right) \mathrm{d}\mu.$$

Similarly to the cotangent module, which is generated by differentials, the tangent module is generated by gradients:

5.5.7 Proposition. The set

$$V := \left\{ X = \sum_{i=1}^n \mathbb{1}_{A_i} \cdot X_i;\ n \in \mathbb{N},\ (A_i) \subset \mathfrak{B}(\mathcal{X}),\ X_i \in \mathrm{Grad}\,(f_i) \text{ with } f_i \in D^{1,p}_{\boldsymbol{\pi}}(\mathcal{X}) \right\}$$

is weakly*-dense in $L^q(T\mathcal{X})$.
In particular, if $L^q(T\mathcal{X})$ is reflexive, then is generated - in the sense of modules - by

$$\bigcup_{f \in D^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})} \mathrm{Grad}(f).$$

Proof. We start by observing that, since the A_i's need not be disjoint, then V is a vector space.
Denote by W the subset of $L^p\,(T^*\mathcal{X})$ given by

$$W := \left\{ \omega = \sum_{i=1}^n \mathbb{1}_{A_i} \cdot \mathrm{d}f_i;\ n \in \mathbb{N},\ (A_i) \subset \mathfrak{B}(\mathcal{X}) \text{ disjoint and } f_i \in D^{1,p}_{\boldsymbol{\pi}}(\mathcal{X}) \right\}.$$

Since $L^p\,(T^*\mathcal{X})$ is generated by $\left\{\mathrm{d}f;\ f \in D^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})\right\}$, see Proposition 5.4.7, by Remark 5.1.13 W is strongly dense in $L^p\,(T^*\mathcal{X})$.
Now given a form $\omega \in W$ we consider $X \in V$. By the definition of gradient,

$$\int_{\mathcal{X}} \omega(X)\mathrm{d}\mu = \|\omega\|^p_{L^p(T^*\mathcal{X})}.$$

Hence, the strong density of W in $L^p\,(T^*\mathcal{X})$ gives that, if $\omega \in L^p\,(T^*\mathcal{X})$ is such that the above integral is zero for all $X \in V$, then $\omega = 0$; in other words, V is weakly*-dense in $L^q\,(T\mathcal{X})$.
Let us turn to the second claim. The reflexivity of $L^p(T\mathcal{X})$ implies that its weak*-topology coincides with the weak one, and then the proof follows by Mazur's Lemma.

□

The following Proposition summarizes the basic calculus rules for gradients:.

5.5.8 Proposition [Gi2, Proposition 2.3.6]. For $f, g \in D^{1,p}_{\pi}(\mathcal{X})$ and $X \in \mathrm{Grad}(f)$ one has

$$\mathbb{1}_{\{f=g\}} \cdot X = \mathbb{1}_{\{f=g\}} \cdot Y$$

μ-almost everywhere for some $Y \in \mathrm{Grad}(g)$. Moreover, if $\mathcal{N} \subset \mathbb{R}$ is a Borel $\mathscr{L}^1$-negligible set and $\varphi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function,

$$\begin{aligned} X = 0, \ \mu - \text{almost everywhere on } f^{-1}(\mathcal{N}), \\ \varphi' \circ f X \in \mathrm{Grad}\,(\varphi \circ f)\,. \end{aligned}$$

□

We now survey *en passant* the duality properties between differentials and gradients; we follow [Gi1, Section 3.1], as an additional reference.

5.5.9 Remark. By the properties of Sobolev-Dirichlet classes discussed in Remark 3.1.5, given $f, g \in D^{1,p}_{\pi}(\mathcal{X})$ one has that the map $\mathbb{R} \ni \varepsilon \mapsto |D(g + \varepsilon f)|$ is convex μ-almost everywhere: in other words, for every $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$ and $\lambda \in [0,1]$ the inequality

$$|D\,(g + (((1-\lambda)\,\varepsilon_1 + \lambda\varepsilon_2)\,f)| \leq (1-\lambda)\,|D\,(g + \varepsilon_1 f)| + \lambda\,|D\,(g + \varepsilon_2 f)|$$

holds μ-almost everywhere. Thus, for $\varepsilon_1 \leq \varepsilon_2$ with $\varepsilon_1, \varepsilon_2 \neq 0$ one has

$$\frac{|D\,(g+\varepsilon_1 f)|^p - |Dg|^p}{p\varepsilon_1 |Dg|^{p-2}} \leq \frac{|D\,(g+\varepsilon_2 f)|^p - |Dg|^p}{p\varepsilon_2 |Dg|^{p-2}} \tag{5.11}$$

and then, in particular,

$$\operatorname*{ess-sup}_{\varepsilon<0} \frac{|D\,(g+\varepsilon f)|^p - |Dg|^p}{p\varepsilon |Dg|^{p-2}} \leq \operatorname*{ess-inf}_{\varepsilon>0} \frac{|D\,(g+\varepsilon f)|^p - |Dg|^p}{p\varepsilon |Dg|^{p-2}}, \tag{5.12}$$

both inequalities being intended μ-almost everywhere. Observe that ess-sup and ess-inf in the above may be replaced by $\lim_{\varepsilon\uparrow 0}$ and $\lim_{\varepsilon\downarrow 0}$ respectively.

5.5.10 Proposition [Gi2, Proposition 2.3.7]. If $f, g \in D^{1,p}_{\pi}(\mathcal{X})$, then for every $X \in \mathrm{Grad}(g)$ one has

$$\mathrm{d}f(X) \leq \operatorname*{ess-inf}_{\varepsilon>0} \frac{|D\,(g+\varepsilon f)|^p - |Dg|^p}{p\varepsilon |Dg|^{p-2}} \tag{5.13}$$

μ-almost everywhere. Moreover, there exists an element in Grad(g), say $X_{f,+}$ for which equality holds.
Similarly, for every $X \in \mathrm{Grad}(g)$ one has

$$\mathrm{d}f(X) \geq \operatorname*{ess-sup}_{\varepsilon<0} \frac{|D\,(g+\varepsilon f)|^p - |Dg|^p}{p\varepsilon|Dg|^{p-2}} \tag{5.14}$$

μ-almost everywhere and, as above, there exists $X_{f,-} \in \mathrm{Grad}(g)$ such that equality holds.

□

5.5.11 Proposition. The following statements are equivalent:

i) for every $g \in D^{1,p}_{\pi}(\mathcal{X})$, Grad$(g)$ has only one element;

ii) for every $f, g \in D^{1,p}_{\pi}(\mathcal{X})$, (5.10) is an equality.

Proof. "*i) ⇒ ii)*" is a direct consequence of Proposition 5.5.10 above: indeed, the two vector fields $X_{f,+}, X_{f,-} \in \mathrm{Grad}(g)$ for which (5.13) and (5.14) become equalities, must coincide.
For the converse implication, fix $g \in D^{1,p}_{\pi}(\mathcal{X})$ and take $X_1, X_2 \in \mathrm{Grad}(g)$; notice that the hypothesis and Proposition 5.5.10 imply that for all $f \in D^{1,p}_{\pi}(\mathcal{X})$ it holds $\mathrm{d}f\,(X_1) = \mathrm{d}f\,(X_2)$ μ-almost everywhere, so for $\omega \in L^p\,(T^*\mathcal{X})$ of the form

$$\omega = \sum_{i=1}^{n} \mathbb{1}_{A_i} \cdot \mathrm{d}f_i$$

for some $f_i \in D^{1,p}_{\pi}(\mathcal{X})$ and $(A_i) \subset \mathfrak{B}(\mathcal{X})$, $i \in \mathbb{N}$, one also has $\omega\,(X_1) = \omega\,(X_2)$ μ-almost everywhere.
Since the ω's as above form a dense subset of $L^p\,(T^*\mathcal{X})$, we conclude that $X_1 = X_2$.

□

5.5.12 Definition. $(\mathcal{X}, d, \mu)$ will be called *infinitesimally strictly convex* whenever the two equivalent conditions in Proposition 5.5.11 are fulfilled.
When infinitesimal strict convexity holds, the only element in Grad(g), $g \in D^{1,p}_{\pi}(\mathcal{X})$, will be denoted with the usual notation ∇g.

With all the above tools at our disposal, we are entitled to define the notion of divergence:

5.5.13 Definition. We introduce the set

$$D\,(\mathrm{div}) := \left\{ X \in L^q(T\mathcal{X});\ \exists\, f \in L^q(\mu) : \int_{\mathcal{X}} fg\mathrm{d}\mu = -\int_{\mathcal{X}} \mathrm{d}g(X)\mathrm{d}\mu \ \forall\, g \in W^{1,p}_{\pi}(\mathcal{X}) \right\}.$$

Clearly, $D(\mathrm{div}) \subset L^q(T\mathcal{X})$; the function f, which is unique by the density of $W^{1,q}_{\boldsymbol{\pi}}(\mathcal{X})$ in $L^q(\mu)$, will be called the *divergence* of the vector field X, $f := \mathrm{div}(X)$.

5.5.14 Remark. The linearity of the differential implies that $D\,(\mathrm{div})$ is a vector space and hence that the divergence is a linear operator. Moreover, the Leibniz rule for differentials immediately yields the same property for the divergence as well: given $X \in D\,(\mathrm{div})$ and $f \in L^\infty \cap D^{1,p}_{\boldsymbol{\pi}}(\mathcal{X},\mu)$ with $|\mathrm{d}f|_* \in L^\infty(\mu)$, one has

$$\begin{aligned} fX &\in D(\mathrm{div}), \\ \mathrm{div}(fX) &= \mathrm{d}f(X) + f\mathrm{div}(\mathrm{X}). \end{aligned}$$

In fact, these hypotheses give $\mathrm{d}f(X) + f\mathrm{div}(X) \in L^p(\mu)$ and for all $g \in W^{1,q}_{\boldsymbol{\pi}}(\mathcal{X})$ one has $fg \in W^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})$, whence

$$-\int_{\mathcal{X}} gf\mathrm{div}(X)\mathrm{d}\mu = \int_{\mathcal{X}} \mathrm{d}(fg)(X)\mathrm{d}\mu = \int_{\mathcal{X}} \left(g\mathrm{d}f(X) + \mathrm{d}g(fX)\right)\mathrm{d}\mu,$$

so the claims hold.

5.5.15 Remark. Combining [Gi2, Proposition 2.3.13] and the results contained in [Ke], we have that if $D(\mathrm{div})$ is dense in $L^q(T\mathcal{X})$ with respect to the strong topology, then $W^{1,p}_{\boldsymbol{\pi}}(\mathcal{X})$ is reflexive.

At this point, we wish to discuss an interesting byproduct of the infinitesimal strict convexity of $\mathcal{X}$: namely, the possibility of characterizing the Laplacian as the "divergence of the gradient", just as in the smooth setting.

In order to do so, we step back for a moment to a few more basic definitions.

5.1.16 Definition. Let $p \in [1,\infty[$. We define the *Cheeger-Dirichlet Energy* $\mathbf{E}_p : L^2(\mathcal{X},\mu) \to [0,\infty]$ as the functional given by

$$\mathbf{E}_p(f) := \begin{cases} \dfrac{1}{p}\displaystyle\int_{\mathcal{X}} |Df|^p \mathrm{d}\mu, & f \in D^{1,p}_{\boldsymbol{\pi}} \cap L^2(\mathcal{X},\mu) \\ \\ +\infty, & \text{otherwise.} \end{cases}$$

By the properties of the minimal weak upper gradient, $\mathbf{E}_p$ is convex and lower semicontinuous; moreover, its domain is dense in $L^2(\mathcal{X},\mu)$.

5.1.17 Definition. Let $p = 2$. The *subdifferential* $\partial^-\mathbf{E}_2(f) \subset L^2(\mathcal{X},\mu)$ of $\mathbf{E}_2$ at $f \in L^2(\mathcal{X},\mu)$ is defined as the empty set whenever $\mathbf{E}_2(f) = +\infty$ and, otherwise, as the possibly empty subset of $L^2(\mathcal{X},\mu)$ given by

$$\partial^-\mathbf{E}_2(f) := \left\{ v \in L^2(\mathcal{X},\mu);\ \mathbf{E}_2(f) + \int_{\mathcal{X}} vg\mathrm{d}\mu \le \mathbf{E}_2(f+g),\ \forall\, g \in L^2(\mathcal{X},\mu) \right\}.$$

The subdifferential is always closed and convex, and the class $\{f \in L^2(\mathcal{X}, \mu);\ \partial^- \mathbf{E}_2(f) \neq \emptyset\}$ is dense in $L^2(\mathcal{X}, \mu)$.
Thus, we define the *domain of the Laplacian* as

$$D(\Delta) := \left\{ f \in L^2(\mathcal{X}, \mu);\ \partial^- \mathbf{E}_2(f) \neq \emptyset \right\} \subset L^2(\mathcal{X}, \mu),$$

and for $f \in D(\Delta)$ we define its *Laplacian* $\Delta f \in L^2(\mathcal{X}, \mu)$ as $\Delta f := -v$, v being the element of minimal norm in $\partial^- \mathbf{E}_2(f)$.

5.5.18 Proposition. For $f \in W^{1,2}_{\boldsymbol{\pi}}(\mathcal{X})$, assume there exists a vector field $X \in \mathrm{Grad}(f) \cap D(\mathrm{div})$. Then, $\mathrm{div}(X) \in \partial^- \mathbf{E}_2(f)$ and in particular $f \in D(\Delta)$.
Vice-versa, if the space is infinitesimally strictly convex and $f \in D(\Delta)$, then $\nabla f \in D(\mathrm{div})$ and $\mathrm{div}(\nabla f) = \Delta f$.

Proof. Take f and X as in the statement. Given an arbitrary $g \in W^{1,2}_{\boldsymbol{\pi}}(\mathcal{X})$, the convexity of the 2-energy yields

$$\mathbf{E}_2(f+g) - \mathbf{E}_2(f) \geq \lim_{\varepsilon \to 0^+} \int_{\mathcal{X}} \frac{|D(f+\varepsilon g)|^2 - |Df|^2}{2\varepsilon} \mathrm{d}\mu \geq \int_{\mathcal{X}} \mathrm{d}g(X) \mathrm{d}\mu = -\int_{\mathcal{X}} g \mathrm{div}(X) \mathrm{d}\mu,$$

where we used (5.14) for $p = 2$. Thus, the first claim follows.
Let us now observe that, by the definition of Laplacian, for every $g \in W^{1,2}_{\boldsymbol{\pi}}(\mathcal{X})$ it holds

$$\mathbf{E}_2\left(f + \varepsilon g\right) - \mathbf{E}_2(f) \geq -\int_{\mathcal{X}} \varepsilon g \Delta f \mathrm{d}\mu$$

for all $\varepsilon \in \mathbb{R}$. If we divide by ε and then let $\varepsilon \to 0^{\pm}$, by Proposition 5.5.11 we find

$$\lim_{\varepsilon \to 0} \frac{\mathbf{E}_2(f+\varepsilon g) - \mathbf{E}_2(f)}{\varepsilon} = -\int_{\mathcal{X}} g \Delta f \mathrm{d}\mu.$$

On the other hand, Proposition 5.5.10 grants that the above limit equals

$$\int_{\mathcal{X}} \mathrm{d}g(\nabla f) \mathrm{d}\mu,$$

which is enough to conclude.

□

The following definition, along with the notion of infinitesimally strictly convex space, will be of crucial importance in the development of our discussion:

5.5.19 Definition. $(\mathcal{X}, d, \mu)$ will be called *infinitesimally Hilbertian* whenever $W^{1,2}_{\boldsymbol{\pi}}(\mathcal{X})$ is a Hilbert space. This is equivalent to ask that the semi-norm $\|\cdot\|_{D^{1,2}_{\boldsymbol{\pi}}(\mathcal{X})}$ satisfies the parallelogram rule, and that the 2-energy $\mathbf{E}_2$ is a Dirichlet form.

Observe that, by definition, infinitesimal Hilbertianity is necessary to ensure that Δ is a linear operator.

At first glance, the above property is of global nature but, actually, it forces a pointwise Hilbertianity of the space, whence the term "infinitesimal":

5.5.20 Proposition [Gi2, Proposition 2.3.17]. The following claims are equivalent:

i) $(\mathcal{X}, d, \mu)$ is infinitesimally Hilbertian;

ii) $(\mathcal{X}, d, \mu)$ is infinitesimally strictly convex and

$$\mathrm{d}f(\nabla g) = \mathrm{d}g(\nabla f) \tag{5.15}$$

μ-almost everywhere for all $f, g \in D^{1,2}_{\boldsymbol{\pi}}(\mathcal{X})$;

iii) $L^2(T^*\mathcal{X})$ and $L^2(T\mathcal{X})$ are Hilbert modules;

iv) $(\mathcal{X}, d, \mu)$ is infinitesimally strictly convex and

$$\nabla(f+g) = \nabla f + \nabla g \tag{5.16}$$

μ-almost everywhere for all $f, g \in D^{1,2}_{\boldsymbol{\pi}}(\mathcal{X})$.

v) $(\mathcal{X}, d, \mu)$ is infinitesimally strictly convex and

$$\nabla(fg) = f\nabla g + g\nabla f \tag{5.17}$$

μ-almost everywhere for all $f, g \in D^{1,2}_{\boldsymbol{\pi}}(\mathcal{X})$.

□

5.5.21 Remark. We already pointed out that, under the hypothesis of infinitesimal Hilbertianity, the 2-energy $\mathbf{E}_2 : L^2(\mu) \to [0, \infty]$ is a Dirichlet form; moreover, by Proposition 5.5.20, $\mathbf{E}_2$ admits a *carré du champ* given by $\langle \nabla f, \nabla g \rangle$ where $\langle \cdot, \cdot \rangle$ denotes the pointwise inner product on $L^2(T\mathcal{X})$.
In particular, the Laplacian and its domain $D(\Delta) \subset W^{1,2}_{\boldsymbol{\pi}}(\mathcal{X})$ can be equivalently characterized in the usual way, namely

$$f \in D(\Delta),\ h = \Delta f \Longleftrightarrow \int_{\mathcal{X}} gh\,\mathrm{d}\mu = -\int_{\mathcal{X}} \langle \nabla g, \nabla f \rangle \mathrm{d}\mu \quad \forall\, g \in W^{1,2}_{\boldsymbol{\pi}}(\mathcal{X}),$$

hence $D(\Delta)$ is a vector space and $\Delta : D(\Delta) \to L^2(\mu)$ is a linear operator.
Infinitesimal Hilbertianity also implies the following Leibniz rule for the Laplacian:

$$\begin{cases} f, g \in D(\Delta) \cap L^\infty(\mu), \\ |\mathrm{d}f|_*, |\mathrm{d}g|_* \in L^\infty(\mu) \end{cases} \Longrightarrow \begin{cases} fg \in D(\Delta) \text{ and} \\ \Delta(fg) = f\Delta g + g\Delta f + 2\langle \nabla f, \nabla g \rangle, \end{cases}$$

see the remarks after [Gi2, Proposition 2.3.17] for a more detailed discussion.

We conclude this chapter with some comments about the machinery we have illustrated so far.

5.5.22 Remark. *i)* As already pointed out in the introductory comments to the present chapter, we preferred to extend the approach of [Gi2] to an arbitrary exponent p for the sake of generality. However, we are faced with the recurring issue of the dependence of weak gradients on p: more precisely, in quite general settings where no structural assumption is given, we would have one cotangent module for each possible value of p. In other words, in such cases $L^p(T^*\mathcal{X})$, as we described it in Section 5.4, would not be well defined. Anyways, if we assume $(\mathcal{X}, d, \mu)$ to be an $\mathsf{RCD}(K, \infty)$ space (see for instance [GH]), or a doubling metric measure space supporting a $(1, p)$-Poincaré inequality (by the already mentioned results in [Ch]), all the above is not an issue anymore and thus there is always one, and only one, cotangent module, regardless of p. In particular, the fundamental Proposition 5.5.20 - which we reported as it is in [Gi2] - holds as well for any $p \neq 2$, with the obvious exception of point *iii)*.

ii) Right before Definition 5.5.3, we made clear that the notion of "derivation" taken from [Gi2] differs from the corresponding ones given in Definition 3.2.1 after [Di1], and in [We] respectively. The main difference is that, both in [Di1] and [We], derivations act on Lipschitz functions, while in our case they act on Sobolev ones. Again, if we assume that $(\mathcal{X}, d, \mu)$ is a doubling metric measure space supporting a $(1, p)$-Poincaré inequality, then - still by [Ch] - $\mathrm{lip}(f) = |Df|$ μ-almost everywhere for every $f \in \mathrm{Lip}(\mathcal{X})$; in other words, the difference between the two approaches becomes negligible, being just a matter of integrability requirements. In more general situations, the derivations as in Definition 5.5.3 are always derivations in both the sense of [Di1] and [We] as well, but the converse is not clear because with our choices the definition makes sense within the context of a metric measure space, while in [We] - and then, also in [Di1] - the assignment of a measure is not actually needed.

In view of the notion of BV functions we shall give in Chapter 6, we anticipate here that the above technicalities will not be an issue, since under our future assumptions - namely an $\mathsf{RCD}(K, \infty)$ space, or a doubling metric measure space supporting a $(1, p)$-Poincaré inequality - Theorem 4.3.5 will be still valid, including also our definition of the BV space.

6 Functions of Bounded Variation $BV_{\mathbf{F}}(\mathcal{X})$

Throughout this chapter, we shall assume that $(\mathcal{X}, d, \mu)$ is a complete and separable metric measure space equipped with a non-negative Radon measure μ. In accordance with the clarifications made in Remark 5.5.22, we start with some preliminaries about $\mathsf{RCD}(K, \infty)$ spaces.

6.1 Preliminaries

Basically speaking, $\mathsf{RCD}(K, \infty)$ spaces are metric measure spaces whose Ricci curvature is bounded from below by some number $K \in \mathbb{R}$ and whose dimension is bounded from above by $+\infty$. These objects first appeared in [AGS2] and were later axiomatized in [AGMR] after the seminal works of J. Lott and C. Villani ([LV2]), and of K. T. Sturm ([St1], [St2]), where the authors attack the issue of describing spaces with controlled sectional curvature and dimension - namely the $\mathsf{CD}(K, N)$ spaces - by means of the theory of Optimal Transportation.

In the works of L. Ambrosio et al. ([AGMR], [AGS2], [AGS3], [AGS4]), the characterization of $\mathsf{RCD}(K, \infty)$ spaces is strongly reminiscent of the optimal transportation approach, but it is combined heavily with the tools from the theory of Gradient Flows to prove eventually that both the descriptions yield equivalent notions.

Below, we shall give an informal and brief overview on the subject of $\mathsf{RCD}(K, \infty)$ spaces. In the manner of [Gi2], the "heat flow" - namely, the gradient flow of the Cheeger-Dirichlet 2-energy $\mathbf{E}_2$ - will be the key tool of our discussion; also, a crucial role will be played by "test objects", which will allow us to rephrase the definition of BV functions in the present setting.

Due to the patchy nature of this overview, besides the aforementioned papers we refer the reader to the monographs [Vi] and [AGS1], which provide deep and extensive surveys on the theories of optimal transportation and gradient flows, respectively.

6.1.1 Definition. The *relative entropy* is defined as the functional $\mathscr{E}_\mu : \mathscr{P}(\mathcal{X}) \to \mathbb{R} \cup \{+\infty\}$ given by

$$\mathscr{E}_\mu(\mathfrak{m}) := \begin{cases} \displaystyle\int_{\mathcal{X}} \rho \log(\rho) \mathrm{d}\mu & \text{if } \mathfrak{m} = \rho\mu \text{ and } (\rho\log(\rho)) \in L^1(\mathcal{X}, \mu) \\ \\ +\infty, & \text{otherwise.} \end{cases}$$

6.1.2 Definition. $(\mathcal{X}, d, \mu)$ will be called an $\mathsf{RCD}(K, \infty)$ *space*, $K \in \mathbb{R}$, if it is infinitesimally Hilbertian and, for every $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathscr{P}_2(\mathcal{X})$ with finite relative entropy, there exists a W_2-geodesic $\tilde{\mathfrak{m}}_t$ with $\mathrm{e}_0(\tilde{\mathfrak{m}}) = \mathfrak{m}_1$ and $\mathrm{e}_1(\tilde{\mathfrak{m}}) = \mathfrak{m}_2$ and such that, for every $t \in [0, 1]$,

$$\mathscr{E}_\mu(\tilde{\mathfrak{m}}_t) \leq (1-t)\mathscr{E}_\mu(\mathfrak{m}_1) + t\mathscr{E}_\mu(\mathfrak{m}_2) - \frac{K}{2} t(1-t) W_2^2(\mathfrak{m}_1, \mathfrak{m}_2),$$

where $(\mathscr{P}_2(\mathcal{X}), W_2)$ denotes the L^2 *Wasserstein Space*, namely the space of probability measures on $\mathcal{X}$ with finite *Wasserstein distance* W_2,

$$W_2^2(\mathfrak{m}_1, \mathfrak{m}_2) := \inf \int_{\mathcal{X}\times\mathcal{X}} d^2(x,y) \mathrm{d}\boldsymbol{\gamma}(x,y),$$

the infimum being taken among all $\boldsymbol{\gamma} \in \mathscr{P}(\mathcal{X} \times \mathcal{X})$ such that $\pi^1_\#(\boldsymbol{\gamma}) = \mathfrak{m}_1$ and $\pi^2_\#(\boldsymbol{\gamma}) = \mathfrak{m}_2$. Here, π^i, $i = 1, 2$, denotes the canonical projection over the i-th component.

6.1.3 Remark. If $(\mathcal{X}, d, \mu)$ is an $\mathsf{RCD}(K, \infty)$ space, then every ball $B_\rho(x) \in \mathscr{B}(\mathcal{X})$ satisfies the volume bound

$$\mu(B_\rho(x)) \le c \cdot e^{c\rho^2}$$

for some constant $c > 0$, see [St1]. This bound entails that if $\mathfrak{m} \in \mathscr{P}_2(\mathcal{X})$ with $\mathfrak{m} = \rho\mu$, then it is always true that $(\rho\log(\rho))^- \in L^1(\mu)$, which makes the relative entropy a lower semicontinuous functional.

6.1.4 Definition. The *heat flow* h_t is the gradient flow of the Cheeger-Dirichlet 2-energy $\mathbf{E}_2$.

6.1.5 Remark. As observed in [Gi2], the theory of gradient flows ensures the existence and uniqueness of the heat flow as a 1-parameter semigroup $(\mathrm{h}_t)_{t\ge 0}$, $\mathrm{h}_t : L^2(\mu) \to L^2(\mu)$, such that for every $f \in L^2(\mu)$ the curve $t \mapsto \mathrm{h}_t(f)$ is continuous on $[0, \infty[$, absolutely continuous on $]0, \infty[$ and moreover fulfills the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{h}_t(f) = \Delta f$$

for almost every $t > 0$, which means $\mathrm{h}_t(f) \in D(\Delta)$ for every $f \in L^2(\mu)$ and for every $t > 0$.

The infinitesimal Hilbertianity of $\mathsf{RCD}(K, \infty)$ spaces grants that, in our setting, $(\mathrm{h}_t)_{t\ge 0}$ defines a semigroup of linear and self-adjoint operators.

Also, from the analysis carried on in [AGS3], we have that for every $p \in [1, \infty]$ it holds

$$\|\mathrm{h}_t f\|_{L^p(\mathcal{X},\mu)} \le \|f\|_{L^p(\mathcal{X},\mu)} \tag{6.1}$$

for every $t \ge 0$ and for every $f \in L^2 \cap L^p(\mathcal{X}, \mu)$. Then, by a density argument we can uniquely extend the heat flow to a family of linear and continuous operators $\mathrm{h}_t : L^p(\mathcal{X}, \mu) \to L^p(\mathcal{X}, \mu)$ of norm bounded by 1 for every $p \in [1, \infty]$, as the contraction results proven in [AGS2] and [AGMR] showed.

A non trivial consequence of the Ricci curvature bound, again proven in [AGS2], is the following regularity result:

$$f \in W^{1,2}_{\boldsymbol{\pi}}(\mathcal{X}),\ |\mathrm{d}f|_* \in L^\infty(\mathcal{X}, \mu) \quad \Longrightarrow \quad \begin{array}{c} f \text{ has a Lipschitz representative } \tilde{f} \\ \text{with } \mathrm{Lip}\left(\tilde{f}\right) \le \||\mathrm{d}f|_*\|_{L^\infty(\mathcal{X},\mu)} . \end{array} \tag{6.2}$$

In regard to our discussion, the most important property of the heat flow is the *Bakry-Émery contraction estimate*:

$$|\nabla \mathrm{h}_t f|^2 \le e^{-2Kt}\mathrm{h}_t\left(|\nabla f|^2\right) \tag{6.3}$$

μ-almost everywhere for every $t \ge 0$ and for every $f \in W^{1,2}_{\boldsymbol{\pi}}(\mathcal{X})$ - see for example [AGS2] and [GKO] for a throughout treatment of this property.
A “self-improvement” of (6.3) was later given by G. Savaré in [Sa] - which generalizes the seminal work by D. Bakry [Ba] -, namely

$$|\nabla \mathrm{h}_t f| \le e^{-Kt}\mathrm{h}_t\left(|\nabla f|\right) \tag{6.4}$$

μ-almost everywhere for every $t \ge 0$ and for every $f \in W^{1,2}_{\boldsymbol{\pi}}(\mathcal{X})$.

Next, we introduce the anticipated “test objects”.

6.1.6 Definition. The class of *test functions* $\mathrm{TestF}(\mathcal{X}) \subset W^{1,2}_{\boldsymbol{\pi}}(\mathcal{X})$ is defined as

$$\mathrm{TestF}(\mathcal{X}) \coloneqq \left\{f \in D(\Delta) \cap L^\infty(\mathcal{X},\mu);\ |\nabla f| \in L^\infty(\mathcal{X},\mu),\ \Delta f \in W^{1,2}(\mathcal{X})\right\}.$$

From (6.2) one infers that any $f \in \mathrm{TestF}(\mathcal{X})$ has a Lipschitz representative $\tilde{f} : \mathcal{X} \to \mathbb{R}$ such that $\mathrm{Lip}\left(\tilde{f}\right) \le \||\nabla f|\|_{L^\infty(\mathcal{X},\mu)}$. Moreover, (6.3) implies that if $f \in L^2 \cap L^\infty(\mathcal{X},\mu)$, then $\mathrm{h}_t f \in \mathrm{TestF}(\mathcal{X})$ for every $t > 0$; as a notable byproduct of the latter, $\mathrm{TestF}(\mathcal{X})$ is dense in $W^{1,2}_{\boldsymbol{\pi}}(\mathcal{X})$.

6.1.7 Definition. By finite linear combinations of test functions and of their gradients, we obtain the class of *test vector fields* $\mathrm{TestV}(\mathcal{X}) \subset L^2(T\mathcal{X})$ as

$$\mathrm{TestV}(\mathcal{X}) \coloneqq \left\{X = \sum_{i=1}^n f_i \nabla g_i;\ n \in \mathbb{N},\ f_i, g_i \in \mathrm{TestF}(\mathcal{X})\ \forall\, i = 1, ..., n\right\}.$$

By the arguments in [Gi2, Section 3.2], $\mathrm{TestV}(\mathcal{X})$ is dense in $L^2(T\mathcal{X})$, while the properties of $\mathrm{TestF}(\mathcal{X})$ yield the inclusions $\mathrm{TestV}(\mathcal{X}) \subset L^1 \cap L^\infty(T\mathcal{X})$ and $\mathrm{TestV}(\mathcal{X}) \subset D(\mathrm{div})$.

6.2 The space $BV_{\mathbf{F}}(\mathcal{X})$

Thanks to the discussion carried forth in Chapter 5, we saw that we are given a notion of vector fields in the abstract metric setting. This argument suggests us the possibility of rephrasing the definition of BV in terms of suprema over the divergence of suitable vector fields, in accordance with the classical characterization of the Euclidean calculus.
For convenience, we shall suppose $(\mathcal{X}, d, \mu)$ to be an $\mathsf{RCD}(K, \infty)$ space; the motivation for this choice will be clear as our analysis goes on.

In order to avoid integrability issues, we slightly modify the definition of test vector fields as follows:

6.2.1 Definition. The class of ∞-*test vector fields* is defined as

$$\mathrm{T\textsc{est}V}^\infty(\mathcal{X}) := \left\{ X = \sum_{i=1}^{n} f_i \nabla g_i;\ f_i, g_i \in \mathrm{T\textsc{est}F}(\mathcal{X})\, \forall\, i = 1, ..., n,\ \Delta g_i \in L^\infty(\mu) \right\}.$$

According to the remarks in [Gi2, Section 3.2] the class $\mathrm{T\textsc{est}V}^\infty(\mathcal{X})$ is weakly-* dense in $L^\infty(T\mathcal{X})$.

6.2.2 Definition. A function $u \in L^1(\mathcal{X}, \mu)$ will be said of *bounded variation* in $\mathcal{X}$, and we shall write $u \in BV_{\mathbf{F}}(\mathcal{X})$, if its *total variation* in $\mathcal{X}$, namely

$$\|Du\|(\mathcal{X}) := \sup\left\{ \int_{\mathcal{X}} u \operatorname{div}(X) \mathrm{d}\mu;\ X \in \mathrm{T\textsc{est}V}^\infty(\mathcal{X}),\ |X| \le 1 \right\},$$

is finite. The notation is reminiscent of the fact that this definition is tailored on vector fields, whence our choice of the label $\mathbf{F}$.

As one may expect, the *perimeter* of any measurable set $E \subset \mathcal{X}$ will be defined in the usual way, namely as the total variation of the characteristic function $\mathbb{1}_E$ on $\mathcal{X}$:

$$\|D\mathbb{1}_E\|(\mathcal{X}) = \sup\left\{ \int_{\mathcal{X}} \operatorname{div}(X) \mathrm{d}\mu;\ X \in \mathrm{T\textsc{est}V}^\infty(\mathcal{X}),\ |X| \le 1 \right\}.$$

6.2.3 Remark. *i)* The $\mathsf{RCD}(K, \infty)$ structure actually allows us to define functions of bounded variation also via a relaxation procedure over sequences of $\mathrm{T\textsc{est}F}(\mathcal{X})$ functions. Namely, $u \in BV(\mathcal{X})$ whenever

$$\|Du\|(\mathcal{X}) := \inf\left\{ \liminf_{j\to+\infty} \int_{\mathcal{X}} |\nabla f_j| \,\mathrm{d}\mu;\ (f_j)_{j\in\mathbb{N}} \subset \mathrm{T\textsc{est}F}(\mathcal{X}),\ f_j \to u \text{ in } L^1_{\mathrm{loc}}(\mathcal{X}, \mu) \right\} < \infty. \tag{6.5}$$

Indeed, let us consider

$$\|Du\|(\mathcal{X}) = \inf\left\{ \liminf_{j\to+\infty} \int_{\mathcal{X}} |\nabla f_j| \,\mathrm{d}\mu;\ (f_j)_{j\in\mathbb{N}} \subset \mathrm{Lip}_b(\mathcal{X}),\ f_j \to u \text{ in } L^1_{\mathrm{loc}}(\mathcal{X}, \mu) \right\}. \tag{6.6}$$

and let us label the quantities in (6.5) and (6.6) by $\text{T-Var}_u(\mathcal{X})$ and $\mathrm{Var}_u(\mathcal{X})$ respectively.

The remarks right after Definition 6.1.6 imply that the class of test functions is contained in $\mathrm{Lip}_b(\mathcal{X})$ and then we obviously have $\mathrm{Var}_u(\mathcal{X}) \le \text{T-Var}_u(\mathcal{X})$; we need to prove the opposite inequality.

Choose $(f_j)_{j\in\mathbb{N}}$ as in the definition of $\mathrm{Var}_u(\mathcal{X})$; then $\mathrm{h}_t f_j \in \mathrm{T\textsc{est}F}(\mathcal{X})$ for all $j \in \mathbb{N}$. Now fix a positive decreasing sequence $(t_j)_{j\in\mathbb{N}}$ such that $t_j \to 0$ and define $g_j := \mathrm{h}_{t_j} f_j$, which again defines a test function for every index j. Moreover, it is clear that $g_j \to u$ in $L^1_{\mathrm{loc}}(\mathcal{X}, \mu)$, a convergence which is guaranteed by the properties of the heat flow.

Now,

$$
\begin{aligned}
\text{T-Var}_u(\mathcal{X}) &\le \liminf_{j\to+\infty} \int_{\mathcal{X}} |\nabla g_j| \,\mathrm{d}\mu \\
&= \liminf_{j\to+\infty} \int_{\mathcal{X}} \left|\nabla \mathrm{h}_{t_j} f_j\right| \mathrm{d}\mu \\
&\le \liminf_{j\to+\infty} \left(e^{-Kt_j} \int_{\mathcal{X}} |\nabla f_j| \,\mathrm{d}\mu \right),
\end{aligned}
$$

where we explicitly used the self-improvement of the Bakry-Émery contraction estimate. Thus, upon passing to the infimum we obtain $\text{T-Var}_u(\mathcal{X}) \le \mathrm{Var}_u(\mathcal{X})$, which shows the two notions to be equivalent.

Of course, in accordance with Theorem 4.3.5 the formula (6.5) defines an equivalent characterization of the total variation given in [Di1].

ii) By Remark 5.5.22, the equivalence shown in *i)* extends to $BV_{\mathbf{F}}(\mathcal{X})$ if one additionally assume the space to be equipped with a doubling measure and to support a Poincaré inequality, because in such case the notions of derivation given in [Gi2] and [Di1] coincide. However, it is not clear to us if, without assuming also the doubling property and a Poincaré inequality, the $\mathsf{RCD}(K,\infty)$ hypothesis alone is enough to ensure an equivalence between the two definitions of derivation. It is known anyways ([LV2], [St1], [St2]) that conditions like $\mathsf{CD}(K,N)$ or $\mathsf{CD}(K,\infty)$ imply Poincaré inequalities, at least of local nature; a sharp global Poincaré inequality was derived in [LV1] for $\mathsf{CD}(K,N)$ spaces.

iii) Of course, one may attempt to define BV functions on a domain $\Omega \subset \mathcal{X}$. In line with Theorem 4.3.4, one modify the class of $\textsc{TestV}^\infty(\mathcal{X})$ vector fields and consider

$$
\mathrm{LipV}_{b,c}(\Omega) \coloneqq \left\{ X = \sum_{i=1}^n f_i \nabla g_i; f_i \in \mathrm{Lip}_b(\Omega), \mathrm{supp}(f) \Subset \Omega, g_i \in \textsc{TestF}(\mathcal{X}), \Delta g_i \in L^\infty(\mu) \right\}
$$

in order to state that $u \in L^1(\Omega,\mu)$ is in $BV_{\mathbf{F}}(\Omega)$ whenever

$$
\|Du\|(\Omega) \coloneqq \sup \left\{ \int_\Omega u \mathrm{div}(X) \mathrm{d}\mu;\ X \in \textsc{TestV}_{b,c}(\Omega),\ |X| \le 1 \right\} < \infty.
$$

With this characterization, we would easily have the Coarea Formula, whose proof would just be the same as in [EG, Theorem 5.9]. However, we are faced again with the same issues as in *ii)*, so it is not enough clear to us which would be the most appropriate choice of vector fields to work with.

7 Traces and Gauss-Green Formulæ

In this concluding chapter, we consider the issue of traces of BV functions in metric measure spaces, with the aim of finding the appropriate classes of domains where Gauss-Green type formulæ hold.

7.1 Integration by parts on Regular Domains

We shall begin with the characterization of "regular domains", a class of open sets which improves the notion of "regular ball" given in [MMS] and allows for an extended Gauss-Green formula which in turn refines the analogue established therein.

Let us start with some preliminary considerations.
We shall assume $(\mathcal{X}, d, \mu)$ to be a complete metric measure space with the property of being geodesic. In other words, for every $x, y \in \mathcal{X}$, one has

$$d(x,y) = \inf\left\{\ell(\gamma);\ \gamma : [0,1] \to \mathcal{X},\ \gamma_0 = x,\ \gamma_1 = y\right\}.$$

If this assumption is satisfied, it is possible to prove that the distance function

$$d_{x_0}(x) = d(x, x_0), \qquad \forall\, x_0 \in \mathcal{X}$$

is a Lipschitz function with upper gradient given by

$$\left|\nabla d_{x_0}\right|(x) = 1$$

μ-almost everywhere in $\mathcal{X}$.

7.1.1 Definition. Given a sequence of finite Radon measures $(\mu_j)_{j\in\mathbb{N}}$ on $\mathcal{X}$, we shall say that $(\mu_j)_{j\in\mathbb{N}}$ is *weakly convergent* in the sense of measures to a Radon measure μ if

$$\lim_{j\to+\infty} \int_{\mathcal{X}} f d\mu_j = \int_{\mathcal{X}} f d\mu, \qquad \forall f \in C_b(\mathcal{X}).$$

7.1.2 Remark. The above notion of convergence is the natural generalization of the weakly-* convergence in the duality of $C(K)$ with its dual $\mathbf{M}(K)$ for any compact set $K \subset \mathcal{X}$.
The duality of $C(K)$ with $\mathbf{M}(K)$ allows us to deduce that for a Radon measure $\mu \in \mathbf{M}(\mathcal{X})$, its total variation in $\mathcal{X}$ is given by

$$|\mu|(\mathcal{X}) = \sup\left\{\int_{\mathcal{X}} f(x)\mu(dx);\ f \in C_b(\mathcal{X}),\ |f(x)| \le 1,\ \forall\, x \in \mathcal{X}\right\}.$$

Since bounded Lipschitz functions, $\mathrm{Lip}_b(\mathcal{X})$, constitute a subalgebra of $C_b(\mathcal{X})$ and its restriction to K for any compact set $K \subset \mathcal{X}$ is a subalgebra of $C(K)$ containing a non–zero constant function and which separates points, it turns out that the restiction of $\mathrm{Lip}_b(\mathcal{X})$ to K is dense in $C(K)$ and then we also obtain that

$$|\mu|(\mathcal{X}) = \sup\left\{\int_{\mathcal{X}} f(x)\mu(dx);\ f \in \mathrm{Lip}_b(\mathcal{X}),\ |f(x)| \leq 1,\ \forall\, x \in \mathcal{X}\right\}.$$

Of course, this argument can be also extended to Radon measures on open domains $\Omega \subset \mathcal{X}$ to deduce that for any $\mu \in \mathbf{M}(\Omega)$

$$|\mu|(\Omega) = \sup\left\{\int_{\Omega} f(x)\mathrm{d}\mu(x);\ f \in \mathrm{Lip}_{b,c}(\Omega),\ |f(x)| \leq 1,\ \forall\, x \in \mathcal{X}\right\},$$

where

$$\mathrm{Lip}_{b,c}(\Omega) \coloneqq \{f \in \mathrm{Lip}_b(\mathcal{X});\ \mathrm{dist}\,(\mathrm{supp}(f), \Omega^c) > 0\}.$$

With these premises at our disposal, we proceed by giving a definition of regularity on domains that will allow us to define some integration by parts formula. To begin, we recall the notion of inner Minkowski content of a set; if we set

$$\Omega_t \coloneqq \{x \in \Omega;\ \mathrm{dist}\,(x, \Omega^c) \geq t\},$$

$t > 0$, then we define

$$\mathfrak{M}_i(\partial\Omega) \coloneqq \limsup_{t\to 0} \frac{\mu\,(\Omega \setminus \Omega_t)}{t}.$$

7.1.3 Definition. An open set $\Omega \subset \mathcal{X}$ is said to be a *regular domain* if it has finite perimeter in $\mathcal{X}$, namely $\mathbb{1}_\Omega \in BV(\mathcal{X})$ and if

$$\|D\mathbb{1}_\Omega\|\,(\mathcal{X}) = \mathfrak{M}_i(\Omega).$$

A first easy consequence of the regularity of the domain Ω is that, since by the lower semicontinuity of the perimeter

$$\|D\mathbb{1}_\Omega\|\,(\mathcal{X}) \leq \liminf_{t\to 0} \frac{\mu\,(\Omega \setminus \Omega_t)}{t},$$

one then has that there exists

$$\lim_{t\to 0} \frac{\mu\,(\Omega \setminus \Omega_t)}{t} = \|D\mathbb{1}_\Omega\|\,(\mathcal{X}).$$

Examples of regular domains are given by balls; it is known that for any $x_0 \in \mathcal{X}$ and for almost every $\rho > 0$, $\Omega = B_\rho(x_0)$ is a regular domain.
We have the following result:

7.1.4 Proposition. Let $\Omega \subset \mathcal{X}$ be a regular domain. Then, $\|D\mathbb{1}_{\Omega_t}\| \rightharpoonup \|D\mathbb{1}_\Omega\|$ (in the sense of measures).

PROOF. We have to prove that, for any $f \in C_b(\mathcal{X})$,

$$\lim_{t\to 0} \int_{\mathcal{X}} f(x) \mathrm{d} \left\| D\mathbb{1}_{\Omega_t} \right\| (x) = \int_{\mathcal{X}} f(x) \mathrm{d} \left\| D\mathbb{1}_{\Omega} \right\| (x).$$

Thanks to [Bo, Theorem 8.2.9], it is sufficient to prove that for any Borel set E such that $\left\| D\mathbb{1}_{\Omega} \right\| (\partial E) = 0$, one has

$$\lim_{t\to 0} \left\| D\mathbb{1}_{\Omega_t} \right\| (E) = \left\| D\mathbb{1}_{\Omega} \right\| (E).$$

We denote by A the interior of E, namely $A = \mathring{E} = E \backslash \partial E$, and by C its closure, $C = \overline{E} = E \cup \partial E$. We obviously have that $A \subset E \subset C$ and $\mu(C \setminus A) = 0$.
We claim that

$$\lim_{t\to 0} \frac{\mu\left((\Omega \backslash \Omega_t) \cap A\right)}{t} = \left\| D\mathbb{1}_{\Omega} \right\| (A);$$

indeed, by Coarea Formula and by the fact that the function $g(x) = \mathrm{dist}\,(x, \Omega^c)$ is Lipschitz with $|\nabla g(x)| = 1$, since $\Omega_t = \{g > t\}$ we obtain

$$\begin{aligned}
\frac{\mu\left((\Omega \setminus \Omega_t) \cap E\right)}{t} &= \frac{1}{t} \int_{(\Omega \backslash \Omega_t) \cap A} \mathrm{d}\mu(x) \\
&= \frac{1}{t} \int_{(\Omega \backslash \Omega_t) \cap A} |\nabla g(x)| \mathrm{d}\mu(x) \\
&= \frac{1}{t} \int_{\mathbb{R}} \left\| D\mathbb{1}_{\{g>s\}} \right\| \left((\Omega \backslash \Omega_t) \cap A\right) \mathrm{d}s \\
&= \frac{1}{t} \int_0^t \left\| D\mathbb{1}_{\Omega_s} \right\| (A) \mathrm{d}s \\
&= \int_0^1 \left\| D\mathbb{1}_{\Omega_{st}} \right\| (A) \mathrm{d}s.
\end{aligned}$$

Hence, by Fatou Lemma and by the lower semicontinuity of the perimeter measure, using the fact that Ω_{st} converges to Ω in measure we get

$$\liminf_{t\to 0} \frac{\mu\left((\Omega \backslash \Omega_t) \cap A\right)}{t} \geq \left\| D\mathbb{1}_{\Omega} \right\| (A).$$

The same argument can be done to prove that

$$\liminf_{t\to 0} \frac{\mu\left((\Omega \backslash \Omega_t) \cap \overline{E}^c\right)}{t} \geq \left\| D\mathbb{1}_{\Omega} \right\| \left(\overline{E}^c\right) = \left\| D\mathbb{1}_{\Omega} \right\| (E^c).$$

Hence, by writing

$$\frac{\mu\left((\Omega \backslash \Omega_t) \cap A\right)}{t} = \frac{\mu\left(\Omega \backslash \Omega_t\right)}{t} - \frac{\mu\left((\Omega \backslash \Omega_t) \cap A^c\right)}{t} \leq \frac{\mu\left(\Omega \backslash \Omega_t\right)}{t} - \frac{\mu\left((\Omega \backslash \Omega_t) \cap E^c\right)}{t},$$

we get that

$$\limsup_{t\to 0} \frac{\mu\left((\Omega \backslash \Omega_t) \cap A\right)}{t} \leq \left\| D\mathbb{1}_{\Omega} \right\| (\mathcal{X}) - \left\| D\mathbb{1}_{\Omega} \right\| (E^c) = \left\| D\mathbb{1}_{\Omega} \right\| (A),$$

which means there exists

$$\lim_{t\to 0}\frac{\mu\left((\Omega\backslash\Omega_t)\cap A\right)}{t}=\left\|D\mathbb{1}_\Omega\right\|(A)$$

for any $A\subset\mathcal{X}$ with $\left\|D\mathbb{1}_\Omega\right\|(\partial A)=0$.
In conclusion, since for any $s\in[0,1]$

$$\left\|D\mathbb{1}_\Omega\right\|(A)\le\liminf_{t\to 0}\left\|D\mathbb{1}_{\Omega_{st}}\right\|(A),$$

we find

$$\left\|D\mathbb{1}_\Omega\right\|(A)\le\liminf_{t\to 0}\int_0^1\left\|D\mathbb{1}_{\Omega_{st}}\right\|(A)\mathrm{d}s=\liminf_{t\to 0}\frac{\mu\left((\Omega\backslash\Omega_t)\cap A\right)}{t}=\left\|D\mathbb{1}_\Omega\right\|(A).$$

This implies that for any $s\in[0,1]$ there exists

$$\lim_{t\to 0}\left\|D\mathbb{1}_{\Omega_{st}}\right\|(A)=\left\|D\mathbb{1}_\Omega\right\|(A),$$

whence the existence of

$$\lim_{t\to 0}\left\|D\mathbb{1}_{\Omega_t}\right\|(A)=\left\|D\mathbb{1}_\Omega\right\|(A).$$

□

7.1.5 Remark. We notice that from Proposition 7.1.4, if we define

$$\varphi_\varepsilon(x):=\begin{cases}0, & x\in\Omega^c\\ \frac{\mathrm{dist}(x,\Omega^c)}{\varepsilon}, & x\in\Omega\backslash\Omega_\varepsilon\\ 1, & x\in\Omega_\varepsilon,\end{cases}\tag{7.1}$$

since

$$\int_A\left|\nabla\varphi_\varepsilon\right|(x)\mathrm{d}\mu(x)=\frac{\mu\left((\Omega\backslash\Omega_\varepsilon)\cap A\right)}{\varepsilon},$$

then also the measures $\mu_\varepsilon:=\left|\nabla\varphi_\varepsilon\right|\mu$ are weakly convergent - always in the sense of measures - to $\left\|D\mathbb{1}_\Omega\right\|$.
The φ_ε's define a sequence $(\varphi_\varepsilon)_{\varepsilon>0}\subset\mathrm{Lip}_c(\Omega)$, which will be also called the *defining sequence* of a regular domain Ω.

In the manner of [MMS], we wish to prove an integration by parts (or better, a Gauss-Green) formula on regular domains, making use of “divergence measure” vector fields; in order to do so, let us start with the following definition:

7.1.6 Definition. We denote by $\mathcal{DM}^\infty(\mathcal{X})$ the set of *divergence measure vector fields*, namely the class of vector fields $F\in L^\infty(T\mathcal{X})$ such that $\mathrm{div}(F)$ is a measure in the distributional sense, that is, there exists a measure $\mu_F\in\mathbf{M}(\mathcal{X})$ such that

$$\int_{\mathcal{X}} \mathrm{d}\varphi(F)\mathrm{d}\mu = -\int_{\mathcal{X}} \varphi d\mu_F, \qquad \forall\, \varphi \in \mathrm{Lip}_b(\mathcal{X}).$$

7.1.7 Theorem. Let $F \in \mathcal{DM}^\infty(\mathcal{X})$ and let $\Omega \subset \mathcal{X}$ be a regular domain; then, there exists a function - which we shall call the *inner normal trace* of F on $\partial\Omega$ - denoted by $(F \cdot \nu)^-_{\partial\Omega} \in L^1(\partial\Omega, \|D\mathbb{1}_\Omega\|)$ such that the following extended Gauss-Green formula holds:

$$\int_\Omega f(x)\mathrm{d}\mu_F(x) + \int_\Omega \mathrm{d}f(F)(x)\mathrm{d}\mu(x) = \int_{\mathcal{X}} f(x)(F\cdot\nu)^-_{\partial\Omega}(x)\mathrm{d}\,\|D\mathbb{1}_\Omega\|\,(x), \tag{7.2}$$

for every $f \in \mathrm{Lip}_b(\mathcal{X})$.

Proof. We shall consider the defining sequence $(\varphi_\varepsilon)_{\varepsilon>0}$ as in (7.1). Then

$$\begin{aligned}\int_{\mathcal{X}} f(x)\mathrm{d}\varphi_\varepsilon(F)(x)\mathrm{d}\mu(x) &= \int_{\mathcal{X}} \mathrm{d}\left(f\varphi_\varepsilon\right)(F)(x)\mathrm{d}\mu(x) - \int_{\mathcal{X}} \varphi_\varepsilon(x)\mathrm{d}f(F)(x)\mathrm{d}\mu(x)\\ &= -\int_{\mathcal{X}} \varphi_\varepsilon(x)f(x)\mathrm{d}\mu_F(x) - \int_{\mathcal{X}} \varphi_\varepsilon(x)\mathrm{d}f(F)(x)\mathrm{d}\mu(x).\end{aligned}$$

By the fact that φ_ε converges to $\mathbb{1}_\Omega$ everywhere, by the Dominated Convergence Theorem we obtain that there exists

$$\lim_{\varepsilon\to 0}\int_{\mathcal{X}} f(x)\mathrm{d}\varphi_\varepsilon(F)(x)\mathrm{d}\mu(x) = -\int_\Omega f(x)\mathrm{d}\mu_F(x) - \int_\Omega \mathrm{d}f(F)(x)\mathrm{d}\mu(x).$$

We have then defined, for any $f \in \mathrm{Lip}_b(\mathcal{X})$, the distribution

$$T_F(f) = \lim_{\varepsilon\to 0} T_F^\varepsilon(f),$$

where

$$T_F^\varepsilon(f) = \int_{\mathcal{X}} f(x)\mathrm{d}\varphi_\varepsilon(F)(x)\mathrm{d}\mu(x).$$

Since we have the estimate

$$|T_F^\varepsilon(f)| \le \|f\|_{L^\infty(\mu)}\|F\|_{L^\infty(T\mathcal{X})}\int_{\mathcal{X}} |\nabla\varphi_\varepsilon|\,\mathrm{d}\mu,$$

we deduce

$$|T_F(f)| \le \|f\|_\infty \|F\|_{L^\infty(T\mathcal{X})}\,\|D\mathbb{1}_\Omega\|\,(\mathcal{X}).$$

From the above we infer the existence of $\sigma_F^\nu \in \mathbf{M}(\mathcal{X})$ such that

$$T_F(f) = \int_{\mathcal{X}} f(x)\mathrm{d}\sigma_F^\nu(x).$$

We claim that σ_F^ν is concentrated on $\partial\Omega$ and that $\sigma_F \ll \|D\mathbb{1}_\Omega\|$. The first assertion follows since if $K \subset \mathcal{X}$ is a compact set such that $K \cap \partial\Omega = \emptyset$, then as $r = \mathrm{dist}(K, \partial\Omega) > 0$, there exists an open set A such that $K \subset A$, $\mathrm{dist}(A, \partial\Omega) > 0$ and $P(\Omega, \partial A) = 0$. Indeed we can consider

$$K \subset \bigcup_{x \in K} B_{\rho(x)}(x),$$

where $0 < \rho(x) < r/2$ is such that $\|D\mathbb{1}_\Omega\| \left(\partial B_{\rho(x)}(x)\right) = 0$ for all $x \in K$. By compactness, there exists a finite number of balls $B_i := B_{\rho((x_i)}(x_i)$, $i = 1, ..., m$, such that

$$K \subset A := \bigcup_{i=1}^m B_i.$$

We then have that if $\varepsilon < r/2$,

$$\int_A |\nabla\varphi_\varepsilon| \mathrm{d}\mu = 0.$$

Thus, for any $f \in \mathrm{Lip}_{b,c}(A)$,

$$\int_A f \mathrm{d}\sigma_F^\nu = \int_{\mathcal{X}} f \mathrm{d}\sigma_F = \lim_{\varepsilon \to 0} \int_{\mathcal{X}} f \mathrm{d}\varphi_\varepsilon(F) \mathrm{d}\mu = 0.$$

In conclusion, we get $|\sigma_F^\nu|(A) = 0$, whence

$$|\sigma_F^\nu(K)| = 0$$

for every compact set K such that $K \cap \partial\Omega = \emptyset$.

Let us now turn to absolute continuity. If K is a Borel set with $\|D\mathbb{1}_\Omega\|(K) = 0$, we have that for any $\eta > 0$ there exists an open set A_η with $\|D\mathbb{1}_\Omega\|(A_\eta) < \eta$ and $\|D\mathbb{1}_\Omega\|(\partial A_\eta) = 0$. Then, for any $f \in \mathrm{Lip}_{b,c}(A_\eta)$,

$$\begin{aligned}\left|\int_{A_\eta} f \mathrm{d}\sigma_F^\nu\right| &\leq \lim_{\varepsilon \to 0} \|f\|_{L^\infty(\mu)} \|F\|_{L^\infty(T\mathcal{X})} \int_{A_\eta} |\nabla\varphi_\varepsilon| \,\mathrm{d}\mu \\ &= \|f\|_{L^\infty(\mu)} \|F\|_{L^\infty(T\mathcal{X})} \|D\mathbb{1}_\Omega\|(A_\eta) < \eta \|f\|_{L^\infty(\mu)} \|F\|_{L^\infty(T\mathcal{X})}.\end{aligned}$$

Hence $\left|\sigma_f^\nu\right|(K) = 0$ for any compact set K such that $\|D\mathbb{1}_\Omega\|(K) = 0$; the fact that the measures are Radon implies the absolute continuity $\sigma_F^\nu \ll \|D\mathbb{1}_\Omega\|$. Thus, by the Radon-Nikodym Theorem there exists a function $(F \cdot \nu)^-_{\partial\Omega} \in L^1(\partial\Omega, \|D\mathbb{1}_\Omega\|)$ such that

$$\int_{\mathcal{X}} f(x)(F \cdot \nu)^-_{\partial\Omega}(x) \mathrm{d}\|D\mathbb{1}_\Omega\|(x) = -\int_\Omega \mathrm{d}f(F)(x) \mathrm{d}\mu(x) - \int_\Omega f(x) \mathrm{d}\mu_F(x).$$

□

The following Proposition is useful to extend the results contained in [MMS] to the case $f \in W^{1,2}_\pi(\mathcal{X})$:

7.1.8 Proposition. Let $F \in \mathcal{DM}^\infty(\mathcal{X})$ and let $B \subset \mathcal{X}$ be a Borel set such that $\mathrm{Cap}_2(B) = 0$; then $|\mu_F|(B) = 0$.

PROOF. We follow the same approach of the proof given in [MMS, Lemma 3.6] for the measure-valued Laplacian. Since μ_F is a signed measure, we can consider its positive and negative parts, namely $\mu_F = \mu_F^+ - \mu_F^-$; we can also decompose $\mathcal{X}$ into two parts $\mathcal{X} = P \cup N$ in such a way that

$$\mu_F^+(E) \geq 0, \qquad \forall E \subset P,$$

and

$$\mu_F^-(E) \geq 0, \qquad \forall E \subset N.$$

Without loss of generality, we may assume B to be a compact set, because both μ_F^+ and μ_F^- are inner measures (as Radon measures) and B is a Borel set.
Let us show that if $B \subset P$ is such that $\mathrm{Cap}_2(B) = 0$, then $\mu_F^+(B) = 0$; We can take a sequence $(\psi_n)_{n\in\mathbb{N}} \in \mathrm{Lip}_b(\Omega)$ such that

$$\mathrm{supp}\,(\psi_n) \subset \bigcup_{x\in B} B_{\frac{1}{n}}(x),$$

$0 \leq \psi_n \leq 1$, $\psi_n = 1$ on B and

$$\|\psi_n\|_{W^{1,2}_{\pi}(\mathcal{X})} \leq \frac{1}{2^n}.$$

With this choice, $\psi_n \to 0$ on $\mathcal{X}\backslash B$ and

$$\left| \int_{\mathcal{X}} \psi_n(x) \mathrm{d}\mu_F(x) \right| = \left| \int_{\mathcal{X}} \mathrm{d}\psi_n(F) \mathrm{d}\mu \right| \leq \|F\|_{L^\infty(T\mathcal{X})} \|\psi_n\|_{W^{1,2}_{\pi}(\mathcal{X})} \mu(\Omega)^{\frac{1}{2}}.$$

Thus, by the Dominated Convergence Theorem

$$0 = \lim_{n\to+\infty} \int_{\mathcal{X}} \psi_n(x) \mathrm{d}\mu_F(x) = \mu_F(B) = \mu_F^+(B).$$

A similar argument shows that if $B \subset N$ is such that $\mathrm{Cap}_2(B) = 0$, then $\mu_F^-(B) = 0$. For the general case, if $B \subset \mathcal{X}$ is such that $\mathrm{Cap}_2(B) = 0$, by decomposing B as $B = (B \cap P) \cup (B \cap N)$, the monotonicity of capacity yields $\mathrm{Cap}_2(B \cap P) = \mathrm{Cap}_2(B \cap N) = 0$ and then

$$|\mu_F|(B) = \mu_F^+(B \cap P) + \mu_F^-(B \cap N) = 0.$$

□

The validity of Theorem 7.1.7 can be extended to any function $f \in N^{1,2}(\mathcal{X})$ in the case $\mathcal{X}$ is a doubling metric measure space supporting a Poincaré inequality; indeed, in this case Theorem 2.4.3 holds and it becomes possible to approximate any $f \in N^{1,2}(\mathcal{X})$ by Lipschitz functions f_j converging to f everywhere except for a set of zero 2–capacity. We repeat here this approximation argument in the Sobolev space $W^{1,2}_{\pi}(\mathcal{X})$.

7.1.9 Proposition. Let $f \in W^{1,2}_{\pi}(\mathcal{X})$; if $(f_j)_{j\in\mathbb{N}} \subset \mathrm{TESTF}(\mathcal{X})$ is a sequence such that $f_j \to f$ in $W^{1,2}_{\pi}(\mathcal{X})$, then there exists a Borel set $B \subset \mathcal{X}$ with $\mathrm{Cap}_2(B) = 0$ such that, up to subsequences, $f_j \to \tilde{f}$ everywhere on $\mathcal{X} \setminus B$, $\tilde{f}$ being a $W^{1,2}_{\pi}(\mathcal{X})$ representative of f.

PROOF. The proof is essentially a repetition of the one contained in [Sh2, Theorem 3.7]; since the sequence $(f_j)_{j\in\mathbb{N}}$ is converging in $W^{1,2}_{\boldsymbol{\pi}}(\mathcal{X})$, it is a Cauchy sequence. Passing to a subsequence if necessary, we may assume that for any $k \in \mathbb{N}$,

$$\|f_k - f_{k+1}\|_{W^{1,2}_{\boldsymbol{\pi}}(\mathcal{X})} \le 2^{-3k/2}, \qquad \||\nabla (f_k - f_{k+1})|\|_{L^2(\mathcal{X},\mu)} \le 2^{-k}.$$

Let us define the set

$$E_k := \left\{ |f_k - f_{k+1}| \ge 2^{-k} \right\}.$$

By the definition of capacity, we then have

$$\mathrm{Cap}_2 (E_k) \le 2^{2k} \|f_k - f_{k+1}\|^2_{W^{1,2}_{\boldsymbol{\pi}}(\mathcal{X})} \le 2^{-k}.$$

So if we define

$$B_j := \bigcup_{k=j}^{\infty} E_k,$$

the properties of capacity give us

$$\mathrm{Cap}_2 (B_j) \le \sum_{j=k}^{\infty} \mathrm{Cap}_2 (E_k) \le 2^{-j+1}.$$

Let us now set

$$B := \bigcap_{j\in\mathbb{N}} B_j.$$

By monotonicity, we find $\mathrm{Cap}_2(B) = 0$. If $x \in \mathcal{X}\backslash B$, then there exists $j \in \mathbb{N}$ such that $x \in \mathcal{X} \setminus B_j$ and so, for every $k \ge j$, $x \in \mathcal{X} \setminus E_k$ and then

$$|f_k(x) - f_{k+1}(x)| \le 2^{-k+1} \le 2^{-j+1}.$$

As a consequence, if $h, k \ge j$, we have that

$$|f_h(x) - f_k(x)| \le 2^{-j+1},$$

which in turn entails

$$\lim_{j\to+\infty} f_j(x) = \tilde{f}(x)$$

for every $x \in \mathcal{X}\backslash B$.

□

7.1.10 Remark. Observe that a particular subclass of elements in $\mathcal{DM}^\infty(\mathcal{X})$ is given by $\mathrm{T\small{EST}}\mathrm{V}(\mathcal{X})$. Let us discuss the particular case $F = f\nabla g$; in this case clearly $F \in L^\infty(T\mathcal{X})$ and regarding the divergence measure we find that it is absolutely continuous with respect to μ with

$$\mathrm{d}\mu_F(x) = (\mathrm{d}f(\nabla g) + f\Delta g)\mathrm{d}\mu(x) = \mathrm{div}(F)\mathrm{d}\mu(x).$$

Hence, we have the following integration by parts formula

$$\int_\Omega \varphi(x)\mathrm{div}F(x)\mathrm{d}\mu(x) = -\int_{\mathcal{X}} \varphi(x)(F\cdot\nu)^-_{\partial\Omega}\mathrm{d}\left\|D\mathbb{1}_\Omega\right\|(x) - \int_\Omega \mathrm{d}\varphi(F)(x)\mathrm{d}\mu(x)$$

for any $\varphi \in \mathrm{Lip}_b(\mathcal{X})$.

We want to discuss now how to generalize this formula to functions $f \in W^{1,2}_{\boldsymbol{\pi}}(\mathcal{X})$; in order to do so, we consider again the following subclass of $\textsc{TestV}(\mathcal{X})$,

$$\textsc{TestV}^\infty(\mathcal{X}) := \left\{X = \sum_{i=1}^n f_i \nabla g_i;\ f_i, g_i \in \textsc{TestF}(\mathcal{X})\ \forall\, i = 1, ..., n,\ \Delta g_i \in L^\infty(\mu)\right\},$$

already introduced in Definition 6.2.1. For the reader's convenience, we explicitly observe once again that, when $(\mathcal{X}, d, \mu)$ is an $\mathsf{RCD}(K,\infty)$ space, the above class is weakly-* dense in $L^\infty(T\mathcal{X})$ by the remarks in [Gi2, Section 3.2].

Using the same notation as in [Di1], we have the following:

7.1.11 Lemma. Let Ω be a set with finite perimeter and such that $\mu(\Omega) < +\infty$, let $F \in \textsc{TestV}^\infty(\mathcal{X})$; then, if B is any Borel set with $\mathrm{Cap}_2(B) = 0$, $\left|L_\Omega(\mathfrak{d}_F)\right|(B) = 0$.

Proof. We can use the fact that any $F \in \textsc{TestV}^\infty(\mathcal{X})$ defines a derivation $\mathfrak{d}_F \in \textsc{Der}_b(\mathcal{X})$,

$$\mathfrak{d}_F(\varphi) := \mathrm{d}\varphi(F),$$

where φ is any function in $\mathrm{Lip}_0(\mathcal{X})$. Then, we may consider a sequence $(\psi_n)_{n\in\mathbb{N}} \subset \mathrm{Lip}_b(\mathcal{X})$ as in the proof of Proposition 7.1.8 to obtain

$$\begin{aligned}\int_{\mathcal{X}} \psi_n(x)\mathrm{d}L_\Omega(\mathfrak{d}_F)(x) &= \int_{\mathcal{X}} \mathrm{d}L_\Omega(\psi_n\mathfrak{d}_F)(x) \\ &= -\int_\Omega \mathrm{d}\psi_n(F)(x)\mathrm{d}\mu(x) - \int_\Omega \psi_n(x)\mathrm{div}F(x)\mathrm{d}\mu(x),\end{aligned}$$

which in turn gives

$$\left|\int_{\mathcal{X}} \psi_n(x)\mathrm{d}L_\Omega(\mathfrak{d}_F)(x)\right| \le \|\psi_n\|_{W^{1,2}_{\boldsymbol{\pi}}(\mathcal{X})}\left(\|F\|_{L^\infty(T\mathcal{X})} + \|\mathrm{div}(F)\|_{L^\infty(\mathcal{X},\mu)}\right)\mu(\Omega)^{\frac{1}{2}}.$$

From this we infer

$$\left|L_\Omega(\mathfrak{d}_F)\right|(B) = 0$$

whenever $\mathrm{Cap}_2(B) = 0$.

□

Putting together the previous results, we have thus proven the following:

7.1.12 Proposition. Let Ω be a regular domain with $\mu(\Omega) < +\infty$; then for any $F \in \mathrm{TestV}^{\infty}(\mathcal{X})$ it holds

$$\int_{\Omega} \varphi(x)\mathrm{div}F(x)\mathrm{d}\mu(x) = \int_{\mathcal{X}} \varphi(x)\mathrm{d}L_{\Omega}\left(\mathfrak{d}_F\right)(x) - \int_{\Omega} \mathrm{d}\varphi(F)(x)\mathrm{d}\mu(x), \qquad \forall \varphi \in W^{1,2}_{\boldsymbol{\pi}}(\mathcal{X}).$$

□

7.2 Rough Trace

In this section, we shall assume $(\mathcal{X}, d, \mu)$ to be a doubling metric measure space supporting a Poincaré inequality. After V. Maz'ya ([Ma]), we re-adapt the notion of "rough trace" of a BV function to the present setting and eventually prove that this tool allows for a Gauss-Green type formula involving indeed the rough trace of such a function. Then, we compare our analysis with the discussion done in [LS], where traces of BV functions are studied by means of the Lebesgue-point characterization, and we determine the conditions under which the two notions coincide.

7.2.1 Definition. Let $\Omega \subset \mathcal{X}$ be a bounded open set, and denote by $\partial^*\Omega$ its essential boundary. Given $u \in BV(\Omega)$, we define its *rough trace* as the quantity

$$u^*(x) := \sup\left\{t \in \mathbb{R};\ \left\|D\mathbb{1}_{E_t}\right\|(\mathcal{X}) < \infty,\ x \in \partial^* E_t\right\},$$

where E_t as usual denotes the super-level sets of u for $t \in \mathbb{R}$. Of course, when u has a limit value at $x \in \partial^*\Omega$, then

$$u^*(x) = \lim_{y \to x} u(y).$$

Below, $\mathcal{S}^h$ denotes as usual the spherical Hausdorff measure generated from the function $h(x)$, see the comments right before Definition 4.1.1.

7.2.2 Lemma. If $\left\|D\mathbb{1}_{\Omega}\right\|(\mathcal{X}) < \infty$ and $u \in BV(\Omega)$, then u^* is $\mathcal{S}^h$-measurable on $\partial^*\Omega$ and

$$\mathcal{S}^h\left(\{x \in \partial^*\Omega;\ u^*(x) \geq t\}\right) = \mathcal{S}^h\left(\partial^*\Omega \cap \partial^* E_t\right) \tag{7.3}$$

for almost every $t \in \mathbb{R}$.

Proof. Instead of the above condition (7.3), we shall prove that, for almost every $t \in \mathbb{R}$ except a countable and dense subset, it holds

$$\mathcal{S}^h\left(\{x \in \partial^*\Omega;\ u^*(x) \geq t\}\,\Delta\partial^* E_t\right) = 0,$$

where $A\Delta B \coloneqq (A\backslash B) \cap (B\backslash A)$ for any two sets A and B.

For brevity, let us set $A_t \coloneqq \{x \in \partial^*\Omega;\ u^*(x) \geq t\}$, $B_t \coloneqq \partial^* E_t$ and $F_t \coloneqq A_t \backslash B_t$. By definition of rough trace, one always has the inclusion $B_t \subset A_t$, so we need to prove $\mathcal{S}^h(F_t) = 0$.

By Coarea, the sets B_t are measurable, while the F_t's are disjoint. Observe that for every $t_0 < t_1$ one has $B_{t_0} \supset B_{t_1}$ and $B_{t_0} \cup F_{t_0} \supset B_{t_1} \cup F_{t_1}$, so $B_{t_0} \supset F_{t_1}$. Thus we have

$$\left(\bigcap_{t<t_1} B_t\right) \backslash B_{t_1} \supset F_{t_1},$$

with the sets $\left(\bigcap_{t<t_1} B_t\right) \backslash B_{t_1}$ being measurable and disjoint; therefore, they have zero $\mathcal{S}^h$ measure for almost every $t_1 \in \mathbb{R}$. As a consequence, the sets F_t, being subsets of $\mathcal{S}^h$-negligible sets, are measurable and $\mathcal{S}^h$-negligible as well. We can then conclude that the B_t's are measurable sets, and the proof follows.

□

The following Lemma combines Lemma 4 and Corollary 2 in [Ma, Section 9.5].

7.2.3 Lemma. For any $u \in BV(\Omega)$ and for $\mathcal{S}^h$-almost every $x \in \partial^*\Omega$, one has

$$-u^*(x) = (-u)^*(x).$$

Consequently, $(u^*)^+ = (u^+)^*$, $(u^*)^- = (u^-)^*$ and then $u^* = (u^+)^* - (u^-)^*$.

□

7.2.4 Theorem. Let $\|D\mathbb{1}_\Omega\|(\mathcal{X}) < \infty$ and assume $\mathcal{S}^h(\partial\Omega\backslash\partial^*\Omega) = 0$. In order for any $u \in BV(\Omega)$ to satisfy

$$\inf_{c\in\mathbb{R}} \int_{\partial\Omega} |u^*(x) - c| \,\mathrm{d}\mathcal{S}^h(x) \leq k \|Du\|(\Omega)$$

with $k > 0$ independent of u, it is necessary and sufficient that

$$\min\left\{\|D\mathbb{1}_E\|(\Omega^c), \left\|D\mathbb{1}_{\Omega\backslash E}\right\|(\Omega^c)\right\} \leq k \|D\mathbb{1}_E\|(\Omega)$$

holds for any $E \subset \Omega$.

Proof. We start with necessity. Let $E \subset \Omega$ be such that $\|D\mathbb{1}_E\|(\Omega) < \infty$, and apply Lemma 1 in [Ma, Section 9.5] to find that $\|D\mathbb{1}_E\|(\mathcal{X}) < \infty$. Then,

$$\begin{aligned}
\inf_{c\in\mathbb{R}} \int_{\partial^*\Omega} |\mathbb{1}^*_E(x) - c| \,\mathrm{d}\mathcal{S}^h(x) &= \min_{c\in\mathbb{R}} \left\{|1-c|\mathcal{S}^h(\partial^*E \cap \partial^*\Omega) + |c|\mathcal{S}^h(\partial^*\Omega\backslash\partial^*E)\right\} \\
&= \min\left\{\mathcal{S}^h(\partial^*E \cap \partial^*\Omega), \mathcal{S}^h(\partial^*\Omega\backslash\partial^*E)\right\} \\
&= \min\left\{\|D\mathbb{1}_E\|(\Omega^c), \left\|D\mathbb{1}_{\Omega\backslash E}\right\|(\Omega^c)\right\}.
\end{aligned}$$

Since by hypothesis

$$\inf_{c\in\mathbb{R}}\int_{\partial\Omega}|\mathbb{1}_E^*(x)-c|\,\mathrm{d}\mathcal{S}^h(x)\le k\,\|Du\|\,(\Omega),$$

we obtain the required inequality.

We then pass to sufficiency. Let $u\in BV(\Omega)$; then for every t, $\mathcal{S}^h\left(\partial\Omega\cap\partial^*E_t\right)$ is a non-increasing function of t. In fact, if $x\in\partial^*\Omega\cap\partial^*E_t$ and $\tau<t$, then $\Omega\supset E_\tau\supset E_t$ and the same holds as well for the essential boundaries); moreover,

$$\Theta_x\left(E_t\right)\le\Theta_x\left(E_\tau\right)\le\Theta_x\left(\Omega\right).$$

This means - by the hypothesis and by the definition of essential boundary - that $\Theta_x\left(E_\tau\right)$ $\neq 0,1$ and then $x\in\partial^*\Omega\cap\partial^*E_\tau$.

In a similar manner we can show that $\mathcal{S}^h\left(\partial\Omega\backslash\partial^*E_t\right)$ is a non-decreasing function of t.

By the Coarea Formula,

$$k\,\|Du\|\,(\Omega)=k\int_{\mathbb{R}}\|D\mathbb{1}_{E_t}\|\,(\Omega)\mathrm{d}t\ge\int_{\mathbb{R}}\min\left\{\mathcal{S}^h\left(\partial\Omega\cap\partial^*E_t\right),\mathcal{S}^h\left(\partial\Omega\backslash\partial^*E_t\right)\right\}\mathrm{d}t.$$

If we now set $t_0:=\sup\left\{t;\ \|D\mathbb{1}_{E_t}\|\,(\mathcal{X})<\infty,\ \mathcal{S}^h\left(\partial\Omega\cap\partial^*E_t\right)\ge\mathcal{S}^h\left(\partial\Omega\backslash\partial^*E_t\right)\right\}$ we get

$$\begin{aligned}
k\,\|Du\|\,(\Omega)&\ge\int_{t_0}^{+\infty}\mathcal{S}^h\left(\partial\Omega\cap\partial^*E_t\right)\mathrm{d}t+\int_{-\infty}^{t_0}\mathcal{S}^h\left(\partial\Omega\backslash\partial^*E_t\right)\mathrm{d}t\\
&=\int_{t_0}^{+\infty}\mathcal{S}^h\left(\{x;\ u^*(x)\ge t\}\cap\partial\Omega\right)\mathrm{d}t+\int_{-\infty}^{t_0}\mathcal{S}^h\left(\{x;\ u^*(x)\le t\}\cap\partial\Omega\right)\mathrm{d}t\\
&=\int_{\partial\Omega}[u^*(x)-t_0]^+\,\mathrm{d}\mathcal{S}^h(x)+\int_{\partial\Omega}[u^*(x)-t_0]^-\,\mathrm{d}\mathcal{S}^h(x)\\
&=\int_{\partial\Omega}|u^*(x)-t_0|\,\mathrm{d}\mathcal{S}^h(x).
\end{aligned}$$

In other words,

$$k\,\|Du\|\,(\Omega)\ge\inf_{c\in\mathbb{R}}\int_{\partial\Omega}|u^*(x)-c|\,\mathrm{d}\mathcal{S}^h(x).$$

□

7.2.5 Definition. Let $A\subset\bar{\Omega}$. We shall denote by $\zeta_A^{(\alpha)}$ the infimum of those $k>0$ such that $\left[\|D\mathbb{1}_E\|\,(\Omega^c)\right]^\alpha\le k\,\|D\mathbb{1}_E\|\,(\Omega)$ for all sets $E\subset\Omega$ which satisfy $\mu\left(E\cap A\right)+\mathcal{S}^h\left(A\cap\partial^*E\right)=0$.

7.2.6 Theorem. Let $\|D\mathbb{1}_\Omega\|\,(\mathcal{X})<\infty$ and assume $\mathcal{S}^h\left(\partial\Omega\backslash\partial^*\Omega\right)=0$. Then, if $A\subset\overline{\Omega}$, for every $u\in BV(\Omega)$ such that $u|_{A\cap\Omega}=0$ and $u^*|_{A\cap\partial^*\Omega}=0$, it holds

$$\int_{\partial\Omega}|u^*(x)|\,\mathrm{d}\mathcal{S}^h(x)\le\zeta_A^{(1)}\,\|Du\|\,(\Omega)$$

and the constant $\zeta_A^{(1)}$ is exact.

PROOF. We know that

$$\int_{\partial\Omega} |u^*(x)| \,\mathrm{d}\mathcal{S}^h(x) = \int_0^{+\infty} \Big[\mathcal{S}^h\left(\{x;\ u^*(x) \geq t\} \cap \partial\Omega\right) + \mathcal{S}^h\left(\{x;\ -u^*(x) \geq t\}\,\partial\Omega\right)\Big]\,\mathrm{d}t.$$

Notice that the integral of the first summand is equal to

$$\int_0^{+\infty} \mathcal{S}^h\left(\partial^* E_t \cap \partial^*\Omega\right)\mathrm{d}t = \int_0^{\infty} \|D\mathbb{1}_{E_t}\|\left(\Omega^c\right)\mathrm{d}t.$$

By our hypotheses, we get $\mu\left(A \cap E_t\right) + \mathcal{S}^h\left(A \cap \partial^* E_t\right) = 0$ for almost every t; so, by the definition of $\zeta_A^{(1)}$,

$$\int_0^{+\infty} \mathcal{S}^h\left(\{x;\ u^*(x) \geq t\}\right)\mathrm{d}t \leq \int_0^{+\infty} \|D\mathbb{1}_{E_t}\|\left(\Omega^c\right)\mathrm{d}t \leq \zeta_A^{(1)} \int_0^{+\infty} \|D\mathbb{1}_{E_t}\|\,(\Omega)\mathrm{d}t,$$

and similarly we find

$$\int_0^{+\infty} \mathcal{S}^h\left(\{x;\ -u^*(x) \geq t\}\right)\mathrm{d}t \leq \int_{-\infty}^{0} \left\|D\mathbb{1}_{\Omega\setminus E_t}\right\|\left(\Omega^c\right)\mathrm{d}t \leq \zeta_A^{(1)} \int_{-\infty}^{0} \|D\mathbb{1}_{E_t}\|\,(\Omega)\mathrm{d}t.$$

To deduce that $\zeta_A^{(1)}$ is sharp, it suffices to substitute u with $\mathbb{1}_E$, taking E as in the above definition.

□

7.2.7 Definition. We set

$$\zeta_\alpha(S) := \sup\left\{\frac{\left[\|D\mathbb{1}_E\|\left(\Omega^c\right)\right]^\alpha}{\|D\mathbb{1}_E\|\left(\Omega\right)};\ E \subset \Omega,\ \|D\mathbb{1}_E\|\left(\Omega\right) > 0,\ \|D\mathbb{1}_E\|\left(\Omega^c\right) \leq S\right\}.$$

With this definition, the following result is straightforward:

7.2.8 Corollary [Ma, Section 9.5.3]. Let $\|D\mathbb{1}_\Omega\|\left(\mathcal{X}\right) < \infty$ and assume $\mathcal{S}^h\left(\partial\Omega\setminus\partial^*\Omega\right) = 0$. Then, for every $u \in BV(\Omega)$ such that $\mathcal{S}^h\left(\{x;\ u^*(x) \neq 0\}\right) \leq S$, it holds

$$\int_{\partial\Omega} |u^*(x)|\,\mathrm{d}\mathcal{S}^h(x) \leq \zeta_1(S)\,\|Du\|\left(\Omega\right).$$

Moreover, the constant $\zeta_1(S)$ is exact.

□

7.2.9 Theorem [Ma, Theorem 9.5.4] Let $\|D\mathbb{1}_\Omega\|(\mathcal{X}) < \infty$ and assume $\mathcal{S}^h(\partial\Omega\backslash\partial^*\Omega) = 0$. Then, every $u \in BV(\Omega)$ satisfies

$$\|u^*\|_{L^1(\partial\Omega)} \leq k\,\|u\|_{BV(\Omega)}$$

with a constant $k > 0$ independent of u, if and only if there exists $\delta > 0$ such that for every measurable set $E \subset \Omega$ with diameter at most equal to δ it holds

$$\|D\mathbb{1}_E\|(\Omega^c) \leq k'\,\|D\mathbb{1}_{\mathrm{E}}\|(\Omega)$$

with a constant $k' > 0$ independent of E.

□

7.2.10 Definition. Suppose that $\|D\mathbb{1}_\Omega\|(\mathcal{X}) < \infty$ and that $\mathcal{S}^h(\partial\Omega\backslash\partial^*\Omega) = 0$. For $c \in \mathbb{R}$, we set

$$u_c(x) := \begin{cases} u(x), & x \in \Omega \\ \\ c, & x \in \Omega^c. \end{cases}$$

7.2.11 Lemma. $\|Du_c\|(\mathcal{X}) = \|Du\|(\Omega) + \|u^* - c\|_{L^1(\partial\Omega)}$.

Proof. We have

$$\begin{aligned} \|Du_c\|(\mathcal{X}) &= \int_0^{+\infty} \left\|D\mathbb{1}_{\{x;\,|u_c-c|>t\}}\right\|(\mathcal{X})\mathrm{d}t \\ &= \int_0^{+\infty} \left[\left\|D\mathbb{1}_{\{x;\,|u-c|>t\}}\right\|(\Omega) + \left\|D\mathbb{1}_{\{x;\,|u-c|>t\}}\right\|(\Omega^c)\right]\mathrm{d}t. \end{aligned}$$

Of course the total variation of u in Ω is given by

$$\|Du\|(\Omega) = \int_0^{+\infty} \left\|D\mathbb{1}_{\{x;\,|u-c|>t\}}\right\|(\Omega)\mathrm{d}t,$$

so by the hypothesis $\mathcal{S}^h(\partial\Omega\backslash\partial^*\Omega) = 0$ we deduce

$$\begin{aligned} &\int_0^{+\infty} \left\|D\mathbb{1}_{\{x;\,|u-c|>t\}}\right\|(\Omega^c)\,\mathrm{d}t = \\ &\qquad = \int_0^{+\infty} \mathcal{S}^h(\{x;\,(u-c)^* > t\})\,\mathrm{d}t + \int_{-\infty}^0 \mathcal{S}^h(\{x;\,(u-c)^* < t\})\,\mathrm{d}t \\ &\qquad = \int_{\partial\Omega} |(u(x)-c)^*|\,\mathrm{d}\mathcal{S}^h(x) = \int_{\partial\Omega} |u^*(x)-c|\,\mathrm{d}\mathcal{S}^h(x) \\ &\qquad = \|u^*-c\|_{L^1(\partial\Omega)}\,. \end{aligned}$$

The proof is complete.

□

Summarizing the previous results, we see that $\|D\mathbb{1}_\Omega\|(\mathcal{X}) < \infty$ and $\mathcal{S}^h(\partial\Omega\backslash\partial^*\Omega) = 0$ are the necessary conditions under which the rough trace $u^*(x)$ of $u \in BV(\Omega)$ is in $L^1(\partial\Omega)$. This conclusion allows us to proceed towards a Gauss-Green formula for functions of bounded variation.

Namely, we have the following result:

7.2.12 Theorem. Let $\Omega \subset \mathcal{X}$ be a bounded open set such that $\|D\mathbb{1}_\Omega\|(\mathcal{X}) < \infty$ and $\mathcal{S}^h(\partial\Omega\backslash\partial^*\Omega) = 0$. Then, for every $u \in BV \cap L^\infty(\Omega)$ and every $F \in \textsc{TestV}(\mathcal{X})$ one has

$$\int_\Omega \mathrm{d}u(F) + \int_\Omega u\mathrm{div}(F)\mathrm{d}\mu = -\int_{\mathcal{X}} \Theta(u^*(x))\,\mathrm{d}\mathbb{1}_\Omega(F)\mathrm{d}\mu(x),$$

where

$$\Theta(u^*(x)) \coloneqq \int_0^{u^*(x)} \mathbb{1}_{E_t} f_{E_t,\Omega}\mathrm{d}t$$

with $f_{E_t,\Omega}$ given by (7.4) below.

Proof. Observe that, in order to drop the assumption that u is also in $L^\infty(\mu)$, one may have to suppose for instance $u \geq 0$, in order to avoid summability issues when working with positive and negative parts of u.

We already know that an integration by parts formula holds for the whole of $\mathcal{X}$, namely

$$\int_{\mathcal{X}} \mathrm{d}u(F) = -\int_{\mathcal{X}} u\mathrm{div}(F)\mathrm{d}\mu.$$

Moreover, clearly,

$$\int_\Omega \mathrm{d}u(F) = \int_{\mathcal{X}} \mathrm{d}u(F) - \int_{\mathcal{X}\backslash\Omega} \mathrm{d}u(F) = -\int_{\mathcal{X}} u\mathrm{div}(F)\mathrm{d}\mu - \int_{\mathcal{X}\backslash\Omega} \mathrm{d}u(F).$$

Now, suppose $u \equiv \mathbb{1}_E$ with E as above. The previous equalities become

$$\int_\Omega \mathrm{d}\mathbb{1}_E(F) = -\int_{\mathcal{X}} \mathbb{1}_E\mathrm{div}(f)\mathrm{d}\mu - \int_{\mathcal{X}\backslash\Omega} \mathrm{d}\mathbb{1}_E(F) = -\int_{\mathcal{X}} \mathbb{1}_E\mathrm{div}(F)\mathrm{d}\mu - \int_{\partial\Omega\cap\partial^* E} \mathrm{d}\mathbb{1}_E(F).$$

We used the fact that - by Theorem 4.1.12 - the perimeter measure is concentrated on the essential boundary of E, so

$$\|D\mathbb{1}_E\|(\Omega) = -\|D\mathbb{1}_E\|(\partial\Omega) = -\|D\mathbb{1}_E\|(\partial\Omega\cap\partial^* E).$$

Using Coarea Formula we now obtain

$$\begin{aligned}
\int_\Omega \mathrm{d}u(F) &= \int_0^{+\infty} \mathrm{d}t \int_\Omega \mathrm{d}\mathbb{1}_{E_t}(F) \\
&= -\int_0^{+\infty} \mathrm{d}t \left(\int_{\mathcal{X}} \mathbb{1}_{E_t}\mathrm{div}(F)\mathrm{d}\mu + \int_{\partial\Omega\cap\partial^* E_t} \mathrm{d}\mathbb{1}_{E_t}(F)\right) \\
&= -\int_0^{+\infty} \mathrm{d}t \left(\int_{\mathcal{X}} \mathbb{1}_{E_t}\mathrm{div}(F)\mathrm{d}\mu + \int_{\partial^*\Omega\cap\partial^* E_t} \mathrm{d}\mathbb{1}_{E_t}(F)\right).
\end{aligned}$$

The pairing $\mathrm{d}\mathbb{1}_E(F)$ actually defines a measure which is absolutely continuous with respect to the perimeter measure: setting

$$\nu_E^F : A \mapsto \int_A \mathrm{d}\mathbb{1}_E(F) = \boldsymbol{\nu}_E^F(A),$$

one has $\left|\nu_E^F\right|(A) \leq \|F\|_{L^\infty(T\mathcal{X})} \|D\mathbb{1}_E\|(A)$ and then again by Theorem 4.1.12

$$\nu_E^F(A) = \int_A \sigma_E^F(x)\mathrm{d}\,\|D\mathbb{1}_E\|(x) = \int_{A\cap\partial^* E} \sigma_E^F\theta_E^F \mathrm{d}\mathcal{S}^h(x).$$

So, $\mathrm{d}\mathbb{1}_E(F) = \sigma_E^F\theta_E^F\mathcal{S}^h\lfloor\partial^* E$ and similarly $\mathrm{d}\mathbb{1}_\Omega(F) = \sigma_\Omega^F\theta_\Omega^F\mathcal{S}^h\lfloor\partial^*\Omega$. Let us set

$$f_{E,\Omega} := \frac{\sigma_E^F\theta_E^F}{\sigma_\Omega^F\theta_\Omega^F} = \frac{\lambda_E^F}{\lambda_\Omega^F}. \tag{7.4}$$

Summing up, we find
$\int_{\partial^*\Omega} \mathrm{d}\mathbb{1}_E(F)$

$$= \int_{\partial^*\Omega\cap\partial^* E} \lambda_E^F \mathrm{d}\mathcal{S}^h(x) = \int_{\partial^*\Omega\cap\partial^* E} \lambda_\Omega^F \mathrm{d}\mathcal{S}^h(x) = \int_{\partial^* E} f_{E,\Omega}\mathrm{d}\mathcal{S}^h(x).$$

Applying the same argument to our case,

$$\begin{aligned}
\int_\Omega \mathrm{d}u(F) &= -\int_0^{+\infty} \mathrm{d}t \left(\int_{\mathcal{X}} \mathbb{1}_{E_t}\mathrm{div}(F)\mathrm{d}\mu + \int_{\partial^* E_t} f_{E_t,\Omega}\mathrm{d}\mathbb{1}_\Omega(F)\right)\\
&= -\int_0^{+\infty} \mathrm{d}t \left(\int_\Omega \mathbb{1}_{E_t}\mathrm{div}(F)\mathrm{d}\mu + \int_{\{u^*\geq t\}} f_{E_t,\Omega}\mathrm{d}\mathbb{1}_\Omega(F)\right)\\
&= -\int_\Omega u\mathrm{div}(F)\mathrm{d}\mu - \int_{\partial\Omega} \Theta\left(u^*(x)\right)\mathrm{d}\mathbb{1}_\Omega(F),
\end{aligned}$$

where

$$\Theta\left(u^*(x)\right) = \int_0^{u^*(x)} \mathbb{1}_{E_t} f_{E_t,\Omega}\mathrm{d}t.$$

□

7.2.13 Definition & Remark. *i)* According to Definition 4.1.13, we shall say that $(\mathcal{X}, d, \mu)$ is *strongly local* if, besides the condition $\theta_E = \theta_\Omega$ $\mathcal{S}^h$-almost everywhere on $\partial^*\Omega\cap\partial^* E$, one also has $\sigma_E^F = \sigma_\Omega^F$ $\mathcal{S}^h$-almost everywhere on $\partial^*\Omega\cap\partial^* E$.
Observe that if $(\mathcal{X}, d, \mu)$ is strongly local, then the function $f_{E,\Omega}$ in (7.4) is identically 1.
ii) If we change the statement of Theorem 7.2.12 assuming that Ω is a regular domain with $\mathcal{S}^h(\partial\Omega\backslash\partial^*\Omega) = 0$, then for every $u \in BV \cap L^\infty(\Omega)$ there exists an operator Tr : $BV\cap L^\infty(\Omega) \to L^1(\partial\Omega, \|D\mathbb{1}_\Omega\|)$ such that for every $F \in \textsc{TestV}(\mathcal{X})$ one has

$$\int_\Omega \mathrm{d}u(F) + \int_\Omega u\mathrm{div}(F)\mathrm{d}\mu = -\int_{\partial\Omega} u^*(x)\left(F\cdot\nu\right)_{\partial\Omega}^- \mathrm{d}\,\|D\mathbb{1}_\Omega\|(x) := \left\langle \mathrm{Tr}(u), (F\cdot\nu)_{\partial\Omega}^-\right\rangle.$$

Indeed, in this case we can use the defining sequence $(\varphi_\varepsilon)_{\varepsilon>0} \subset \mathrm{Lip}_c(\Omega)$ of the regular domain Ω and we are entitled to repeat the proof of Theorem 7.1.7.

We now pass to the comparison between the rough trace and the trace defined in terms of Lebesgue points. In order to do so, we step back for a moment to a few more basic definitions and then recall quickly the salient results of [LS].

7.2.14 Definition. For any measurable function $u : \mathcal{X} \to \mathbb{R}$, we define its lower and upper *approximate limits* as

$$u^\wedge(x) := \sup \left\{ t \in \overline{\mathbb{R}};\ \lim_{\rho\to 0^+} \frac{\mu\left(B_\rho(x) \cap E_t^c\right)}{\mu\left(B_\rho(x)\right)} = 0 \right\}$$

and

$$u^\vee(x) := \inf \left\{ t \in \overline{\mathbb{R}};\ \lim_{\rho\to 0^+} \frac{\mu\left(B_\rho(x) \cap E_t\right)}{\mu\left(B_\rho(x)\right)} = 0 \right\},$$

where E_t as usual denotes the super-level sets of u. The arithmetic average of the approximate limits will by denoted by $\tilde{u}$.

7.2.15 Definition. Let $\Omega \subset \mathcal{X}$ be an open set and let u be a μ-measurable function on Ω. Then, we shall say that a function $\mathrm{T}u : \partial\Omega \to \mathbb{R}$ is a *trace* of u if for $\mathcal{S}^h$-almost every $x \in \partial\Omega$ one has

$$\lim_{\rho\to 0^+} \fint_{\Omega\cap B_\rho(x)} |u - \mathrm{T}u(x)| \,\mathrm{d}\mu = 0.$$

Recall that the zero extension of μ from Ω to $\overline{\Omega}$, $\bar{\mu}$, is given by $\bar{\mu}(A) := \mu(A \cap \Omega)$ whenever $A \subset \overline{\Omega}$.

7.2.16 Proposition [LS, Proposition 3.3]. Let $\Omega \subset \mathcal{X}$ be a bounded open set supporting a $(1,1)$-Poincaré inequality and assume that μ is doubling on Ω. Let $\overline{\Omega}$ be equipped with the extended measure $\bar{\mu}$. If $u \in BV(\Omega)$, then its zero-extension $u^{\overline{\Omega}} := \bar{u}$ to $\overline{\Omega}$ is such that $\|\bar{u}\|_{BV(\overline{\Omega})} = \|u\|_{BV(\Omega)}$, whence $\|D\bar{u}\|(\partial\Omega) = 0$.

□

7.2.17 Remark. The condition $\|D\bar{u}\|(\Omega) = 0$ entails the equality

$$\bar{u}^\wedge(x) = \bar{u}^\vee(x)$$

for $\mathcal{S}^h$-almost every $x \in \partial\Omega$.

7.2.18 Definition [LS, (3.2)]. We say that an open set Ω satisfies a *measure-density condition* if there exists a constant $C > 0$ such that

$$\mu\left(B_\rho(x)\cap\Omega\right)\geq C\mu\left(B_\rho(x)\right) \tag{7.5}$$

for $\mathcal{S}^h$-almost every $x\in\partial\Omega$ and for every $\rho\in]0,\mathrm{diam}(\Omega)[$.

7.2.19 Theorem [LS, Theorem 3.4]. Let Ω and μ be as in Proposition 7.2.16. Then, denoting by $\overline{\mathcal{S}}^h$ the zero-extension to $\overline{\Omega}$ of $\mathcal{S}^h$, there exists a linear trace operator T on $BV(\Omega)$ such that, given $u\in BV(\Omega)$, for $\overline{\mathcal{S}}^h$-almost every $x\in\partial\Omega$ one has

$$\lim_{\rho\to 0^+}⨍_{\Omega\cap B_\rho(x)}|u-\mathrm{T}u(x)|^{\frac{s}{s-1}}\,\mathrm{d}\mu=0.$$

Moreover, if Ω satisfies also (7.5), the above holds for $\mathcal{S}^h$-almost every $x\in\partial\Omega$.

□

7.2.20 Remark. Putting together Proposition 7.2.16, Remark 7.2.17 and Theorem 7.2.19, one obtains that $\mathrm{T}u(x)=\bar{u}^\wedge(x)=\bar{u}^\vee(x)$ for $\mathcal{S}^h$-almost every $x\in\partial\Omega$. This is a direct consequence of the fact that the BV-energy associated to $\bar{u}$ does not charge the boundary of Ω, $\|D\bar{u}\|(\partial\Omega)=0$.

To conclude, we give an extension result in comparison with Lemma 7.2.11 and then show that $u^*(x)=\mathrm{T}u(x)$ for $\mathcal{S}^h$-almost every $x\in\partial\Omega$.

7.2.21 Proposition. Suppose $\Omega\subset\mathcal{X}$ is an open set such that $\mathcal{S}^h(\partial\Omega)<\infty$ and $\mathcal{S}^h\left(\partial\Omega\backslash\partial^*\Omega\right)=0$; let $E\subset\Omega$ be a set of finite perimeter in Ω. Then, $\bar{\mathbb{1}}_E\in BV(\mathcal{X})$.
Under the same hypotheses, for any $u\in BV\cap L^\infty(\Omega)$ one has $\bar{u}\in BV(\mathcal{X})$.

Proof. As in the proof of Proposition 7.2.12, we remark that in order to drop the assumption that u is also in $L^\infty(\mu)$, one may have to suppose for instance $u\geq 0$, in order to avoid summability issues when working with positive and negative parts of u.
We can write

$$\begin{aligned}\|D\mathbb{1}_E\|(\mathcal{X})&=\|D\mathbb{1}_E\|(\Omega)+\|D\mathbb{1}_E\|(\mathcal{X}\backslash\Omega)\\&=\|D\mathbb{1}_E\|(\Omega)+\|D\mathbb{1}_E\|(\partial\Omega)\\&=\|D\mathbb{1}_E\|(\Omega)+\|D\mathbb{1}_E\|(\partial^*\Omega)\leq\|D\mathbb{1}_E\|(\Omega)+\mathcal{S}^h\left(\partial^*\Omega\cap\partial^*E\right),\end{aligned}$$

which is finite by our assumptions. Then the assertion follows for $\mathbb{1}_E\in BV(\Omega)$.
Let us now take $u\in BV(\Omega)$; we shall make use of the above argument and of the Coarea Formula. For simplicity, assume $u\geq 0$. Then

$$\|D\bar{u}\|(\mathcal{X})=\int_0^{+\infty}\left\|D\bar{\mathbb{1}}_{\bar{E}_t}\right\|\mathrm{d}t\leq\int_0^{\infty}\Big[\|D\mathbb{1}_{E_t}\|(\Omega)+\mathcal{S}^h\left(\partial^*\Omega\cap\partial^*E_t\right)\Big]\,\mathrm{d}t,$$

by the previous inequality. We already know that $\int_0^{+\infty}\|D\mathbb{1}_{E_t}\|\,\mathrm{d}t$ is finite since $u\in BV(\Omega)$; let us then estimate $\int_0^{+\infty}\mathcal{S}^h\left(\partial^*\Omega\cap\partial^*E_t\right)\mathrm{d}t$. Using Cavalieri's Principle,

$$\int_0^{+\infty} \mathcal{S}^h \left(\partial^*\Omega \cap \partial^* E_t\right) \mathrm{d}t = \int_0^{+\infty} \mathrm{d}t \int_{\partial^*\Omega} \mathbb{1}_{\partial^* E_t} \mathrm{d}\mathcal{S}^h(x) = \int_{\partial^*\Omega} \mathrm{d}\mathcal{S}^h(x) \int_0^{+\infty} \mathbb{1}_{\partial^* E_t} \mathrm{d}t.$$

Now observe that since we are integrating over $\partial^*\Omega \cap \partial^* E_t$, then $t \in [0, \bar{u}^\vee(x)]$. Indeed, if $t > \bar{u}^\vee(x)$, then we would have $\Theta^*_{\bar{\mu}}(E_t, x) = 0$ and thus $\Theta^*_\mu(E_t, x) = 0$ - see Definition 4.1.11 for the notion of density of a set at a point - implying $E_t^{(0)} \ni x \notin \partial^* E_t$, which is a contradiction. If $0 < t < \bar{u}^\vee(x)$, then the upper density associated with $\bar{\mu}$, $\Theta^*_{\bar{\mu}}(E_t, x)$, is positive and then also $\Theta^*_\mu(E_t, x) > 0$.
Moreover, if $x \in \partial^*\Omega$ then we find

$$\begin{aligned} 0 < \limsup_{\rho\to 0} \frac{\mu\left(B_\rho(x) \cap E_t\right)}{\mu\left(B_\rho(x)\right)} &= \limsup_{\rho\to 0} \frac{\mu\left(B_\rho(x) \cap E_t\right)}{\mu\left(B_\rho(x) \cap \Omega\right)} \frac{\mu\left(B_\rho(x) \cap \Omega\right)}{\mu\left(B_\rho(x)\right)} \\ &\le \limsup_{\rho\to 0} \frac{\mu\left(B_\rho(x) \cap E_t\right)}{\mu\left(B_\rho(x) \cap \Omega\right)} < 1. \end{aligned}$$

This means $x \in \partial^* E_t$. In other words, for $x \in \partial^*\Omega$ we have shown the following:

i) if $x \in \partial^* E_t$ then $0 \le t \le \bar{u}^\vee(x)$;

ii) if $0 < t < \bar{u}^\vee(x)$ then $x \in \partial^* E_t$.

Thus,

$$\begin{aligned} \int_{\partial^*\Omega} \mathrm{d}\mathcal{S}^h(x) \int_0^{+\infty} \mathbb{1}_{\partial^* E_t} \mathrm{d}t &= \int_{\partial^*\Omega} \mathrm{d}\mathcal{S}^h(x) \int_0^{\bar{u}^\vee(x)} \mathbb{1}_{\partial^* E_t} \mathrm{d}t \\ &= \int_{\partial^*\Omega} \bar{u}^\vee(x) \mathrm{d}\mathcal{S}^h(x) \\ &= \int_{\partial^*\Omega} u^*(x) \mathrm{d}\mathcal{S}^h(x) = \int_{\partial^*\Omega} Tu(x) \mathrm{d}\mathcal{S}^h(x). \end{aligned}$$

Summarizing, we have found

$$\|D\bar{u}\|(\mathcal{X}) \le \|Du\|(\Omega) + c \int_{\partial^*\Omega} Tu(x) \mathrm{d}\mathcal{S}^h(x).$$

The Proposition is proven.

$\square$

7.2.22 Proposition. Let $\Omega \subset \mathcal{X}$ be an open set. Then, for every $x \in \partial^*\Omega$ and $u \in BV(\Omega)$, assuming $\bar{u}^\wedge(x), \bar{u}^\vee(x) \in \mathbb{R}$ one has $\bar{u}^\wedge(x) \le u^*(x) \le \bar{u}^\vee(x)$.

Proof. Assume $u \ge 0$ for simplicity. By definition, $\bar{u} := u|^{\mathcal{X}}$ is the zero-extension of u to the whole of $\mathcal{X}$. Recall that, by definition, $u^*(x)$ is the supremum of those t for which $x \in \partial^* E_t$. We first assume $t < \bar{u}^\wedge(x)$. This gives, by the definition of approximate limits and of extended measure,

$$
\begin{aligned}
1 &= \Theta_{*,\bar{\mu}}\left(E_t, x\right) \\
&= \liminf_{\rho \to 0} \frac{\mu\left(B_\rho(x) \cap E_t \cap \Omega\right)}{\mu\left(B_\rho(x) \cap \Omega\right)} \\
&= \liminf_{\rho \to 0} \frac{\mu\left(B_\rho(x) \cap E_t\right)}{\mu\left(B_\rho(x) \cap \Omega\right)} = \liminf_{\rho \to 0} \frac{\mu\left(B_\rho(x) \cap E_t\right)}{\mu\left(B_\rho(x)\right)} \frac{\mu\left(B_\rho(x)\right)}{\mu\left(B_\rho(x) \cap \Omega\right)} \\
&> \liminf_{\rho \to 0} \frac{\mu\left(B_\rho(x) \cap E_t\right)}{\mu\left(B_\rho(x)\right)} = \Theta_{*,\mu}\left(E_t, x\right),
\end{aligned}
$$

hence $x \notin E_t^{(1)}$. Let us then show that $x \notin E_t^{(0)}$. Since by hypothesis $x \in \partial^* \Omega$, there exists a constant $c > 0$ such that $c \leq \Theta_{*,\mu}\left(\Omega, x\right) \leq 1 - c$. Then, arguing in the same manner as above we find

$$
\liminf_{\rho \to 0} \frac{\mu\left(B_\rho(x) \cap E_t\right)}{\mu\left(B_\rho(x)\right)} \geq c,
$$

forcing $x \notin E_t^{(0)}$. In other words, $x \in \partial^* E_t$ and $t \leq u^*(x)$.

Now, assume $t > \bar{u}^\vee(x)$. Using the previous arguments,

$$
\begin{aligned}
0 &= \Theta^*_{\bar{\mu}}\left(E_t, x\right) \\
&= \limsup_{\rho \to 0} \frac{\mu\left(B_\rho(x) \cap E_t \cap \Omega\right)}{\mu\left(B_\rho(x) \cap \Omega\right)} \\
&> \limsup_{\rho \to 0} \frac{\mu\left(B_\rho(x) \cap E_t\right)}{\mu\left(B_\rho(x)\right)} = \Theta^*_{\mu}\left(E_t, x\right),
\end{aligned}
$$

but this would force $\Theta^*_\mu\left(E_t, x\right) = 0$. Hence, $x \in E_t^{(0)}$ implying $x \notin \partial^* E_t$. This means $t \geq u^*(x)$, and then $u^*(x) \leq \bar{u}^\vee(x)$.

□

7.2.23 Remark. Observe that, when $t > \bar{u}^\vee(x)$ we may actually take x to be an arbitrary point of $\mathcal{X}$.

If we combine Proposition 7.2.22 with Theorem 7.2.19 and Remark 7.2.20, we obtain $u^*(x) = \mathrm{T}u(x)$ $\mathcal{S}^h$-almost everywhere on $\partial\Omega$; an important consequence of this equality is that $u^*(x)$ defines *a fortiori* a linear operator.

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