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Hyperbolic Problems: Theory, Numerics, Applications

Fabio Ancona
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Editors

American Institute of Mathematical Sciences

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Hyperbolic Problems: Theory, Numerics, Applications

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of the Fourteenth International Conference
on Hyperbolic Problems
held in Padova,
June 25-29, 2012

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PREFACE

This volume contains the Proceedings of the HYP2012 International Conference devoted to Theory, Numerics and Applications of Hyperbolic Problems, held in Padova, June 24–29, 2012. This was the fourteenth in a highly successful series of bi-annual meetings, which brought together several leading experts in the field, practitioners, and young researchers, discussing the latest theoretical advances and the most relevant applications.[1]

Hyperbolic conservation laws is a mathematical discipline deeply rooted in the tradition of classical continuum mechanics, and yet replenished with challenging open problems. It has experienced continued growth in recent years, thanks to the introduction of new ideas and techniques, and a wealth of new applications. The HYP2012 meeting highlighted a number of topics where recent progress has been particularly significant: singular limits and dispersive equations in mathematical physics, nonlinear wave patterns in several space dimensions, particle dynamics, multiphase flow and interfaces, transport in complex environments, control problems for hyperbolic PDEs and related Hamilton-Jacobi equations, general relativity and geometric PDEs.

The conference was attended by 340 participants from 30 different countries. The social program included a boat excursion to the historical Villa Pisani and to Villa Foscari - La Malcontenta, and a conference banquet in the great hall of the 13-th century building "Palazzo della Ragione" in Padova, which was once the seat of the City Council, with frescoes from the Giotto school. During the dinner, Professor Constantine Dafermos, Professor James Glimm, and Professor Tai Ping Liu were honored with the "Galileo medal" for scientific excellence conferred by the Mayor of Padova, Flavio Zanonato. A keynote speech was delivered by Professor James Glimm. Professor Glimm is credited with many pioneering contributions in the general area of the theory and numerics of hyperbolic equations. His speech provided an overview of the field, from its early days to the present time, with an outlook toward the role of hyperbolic PDE models in interdisciplinary science. The conference banquet also featured the brilliant performance of the Marco Castelli quartet, one of the most talented Italian jazz groups, introduced by Silvia Faggian from the University of Venice.

The present volume of proceedings contains 7 papers from plenary speakers, 9 from invited speakers, and 100 papers related to contributed talks. These contributions cover a wide range of topics. A very partial list includes: new methods for constructing turbulent solutions to multi-dimensional systems of conservation laws based on Baire category, transport equations with non-Lipschitz velocity fields, relative entropy functionals and the stability of fluid systems, numerical methods for hyperbolic systems with stiff relaxation terms and for multiphase flow, new advances

[1] The detailed program and the slides of all speakers as well as most of the video of the plenary speakers of the HYP2012 conference can be found on the website `http://www.hyp2012.eu/` .The HYP2012 website will be accessible at this address untill 2020.

in homogenization theory, optimal sensor location for solutions to multidimensional wave equations, singularities in general relativity.

We believe that this volume will provide a timely survey of the state of the art, and a stimulus for further progress in this exciting field.

We take this opportunity to thank the members of the HYP2012 Scientific Committee (listed at `http://www.hyp2012.eu/organization/scientific-committee`) for their expertise in the selection of the plenary and invited speakers of the conference and for their contribution in reviewing the papers of the volume. We would like also to express our warm appreciation to all other members of the HYP2012 Organizing Committee (listed at `http://www.hyp2012.eu/organization/organizing-committee`) that in various ways have contributed to the successful realization of this event. Finally, we are extremely thankful to the many graduate students and post-docs of the Dipartimento di Matematica of Università di Padova, coordinated by Khai T. Nguyen and Fabio S. Priuli, for their assistance and dedicated work throughout the conference.

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CONTENTS

Part 3: Contributed Talks 233–1066

ASYMPTOTICALLY IMPLICIT SCHEMES FOR THE HYPERBOLIC HEAT EQUATION

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Abstract. The main concern of this paper is to derive numerical schemes for the solution of kinetic equations with diffusive scaling which works efficiently for a wide range of the scaling parameter ε. We will concentrate on the simple Goldstain-Taylor model from kinetic theory and propose a resolution method based on the reformulation first introduced in [8]. We show how this reformulation corresponds to the use of interpolated fluxes and then we adopt the penalized implicit-explicit Runge-Kutta approach recently introduced in [1] to overcome the parabolic time step restriction in the diffusive regime. The resulting schemes permit to choose a time step $\Delta t = O(\Delta x)$, independent from ε, in all regimes. Some numerical examples show the efficiency and accuracy of the proposed methods.

1. **Introduction.** Kinetic theory of rarefied gases has been used for studies in various and very important fields of research and nowadays it still represents one of the most powerful tools for simulations and applications for cutting edge sectors of academic and industrial research.

The aim of this paper is to derive a numerical scheme for the solution of kinetic equations with diffusive scaling which works efficiently for a wide range of the scaling parameter ε. The Goldstain-Taylor model, despite its simplicity, is a prototype kinetic equation which contains some of the major difficulties encountered when dealing with more sophisticated kinetic models in the diffusive scaling. For this reason it is often used for the derivation and the analysis of robust numerical schemes for the diffusion limit [2, 4, 10, 11]. In particular, for small values of the scaling parameter, the kinetic model is well approximated by a standard heat equation for the mass density. This is the rationale behind the use of the terminology *hyperbolic heat equation* when considering the Goldstain-Taylor model in the diffusive scaling.

More in general when dealing with kinetic equations in the diffusive scaling, it is necessary to perform numerical simulations in both resolved (when spatial grid size is smaller than the mean free path) and under-resolved (when spatial grid size is larger than the mean free path) regimes. In the former case, standard numerical methods for kinetic equations work effectively. In the latter case, one expects that the best possible numerical solution is the approximation of the diffusion equation.

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Key words and phrases. Asymptotically implicit schemes, hyperbolic heat equation, IMEX schemes, Goldstain-Taylor model, diffusive scaling.

However, even to achieve this goal is nontrivial for standard numerical methods when applied to kinetic equations. Earlier studies on numerical methods for transport or kinetic equations indicate that, in order for the under-resolved numerical approximation to capture the correct diffusive behavior, the scheme should be *asymptotic preserving* (AP), in the sense that the asymptotic limit that leads from the transport or kinetic equations to the diffusion equations should be preserved at the discrete level [6]. Moreover, an efficient method should be able to numerically go along with the physics of the system which goes from an hyperbolic behavior underlying the kinetic regime to a parabolic behavior in the limiting regime.

Here we described a general approach to tackle this kind of problems. For the construction of the method first we rely on the problem reformulation presented in [8] and show that the method can be seen as a natural way to construct interpolated fluxes for the transport part. It is well-known that this reformulation in combination with a splitting method or an Implicit-Explicit (IMEX) time discretization provides a consistent way to approximate the kinetic model in the different regimes avoiding the time constraints $\Delta t = O(\varepsilon \Delta x)$ for small values of ε.

In fact, since the characteristic speeds of the hyperbolic part are of order $1/\varepsilon$, standard approaches developed for hyperbolic systems with stiff relaxation become useless in such parabolic scaling, because the CFL condition would require $\Delta t = O(\varepsilon\, \Delta x)$. Of course, in the diffusive regime where $\varepsilon \ll \Delta x$, this is too much restrictive since also for an explicit method a parabolic condition $\Delta t = O(\Delta x^2)$ would suffice.

Most previous works on asymptotic preserving schemes for hyperbolic systems and kinetic equations with diffusive relaxation focus on schemes which, in the limit of infinite stiffness, become consistent explicit schemes for the diffusive limit equation [2, 4, 5, 8, 11]. Such explicit (in the limit) schemes clearly suffer from the usual parabolic stability restriction $\Delta t = O(\Delta x^2)$. Here, following the strategy presented in [1], we construct schemes that work uniformly in ε and that, in the diffusion limit, originate a fully implicit solver for the diffusion equation. Therefore the resulting schemes permit to choose a time step $\Delta t = O(\Delta x)$ in all regimes.

The paper is organized as follows: in the next section we present the prototype system we are going to solve together with the proposed techniques used to implement our strategy. Later section is devoted to the presentation of system's discretization and is followed by the section of the numerical results. A concluding section ends the paper.

2. The Goldstain-Taylor model.

A two-velocity model of the Boltzmann equation describes the behavior of a fictitious gas of two kind of particles that move parallel to the $x-$axis with constant and equal speed. We can consider at time t the particles with a density $f(x,t)$, which move in the positive $x-$direction, and the particles which move in the negative $x-$direction with a density $g(x,t)$. The simplest two-velocity gas, which is in local equilibrium when $f = g$, is described by the following hyperbolic system

$$\begin{aligned} f_t + c f_x &= k(g - f), \\ g_t - c g_x &= k(f - g), \end{aligned} \tag{1}$$

where $c > 0$ and $k > 0$ characterize respectively the velocities and the interactions of particles.

In the *diffusive scaling* we consider the system of equations (1) in the form

$$\begin{aligned} f_t + \frac{c}{\varepsilon} f_x &= \frac{k}{\varepsilon^2}(g-f), \\ g_t - \frac{c}{\varepsilon} g_x &= \frac{k}{\varepsilon^2}(f-g). \end{aligned} \tag{2}$$

In (2) ε is called the *relaxation time* and the limit problem for $\varepsilon \to 0$ is called *diffusive limit*.

Introducing the macroscopic variables $u = f+g$ and $v = c(f-g)/\varepsilon$, corresponding to the mass density and the flux, the model is rewritten in the form of a relaxation system

$$\begin{aligned} u_t + v_x &= 0, \\ v_t + \frac{c^2}{\varepsilon^2} u_x &= -\frac{2k}{\varepsilon^2} v. \end{aligned} \tag{3}$$

In the limit $\varepsilon = 0$ formally we obtain the local equilibrium $2kv = -c^2 u_x$ and the system reduces to the heat equation

$$u_t = \frac{c^2}{2k} u_{xx}.$$

For notation simplicity in the sequel we will assume $c = 1$ and $k = 1/2$, so that the limiting heat equation has a constant diffusion coefficient equal to one.

We will introduce the spatial grid points $x_{i+1/2}$, $i = \dots, -1, 0, 1, \dots$ with uniform mesh width $\Delta x = x_{i+1/2} - x_{i-1/2}$. As usual we denote by $U_{i+1/2}(t) = U(x_{i+1/2}, t)$ the nodal values and by $U_i(t)$ the cell averages of U in the cell $[x_{i-1/2}, x_{i+1/2}]$ at time t

$$U_i(t) = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} U(x,t)\, dx.$$

Thus we consider the following semi-discrete system in conservative form

$$\begin{aligned} (u_i)_t + \frac{v_{i+1/2} - v_{i-1/2}}{\Delta x} &= 0, \\ (v_i)_t + \frac{1}{\varepsilon^2} \frac{u_{i+1/2} - u_{i-1/2}}{\Delta x} &= -\frac{1}{\varepsilon^2} v_i. \end{aligned} \tag{4}$$

As observed in [11] a standard upwind selection of the numerical fluxes in (4) originates a numerical dissipation which may dominate the physical one. For example, the first order upwinding yields

$$\begin{aligned} u_{i\pm1/2} &= \frac{1}{2}(u_i + u_{i\pm1}) \pm \frac{\varepsilon}{2}(v_i - v_{i\pm1}), \\ v_{i\pm1/2} &= \frac{1}{2}(v_i + v_{i\pm1}) \pm \frac{1}{2\varepsilon}(u_i - u_{i\pm1})\,. \end{aligned} \tag{5}$$

It is easy to verify that the use of (5) in the discrete equation (4) for small values of ε leads to the semi-discrete approximation

$$(u_i)_t - \frac{u_{i+2} + u_{i-2} - 2u_i}{(2\Delta x)^2} - \frac{\Delta x}{2\varepsilon} \frac{u_{i+1} + u_{i-1} - 2u_i}{\Delta x^2} = 0, \tag{6}$$

for which the numerical dissipation dominates the physical one unless $\Delta x \ll \varepsilon$. Note also that we obtain a wider stencil for the second order derivative in the heat equation with respect to the classical one. There are several alternatives to deal with this problem, see for example [4, 7, 10, 11].

The upwind selection (5) can be modified in another way so as to capture the proper parabolic behavior. The idea now is to apply the upwind selection only in the hyperbolic regime and to use a standard central discretization in the parabolic one. This can be better understood by rewriting system (3) in the form [8]

$$\begin{aligned} &u_t + v_x = 0 \\ &v_t + \phi^2 u_x = -\frac{1}{\varepsilon^2}\left(v + (1-\phi^2\varepsilon^2)u_x\right), \end{aligned} \tag{7}$$

where $\phi = \phi(\varepsilon)$ is a suitable function such that $\phi = O(1)$ for small values of ε. The above reformulation is equivalent to rewrite

$$u_x = \underbrace{\phi^2\varepsilon^2\, u_x}_{\text{hyperbolic regime}} + \underbrace{\left(1-\phi^2\varepsilon^2\right) u_x}_{\text{parabolic regime}}, \tag{8}$$

the key point now is to combine two different discretization for the different regimes. For example in [8] it was proposed to consider an upwind discretization on the variables $(u \pm v)/2$ for the first derivative combined with a central scheme for the second term. This interpolation is also relevant in term of the time discretization of the system, since the hyperbolic flux can be evaluated explicitly whereas for the parabolic flux it is desirable to have an implicit integrator.

In the simplest case, taking $\phi \equiv 1$ this idea leads to the modified fluxes

$$\begin{aligned} u_{i\pm1/2} &= \frac{1}{2}(u_i + u_{i\pm1}) \pm \frac{\varepsilon^2}{2}(v_i - v_{i\pm1}), \\ v_{i\pm1/2} &= \frac{1}{2}(v_i + v_{i\pm1}) \pm \frac{1}{2}(u_i - u_{i\pm1}). \end{aligned} \tag{9}$$

With these fluxes, for small values of ε, the semi-discrete system (4) gives

$$(u_i)_t - \frac{u_{i+2} + u_{i-2} - 2u_i}{(2\Delta x)^2} - \frac{\Delta x}{2}\frac{u_{i+1} + u_{i-1} - 2u_i}{\Delta x^2} = 0, \tag{10}$$

which is an approximation of the equilibrium heat equation with an accuracy of $O(\Delta x/2)$. Second order extension can be obtained easily simply increasing to second order the choice of the flux used for the hyperbolic derivative. By combining this with standard implicit-explicit discretizations one obtain a scheme that work uniformly in ε with a CFL condition of the type $\Delta t = O(\Delta x^2)$ in the limit $\varepsilon \to 0$.

3. **Asymptotically implicit IMEX schemes.** In order to write a stable discretization to system (3) we should use implicit temporal integrators on the stiff terms. Because the stiffness of the convection term depends on the size of the two eigenvalues $\pm 1/\varepsilon$ of the Riemann invariant form (2) of system (3), both convection terms have to be implicit. However a fully implicit scheme causes global data dependencies, a severe disadvantage if we consider second order nonlinear schemes using slope limiters. Furthermore, the gain of stability is partially offset by the loss of accuracy typical of implicit schemes in the context of wave-propagation phenomena. The approach just introduced permits to overcome this problem since we can apply an Implicit-Explicit discretization in the form

$$\begin{aligned} &u_t + \underbrace{v_x}_{\text{Explicit}} = 0, \\ &v_t + \underbrace{\phi^2 u_x}_{\text{Explicit}} = \underbrace{-\frac{1}{\varepsilon^2}(v + (1-\phi^2\varepsilon^2)u_x)}_{\text{Implicit}} \end{aligned} \tag{11}$$

This approach was proposed in [8] and originates an explicit discretization of the heat equation in the limit. We refer to this standard approach as *asymptotically explicit* IMEX method.

Here we adopt a different technique which has been proposed recently in [1] which permits to overcome the parabolic time step limitation for small regimes of ε. The method is based on a penalization technique consisting in adding and subtracting to the first equation in system (3) an appropriate term, which will establish the correct limiting diffusion equation. This lead to the modified system

$$\begin{aligned} u_t + \Big(v + \mu\, u_x\Big)_x &= \mu\, u_{xx}, \\ v_t + \frac{1}{\varepsilon^2} u_x &= -\frac{1}{\varepsilon^2} v, \end{aligned} \tag{12}$$

where $\mu = \mu(\varepsilon)$ is such that $\mu(0) = 1$. Finally, using the interpolated fluxes approach we end up with the following system

$$\begin{aligned} u_t + \Big(v + \mu\, u_x\Big)_x &= \mu\, u_{xx}, \\ v_t + \phi^2 u_x &= -\frac{1}{\varepsilon^2}\left(v + (1-\phi^2\varepsilon^2) u_x\right), \end{aligned} \tag{13}$$

which is now tackled with an IMEX scheme in the form

$$\begin{aligned} u_t + \underbrace{\Big(v + \mu\, u_x\Big)_x}_{\text{Explicit}} &= \underbrace{\mu\, u_{xx}}_{\text{Implicit}}, \\ v_t + \underbrace{\phi^2 u_x}_{\text{Explicit}} &= \underbrace{-\frac{1}{\varepsilon^2}\left(v + (1-\phi^2\varepsilon^2) u_x\right)}_{\text{Implicit}} \end{aligned} \tag{14}$$

In addition to the space discretizations of the first order derivatives described in the last section let us specify that in (14) the second order space derivative in the first equation is discretized using a standard second order central discretization. Note that in the limit $\varepsilon \to 0$ we obtain a fully implicit discretization of the heat equation with a standard compact stencil instead of the explicit discretization on a wider stencil described before. The only additional cost is due to the inversion of the linear system originated by the implicit discretization of the heat equation. We refer to this new approach as *asymptotically implicit* IMEX method.

Let us finally recall the general structure of the IMEX-RK scheme adopted. Here we omit the space derivatives for simplicity of notation. For the internal stages for $k = 1, \dots, \nu$ we have

$$U^{(k)} = u^n - \Delta t \sum_{j=0}^{k-1} \widetilde{a}_{kj} \left(V^{(j)} + \mu U_x^{(j)}\right)_x + \mu \Delta t \sum_{j=1}^{k} a_{kj} U_{xx}^{(j)}$$

$$V^{(k)} = v^n - \Delta t \phi^2 \sum_{j=0}^{k-1} \widetilde{a}_{kj} U_x^{(j)} - \frac{\Delta t}{\varepsilon^2} \sum_{j=1}^{k} a_{kj} \left[V^{(j)} + (1-\phi^2\varepsilon^2) U_x^{(j)}\right].$$

For the numerical solution we simply have $u^{n+1} = U^\nu$ and $v^{n+1} = V^\nu$ since we restrict to globally stiffly accurate IMEX schemes [1]. In all considered IMEX schemes, matrix $\widetilde{A}$ is lower triangular with zero diagonal, while matrix A is lower triangular, i.e. the implicit scheme is a Diagonally Implicit Runge-Kutta (DIRK) scheme. This choice guarantees that implicit terms are, indeed, always explicitly

evaluated. For more details on properties and requirements of IMEX-RK schemes when applied to kinetic equations in the diffusion limit we refer to [1] and the references therein.

4. **Numerical results.** We present some numerical results in order to show the behavior of the proposed scheme. In the computed results second order accuracy is obtained using a WENO scheme for the hyperbolic derivative whereas the parabolic flux is always evaluated by central difference. First order Implicit-Explicit Euler scheme and second order ARS$(2,2,2)$ scheme are adopted for the time discretization (see [1]). In all test cases the initial conditions are given by

$$u_0(x) = 1 \text{ and } v_0(x) = 0 \text{ if } x \leq 0,$$
$$u_0(x) = 0 \text{ and } v_0(x) = 0 \text{ if } x > 0.$$

The space variable x ranges in the computational domain $[-1,1]$ and we choose N_x grid points.

Test 1. In this case computations are performed in the hyperbolic regime for $\varepsilon = 1$ and consider system (13) with $\phi \equiv 1$ and $\mu \equiv 0$. For this test case, we assume $\Delta t = \Delta x/2$ and $N_x = 100$. In fig. 4 we show the results of the computation at time $T_f = 0.3$ of the u component. In this non equilibrium regime we can see that the transport term, which dominant with respect to the source term, is properly approximated.

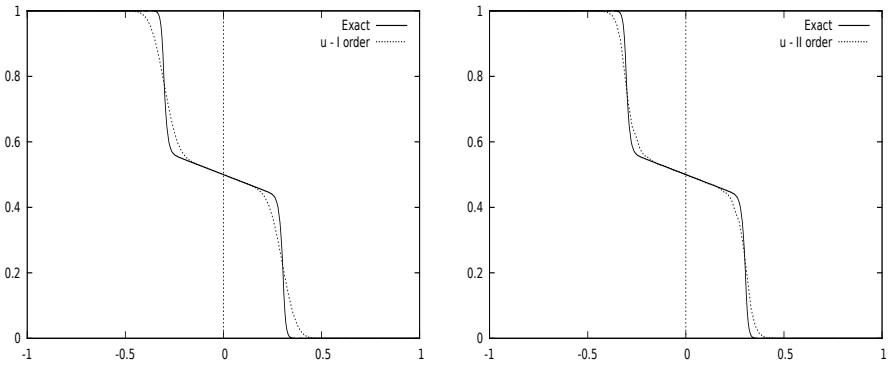

FIGURE 1. Test 1: Hyperbolic regime. Solution for $u(x)$ at time $T_f = 0.3$ for $\varepsilon = 1$. Left: first order method. Right: second order method.

Test 2. Here we consider the parabolic regime for $\varepsilon = 10^{-3}$, and approximate system (13) with $\phi \equiv 1$ and $\mu \equiv 1$. For this test case, we can assume $\Delta t = \Delta x$ and $N_x = 100$. In fig. 4 we show the result of the computation at time $T_f = 0.1$ of the u component. We stop the computation before the stationary state is reached, thus showing a plot which shows we catch the correct diffusive behavior; in this case, the contribution given by the transport term is less important then that of the relaxation term.

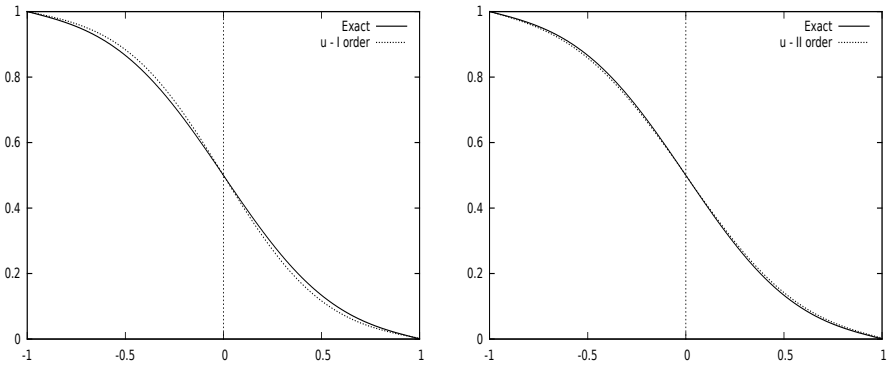

FIGURE 2. Test 2: Parabolic regime. Solution for $u(x)$ at time $T_f = 0.1$ for $\varepsilon = 10^{-3}$. Left: first order method. Right: second order method.

Test 3. In the last test case we consider a mixing regime where on the left of the computational domain for $x \leq 0$ we are in the hyperbolic regime with $\varepsilon = \varepsilon_L = 0.2$ while on the right for $x > 0$ we are in the parabolic regime with $\varepsilon = \varepsilon_R = 0.01$. For this last test case we consider $N_x = 60$ points as in [2]. We set $\mu \equiv 1$ while we set $\phi = 1/\varepsilon_L$ for $x \leq 0$ and $\phi = 1$ for $x > 0$ (we consider the correct "physical" velocity in the kinetic regime). For this test case, we set $\Delta t = \varepsilon_L \Delta x/5$. In fig. 4 we show the result of the computation at time $T_f = 0.05$ of the u component.

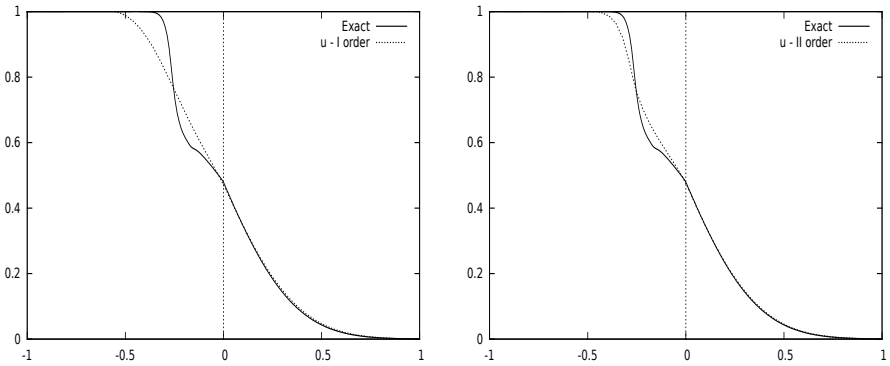

FIGURE 3. Test 3: Mixing regime. Solution for $u(x)$ at time $T_f = 0.05$ for $\varepsilon = 0.2$ for $x < 0$ and $\varepsilon = 10^{-3}$ for $x > 0$. Left: first order method. Right: second order method.

We observe that our scheme is able to efficiently compute the solution, with a good behavior also at the interface between the two regimes. We refer to [2] for comparisons.

5. **Conclusions.** We have presented a general approach to tackle kinetic equations in the diffusive scaling which leads to a fully implicit discretization of the limiting diffusion terms. The method is based on combining the strategies presented in [8]

for the space derivatives together with the time discretization presented in [1]. We emphasize that different choices of the space discretizations may originate schemes with different stability properties. In particular the optimal choice of the schemes parameters ϕ and μ which permits to switch between the hyperbolic and the parabolic flux and the explicit and implicit integrators deserves further investigations.

REFERENCES

[1] S. Boscarino, L. Pareschi and G. Russo, *Implicit-Explicit Runge-Kutta schemes for hyperbolic Systems and kinetic equations in the diffusion limit*, SIAM J. Sci. Comp. to appear (arXiv:abs/1110.4375v2).

[2] A.C. Boulanger, C. Cancès, H. Mathis, K. Saleh and N. Seguin, *OSAMOAL: Optimized Simulations by Adapted Models using Asymptotic Limits*, ESAIM: proceedings, (2012) to appear.

[3] G. Dimarco, L. Pareschi and V. Rispoli, *Implicit-Explicit Runge Kutta schemes for the Boltzmann-Poisson equation for semiconductors*, preprint (2012).

[4] L. Gosse and G. Toscani, *An asymptotic-preserving well-balanced scheme for the hyperbolic heat equations*, C. R. Math. Acad. Sci. Paris, **334** (2002), 337–342.

[5] L. Gosse and G. Toscani, *Space localization and well-balanced schemes for discrete kinetic models in diffusive regimes*, SIAM J. Numer. Anal., **41** (2003), 641–658.

[6] S. Jin, *Efficient asymptotic-preserving (AP) schemes for some multiscale kinetic equations*, SIAM J. Sci. Comput., **21** (1999), 441–454.

[7] S. Jin and D. Levermore, *Numerical schemes for hyperbolic conservation laws with stiff relaxation*, J. Comp. Phys. **126** (1996), 449–467.

[8] S. Jin, L. Pareschi and G. Toscani, *Diffusive Relaxation Schemes for Multiscale Discrete-Velocity Kinetic Equations*, SIAM Journal on Numerical Analysis, **35** (1998), 2405–2439.

[9] S. Jin and Z. Xin, *The relaxation schemes for systems of conservation laws in arbitrary space dimensions*, Comm. Pure Appl. Math. **48** (1995), 235–277.

[10] R.B. Lowrie and J.E. Morel, *Methods for Hyperbolic Systems with Stiff Relaxation*, Int. J. Num. Meth. Fluids, **40** (2002), 413–423.

[11] G.Naldi, L.Pareschi, *Numerical schemes for kinetic equations in diffusive regimes*, Applied Math. Letters, **11** (1998), 29–35.

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STEADY SELF-SIMILAR INVISCID FLOW

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ABSTRACT. We consider admissible L^∞ solutions to 2-dimensional $m \times m$ systems of hyperbolic conservation laws that are steady in time, constant along rays emanating from the origin, and sufficiently small perturbations of a constant background state. We classify the possible structures of these solutions and prove that they must be BV. As a special case, we obtain uniqueness in this class of L^∞ functions for admissible forward in time solutions to 1-dimensional Riemann problems and show that backward in time solutions also must be BV.

1. **Introduction.** We consider a 2-dimensional system of conservation laws

$$U_t + f^x(U)_x + f^y(U)_y = 0. \tag{1}$$

Here the unknown U and the flux functions f^x and f^y take values in $\mathbb{R}^m$. The fluxes are assumed to be smooth and possess an entropy-entropy flux pair $(\eta, \vec{\psi}) = (\eta, \psi^x, \psi^y)$ with uniformly convex η on some open nonempty set in state space. The entropy inequality is then

$$\eta(U)_t + \psi^x(U)_x + \psi^y(U)_y \leq 0. \tag{2}$$

In multidimensional inviscid Euler flow, there are well studied cases in which there exists a distinguished point around which the flow is, to first order, constant along rays starting at this point. These include regular reflection (four shock waves meeting at a point) (see [7], [6], [3]) and Mach reflection (three shocks and a contact meeting at a point) (see [1]). However, other configurations such as triple points (three shocks with no other waves in between) cannot occur for most commonly used equations of state (see [13],[4], [12]). Beyond these and some other special cases, the possible configurations of such waves meeting at a point have not been classified.

The flow is steady from the point of view of an observer moving with the distinguished point described above. Therefore, we consider solutions of the form

$$U(t, x, y) = U(\phi), \text{ with } \phi = \angle(x, y) \in [0, 2\pi). \tag{3}$$

We are also motivated by a possible nonuniqueness result found in [5], in which an unsteady numerical solution was observed which took a steady self-similar solution as initial data. Perhaps analysis of the steady problem will lead to an analytical example of nonuniqueness of this form.

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Hyperbolic Problems: Theory, Numerics, Applications

The International Conference devoted to Theory, Numerics and Applications of Hyperbolic Problems, HYP2012, was held in Padova on June 24–29, 2012. The conference was the fourteenth in a highly successful series of bi-annual meetings that has become one of the most important international events in Applied Mathematics. The volume contains more than 110 contributions that were presented in this conference, including plenary presentations by C. De Lellis, E. Feireisl, N. Masmoudi, S. Mishra, G. Russo, J. Sethian, E. Zuazua, and a contribution by the keynote speaker J. Glimm. These contributions cover a wide range of topics. A very partial list includes: new methods for constructing turbulent solutions to multi-dimensional systems of conservation laws based on Baire category, transport equations with non-Lipschitz velocity fields, relative entropy functionals and the stability of fluid systems, numerical methods for hyperbolic systems with stiff relaxation terms and for multiphase flow, new advances in homogenization theory, optimal sensor location for solutions to multidimensional wave equations, singularities in general relativity. The volume should appeal to researchers, students and practitioners with general interest in time-dependent problems governed by hyperbolic equations.

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