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Hyperbolic Problems: Theory, Numerics, Applications

Fabio Ancona
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Editors



American Institute of Mathematical Sciences

Hyperbolic Problems: Theory, Numerics, Applications

Proceedings
of the Fourteenth International Conference
on Hyperbolic Problems
held in Padova,
June 25-29, 2012

Fabio Ancona,
Alberto Bressan,
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Andrea Marson
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American Institute of Mathematical Sciences

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PREFACE

This volume contains the Proceedings of the HYP2012 International Conference devoted to Theory, Numerics and Applications of Hyperbolic Problems, held in Padova, June 24–29, 2012. This was the fourteenth in a highly successful series of bi-annual meetings, which brought together several leading experts in the field, practitioners, and young researchers, discussing the latest theoretical advances and the most relevant applications.¹

Hyperbolic conservation laws is a mathematical discipline deeply rooted in the tradition of classical continuum mechanics, and yet replenished with challenging open problems. It has experienced continued growth in recent years, thanks to the introduction of new ideas and techniques, and a wealth of new applications. The HYP2012 meeting highlighted a number of topics where recent progress has been particularly significant: singular limits and dispersive equations in mathematical physics, nonlinear wave patterns in several space dimensions, particle dynamics, multiphase flow and interfaces, transport in complex environments, control problems for hyperbolic PDEs and related Hamilton-Jacobi equations, general relativity and geometric PDEs.

The conference was attended by 340 participants from 30 different countries. The social program included a boat excursion to the historical Villa Pisani and to Villa Foscari - La Malcontenta, and a conference banquet in the great hall of the 13-th century building “Palazzo della Ragione” in Padova, which was once the seat of the City Council, with frescoes from the Giotto school. During the dinner, Professor Constantine Dafermos, Professor James Glimm, and Professor Tai Ping Liu were honored with the “Galileo medal” for scientific excellence conferred by the Mayor of Padova, Flavio Zanonato. A keynote speech was delivered by Professor James Glimm. Professor Glimm is credited with many pioneering contributions in the general area of the theory and numerics of hyperbolic equations. His speech provided an overview of the field, from its early days to the present time, with an outlook toward the role of hyperbolic PDE models in interdisciplinary science. The conference banquet also featured the brilliant performance of the Marco Castelli quartet, one of the most talented Italian jazz groups, introduced by Silvia Faggian from the University of Venice.

The present volume of proceedings contains 7 papers from plenary speakers, 9 from invited speakers, and 100 papers related to contributed talks. These contributions cover a wide range of topics. A very partial list includes: new methods for constructing turbulent solutions to multi-dimensional systems of conservation laws based on Baire category, transport equations with non-Lipschitz velocity fields, relative entropy functionals and the stability of fluid systems, numerical methods for hyperbolic systems with stiff relaxation terms and for multiphase flow, new advances

¹The detailed program and the slides of all speakers as well as most of the video of the plenary speakers of the HYP2012 conference can be found on the website <http://www.hyp2012.eu/>. The HYP2012 website will be accessible at this address until 2020.

in homogenization theory, optimal sensor location for solutions to multidimensional wave equations, singularities in general relativity.

We believe that this volume will provide a timely survey of the state of the art, and a stimulus for further progress in this exciting field.

We take this opportunity to thank the members of the HYP2012 Scientific Committee (listed at <http://www.hyp2012.eu/organization/scientific-committee>) for their expertise in the selection of the plenary and invited speakers of the conference and for their contribution in reviewing the papers of the volume. We would like also to express our warm appreciation to all other members of the HYP2012 Organizing Committee (listed at <http://www.hyp2012.eu/organization/organizing-committee>) that in various ways have contributed to the successful realization of this event. Finally, we are extremely thankful to the many graduate students and post-docs of the Dipartimento di Matematica of Università di Padova, coordinated by Khai T. Nguyen and Fabio S. Priuli, for their assistance and dedicated work throughout the conference.

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CONTENTS

Preface	i
Part 1: Plenary Lectures 1–107	
Camillo De Lellis, Elisabetta Chiodaroli and Ondřej Kreml Surprising solutions to the isentropic Euler system of gas dynamics	1
Eduard Feireisl Relative entropies, dissipative solutions, and singular limits of complete fluid systems	11
David Gérard-Varet and Nader Masmoudi Recent progress in the theory of homogenization with oscillating Dirichlet data	29
Franz Fuchs, Andrew McMurry, Nils Henrik Risebro and Siddhartha Mishra Explicit and implicit finite volume schemes for radiation MHD and the effects of radiation on wave propagation in stratified atmospheres	41
Sebastiano Boscarino and Giovanni Russo Implicit-explicit Runge-Kutta schemes for hyperbolic systems with stiff relaxation and applications	61
Robert Saye and James Sethian New interface methods for tracking multiphase multiphysics	81
Yannick Privat, Emmanuel Trélat and Enrique Zuazua Optimal sensor location for wave and Schrödinger equations	89
Part 2: Invited Lectures 109–232	
Gianluca Crippa Ordinary differential equations and singular integrals	109
Jean-Luc Guermond and Bojan Popov Entropy viscosity for the Euler equations and questions regarding parabolic regularization	119
Soohyun Bae, Sun-Ho Choi and Seung-Yeal Ha Dynamic instability of the Vlasov-Poisson-Boltzmann system in high dimensions	125
Rinaldo M. Colombo and Michael Herty Nodal conditions for hyperbolic systems of balance laws	147
Corrado Lattanzio and Athanasios E. Tzavaras Relative entropy methods for hyperbolic and diffusive limits	163
Pierre Degond, Amic Frouvelle and Jian-Guo Liu A note on phase transitions for the Smoluchowski equation with dipolar potential	179
Katrin Grunert, Helge Holden and Xavier Raynaud Lipschitz metric for the two-component Camassa–Holm system	193

Moritz Reintjes	
Points of shock wave interaction are ‘regularity singularities’ in spacetime	209
Michael Shearer	
Two fluid flow in porous media	221
Part 3: Contributed Talks 233–1066	
Eduardo Abreu	
Numerical simulation of three-phase flow in heterogeneous media with spatially varying nonlinear hyperbolic-parabolic flux functions	233
Franz Achleitner, Sabine Hittmeir and Christian Schmeiser	
On nonlinear conservation laws regularized by a Riesz-Feller operator	241
Fabio Ancona, Olivier Glass and Khai T. Nguyen	
On quantitative compactness estimates for hyperbolic conservation laws	249
Boris Andreianov	
One-dimensional conservation law with boundary conditions: General results and spatially inhomogeneous case	259
Denise Aregba-Driollet	
The Riemann problem for Kerr equations and non-uniqueness of selfsimilar entropy solutions	269
Thomas Auphan, Philippe Angot and Olivier Guès	
Penalty methods for edge plasma transport in a tokamak	277
Hantaek Bae and Konstantina Trivisa	
On the Doi model for the suspensions of rod-like molecules in compressible fluids	285
Luís Almeida and Patrizia Bagnerini	
Multi-scale tissular-cellular model for wound healing	293
Joshua Ballew	
Low Mach number limits to the Navier-Stokes-Smoluchowski system	301
Caroline Bauzet	
Method for solving a stochastic conservation law	309
Stefan Berres, Frank Huth, Hartmut Schwandt and Günter Bärwolff	
A multi-phase model for pedestrian flow with strategic and tactic component .	317
Luigi C. Berselli	
Some results on the two-dimensional dissipative Euler equations	325
Raul Borsche and Jochen Kall	
ADER-schemes on networks of scalar conservation laws	333
Laurent Boudin and Julien Mathiaud	
Asymptotic behavior of a diffusive scheme solving the inviscid one-dimensional pressureless gases system	341
Benjamin Boutin, Frédéric Coquel and Philippe G. LeFloch	
Coupling techniques for nonlinear hyperbolic equations	349
Alberto Bressan	
Globally optimal and Nash equilibrium solutions for traffic flow on networks .	357

M. Briani, D. Aregba-Driollet and R. Natalini How to improve the decay of the numerical error for large times: The case of dissipative BGK systems	365
Buğra Kabil On the asymptotics of solutions to resonator equations	373
Raimund Bürger, Pep Mulet and Luis M. Villada Adaptive mesh refinement for spectral WENO schemes for efficient simulation of polydisperse sedimentation processes	381
Sunčica Čanić and Boris Muha A nonlinear moving-boundary problem of parabolic-hyperbolic-hyperbolic type arising in fluid-multi-layered structure interaction problems	389
Giovanni Alberti, Stefano Bianchini and Laura Caravenna Reduction on characteristics for continuous solutions of a scalar balance law ..	399
Filipe Carvalho and Ana Jacinta Soares Detonation wave problems: Modeling, numerical simulations and linear stability	407
Pablo Castañeda, Dan Marchesin and Frederico Furtado On singular points for convex permeability models	415
Fausto Cavalli Linearly implicit schemes for convection-diffusion equations	423
Nikolai Chemetov and Fernanda Cipriano The inviscid limit for slip boundary conditions	431
Koottungal Revi Arun, Guoxian Chen and Sebastian Noelle A finite volume evolution Galerkin scheme for acoustic waves in heterogeneous media	439
Fabio Ancona and Giuseppe Maria Coclite On the asymptotic stabilization of a generalized hyperelastic-rod wave equation	447
Armando Coco and Giovanni Russo Boundary treatment in ghost point finite difference methods for compressible gas dynamics in domain with moving boundaries	455
Rinaldo M. Colombo, Mauro Garavello, Magali Lécureux-Mercier and Nikolay Pogodaev Conservation laws in the modeling of moving crowds	467
Andrea Corli and Haitao Fan A hyperbolic model for phase transitions in porous media	475
Kristian Debrabant and Espen R. Jakobsen Semi-Lagrangian schemes for linear and fully non-linear Hamilton-Jacobi- Bellman equations	483
A.I. Delis and I.K. Nikolos On a solution reconstruction and limiting procedure for unstructured finite volumes	491
Maria Laura Delle Monache and Paola Goatin A strongly coupled PDE-ODE system modeling moving density	

constraints in traffic flow	501
Bruno Després, Christophe Buet and Emmanuel Franck	
A priori analysis of asymptotic preserving schemes with the modified equation	509
Liviu Florin Dinu	
Wave-wave interactions of a gasdynamic type	515
Donatella Donatelli and Pierangelo Marcati	
Analysis of oscillations and defect measures in plasma physics	525
Renjun Duan	
Asymptotic stability of kinetic plasmas for general collision potentials	533
Arnaud Duran, Fabien Marche, Christophe Berthon and Rodolphe Turpault	
Numerical discretizations for shallow water equations with source terms on unstructured meshes	541
Jan Ernest	
Schemes with Well-Controlled Dissipation (WCD) for scalar conservation laws with pseudo-parabolic regularization	551
Adriano Festa and Maurizio Falcone	
L^1 convergence of a SL scheme for the eikonal equation with discontinuous coefficients	559
Hermano Frid	
A brief note on recent developments on divergence-measure fields	567
Raul Borsche, Rinaldo M. Colombo and Mauro Garavello	
Mixed systems with boundaries	575
Mohamed-Gazibo Karimou	
Degenerate convection-diffusion equation with a Robin boundary condition ...	583
Christoph Gersbacher	
Explicit higher order schemes for the coupling of dimensionally heterogeneous free-surface flow models	591
Jan Giesselmann and Athanasios E. Tzavaras	
On cavitation in elastodynamics	599
Christophe Chalons, Mathieu Girardin and Samuel Kokh	
Operator-splitting-based asymptotic preserving scheme for the gas dynamics equations with stiff source terms	607
Christian Bourdarias, Marguerite Gisclon and Stéphane Junca	
Blow-up or not blow-up at the hyperbolic boundary for a system from chemistry?	615
Hyunkyung Lim, Yijie Zhou, Valmor F. de Almeida and James Glimm	
Fully developed turbulent mixing in an annular sector	623
Frédéric Coquel, Edwige Godlewski and Nicolas Seguin	
Fluid systems: Approximation by relaxation and coupling	631

Debora Amadori, Rinaldo M. Colombo, Graziano Guerra and Wen Shen	
Slow erosion of granular flow: Continuous and discontinuous profiles	641
Maren Hantke and Ferdinand Thein	
Numerical solutions to the Riemann problem for compressible isothermal Euler equations for two phase flows with and without phase transition	651
Itsuko Hashimoto	
Asymptotic behavior of solutions for damped wave equations with non-convex convection term on the half line	659
Boris Haspot	
Porous media equations, fast diffusions equations and the existence of global weak solution for the quasi-solutions of compressible Navier-Stokes equations ..	667
Christiane Helzel and Marsha J. Berger	
Cartesian grid embedded boundary methods for hyperbolic problems	675
Zhenying Hong, Guangwei Yuan and Xuedong Fu	
Time and space discrete scheme to suppress numerical solution oscillation for the neutron transport equations	685
François James and Nicolas Vauchelet	
Numerical simulation of a hyperbolic model for chemotaxis after blow up	693
Quansen Jiu, Yi Wang and Zhouping Xin	
Global well-posedness of 2D compressible Navier-Stokes equations with large data and vacuum	701
Pierre Castelli and Stéphane Junca	
Oscillating waves and optimal smoothing effect for one-dimensional nonlinear scalar conservation laws	709
Dante Kalise	
A WENO-TVD finite volume scheme for the approximation of atmospheric phenomena	717
Friedemann Kemm	
An enhancement to the AUFS flux splitting scheme by Sun and Takayama ...	725
K. Sudarshan Kumar, G. D. Veerappa Gowda and C. Praveen	
On polymer flooding problem in oil reservoir simulation	733
Christian Klingenberg and Ujjwal Koley	
An error estimate for viscous approximate solutions to degenerate anisotropic convection-diffusion equations	741
Michael Benfield, Helge Kristian Jenssen and Irina A. Kogan	
1-D conservative systems: A geometric approach	749
Ujjwal Koley and Nils Henrik Risebro	
Convergence of a finite difference scheme for 2×2 Keyfitz-Kranzer system ...	759
Mirko Kränkel	
A local discontinuous Galerkin scheme for compressible phase field flow	767
Gerhard Dziuk, Dietmar Kröner and Thomas Müller	
Existence and uniqueness for scalar conservation laws on moving hypersurfaces	775

Alexei A. Mailybaev	
Renormalization and universal structure of blowup in 1D conservation laws . . .	783
Vítor Matos, Pablo Castañeda and Dan Marchesin	
Classification of the umbilic point in immiscible three-phase flow in porous media	791
Thomas Müller and Axel Pfeiffer	
Well-balanced simulation of geophysical flows via the shallow water equations with bottom topography: Consistency and numerical computation	801
Martin Nolte	
Approximation of the effective Hamiltonian through a degenerate elliptic problem	809
Shinya Nishibata, Masashi Ohnawa and Masahiro Suzuki	
Nonlinear stability of boundary layer solutions to the Euler-Poisson equations in plasma physics	817
Sigrun Ortleb and Andreas Meister	
A well-balanced DG scheme with unconditionally positive implicit time integration	823
Evgeniy Yu. Panov	
On decay of periodic renormalized solutions to scalar conservation laws	831
Marica Pelanti and Keh-Ming Shyue	
A mixture-energy-consistent numerical approximation of a two-phase flow model for fluids with interfaces and cavitation	839
Carlo Morosi, Mario Pernici and Livio Pizzocchero	
A posteriori estimates for Euler and Navier-Stokes equations	847
Ramón G. Plaza	
On the stability of degenerate viscous shock profiles	857
Giacomo Dimarco, Lorenzo Pareschi and Vittorio Rispoli	
Asymptotically implicit schemes for the hyperbolic heat equation	865
Volker Elling and Joseph Roberts	
Steady self-similar inviscid flow	873
Panters Rodríguez-Bermúdez and Dan Marchesin	
Loss of strict hyperbolicity for vertical three-phase flow in porous media	881
Rinaldo M. Colombo, Paola Goatin and Massimiliano D. Rosini	
On the management of vehicular and pedestrian flows	889
Olga Rozanova	
Exact solutions with singularities to ideal hydrodynamics of inelastic gases . . .	899
Ingenuin Gasser, Martin Rybicki and Winnifried Wollner	
Modelling, simulation and optimization of gas dynamics in an exhaust pipe	907
Jochen Schütz, Michael Woopen and Georg May	
A combined hybridized discontinuous Galerkin / hybrid mixed method for viscous conservation laws	915
Gabriella Puppo and Matteo Semplice	
Finite volume schemes on 2D non-uniform grids	923

Maria Lukáčová-Medvid'ová and Nikolaos Sfakianakis	
Theoretical study of entropy dissipation of moving meshes	931
Zhi-Qiang Shao	
Almost global existence of classical discontinuous solutions to general quasilinear hyperbolic systems of conservation laws with small BV initial data	941
Julio Daniel Silva and Dan Marchesin	
Riemann solutions without intermediate constant states for a system of conservation laws	949
Gianluca Crippa, Carlotta Donadello and Laura V. Spinolo	
A note on the initial-boundary value problem for continuity equations with rough coefficients	957
Stefano Spirito and Luigi C. Berselli	
On inviscid limits for the Navier-Stokes equations with slip boundary conditions involving the vorticity	967
Marta Strani	
Metastable dynamics of interfaces for the hyperbolic Jin-Xin system	975
Andrea Terracina	
Entropy formulation for forward-backward parabolic equations	983
Dmitry Tkachev and Aleksander Blokhin	
Stability of a supersonic flow about a wedge with a weak shock wave satisfying the Lopatinski condition	991
Yuri Trakhinin	
Existence and stability of relativistic plasma-vacuum interfaces	999
Alessandro Morando, Paola Trebeschi and Yuri Trakhinin	
The linearized plasma-vacuum interface problem in ideal incompressible MHD	1007
Knut Waagan	
Relaxation-based approximate Riemann solvers for elastic rods	1015
Jean-Michel Coron and Zhiqiang Wang	
Control of a scalar conservation law with a nonlocal velocity	1023
Nils Henrik Risebro and Franziska Weber	
Burgers' equation with a filtered velocity	1031
Chi-Kun Lin and Kung-Chien Wu	
Anelastic approximation of the Gross-Pitaevskii equation for general initial data	1039
Dmitry Kotov, H.C. Yee and Bjorn Sjögren	
Comparative study of high-order positivity-preserving WENO schemes	1047
Lorenzo Zanelli	
Mather measures in semiclassical analysis	1059

ASYMPTOTICALLY IMPLICIT SCHEMES FOR THE HYPERBOLIC HEAT EQUATION

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ABSTRACT. The main concern of this paper is to derive numerical schemes for the solution of kinetic equations with diffusive scaling which works efficiently for a wide range of the scaling parameter ε . We will concentrate on the simple Goldstain-Taylor model from kinetic theory and propose a resolution method based on the reformulation first introduced in [8]. We show how this reformulation corresponds to the use of interpolated fluxes and then we adopt the penalized implicit-explicit Runge-Kutta approach recently introduced in [1] to overcome the parabolic time step restriction in the diffusive regime. The resulting schemes permit to choose a time step $\Delta t = O(\Delta x)$, independent from ε , in all regimes. Some numerical examples show the efficiency and accuracy of the proposed methods.

1. Introduction. Kinetic theory of rarefied gases has been used for studies in various and very important fields of research and nowadays it still represents one of the most powerful tools for simulations and applications for cutting edge sectors of academic and industrial research.

The aim of this paper is to derive a numerical scheme for the solution of kinetic equations with diffusive scaling which works efficiently for a wide range of the scaling parameter ε . The Goldstain-Taylor model, despite its simplicity, is a prototype kinetic equation which contains some of the major difficulties encountered when dealing with more sophisticated kinetic models in the diffusive scaling. For this reason it is often used for the derivation and the analysis of robust numerical schemes for the diffusion limit [2, 4, 10, 11]. In particular, for small values of the scaling parameter, the kinetic model is well approximated by a standard heat equation for the mass density. This is the rationale behind the use of the terminology *hyperbolic heat equation* when considering the Goldstain-Taylor model in the diffusive scaling.

More in general when dealing with kinetic equations in the diffusive scaling, it is necessary to perform numerical simulations in both resolved (when spatial grid size is smaller than the mean free path) and under-resolved (when spatial grid size is larger than the mean free path) regimes. In the former case, standard numerical methods for kinetic equations work effectively. In the latter case, one expects that the best possible numerical solution is the approximation of the diffusion equation.

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Key words and phrases. Asymptotically implicit schemes, hyperbolic heat equation, IMEX schemes, Goldstain-Taylor model, diffusive scaling.

However, even to achieve this goal is nontrivial for standard numerical methods when applied to kinetic equations. Earlier studies on numerical methods for transport or kinetic equations indicate that, in order for the under-resolved numerical approximation to capture the correct diffusive behavior, the scheme should be *asymptotic preserving* (AP), in the sense that the asymptotic limit that leads from the transport or kinetic equations to the diffusion equations should be preserved at the discrete level [6]. Moreover, an efficient method should be able to numerically go along with the physics of the system which goes from an hyperbolic behavior underlying the kinetic regime to a parabolic behavior in the limiting regime.

Here we described a general approach to tackle this kind of problems. For the construction of the method first we rely on the problem reformulation presented in [8] and show that the method can be seen as a natural way to construct interpolated fluxes for the transport part. It is well-known that this reformulation in combination with a splitting method or an Implicit-Explicit (IMEX) time discretization provides a consistent way to approximate the kinetic model in the different regimes avoiding the time constraints $\Delta t = O(\varepsilon \Delta x)$ for small values of ε .

In fact, since the characteristic speeds of the hyperbolic part are of order $1/\varepsilon$, standard approaches developed for hyperbolic systems with stiff relaxation become useless in such parabolic scaling, because the CFL condition would require $\Delta t = O(\varepsilon \Delta x)$. Of course, in the diffusive regime where $\varepsilon \ll \Delta x$, this is too much restrictive since also for an explicit method a parabolic condition $\Delta t = O(\Delta x^2)$ would suffice.

Most previous works on asymptotic preserving schemes for hyperbolic systems and kinetic equations with diffusive relaxation focus on schemes which, in the limit of infinite stiffness, become consistent explicit schemes for the diffusive limit equation [2, 4, 5, 8, 11]. Such explicit (in the limit) schemes clearly suffer from the usual parabolic stability restriction $\Delta t = O(\Delta x^2)$. Here, following the strategy presented in [1], we construct schemes that work uniformly in ε and that, in the diffusion limit, originate a fully implicit solver for the diffusion equation. Therefore the resulting schemes permit to choose a time step $\Delta t = O(\Delta x)$ in all regimes.

The paper is organized as follows: in the next section we present the prototype system we are going to solve together with the proposed techniques used to implement our strategy. Later section is devoted to the presentation of system's discretization and is followed by the section of the numerical results. A concluding section ends the paper.

2. The Goldstain-Taylor model. A two-velocity model of the Boltzmann equation describes the behavior of a fictitious gas of two kind of particles that move parallel to the x -axis with constant and equal speed. We can consider at time t the particles with a density $f(x, t)$, which move in the positive x -direction, and the particles which move in the negative x -direction with a density $g(x, t)$. The simplest two-velocity gas, which is in local equilibrium when $f = g$, is described by the following hyperbolic system

$$\begin{aligned} f_t + cf_x &= k(g - f), \\ g_t - cg_x &= k(f - g), \end{aligned} \tag{1}$$

where $c > 0$ and $k > 0$ characterize respectively the velocities and the interactions of particles.

In the *diffusive scaling* we consider the system of equations (1) in the form

$$\begin{aligned} f_t + \frac{c}{\varepsilon} f_x &= \frac{k}{\varepsilon^2} (g - f), \\ g_t - \frac{c}{\varepsilon} g_x &= \frac{k}{\varepsilon^2} (f - g). \end{aligned} \quad (2)$$

In (2) ε is called the *relaxation time* and the limit problem for $\varepsilon \rightarrow 0$ is called *diffusive limit*.

Introducing the macroscopic variables $u = f + g$ and $v = c(f - g)/\varepsilon$, corresponding to the mass density and the flux, the model is rewritten in the form of a relaxation system

$$\begin{aligned} u_t + v_x &= 0, \\ v_t + \frac{c^2}{\varepsilon^2} u_x &= -\frac{2k}{\varepsilon^2} v. \end{aligned} \quad (3)$$

In the limit $\varepsilon = 0$ formally we obtain the local equilibrium $2kv = -c^2 u_x$ and the system reduces to the heat equation

$$u_t = \frac{c^2}{2k} u_{xx}.$$

For notation simplicity in the sequel we will assume $c = 1$ and $k = 1/2$, so that the limiting heat equation has a constant diffusion coefficient equal to one.

We will introduce the spatial grid points $x_{i+1/2}$, $i = \dots, -1, 0, 1, \dots$ with uniform mesh width $\Delta x = x_{i+1/2} - x_{i-1/2}$. As usual we denote by $U_{i+1/2}(t) = U(x_{i+1/2}, t)$ the nodal values and by $U_i(t)$ the cell averages of U in the cell $[x_{i-1/2}, x_{i+1/2}]$ at time t

$$U_i(t) = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} U(x, t) dx.$$

Thus we consider the following semi-discrete system in conservative form

$$\begin{aligned} (u_i)_t + \frac{v_{i+1/2} - v_{i-1/2}}{\Delta x} &= 0, \\ (v_i)_t + \frac{1}{\varepsilon^2} \frac{u_{i+1/2} - u_{i-1/2}}{\Delta x} &= -\frac{1}{\varepsilon^2} v_i. \end{aligned} \quad (4)$$

As observed in [11] a standard upwind selection of the numerical fluxes in (4) originates a numerical dissipation which may dominate the physical one. For example, the first order upwinding yields

$$\begin{aligned} u_{i\pm 1/2} &= \frac{1}{2} (u_i + u_{i\pm 1}) \pm \frac{\varepsilon}{2} (v_i - v_{i\pm 1}), \\ v_{i\pm 1/2} &= \frac{1}{2} (v_i + v_{i\pm 1}) \pm \frac{1}{2\varepsilon} (u_i - u_{i\pm 1}). \end{aligned} \quad (5)$$

It is easy to verify that the use of (5) in the discrete equation (4) for small values of ε leads to the semi-discrete approximation

$$(u_i)_t - \frac{u_{i+2} + u_{i-2} - 2u_i}{(2\Delta x)^2} - \frac{\Delta x}{2\varepsilon} \frac{u_{i+1} + u_{i-1} - 2u_i}{\Delta x^2} = 0, \quad (6)$$

for which the numerical dissipation dominates the physical one unless $\Delta x \ll \varepsilon$. Note also that we obtain a wider stencil for the second order derivative in the heat equation with respect to the classical one. There are several alternatives to deal with this problem, see for example [4, 7, 10, 11].

The upwind selection (5) can be modified in another way so as to capture the proper parabolic behavior. The idea now is to apply the upwind selection only in the hyperbolic regime and to use a standard central discretization in the parabolic one. This can be better understood by rewriting system (3) in the form [8]

$$\begin{aligned} u_t + v_x &= 0 \\ v_t + \phi^2 u_x &= -\frac{1}{\varepsilon^2} (v + (1 - \phi^2 \varepsilon^2) u_x), \end{aligned} \quad (7)$$

where $\phi = \phi(\varepsilon)$ is a suitable function such that $\phi = O(1)$ for small values of ε . The above reformulation is equivalent to rewrite

$$u_x = \underbrace{\phi^2 \varepsilon^2 u_x}_{\text{hyperbolic regime}} + \underbrace{(1 - \phi^2 \varepsilon^2) u_x}_{\text{parabolic regime}}, \quad (8)$$

the key point now is to combine two different discretization for the different regimes. For example in [8] it was proposed to consider an upwind discretization on the variables $(u \pm v)/2$ for the first derivative combined with a central scheme for the second term. This interpolation is also relevant in term of the time discretization of the system, since the hyperbolic flux can be evaluated explicitly whereas for the parabolic flux it is desirable to have an implicit integrator.

In the simplest case, taking $\phi \equiv 1$ this idea leads to the modified fluxes

$$\begin{aligned} u_{i\pm 1/2} &= \frac{1}{2} (u_i + u_{i\pm 1}) \pm \frac{\varepsilon^2}{2} (v_i - v_{i\pm 1}), \\ v_{i\pm 1/2} &= \frac{1}{2} (v_i + v_{i\pm 1}) \pm \frac{1}{2} (u_i - u_{i\pm 1}). \end{aligned} \quad (9)$$

With these fluxes, for small values of ε , the semi-discrete system (4) gives

$$(u_i)_t - \frac{u_{i+2} + u_{i-2} - 2u_i}{(2\Delta x)^2} - \frac{\Delta x}{2} \frac{u_{i+1} + u_{i-1} - 2u_i}{\Delta x^2} = 0, \quad (10)$$

which is an approximation of the equilibrium heat equation with an accuracy of $O(\Delta x/2)$. Second order extension can be obtained easily simply increasing to second order the choice of the flux used for the hyperbolic derivative. By combining this with standard implicit-explicit discretizations one obtain a scheme that work uniformly in ε with a CFL condition of the type $\Delta t = O(\Delta x^2)$ in the limit $\varepsilon \rightarrow 0$.

3. Asymptotically implicit IMEX schemes. In order to write a stable discretization to system (3) we should use implicit temporal integrators on the stiff terms. Because the stiffness of the convection term depends on the size of the two eigenvalues $\pm 1/\varepsilon$ of the Riemann invariant form (2) of system (3), both convection terms have to be implicit. However a fully implicit scheme causes global data dependencies, a severe disadvantage if we consider second order nonlinear schemes using slope limiters. Furthermore, the gain of stability is partially offset by the loss of accuracy typical of implicit schemes in the context of wave-propagation phenomena. The approach just introduced permits to overcome this problem since we can apply an Implicit-Explicit discretization in the form

$$\begin{aligned} u_t + \underbrace{v_x}_{\text{Explicit}} &= 0, \\ v_t + \underbrace{\phi^2 u_x}_{\text{Explicit}} &= -\underbrace{\frac{1}{\varepsilon^2} (v + (1 - \phi^2 \varepsilon^2) u_x)}_{\text{Implicit}} \end{aligned} \quad (11)$$

This approach was proposed in [8] and originates an explicit discretization of the heat equation in the limit. We refer to this standard approach as *asymptotically explicit* IMEX method.

Here we adopt a different technique which has been proposed recently in [1] which permits to overcome the parabolic time step limitation for small regimes of ε . The method is based on a penalization technique consisting in adding and subtracting to the first equation in system (3) an appropriate term, which will establish the correct limiting diffusion equation. This lead to the modified system

$$\begin{aligned} u_t + \left(v + \mu u_x \right)_x &= \mu u_{xx}, \\ v_t + \frac{1}{\varepsilon^2} u_x &= -\frac{1}{\varepsilon^2} v, \end{aligned} \quad (12)$$

where $\mu = \mu(\varepsilon)$ is such that $\mu(0) = 1$. Finally, using the interpolated fluxes approach we end up with the following system

$$\begin{aligned} u_t + \left(v + \mu u_x \right)_x &= \mu u_{xx}, \\ v_t + \phi^2 u_x &= -\frac{1}{\varepsilon^2} \left(v + (1 - \phi^2 \varepsilon^2) u_x \right), \end{aligned} \quad (13)$$

which is now tackled with an IMEX scheme in the form

$$\begin{aligned} \underbrace{u_t + \left(v + \mu u_x \right)_x}_{\text{Explicit}} &= \underbrace{\mu u_{xx}}_{\text{Implicit}}, \\ v_t + \underbrace{\phi^2 u_x}_{\text{Explicit}} &= -\frac{1}{\varepsilon^2} \underbrace{\left(v + (1 - \phi^2 \varepsilon^2) u_x \right)}_{\text{Implicit}} \end{aligned} \quad (14)$$

In addition to the space discretizations of the first order derivatives described in the last section let us specify that in (14) the second order space derivative in the first equation is discretized using a standard second order central discretization. Note that in the limit $\varepsilon \rightarrow 0$ we obtain a fully implicit discretization of the heat equation with a standard compact stencil instead of the explicit discretization on a wider stencil described before. The only additional cost is due to the inversion of the linear system originated by the implicit discretization of the heat equation. We refer to this new approach as *asymptotically implicit* IMEX method.

Let us finally recall the general structure of the IMEX-RK scheme adopted. Here we omit the space derivatives for simplicity of notation. For the internal stages for $k = 1, \dots, \nu$ we have

$$\begin{aligned} U^{(k)} &= u^n - \Delta t \sum_{j=0}^{k-1} \tilde{a}_{kj} \left(V^{(j)} + \mu U_x^{(j)} \right)_x + \mu \Delta t \sum_{j=1}^k a_{kj} U_{xx}^{(j)} \\ V^{(k)} &= v^n - \Delta t \phi^2 \sum_{j=0}^{k-1} \tilde{a}_{kj} U_x^{(j)} - \frac{\Delta t}{\varepsilon^2} \sum_{j=1}^k a_{kj} \left[V^{(j)} + (1 - \phi^2 \varepsilon^2) U_x^{(j)} \right]. \end{aligned}$$

For the numerical solution we simply have $u^{n+1} = U^\nu$ and $v^{n+1} = V^\nu$ since we restrict to globally stiffly accurate IMEX schemes [1]. In all considered IMEX schemes, matrix \tilde{A} is lower triangular with zero diagonal, while matrix A is lower triangular, i.e. the implicit scheme is a Diagonally Implicit Runge-Kutta (DIRK) scheme. This choice guarantees that implicit terms are, indeed, always explicitly

evaluated. For more details on properties and requirements of IMEX-RK schemes when applied to kinetic equations in the diffusion limit we refer to [1] and the references therein.

4. Numerical results. We present some numerical results in order to show the behavior of the proposed scheme. In the computed results second order accuracy is obtained using a WENO scheme for the hyperbolic derivative whereas the parabolic flux is always evaluated by central difference. First order Implicit-Explicit Euler scheme and second order ARS(2, 2, 2) scheme are adopted for the time discretization (see [1]). In all test cases the initial conditions are given by

$$\begin{aligned} u_0(x) &= 1 \text{ and } v_0(x) = 0 \text{ if } x \leq 0, \\ u_0(x) &= 0 \text{ and } v_0(x) = 0 \text{ if } x > 0. \end{aligned}$$

The space variable x ranges in the computational domain $[-1, 1]$ and we choose N_x grid points.

Test 1. In this case computations are performed in the hyperbolic regime for $\varepsilon = 1$ and consider system (13) with $\phi \equiv 1$ and $\mu \equiv 0$. For this test case, we assume $\Delta t = \Delta x/2$ and $N_x = 100$. In fig. 4 we show the results of the computation at time $T_f = 0.3$ of the u component. In this non equilibrium regime we can see that the transport term, which dominant with respect to the source term, is properly approximated.

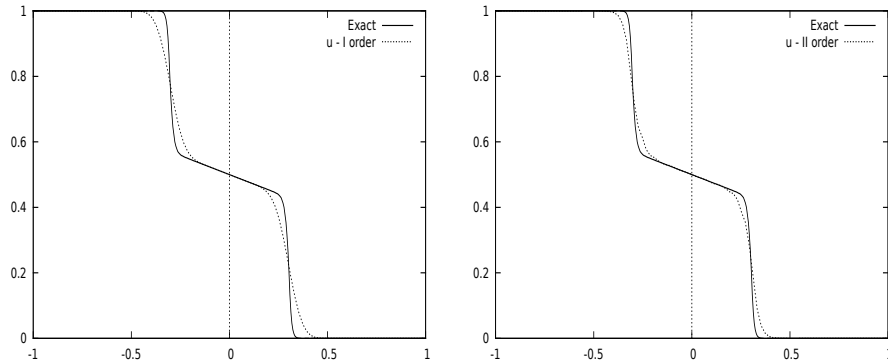


FIGURE 1. Test 1: Hyperbolic regime. Solution for $u(x)$ at time $T_f = 0.3$ for $\varepsilon = 1$. Left: first order method. Right: second order method.

Test 2. Here we consider the parabolic regime for $\varepsilon = 10^{-3}$, and approximate system (13) with $\phi \equiv 1$ and $\mu \equiv 1$. For this test case, we can assume $\Delta t = \Delta x$ and $N_x = 100$. In fig. 4 we show the result of the computation at time $T_f = 0.1$ of the u component. We stop the computation before the stationary state is reached, thus showing a plot which shows we catch the correct diffusive behavior; in this case, the contribution given by the transport term is less important than that of the relaxation term.

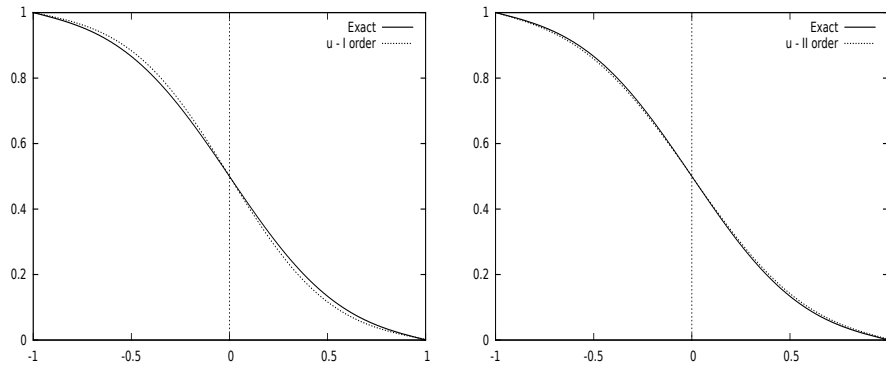


FIGURE 2. Test 2: Parabolic regime. Solution for $u(x)$ at time $T_f = 0.1$ for $\varepsilon = 10^{-3}$. Left: first order method. Right: second order method.

Test 3. In the last test case we consider a mixing regime where on the left of the computational domain for $x \leq 0$ we are in the hyperbolic regime with $\varepsilon = \varepsilon_L = 0.2$ while on the right for $x > 0$ we are in the parabolic regime with $\varepsilon = \varepsilon_R = 0.01$. For this last test case we consider $N_x = 60$ points as in [2]. We set $\mu \equiv 1$ while we set $\phi = 1/\varepsilon_L$ for $x \leq 0$ and $\phi = 1$ for $x > 0$ (we consider the correct “physical” velocity in the kinetic regime). For this test case, we set $\Delta t = \varepsilon_L \Delta x / 5$. In fig. 4 we show the result of the computation at time $T_f = 0.05$ of the u component.

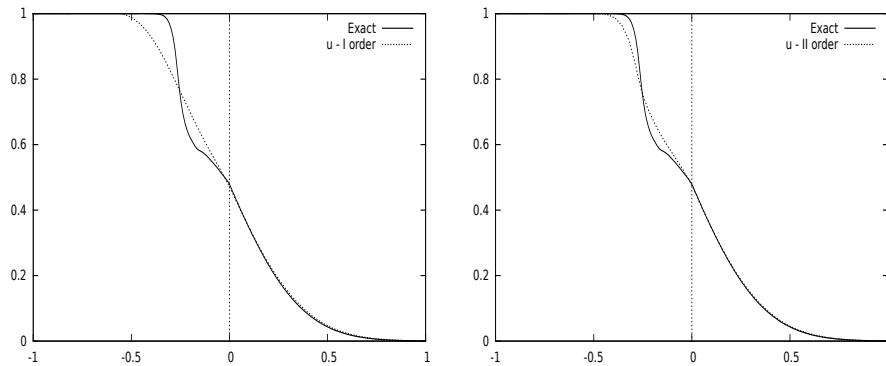


FIGURE 3. Test 3: Mixing regime. Solution for $u(x)$ at time $T_f = 0.05$ for $\varepsilon = 0.2$ for $x < 0$ and $\varepsilon = 10^{-3}$ for $x > 0$. Left: first order method. Right: second order method.

We observe that our scheme is able to efficiently compute the solution, with a good behavior also at the interface between the two regimes. We refer to [2] for comparisons.

5. Conclusions. We have presented a general approach to tackle kinetic equations in the diffusive scaling which leads to a fully implicit discretization of the limiting diffusion terms. The method is based on combining the strategies presented in [8]

for the space derivatives together with the time discretization presented in [1]. We emphasize that different choices of the space discretizations may originate schemes with different stability properties. In particular the optimal choice of the schemes parameters ϕ and μ which permits to switch between the hyperbolic and the parabolic flux and the explicit and implicit integrators deserves further investigations.

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STEADY SELF-SIMILAR INVISCID FLOW

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ABSTRACT. We consider admissible L^∞ solutions to 2-dimensional $m \times m$ systems of hyperbolic conservation laws that are steady in time, constant along rays emanating from the origin, and sufficiently small perturbations of a constant background state. We classify the possible structures of these solutions and prove that they must be *BV*. As a special case, we obtain uniqueness in this class of L^∞ functions for admissible forward in time solutions to 1-dimensional Riemann problems and show that backward in time solutions also must be *BV*.

1. **Introduction.** We consider a 2-dimensional system of conservation laws

$$U_t + f^x(U)_x + f^y(U)_y = 0. \quad (1)$$

Here the unknown U and the flux functions f^x and f^y take values in \mathbb{R}^m . The fluxes are assumed to be smooth and possess an entropy-entropy flux pair $(\eta, \vec{\psi}) = (\eta, \psi^x, \psi^y)$ with uniformly convex η on some open nonempty set in state space. The entropy inequality is then

$$\eta(U)_t + \psi^x(U)_x + \psi^y(U)_y \leq 0. \quad (2)$$

In multidimensional inviscid Euler flow, there are well studied cases in which there exists a distinguished point around which the flow is, to first order, constant along rays starting at this point. These include regular reflection (four shock waves meeting at a point) (see [7], [6], [3]) and Mach reflection (three shocks and a contact meeting at a point) (see [1]). However, other configurations such as triple points (three shocks with no other waves in between) cannot occur for most commonly used equations of state (see [13],[4], [12]). Beyond these and some other special cases, the possible configurations of such waves meeting at a point have not been classified.

The flow is steady from the point of view of an observer moving with the distinguished point described above. Therefore, we consider solutions of the form

$$U(t, x, y) = U(\phi), \text{ with } \phi = \angle(x, y) \in [0, 2\pi). \quad (3)$$

We are also motivated by a possible nonuniqueness result found in [5], in which an unsteady numerical solution was observed which took a steady self-similar solution as initial data. Perhaps analysis of the steady problem will lead to an analytical example of nonuniqueness of this form.

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Key words and phrases. Conservation law, Riemann problem, bounded variation, self-similar, shocks.

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F. Ancona, A. Bressan, P. Marcati and A. Marson (Editors)
Hyperbolic Problems: Theory, Numerics, Applications

The International Conference devoted to Theory, Numerics and Applications of Hyperbolic Problems, HYP2012, was held in Padova on June 24–29, 2012. The conference was the fourteenth in a highly successful series of bi-annual meetings that has become one of the most important international events in Applied Mathematics. The volume contains more than 110 contributions that were presented in this conference, including plenary presentations by C. De Lellis, E. Feireisl, N. Masmoudi, S. Mishra, G. Russo, J. Sethian, E. Zuazua, and a contribution by the keynote speaker J. Glimm. These contributions cover a wide range of topics. A very partial list includes: new methods for constructing turbulent solutions to multi-dimensional systems of conservation laws based on Baire category, transport equations with non-Lipschitz velocity fields, relative entropy functionals and the stability of fluid systems, numerical methods for hyperbolic systems with stiff relaxation terms and for multiphase flow, new advances in homogenization theory, optimal sensor location for solutions to multidimensional wave equations, singularities in general relativity. The volume should appeal to researchers, students and practitioners with general interest in time-dependent problems governed by hyperbolic equations.