

# PRE-TORSORS AND GALOIS COMODULES OVER MIXED DISTRIBUTIVE LAWS

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**ABSTRACT.** We study comodule functors for comonads arising from mixed distributive laws. Their Galois property is reformulated in terms of a (so-called) regular arrow in Street's bicategory of comonads. Between categories possessing equalizers, we introduce the notion of a regular adjunction. An equivalence is proven between the category of pre-torsors over two regular adjunctions  $(N_A, R_A)$  and  $(N_B, R_B)$  on one hand, and the category of regular comonad arrows  $(R_A, \xi)$  from some equalizer preserving comonad  $\mathbb{C}$  to  $N_B R_B$  on the other. This generalizes a known relationship between pre-torsors over equal commutative rings and Galois objects of coalgebras. Developing a bi-Galois theory of comonads, we show that a pre-torsor over regular adjunctions determines also a second (equalizer preserving) comonad  $\mathbb{D}$  and a co-regular comonad arrow from  $\mathbb{D}$  to  $N_A R_A$ , such that the comodule categories of  $\mathbb{C}$  and  $\mathbb{D}$  are equivalent.

## 1. INTRODUCTION

In order to generalize Hopf Galois extensions to Galois extensions by coalgebras, so called *entwining structures* were introduced in [BrMa]. These are examples of Beck's *mixed distributive laws* [Be], for a monad  $(-) \otimes_k \mathbf{T}$  induced by an algebra  $\mathbf{T}$ , and a comonad  $(-) \otimes_k \mathbf{C}$  induced by a coalgebra  $\mathbf{C}$ , on the category of  $k$ -modules, for a commutative ring  $k$ .

The basis of the generalization is an observation that a right comodule algebra  $\mathbf{T}$  of a Hopf algebra  $\mathbf{H}$  is entwined with the coalgebra underlying  $\mathbf{H}$ . Moreover,  $\mathbf{T}$  is a comodule for the lifted comonad  $(-) \otimes_{\mathbf{T}} (\mathbf{T} \otimes_k \mathbf{H}) \cong (-) \otimes_k \mathbf{H}$  on the category of  $\mathbf{T}$ -modules (whose comodules are usually called Hopf modules). Denoting by  $\mathbf{B}$  the endomorphism algebra of  $\mathbf{T}$  as a Hopf module, the  $\mathbf{H}$ -Galois property of the algebra extension  $\mathbf{B} \subseteq \mathbf{T}$  can be formulated as a Galois property of the functor  $(-) \otimes_{\mathbf{B}} \mathbf{T} : \text{Mod-}\mathbf{B} \rightarrow \text{Mod-}\mathbf{T}$ . This latter property means that the canonical comonad morphism

$$(-) \otimes_{\mathbf{T}} (\mathbf{T} \otimes_{\mathbf{B}} \mathbf{T}) \rightarrow (-) \otimes_{\mathbf{T}} (\mathbf{T} \otimes_k \mathbf{H}), \quad m \otimes_{\mathbf{T}} a' \otimes_{\mathbf{B}} a \mapsto m \otimes_{\mathbf{T}} a' \varrho(a)$$

is an isomorphism, where  $\varrho : \mathbf{T} \rightarrow \mathbf{T} \otimes_k \mathbf{H}$  denotes the coaction.

Although entwining structures were originally introduced to develop Galois theory for coalgebras in [BrMa], this method turned out to have a wider application. Using mixed distributive laws of a monad and a comonad on the category of modules over an arbitrary algebra  $\mathbf{R}$ , also Galois extensions by bialgebroids, and more generally by corings, over  $\mathbf{R}$  fit this scenario. By these motivations, the first aim of this paper

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is to study Galois functors for comonads that arise from arbitrary mixed distributive laws.

A Galois comodule algebra  $\mathbf{T}$  of a  $k$ -Hopf algebra  $\mathbf{H}$ , for which the Hopf module endomorphism algebra  $\mathbf{B}$  of  $\mathbf{T}$  is trivial (i.e. it contains only multiplication by  $k$ ), is called an  $\mathbf{H}$ -Galois object. As observed by Grunspan [G] and Schauenburg [Sch1], [Sch2], [Sch4], *faithfully flat* Hopf-Galois objects can be described equivalently, without explicit mention of the coacting Hopf algebra  $\mathbf{H}$ , in terms of *torsors*. A torsor means a certain map  $\mathbf{T} \rightarrow \mathbf{T} \otimes_k \mathbf{T} \otimes_k \mathbf{T}$ , from which the *flat* Hopf algebra  $\mathbf{H}$  can be reconstructed uniquely up to isomorphism. (For a study of the case when  $\mathbf{T} = \mathbf{H}$  as algebras, not necessarily faithfully flat over  $k$ , see also [Šk]). This observation was generalized to Hopf Galois extensions of arbitrary algebras  $\mathbf{B}$  (using the notion of a  $\mathbf{B}$ -torsor) in [Sch3], to Galois extensions by bialgebroids (using the notion of an  $\mathbf{A}$ - $\mathbf{B}$ -torsor) in [H] and [BB], to Galois extensions by corings (using the notion of a *pre-torsor*) in [BB] and to Galois comodules of corings arising from entwining structures (using the notion of a *bimodule herd*) in [BV].

Note that the definition of a torsor has the symmetry of reversing the order of the tensor factors in the codomain. This symmetry leads to the interesting fact [Sch2] that a faithfully flat torsor  $\mathbf{T}$  determines a second *flat* Hopf algebra  $\mathbf{H}'$ , such that  $\mathbf{T}$  is a left  $\mathbf{H}'$ -comodule algebra and an  $\mathbf{H}'$ -Galois extension of  $k$ . Moreover, the Hopf algebras  $\mathbf{H}$  and  $\mathbf{H}'$ , coacting on  $\mathbf{T}$  on the right and on the left, respectively, are *Morita-Takeuchi equivalent*, i.e. they have equivalent categories of comodules.

Under sufficiently strong assumptions, the construction of two corings (over respective base algebras  $\mathbf{A}$  and  $\mathbf{B}$ ) from an  $\mathbf{A}$ - $\mathbf{B}$  pre-torsor or even from a bimodule herd, has been carried out in [BB] and [BV], respectively. However, the resulting corings are not known to be flat, and their comodule categories do not seem to be equivalent without further, somewhat intricate assumptions, see [BB, Remark 4.7]. Placing the problem in a more general categorical context, in this paper we give an explanation of the origin of this difficulty. Namely, we show that a pre-torsor (or a finitely generated projective bimodule herd) that is faithfully flat as a left module for both base algebras  $\mathbf{A}$  and  $\mathbf{B}$ , determines (uniquely up to natural isomorphisms) two comonads  $\mathbb{C}$  and  $\mathbb{D}$  on the category of  $\mathbf{A}$ -, and  $\mathbf{B}$ -modules, respectively, whose underlying functors preserve kernels. These two comonads have equivalent comodule categories. However, even in the situation when one can associate an  $\mathbf{A}$ -coring  $\mathbf{C}$  and a  $\mathbf{B}$ -coring  $\mathbf{D}$  to a pre-torsor or a bimodule herd, as in [BB] or [BV], it is not guaranteed that the comonads  $\mathbb{C}$  and  $\mathbb{D}$  are induced by these corings  $\mathbf{C}$  and  $\mathbf{D}$ . This holds exactly if the corings  $\mathbf{C}$  and  $\mathbf{D}$  are *flat* left modules over their respective base algebras, i.e. in the situation discussed in [BB, Remark 4.7].

The paper is organized as follows.

In Section 2 we recall some results from category theory that we need as background.

Section 3 is devoted to a study of Galois functors (in the sense of [MW, Definition 4.5]) for a comonad arising from a mixed distributive law. The category of Galois functors (with domain category  $\mathcal{B}$ ) for a comonad arising from a mixed distributive law (of functors on a category  $\mathcal{A}$ ), is described as a suitable subcategory of a newly

defined category  $\text{Arr}(\mathcal{A}, \mathcal{B})$ . The objects of  $\text{Arr}(\mathcal{A}, \mathcal{B})$  consist of two adjunctions

$$(1.1) \quad \mathcal{A} \begin{array}{c} \xleftarrow{N_A} \\ \xrightarrow{R_A} \end{array} \mathcal{T} \begin{array}{c} \xleftarrow{N_B} \\ \xrightarrow{R_B} \end{array} \mathcal{B} ,$$

together with a comonad  $\mathbb{C}$  on  $\mathcal{A}$  and a comonad arrow  $(R_A, \xi)$  from  $\mathbb{C}$  to the comonad  $N_B R_B$ .

Generalizing pre-torsors in [BB] (thus in particular generalizing Grunspan- Schauenburg torsors), in Section 4 we define pre-torsors over two adjunctions as in (1.1). A pre-torsor is a natural transformation  $R_A N_B \rightarrow R_A N_B R_B N_A R_A N_B$ , subject to compatibility conditions with the units and counits of the adjunctions. We consider a full subcategory of so called *regular* pre-torsors that is shown to be equivalent to a full subcategory of  $\text{Arr}(\mathcal{A}, \mathcal{B})$ . Since in Section 3 we describe Galois functors (of comonads arising from mixed distributive laws) via objects in  $\text{Arr}(\mathcal{A}, \mathcal{B})$ , this equivalence relates in particular such Galois functors to pre-torsors, provided that they obey our regularity assumptions. Note that all these regularity assumptions hold for a ( $\mathbf{B}$ -)torsor corresponding to a faithfully flat Galois extension  $\mathbf{B} \subseteq \mathbf{T}$  by a Hopf algebra over a field  $k$ , i.e. when

$$\mathcal{A} = \text{Vec-}k \begin{array}{c} \xleftarrow{N_A=(-)\otimes_k \mathbf{T}} \\ \xrightarrow{R_A=\text{Hom}_{\mathbf{T}}(\mathbf{T},-)} \end{array} \mathcal{T} = \text{Mod-}\mathbf{T} \begin{array}{c} \xleftarrow{N_B=(-)\otimes_{\mathbf{B}} \mathbf{T}} \\ \xrightarrow{R_B=\text{Hom}_{\mathbf{T}}(\mathbf{T},-)} \end{array} \mathcal{B} = \text{Mod-}\mathbf{B} .$$

More generally, these assumptions hold for bimodule herds corresponding to (left) faithfully flat Galois comodules for entwining structures.

By the equivalences in Section 4, we associate in particular two comonads  $\mathbb{C}$  on  $\mathcal{A}$  and  $\mathbb{D}$  on  $\mathcal{B}$  to a pre-torsor satisfying our regularity assumptions. Generalizing the Morita Takeuchi equivalence of two Hopf algebras in a faithfully flat bi-Galois object, in the final Section 5 the two comonads  $\mathbb{C}$  and  $\mathbb{D}$  are shown to have equivalent comodule categories.

## 2. PRELIMINARIES

In this section we recall some categorical preliminaries that will be used later on.

**2.1. Notations.** Throughout the paper, in the 2-category CAT of categories – functors – natural morphisms, the following notations are used. The identity functor on a category  $\mathcal{C}$  is denoted by the same symbol  $\mathcal{C}$ . Similarly, for any functor  $F$ , the identity natural morphism  $F \rightarrow F$  is denoted by the same symbol  $F$ . We denote vertical composition by  $\circ$  and horizontal composition by juxtaposition. For three parallel functors  $F, F', F'' : \mathcal{C} \rightarrow \mathcal{D}$  and natural morphisms  $\alpha : F \rightarrow F'$  and  $\beta : F' \rightarrow F''$ , for the composite natural morphism  $F \rightarrow F''$  we write  $\beta \circ \alpha$ . For consecutive functors  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $G : \mathcal{D} \rightarrow \mathcal{E}$ , the composite functor is denoted by  $GF : \mathcal{C} \rightarrow \mathcal{E}$ . Moreover, for functors  $F, F' : \mathcal{C} \rightarrow \mathcal{D}$  and  $G, G' : \mathcal{D} \rightarrow \mathcal{E}$ , and natural morphisms  $\alpha : F \rightarrow F'$ ,  $\beta : G \rightarrow G'$ , the Godement product  $(F'\beta) \circ (\alpha G) = (\alpha G') \circ (F\beta) : FG \rightarrow F'G'$  is denoted simply by  $\alpha\beta : FG \rightarrow F'G'$ . For a natural morphism  $\alpha : F \rightarrow F'$ , between functors  $F, F' : \mathcal{C} \rightarrow \mathcal{D}$ , we denote by  $\alpha X$  the morphism in  $\mathcal{D}$  obtained by evaluating  $\alpha$  at an object  $X$  of  $\mathcal{C}$ .

The vertical category of CAT, i.e. the category of functors – natural morphisms, is denoted by Fun.

In any category  $\mathcal{K}$ , the equalizer (resp. coequalizer) of two parallel morphisms  $f$  and  $g$  (if it exists) is denoted by  $\text{Equ}_{\mathcal{K}}(f, g)$  (resp.  $\text{Coequ}_{\mathcal{K}}(f, g)$ ).

**2.2. Equalizers in functor categories.** We study equalizers in the category of functors  $\mathcal{C} \rightarrow \mathcal{K}$ , where  $\mathcal{K}$  is assumed to have equalizers.

**LEMMA 2.1.** *Let  $\mathcal{C}$  and  $\mathcal{K}$  be categories, let  $G, G' : \mathcal{C} \rightarrow \mathcal{K}$  be functors and  $\gamma, \theta : G \rightarrow G'$  be natural morphisms. If, for every  $X \in \mathcal{C}$ , there exists  $\text{Equ}_{\mathcal{K}}(\gamma X, \theta X)$ , then there exists the equalizer  $(E, i) = \text{Equ}_{\text{Fun}}(\gamma, \theta)$  in the category of functors. Moreover, for any object  $X$  in  $\mathcal{C}$ ,  $(EX, iX) = \text{Equ}_{\mathcal{K}}(\gamma X, \theta X)$ .*

*Proof.* Define a functor  $E : \mathcal{C} \rightarrow \mathcal{K}$  with object map  $(EX, i_X) = \text{Equ}_{\mathcal{K}}(\gamma X, \theta X)$ . For a morphism  $f : X \rightarrow X'$  in  $\mathcal{C}$ , naturality of  $\gamma$  and  $\theta$  implies that  $(Gf) \circ i_X$  equalizes the parallel morphisms  $\gamma X'$  and  $\theta X'$ . In light of this fact,  $Ef$  is defined as the unique morphism in  $\mathcal{K}$  such that  $i_{X'} \circ (Ef) = (Gf) \circ i_X$ . By construction,  $i$  is a natural transformation  $E \rightarrow G$  such that  $\gamma \circ i = \theta \circ i$ . It remains to prove universality of  $i$ . Let  $H : \mathcal{C} \rightarrow \mathcal{K}$  be a functor and  $\chi : H \rightarrow G$  be a natural morphism such that  $\gamma \circ \chi = \theta \circ \chi$ . Then, for any object  $X$  in  $\mathcal{C}$ ,  $(\gamma X) \circ (\chi X) = (\theta X) \circ (\chi X)$ . Since  $(EX, iX) = \text{Equ}_{\mathcal{K}}(\gamma X, \theta X)$ , there is a unique morphism  $\xi_X : HX \rightarrow EX$  such that  $(iX) \circ \xi_X = \chi X$ . The proof is completed by proving naturality of  $\xi_X$  in  $X$ . Take a morphism  $f : X \rightarrow X'$  in  $\mathcal{C}$ . Since  $i$  and  $\chi$  are natural,

$$\begin{aligned} (iX') \circ \xi_{X'} \circ (Hf) &= (\chi X') \circ (Hf) = (Gf) \circ (\chi X) \\ &= (Gf) \circ (iX) \circ \xi_X = (iX') \circ (Ef) \circ \xi_X. \end{aligned}$$

Since  $iX'$  is a monomorphism, this proves naturality of  $\xi$ .  $\square$

In the case when  $\mathcal{K}$  has coequalizers, the opposite category  $\mathcal{K}^{op}$  has equalizers. Thus applying Lemma 2.1 to the opposite functors  $\mathcal{C}^{op} \rightarrow \mathcal{K}^{op}$ , we conclude that coequalizers of natural transformations between functors of codomain  $\mathcal{K}$  exist, and can be computed ‘objectwise’.

As an immediate consequence of Lemma 2.1, we obtain

**LEMMA 2.2.** *Let  $G, G' : \mathcal{C} \rightarrow \mathcal{K}$  be functors, and let  $\gamma, \theta : G \rightarrow G'$  be natural morphisms. Assume that every pair of parallel morphisms in  $\mathcal{K}$  has an equalizer and let  $(E, i) = \text{Equ}_{\text{Fun}}(\gamma, \theta)$ . Under these assumptions, for any functor  $P : \mathcal{D} \rightarrow \mathcal{C}$ ,  $\text{Equ}_{\text{Fun}}(\gamma P, \theta P) = (EP, iP)$ .*

**LEMMA 2.3.** *For any adjunction  $(N, R)$ , with unit  $\eta$  and counit  $\epsilon$ , the following diagrams are split equalizers in the category of functors.*

$$(1) \quad R \xrightarrow{\eta R} RNR \begin{array}{c} \xrightarrow{\eta RNR} \\ \xrightarrow{RN\eta R} \end{array} \Rrightarrow RNRNR;$$

$$(2) \quad N \xrightarrow{N\eta} NRN \begin{array}{c} \xrightarrow{N\eta RN} \\ \xrightarrow{NRN\eta} \end{array} \Rrightarrow NRNRN.$$

*Proof.* By naturality,  $(\eta RN) \circ \eta = (RN\eta) \circ \eta$ , so both diagrams are commutative forks.

(1) A natural morphism  $f : F \rightarrow RNR$ , such that  $(\eta RNR) \circ f = (RN\eta R) \circ f$ , factorizes uniquely through  $\eta R$  and the morphism  $(R\epsilon) \circ f$ . Thus (1) is an equalizer. It is split by the morphism  $RNR\epsilon$ , i.e. the identities

$$\begin{aligned} (RNR\epsilon) \circ (RN\eta R) &= RNR & \text{and} \\ (RN\eta R) \circ (RNR\epsilon) \circ (\eta RNR) &= (\eta RNR) \circ (RNR\epsilon) \circ (\eta RNR) \end{aligned}$$

hold.

(2) A natural morphism  $f : F \rightarrow NRN$ , such that  $(N\eta RN) \circ f = (NRN\eta) \circ f$ , factorizes uniquely through  $N\eta$  and the morphism  $(\epsilon N) \circ f$ . Thus (2) is an equalizer. It is split by  $\epsilon NRN$ .  $\square$

Consider an adjoint pair of functors  $(N : \mathcal{K} \rightarrow \mathcal{C}, R : \mathcal{C} \rightarrow \mathcal{K})$  with unit  $\eta$  and counit  $\epsilon$ . Since split equalizers are preserved by any functor, cf. [BW, p 110 Proposition 2], for any categories  $\mathcal{D}$  and  $\mathcal{D}'$  and functors  $P : \mathcal{C} \rightarrow \mathcal{D}$  and  $Q : \mathcal{K} \rightarrow \mathcal{D}'$ ,  $Q\eta R = \text{Equ}_{\text{Fun}}(Q\eta RNR, QRN\eta R)$  and  $PN\eta = \text{Equ}_{\text{Fun}}(PN\eta RN, PNRN\eta)$ .

LEMMA 2.4. *Let  $\mathcal{K}$  be a category in which all equalizers exist and let  $\mathcal{C}$  be any category. Consider an adjoint pair of functors  $(N : \mathcal{K} \rightarrow \mathcal{C}, R : \mathcal{C} \rightarrow \mathcal{K})$  with unit  $\eta$  and counit  $\epsilon$ . Then*

$$(2.1) \quad (\mathcal{K}, \eta) = \text{Equ}_{\text{Fun}}(RN\eta, \eta RN)$$

if and only if  $\eta$  is a regular natural monomorphism.

*Proof.* If (2.1) holds then  $\eta$  is obviously a regular natural monomorphism.

Conversely, assume that  $\eta$  is a regular natural monomorphism. Then we deduce from Lemma 2.1 that  $\eta A$  is a regular monomorphism in  $\mathcal{K}$ , for any object  $A \in \mathcal{K}$ . By [BW, p 115 Lemma 6] we conclude that

$$(A, \eta A) = \text{Equ}_{\mathcal{K}}(RN\eta A, \eta RNA),$$

for any object  $A$  in  $\mathcal{K}$ . Equality (2.1) follows by applying Lemma 2.1 again.  $\square$

Recall from [BW, p 111] that a diagram like in Lemma 2.5 is said to be *serially commutative* if the squares that are bordered by parallel arrows are commutative with either *simultaneous* choice of the upper or lower (or left or right) arrows.

LEMMA 2.5. *Consider the following serially commutative diagram in an arbitrary category  $\mathcal{K}$ .*

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \xrightleftharpoons[f]{g} & C \\ e \downarrow & & e' \downarrow & & e'' \downarrow \\ A' & \xrightarrow{i'} & B' & \xrightleftharpoons[f']{g'} & C' \\ n \downarrow m & & n' \downarrow m' & & n'' \downarrow m'' \\ A'' & \xrightarrow{i''} & B'' & \xrightleftharpoons[f'']{g''} & C'' \end{array}$$

Assume that all columns are equalizers and also the second and third rows are equalizers. Then the first row is an equalizer too.

*Proof.* In order to see that the first row is a fork, note that, by commutativity of the diagram and fork property of the second row,

$$e'' \circ f \circ i = f' \circ e' \circ i = f' \circ i' \circ e = g' \circ i' \circ e = g' \circ e' \circ i = e'' \circ g \circ i.$$

Since  $e''$  is a monomorphism, this proves that the first row is a fork. Take any morphism  $x : X \rightarrow B$  such that  $f \circ x = g \circ x$ . Then

$$f' \circ e' \circ x = e'' \circ f \circ x = e'' \circ g \circ x = g' \circ e' \circ x.$$

Since the second row is an equalizer by assumption, there is a unique morphism  $y : X \rightarrow A'$  such that

$$(2.2) \quad i' \circ y = e' \circ x.$$

Then  $i'' \circ n \circ y = i'' \circ m \circ y$  and, since  $i''$  is monic,  $n \circ y = m \circ y$ . Since the first column is an equalizer, there exists a unique morphism  $z : X \rightarrow A$  such that

$$(2.3) \quad e \circ z = y.$$

Then  $e' \circ i \circ z = e' \circ x$  and since  $e'$  is monic,  $i \circ z = x$ . Since  $e' \circ i = i' \circ e$  and  $e', e, i'$  are monic, we deduce that  $i$  is monic.  $\square$

**COROLLARY 2.6.** *Let  $G, G' : \mathcal{C} \rightarrow \mathcal{K}$  be functors and  $\gamma, \theta : G \rightarrow G'$  be natural morphisms. Assume that in  $\mathcal{K}$  there exists the equalizer of any parallel pair of morphisms hence there exists  $(C, i) = \text{Equ}_{\text{Fun}}(\gamma, \theta)$ , cf. Lemma 2.1. Then the functor  $C$  preserves equalizers provided that  $G$  and  $G'$  preserve equalizers.*

*Proof.* Consider an equalizer  $(E, e) = \text{Equ}_{\mathcal{C}}(f, g)$  of morphisms  $f, g : X \rightarrow Y$  in  $\mathcal{C}$ . The following diagram (in  $\mathcal{K}$ ) is serially commutative by naturality.

$$\begin{array}{ccccc} CE & \xrightarrow{C_e} & CX & \begin{array}{c} \xrightarrow{C_f} \\ \xrightarrow{C_g} \end{array} & CY \\ \downarrow i_E & & \downarrow i_X & & \downarrow i_Y \\ GE & \xrightarrow{G_e} & GX & \begin{array}{c} \xrightarrow{G_f} \\ \xrightarrow{G_g} \end{array} & GY \\ \downarrow \gamma_E \parallel \theta_E & & \downarrow \gamma_X \parallel \theta_X & & \downarrow \gamma_Y \parallel \theta_Y \\ G'E & \xrightarrow{G'_e} & G'X & \begin{array}{c} \xrightarrow{G'_f} \\ \xrightarrow{G'_g} \end{array} & G'Y \end{array}$$

The columns are equalizers by Lemma 2.1. The second and third rows are equalizers by assumption. Thus the first row is an equalizer by Lemma 2.5.  $\square$

**2.3. (Co)monads and their (co)modules.** We recall some basic facts about monads and their modules, mainly to fix notation and terminology.

**DEFINITION 2.7.** (1) A *monad* on a category  $\mathcal{K}$  is a triple  $\mathbb{T} = (T, m, u)$  where  $T : \mathcal{K} \rightarrow \mathcal{K}$  is a functor and  $m : TT \rightarrow T$ ,  $u : \mathcal{K} \rightarrow T$  are natural morphisms, called the *product* and *unit*, respectively, such that

$$m \circ (Tm) = m \circ (mT) \quad \text{and} \quad m \circ (Tu) = T = m \circ (uT).$$

(2) A *morphism* between two monads  $\mathbb{T} = (T, m, u)$  and  $\mathbb{T}' = (T', m', u')$  on the same category  $\mathcal{K}$  is a natural morphism  $\varphi : T \rightarrow T'$  such that

$$\varphi \circ m = m' \circ (\varphi\varphi) \quad \text{and} \quad \varphi \circ u = u'.$$

(3) A  $\mathbb{T}$ -module over a monad  $\mathbb{T} = (T, m, u)$  on  $\mathcal{K}$  is a pair  $(M, \mu)$  where  $M$  is an object and  $\mu : TM \rightarrow M$  is a morphism in  $\mathcal{K}$  such that

$$\mu \circ (mM) = \mu \circ (T\mu) \quad \text{and} \quad \mu \circ (uM) = M.$$

(4) A *morphism* between two  $\mathbb{T}$ -modules  $(M, \mu)$  and  $(M', \mu')$  is a morphism  $f : M \rightarrow M'$  in  $\mathcal{K}$  such that

$$\mu' \circ (Tf) = f \circ \mu.$$

We denote by  $\mathcal{K}_{\mathbb{T}}$  the category of  $\mathbb{T}$ -modules and their morphisms.

(5) Corresponding to a monad  $\mathbb{T} = (T, m, u)$  on  $\mathcal{K}$ , there is an adjunction

$$F_{\mathbb{T}} : \mathcal{K} \rightarrow \mathcal{K}_{\mathbb{T}} \quad U_{\mathbb{T}} : \mathcal{K}_{\mathbb{T}} \rightarrow \mathcal{K},$$

where  $U_{\mathbb{T}}$  is the *forgetful functor*, with object map  $(M, \mu) \mapsto M$  and acting on the morphisms as the identity map.  $F_{\mathbb{T}}$  is the so called *free functor*, with object map  $X \mapsto (TX, mX)$  and acting on the morphisms as  $f \mapsto Tf$ . Note that  $U_{\mathbb{T}}F_{\mathbb{T}} = T$ . The unit of the adjunction is given by

$$u : \mathcal{K} \rightarrow U_{\mathbb{T}}F_{\mathbb{T}} = T.$$

For a  $\mathbb{T}$ -module  $(M, \mu)$ , the counit of the adjunction is given by

$$\mu : F_{\mathbb{T}}U_{\mathbb{T}}(M, \mu) = (TM, m) \rightarrow (M, \mu).$$

We will use the notation  $\lambda^T$  for the natural transformation  $: F_{\mathbb{T}}U_{\mathbb{T}} \rightarrow \mathcal{K}_{\mathbb{T}}$ , for which  $U_{\mathbb{T}}\lambda^T(M, \mu) = \mu$ .

(6) A *comonad* on  $\mathcal{K}$  is a monad on the opposite category  $\mathcal{K}^{op}$ . That is, a comonad  $\mathbb{C}$  consists of a functor  $C : \mathcal{K} \rightarrow \mathcal{K}$ , and two natural transformations  $\Delta : C \rightarrow CC$  and  $\varepsilon : C \rightarrow \mathcal{K}$ , called the *coproduct* and *counit*, respectively, subject to coassociativity and counitality constraints. Morphisms, comodules and morphisms of comodules for a comonad are defined as morphisms, modules and morphisms of modules, respectively, for the corresponding monad on  $\mathcal{K}^{op}$ . In particular, the category of  $\mathbb{C}$ -comodules is denoted by  $\mathcal{K}^{\mathbb{C}}$ . The forgetful functor  $U^{\mathbb{C}} : \mathcal{K}^{\mathbb{C}} \rightarrow \mathcal{K}$  has a *right* adjoint  $F^{\mathbb{C}}$  with object and morphism maps

$$N \mapsto (CN, \Delta N) \quad \text{and} \quad f \mapsto Cf,$$

respectively. The unit of the adjunction is given by the coaction  $\varrho : M \rightarrow CM$ , for any object  $(M, \varrho) \in \mathcal{K}^{\mathbb{C}}$  and it will be denoted by  $\gamma^{\mathbb{C}}$ . That is,  $U^{\mathbb{C}}\gamma^{\mathbb{C}}(M, \varrho) = \varrho$ . The counit is given by  $\varepsilon N : CN \rightarrow N$ , for all  $N \in \mathcal{K}$ .

**PROPOSITION 2.8.** *Let  $\mathcal{K}$  be a category with equalizers and  $\mathbb{T} = (T, m, u)$  be a monad on  $\mathcal{K}$ . Then any parallel pair of morphisms in  $\mathcal{K}_{\mathbb{T}}$  possesses an equalizer. Moreover, the forgetful functor  $U_{\mathbb{T}} : \mathcal{K}_{\mathbb{T}} \rightarrow \mathcal{K}$  preserves and reflects equalizers.*

*Proof.* Consider two parallel morphisms  $f, g : (X, x) \rightarrow (Y, y)$  in  $\mathcal{K}_{\mathbb{T}}$  and denote  $(E, i) := \text{Equ}_{\mathcal{K}}(U_{\mathbb{T}}f, U_{\mathbb{T}}g)$ . Since  $f$  and  $g$  are  $\mathbb{T}$ -module morphisms,  $x \circ (Ti)$  equalizes  $f$  and  $g$ . So there exists a unique morphism  $e : TE \rightarrow E$  in  $\mathcal{K}$ , such that  $i \circ e = x \circ (Ti)$ . By associativity and unitality of the  $\mathbb{T}$ -action  $x$ ,

$$\begin{aligned} i \circ e \circ (Te) &= x \circ (Ti) \circ (Te) = x \circ (Tx) \circ (TTi) = x \circ (mX) \circ (TTi) \\ &= x \circ (Ti) \circ (mE) = i \circ e \circ (mE), \\ i \circ e \circ (uE) &= x \circ (Ti) \circ (uE) = x \circ (uX) \circ i = i. \end{aligned}$$

Since  $i$  is monic, we conclude that  $(E, e)$  is a  $\mathbb{T}$ -module and  $i$  lifts to a  $\mathbb{T}$ -module morphism  $\widehat{i}$ , such that  $f \circ \widehat{i} = g \circ \widehat{i}$ . It remains to prove universality of  $\widehat{i}$ . Consider a  $\mathbb{T}$ -module morphism  $h : (Z, z) \rightarrow (X, x)$ , such that  $f \circ h = g \circ h$ . Then  $(U_{\mathbb{T}}f) \circ (U_{\mathbb{T}}h) = (U_{\mathbb{T}}g) \circ (U_{\mathbb{T}}h)$ , so by universality of  $i$ , there exists a unique morphism  $k : Z \rightarrow E$  in  $\mathcal{K}$ , such that  $i \circ k = U_{\mathbb{T}}h$ . Since  $h$  is a  $\mathbb{T}$ -module morphism,

$$i \circ e \circ (Tk) = x \circ (Ti) \circ (Tk) = x \circ (TU_{\mathbb{T}}h) = (U_{\mathbb{T}}h) \circ z = i \circ k \circ z.$$

Since  $i$  is monic, this proves that  $k$  lifts to a  $\mathbb{T}$ -module morphism  $(Z, z) \rightarrow (E, e)$ . Since  $U_{\mathbb{T}}$  is a right adjoint, it preserves equalizers. (Note that it is manifest also by the above construction of  $\widehat{i}$  that  $U_{\mathbb{T}}$  preserves equalizers.) It follows by the faithfulness of  $U_{\mathbb{T}}$  that it also reflects equalizers.  $\square$

From Proposition 2.8 and Lemma 2.1, we obtain

**COROLLARY 2.9.** *Let  $\mathcal{K}$  be a category with equalizers and let  $\mathbb{T} = (T, m, u)$  be a monad on  $\mathcal{K}$ . Let  $G, G' : \mathcal{C} \rightarrow \mathcal{K}_{\mathbb{T}}$  be functors, and let  $\gamma, \theta : G \rightarrow G'$  be natural morphisms. Then there exists  $\text{Equ}_{\text{Fun}}(\gamma, \theta)$  and  $U_{\mathbb{T}}\text{Equ}_{\text{Fun}}(\gamma, \theta) = \text{Equ}_{\text{Fun}}(U_{\mathbb{T}}\gamma, U_{\mathbb{T}}\theta)$ .*

*Proof.* By Lemma 2.1, there exists  $(E', \iota') := \text{Equ}_{\text{Fun}}(U_{\mathbb{T}}\gamma, U_{\mathbb{T}}\theta)$ . By Proposition 2.8 and Lemma 2.1, there exists  $(E, \iota) := \text{Equ}_{\text{Fun}}(\gamma, \theta)$ . Moreover, for any object  $X$  in  $\mathcal{C}$ ,

$$\iota'X = \text{Equ}_{\text{Fun}}(U_{\mathbb{T}}\gamma X, U_{\mathbb{T}}\theta X) = U_{\mathbb{T}}\text{Equ}_{\text{Fun}}(\gamma X, \theta X) = U_{\mathbb{T}}\iota X,$$

where in the second equality we used that  $U_{\mathbb{T}}$  preserves equalizers. Thus the claim follows by Lemma 2.1.  $\square$

A statement of similar generality does not hold for coequalizers. Instead, we have the following

**PROPOSITION 2.10.** *Let  $\mathcal{K}$  be a category with coequalizers and  $\mathbb{T} = (T, m, u)$  be a monad on  $\mathcal{K}$  such that  $T$  preserves coequalizers. Then every parallel pair of morphisms has a coequalizer in  $\mathcal{K}_{\mathbb{T}}$ . Moreover,  $U_{\mathbb{T}}$  preserves and reflects coequalizers.*

*Proof.* For two parallel morphisms  $f, g : (Y, y) \rightarrow (Z, z)$  in  $\mathcal{K}_{\mathbb{T}}$ , denote  $(P, \pi) = \text{Coequ}_{\mathcal{K}}(U_{\mathbb{T}}f, U_{\mathbb{T}}g)$ . Since  $f$  and  $g$  are  $\mathbb{T}$ -module morphisms, the morphism  $\pi \circ z$  coequalizes  $TU_{\mathbb{T}}f$  and  $TU_{\mathbb{T}}g$ . Therefore, by universality of the coequalizer  $(TP, T\pi)$ , there exists a unique morphism  $\nu : TP \rightarrow P$  such that

$$(2.4) \quad \nu \circ (T\pi) = \pi \circ z.$$

By associativity of  $z$ ,  $\nu \circ (T\nu) \circ (TT\pi) = \nu \circ (mP) \circ (TT\pi)$ . Since  $T$  preserves coequalizers,  $TT\pi$  is an epimorphism so we conclude that  $\nu$  is an associative  $T$ -action on  $P$ . Similarly, by unitality of  $z$ ,  $\nu \circ (uP) \circ \pi = \pi$ , hence the action  $\nu$  is also unital. Therefore, there is a  $\mathbb{T}$ -module morphism  $\widehat{\pi} : (Z, z) \rightarrow (P, \nu)$ , such that  $U_{\mathbb{T}}\widehat{\pi} = \pi$ , cf. (2.4).

We claim that  $((P, \nu), \widehat{\pi}) = \text{Coequ}_{\mathcal{K}_{\mathbb{T}}}(f, g)$ . Let  $h : (Z, z) \rightarrow (W, w)$  be a morphism in  $\mathcal{K}_{\mathbb{T}}$  such that  $h \circ f = h \circ g$ . Then there exists a unique morphism  $t : P \rightarrow W$  in  $\mathcal{K}$  such that  $t \circ \pi = U_{\mathbb{T}}h$ . Since  $h$  is a morphism in  $\mathcal{K}_{\mathbb{T}}$ ,

$$t \circ \nu \circ (T\pi) = w \circ (Tt) \circ (T\pi).$$

Thus, since  $T\pi$  is epi,  $t$  gives rise to a morphism  $\widehat{t} : (P, \nu) \rightarrow (W, w)$ . From the uniqueness of  $t$  in  $\mathcal{K}$  one obviously gets the uniqueness of  $\widehat{t}$  in  $\mathcal{K}_{\mathbb{T}}$ . It is clear from the above construction that  $U_{\mathbb{T}}$  preserves coequalizers, so we conclude by uniqueness of a coequalizer and faithfulness of  $U_{\mathbb{T}}$  that  $U_{\mathbb{T}}$  also reflects coequalizers.  $\square$

Proposition 2.10 has the following immediate consequence. Consider a category  $\mathcal{K}$  with coequalizers and a monad  $\mathbb{T}$  on  $\mathcal{K}$  such that the underlying functor preserves coequalizers. Let  $G$  and  $G'$  be functors from any category  $\mathcal{C}$  to  $\mathcal{K}_{\mathbb{T}}$  and  $\varphi, \gamma : G \rightarrow G'$  be natural morphisms. By Proposition 2.10, for any object  $Z$  in  $\mathcal{C}$ , there exists  $(K_Z, \pi_Z) := \text{Coequ}_{\mathcal{K}_{\mathbb{T}}}(\varphi Z, \gamma Z)$ . It follows by the dual form of Lemma 2.1 that this construction defines a functor  $K : \mathcal{C} \rightarrow \mathcal{K}_{\mathbb{T}}$  and a natural morphism  $\pi : G' \rightarrow K$ , such that  $(K, \pi) = \text{Coequ}_{\text{Fun}}(\varphi, \gamma)$ .

**DEFINITION 2.11.** (1) A *left comodule functor* for a comonad  $\mathbb{C} = (C, \Delta, \varepsilon)$  on a category  $\mathcal{A}$  is a pair  $(L, l)$ , where  $L : \mathcal{B} \rightarrow \mathcal{A}$  is a functor from any category  $\mathcal{B}$  to  $\mathcal{A}$  and  $l : L \rightarrow CL$  is a natural morphism (called a *coaction*) satisfying the counitality and coassociativity constraints

$$(\varepsilon L) \circ l = L \quad \text{and} \quad (\Delta L) \circ l = (Cl) \circ l.$$

(2) Symmetrically, a *right  $\mathbb{C}$ -comodule functor* is a pair  $(R, r)$ , where  $R : \mathcal{A} \rightarrow \mathcal{B}$  is a functor from  $\mathcal{A}$  to any category  $\mathcal{B}$  and  $r : R \rightarrow RC$  is a natural morphism satisfying the counitality and coassociativity constraints

$$(R\varepsilon) \circ r = R \quad \text{and} \quad (R\Delta) \circ r = (rC) \circ r.$$

(3) For two comonads  $\mathbb{C}$  on  $\mathcal{A}$  and  $\mathbb{D}$  on  $\mathcal{B}$ , a *bicomodule functor* is a triple  $(Q, l, r)$ , where  $Q : \mathcal{B} \rightarrow \mathcal{A}$  is a functor,  $l$  is a left  $\mathbb{C}$ -coaction and  $r$  is a right  $\mathbb{D}$ -coaction on  $Q$ , such that

$$(Cr) \circ l = (lD) \circ r.$$

(4) *Module functors* for a monad are defined as comodule functors for the corresponding comonad on the opposite category.

**THEOREM 2.12.** *Let  $\mathbb{A} = (A, m^A, u^A)$  and  $\mathbb{T} = (T, m^T, u^T)$  be monads on a category  $\mathcal{K}$  and  $\alpha : \mathbb{A} \rightarrow \mathbb{T}$  be a morphism of monads. Then there exists a functor  $R_A : \mathcal{K}_{\mathbb{T}} \rightarrow \mathcal{K}_{\mathbb{A}}$  such that*

$$(2.5) \quad U_{\mathbb{A}} R_A = U_{\mathbb{T}} \quad \text{and} \quad U_{\mathbb{A}} \lambda^A R_A = (U_{\mathbb{T}} \lambda^T) \circ (\alpha U_{\mathbb{T}}).$$

*Moreover, if there exist coequalizers in  $\mathcal{K}$  and  $T$  preserves coequalizers, then  $R_A$  has a left adjoint.*

*Proof.* In order to construct a functor  $R_A$ , note that for  $(X, x) \in \mathcal{K}_{\mathbb{T}}$ ,  $(X, x \circ (\alpha X)) \in \mathcal{K}_{\mathbb{A}}$ . A morphism  $f : (X, x) \rightarrow (X', x')$  in  $\mathcal{K}_{\mathbb{T}}$  can be regarded as a morphism  $(X, (x \circ \alpha X)) \rightarrow (X', x' \circ (\alpha X'))$  in  $\mathcal{K}_{\mathbb{A}}$ . Therefore we introduce the functor  $R_A : \mathcal{K}_{\mathbb{T}} \rightarrow \mathcal{K}_{\mathbb{A}}$  by setting (2.5).

Assume now that the category  $\mathcal{K}$  has coequalizers. We can define a functor  $N_A : \mathcal{K}_{\mathbb{A}} \rightarrow \mathcal{K}_{\mathbb{T}}$  as an ‘ $\mathbb{A}$ -module product’ of the right  $\mathbb{A}$ -module functor  $(F_{\mathbb{T}}, (\lambda^T F_{\mathbb{T}}) \circ (F_{\mathbb{T}} \alpha))$  and the left  $\mathbb{A}$ -module functor  $(U_{\mathbb{A}}, U_{\mathbb{A}} \lambda^A)$ . That is, we define  $N_A$  via the coequalizer

$$(2.6) \quad (N_A, \chi^A) = \text{Coequ}_{\text{Fun}}((\lambda^T F_{\mathbb{T}} U_{\mathbb{A}}) \circ (F_{\mathbb{T}} \alpha U_{\mathbb{A}}), F_{\mathbb{T}} U_{\mathbb{A}} \lambda^A).$$

It takes a morphism  $f : (Y, y) \rightarrow (Y', y')$  in  $\mathcal{K}_{\mathbb{A}}$  to the the unique morphism  $N_A f$  in  $\mathcal{K}_{\mathbb{T}}$  for which

$$(N_A f) \circ (\chi^A(Y, y)) = (\chi^A(Y', y')) \circ (F_{\mathbb{T}} U_{\mathbb{A}} f).$$

Under the assumption that  $T$  (hence by Proposition 2.10 also  $U_{\mathbb{T}}$ ) preserves coequalizers, we prove next that  $N_A$  and  $R_A$  are adjoint functors. By the construction of  $R_A$ , the morphism  $\lambda^T$  coequalizes the parallel morphisms  $(\lambda^T F_{\mathbb{T}} U_{\mathbb{A}} R_A) \circ (F_{\mathbb{T}} \alpha U_{\mathbb{A}} R_A)$  and  $F_{\mathbb{T}} U_{\mathbb{A}} \lambda^A R_A$ . Thus it follows by universality of the coequalizer  $\chi^A R_A$  (cf. dual form of Lemma 2.2) that there exists a unique natural morphism  $\epsilon^A : N_A R_A \rightarrow \mathcal{K}_{\mathbb{T}}$  such that

$$(2.7) \quad \epsilon^A \circ (\chi^A R_A) = \lambda^T.$$

Since  $(U_{\mathbb{T}} \chi^A) \circ (u^T U_{\mathbb{A}})$  is an  $\mathbb{A}$ -module morphism in the sense that

$$(U_{\mathbb{T}} \lambda^T N_A) \circ (\alpha U_{\mathbb{T}} N_A) \circ (A U_{\mathbb{T}} \chi^A) \circ (A u^T U_{\mathbb{A}}) = (U_{\mathbb{T}} \chi^A) \circ (u^T U_{\mathbb{A}}) \circ (U_{\mathbb{A}} \lambda^A),$$

it lifts to a natural morphism  $\eta^A : \mathcal{K}_{\mathbb{A}} \rightarrow R_A N_A$  in the sense that

$$(2.8) \quad U_{\mathbb{A}} \eta^A = (U_{\mathbb{T}} \chi^A) \circ (u^T U_{\mathbb{A}}).$$

Note that (2.8) immediately implies

$$(2.9) \quad (\lambda^T N_A) \circ (F_{\mathbb{T}} U_{\mathbb{A}} \eta^A) = \chi^A.$$

From (2.7) and (2.9) we deduce that  $(U_{\mathbb{T}} \epsilon^A N_A) \circ (U_{\mathbb{T}} N_A \eta^A) \circ (U_{\mathbb{T}} \chi^A) = U_{\mathbb{T}} \chi^A$ . Since  $U_{\mathbb{T}} \chi^A$  is epi and  $U_{\mathbb{T}}$  is faithful, this implies  $(\epsilon^A N_A) \circ (N_A \eta^A) = N_A$ . Similarly,  $(U_{\mathbb{A}} R_A \epsilon^A) \circ (U_{\mathbb{A}} \eta^A R_A) = U_{\mathbb{A}} R_A$ , so by faithfulness of  $U_{\mathbb{A}}$ ,  $(R_A \epsilon^A) \circ (\eta^A R_A) = R_A$ . Thus we proved that  $(N_A, R_A)$  is an adjunction.  $\square$

Theorem 2.12 implies, in particular, that  $(N_A R_A, N_A \eta^A R_A, \epsilon^A)$  is a comonad on  $\mathcal{K}_{\mathbb{T}}$  and  $(R_A N_A, R_A \epsilon^A N_A, \eta^A)$  is a monad on  $\mathcal{K}_{\mathbb{A}}$ .

For the adjoint functors in Theorem 2.12, induced by a monad morphism  $\alpha$ , we use also the notation  $N_A = \alpha^*$  and  $R_A = \alpha_*$ . Note that, for two monad morphisms  $\alpha : \mathbb{A} \rightarrow \mathbb{T}$  and  $\varphi : \mathbb{T} \rightarrow \mathbb{T}'$ ,  $(\varphi \circ \alpha)_* = \alpha_* \circ \varphi_*$  and  $(\varphi \circ \alpha)^* = \varphi^* \circ \alpha^*$ . The identity functor on any category  $\mathcal{K}$  is a monad, via the multiplication and unit given by the identity natural morphism  $\mathcal{K}$ . Given any monad  $\mathbb{T} = (T, m, u)$  on  $\mathcal{K}$ ,  $u$  gives rise to a monad morphism  $\mathcal{K} \rightarrow \mathbb{T}$  and  $U_{\mathbb{T}} = u_*$  while  $F_{\mathbb{T}} = u^*$ .

**THEOREM 2.13.** *Let  $\mathcal{K}$  be a category with coequalizers and let  $\alpha : \mathbb{A} = (A, m^A, u^A) \rightarrow \mathbb{T} = (T, m^T, u^T)$  be a morphism of monads on  $\mathcal{K}$ . Assume that  $A$  and  $T$  preserve coequalizers. Consider the canonical adjunction  $(N_A, R_A)$ , associated to  $\alpha$  in Theorem 2.12. Then the Eilenberg-Moore comparison functor  $K : \mathcal{K}_{\mathbb{T}} \rightarrow (\mathcal{K}_{\mathbb{A}})_{R_A N_A}$  is an isomorphism.*

*Proof.* Denote by  $U$  forgetful functor  $(\mathcal{K}_{\mathbb{A}})_{R_A N_A} \rightarrow \mathcal{K}_{\mathbb{A}}$ . Recall that the comparison functor  $K$  has the explicit form

$$K(X, x) = (R_A(X, x), R_A \epsilon^A(X, x)) = ((X, x \circ (\alpha X)), R_A \epsilon^A(X, x)),$$

for any object  $(X, x)$  of  $\mathcal{K}_{\mathbb{T}}$ , and  $Kf = R_A f$ , for each morphism  $f$  in  $\mathcal{K}_{\mathbb{T}}$ . In what follows we construct the inverse  $\tilde{K}$  of  $K$ . A direct computation shows that, for any object  $((X, x), x') \in (\mathcal{K}_{\mathbb{A}})_{R_A N_A}$ , the morphism  $(U_{\mathbb{A}} x') \circ (U_{\mathbb{T}} \chi^A(X, x))$  is a  $\mathbb{T}$ -action

on  $X$ . Moreover, with respect to this  $\mathbb{T}$ -action, any morphism in  $(\mathcal{K}_A)_{R_A N_A}$  is a morphism of  $\mathbb{T}$ -modules. Thus, for any object  $((X, x), x') \in (\mathcal{K}_A)_{R_A N_A}$  we can put

$$\tilde{K}((X, x), x') = (X, (U_{\mathbb{A}}x') \circ (U_{\mathbb{T}}\chi^A(X, x))).$$

For a morphism  $f$  in  $(\mathcal{K}_A)_{R_A N_A}$ ,  $\tilde{K}f$  is defined as the unique morphism in  $\mathcal{K}_{\mathbb{T}}$  such that  $U_{\mathbb{T}}\tilde{K}f = U_{\mathbb{A}}Uf$ . The equality  $\tilde{K}K = \mathcal{K}_{\mathbb{T}}$  is obvious. The proof of  $K\tilde{K} = (\mathcal{K}_A)_{R_A N_A}$  is slightly longer but straightforward, so it is left to the reader too.  $\square$

DEFINITION 2.14. Let  $\mathbb{A} = (A, m, u)$  be a monad and  $\mathbb{C} = (C, \Delta, \varepsilon)$  be a comonad on the same category  $\mathcal{K}$ . A natural morphism  $\Psi : AC \rightarrow CA$  is called a *mixed distributive law* (or in some papers an *entwining*) if

- $\Psi \circ (mC) = (Cm) \circ (\Psi A) \circ (A\Psi)$  and  $\Psi \circ (uC) = Cu$ ,
- $(\Delta A) \circ \Psi = (C\Psi) \circ (\Psi C) \circ (A\Delta)$  and  $(\varepsilon A) \circ \Psi = A\varepsilon$ .

THEOREM 2.15. [Be] *Let  $\mathbb{A} = (A, m, u)$  be a monad and  $\mathbb{C} = (C, \Delta, \varepsilon)$  be a comonad on a category  $\mathcal{K}$ . There is a bijection between*

- liftings of  $\mathbb{C}$  to a comonad  $\tilde{\mathbb{C}}$  on  $\mathcal{K}_{\mathbb{A}}$ , i.e. comonads  $\tilde{\mathbb{C}} = (\tilde{C}, \tilde{\Delta}, \tilde{\varepsilon})$  on  $\mathcal{K}_{\mathbb{A}}$ , such that  $U_{\mathbb{A}}\tilde{C} = CU_{\mathbb{A}}$ ,  $U_{\mathbb{A}}\tilde{\Delta} = \Delta U_{\mathbb{A}}$  and  $U_{\mathbb{A}}\tilde{\varepsilon} = \varepsilon U_{\mathbb{A}}$ ;
- mixed distributive laws  $\Psi : AC \rightarrow CA$ .

*Proof.* It is a standard result due to Johnstone [J] that a mixed distributive law  $AC \rightarrow CA$  determines a functor  $\tilde{C} : \mathcal{K}_{\mathbb{A}} \rightarrow \mathcal{K}_{\mathbb{A}}$ , with object map

$$\tilde{C}(X, x) = (CX, (Cx) \circ (\Psi X))$$

and morphism map satisfying  $U_{\mathbb{A}}\tilde{C}f = CU_{\mathbb{A}}f$ , for every morphism  $f \in \mathcal{K}_{\mathbb{A}}$ . It is proven by a slightly twisted version of Beck's arguments [Be] (cf. [W, 5.1]) that it is a comonad, with the stated coproduct and counit.

Conversely, if  $\tilde{\mathbb{C}}$  is a lifting of  $\mathbb{C}$  then a mixed distributive law is constructed as

$$AC \xrightarrow{ACu} ACA = U_{\mathbb{A}}F_{\mathbb{A}}U_{\mathbb{A}}\tilde{C}F_{\mathbb{A}} \xrightarrow{U_{\mathbb{A}}\lambda^A\tilde{C}F_{\mathbb{A}}} U_{\mathbb{A}}\tilde{C}F_{\mathbb{A}} = CA.$$

$\square$

### 3. GALOIS FUNCTORS FOR MIXED DISTRIBUTIVE LAWS

In this section we study particular kinds of comonads – those arising from mixed distributive laws – including the comonads arising from entwining structures of algebras and coalgebras. Our aim is to reformulate the Galois property of a comodule functor of such a comonad, in terms of so called regular comonad arrows. Comonad arrows (not only those corresponding to Galois functors for mixed distributive laws) will play an important role in later sections: They will be related to pre-torsors. Together with the results of the current section, this implies a relation between Galois functors and pre-torsors.

**3.1. Galois functors and regular comonad arrows.** Under various names, functors that we term *Galois functors* (following [MW, Definition 4.5]), have been discussed by several authors, see e.g. [D], [KS], [GT], [MW].

DEFINITION 3.1. If, for a left  $\mathbb{C}$ -comodule functor  $(L, l)$ , the underlying functor  $L$  has a right adjoint  $R$ , then there is a canonical comonad morphism

$$(3.1) \quad \text{can} := (C\epsilon) \circ (lR) : LR \rightarrow C,$$

where  $\epsilon$  denotes the counit of the adjunction, see [KS, Proposition 3.3]. A  $\mathbb{C}$ -Galois functor is, by definition, a left  $\mathbb{C}$ -comodule functor  $(L, l)$ , such that the underlying functor  $L$  possesses a right adjoint  $R$  and the canonical comonad morphism (3.1) is an isomorphism.

The bicategory of (co)monads was introduced in the paper [S]. Its 1-cells, recalled under the name *comonad arrow* in Definition 3.2, generalize morphisms of comonads.

DEFINITION 3.2. Consider two comonads  $\mathbb{C} = (C, \Delta, \epsilon)$  and  $\mathbb{C}' = (C', \Delta', \epsilon')$ , on respective categories  $\mathcal{A}$  and  $\mathcal{A}'$ . A *comonad arrow* from  $\mathbb{C}$  to  $\mathbb{C}'$  is a pair  $(F, \xi)$ , where  $F : \mathcal{A}' \rightarrow \mathcal{A}$  is a functor and  $\xi : CF \rightarrow FC'$  is a natural morphism subject to the conditions

$$(3.2) \quad (F\epsilon') \circ \xi = \epsilon F \quad \text{and} \quad (F\Delta') \circ \xi = (\xi C') \circ (C\xi) \circ (\Delta F).$$

A comonad arrow  $(F, \xi)$  is said to be *regular* provided that  $\xi$  is an isomorphism.

A comonad arrow  $(F, \xi)$  is termed *co-regular* if  $F$  has a left adjoint  $G$  (with unit  $\eta$  and counit  $\epsilon$  of the adjunction) and  $\bar{\xi} := (\epsilon C'G) \circ (G\xi G) \circ (GC\eta) : GC \rightarrow C'G$  is an isomorphism.

In the following theorem, the functor

$$\mathcal{A}_{\mathbb{N}'} \rightarrow \mathcal{A}_{\mathbb{N}}, \quad (X, x) \mapsto (X, x \circ (\varphi X)),$$

induced by a morphism  $\varphi : \mathbb{N} \rightarrow \mathbb{N}'$  of monads on the category  $\mathcal{A}$  (cf. Theorem 2.12), is denoted by  $\varphi_*$ . For an adjunction  $(L, R)$ , the unit and counit are denoted by  $\eta$  and  $\epsilon$ , respectively. For a second adjunction  $(L', R')$ , primed symbols  $\eta'$  and  $\epsilon'$  are used. The symbol  $\text{can}$  (resp.  $\text{can}'$ ) denotes the comonad morphism (3.1) corresponding to a comodule functor  $L$  (resp.  $L'$ ).

THEOREM 3.3. (1) Let  $\mathbb{N} = (N, m^N, u^N)$  be a monad and  $\mathbb{C} = (C, \Delta^C, \epsilon^C)$  be a comonad on a category  $\mathcal{A}$ . For any category  $\mathcal{B}$  and any adjoint pair of functors  $(L : \mathcal{B} \rightarrow \mathcal{A}_{\mathbb{N}}, R : \mathcal{A}_{\mathbb{N}} \rightarrow \mathcal{B})$ , there is a bijective correspondence between the sets of the following data.

(a) Liftings of  $\mathbb{C}$  to a comonad  $\tilde{\mathbb{C}}$  on  $\mathcal{A}_{\mathbb{N}}$  together with a  $\tilde{\mathbb{C}}$ -Galois functor structure on  $L$ ;

(b) regular comonad arrows  $(U_{\mathbb{N}}, \xi)$  from  $\mathbb{C}$  to the comonad  $LR$ .

(2) Let  $\mathbb{N}$  and  $\mathbb{N}'$  be monads on a category  $\mathcal{A}$ . Let  $\mathbb{C}$  and  $\mathbb{C}'$  be comonads on  $\mathcal{A}$ , such that  $\mathbb{C}$  lifts to a comonad  $\tilde{\mathbb{C}}$  on  $\mathcal{A}_{\mathbb{N}}$  and  $\mathbb{C}'$  lifts to a comonad  $\tilde{\mathbb{C}}'$  on  $\mathcal{A}_{\mathbb{N}'}$ . Let  $(L, l)$  be a  $\tilde{\mathbb{C}}$ -Galois functor with right adjoint  $R$  and  $(L', l')$  be a  $\tilde{\mathbb{C}}'$ -Galois functor with right adjoint  $R'$ . In this setting, for any monad morphism  $\varphi : \mathbb{N} \rightarrow \mathbb{N}'$  and comonad morphism  $\theta : \mathbb{C} \rightarrow \mathbb{C}'$ , the following groups of statements are equivalent.

(a)•  $R\varphi_* = R'$ ,

- there is a comonad arrow  $(\varphi_*, \tilde{\theta})$  from  $\tilde{\mathbb{C}}$  to  $\tilde{\mathbb{C}}'$ , such that  $U_{\mathbb{N}}\tilde{\theta} = \theta U_{\mathbb{N}'}$ ,
  - the natural morphism  $\tilde{\varphi} := (\epsilon\varphi_*L') \circ (L\eta') : L \rightarrow \varphi_*L'$  satisfies  $(\varphi_*l') \circ \tilde{\varphi} = (\tilde{\theta}L') \circ (\tilde{\mathbb{C}}\tilde{\varphi}) \circ l$ .
- (b)•  $R\varphi_* = R'$ ,
- $(U_{\mathbb{N}'}\text{can}') \circ (U_{\mathbb{N}}\epsilon\varphi_*L'R') \circ (U_{\mathbb{N}}L\eta'R') = (\theta U_{\mathbb{N}'}) \circ (U_{\mathbb{N}}\text{can}\varphi_*)$ .

*Proof.* (1) Consider first data as in part (a) and put  $\xi := U_{\mathbb{N}}\text{can}^{-1}$ . It is obviously an isomorphism and the identities (3.2) follow by using that  $\text{can} : LR \rightarrow \tilde{\mathbb{C}}$  is a comonad morphism.

Conversely, in terms of the data in part (b), a mixed distributive law  $NC \rightarrow CN$  is given by the natural morphism

$$\psi := (\xi^{-1}F_{\mathbb{N}}) \circ (U_{\mathbb{N}}\lambda^N LRF_{\mathbb{N}}) \circ (N\xi F_{\mathbb{N}}) \circ (NCu^N).$$

This proves that  $\mathbb{C}$  lifts to a comonad  $\tilde{\mathbb{C}}$  on  $\mathcal{A}_{\mathbb{N}}$ , cf. Theorem 2.15. Moreover,  $\psi$  induces an  $\mathbb{N}$ -action

$$(CU_{\mathbb{N}}\lambda^N) \circ (\psi U_{\mathbb{N}}) : NCU_{\mathbb{N}} \rightarrow CU_{\mathbb{N}}$$

on  $CU_{\mathbb{N}}$ . It is easy to check that (by naturality and the adjunction relations)  $\xi$  is an  $\mathbb{N}$ -module morphism in the sense that

$$\xi \circ (CU_{\mathbb{N}}\lambda^N) \circ (\psi U_{\mathbb{N}}) = (U_{\mathbb{N}}\lambda^N LR) \circ (N\xi).$$

This means that  $\xi$  gives rise to a morphism  $\hat{\xi}$  in  $\mathcal{A}_{\mathbb{N}}$ , such that  $U_{\mathbb{N}}\hat{\xi} = \xi$ . Since the forgetful functor  $U_{\mathbb{N}}$  reflects isomorphisms,  $\hat{\xi}$  is an isomorphism in  $\mathcal{A}_{\mathbb{N}}$ . This enables us to equip  $L$  with a  $\tilde{\mathbb{C}}$ -coaction by putting

$$(3.3) \quad l := (\hat{\xi}^{-1}L) \circ (L\eta) : L \rightarrow \tilde{\mathbb{C}}L.$$

Using identities (3.2), naturality and the adjunction relations, we find that

$$(\epsilon^C U_{\mathbb{N}}L) \circ (U_{\mathbb{N}}l) = U_{\mathbb{N}}L \quad \text{and} \quad (\Delta^C U_{\mathbb{N}}L) \circ (U_{\mathbb{N}}l) = (CU_{\mathbb{N}}l) \circ (U_{\mathbb{N}}l).$$

Since  $\tilde{\mathbb{C}}$  is the lifting of  $\mathbb{C}$  and  $U_{\mathbb{N}}$  is faithful, this implies coassociativity and counitality of the coaction  $l$ . Moreover, by (3.3), naturality and the adjunction relations,

$$U_{\mathbb{N}}\text{can} = (CU_{\mathbb{N}}\epsilon) \circ (U_{\mathbb{N}}lR) = \xi^{-1} = U_{\mathbb{N}}\hat{\xi}^{-1}.$$

Thus by faithfulness of  $U_{\mathbb{N}}$ , we conclude that  $\text{can} = \hat{\xi}^{-1}$  is an isomorphism, hence  $(L, l)$  is a  $\tilde{\mathbb{C}}$ -Galois functor.

Above constructions are easily checked to yield a bijective correspondence between the data in parts (a) and (b).

(2) Assume that the conditions in part (a) hold. By (3.1), the second property in part (b) is equivalent to

$$(C'U_{\mathbb{N}'}\epsilon') \circ (U_{\mathbb{N}'}l'R') \circ (U_{\mathbb{N}}\tilde{\varphi}R') = (\theta U_{\mathbb{N}'}) \circ (CU_{\mathbb{N}}\epsilon\varphi_*) \circ (U_{\mathbb{N}}lR\varphi_*),$$

which is proven by using the third condition in part (a), the identity  $U_{\mathbb{N}}\tilde{\theta} = \theta U_{\mathbb{N}'}$  and the fact that  $\tilde{\mathbb{C}}$  is the lifting of  $\mathbb{C}$ , naturality and definition of  $\tilde{\varphi}$ , adjunction relations and finally the first identity in part (a).

Conversely, assume that assertion (b) holds. The natural morphism  $\theta$  is checked to have a lifting  $\tilde{\theta}$  if and only if the mixed distributive laws  $\psi$  and  $\psi'$ , that are responsible for the lifting of  $\mathbb{C}$  to  $\tilde{\mathbb{C}}$  and the lifting of  $\mathbb{C}'$  to  $\tilde{\mathbb{C}}'$ , respectively, satisfy

$$(3.4) \quad \psi' \circ (\varphi\theta) = (\theta\varphi) \circ \psi.$$

In order to prove (3.4), note that both  $N$  and  $N'$  are left  $\mathbb{N}$ -module functors, via the actions provided by the multiplication  $m^N$  in  $\mathbb{N}$ , and  $m^{N'} \circ (\varphi N')$ , respectively. With respect to these actions,  $\varphi$  is an  $\mathbb{N}$ -module morphism. Hence there exists a (unique) morphism  $\widehat{\varphi} : F_{\mathbb{N}} \rightarrow \varphi_* F_{\mathbb{N}'}$  (explicitly given by  $\widehat{\varphi} = (\lambda^N \varphi_* F_{\mathbb{N}'}) \circ (F_{\mathbb{N}} u^{N'})$ ) such that  $U_{\mathbb{N}} \widehat{\varphi} = \varphi$ . Moreover, by the definition of the functor  $\varphi_*$ , the identity  $(U_{\mathbb{N}'} \lambda^{N'}) \circ (\varphi U_{\mathbb{N}'}) = U_{\mathbb{N}} \lambda^N \varphi_*$  holds. Making use of these observations, (3.4) follows by recalling the form of  $\psi$  and  $\psi'$  from part (1), repeated use of the second condition in part (b), the monad morphism property of  $\varphi$  and naturality. This proves the existence of  $\tilde{\theta}$ . Since  $\theta$  is a comonad morphism,  $\tilde{\theta}$  satisfies (3.2).

The third condition in part (a) follows by (3.3) and the analogous formula for  $l'$ , the second condition in part (b), naturality and adjunction relations.  $\square$

Theorem 3.3 can be rephrased as a statement about the existence of a certain functor, between categories defined below.

**DEFINITION 3.4.** For any two categories  $\mathcal{A}$  and  $\mathcal{B}$ , the category  $\text{Adj}(\mathcal{A}, \mathcal{B})$  is defined as follows.

*Objects* are triples  $(\mathcal{T}, (N_A, R_A), (N_B, R_B))$ , where  $\mathcal{T}$  is a category and

$$(N_A : \mathcal{A} \rightarrow \mathcal{T}, R_A : \mathcal{T} \rightarrow \mathcal{A}) \quad \text{and} \quad (N_B : \mathcal{B} \rightarrow \mathcal{T}, R_B : \mathcal{T} \rightarrow \mathcal{B})$$

are adjunctions. We denote the respective units of the adjunctions by  $\eta^A$  and  $\eta^B$  and the counits by  $\epsilon^A$  and  $\epsilon^B$ .

*Morphisms*  $(\mathcal{T}, (N_A, R_A), (N_B, R_B)) \rightarrow (\mathcal{T}', (N'_A, R'_A), (N'_B, R'_B))$  are functors  $F : \mathcal{T}' \rightarrow \mathcal{T}$  such that  $R_A F = R'_A$  and  $R_B F = R'_B$ .

Note that a morphism  $F$  in  $\text{Adj}(\mathcal{A}, \mathcal{B})$  comes equipped with natural morphisms  $a := (\epsilon^A F N'_A) \circ (N_A \eta'^A) : N_A \rightarrow F N'_A$  and  $b := (\epsilon^B F N'_B) \circ (N_B \eta'^B) : N_B \rightarrow F N'_B$  such that the following compatibility conditions hold.

$$(3.5) \quad \begin{aligned} (R_A a) \circ \eta^A &= \eta'^A & \text{and} & & (R_B b) \circ \eta^B &= \eta'^B, \\ (F \epsilon'^A) \circ (a R'_A) &= \epsilon^A F & \text{and} & & (F \epsilon'^B) \circ (b R'_B) &= \epsilon^B F. \end{aligned}$$

In fact,  $a$  and  $b$  are unique natural morphisms satisfying these identities.

An object  $(\mathcal{T}, (N_A, R_A), (N_B, R_B))$  of  $\text{Adj}(\mathcal{A}, \mathcal{B})$  determines two comonads on  $\mathcal{T}$ ,

$$(N_A R_A, N_A \eta^A R_A, \epsilon^A) \quad \text{and} \quad (N_B R_B, N_B \eta^B R_B, \epsilon^B).$$

**DEFINITION 3.5.** The category  $\text{Arr}(\mathcal{A}, \mathcal{B})$  is defined to have *objects* of the form  $(\mathcal{T}, (N_A, R_A), (N_B, R_B), \mathbb{C}, \xi)$ , where  $(\mathcal{T}, (N_A, R_A), (N_B, R_B))$  is an object in  $\text{Adj}(\mathcal{A}, \mathcal{B})$ ,  $\mathbb{C}$  is a comonad on  $\mathcal{A}$  and  $(R_A, \xi)$  is a comonad arrow from  $\mathbb{C}$  to  $N_B R_B$ .

A *morphism* in  $\text{Arr}(\mathcal{A}, \mathcal{B})$

$$(\mathcal{T}, (N_A, R_A), (N_B, R_B), \mathbb{C}, \xi) \rightarrow (\mathcal{T}', (N'_A, R'_A), (N'_B, R'_B), \mathbb{C}', \xi')$$

consists of a morphism  $F : (\mathcal{T}, (N_A, R_A), (N_B, R_B)) \rightarrow (\mathcal{T}', (N'_A, R'_A), (N'_B, R'_B))$  in  $\text{Adj}(\mathcal{A}, \mathcal{B})$  and a comonad morphism  $t : \mathbb{C} \rightarrow \mathbb{C}'$ , such that

$$(3.6) \quad (R_A b R'_B) \circ (\xi F) = \xi' \circ (t R'_A),$$

where the morphism  $b$  was introduced in (3.5).

The full subcategory of  $\text{Arr}(\mathcal{A}, \mathcal{B})$  of objects  $(\mathcal{T}, (N_A, R_A), (N_B, R_B), \mathbb{C}, \xi)$ , such that the comonad arrow  $(R_A, \xi)$  is *regular*, will be denoted by  $\text{RArr}(\mathcal{A}, \mathcal{B})$ .

The full subcategory of  $\text{Arr}(\mathcal{A}, \mathcal{B})$  of objects  $(\mathcal{T}, (N_A, R_A), (N_B, R_B), \mathbb{C}, \xi)$ , such that the comonad arrow  $(R_A, \xi)$  is *co-regular*, will be denoted by  $\overline{\text{RArr}}(\mathcal{A}, \mathcal{B})$ .

Obviously, composites of morphisms in  $\text{Arr}(\mathcal{A}, \mathcal{B})$  are again morphisms in  $\text{Arr}(\mathcal{A}, \mathcal{B})$ .

The other category occurring in Theorem 3.3 is the following.

**DEFINITION 3.6.** For any two categories  $\mathcal{A}$  and  $\mathcal{B}$ , *objects* of the category  $\text{Gal}(\mathcal{A}, \mathcal{B})$  are quintuples  $(\mathbb{N}, \mathbb{C}, \psi, L, l)$ , where  $\mathbb{N}$  is a monad and  $\mathbb{C}$  is a comonad on  $\mathcal{A}$ , and  $\psi$  is a mixed distributive law between them.  $L : \mathcal{B} \rightarrow \mathcal{A}_{\mathbb{N}}$  is a Galois functor, with coaction  $l$ , for the lifted comonad  $\tilde{\mathbb{C}}$  on  $\mathcal{A}_{\mathbb{N}}$ , determined by the mixed distributive law  $\psi$ .

A *morphism*  $(\mathbb{N}, \mathbb{C}, \psi, L, l) \rightarrow (\mathbb{N}', \mathbb{C}', \psi', L', l')$  is a pair  $(\varphi, \theta)$ , consisting of a monad morphism  $\varphi : \mathbb{N} \rightarrow \mathbb{N}'$  and a comonad morphism  $\theta : \mathbb{C} \rightarrow \mathbb{C}'$ , subject to the conditions in Theorem 3.3 (2)(b).

**COROLLARY 3.7.** *By Theorem 3.3, there is a functor  $I : \text{Gal}(\mathcal{A}, \mathcal{B}) \rightarrow \text{RArr}(\mathcal{A}, \mathcal{B})$ , that is faithful and injective also on the objects. The object and morphism maps of  $I$  are*

$$(\mathbb{N}, \mathbb{C}, \psi, L, l) \mapsto (\mathcal{A}_{\mathbb{N}}, (F_{\mathbb{N}}, U_{\mathbb{N}}), (L, R), \mathbb{C}, U_{\mathbb{N}} \text{can}^{-1}) \quad \text{and} \quad (\varphi, \theta) \mapsto (\varphi_*, \theta),$$

where  $R$  is the right adjoint of  $L$ . Moreover, any object in  $\text{RArr}(\mathcal{A}, \mathcal{B})$  that is of the form  $(\mathcal{A}_{\mathbb{N}}, (F_{\mathbb{N}}, U_{\mathbb{N}}), (L, R), \mathbb{C}, \xi)$  arises as the image of an (unique) object in  $\text{Gal}(\mathcal{A}, \mathcal{B})$  under the functor  $I$ .

**3.2. Examples from bimodules.** For an associative and unital algebra  $\mathbf{A}$ , consider an  $\mathbf{A}$ -ring  $\mathbf{T}$  and an  $\mathbf{A}$ -coring  $\mathbf{C}$ , that are entwined by  $\psi : \mathbf{C} \otimes_{\mathbf{A}} \mathbf{T} \rightarrow \mathbf{T} \otimes_{\mathbf{A}} \mathbf{C}$ . (For a review of these structures we refer to Sections A.1, A.2 and A.4 of [ABM].) Denote the induced  $\mathbf{T}$ -coring  $\mathbf{T} \otimes_{\mathbf{A}} \mathbf{C}$  (cf. [ABM, Section A.4]) by  $\tilde{\mathbf{C}}$ . These data determine a monad  $\mathbb{N} = (-) \otimes_{\mathbf{A}} \mathbf{T}$  and a comonad  $\mathbb{C} = (-) \otimes_{\mathbf{A}} \mathbf{C}$  on the category  $\mathcal{A} := \text{Mod-}\mathbf{A}$  and also a lifted comonad  $\tilde{\mathbb{C}} = (-) \otimes_{\mathbf{T}} \tilde{\mathbf{C}} \cong (-) \otimes_{\mathbf{A}} \mathbf{C}$  on  $\mathcal{A}_{\mathbb{N}} \cong \text{Mod-}\mathbf{T}$ .

Take now a right  $\tilde{\mathbf{C}}$ -comodule (i.e. entwined module)  $\Sigma$ , and let  $\mathbf{B}$  be any subalgebra of  $\text{End}^{\tilde{\mathbf{C}}}(\Sigma)$ . Then the functor  $(-) \otimes_{\mathbf{B}} \Sigma$ , from the category  $\mathcal{B} = \text{Mod-}\mathbf{B}$  to  $\text{Mod-}\mathbf{T}$ , is a  $\tilde{\mathbf{C}}$ -comodule functor, which is  $\tilde{\mathbf{C}}$ -Galois provided that  $\Sigma$  is a (not necessarily finite) Galois comodule. The corresponding object in  $\text{RArr}(\text{Mod-}\mathbf{A}, \text{Mod-}\mathbf{B})$  is

$$(\text{Mod-}\mathbf{T}, ((-) \otimes_{\mathbf{A}} \mathbf{T}, \text{Hom}_{\mathbf{T}}(\mathbf{T}, -)), ((-) \otimes_{\mathbf{B}} \Sigma, \text{Hom}_{\mathbf{T}}(\Sigma, -)), (-) \otimes_{\mathbf{A}} \mathbf{C}, \beta^{-1}),$$

where  $\beta$  is the natural isomorphism, given in terms of the  $\mathbf{C}$ -coaction  $x \mapsto x_{(0)} \otimes_{\mathbf{A}} x_{(1)}$  on  $\Sigma$  as

$$\text{Hom}_{\mathbf{T}}(\Sigma, -) \otimes_{\mathbf{B}} \Sigma \rightarrow (-) \otimes_{\mathbf{A}} \mathbf{C}, \quad f \otimes_{\mathbf{A}} x \mapsto f(x_{(0)}) \otimes_{\mathbf{A}} x_{(1)}.$$

In general, objects of the category  $\overline{\text{RArr}}(\mathcal{A}, \mathcal{B})$  seem to have no interpretation similar to Corollary 3.7. Any object  $(\mathcal{T}, (N_A, R_A), (N_B, R_B), \mathbb{C}, \xi)$  of  $\overline{\text{RArr}}(\mathcal{A}, \mathcal{B})$  determines a natural morphism

$$\Phi := (R_A \bar{\xi}^{-1}) \circ (\xi N_A) : CR_A N_A \rightarrow R_A N_A C,$$

where  $\bar{\xi} = (\epsilon^A N_B R_B N_A) \circ (N_A \xi N_A) \circ (N_A C \eta^A)$  is an isomorphism by assumption. In terms of the structure morphisms of the comonad  $\mathbb{C} = (C, \Delta, \varepsilon)$ , this morphism  $\Phi$  satisfies

$$\begin{aligned} (R_A N_A \Delta) \circ \Phi &= (\Phi C) \circ (C \Phi) \circ (\Delta R_A N_A) \\ (R_A N_A \varepsilon) \circ \Phi &= \varepsilon R_A N_A \\ \Phi \circ (C R_A \epsilon^A N_A) &= (R_A \epsilon^A N_A C) \circ (R_A N_A \Phi) \circ (\Phi R_A N_A) \\ \Phi \circ (C \eta^A) &= \eta^A C. \end{aligned}$$

Although these conditions are reminiscent to Definition 2.14 of a mixed distributive law, in general  $\Phi$  has no interpretation in terms of a corresponding lifting of  $\mathbb{C}$  to the category of modules for the monad  $R_A N_A$ . (But note that if both  $\xi$  and  $\bar{\xi}$  are isomorphisms then so is  $\Phi$  and its inverse is a mixed distributive law in the sense of Definition 2.14.)

However, there are interesting (every-day-seen) examples of objects in  $\overline{\text{RArr}}(\mathcal{A}, \mathcal{B})$  that can be interpreted as Galois functors. Namely, for two algebras  $\mathbf{A}$  and  $\mathbf{B}$ , an  $\mathbf{A}$ -ring  $\mathbf{T}$  and a  $\mathbf{B}$ - $\mathbf{T}$  bimodule  $\Sigma$ , such that  $\Sigma$  is a finitely generated and projective right  $\mathbf{T}$ -module, there are adjunctions

$$\begin{aligned} (\mathbf{T} \otimes_{\mathbf{A}} (-) : \mathbf{A}\text{-Mod} \rightarrow \mathbf{T}\text{-Mod}, \mathbf{T} \otimes_{\mathbf{T}} (-) : \mathbf{T}\text{-Mod} \rightarrow \mathbf{A}\text{-Mod}) \\ (\Sigma^* \otimes_{\mathbf{B}} (-) : \mathbf{B}\text{-Mod} \rightarrow \mathbf{T}\text{-Mod}, \Sigma \otimes_{\mathbf{T}} (-) : \mathbf{T}\text{-Mod} \rightarrow \mathbf{B}\text{-Mod}). \end{aligned}$$

Moreover, for any  $\mathbf{A}$ -coring  $\mathbf{C}$ ,

$$(\text{Mod-}\mathbf{T}, ((-) \otimes_{\mathbf{A}} \mathbf{T}, (-) \otimes_{\mathbf{T}} \mathbf{T}), ((-) \otimes_{\mathbf{B}} \Sigma, (-) \otimes_{\mathbf{T}} \Sigma^*), (-) \otimes_{\mathbf{A}} \mathbf{C}, \xi = (-) \otimes_{\mathbf{T}} \xi \mathbf{T})$$

is an object of  $\overline{\text{RArr}}(\text{Mod-}\mathbf{A}, \text{Mod-}\mathbf{B})$  if and only if

$$(\mathbf{T}\text{-Mod}, (\mathbf{T} \otimes_{\mathbf{A}} (-), \mathbf{T} \otimes_{\mathbf{T}} (-)), (\Sigma^* \otimes_{\mathbf{B}} (-), \Sigma \otimes_{\mathbf{T}} (-)), \mathbf{C} \otimes_{\mathbf{A}} (-), \bar{\xi} \mathbf{A} \otimes_{\mathbf{T}} (-))$$

is an object of  $\text{RArr}(\mathbf{A}\text{-Mod}, \mathbf{B}\text{-Mod})$ . That is, if and only if  $\Sigma^* \otimes_{\mathbf{B}} (-) : \mathbf{B}\text{-Mod} \rightarrow \mathbf{T}\text{-Mod}$  is a Galois functor for a lifted comonad  $\mathbf{C} \otimes_{\mathbf{A}} (-)$  on  $\mathbf{T}\text{-Mod}$ .

Thus, as objects of the category  $\text{RArr}(\mathcal{A}, \mathcal{B})$  generalize (finite) right Galois comodules of corings arising from *right* entwining structures, objects of  $\overline{\text{RArr}}(\mathcal{A}, \mathcal{B})$  generalize, in a sense, finite left Galois comodules of corings coming from *left* entwining structures. Objects that belong to both categories  $\text{RArr}(\mathcal{A}, \mathcal{B})$  and  $\overline{\text{RArr}}(\mathcal{A}, \mathcal{B})$  generalize dual pairs of finite left and right Galois comodules of the two (isomorphic) corings arising from *bijective* entwining structures.

**3.3. Examples from monad morphisms.** More exotic examples of objects in a category  $\text{RArr}(\mathcal{A}, \mathcal{B})$  are obtained from *Galois extensions of monads by a comonad*.

Recall that in an adjunction  $(L, R)$  – with unit  $\eta$  and counit  $\epsilon -$ , the right adjoint  $R$  is a right comodule functor for some comonad  $\mathbb{G}$ , with coaction  $g : R \rightarrow RG$  if

and only if  $L$  is a left  $\mathbb{G}$ -comodule functor via the coaction

$$(3.7) \quad \bar{g} := (\epsilon GL) \circ (LgL) \circ (L\eta) : L \rightarrow GL.$$

**PROPOSITION 3.8.** *Consider an adjoint pair of functors  $(L : \mathcal{K} \rightarrow \mathcal{C}, R : \mathcal{C} \rightarrow \mathcal{K})$ , with unit  $\eta$  and counit  $\epsilon$  of the adjunction. Assume that  $(R, g)$  is a right comodule functor for some comonad  $\mathbb{G}$  on  $\mathcal{C}$  (equivalently,  $(L, \bar{g})$  is a left  $\mathbb{G}$ -comodule functor, with coaction  $\bar{g}$  in (3.7)). Then, if there exists the equalizer*

$$(3.8) \quad (B, \beta) = \text{Equ}_{\text{Fun}}(gL, R\bar{g}),$$

then there exists a monad  $\mathbb{B} = (B, m^B, u^B)$  on  $\mathcal{K}$  such that  $\beta$  gives rise to a morphism of monads  $\beta : \mathbb{B} \hookrightarrow RL$ . Moreover,  $\mathbb{B}$  is the unique monad with this property.

*Proof.* By naturality and the adjunction relation,  $(R\bar{g}) \circ \eta = (gL) \circ \eta$ . Thus by the universality of the equalizer (3.8), there exists a unique natural morphism  $u^B : \mathcal{K} \rightarrow B$ , such that  $\beta \circ u^B = \eta$ .

Similarly,  $(G\epsilon) \circ (\bar{g}R) = (\epsilon G) \circ (Lg)$ . With this identity at hand and using the fork property of (3.8) (twice), one checks that  $(R\epsilon L) \circ (\beta\beta)$  equalizes the parallel morphisms in (3.8). Hence there exists a unique natural morphism  $m^B : BB \rightarrow B$ , such that  $\beta \circ m^B = (R\epsilon L) \circ (\beta\beta)$ . Associativity and unitality of the monad  $\mathbb{B} := (B, m^B, u^B)$  are obvious by respective properties of the monad  $(RL, R\epsilon L, \eta)$ . The morphism  $\beta$  is compatible with the monad structures by construction.  $\square$

Consider a monad  $\mathbb{T}$  on a category  $\mathcal{K}$ . We can apply Proposition 3.8 to the associated adjunction  $(L = F_{\mathbb{T}}, R = U_{\mathbb{T}})$ , and a comonad  $\mathbb{G}$  on  $\mathcal{K}_{\mathbb{T}}$ . Note that the resulting notion of a  $\mathbb{G}$ -coaction  $g$  on  $U_{\mathbb{T}}$  generalizes the notion of a *grouplike element* in a coring. In this setting, if there exists the monad  $\mathbb{B} = (B, m^B, u^B)$  on  $\mathcal{K}$  that is described in Proposition 3.8, then it will be denoted by  $\mathbb{T}^{Co(G,g)}$  and it will be called the *coinvariant monad* of  $\mathbb{T}$  with respect to  $\mathbb{G}$  and  $g$ . Note that the coinvariant monad  $\mathbb{T}^{Co(G,g)}$  exists whenever  $\mathcal{K}$  has equalizers.

**PROPOSITION 3.9.** *Let  $\mathcal{K}$  be a category with equalizers and coequalizers. Let  $\mathbb{T} = (T, m, u)$  be a monad on  $\mathcal{K}$  and  $\mathbb{G} = (G, \Delta, \epsilon)$  be a comonad on  $\mathcal{K}_{\mathbb{T}}$ . Assume that  $T$  preserves coequalizers and that there exists a right  $\mathbb{G}$ -coaction  $g : U_{\mathbb{T}} \rightarrow U_{\mathbb{T}}G$ . Denote  $\mathbb{B} := \mathbb{T}^{Co(G,g)}$ . Let  $(N_B, R_B)$  be the canonical adjunction, with unit  $\eta^B$  and counit  $\epsilon^B$ , associated as in Theorem 2.12 to the canonical inclusion  $\beta : \mathbb{B} \hookrightarrow \mathbb{T}$ . Under these assumptions,  $N_B$  can be equipped with the structure of a left  $\mathbb{G}$ -comodule functor.*

*Proof.* Consider the left  $\mathbb{G}$ -coaction  $\bar{g} = (\lambda^T GF_{\mathbb{T}}) \circ (F_{\mathbb{T}}gF_{\mathbb{T}}) \circ (F_{\mathbb{T}}u^T) : F_{\mathbb{T}} \rightarrow GF_{\mathbb{T}}$ . Recall that  $F_{\mathbb{T}}$  is also a right  $\mathbb{B}$ -module functor, via the action  $f := (\lambda^T F_{\mathbb{T}}) \circ (F_{\mathbb{T}}\beta)$ . Moreover, the  $\mathbb{B}$ -action and the  $\mathbb{G}$ -coaction on  $F_{\mathbb{T}}$  commute in the sense that  $(Gf) \circ (\bar{g}B) = \bar{g} \circ f$ . This implies that  $(G\chi^B) \circ (\bar{g}U_{\mathbb{B}})$  coequalizes the parallel morphisms (given by the  $\mathbb{B}$ -actions) in  $(N_B, \chi^B) = \text{Coequ}_{\text{Fun}}(fU_{\mathbb{B}}, F_{\mathbb{T}}U_{\mathbb{B}}\lambda^B)$ . So there exists a unique natural morphism  $\bar{h} : N_B \rightarrow GN_B$ , such that

$$(3.9) \quad \bar{h} \circ \chi^B = (G\chi^B) \circ (\bar{g}U_{\mathbb{B}}).$$

Coassociativity and counitality of  $\bar{h}$  are obvious by the analogous properties of  $\bar{g}$ .  $\square$

The comodule functor  $(N_B, \bar{h})$  in Proposition 3.9 is a Galois functor provided that the canonical comonad morphism

$$(3.10) \quad \text{can} := (G\epsilon^B) \circ (\bar{h}R_B) : (N_B R_B, N_B \eta^B R_B, \epsilon^B) \rightarrow (G, \Delta, \varepsilon)$$

is an isomorphism.

**THEOREM 3.10.** *Let  $\mathcal{K}$  be a category with equalizers and coequalizers and let  $\alpha : \mathbb{A} \rightarrow \mathbb{T}$  and  $\beta : \mathbb{B} \rightarrow \mathbb{T}$  be morphisms of monads on  $\mathcal{K}$ . Denote their canonical adjunctions (cf. Theorem 2.12) by  $(N_A, R_A)$  and  $(N_B, R_B)$ , respectively. Assume that the underlying functors  $A$  and  $T$  preserve coequalizers. Assume furthermore that the unit  $\eta^B$  of the adjunction  $(N_B, R_B)$  is a regular natural monomorphism. Then there is a bijective correspondence between the following sets of data.*

- (a) objects  $(\mathcal{K}_{\mathbb{T}}, (N_A, R_A), (N_B, R_B), \mathbb{C}, \xi)$  of  $\text{RArr}(\mathcal{K}_{\mathbb{A}}, \mathcal{K}_{\mathbb{B}})$ ,
- (b) right coactions  $g : U_{\mathbb{T}} \rightarrow U_{\mathbb{T}}G$  for a lifting of a comonad  $\mathbb{C}$  on  $\mathcal{K}_{\mathbb{A}}$  to a comonad  $\mathbb{G}$  on  $\mathcal{K}_{\mathbb{T}} \cong (\mathcal{K}_{\mathbb{A}})_{R_A N_A}$ , subject to the following conditions.
  - $\mathbb{B} = \mathbb{T}^{C \circ (G.g)}$  and  $\beta$  is the canonical inclusion  $\mathbb{B} = \mathbb{T}^{C \circ (G.g)} \hookrightarrow \mathbb{T}$ ,
  - the canonical comonad morphism (3.10) is an isomorphism.

*Proof.* By Theorem 2.13, the module categories  $\mathcal{K}_{\mathbb{T}}$  and  $(\mathcal{K}_{\mathbb{A}})_{R_A N_A}$  are isomorphic. Hence the data in part (a) are in bijective correspondence with the objects

$$(3.11) \quad ((\mathcal{K}_{\mathbb{A}})_{R_A N_A}, (N'_A, R'_A), (N'_B, R'_B), \mathbb{C}, \xi')$$

of  $\text{RArr}(\mathcal{K}_{\mathbb{A}}, \mathcal{K}_{\mathbb{B}})$ , where the primed functors are obtained by composing with the isomorphism  $\mathcal{K}_{\mathbb{T}} \cong (\mathcal{K}_{\mathbb{A}})_{R_A N_A}$  on the appropriate side. Note that  $R'_A : (\mathcal{K}_{\mathbb{A}})_{R_A N_A} \rightarrow \mathcal{K}_{\mathbb{A}}$  is the forgetful functor corresponding to the monad  $R_A N_A$ . Therefore, by Theorem 3.3 there is a bijection between the data in (3.11) and liftings of  $\mathbb{C}$  to a comonad  $\tilde{\mathbb{C}}$  on  $(\mathcal{K}_{\mathbb{A}})_{R_A N_A}$  together with a  $\tilde{\mathbb{C}}$ -Galois structure on the functor  $N'_B$ . The isomorphism  $\mathcal{K}_{\mathbb{T}} \cong (\mathcal{K}_{\mathbb{A}})_{R_A N_A}$  takes a comonad  $\tilde{\mathbb{C}}$  on  $(\mathcal{K}_{\mathbb{A}})_{R_A N_A}$  to a comonad  $\mathbb{G}$  on  $\mathcal{K}_{\mathbb{T}}$ . Clearly,  $N_B$  is a  $\tilde{\mathbb{C}}$ -Galois functor if and only if  $N'_B$  is a  $\mathbb{G}$ -Galois functor. Thus the data in part (a) are in bijective correspondence with liftings of  $\mathbb{C}$  to a comonad  $\mathbb{G}$  on  $\mathcal{K}_{\mathbb{T}}$  together with a  $\mathbb{G}$ -Galois structure on  $N_B$ .

The data in part (b) determine a  $\mathbb{G}$ -Galois structure on  $N_B$  by Proposition 3.9. Conversely, a  $\mathbb{G}$ -Galois functor  $(N_B, \bar{h})$  determines a right  $\mathbb{G}$ -coaction  $h := (R_B G \epsilon^B) \circ (R_B \bar{h} R_B) \circ (\eta^B R_B)$  on  $R_B$  and a right  $\mathbb{G}$ -coaction  $g := U_{\mathbb{B}} h = (U_{\mathbb{T}} \text{can}) \circ (U_{\mathbb{B}} \eta^B R_B)$  on  $U_{\mathbb{T}} = U_{\mathbb{B}} R_B$ , where  $\text{can}$  is the isomorphism (3.10). It remains to show that for this coaction  $g$ ,

$$(3.12) \quad \begin{aligned} (B, \beta) &= \text{Equ}_{\text{Fun}}(gF_{\mathbb{T}}, (U_{\mathbb{T}} \lambda^T G F_{\mathbb{T}}) \circ (TgF_{\mathbb{T}}) \circ (Tu^T)) \\ &\equiv \text{Equ}_{\text{Fun}}((U_{\mathbb{B}} \eta^B R_B F_{\mathbb{T}}), (U_{\mathbb{T}} \lambda^T N_B R_B F_{\mathbb{T}}) \circ (TU_{\mathbb{B}} \eta^B R_B F_{\mathbb{T}}) \circ (Tu^T)). \end{aligned}$$

Since both  $F_{\mathbb{T}}$  and  $N_B F_{\mathbb{B}}$  are left adjoints of  $U_{\mathbb{T}} = U_{\mathbb{B}} R_B$ , there is a natural isomorphism  $\gamma : N_B F_{\mathbb{B}} \rightarrow F_{\mathbb{T}}$ . Clearly, (3.12) is equivalent to

$$(3.13) \quad \begin{aligned} (B, (U_{\mathbb{T}} \gamma^{-1}) \circ \beta) &= \\ &\text{Equ}_{\text{Fun}}((U_{\mathbb{T}} N_B R_B \gamma^{-1}) \circ (U_{\mathbb{B}} \eta^B R_B F_{\mathbb{T}}) \circ (U_{\mathbb{T}} \gamma), \\ &(U_{\mathbb{T}} N_B R_B \gamma^{-1}) \circ (U_{\mathbb{T}} \lambda^T N_B R_B F_{\mathbb{T}}) \circ (TU_{\mathbb{B}} \eta^B R_B F_{\mathbb{T}}) \circ (Tu^T) \circ (U_{\mathbb{T}} \gamma)). \end{aligned}$$

Using the explicit form of  $\gamma = (\epsilon^B F_{\mathbb{T}}) \circ (N_B \lambda^B R_B F_{\mathbb{T}}) \circ (N_B F_{\mathbb{B}} u^T)$ , it is straightforward to check that

$$\begin{aligned} (U_{\mathbb{T}} \gamma^{-1}) \circ \beta &= U_{\mathbb{B}} \eta^B F_{\mathbb{B}}, \\ (U_{\mathbb{T}} N_B R_B \gamma^{-1}) \circ (U_{\mathbb{B}} \eta^B R_B F_{\mathbb{T}}) \circ (U_{\mathbb{T}} \gamma) &= U_{\mathbb{B}} \eta^B R_B N_B F_{\mathbb{B}}, \\ (U_{\mathbb{T}} N_B R_B \gamma^{-1}) \circ (U_{\mathbb{T}} \lambda^T N_B R_B F_{\mathbb{T}}) \circ (T U_{\mathbb{B}} \eta^B R_B F_{\mathbb{T}}) \circ (T u^T) \circ (U_{\mathbb{T}} \gamma) &= U_{\mathbb{B}} R_B N_B \eta^B F_{\mathbb{B}}. \end{aligned}$$

Thus (3.13) is equivalent to

$$(3.14) \quad (B, U_{\mathbb{B}} \eta^B F_{\mathbb{B}}) = \text{Equ}_{\text{Fun}} (U_{\mathbb{B}} \eta^B R_B N_B F_{\mathbb{B}}, U_{\mathbb{B}} R_B N_B \eta^B F_{\mathbb{B}}).$$

By the assumption that  $\eta^B$  is an equalizer, Lemma 2.4 and Lemma 2.2, we conclude that  $\eta^B F_{\mathbb{B}}$  is the equalizer of the parallel morphisms  $\eta^B R_B N_B F_{\mathbb{B}}$  and  $R_B N_B \eta^B F_{\mathbb{B}}$ . Since  $U_{\mathbb{B}}$  is a right adjoint functor, it preserves equalizers. This proves (3.14) hence (3.12).

Bijectivity of the constructed maps between the data in part (a) and part (b) is checked by a straightforward computation, making use of (2.7) and (2.8).  $\square$

#### 4. EQUIVALENCE BETWEEN REGULAR COMONAD ARROWS AND PRE-TORSORS

In this section we study the category  $\text{RArr}(\mathcal{A}, \mathcal{B})$  introduced in Section 3 (possessing the subcategory  $\text{Gal}(\mathcal{A}, \mathcal{B})$ ). Our main aim is to find a full subcategory  $\text{RArr}_{\text{reg}}(\mathcal{A}, \mathcal{B})$  that is equivalent to a full subcategory of the category of pre-torsors, introduced below. The subcategory  $\text{RArr}_{\text{reg}}(\mathcal{A}, \mathcal{B})$  has some intersection with the subcategory  $\text{Gal}(\mathcal{A}, \mathcal{B})$  of  $\text{RArr}(\mathcal{A}, \mathcal{B})$ . Hence the intersection (let's denote it by  $\text{Gal}_{\text{reg}}(\mathcal{A}, \mathcal{B})$ ), is equivalent to a subcategory of the category of pre-torsors. Since any torsor corresponding to a faithfully flat Hopf Galois object belongs to  $\text{Gal}_{\text{reg}}(\mathcal{A}, \mathcal{B})$ , the results of this section not only generalize the relation between faithfully flat Hopf Galois objects and torsors over commutative rings, but also yield a deeper explanation of it. It is an open question how to interpret those objects of  $\text{RArr}_{\text{reg}}(\mathcal{A}, \mathcal{B})$  that do not belong to  $\text{Gal}(\mathcal{A}, \mathcal{B})$ .

**4.1. Pre-torsors.** In this section we introduce a further key notion of these notes, *pre-torsors* over two adjunctions. They provide examples of herd functors in [BV, Appendix].

**DEFINITION 4.1.** *Objects* in the category  $\text{PreTor}(\mathcal{A}, \mathcal{B})$ , called *pre-torsors*, consist of an object  $(\mathcal{T}, (N_A, R_A), (N_B, R_B))$  of  $\text{Adj}(\mathcal{A}, \mathcal{B})$  together with a natural morphism  $\tau : R_A N_B \rightarrow R_A N_B R_B N_A R_A N_B$ , subject to the following conditions.

- $(R_A N_B R_B \epsilon^A N_B) \circ \tau = R_A N_B \eta^B$ ,
- $(R_A \epsilon^B N_A R_A N_B) \circ \tau = \eta^A R_A N_B$ ,
- $(R_A N_B R_B N_A \tau) \circ \tau = (\tau R_B N_A R_A N_B) \circ \tau$ ,

where  $\eta^A$  and  $\epsilon^A$  denote the unit and counit of the adjunction  $(N_A, R_A)$  and analogous notations  $\eta^B$  and  $\epsilon^B$  are used for  $(N_B, R_B)$ .

A *morphism* of pre-torsors

$$(\mathcal{T}, (N_A, R_A), (N_B, R_B), \tau) \rightarrow (\mathcal{T}', (N'_A, R'_A), (N'_B, R'_B), \tau')$$

is a morphism  $F : (\mathcal{T}, (N_A, R_A), (N_B, R_B)) \rightarrow (\mathcal{T}', (N'_A, R'_A), (N'_B, R'_B))$  in  $\text{Adj}(\mathcal{A}, \mathcal{B})$ , such that

$$(4.1) \quad (R_A b R_B a R_A b) \circ \tau = \tau' \circ (R_A b),$$

where the natural morphisms  $a$  and  $b$  were defined in (3.5).

Note that  $R_A b$  is a natural morphism  $R_A N_B \rightarrow R'_A N'_B$  and  $R_B a$  is a natural morphism  $R_B N_A \rightarrow R'_B N'_A$ . We leave it to the reader to check that the composite of two morphisms of pre-torsors is a morphism of pre-torsors again.

**EXAMPLE 4.2.** For a commutative ring  $k$ , consider  $k$ -algebra homomorphisms  $\alpha : \mathbf{A} \rightarrow \mathbf{T}$  and  $\beta : \mathbf{B} \rightarrow \mathbf{T}$ . Denote by  $(N_A, R_A)$  (resp.  $(N_B, R_B)$ ) the corresponding ‘extension of scalars’ and ‘restriction of scalars’ functors between the module categories of these algebras. A pre-torsor  $(\alpha, \beta, \tau)$ , over  $k$ -algebras  $\mathbf{A}$  and  $\mathbf{B}$  as in [BB, Definition 3.1], induces a pre-torsor  $(\text{Mod-}\mathbf{T}, (N_A, R_A), (N_B, R_B), (-) \otimes_{\mathbf{B}} \tau)$  in the sense of Definition 4.1.

**EXAMPLE 4.3.** More generally, for two associative and unital algebras  $\mathbf{A}$  and  $\mathbf{B}$ , consider an  $\mathbf{A}$ -ring  $\mathbf{T}$  and a  $\mathbf{B}$ - $\mathbf{T}$  bimodule  $\Sigma$ . As we have seen in Section 3.2, these data determine two pairs of adjoint functors:

$$\begin{aligned} N_A &= (-) \otimes_{\mathbf{A}} \mathbf{T} : \text{Mod-}\mathbf{A} \rightarrow \text{Mod-}\mathbf{T} & R_A &= \text{Hom}_{\mathbf{T}}(\mathbf{T}, -) : \text{Mod-}\mathbf{T} \rightarrow \text{Mod-}\mathbf{A} \\ N_B &= (-) \otimes_{\mathbf{B}} \Sigma : \text{Mod-}\mathbf{B} \rightarrow \text{Mod-}\mathbf{T} & R_B &= \text{Hom}_{\mathbf{T}}(\Sigma, -) : \text{Mod-}\mathbf{T} \rightarrow \text{Mod-}\mathbf{B}. \end{aligned}$$

Then  $R_A N_B = (-) \otimes_{\mathbf{B}} \Sigma$ . If  $\Sigma$  is finitely generated and projective as a right  $\mathbf{T}$ -module, then  $R_B N_A = (-) \otimes_{\mathbf{A}} \Sigma^*$ , where the notation  $\Sigma^* := \text{Hom}_{\mathbf{T}}(\Sigma, \mathbf{T})$  is used. Hence a pre-torsor is induced by any  $\mathbf{B}$ - $\mathbf{A}$ -bimodule map

$$\tau : \Sigma \rightarrow \Sigma \otimes_{\mathbf{A}} \Sigma^* \otimes_{\mathbf{B}} \Sigma, \quad x \mapsto x^{(1)} \otimes_{\mathbf{A}} x_{(2)} \otimes_{\mathbf{B}} x^{(3)},$$

(with implicit summation understood), which is subject to the following conditions, for all  $x \in \Sigma$ .

$$\begin{aligned} x^{(1)} x_{(2)} (-) \otimes_{\mathbf{B}} x^{(3)} &= \text{Id}_{\Sigma} \otimes_{\mathbf{B}} x && \in \text{End}_{\mathbf{T}}(\Sigma) \otimes_{\mathbf{B}} \Sigma, \\ x^{(1)} \otimes_{\mathbf{A}} x_{(2)} (x^{(3)}) &= x \otimes_{\mathbf{A}} 1_{\mathbf{T}} && \in \Sigma \otimes_{\mathbf{A}} \mathbf{T}, \\ x^{(1)} \otimes_{x_{(2)}} \otimes x^{(3)(1)} \otimes x^{(3)}_{(2)} \otimes x^{(3)(3)} &= x^{(1)(1)} \otimes x^{(1)}_{(2)} \otimes x^{(1)(3)} \otimes x_{(2)} \otimes x^{(3)} \\ &&& \in \Sigma \otimes_{\mathbf{A}} \Sigma^* \otimes_{\mathbf{B}} \Sigma \otimes_{\mathbf{A}} \Sigma^* \otimes_{\mathbf{B}} \Sigma. \end{aligned}$$

This structure is an example of a *bimodule* herd, introduced in [BV, Definition 2.4], over the ring maps  $\mathbf{A} \rightarrow \mathbf{T}$  and  $\mathbf{B} \rightarrow \text{End}_{\mathbf{T}}(\Sigma)$ .

**4.2. From regular comonad arrows to pre-torsors.** Our next aim is to find an equivalence between (certain) objects in  $\text{RArr}(\mathcal{A}, \mathcal{B})$  on one hand, and (certain) pre-torsors on the other hand. In the same way as it happens with the relation between Galois extensions by corings and pre-torsors over rings, we will see in this section that any object in  $\text{RArr}(\mathcal{A}, \mathcal{B})$  (or  $\overline{\text{RArr}}(\mathcal{B}, \mathcal{A})$ ) determines a pre-torsor. In the next section we ask about a converse construction and look for conditions under which a pre-torsor determines an object in  $\text{RArr}(\mathcal{A}, \mathcal{B})$  (or  $\overline{\text{RArr}}(\mathcal{B}, \mathcal{A})$ ). Our main result is a category equivalence in Section 4.4.

**THEOREM 4.4.** (1) Consider an object  $(\mathcal{T}, (N_A, R_A), (N_B, R_B), \mathbb{C}, \xi)$  in  $\text{RArr}(\mathcal{A}, \mathcal{B})$ . Then  $(\mathcal{T}, (N_A, R_A), (N_B, R_B), \tau)$  is a pre-torsor, with pre-torsor map

$$\tau = (\xi N_A R_A N_B) \circ (C \eta^A R_A N_B) \circ (\xi^{-1} N_B) \circ (R_A N_B \eta^B).$$

(2) For a morphism

$$(F, t) : (\mathcal{T}, (N_A, R_A), (N_B, R_B), \mathbb{C}, \xi) \rightarrow (\mathcal{T}', (N'_A, R'_A), (N'_B, R'_B), \mathbb{C}', \xi')$$

in  $\text{RArr}(\mathcal{A}, \mathcal{B})$ ,  $F$  is a morphism of the corresponding pre-torsors in part (1).

*Proof.* Part (1) is verified by a direct computation, using the definition of an object in  $\text{RArr}(\mathcal{A}, \mathcal{B})$ , naturality and the adjunction relations. Similarly, to check claim (2), one has to use the definition of a morphism in  $\text{RArr}(\mathcal{A}, \mathcal{B})$ , together with naturality and the adjunction relations.  $\square$

Symmetrically to Theorem 4.4, we have

**THEOREM 4.5.** (1) Consider an object  $(\mathcal{T}, (N_B, R_B), (N_A, R_A), \mathbb{D}, \zeta)$  in  $\overline{\text{RArr}}(\mathcal{B}, \mathcal{A})$ . Then  $(\mathcal{T}, (N_A, R_A), (N_B, R_B), \tau)$  is a pre-torsor, with pre-torsor map

$$\tau = (R_A N_B R_B \bar{\zeta}) \circ (R_A N_B \eta^B D) \circ (R_A \bar{\zeta}^{-1}) \circ (\eta^A R_A N_B),$$

where  $\bar{\zeta} := (\epsilon^B N_A R_A N_B) \circ (N_B \zeta N_B) \circ (N_B D \eta^B)$ .

(2) For a morphism

$$(F, t) : (\mathcal{T}, (N_B, R_B), (N_A, R_A), \mathbb{D}, \zeta) \rightarrow (\mathcal{T}', (N'_B, R'_B), (N'_A, R'_A), \mathbb{D}', \zeta')$$

in  $\overline{\text{RArr}}(\mathcal{B}, \mathcal{A})$ ,  $F$  is a morphism of the corresponding pre-torsors in part (1).

In Theorem 4.4 we constructed, in fact, a functor from  $\text{RArr}(\mathcal{A}, \mathcal{B})$  to  $\text{PreTor}(\mathcal{A}, \mathcal{B})$  and in Theorem 4.5 we constructed a functor from  $\overline{\text{RArr}}(\mathcal{B}, \mathcal{A})$  to  $\text{PreTor}(\mathcal{A}, \mathcal{B})$ . In later sections of these notes we want to see on which objects are these functors equivalences. The problem will be divided to two steps. In Section 4.3, we investigate on what subcategory of the category of pre-torsors we can define functors to  $\text{RArr}(\mathcal{A}, \mathcal{B})$  and  $\overline{\text{RArr}}(\mathcal{B}, \mathcal{A})$ . After that, in Section 4.4, we prove that the functors in Sections 4.2 and 4.3 give rise to inverse equivalences between appropriate subcategories, indeed.

**4.3. From pre-torsors to regular comonad arrows.** The aim of this section is to find criteria on a pre-torsor, under which it determines an object in  $\text{RArr}(\mathcal{A}, \mathcal{B})$  (or  $\overline{\text{RArr}}(\mathcal{B}, \mathcal{A})$ ). The main result of the section is the following.

**THEOREM 4.6.** Consider two categories  $\mathcal{A}$  and  $\mathcal{B}$  both of which possess equalizers, and an object  $(\mathcal{T}, (N_A, R_A), (N_B, R_B), \tau)$  in  $\text{PreTor}(\mathcal{A}, \mathcal{B})$ , such that

- the unit  $\eta^A$  of the adjunction  $(N_A, R_A)$  and the unit  $\eta^B$  of the adjunction  $(N_B, R_B)$  are regular natural monomorphisms;
- the functors  $N_A$  and  $N_B$  preserve equalizers.

Under these assumptions, the following assertions hold.

(1) The equalizer  $(C, i) = \text{Equ}_{\text{Fun}}(\omega^l, \omega^r)$  of the natural morphisms

$$\omega^l := (R_A N_B R_B N_A R_A \epsilon^B N_A) \circ (\tau R_B N_A) \quad \text{and} \quad \omega^r := R_A N_B R_B N_A \eta^A$$

defines a comonad  $\mathbb{C} = (C, \Delta^C, \varepsilon^C)$  on  $\mathcal{A}$  such that the functor  $C$  preserves equalizers.

(2) There is an object  $(\mathcal{T}, (N_A, R_A), (N_B, R_B), \mathbb{C}, \xi) \in \text{RArr}(\mathcal{A}, \mathcal{B})$ , where the comonad  $\mathbb{C}$  was constructed in part (1).

(3) For a morphism  $F$ , between pre-torsors  $(\mathcal{T}, (N_A, R_A), (N_B, R_B), \tau)$  and  $(\mathcal{T}', (N'_A, R'_A), (N'_B, R'_B), \tau')$  both of which satisfy the conditions in the theorem, there exists a unique morphism  $t : \mathbb{C} \rightarrow \mathbb{C}'$  between the comonads in part (1) such that  $(F, t)$  is a morphism between the objects of  $\text{RArr}(\mathcal{A}, \mathcal{B})$  in part (2).

Before turning to prove Theorem 4.6, we give a motivating example of a situation in which the assumptions of the theorem hold.

EXAMPLE 4.7. Consider an object

$$(\text{Mod-}\mathbf{T}, (- \otimes_A \mathbf{T}, \text{Hom}_{\mathbf{T}}(\mathbf{T}, -)), (- \otimes_B \mathbf{T}, \text{Hom}_{\mathbf{T}}(\mathbf{T}, -)), - \otimes_B \tau)$$

in  $\text{PreTor}(\text{Mod-}\mathbf{A}, \text{Mod-}\mathbf{B})$  as in Example 4.2, induced by a pre-torsor  $(\alpha, \beta, \tau)$  in [BB, Definition 3.1]. If  $\mathbf{T}$  is a faithfully flat left  $\mathbf{A}$ -module (via  $\alpha$ ) and a faithfully flat left  $\mathbf{B}$ -module (via  $\beta$ ) then all assumptions in Theorem 4.6 hold.

In the more general situation in Example 4.3, the assumptions in Theorem 4.6 hold provided that  $\mathbf{T}$  is a faithfully flat left  $\mathbf{A}$ -module and  $\Sigma$  is a faithfully flat left  $\mathbf{B}$ -module.

Part (1) of Theorem 4.6 holds in a more general situation in the following lemma.

LEMMA 4.8. Consider two categories  $\mathcal{A}$  and  $\mathcal{B}$  possessing equalizers, and equalizer preserving functors as in the (non-commutative) diagram

$$\begin{array}{ccc} & P & \\ \curvearrowright & \text{---} & \curvearrowright \\ R & \text{---} & S \\ \curvearrowleft & \text{---} & \curvearrowleft \\ & Q & \end{array} \quad \mathcal{A} \quad \mathcal{B} \quad .$$

Let

$$r : \mathcal{A} \rightarrow R, \quad s : \mathcal{B} \rightarrow S, \quad w : QP \rightarrow R, \quad z : PQ \rightarrow S, \quad \tau : Q \rightarrow QPQ$$

be natural morphisms, subject to the following conditions.

- (i)  $r = \text{Equ}_{\text{Fun}}(rR, Rr)$  and  $s = \text{Equ}_{\text{Fun}}(sS, Ss)$ ;
- (ii)  $(QP\tau) \circ \tau = (\tau PQ) \circ \tau$ ;
- (iii)  $(Qz) \circ \tau = Qs$  and  $(wQ) \circ \tau = rQ$ .

Then the following assertions hold.

(1) There is a comonad  $\mathbb{C} = (C, \Delta^C, \varepsilon^C)$  on  $\mathcal{A}$ , such that  $C$  preserves equalizers and  $Q$  is a left  $\mathbb{C}$ -comodule functor;

(2) There is a comonad  $\mathbb{D} = (D, \Delta^D, \varepsilon^D)$  on  $\mathcal{B}$ , such that  $D$  preserves equalizers and  $Q$  is a right  $\mathbb{D}$ -comodule functor;

(3)  $Q$  is a  $\mathbb{C}$ - $\mathbb{D}$  bicomodule functor.

*Proof.* (1) Consider the equalizer of natural morphisms

$$(4.2) \quad C \xrightarrow{i} QP \xrightleftharpoons[\omega^r]{\omega^l} QPR,$$

where  $\omega^l = (QPw) \circ (\tau P)$  and  $\omega^r = QPr$ . (This equalizer exists by Lemma 2.1.) By Corollary 2.6,  $C$  preserves equalizers. By assumptions (ii) and (iii),  $(\omega^l Q) \circ \tau =$

$(\omega^r Q) \circ \tau$ . Hence by Lemma 2.2, there is a unique natural morphism  $c : Q \rightarrow CQ$  such that

$$(4.3) \quad (iQ) \circ c = \tau.$$

Moreover, use (4.3), assumption (ii), definition of the morphism  $i$  and naturality to derive

$$(iQPR) \circ (C\omega^l) \circ (cP) \circ i = (iQPR) \circ (C\omega^r) \circ (cP) \circ i.$$

Since  $iQPR$  is a monomorphism by Lemma 2.2, and  $Ci$  is the equalizer of  $C\omega^l$  and  $C\omega^r$ , this implies the existence of a unique natural morphism  $\Delta^C : C \rightarrow CC$ , such that  $(Ci) \circ \Delta^C = (cP) \circ i$ . Equivalently,

$$(4.4) \quad (ii) \circ \Delta^C = (\tau P) \circ i.$$

Finally, by the definition of the morphism  $i$ , assumption (iii) and naturality,

$$(Rr) \circ w \circ i = (rR) \circ w \circ i.$$

By assumption (i) this implies the existence of a unique natural morphism  $\varepsilon^C : C \rightarrow \mathcal{A}$  such that

$$(4.5) \quad r \circ \varepsilon^C = w \circ i.$$

By (4.4) and assumption (ii),  $\Delta^C$  is coassociative. By (4.4), (4.5) and assumption (iii) on one hand, and definition of  $i$  via (4.2) on the other hand,  $\varepsilon^C$  is counit of  $\Delta^C$ . By (4.3), (4.4) and assumption (ii),  $c$  is a coassociative coaction. By (4.3), (4.5) and assumption (iii), it is counital.

Part (2) is proven symmetrically. That is, we define an (equalizer preserving) functor  $D$  as the equalizer

$$(4.6) \quad D \xrightarrow{j} PQ \xrightleftharpoons[\theta^r]{\theta^l} SPQ,$$

where  $\theta^l = (zPQ) \circ (P\tau)$  and  $\theta^r = sPQ$ . The to-be-coaction  $d : Q \rightarrow QD$  is the unique morphism such that

$$(4.7) \quad (Qj) \circ d = \tau.$$

The coproduct and counit of the comonad  $\mathbb{D}$  are defined via the respective conditions

$$(4.8) \quad (jj) \circ \Delta^D = (P\tau) \circ j \quad \text{and} \quad s \circ \varepsilon^D = z \circ j.$$

(3) The identity  $(Cd) \circ c = (cD) \circ d$  follows by (4.3), (4.7) and assumption (ii).  $\square$

LEMMA 4.9. *Under the assumptions and using the notations of Theorem 4.6,*

$$\xi := (R_A N_B R_B \epsilon^A) \circ (iR_A) : CR_A \rightarrow R_A N_B R_B$$

*is a natural isomorphism.*

*Proof.* It follows easily by the pre-torsor axioms that

$$(4.9) \quad (\omega^l R_A) \circ (R_A N_B R_B N_A R_A \epsilon^B) \circ (\tau R_B) = (\omega^r R_A) \circ (R_A N_B R_B N_A R_A \epsilon^B) \circ (\tau R_B).$$

Since the category  $\mathcal{A}$  has equalizers by assumption, it follows by Lemma 2.2 that  $iR_A$  is the equalizer of  $\omega^l R_A$  and  $\omega^r R_A$ . So by its universality there exists a unique natural morphism  $\xi' : R_A N_B R_B \rightarrow CR_A$ , such that

$$(4.10) \quad (iR_A) \circ \xi' = (R_A N_B R_B N_A R_A \epsilon^B) \circ (\tau R_B).$$

A direct computation verifies that  $\xi'$  is the two-sided inverse of  $\xi$ .  $\square$

We are ready to prove Theorem 4.6, which is the main result of the section.

*Proof of Theorem 4.6.* (1) The functors  $R_A$  and  $R_B$  are right adjoints, hence they preserve equalizers. The left adjoint functors  $N_A$  and  $N_B$  preserve equalizers by assumption. Hence part (1) follows by substituting in Lemma 4.8  $Q = R_A N_B$ ,  $P = R_B N_A$ ,  $R = R_A N_A$ ,  $S = R_B N_B$ ,  $r = \eta^A$ ,  $s = \eta^B$ ,  $w = R_A \epsilon^B N_A$  and  $z = R_B \epsilon^A N_B$ .

(2) We claim that  $(\mathcal{T}, (N_A, R_A), (N_B, R_B), \mathbb{C}, \xi)$  is an object in  $\text{RArr}(\mathcal{A}, \mathcal{B})$ , where  $\mathbb{C}$  is the comonad in part (1) and  $\xi$  is the isomorphism constructed in Lemma 4.9. For that, we need to show that  $\xi$  satisfies conditions (3.2). This is a consequence of its construction, naturality, (4.4) and (4.5), the pre-torsor axioms and the adjunction relations.

(3) Take a morphism

$$F : (\mathcal{T}, (N_A, R_A), (N_B, R_B), \tau) \rightarrow (\mathcal{T}', (N'_A, R'_A), (N'_B, R'_B), \tau')$$

of pre-torsors, with corresponding natural morphisms  $a$  and  $b$  in (3.5). For the morphisms  $\omega^l$  and  $\omega^r$ , defined for the pre-torsor  $(\mathcal{T}, (N_A, R_A), (N_B, R_B), \tau)$  in part (1), and the analogous morphisms  $\omega'^l$  and  $\omega'^r$  for  $(\mathcal{T}', (N'_A, R'_A), (N'_B, R'_B), \tau')$ , we claim that

$$(4.11) \quad \begin{aligned} \omega'^l \circ (R_A b R_B a) &= (R_A b R_B a R_A a) \circ \omega^l & \text{and} \\ \omega'^r \circ (R_A b R_B a) &= (R_A b R_B a R_A a) \circ \omega^r. \end{aligned}$$

The first identity follows by the explicit form of the morphisms  $\omega^l$  and  $\omega'^l$ , combining with the fact that  $F$  is a morphism of pre-torsors. In light of the form of the morphisms  $\omega^r$  and  $\omega'^r$ , the second identity is a consequence of naturality and the compatibility of  $a$  with the units of the adjunctions  $(N_A, R_A)$  and  $(N'_A, R'_A)$ , cf. (3.5). Composing both identities in (4.11) by the equalizer  $(C, i) := \text{Equ}_{\text{Fun}}(\omega^l, \omega^r)$ , we obtain

$$\omega'^r \circ (R_A b R_B a) \circ i = \omega^l \circ (R_A b R_B a) \circ i.$$

Thus by universality of the equalizer  $(C', i') := \text{Equ}_{\text{Fun}}(\omega'^l, \omega'^r)$ , there exists a natural morphism  $t : C \rightarrow C'$  such that

$$(4.12) \quad i' \circ t = (R_A b R_B a) \circ i.$$

We prove next that  $t$  is a comonad morphism. Using definitions (4.5) of the counit  $\varepsilon^C$  and (4.12) of the morphism  $t$  together with (3.5), one checks that  $\eta'^A \circ \varepsilon^C = \eta'^A \circ \varepsilon^{C'} \circ t$ . So by the monomorphism property of  $\eta'^A$ ,  $\varepsilon^C = \varepsilon^{C'} \circ t$ . Using definitions (4.12) of  $t$  and (4.4) of the coproduct  $\Delta^C$  together with the fact that  $F$  is a morphism of pre-torsors, we deduce that

$$(i'i') \circ (tt) \circ \Delta^C = (i'i') \circ \Delta^{C'} \circ t.$$

Since  $i'i' = (R'_A N'_B R'_B N'_A i') \circ (i'C')$  is monic, we conclude that  $(tt) \circ \Delta^C = \Delta^{C'} \circ t$ . Condition (3.6) follows by constructions of  $\xi'$  and  $t$ , and (3.5).  $\square$

Symmetrically to Theorem 4.6, the following holds.

**THEOREM 4.10.** *Consider two categories  $\mathcal{A}$  and  $\mathcal{B}$  both of which possess equalizers, and an object  $(\mathcal{T}, (N_A, R_A), (N_B, R_B), \tau)$  in  $\text{PreTor}(\mathcal{A}, \mathcal{B})$ , that satisfies the assumptions in Theorem 4.6. Then the following assertions hold.*

(1) *The equalizer  $(D, j) = \text{Equ}_{\text{Fun}}(\theta^l, \theta^r)$  of the natural morphisms*

$$\theta^l := (R_B \epsilon^A N_B R_B N_A R_A N_B) \circ (R_B N_A \tau) \quad \text{and} \quad \theta^r := \eta^B R_B N_A R_A N_B$$

*defines a comonad  $\mathbb{D} = (D, \Delta^D, \varepsilon^D)$  on  $\mathcal{B}$  such that the functor  $D$  preserves equalizers.*

(2) *There is an object  $(\mathcal{T}, (N_B, R_B), (N_A, R_A), \mathbb{D}, \zeta) \in \overline{\text{RArr}}(\mathcal{B}, \mathcal{A})$  where the comonad  $\mathbb{D}$  was constructed in part (1).*

(3) *For a morphism  $F$  between pre-torsors  $(\mathcal{T}, (N_A, R_A), (N_B, R_B), \tau)$  and  $(\mathcal{T}', (N'_A, R'_A), (N'_B, R'_B), \tau')$ , both of which satisfy the conditions in Theorem 4.6, there exists a unique morphism  $t : \mathbb{D} \rightarrow \mathbb{D}'$  between the comonads in part (1) such that  $(F, t)$  is a morphism between the objects of  $\overline{\text{RArr}}(\mathcal{B}, \mathcal{A})$  in part (2).*

**4.4. The category equivalence.** In Section 4.2 we constructed functors from the categories  $\text{RArr}(\mathcal{A}, \mathcal{B})$  and  $\overline{\text{RArr}}(\mathcal{B}, \mathcal{A})$  to  $\text{PreTor}(\mathcal{A}, \mathcal{B})$ . In Section 4.3 we constructed functors in the opposite direction, from an appropriate subcategory of pre-torsors to the categories  $\text{RArr}(\mathcal{A}, \mathcal{B})$  and  $\overline{\text{RArr}}(\mathcal{B}, \mathcal{A})$ . The aim of this section is to find subcategories in both categories, in such a way that the functors in Section 4.2 and Section 4.3 establish equivalences between them.

Motivated by the premises of Theorem 4.6, we impose the following definition.

**DEFINITION 4.11.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be categories with equalizers and let  $\mathcal{T}$  be any category. An adjunction  $(N : \mathcal{A} \rightarrow \mathcal{T}, R : \mathcal{T} \rightarrow \mathcal{A})$  is called *regular* whenever the following conditions hold.

- The unit  $\eta$  of the adjunction is a regular natural monomorphism;
- the functor  $N$  preserves equalizers.

An object  $(\mathcal{T}, (N_A, R_A), (N_B, R_B), \tau)$  of  $\text{PreTor}(\mathcal{A}, \mathcal{B})$  is said to be *regular* if both adjunctions  $(N_A, R_A)$  and  $(N_B, R_B)$  are regular. We denote the full subcategory of regular pre-torsors by  $\text{PreTor}_{\text{reg}}(\mathcal{A}, \mathcal{B})$ .

An object  $(\mathcal{T}, (N_A, R_A), (N_B, R_B), \mathbb{C}, \xi)$  of  $\text{Arr}(\mathcal{A}, \mathcal{B})$  is said to be *(co)-regular* if  $(R_A, \xi)$  is a (co)-regular comonad arrow, both adjunctions  $(N_A, R_A)$  and  $(N_B, R_B)$  are regular and the functor underlying  $\mathbb{C}$  preserves equalizers. We denote the full subcategories of regular and co-regular objects in  $\text{Arr}(\mathcal{A}, \mathcal{B})$  by  $\text{RArr}_{\text{reg}}(\mathcal{A}, \mathcal{B})$  and  $\overline{\text{RArr}}_{\text{reg}}(\mathcal{A}, \mathcal{B})$ , respectively.

By corestriction, Theorem 4.6 provides us with a functor  $\Gamma : \text{PreTor}_{\text{reg}}(\mathcal{A}, \mathcal{B}) \rightarrow \text{RArr}_{\text{reg}}(\mathcal{A}, \mathcal{B})$ . By restriction and corestriction, Theorem 4.4 yields a functor  $\Omega : \text{RArr}_{\text{reg}}(\mathcal{A}, \mathcal{B}) \rightarrow \text{PreTor}_{\text{reg}}(\mathcal{A}, \mathcal{B})$ . Symmetrically, by Theorem 4.10 and Theorem 4.5, we have functors between the categories  $\text{PreTor}_{\text{reg}}(\mathcal{A}, \mathcal{B})$  and  $\overline{\text{RArr}}_{\text{reg}}(\mathcal{B}, \mathcal{A})$ . Our main result states that both pairs of functors are inverse equivalences.

**THEOREM 4.12.** *For two categories  $\mathcal{A}$  and  $\mathcal{B}$  with equalizers, the following categories are equivalent.*

- (i)  $\text{PreTor}_{\text{reg}}(\mathcal{A}, \mathcal{B})$ ;
- (ii)  $\text{RArr}_{\text{reg}}(\mathcal{A}, \mathcal{B})$ ;

(iii)  $\overline{\text{RArr}}_{reg}(\mathcal{B}, \mathcal{A})$ .

*Proof. Equivalence of (i) and (ii).* Consider the functors  $\Gamma$  and  $\Omega$  in the paragraph preceding the theorem. First we construct a natural isomorphism  $\text{RArr}_{reg}(\mathcal{A}, \mathcal{B}) \rightarrow \Gamma\Omega$ . That is, we associate an isomorphism

$$(\mathcal{T}, N_A, N_B, w) : (\mathcal{T}, (N_A, R_A), (N_B, R_B), \mathbb{C}, \xi) \rightarrow \Gamma\Omega(\mathcal{T}, (N_A, R_A), (N_B, R_B), \mathbb{C}, \xi) =: (\mathcal{T}, (N_A, R_A), (N_B, R_B), \tilde{\mathbb{C}}, \tilde{\xi})$$

to any object  $(\mathcal{T}, (N_A, R_A), (N_B, R_B), \mathbb{C}, \xi)$  of  $\text{RArr}_{reg}(\mathcal{A}, \mathcal{B})$ . The functor  $\tilde{C}$ , underlying the comonad  $\tilde{\mathbb{C}}$ , is defined as the equalizer of the morphisms  $\omega^l$  and  $\omega^r$  in Theorem 4.6 (1), associated to the pre-torsor in Theorem 4.4. Equivalently,  $\tilde{C}$  is the equalizer of

$$\begin{aligned} (\xi^{-1}N_A R_A N_A) \circ \omega^l &= (C\eta^A R_A N_A) \circ (\xi^{-1}N_A) \quad \text{and} \\ (\xi^{-1}N_A R_A N_A) \circ \omega^r &= (C R_A N_A \eta^A) \circ (\xi^{-1}N_A). \end{aligned}$$

On the other hand, by regularity of  $(\mathcal{T}, (N_A, R_A), (N_B, R_B), \mathbb{C}, \xi)$ , the functor  $C$  underlying the comonad  $\mathbb{C}$  preserves the equalizer  $\eta^A$ . So we conclude that also

$$\begin{aligned} (C, (\xi N_A) \circ (C\eta^A)) &= \text{Equ}_{\text{Fun}}((C\eta^A R_A N_A) \circ (\xi^{-1}N_A), (C R_A N_A \eta^A) \circ (\xi^{-1}N_A)) \\ &= \text{Equ}_{\text{Fun}}(\omega^l, \omega^r). \end{aligned}$$

Thus by uniqueness of an equalizer up to isomorphism, there exists a natural isomorphism  $w : C \rightarrow \tilde{C}$ , such that the natural monomorphism  $\tilde{i} : \tilde{C} \rightarrow R_A N_B R_B N_A$  satisfies

$$(4.13) \quad \tilde{i} \circ w = (\xi N_A) \circ (C\eta^A).$$

The coproduct  $\Delta^{\tilde{C}}$  of the comonad  $\tilde{\mathbb{C}}$  is defined as the unique morphism satisfying (4.4), i.e. the equation

$$(C\eta^A C\eta^A) \circ (w^{-1}w^{-1}) \circ \Delta^{\tilde{C}} = (C\eta^A C\eta^A) \circ \Delta^C \circ w^{-1}.$$

Since  $C$  preserves equalizers by the regularity assumption,  $C\eta^A C\eta^A$  is a monomorphism. Thus we conclude that  $(ww) \circ \Delta^C = \Delta^{\tilde{C}} \circ w$ . Similarly, the counit  $\varepsilon^{\tilde{C}}$  of  $\tilde{\mathbb{C}}$  is the unique morphism satisfying (4.5), that is, the equality

$$\eta^A \circ \varepsilon^{\tilde{C}} = \eta^A \circ \varepsilon^C \circ w^{-1}.$$

Therefore, by monomorphism property of  $\eta^A$ , it follows that  $\varepsilon^C = \varepsilon^{\tilde{C}} \circ w$ , so  $w$  is an isomorphism of comonads, as required.

By (4.13), naturality and an adjunction relation,  $\xi = \tilde{\xi} \circ (wR_A)$ . That is,  $(\mathcal{T}, N_A, N_B, w)$  is a morphism in  $\text{RArr}_{reg}(\mathcal{A}, \mathcal{B})$ .

It remains to prove that the isomorphism  $(\mathcal{T}, N_A, N_B, w)$  is natural. That is, given a morphism  $(F, t) : (\mathcal{T}, (N_A, R_A), (N_B, R_B), \mathbb{C}, \xi) \rightarrow (\mathcal{T}', (N'_A, R'_A), (N'_B, R'_B), \mathbb{C}', \xi')$  in  $\text{RArr}_{reg}(\mathcal{A}, \mathcal{B})$  (with corresponding natural morphisms  $a$  and  $b$  in (3.5)), the commutativity condition

$$(4.14) \quad w' \circ t = \tilde{t} \circ w$$

holds, where  $\tilde{t} : \tilde{C} \rightarrow \tilde{C}'$  is defined by  $(F, \tilde{t}) := \Gamma\Omega(F, t)$ . In order to prove (4.14), compose both sides on the left by the canonical monomorphism  $\tilde{i}' : \tilde{C}' \rightarrow$

$R'_A N'_B R'_B N'_A$ . The resulting equivalent condition follows by the construction of  $\tilde{t}$  via the identity  $\tilde{t}' \circ \tilde{t} = (R_A b R_B a) \circ \tilde{t}$ , (4.13) and the fact that  $(F, t)$  is a morphism in  $\text{RArr}(\mathcal{A}, \mathcal{B})$ . This completes the proof of the claim that the functor  $\Omega$  is naturally isomorphic to the identity functor  $\text{RArr}_{reg}(\mathcal{A}, \mathcal{B})$ .

In the converse order, we claim that  $\Omega\Gamma$  is equal to the identity functor on  $\text{PreTor}_{reg}(\mathcal{A}, \mathcal{B})$ . For an object  $(\mathcal{T}, (N_A, R_A), (N_B, R_B), \tau)$  in  $\text{PreTor}_{reg}(\mathcal{A}, \mathcal{B})$ , denote

$$(\mathcal{T}, (N_A, R_A), (N_B, R_B), \hat{\tau}) := \Omega\Gamma(\mathcal{T}, (N_A, R_A), (N_B, R_B), \tau).$$

By Theorem 4.4 (1) and Theorem 4.6 (2),

$$\hat{\tau} = (\xi N_A R_A N_B) \circ (C\eta^A R_A N_B) \circ (\xi^{-1} N_B) \circ (R_A N_B \eta^B),$$

where  $\xi$  is the isomorphism constructed in Lemma 4.9. Using the explicit form of  $\xi$  and the relation defining its inverse, naturality and the adjunction relations,  $\hat{\tau}$  is checked to be equal to  $\tau$ . On the morphisms  $\Omega\Gamma$  obviously acts as the identity map.

Equivalence of (i) and (iii) is proven symmetrically.  $\square$

We can apply the equivalence functor obtained between  $\text{PerTor}_{reg}(\mathcal{A}, \mathcal{B})$  and  $\text{RArr}_{reg}(\mathcal{A}, \mathcal{B})$  to objects of the subcategory  $\text{Gal}_{reg}(\mathcal{A}, \mathcal{B})$  of  $\text{RArr}_{reg}(\mathcal{A}, \mathcal{B})$ , cf. Section 3.

**COROLLARY 4.13.** (1) Consider a mixed distributive law of a monad  $\mathbb{N}$  and a comonad  $\mathbb{C}$  on a category  $\mathcal{A}$  and denote the corresponding lifting of  $\mathbb{C}$  to a comonad on  $\mathcal{A}_{\mathbb{N}}$  by  $\tilde{\mathbb{C}}$ . Let  $L$  be a  $\tilde{\mathbb{C}}$ -Galois functor with right adjoint  $R$ . Assume that both adjunctions  $(F_{\mathbb{N}}, U_{\mathbb{N}})$  and  $(L, R)$  are regular and the functor underlying  $\mathbb{C}$  preserves equalizers. Then there is a regular pre-torsor

$$(4.15) \quad (\mathcal{A}_{\mathbb{N}}, (F_{\mathbb{N}}, U_{\mathbb{N}}), (L, R), \tau).$$

(2) Conversely, consider a monad  $\mathbb{N}$  on a category  $\mathcal{A}$  and an adjoint pair of functors  $(L : \mathcal{B} \rightarrow \mathcal{A}_{\mathbb{N}}, R : \mathcal{A}_{\mathbb{N}} \rightarrow \mathcal{B})$ . Then any regular pre-torsor of the form (4.15) determines a comonad  $\mathbb{C}$  on  $\mathcal{A}$  such that

- the functor underlying  $\mathbb{C}$  preserves equalizers,
- $\mathbb{C}$  lifts to a comonad  $\tilde{\mathbb{C}}$  on  $\mathcal{A}_{\mathbb{N}}$ ,
- $L$  carries the structure of a  $\tilde{\mathbb{C}}$ -Galois functor.

More specifically, we can consider the examples in Section 3.2. Consider an entwining structure  $(\mathbf{T}, \mathbf{C}, \psi)$  over an associative and unital algebra  $\mathbf{A}$ . Denote the induced  $\mathbf{T}$ -coring  $\mathbf{T} \otimes_{\mathbf{A}} \mathbf{C}$  by  $\tilde{\mathbf{C}}$ . Then any Galois right  $\tilde{\mathbf{C}}$ -comodule  $\Sigma$  determines a (not necessarily regular) object in  $\text{RArr}(\text{Mod-}\mathbf{A}, \text{Mod-}\mathbf{B})$ , hence by Theorem 4.4 also a pre-torsor

$$(4.16) \quad (\text{Mod-}\mathbf{T}, ((-) \otimes_{\mathbf{A}} \mathbf{T}, \text{Hom}_{\mathbf{T}}(\mathbf{T}, -)), ((-) \otimes_{\mathbf{B}} \Sigma, \text{Hom}_{\mathbf{T}}(\Sigma, -)), \tau),$$

which is regular provided that  $\mathbf{T}$  is a faithfully flat left  $\mathbf{A}$ -module and  $\Sigma$  is a faithfully flat left  $\mathbf{B} = \text{End}^{\tilde{\mathbf{C}}}(\Sigma)$ -module. Note, however, that not every (regular) pre-torsor over the given adjunctions in (4.16) arises from an entwining structure.

For two algebras  $\mathbf{A}$  and  $\mathbf{B}$ , consider an  $\mathbf{A}$ -ring  $\mathbf{T}$  and a  $\mathbf{B}$ - $\mathbf{T}$  bimodule  $\Sigma$ . Then any regular pre-torsor of the form (4.16) determines a comonad  $\mathbb{C}$  on  $\text{Mod-}\mathbf{A}$  such that

- the functor underlying  $\mathbb{C}$  preserves equalizers,
- $\mathbb{C}$  lifts to a comonad  $\tilde{\mathbb{C}}$  on  $\text{Mod-}\mathbf{T}$ ,
- $(-)\otimes_{\mathbf{B}}\Sigma$  possesses a  $\tilde{\mathbb{C}}$ -Galois functor structure.

However, the comonad  $\mathbb{C}$  is not known to be induced by a coring (i.e. the underlying functor is not known to be a left adjoint).

Let us explain here the most important difference between the approaches to the relation between Galois comodules and pre-torsors in the current paper on one hand, and in [BB] and [BV] on the other hand.

In [BB, Theorem 3.4] pre-torsors of the form (4.16) are considered in the particular case when the  $\mathbf{B-T}$  bimodule  $\Sigma$  is a free rank 1 right  $\mathbf{T}$ -module (hence  $\mathbf{T}$  is also a  $\mathbf{B}$ -ring). More generally, in [BV, Theorem 2.16] the case is discussed when  $\Sigma$  is a finitely generated and projective right  $\mathbf{T}$ -module. Following the methods in [BV, Theorem 2.16], in the case when  $\Sigma$  is in addition a faithfully flat right  $\mathbf{A}$ -module and  $\mathbf{T}$  is a faithfully flat left  $\mathbf{A}$ -module, a pre-torsor of the form (4.16) can be shown to determine an  $\mathbf{A}$ -coring  $\mathbf{C}$ . However, note that the assumptions made on  $\Sigma$  in this approach, do not imply the regularity conditions in Definition 4.11.

On the other hand, following Theorem 4.6 (1), one can assume that in (4.16)  $\Sigma$  is a faithfully flat left  $\mathbf{B}$ -module, and  $\mathbf{T}$  is a faithfully flat left  $\mathbf{A}$ -module. Under these assumptions we associated a comonad  $\mathbb{C}$  on  $\text{Mod-}\mathbf{A}$  to a pre-torsor (4.16). Thus in the cases when both groups of assumptions hold, both the coring  $\mathbf{C}$  and the comonad  $\mathbb{C}$  can be constructed. However, there is no reason to expect that the comonad  $\mathbb{C}$  is induced by the coring  $\mathbf{C}$ : The comonad  $\mathbb{C}$  is unique (up to a natural isomorphism) with the property that the underlying functor preserves equalizers – a property the comonad  $(-)\otimes_{\mathbf{A}}\mathbf{C}$  is not known to obey (unless  $\mathbf{C}$  is a flat left  $\mathbf{A}$ -module, e.g. because we work with equal commutative base rings  $\mathbf{A}$  and  $\mathbf{B}$  and their symmetrical modules, cf. [BB, Remark 4.7]).

This deviation between the constructions in Section 4 on one hand, and in the works [H], [BB] and [BV] on the other hand, shows that there is a conceptual ambiguity how to generalize faithfully flat Hopf bi-Galois objects to non-commutative base algebras. Following the (more conventional) approach in [H], [BB] and [BV], the coacting symmetry objects can be described by *two corings*. However, as it was observed in [BB], Morita Takeuchi equivalence of these corings can not be proven in general. Here we would like to point out an alternative strategy: One can allow for the coacting symmetry structures to be *two comonads*, whose underlying functors are not necessarily left adjoints but, as a gain, they preserve kernels. As it is proven in Section 5, in this setting Morita Takeuchi equivalence of the two comonads is easily proven.

## 5. EQUIVALENCE OF COMODULE CATEGORIES

In Section 4 we proved equivalences between three categories  $\text{RArr}_{reg}(\mathcal{A}, \mathcal{B})$ ,  $\overline{\text{RArr}}_{reg}(\mathcal{B}, \mathcal{A})$  and  $\text{PreTor}_{reg}(\mathcal{A}, \mathcal{B})$ , for two categories  $\mathcal{A}$  and  $\mathcal{B}$  possessing equalizers. In this way we associated in particular two comonads,  $\mathbb{C}$  on  $\mathcal{A}$  and  $\mathbb{D}$  on  $\mathcal{B}$ , to any object of  $\text{PreTor}_{reg}(\mathcal{A}, \mathcal{B})$ . In this section we prove that the comonads  $\mathbb{C}$  and  $\mathbb{D}$  have equivalent comodule categories, what generalizes the result in [Sch2] about Morita

Takeuchi equivalence of the two Hopf algebras associated to a bi-Galois object. The proof is presented at the level of generality in Lemma 4.8.

Note that, for any comonad  $\mathbb{D}$  on a category  $\mathcal{B}$ , the forgetful functor  $U^{\mathbb{D}}$  is a left  $\mathbb{D}$ -comodule functor via the coaction  $U^{\mathbb{D}}\gamma^{\mathbb{D}}$  (notation introduced in Definition 2.7 (6)). In Proposition 5.1, a ‘cotensor product’ of  $U^{\mathbb{D}}$  with a right  $\mathbb{D}$ -comodule functor occurs.

**PROPOSITION 5.1.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be categories possessing equalizers,  $\mathbb{C} = (C, \Delta^C, \varepsilon^C)$  be a comonad on  $\mathcal{A}$ ,  $\mathbb{D} = (D, \Delta^D, \varepsilon^D)$  be a comonad on  $\mathcal{B}$  and  $(Q, c, d)$  be a  $\mathbb{C}$ - $\mathbb{D}$  bicomodule functor. If the functor  $C$  preserves equalizers, then there is a functor  $I_Q : \mathcal{B}^{\mathbb{D}} \rightarrow \mathcal{A}^{\mathbb{C}}$  such that*

$$U^{\mathbb{C}}I_Q \longrightarrow QU^{\mathbb{D}} \xrightarrow[QU^{\mathbb{D}}\gamma^{\mathbb{D}}]{dU^{\mathbb{D}}} QDU^{\mathbb{D}}$$

is an equalizer.

*Proof.* Since  $\mathcal{A}$  is assumed to have equalizers, by Lemma 2.1 there exists the equalizer

$$I_0 \xrightarrow{e} QU^{\mathbb{D}} \xrightarrow[QU^{\mathbb{D}}\gamma^{\mathbb{D}}]{dU^{\mathbb{D}}} QDU^{\mathbb{D}}.$$

We construct  $I_Q : \mathcal{B}^{\mathbb{D}} \rightarrow \mathcal{A}^{\mathbb{C}}$  by equipping  $I_0$  with a left  $\mathbb{C}$ -coaction  $c_0 : I_0 \rightarrow CI_0$  and putting  $I_Q X := (I_0 X, c_0 X)$ , for any object  $X$  in  $\mathcal{B}^{\mathbb{D}}$ , and  $I_Q f := I_0 f$ , for any morphism  $f$ .

By naturality, the definition of  $e$  and the bicomodule property of  $Q$ , we conclude that

$$(CQU^{\mathbb{D}}\gamma^{\mathbb{D}}) \circ (cU^{\mathbb{D}}) \circ e = (CdU^{\mathbb{D}}) \circ (cU^{\mathbb{D}}) \circ e.$$

Since  $Ce$  is the equalizer of  $CdU^{\mathbb{D}}$  and  $CQU^{\mathbb{D}}\gamma^{\mathbb{D}}$ , its universality implies the existence of a natural morphism  $c_0 : I_0 \rightarrow CI_0$  such that

$$(5.1) \quad (Ce) \circ c_0 = (cU^{\mathbb{D}}) \circ e.$$

Its coassociativity and counitality are immediate by coassociativity and counitality of  $c$ .  $\square$

In light of Proposition 5.1, a functor  $Q$  in Lemma 4.8 induces a functor  $I_Q : \mathcal{B}^{\mathbb{D}} \rightarrow \mathcal{A}^{\mathbb{C}}$  for the comonads  $\mathbb{C}$  and  $\mathbb{D}$  constructed in parts (1) and (2) of Lemma 4.8, respectively. Our next task is to construct a  $\mathbb{D}$ - $\mathbb{C}$  bicomodule functor  $\overline{Q}$  such that the induced functor  $I_{\overline{Q}} : \mathcal{A}^{\mathbb{C}} \rightarrow \mathcal{B}^{\mathbb{D}}$  yields the inverse of  $I_Q$ .

**LEMMA 5.2.** *In the setting of Lemma 4.8 and using the notations in its proof, define functors  $\overline{Q}$  and  $\overline{Q}' : \mathcal{A} \rightarrow \mathcal{B}$  via the respective equalizers*

$$\overline{Q} \xrightarrow{q} PC \xrightarrow[(\theta^r P) \circ (P_i)]{(\theta^l P) \circ (P_i)} SPQP \quad \text{and} \quad \overline{Q}' \xrightarrow{q'} DP \xrightarrow[(P\omega^r) \circ (jP)]{(P\omega^l) \circ (jP)} PQPR.$$

The following statements hold.

(1) The functors  $\overline{Q}$  and  $\overline{Q}'$  are naturally isomorphic (hence they can be chosen equal).

(2) The functor  $\overline{Q}$  can be equipped with the structure of a  $\mathbb{D}$ - $\mathbb{C}$  bicomodule.

*Proof.* (1) By construction,  $(jP) \circ q'$  equalizes  $P\omega^l$  and  $P\omega^r$ . Thus by universality of the equalizer  $Pi$ , there exists a natural morphism  $\nu_0 : \overline{Q}' \rightarrow PC$  such that

$$(5.2) \quad (Pi) \circ \nu_0 = (jP) \circ q'.$$

Moreover, since  $j$  equalizes  $\theta^l$  and  $\theta^r$  by definition,  $(jP) \circ q'$  equalizes  $\theta^l P$  and  $\theta^r P$ . Therefore (5.2) implies that  $\nu_0$  equalizes  $(\theta^l P) \circ (Pi)$  and  $(\theta^r P) \circ (Pi)$ . Thus by universality of the equalizer  $q$ , there is a natural morphism  $\nu : \overline{Q}' \rightarrow \overline{Q}$  such that  $q \circ \nu = \nu_0$ . Equivalently,

$$(5.3) \quad (Pi) \circ q \circ \nu = (jP) \circ q'.$$

By a symmetrical reasoning, there is a morphism  $\nu' : \overline{Q} \rightarrow \overline{Q}'$ , such that

$$(5.4) \quad (jP) \circ q' \circ \nu' = (Pi) \circ q.$$

By (5.3) and (5.4),  $(jP) \circ q' \circ \nu' \circ \nu = (jP) \circ q'$ . Since  $jP$  and  $q'$  are monomorphisms, this implies  $\nu' \circ \nu = \overline{Q}'$ . A symmetrical reasoning justifies  $\nu \circ \nu' = \overline{Q}$ .

Since an equalizer is defined up to isomorphism, we may choose  $\overline{Q} = \overline{Q}'$ , resulting in the identity

$$(5.5) \quad (jP) \circ q' = (Pi) \circ q.$$

(2) By assumption (ii) in Lemma 4.8 and naturality,

$$(\theta^l P Q P) \circ (P \tau P) = (S P \tau P) \circ (\theta^l P) \quad \text{and} \quad (\theta^r P Q P) \circ (P \tau P) = (S P \tau P) \circ (\theta^r P).$$

Together with (5.5) and (4.4), this implies

$$(5.6) \quad (S P Q P i) \circ (\theta^l P C) \circ (P i C) \circ (P \Delta^C) \circ q = (S P Q P i) \circ (\theta^r P C) \circ (P i C) \circ (P \Delta^C) \circ q.$$

Since  $S P Q P i$  is a monomorphism, we conclude by universality of the equalizer  $qC$  that there exists a unique morphism  $\bar{c} : \overline{Q} \rightarrow \overline{Q}C$  such that

$$(5.7) \quad (qC) \circ \bar{c} = (P \Delta^C) \circ q.$$

Obviously,  $\bar{c}$  is a coassociative and counital coaction. Symmetrically, a  $\mathbb{D}$ -coaction  $\bar{d} : \overline{Q} \rightarrow D\overline{Q}$  is defined by the condition

$$(5.8) \quad (Dq') \circ \bar{d} = (\Delta^D P) \circ q'.$$

The  $\mathbb{C}$ -, and  $\mathbb{D}$ -coactions on  $\overline{Q}$  commute by (4.4) and the analogous formula for  $\Delta^D$  in (4.8), assumption (ii) in Lemma 4.8 and (5.5).  $\square$

**THEOREM 5.3.** *In the setting of Lemma 4.8, consider the  $\mathbb{C}$ - $\mathbb{D}$  bicomodule  $Q$  in Lemma 4.8 (3) and the  $\mathbb{D}$ - $\mathbb{C}$  bicomodule  $\overline{Q}$  in Lemma 5.2. Then the induced functors  $I_Q : \mathcal{B}^{\mathbb{D}} \rightarrow \mathcal{A}^{\mathbb{C}}$  and  $I_{\overline{Q}} : \mathcal{A}^{\mathbb{C}} \rightarrow \mathcal{B}^{\mathbb{D}}$  (cf. Proposition 5.1) are inverse equivalences.*

*Proof.* Recall that  $I_{\overline{0}} = U^{\mathbb{D}} I_{\overline{Q}}$  fits the equalizer

$$(5.9) \quad I_{\overline{0}} \xrightarrow{\bar{e}} \overline{Q}U^{\mathbb{C}} \xrightleftharpoons[\overline{Q}U^{\mathbb{C}, \gamma^{\mathbb{C}}}]^{\bar{c}U^{\mathbb{C}}} \overline{Q}CU^{\mathbb{C}},$$

where the  $\mathbb{C}$ -coaction  $\bar{c}$  on  $\overline{Q}$  was constructed in (5.7).

First we construct a natural isomorphism between  $I_{\overline{0}}I_Q$  and  $U^{\mathbb{D}}$  and then show that it lifts to a natural isomorphism between  $I_{\overline{Q}}I_Q$  and the identity functor  $\mathcal{B}^{\mathbb{D}}$ . By (5.9),

$$(\overline{c}I_0) \circ (\overline{e}I_Q) = (\overline{Q}c_0) \circ (\overline{e}I_Q),$$

where the  $\mathbb{C}$ -coaction  $c_0 = U^{\mathbb{C}}\gamma^{\mathbb{C}}I_Q$  on  $I_0$  was constructed in (5.1). Compose both sides of this equality on the left by  $(P\varepsilon^{\mathbb{C}}ie) \circ (qCI_0)$ , and use (5.7), (5.1) and (4.3) to conclude that

$$(5.10) \quad (PiQU^{\mathbb{D}}) \circ (qe) \circ (\overline{e}I_Q) = (P\tau U^{\mathbb{D}}) \circ (P\varepsilon^{\mathbb{C}}QU^{\mathbb{D}}) \circ (qe) \circ (\overline{e}I_Q).$$

By the counit formula in (4.8), (5.5), fork property of (4.2), naturality, (5.5) again, fork property of (4.6), (5.5) and (4.5), it follows that  $(sPr) \circ (\varepsilon^{\mathbb{D}}P) \circ q' = (sPr) \circ (P\varepsilon^{\mathbb{C}}) \circ q$ . Since  $sPr$  is a monomorphism, this implies

$$(\varepsilon^{\mathbb{D}}P) \circ q' = (P\varepsilon^{\mathbb{C}}) \circ q.$$

These identities, together with (5.5), (4.5) and the analogous formula for  $\varepsilon^{\mathbb{D}}$  in (4.8) imply that  $\alpha_0 := (P\varepsilon^{\mathbb{C}}QU^{\mathbb{D}}) \circ (qe) \circ (\overline{e}I_Q)$  equalizes  $\theta^l U^{\mathbb{D}}$  and  $\theta^r U^{\mathbb{D}}$ . Thus by universality of the equalizer  $jU^{\mathbb{D}}$  (cf. Lemma 2.2), there exists a natural morphism  $\alpha : I_{\overline{0}}I_Q \rightarrow DU^{\mathbb{D}}$  such that  $(jU^{\mathbb{D}}) \circ \alpha = \alpha_0$ . Note that by (5.10),  $(P\tau U^{\mathbb{D}}) \circ \alpha_0$  is a monomorphism. This implies that  $\alpha_0$  is a monomorphism and hence  $\alpha$  is a monomorphism.

By the coproduct formula in (4.8), (4.7) and the definition of  $e$ , it follows that

$$(jjU^{\mathbb{D}}) \circ (\Delta^{\mathbb{D}}U^{\mathbb{D}}) \circ \alpha = (jjU^{\mathbb{D}}) \circ (DU^{\mathbb{D}}\gamma^{\mathbb{D}}) \circ \alpha.$$

Since  $jjU^{\mathbb{D}}$  is a monomorphism and  $U^{\mathbb{D}}\gamma^{\mathbb{D}}$  is the equalizer of  $\Delta^{\mathbb{D}}U^{\mathbb{D}}$  and  $DU^{\mathbb{D}}\gamma^{\mathbb{D}}$ , there exists a natural morphism  $\tilde{\beta} : I_{\overline{0}}I_Q \rightarrow U^{\mathbb{D}}$ , such that

$$(5.11) \quad (U^{\mathbb{D}}\gamma^{\mathbb{D}}) \circ \tilde{\beta} = \alpha.$$

We prove that  $\tilde{\beta}$  is an isomorphism by constructing its inverse. Consider the morphism  $\beta_1 := (PcU^{\mathbb{D}}) \circ (jU^{\mathbb{D}}) \circ (U^{\mathbb{D}}\gamma^{\mathbb{D}}) : U^{\mathbb{D}} \rightarrow PCQU^{\mathbb{D}}$ . We claim that there exists a morphism  $\beta : U^{\mathbb{D}} \rightarrow I_{\overline{0}}I_Q$ , such that  $(qe) \circ (\overline{e}I_Q) \circ \beta = \beta_1$ . Equivalently,

$$(5.12) \quad \alpha \circ \beta = U^{\mathbb{D}}\gamma^{\mathbb{D}}.$$

This will be verified in three steps. First observe that by (4.7), (4.3), assumption (ii) in Lemma 4.8 and coassociativity of the  $\mathbb{D}$ -coaction  $\gamma^{\mathbb{D}} : U^{\mathbb{D}} \rightarrow DU^{\mathbb{D}}$ , it follows that

$$(PiQjU^{\mathbb{D}}) \circ (PCdU^{\mathbb{D}}) \circ \beta_1 = (PiQjU^{\mathbb{D}}) \circ (PCQU^{\mathbb{D}}\gamma^{\mathbb{D}}) \circ \beta_1.$$

Since  $PiQjU^{\mathbb{D}}$  is monic, it follows by universality of the equalizer  $PCe$  that there exists  $\beta_2 : U^{\mathbb{D}} \rightarrow PCI_0$  such that  $(PCe) \circ \beta_2 = \beta_1$ . Next one checks with similar steps that  $\beta_2$  equalizes  $(\theta^l PI_0) \circ (PiI_0)$  and  $(\theta^r PI_0) \circ (PiI_0)$ , hence by universality of the equalizer  $qI_0$  there exists  $\beta_3 : U^{\mathbb{D}} \rightarrow \overline{Q}I_0$ , such that  $(qI_0) \circ \beta_3 = \beta_2$ . Finally,  $\beta_3$  is checked to equalize  $\overline{Q}c_0$  and  $\overline{e}I_0$ , hence by universality of the equalizer  $\overline{e}I_Q$  there exists  $\beta : U^{\mathbb{D}} \rightarrow I_{\overline{0}}I_Q$  such that  $(\overline{e}I_Q) \circ \beta = \beta_3$ . This morphism  $\beta$  clearly satisfies (5.12).

Since both  $\alpha$  and  $U^{\mathbb{D}}\gamma^{\mathbb{D}}$  are monomorphisms, (5.11) and (5.12) imply that  $\beta$  is the inverse of  $\tilde{\beta}$ .

Note that composing both sides of it on the left by the monomorphism  $PiQU^{\mathbb{D}}$ , (5.12) can be written equivalently in the form

$$(5.13) \quad (PiQU^{\mathbb{D}}) \circ (qe) \circ (\bar{e}I_Q) \circ \beta = (P\tau U^{\mathbb{D}}) \circ (jU^{\mathbb{D}}) \circ (U^{\mathbb{D}}\gamma^D).$$

The natural isomorphism  $I_{\bar{Q}}I_Q \cong \mathcal{B}^{\mathbb{D}}$  is proven by showing that  $\beta$  is a  $\mathbb{D}$ -comodule morphism, i.e.

$$(5.14) \quad (\bar{d}_0 I_Q) \circ \beta = (D\beta) \circ (U^{\mathbb{D}}\gamma^D),$$

where the  $\mathbb{D}$ -coaction  $\bar{d}_0 : I_{\bar{Q}} \rightarrow DI_{\bar{Q}}$  is defined via the condition

$$(5.15) \quad (D\bar{e}) \circ \bar{d}_0 = (\bar{d}U^{\mathbb{C}}) \circ \bar{e}.$$

In order to prove (5.14), compose both sides of it on the left by the monomorphism  $(jPiQU^{\mathbb{D}}) \circ (Dqe) \circ (D\bar{e}I_Q)$ , and use (5.15), (5.5), (5.8), (4.8), (5.13), assumption (ii) in Lemma 4.8 and then again (4.8) and (5.13).

The natural isomorphism  $I_Q I_{\bar{Q}} \cong \mathcal{A}^{\mathbb{C}}$  is verified by similar steps: First an isomorphism  $I_0 I_{\bar{Q}} \cong U^{\mathbb{C}}$  is constructed, and it is proven to lift to an isomorphism  $I_Q I_{\bar{Q}} \cong \mathcal{A}^{\mathbb{C}}$ . Consider the natural monomorphism  $\nu_0 := (QP\varepsilon^{\mathbb{C}}U^{\mathbb{C}}) \circ (QqU^{\mathbb{C}}) \circ (Q\bar{e}) \circ (eI_{\bar{Q}}) : I_0 I_{\bar{Q}} \rightarrow QPU^{\mathbb{C}}$ . It is checked to equalize  $\omega^l U^{\mathbb{C}}$  and  $\omega^r U^{\mathbb{C}}$ , hence by universality of the equalizer  $iU^{\mathbb{C}}$  it determines a monomorphism  $\nu : I_0 I_{\bar{Q}} \rightarrow CU^{\mathbb{C}}$  such that  $(iU^{\mathbb{C}}) \circ \nu = \nu_0$ . Furthermore,  $\nu$  is checked to equalize  $\Delta^{\mathbb{C}}U^{\mathbb{C}}$  and  $CU^{\mathbb{C}}\gamma^{\mathbb{C}}$ . Since  $U^{\mathbb{C}}\gamma^{\mathbb{C}}$  is the equalizer of  $\Delta^{\mathbb{C}}U^{\mathbb{C}}$  and  $CU^{\mathbb{C}}\gamma^{\mathbb{C}}$ , there exists a natural morphism  $\tilde{\kappa} : I_0 I_{\bar{Q}} \rightarrow U^{\mathbb{C}}$  such that

$$(5.16) \quad (U^{\mathbb{C}}\gamma^{\mathbb{C}}) \circ \tilde{\kappa} = \nu.$$

The inverse of  $\tilde{\kappa}$  is constructed in three steps. First the morphism  $\kappa_1 := (iCU^{\mathbb{C}}) \circ (\Delta^{\mathbb{C}}U^{\mathbb{C}}) \circ (U^{\mathbb{C}}\gamma^{\mathbb{C}})$  is checked to equalize  $(Q\theta^l PU^{\mathbb{C}}) \circ (QP iU^{\mathbb{C}})$  and  $(Q\theta^r PU^{\mathbb{C}}) \circ (QP iU^{\mathbb{C}})$ , hence by universality of the equalizer  $QqU^{\mathbb{C}}$ , it determines a morphism  $\kappa_2 : U^{\mathbb{C}} \rightarrow Q\bar{Q}U^{\mathbb{C}}$ , such that  $(QqU^{\mathbb{C}}) \circ \kappa_2 = \kappa_1$ . Next  $\kappa_2$  is checked to equalize  $Q\bar{e}$  and  $Q\bar{Q}U^{\mathbb{C}}\gamma^{\mathbb{C}}$ . Hence by universality of the equalizer  $Q\bar{e}$ , it determines a morphism  $\kappa_3 : U^{\mathbb{C}} \rightarrow QI_{\bar{Q}}$ , such that  $(Q\bar{e}) \circ \kappa_3 = \kappa_2$ . Finally  $\kappa_3$  is shown to equalize  $dI_{\bar{Q}}$  and  $Q\bar{d}_0$  (where  $\bar{d}_0 : I_{\bar{Q}} \rightarrow DI_{\bar{Q}}$  is the  $\mathbb{D}$ -coaction). Hence by universality of the equalizer  $eI_{\bar{Q}}$ , it determines a morphism  $\kappa : U^{\mathbb{C}} \rightarrow I_0 I_{\bar{Q}}$  such that  $(eI_{\bar{Q}}) \circ \kappa = \kappa_3$ . Equivalently,

$$(5.17) \quad \nu \circ \kappa = U^{\mathbb{C}}\gamma^{\mathbb{C}}.$$

We conclude by (5.16) and (5.17) that  $\kappa$  and  $\tilde{\kappa}$  are mutual inverses. By very similar steps to those used in the case of  $\beta$ , also  $\kappa$  is checked to be a  $\mathbb{C}$ -comodule morphism, i.e. to satisfy  $(e_0 I_{\bar{Q}}) \circ \kappa = (C\kappa) \circ (U^{\mathbb{C}}\gamma^{\mathbb{C}})$ . This proves that  $\kappa$  lifts to the stated isomorphism  $I_Q I_{\bar{Q}} \cong \mathcal{A}^{\mathbb{C}}$ .  $\square$

The following corollary is immediate by Theorem 5.3.

**COROLLARY 5.4.** *Consider an object in  $\text{PreTor}_{\text{reg}}(\mathcal{A}, \mathcal{B})$  and its images in  $\text{RArr}_{\text{reg}}(\mathcal{A}, \mathcal{B})$  and  $\overline{\text{RArr}}_{\text{reg}}(\mathcal{B}, \mathcal{A})$ , respectively, under the equivalences in Theorem 4.12. Then the occurring comonads  $\mathbb{C}$  on  $\mathcal{A}$  and  $\mathbb{D}$  on  $\mathcal{B}$  have equivalent categories of comodules.*

The  $\mathbf{A-B}$  pre-torsors in [BB, Corollary 4.8] induce regular pre-torsors in the sense of Definition 4.11. Moreover, the associated comonads on  $\text{Mod-}\mathbf{A}$  and  $\text{Mod-}\mathbf{B}$  are induced by the corings in [BB, Theorem 3.4]. Therefore, Corollary 5.4 extends [BB, Corollary 4.8].

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