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Fine properties of Cheeger sets and the Prescribed Mean Curvature problem in weakly regular domains

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Abstract

The two main problems we study are the *Cheeger problem* and the *Prescribed Mean Curvature problem*. The former consists in finding the subsets E of a given ambient set Ω that realize the Cheeger constant, i.e. such that

$$\frac{P(E)}{|E|} = \inf \left\{ \frac{P(A)}{|A|} \right\} = h_1(\Omega)\,,$$

where the infimum is sought amongst all subsets of Ω with positive volume; the latter is the non-linear partial differential equation given by

$$\mathrm{div}\,(Tu) = \mathrm{div}\,\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) = H\,,$$

which consists in finding functions u whose graph has mean curvature H. At a first sight these two problems do not seem to be related but in the special case of a positive, constant prescribed mean curvature H on Ω, a necessary and sufficient condition to existence of solutions and uniqueness up to translations is that H equals the Cheeger constant of Ω and Ω is a minimal Cheeger set.

On one hand, we study a generalization of the Cheeger problem considering volumes with positive, non-vanishing L^∞ weights and perimeters weighted through a function $g(x, \nu_\Omega(x))$ depending both on the point $x \in \partial\Omega$ and the outer normal to Ω at x. Then, we prove that any connected minimizer admits a Poincaré trace inequality, as well as the standard Sobolev embeddings. On the other hand, in the case of the standard Cheeger problem in dimension 2 we show that, for simply connected sets Ω that satisfy a "no-bottleneck" condition, the maximal Cheeger set E equals the union of all balls contained in Ω whose radius is $r = h_1^{-1}(\Omega)$. Moreover, the inner Cheeger formula $|[\Omega]^r| = \pi r^2$ holds, where $[\Omega]^r$ denotes the set of points of Ω at distance greater or equal than r from $\partial\Omega$. This result generalizes a property so far proved only for convex sets and planar strips.

Concerning the Prescribed Mean Curvature problem, we show existence and uniqueness of solutions of the PMC equation only assuming that Ω is a *weakly regular* open set, i.e., when Ω satisfies a Poincaré trace inequality and its perimeter agrees with the $(n-1)$-dimensional Hausdorff measure of the topological boundary. Under these assumptions, we show that uniqueness up to vertical translations is equivalent to several other properties. Namely, that the domain is maximal, i.e. no solutions for the same prescribed datum H can exist in any set $\widetilde{\Omega}$ strictly containing Ω; that Ω is critical, i.e. among all its subsets, it is the only one for which the inequality

$|\int_A H| \leq P(A)$ becomes an equality; that there exists a solution which solves the capillarity problem in a tube of cross-section Ω with vertical contact angle, i.e. that it satisfies a tangential boundary condition in an integral sense or in a "weak trace" sense. Moreover, whenever the perimeter of Ω agrees with the inner Minkowski content of Ω, this tangential "weak trace" condition assumes the stronger form $Tu(x) \to \nu_\Omega(z)$ in a measure-theoretic sense, as $x \in \Omega$ approaches a point z in the "super-reduced boundary". Finally, when the prescribed datum H is positive and non-vanishing, we observe again the link between the Cheeger problem and the Prescribed Mean Curvature problem, as being critical corresponds to Ω being a minimal Cheeger set with Cheeger constant 1, for the Cheeger problem with the standard perimeter and volume weighted through H.

Sunto

I due principali problemi che studiamo sono il *problema di Cheeger* ed il *problema di curvatura media prescritta.* Il primo consiste nel trovare i sottoinsiemi E di un certo insieme ambiente Ω che realizzano la costante di Cheeger, ovvero tali che

$$\frac{P(E)}{|E|} = \inf \left\{ \frac{P(A)}{|A|} \right\} = h_1(\Omega)\,,$$

dove l'estremo inferiore è fra tutti i sottoinsiemi di Ω con volume positivo; il secondo problema è l'equazione alle derivate parziali non lineare data da

$$\mathrm{div}\,(Tu) = \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = H\,,$$

che consiste nel trovare delle funzioni u il cui grafico abbia curvatura media H. A prima vista questi due problemi sembrano indipendenti, ma nel caso speciale di una curvatura media prescritta H costante e positiva in Ω, una condizione necessaria e sufficiente all'esistenza di soluzioni e all'unicità a meno di traslazioni, è che H sia uguale della costante di Cheeger e che Ω sia un insieme di Cheeger minimale.

Da un lato, studiamo una generalizzazione del problema di Cheeger considerando dei volumi con pesi L^∞ e dei perimetri pesati tramite funzioni $g(x, \nu_\Omega(x))$ che dipendono sia dal punto $x \in \partial\Omega$ sia dalla normale esterna ad Ω nel punto x. Mostriamo che gli insiemi minimi connessi ammettono una disuguaglianza di traccia di Poincaré e le classiche immersioni di Sobolev. Dall'altro lato, nel caso del problema di Cheeger classico in 2 dimensioni, mostriamo che, per insiemi Ω semplicemente connessi che non presentano "colli di bottiglia", l'insieme di Cheeger massimale E è l'unione di tutte le palle contenute in Ω di raggio $r = h_1^{-1}(\Omega)$. Inoltre, vale la inner Cheeger formula $|[\Omega]^r = \pi r^2$, dove $[\Omega]^r$ indica l'insieme dei punti di Ω che sono a una distanza maggiore o uguale ad r da $\partial\Omega$. Questo risultato generalizza una proprietà finora dimostrata solo per insieme convessi e strisce.

Riguardo al problema di curvatura media prescritta, mostriamo esistenza ed unicità di soluzioni per l'equazione soltanto richiedendo che l insieme Ω sia un aperto "debolmente regolare", ovvero che soddisfi una disuguaglianza di traccia di Poincaré e che il suo perimetro coincida con la misura di Hausdorff $(n-1)$-dimensionale del suo bordo topologico. Sotto tali ipotesi, dimostriamo che l'unicità, a meno di traslazioni, è equivalente a diverse altre proprietà. In particolare,

alla massimalità del dominio, ovvero non esistono soluzioni per la stessa curvatura prescritta H in nessun insieme $\widetilde{\Omega}$ che contiene strettamente Ω; alla criticalità di Ω, ovvero che Ω, fra tutti i suoi sottoinsiemi è l'unico per cui la disuguaglianza $|\int_A H| \leq P(A)$ diventa un'uguaglianza; all'esistenza di una soluzione che risolve il problema di capillarità in un cilindro di sezione Ω con angolo di contatto verticale, ovvero con una condizione al bordo tangenziale, assunta in un senso integrale o di "traccia debole". Inoltre, questa condizione al bordo di "traccia debole", quando il perimetro di Ω coincide con il contenuto interno di Minkoswki di Ω, assume la forma più forte $Tu(x) \to \nu_\Omega(z)$ in misura, per $x \in \Omega$ che tendono a un punto z nella frontiera "super-ridotta". Infine, quando la curvatura prescritta H è positiva e non identicamente nulla, si osserva di nuovo il legame fra il problema di Cheeger e di curvatura media prescritta, in quanto la criticalità di Ω è equivalente a dire che la costante di Cheeger pesata tramite H e con perimetro classico è 1 e che Ω è un insieme minimale di Cheeger.

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Contents

Introduction

In this thesis we mainly deal with two problems: the *Cheeger problem* and the *Prescribed Mean Curvature problem*. The Cheeger problem consists in finding the subsets of a given ambient set that minimize the ratio of perimeter over volume. More precisely, denoted by $P(S)$ the perimeter of S in $\mathbb{R}^n$ in the De Giorgi's sense and by $|S|$ the n-Lebesgue measure of S, given an open set $\Omega \subset \mathbb{R}^n$ one asks whether the infimum

$$h_1(\Omega) := \inf_{S \subseteq \Omega} \frac{P(S)}{|S|}, \tag{0.1}$$

is attained or not. Minimizers of (0.1) are called *Cheeger sets* of Ω. On the other hand given a function H on a domain Ω, i.e. an open, bounded and connected set, the Prescribed Mean Curvature problem consists in finding solutions to the so-called *prescribed mean curvature equation* that is the non-linear partial differential equation

$$\operatorname{div}(Tu) = \operatorname{div}\left(\frac{\nabla u(x)}{\sqrt{1+|\nabla u(x)|^2}}\right) = H(x)\,. \tag{PMC}$$

This one, when H is a constant, and when coupled with the Neumann-like boundary condition

$$Tu \cdot \nu = \cos\gamma$$

for a constant angle $\gamma \in [0, \pi/2]$, represents the physical situation of capillarity in a gravity-free environment in a cylinder of cross-section Ω.

At a first sight the Cheeger problem and the Prescribed Mean Curvature problem do not seem to be related, but when the datum H is a positive constant, and the domain Ω is Lipschitz, it is easy to see by the divergence theorem that a necessary condition to existence of solutions of (PMC) is that $H \le h_1(\Omega)$ and no proper subset of Ω is a minimizer of (0.1). Such a condition is as well sufficient and on top of that, for C^2-regular domains, uniqueness up to translations is equivalent to $H = h_1(\Omega)$ and Ω is a minimal Cheeger set, i.e. the unique minimizer of (0.1) in Ω.

We shall show the above connection to hold whenever H is a positive, non-vanishing Lipschitz function, by considering a generalized Cheeger problem that takes into account the presence of weights at volume. Moreover, we will weaken the classical regularity hypotheses on the boundary of the set Ω to allow (PMC) to have solutions. More precisely, we shall require the domain Ω to be *weakly regular* i.e. with finite perimeter such that

$$P(\Omega) = \mathcal{H}^{n-1}(\partial\Omega)\,, \tag{0.2}$$

and such that there exists a constant $k = k(\Omega) > 0$ for which, for any $E \subset \Omega$ one has the Poincaré trace inequality

$$\min\{P(E;\Omega^c), P(\Omega\setminus E;\Omega^c)\} \le k\, P(E;\Omega)\,. \tag{0.3}$$

An interesting feature is that any connected minimizer of the generalized Cheeger problem studied in Section 3.2 will automatically satisfy (0.3), while we shall show that there are Cheeger sets for which (0.2) does not hold. Clearly, knowing whether a set Ω is Cheeger in itself or not, is interesting in view of the above-mentioned link between uniqueness up to translations of solutions and Ω being a minimal Cheeger set. Thus, we study the maximal Cheeger set for planar and simply connected sets in the classical problem (0.1), providing a characterization.

Roughly, the work is organized as follows. Chapter 1 contains the basic notions, definitions, tools and results of Calculus of Variations and Geometric Measure Theory that we will be using throughout the thesis. Chapter 2 is devoted to prove various technical results that are later needed in the following parts. Chapter 3 is devoted to the Cheeger problem. Finally, Chapter 4 is devoted to the Prescribed Mean Curvature problem and the capillarity problem. A more detailed description of the contents and results of this thesis is given in the next two sections.

The Cheeger problem

Chapter 3 deals with two different aspects of the Cheeger problem. On one hand, in Section 3.1 we show a characterization of maximal Cheeger sets for the planar classical Cheeger problem. On the other hand, in Section 3.2 we study a problem similar to (0.1) and the isoperimetric properties of the minimizers.

More specifically, in Section 3.1, in Theorem 3.8 we prove a precise characterization of the maximal Cheeger set E for planar sets Ω in a special class. Namely,

Theorem. *Let Ω be a bounded open set in $\mathbb{R}^2$ such that*

$$\partial\Omega \quad \textit{is locally homeomorphic to an interval and } |\partial\Omega| = 0, \tag{T}$$

$$\Omega \quad \textit{is simply connected,} \tag{SC}$$

$$\Omega \quad \textit{has no bottlenecks of radius } r = 1/h(\Omega). \tag{NB}$$

Then, denoted by $[\Omega]^r$ the inner parallel set of Ω at distance r, the maximal Cheeger set E of Ω is given by

$$E = \bigcup_{x\in[\Omega]^r} B_r(x)\,,$$

where $r = 1/h_1(\Omega)$*. Moreover, the inner Cheeger formula* $|[\Omega]^r| = \pi r^2$ *holds.*

Roughly speaking, by requiring that $\partial\Omega$ is *locally homeomorphic to an interval* we mean that $\partial\Omega$ is the union of disjoint Jordan curves; instead, requiring (NB) amounts to the possibility to drag any ball of radius $1/h_1(\Omega)$ contained in Ω by its center onto another ball of same radius in Ω along a $C^{1,1}$ curve with curvature bounded by $h_1(\Omega)$, while continuously remaining inside Ω. Specifically, this latter allows to exploit the rolling ball property proved in [LP16], which ensures that the maximal Cheeger set contains the union of all those balls. An apparent drawback of this theorem is that it requires the a-priori knowledge of $h_1(\Omega)$. Thus, we prove Corollary 3.9, where it is required the stronger condition of no bottlenecks in a range of radii independent from $h_1(\Omega)$. This result will appear in the forthcoming paper [LNS17] and extends the characterization for planar convex sets given in [KLR06] and for planar strips (but for annuli) given in [KP11, LP16]. In order to show the above theorem we had to prove the following intermediate result which is of independent interest (see Section 2.4, Theorem 2.20).

Theorem. *Suppose* $E \subset \mathbb{R}^2$ *is simply connected and such that* ∂E *is locally homeomorphic to an interval. If the curvature of* ∂E *is bounded from above by* $h > 0$ *in the viscosity sense, then* E *contains a ball of radius* $1/h$*.*

This result, which will appear in [LNS17], generalizes [IP59, HT95] where a set of much stronger hypotheses was required, namely the C^2-regularity of the boundary and the upper bound on the classical signed curvature.
On the other hand, in Section 3.2 we present the results of [Sar16] on the following problem

$$h^{\alpha}_{f,g}(\Omega) := \inf\left\{\frac{P_g(E)}{|E|_f^{1/\alpha}}\right\}, \tag{0.4}$$

where $\alpha \in [1, 1^*)$,

$$|E|_f := \int_E f\,dx$$

for a positive not identically vanishing L^∞ weight f and

$$P_g(E;\Omega) := \int_{(\partial^* E)\cap\Omega} g(x,\nu_E(x))\,d\mathcal{H}^{n-1}(x),$$

for a weight $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ that is lower semicontinuous, convex and positively 1-homogeneous in the second variable and for which it exists $C > 0$ such that

$$\frac{1}{C}|v| \le g(x,v) \le C|v|,$$

for all $(x,v) \in \mathbb{R}^n \times \mathbb{R}^n$. This can be seen as a generalization of the Cheeger problem as the two coincide for the triplet $(f(x), g(x,v), \alpha) = (1, |v|, 1)$. Additionally, it encloses several other variants of the Cheeger problem, such as those of [ILR05] corresponding to the triplet $(f(x), g(x)|v|, 1)$ (for bounded and continuous f, g) and of [PS17] corresponding to the triplet

$(1,|v|,\alpha)$. On top of proving in Theorem 3.23 that the family $\mathcal{C}^{\alpha}_{f,g}(\Omega)$ of sets realizing the infimum in (0.4) is nonempty for open, bounded sets Ω, we show in Subsection 3.2.2 several isoperimetric properties for (connected) minimizers which, up to our knowledge, had not been discussed anywhere else even in the less general frameset apart from [DS98] where the authors deal with quasi-minimizers with respect to any variation. In Proposition 3.24 we show that any connected minimizer satisfies the Poincaré trace inequality. Without loss of generality, up to staging the Cheeger problem in Ω in one of its Cheeger sets, we can suppose directly that $\Omega \in \mathcal{C}^{\alpha}_{f,g}(\Omega)$, so that one has

Proposition. *Let $\Omega \in \mathcal{C}^{\alpha}_{f,g}(\Omega)$ be connected. Then there exists $k = k(n, \Omega) > 0$ such that for every $E \subset \Omega$ one has*

$$\min\{P(E;\Omega^{c}), P(\Omega \setminus E;\Omega^{c})\} \leq k\, P(E;\Omega).$$

Thus, by the standard theory of Sobolev and BV spaces of [Maz11], we infer in Corollary 3.27 that the Sobolev continuous (compact) embeddings of $W^{1,p}$ into L^{p^*} (L^q, for $q < p^*$) hold. Moreover, whenever Ω is weakly regular, then one has Corollary 3.30 which gives the existence of a trace operator from $BV(\Omega)$ ($L^p(\Omega)$) onto $L^1(\partial\Omega)$ (the appropriate fractionary Lebesgue space). Finally in Section 3.3 we exhibit an example of planar, connected Cheeger set Ω that as part of its boundary has a fat Cantor set C^{ε} made of points of density 1 for Ω, i.e. $C^{\varepsilon} \subset \Omega^{(1)}$, thus showing that being a minimizer of (0.4) implies neither $\Omega = \Omega^{(1)}$ nor $P(\Omega) = \mathcal{H}^{n-1}(\partial\Omega)$. The set, which will appear in the forthcoming paper [LS17], is built by taking off from the unitary disk B_1 a set bounded in B_{ε} with constant curvature and constructed on the complement set of a fat Cantor set on the line $(-\varepsilon, \varepsilon) \times \{0\}$. This shows that there are minimal Cheeger sets Ω that are not weakly regular, thus for which there is no BV trace at the boundary and that do not satisfy the strict interior approximation hypotheses of [Sch14a]. In Proposition 3.29, under the additional assumption that $\Omega = \Omega^{(1)}$, we show though that a minimal Cheeger set is weakly regular.

There are several open questions that could be addressed in the future. Regarding the hypotheses of the characterization of the maximal Cheeger set, on one hand it is unclear if the topological assumption (T) must be required; on the other hand hypothesis (NB) can not be completely dropped but we expect to be able to relax and to weaken the the curve's regularity down to the lone continuity. Finally, it would be worth to see what happens when one drops hypothesis (SC). In that case, one would face the further task of determining the genus g (roughly speaking the number of "holes" or "handles" of the set) of the maximal Cheeger set E which may or may not coincide with the one of Ω. Once determined g, and once shown that E is the union of balls of a suitable radius and has no bottlenecks for that radius, by Steiner's formula one would expect as inner Cheeger formula

$$|[\Omega]^r| = (1-g)\pi r^2\,,$$

which clearly can be valid only for $g = 0, 1$. Therefore for Cheeger sets with genus $g \geq 2$ we expect either the no-bottleneck condition to fail or that one can not fit any ball of radius r in Ω. It is of interests checking if (T) and (NB) yield the same theorem with the appropriate inner Cheeger formula when the maximal Cheeger set has genus $g = 0, 1$, independently from the genus of the ambient set Ω.

An open question that could be addressed regards the regularity and the structural properties of sets in $\mathcal{C}^{\alpha}_{f,g}$. These are well-known in the standard case, and we should check if they transpose painlessly to the more general problem. On one hand, we expect that the regularity properties of the boundary of the minimizers are retained, as it is already known for sets in $\mathcal{C}^{\alpha}_{1,g}$, for $g(x,v) = |v|$, as proved in [PS17]. On the other hand, in that same paper it is proved that some structural properties fail, such as the one of the maximal Cheeger set being the union of all balls of a fixed radius, so that not every structural property should be expected to remain valid.

The Prescribed Mean Curvature problem

In Chapter 4, the main results of [LS16] are presented. In there, we study the Prescribed Mean Curvature equation, i.e. (PMC) which for our convenience we recall to be

$$\operatorname{div}(Tu) = \operatorname{div}\left(\frac{\nabla u(x)}{\sqrt{1+|\nabla u(x)|^2}}\right) = H(x)\,.$$

By the divergence theorem it is straightforward that a necessary condition to existence is that

$$\left|\int_A H\,dx\right| < P(A) \tag{0.5}$$

for all smooth subsets $A \subset\subset \Omega$. As soon as one is able to approximate Ω with smooth subsets both in perimeter and volume one gets the validity of the above inequality for all proper subsets of Ω, i.e. those such that $0 < |A| < |\Omega|$, as proved in Proposition 4.1. Owing to [Sch14a] this is possible whenever $P(\Omega) = \mathcal{H}^{n-1}(\partial\Omega)$. In order to prove the existence theorem, we need to ensure the validity of a Gauss-Green formula on Ω between pairings of $BV(\Omega)$ and $X(\Omega)$, where this latter is defined as the space of vector fields ξ such that $\xi \in L^\infty(\Omega;\mathbb{R}^n) \cap C^0(\Omega;\mathbb{R}^n)$ and $\operatorname{div}\xi \in L^\infty(\Omega)$. We proved such a tailored formula in [LS16] for weakly regular domains. Namely, in Section 2.2, we show Theorem 2.7 which states the following.

Theorem. *Let $\Omega \subset \mathbb{R}^n$ be connected and weakly regular. For any $\xi \in X(\Omega)$ there exists an element of $L^\infty(\partial\Omega)$, called the weak normal trace and denoted by $[\xi\cdot\nu]$, such that for any $\varphi \in BV(\Omega)$,*

$$\int_\Omega \varphi \operatorname{div}\xi + \int_\Omega \xi\cdot D\varphi = \int_{\partial\Omega} \varphi\,[\xi\cdot\nu]\,d\mathcal{H}^{n-1}\,.$$

Moreover, $\|[\xi\cdot\nu]\|_{L^\infty(\partial\Omega)} \leq \|\xi\|_{L^\infty(\Omega)}$.

The notion of weak normal trace already appears in [Anz83], in the special case of Ω being an open bounded set with Lipschitz boundary. Another paper finding an analogous formula is [CTZ09] where a crucial step though is the approximation of Ω by smooth sets which are "mostly" contained in the measure-theoretic interior of Ω with respect to the measure $\mu = \operatorname{div}\xi$. It is worth noting that whenever the approximate limit of the vector field ξ exists for $x \to z \in \partial^*\Omega$, then the weak trace at z can be consistently defined as the scalar product between the approximate limit and the outward normal, as proved in Proposition 2.8. The other way round in general is not true, as the weak normal trace $[\xi \cdot \nu]$ of a vector field does not necessarily coincide to any pointwise, almost-everywhere, or measure-theoretic limit of the scalar product $\xi(y) \cdot \nu(x)$, as $y \to x$ as shown in Example 2.9.

As soon as one ensures the validity of the appropriate divergence theorem on Ω, one is able to relate the perimeter of Ω and the integral of H over Ω, in the same spirit of (0.5). Indeed, one would get $|\int_\Omega H\,dx| \le \int_{\partial\Omega} |[Tu \cdot \nu]|\,d\mathcal{H}^{n-1}$ and thus, since $|[Tu\cdot\nu]| \le 1$ and $P(\Omega) = \mathcal{H}^{n-1}(\partial\Omega)$, either Ω satifies (0.5) as well or

$$\left|\int_\Omega H\,dx\right| = P(\Omega)\,. \tag{0.6}$$

Since the well-posedness of the problem depends both on the domain Ω and on the prescribed mean curvature H, we shall consider the pair (Ω, H). If (0.5) is satisfied we will call the pair *admissible*. If additionally (0.6) holds, we will call the pair *extremal*. One here can again appreciate the link between the Prescribed Mean Curvature problem and the weighted (through H) Cheeger problem. Indeed, whenever H is chosen to be positive and not identically vanishing, fixed $g(x,v) = |v|$, being admissible can be read as $h^1_{H,g}(\Omega) \ge 1$, and no proper subset of Ω is in $\mathcal{C}^1_{H,g}(\Omega)$. Moreover, extremality becomes equivalent to ask that $h^1_{H,g}(\Omega) = 1$ and Ω is a minimal Cheeger set.

Whenever Ω is connected and weakly regular, we prove that a solution exists both in the admissible non-extremal case (Theorem 4.3) and in the extemal one (Theorem 4.7). This result weakens the Lipschitz regularity hypothesis made in [Giu78]. The main result of Chapter 4 is Theorem 4.8, which says that for a pair being extremal is equivalent to several other properties: among these, the uniqueness (up to translations) of the solution for (PMC), and the existence of a solution that satisfies a tangential boundary condition either in a weak trace sense or in an integral sense. More precisely,

Theorem. *Let the pair (Ω, H) be admissible for a connected, weakly regular set Ω and for a Lipschitz function H. Then, the following are equivalent*

- *(E) (Extremality) The pair (Ω, H) is extremal, i.e., $\left|\int_\Omega H\,dx\right| = P(\Omega)$.*
- *(U) (Uniqueness) The solution of (PMC) is unique up to vertical translations.*
- *(M) (Maximality) Ω is maximal, i.e. no solution of (PMC) can exist in any domain strictly containing Ω.*

(V) (weak Verticality) There exists a solution u of (PMC) which is weakly vertical at $\partial\Omega$, i.e.

$$[Tu \cdot \nu] = 1 \qquad \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega\,,$$

where $[Tu \cdot \nu]$ is the weak normal trace of Tu on $\partial\Omega$.

(V') (integral Verticality) There exists a solution u of (PMC) and a sequence $\{\Omega_i\}_i$ of smooth subdomains, such that $\Omega_i \subset\subset \Omega$, $|\Omega \setminus \Omega_i| \to 0$, $P(\Omega_i) \to P(\Omega)$, and

$$\lim_{i\to\infty} \int_{\partial\Omega_i} Tu(x) \cdot \nu_{\Omega_i}(x)\, d\mathcal{H}^{n-1} = P(\Omega),$$

as $i \to \infty$.

This theorem generalizes the one proved in [Giu78] under C^2-regularity of Ω, which was subsequently slightly improved to C^1 up to a $\mathcal{H}^{n-1}$-negligible set (see [Fin86]). In order to account for the lack of regularity, some properties had to be changed and the proof required several technical points to be addressed. Besides using the Gauss-Green theorem of Section 2.2, another tool that proved useful is the weak's Young law for (Λ, r_0)-perimeter minimizers. Specifically, in Theorem 2.12 of Section 2.3, we prove that (Λ, r_0)-perimeter minimizers tangentially touch the ambient set Ω, i.e. whenever the touch point is in the reduced boundary of Ω it is as well in the reduced boundary of the minimizer and the exterior normals coincide. This result generalizes the well-known fact for perimeter minimizers inside regular sets (see [GMT81] for instance) and the more recent result in [LP16] valid for perimeter minimizers inside any set.

Points (V) and (V') of Theorem 4.8 represent Neumann-like boundary conditions. Thus, whenever a pair is extremal, the solution u solves the capillarity problem for perfectly wetting fluids, i.e. with tangential boundary conditions (attained in an integral and weak trace sense). The geometric condition imposed by the extremality ensures as well that (PMC) coupled with a non tangential boundary condition, $Tu \cdot \nu = \text{const} < 1$, is solved (in an integral and weak trace sense). Notably, the lack of regularity forces a weaker condition (V) than the one in [Giu78], where the author was able to recover a pointwise limit, while in our case we obtain a condition on the weak trace $[Tu \cdot \nu]$ which, as noted before, may not coincide with any pointwise or measure-theoretic limit of Tu. In order to address this point and recover a stronger boundary datum we proved Theorem 2.11.

Theorem. *Let $\Omega \subset \mathbb{R}^n$ be connected and weakly regular. We additionally assume that $P(\Omega) = \mathcal{M}_-(\Omega)$. Then for any $\xi \in X(\Omega)$ and for $\mathcal{H}^{n-1}$-almost every $x_0 \in \partial^{**}\Omega$ such that $[\xi \cdot \nu](x_0) = \|\xi\|_{L^\infty(\Omega)}$, we have*

$$\operatorname*{ap-lim}_{x\to x_0} \xi(x) = \|\xi\|_{L^\infty(\Omega)} \nu(x_0)\,.$$

As a Corollary it follows a stronger condition (V) whenever $P(\Omega) = \mathcal{M}_-(\Omega)$, namely the weak trace coincides with a measure-theoretic limit in the points of the so-called *super-reduced* *boundary*. This notion (see [LS16]) can be defined as follows: we say that a point $x \in \partial^*\Omega$ is in the super-reduced boundary, $\partial^{**}\Omega$, if $d_{\Omega_{x,r}}$, the signed distances associated to subsequent scalings $\partial\Omega_{x,r} = (\partial\Omega - x)/r$, converge as $r \to 0$ to d_{H_x}, the signed distance associated to the tangent half-space in x. The precise definition can be found in Section 2.1, Definition 2.1 along with a geometric characterization that appears in Proposition 2.2. This latter says that a point $x \in \partial^*\Omega$ is in the super-reduced boundary if and only if, for all $\varepsilon, R > 0$ there exists a threshold $r_0 > 0$ such that for all $r \in (0, r_0)$ and for all $y \in \partial\Omega_{x,r} \cap H_x$ with $|y| \leq R$ one has that $-d_{H_x}(y) \leq \varepsilon|y|$. In this sense, this new concept can be seen as a regularity for a boundary point midway through being in the reduced boundary and being Lipschitz. In the section, examples of sets Ω such that $\partial^*\Omega \setminus \partial^{**}\Omega$ is nonempty as well as $\partial^*\Omega = \partial^{**}\Omega$ but $\partial\Omega$ is not Lipschitz or even locally a graph, are given.

Finally, in Proposition 4.13 is discussed the stability of the problem. Given an extremal pair (Ω, H), it is known that for any $\varepsilon > 0$ one can make smooth perturbations of Ω smaller than ε both in volume and perimeter, obtaining a domain Ω_ε for which there exists no H_ε such that $(\Omega_\varepsilon, H_\varepsilon)$ is extremal. On the other hand, there might be non-smooth perturbations of the pair (Ω, H), say (Ω_j, H_j), converging in some sense to (Ω, H) that are extremal for any j. Moreover, if the convergence of the pairs is in the right topology, one can ensure the convergence of solutions in the sense of the $L^1_{loc}(\mathbb{R}^{n+1})$-convergence of the epigraphs. Furthermore a family of pairs (Ω_j, H_j), in the special case $H_j = P(\Omega_j)/|\Omega_j|$, that fulfils the hypotheses of the stability theorem is given in Section 4.5. This family is built by taking off from the unit disk B_1 smaller and smaller disks accumulating toward ∂B_1. By controlling their centers and their radii in a suitable way, one can show that this set is a weakly regular minimal Cheeger set thus yielding the extremal pair $(\Omega, P(\Omega)/|H|)$. Particularly, it provides a set for which the existence of solutions was not covered before.

There are still open questions that could be addressed and will be object of our future research. Firstly, we should try to understand if the weakly regularity hypothesis is sharp or the necessity to ask it derives purely from our approach. Secondly, one should try to clarify whether condition (V) can hold in the stronger measure-theoretic sense only when $P(\Omega) = \mathcal{M}_-(\Omega)$ and only in points of the super-reduced boundary points or if the stronger condition can be recovered by other means in all the points of the boundary. Finally, we noticed that conditions $P(\Omega) = \mathcal{M}_-(\Omega)$ and $\partial\Omega = \partial^{**}\Omega\,, \mathcal{H}^{n-1}$-a.e. seem to be related as all the examples we produced either satisfy both or none. Thus, we expect to investigate further and clarify whether or not there is a link between the two conditions.

CHAPTER 1

Preliminaries

1.1 Measures

Radon measures are a key tool in the study of sets of finite perimeter. The theory is well established and we here only recall the basic definitions and main results without giving any proof. For further details and complete proofs one can refer to [AFP00, DU77, Mag12, Rud76]. For sake of notation and simplicity, the following definitions and theorems are given in the special setting of $\Omega \subset \mathbb{R}^n$, for an open set Ω endowed with the induced euclidean topology, since this section is only devoted to introduce the space of BV functions on an open set Ω.

1.1.1. Positive Radon measures

Definition 1.1. Let X be a set and $\mathcal{S}$ a collection of subsets $\{V_i\}$ of X. We say that $\mathcal{S}$ is a *σ-algebra* for X if

- $X \in \mathcal{S}$;
- if $V \in \mathcal{S}$, then $X \setminus V \in \mathcal{S}$;
- if $\{V_j\}_j$ is a countable family of elements of $\mathcal{S}$, then $\bigcup_j V_j \in \mathcal{S}$.

For an open set $\Omega \subset \mathbb{R}^n$ with the standard euclidean topology, we call the *Borel σ-algebra* and denote it by $\mathcal{B}(\Omega)$ the smallest σ-algebra containing all the open sets of Ω.

Definition 1.2. A set function $\mu : \mathcal{P}(\Omega) \to [0, \infty]$ is said to be a *Borel measure* if $\mu(\emptyset) = 0$ and for any countable family of pairwise disjoint $\{V_j\}_j \subset \mathcal{B}(\Omega)$ one has

$$\mu\left(\bigcup_j V_j\right) = \sum_j \mu(V_j)\,.$$

Definition 1.3. Let μ a Borel measure and $U \subset \Omega$. U is said to be μ-negligible if there exists $V \in \mathcal{B}(\Omega)$ containing U such that $\mu(V) = 0$. A property P is said to hold for μ-almost every x (μ-a.e. x) if the set F where it fails is μ-negligible.

Definition 1.4 (Regular Borel measure)**.** A Borel measure is said to be *regular* if for any set $U \subset \Omega$ there exists a set $V \in \mathcal{B}(\Omega)$ containing U such that $\mu(U) = \mu(V)$.

Definition 1.5 (Locally finite Borel measure)**.** A Borel measure is said to be *locally finite* if for any compact set $K \subset \Omega$, one has $\mu(K) < \infty$.

Definition 1.6. A locally finite, regular Borel measure μ is called *positive Radon measure.*

Theorem 1.7 (Outer and inner approximation)**.** *Given a positive Radon measure μ and a Borel set V one has*

$$\begin{aligned}\mu(V) &= \inf\{\mu(A)\ : V \subset A\,, A \textit{ is open}\}\\ &= \sup\{\mu(K)\ : K \subset V\,, K \textit{ is compact}\}\,.\end{aligned}$$

1.1.2. Lebesgue measure

The Lebesgue measure of a set $U \subset \mathbb{R}^n$ is defined as

$$\mathcal{L}^n(U) := \inf_{\mathcal{F}} \sum_{Q\in\mathcal{F}} l(Q)^n\,,$$

where $\mathcal{F}$ is a countable covering of U by open cubes Q with sides parallel to the coordinate axes and $l(Q)$ denotes the side length of Q. $\mathcal{L}^n$ is translation-invariant, i.e. $\mathcal{L}^n(x+U) = \mathcal{L}^n(U)$ for every $U \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$ while it is not scaling invariant, rather it follows the scaling law $\mathcal{L}^n(\lambda U) = \lambda^n \mathcal{L}^n(U)$ for any $\lambda > 0$, i.e. it is n-homogeneous. The Lebesgue measure of a set U is usually called the n-dimensional volume of U. We shall set

$$\mathcal{L}^n(U) = |U|\,,$$

and refer to $|U|$ as to the *volume* or *mass* of U. Finally, we set $\omega_n = |B_1|$, where B_1 is any unitary ball in $\mathbb{R}^n$ in the euclidean topology. The Lebesgue measure restricted to $\mathcal{B}(\mathbb{R}^n)$ is a positive Radon measure.

1.1.3. Hausdorff measure

Let $n, k \in \mathbb{N}$ and $\delta > 0$, the k-dimensional Hausdorff pre-measure of step δ of a set $U \subset \mathbb{R}^n$ is defined as

$$\mathcal{H}^k_\delta(U) = \inf_{\mathcal{F}} \sum_{V\in\mathcal{F}} \omega_k \left(\frac{\mathrm{diam}(V)}{2}\right)^k\,, \tag{1.1}$$

where $\mathcal{F}$ is a countable covering of U by sets $V \subset \mathbb{R}^n$ such that $\mathrm{diam}(V) < \delta$. The k-dimensional Hausdorff measure is defined as

$$\mathcal{H}^k(U) := \lim_{\delta \to 0^+} \mathcal{H}^k_\delta(U)\,. \tag{1.2}$$

The restriction to $\mathcal{B}(\mathbb{R}^n)$ of the k-dimensional Hausdorff measure is a positive Radon measure which is translation-invariant and scales as $\mathcal{H}^k(\lambda U) = \lambda^k \mathcal{H}^k(U)$ for any $\lambda > 0$, i.e. it is k-homogeneous. One could define the s-dimensional Hausdorff measure in a similar way for any $s \le 0$ by using ω_s defined through the Euler Gamma function in (1.1) to interpolate the volumes of the n-dimensional unitary balls.

Definition 1.8 (Hausdorff dimension)**.** For a set $U \subset \mathbb{R}^n$, we define its *Hausdorff dimension* as

$$\dim(U) := \inf\{s \in [0,\infty) \,:\, \mathcal{H}^s(U) = 0\}\,.$$

The k-dimensional Hausdorff measure $\mathcal{H}^k$ and for the Hausdorff dimension of a set satisfy the following list of properties.

1. If $U \subset \mathbb{R}^n$, then $\dim(U) \in [0,n]$. In particular $\mathcal{H}^s(U) = \infty$ for all $s < \dim(U)$ and $\mathcal{H}^s(U) = 0$ for all $s > \dim(U)$;
2. For all $s \in [0,n]$ there exists a compact set $K \subset \mathbb{R}^n$ such that $\dim(K) = s$;
3. If $U \subset \mathbb{R}^n$, then $\mathcal{H}^n(U) = \mathcal{L}^n(U)$;
4. If $1 \le k \le n-1$, $k \in \mathbb{N}$ and U is a k-dimensional C^1 surface, then $\mathcal{H}^k(U)$ coincides with the classical k-dimensional area of U.

1.1.4. Radon measures

Definition 1.9 (Definition 1.4 and 1.40 of [AFP00])**.** A set function $\mu : \mathcal{P}(\Omega) \to \mathbb{R}^m$ is said to be a *finite*, *real* (if $m = 1$) or *vector valued Radon measure* (if $m > 1$) if

- $\mu(\emptyset) = 0$;
- for any countable family of pairwise disjoint set $\{V_j\}_j \subset \mathcal{B}(\Omega)$ one has

$$\mu\left(\bigcup_j V_j\right) = \sum_j \mu(V_j)\,.$$

We shall denote by $\mathcal{M}(\Omega, \mathbb{R}^m)$ the space of all finite Radon measures.

Definition 1.10 (Total variation)**.** Let μ be a finite Radon measure. We define its total variation as the set function $|\mu| : \mathcal{B}(\Omega) \to [0,\infty]$ such that

$$|\mu|(V) := \sup_i \sum_i |\mu(V_i)|$$

where the supremum is taken over pairwise disjoint families $\{V_i\} \subset \mathcal{B}(\Omega)$ such that $V = \bigcup_i V_i$.

Theorem 1.11. *Let μ be a vector valued Radon measure; its total variation $|\mu|$ is a positive Radon measure.*

Theorem 1.12. *The total variation defines a norm on $\mathcal{M}(\Omega, \mathbb{R}^m)$. Moreover the space $\mathcal{M}(\Omega, \mathbb{R}^m)$ endowed with such a norm is a Banach space.*

1.1.5. Weak-star convergence of Radon measures

Theorem 1.13 (Riesz's theorem)**.** *The Banach space $\mathcal{M}(\Omega, \mathbb{R}^m)$ is the dual space of $C_c^0(\Omega, \mathbb{R}^m)$ equipped with the usual norm.*

Hence, we define the weak-star convergence of finite Radon measures in the usual duality way as recalled in the following definition.

Definition 1.14 (Weak-star convergence)**.** Let μ be in $\mathcal{M}(\Omega, \mathbb{R}^m)$. We say that a sequence $\{\mu_i\}_i \subset \mathcal{M}(\Omega, \mathbb{R}^m)$ weakly-star converges in $\mathcal{M}(\Omega, \mathbb{R}^m)$ to μ, and denote it by $\mu_i \xrightharpoonup[i\to\infty]{*} \mu$, if

$$\int_\Omega \varphi \, d\, \mu_i \to \int_\Omega \varphi \, d\, \mu \,,$$

for all $\varphi \in C_c^0(\Omega, \mathbb{R}^m)$.

Proposition 1.15. *Given a family $\{\mu_i\}_i$ and μ in $\mathcal{M}(\Omega, \mathbb{R}^m)$ the following are equivalent.*

- $\mu_i \xrightharpoonup[i\to\infty]{*} \mu$;
- *If K is compact and A is open then*

$$\mu(K) \geq \limsup_{i\to\infty} \mu_i(K) \,,$$
$$\mu(A) \leq \liminf_{i\to\infty} \mu_i(A) \,;$$

- *If E is a Borel set such that $\mu(\partial E) = 0$ then*

$$\mu(E) = \lim_{i\to\infty} \mu_i(E) \,.$$

1.1.6. Radon-Nikodým derivative and Besicovitch Theorem

We here recall the Radon-Nikodým derivative of a measure and the Besicovitch derivation theorem.

Definition 1.16. Let ν be a positive Radon measure and μ a finite Radon measure. We say that μ is absolutely continuous with respect to ν, and write $\mu << \nu$ if the following implication holds

$$\nu(B) \implies |\mu|(B) = 0 \,.$$

We here report (part of) the statement of Radon-Nikodým's and Besicovitch's theorems.

Theorem 1.17 (Radon-Nikodým). *Let μ be absolutely continuous with respect to ν. Then, there exists a unique function f such that $\mu = f\nu$. The function f is said to be the* Radon-Nikodým derivative *of μ with respect to ν and we shall denote*[1] *it by μ/ν.*

Theorem 1.18 (Besicovitch). *Let μ be absolutely continuous with respect to ν. Then, for ν-almost every x*

$$\frac{\mu}{\nu}(x) = \lim_{r\to 0^+} \frac{\mu(B_r(x))}{\nu(B_r(x))}\,.$$

1.1.7. Reshetnyak Theorem

The following theorem was firstly proved in [Res68] and it can be found along with a short proof in [AFP00]. It states the lower semicontinuity of functionals of the kind $\int_\Omega f(x, v/|v|)\, d|v|$, where v is a Radon measure and f is a suitable Borel function.

Theorem 1.19 (Reshetnyak lower semicontinuity). *Let Ω be an open set in $\mathbb{R}^n$ and μ a $\mathbb{R}^m$-valued Radon measure. Let a sequence $\{\mu_i\}_i \subset \mathcal{M}(\Omega, \mathbb{R}^m)$ be weakly-star converging to μ. Then,*

$$\int_\Omega f\left(x\,, \frac{\mu}{|\mu|}(x)\right) d\,|\mu|(x) \leq \liminf_{i\to\infty} \int_\Omega f\left(x\,, \frac{\mu_i}{|\mu_i|}(x)\right) d\,|\mu_i|(x) \tag{1.3}$$

for all lower semicontinuous functions $f : \Omega\times\mathbb{R}^m \to [0,\infty]$, positively 1-homogeneous and convex in the second variable.

1.1.8. Approximate limits

Definition 1.20 (Points of density α). Let E be a Borel set in $\mathbb{R}^n$, $x \in \mathbb{R}^n$ and denote by ω_n the volume of the unitary n-dimensional ball. If the limit

$$\theta(E)(x) := \lim_{r\to 0^+} \frac{|E \cap B_r(x)|}{\omega_n r^n}$$

exists, it is called the density of E at x. In general $\theta(E)(x) \in [0,1]$, hence, we define the set of points of density $\alpha \in [0,1]$ for E as

$$E^{(\alpha)} := \{x \in \mathbb{R}^n \,:\, \theta(E)(x) = \alpha\}\ .$$

Definition 1.21 (Approximate limit). Let f be a measurable function or vector field defined on Ω. Given $z \in \overline{\Omega}$ we write

$$\operatorname*{ap-lim}_{x\to z} f(x) = w$$

if for every $\alpha > 0$ the set $\{x \in \Omega :\ |f(x) - w| \geq \alpha\}$ has density 0 at z.

[1] In some texts the notation used is $d\mu/d\nu$.

1.2 BV functions

The theory of BV functions is now well established and several monographs presenting it are available. We recommend for instance [AFP00].

Definition 1.22. Given a function $u \in L^1(\Omega)$ we say that it is of *bounded variation* if there exists a finite $\mathbb{R}^n$-valued Radon measure μ such that

$$\int_\Omega u\mathrm{div}\,(\varphi)\,dx = -\int_\Omega \varphi\,d\,\mu\,,$$

for all $\varphi \in C^1_c(\Omega, \mathbb{R}^n)$. We shall denote μ by Du and the vector space of all functions of bounded variation by $BV(\Omega)$.

Definition 1.23. For a function $u \in L^1(\Omega)$ we define its total variation in Ω, $V(u, \Omega)$, as follows

$$V(u,\Omega) := \sup\left\{\int_\Omega u\mathrm{div}\,(\varphi)\,dx\ :\ \varphi \in C^1_c(\Omega,\mathbb{R}^n)\,,\ \|\varphi\|_\infty \le 1\right\}.$$

Proposition 1.24. *Let $u \in L^1(\Omega)$. Then, $u \in BV(\Omega)$ if and only if $V(u,\Omega) < \infty$. Moreover, $V(u,\Omega) = |Du|(\Omega)$ and the map $u \mapsto |Du|(\Omega)$ is lower semicontinuous in $BV(\Omega)$ with respect to the $L^1(\Omega)$ topology.*

Proposition 1.25. *The set of BV functions endowed with the norm*

$$\|u\|_{BV} := \|u\|_{L^1} + |Du|(\Omega)\,,$$

is a Banach space.

Theorem 1.26 (Teorema 1 and Corollario 1 of [AG78])**.** *Let u be a function in $L^1(\Omega)$. Then, $u \in BV(\Omega)$ if and only if there exists a sequence $\{u_i\}_i$ of smooth functions converging to u in $L^1(\Omega)$ and such that*

$$L := \lim_{i\to\infty}\int_\Omega |\nabla u_i|\,dx < \infty\,. \tag{1.4}$$

Moreover, the least costant L in (1.4) is $|Du|(\Omega)$.

The natural convergence on BV induced by the BV-norm is usually too strong for many applications. Therefore, we introduce two other notion of convergence which are much more used.

Definition 1.27 (Weak-star convergence)**.** Let u be a function in $BV(\Omega)$. We say that a sequence $\{u_i\}_i \subset BV(\Omega)$ weakly-star converges in BV to u if $u_h \xrightarrow{L^1} u$ and $\{Du_i\}_i$ weakly-star converges to Du in the sense of Definition 1.14, i.e. if

$$\int_\Omega \varphi\,d\,Du_i \to \int_\Omega \varphi\,d\,Du\,,$$

for all $\varphi \in C^0_c(\Omega, \mathbb{R}^n)$.

Definition 1.28 (Strict convergence)**.** Let u be a function in $BV(\Omega)$. We say that a sequence $\{u_i\}_i \subset BV(\Omega)$ strictly converges in BV to u if $u_h \xrightarrow{L^1} u$ and $\{|Du_i|(\Omega)\}_i$ converges to $|Du|(\Omega)$.

It is worth noting that the norm convergence implies the strict convergence which in turns implies the weak* convergence.

1.3 Sets of finite perimeter

Geometric Measure Theory and the theory of sets of finite perimeter has its roots in a series of papers of of Caccioppoli [Cac52a, Cac52b], De Giorgi [DG54, DG55, DG58, DG61] and Federer [Fed58, Fed59]. We here recall the basic definitions and results of the now well estabilished theory according to [Mag12]. We shall use the notation χ_E to denote the characteristic function of the set E.

Definition 1.29 (Perimeter)**.** Let E be a Borel set in $\mathbb{R}^n$. We define the perimeter of E in an open set $\Omega \subset \mathbb{R}^n$ as

$$P(E;\Omega) := \sup\left\{ \int_\Omega \chi_E(x) \operatorname{div} h(x)\, dx \,:\, h \in C^1_c(\Omega;\mathbb{R}^n)\,, \|h\|_\infty \leq 1 \right\}. \tag{1.5}$$

We set $P(E) = P(E;\mathbb{R}^n)$. If $P(E;\Omega) < \infty$ we say that E is a set of finite perimeter in Ω.

Proposition 1.30. *Let E be a set of finite perimeter in Ω. Then, χ_E, belongs to $BV(\Omega)$ and $P(E;\Omega) = |D\chi_E|(\Omega)$.*

Notice that this alternative definition of perimeter agrees with the $\mathcal{H}^{n-1}$ measure of the topological boundary, whenever the set E has Lipschitz boundary, as stated in the following proposition.

Proposition 1.31 (Remark 9.5 and Example 12.6 of [Mag12])**.** *Let $E \subset \mathbb{R}^n$ be an open Lipschitz set. Then, $P(E) = \mathcal{H}^{n-1}(\partial E)$.*

Proposition 1.32 (Properties of the perimeter)**.** *Let E, F and A be Borel sets. Then, the following are true.*

(i) *The function $E \mapsto P(E;A)$ is lower semicontinuous with respect to the L^1 convergence of the characteristic functions;*

(ii) *If A is open the function $E \mapsto P(E;A)$ is local, i.e. $P(E;A) = P(F;A)$ whenever $|A \cap (E\Delta F)| = 0$;*

(iii) *$P(E;A) = P(\mathbb{R}^n \setminus E;A)$ and*

$$P(E\cup F;A) + P(E\cap F;A) \leq P(E;A) + P(F;A)\,;$$

(iv) The derivative of the function $m(r) = |A \cap B_r(x)|$ is defined for almost every r and

$$m'(r) = P(B_r(x); A)\,.$$

1.3.1. L^1 topology and compactness

We shall say that a sequence of sets $\{E_i\}_i$ converges $L^1_{\mathrm{loc}}(\Omega)$ as i goes to infinity to a set E and denote it by

$$E_i \xrightarrow{L^1_{\mathrm{loc}}(\Omega)} E$$

if

$$\chi_{E_i} \xrightarrow{L^1_{\mathrm{loc}}(\Omega)} \chi_E\,,$$

or equivalently if for every compact set $K \subset \Omega$ one has

$$|(E_i \Delta E) \cap K| \to 0\,.$$

Most of the time we will deal with a stronger convergence, that is the L^1 convergence. We shall say that a sequence of sets $\{E_i\}_i$ converges $L^1(\Omega)$ to a set E as i goes to infinity to a set E and denote it by

$$E_i \xrightarrow{L^1(\Omega)} E$$

if

$$\chi_{E_i} \xrightarrow{L^1(\Omega)} \chi_E\,,$$

or equivalently if one has

$$|(E_i \Delta E)| \to 0\,.$$

The following classic results states that families of sets of finite perimeter with bounded BV norm are precompact in L^1 topology. It holds more generally for families of $BV(\Omega)$ functions, under some assumptions on Ω (see Section 1.5).

Theorem 1.33. *Let $\{E_i\}_i$ be a sequence of sets of finite perimeter such that*

(i) $\sup_i \{P(E_i)\} < \infty$

(ii) there exists $R > 0$ such that $E_i \subset B_R$ for all i.

Then, up to subsequences E_h converges in L^1 to some set $E \subset B_R$ of finite perimeter and $D\chi_{E_i} \xrightharpoonup{*} D\chi_E$.

1.3.2. Coarea formula

We here recall another basic tool of Geometric Measure Theory which can be seen as a generalization of Fubini's theorem. The coarea formula states that the total variation of a BV function u over a set E can be computed in terms of the integral of the perimeter of its level sets.

Proposition 1.34. *For any $u \in BV(\Omega)$, the set $\{u > t\}$ has finite perimeter in Ω for almost every $t \in \mathbb{R}$ and for any Borel set $E \subset \Omega$ one has*

$$|Du|(E) = \int_{-\infty}^{\infty} P(\{u > t\}; E)\, dt\,.$$

1.3.3. De Giorgi's and Federer's theorems

We here introduce the concept of $\mathcal{H}^k$-rectifiability which is a pillar in the theory of sets of finite perimeter by De Giorgi. Roughly, a set is said to be $\mathcal{H}^k$-rectifiable if it can ben covered by the images of countably many Lipschitz maps defined on $\mathbb{R}^k$, up to a null measure set. We shall use the following definitions as in [AFP00].

Definition 1.35. Let $E \subset \mathbb{R}^n$ be a $\mathcal{H}^k$-measurable set. We say that E

- is countably k-rectifiable if there exist countably many Lipschitz maps $f_i : \mathbb{R}^k \to \mathbb{R}^n$ such that
$$E \subset \bigcup f_i(\mathbb{R}^k)\,.$$
- is countably $\mathcal{H}^k$-rectifiable if there exist countably many Lipschitz maps $f_i : \mathbb{R}^k \to \mathbb{R}^n$ such that
$$\mathcal{H}^k\left(E \setminus \bigcup f_i(\mathbb{R}^k)\right) = 0\,.$$
- is $\mathcal{H}^k$-rectifiable if it is countably $\mathcal{H}^k$-rectifiable and $\mathcal{H}^k(E) < \infty$.

The notion of countably $\mathcal{H}^k$-rectifiability is important because it is the regularity displayed by the *reduced boundary* of sets of finite perimeter, which is defined below along with several properties.

Theorem 1.36 (De Giorgi Structure Theorem)**.** *Let E be a set of finite perimeter and let $\partial^* E$ be the reduced boundary of E defined as*

$$\partial^* E := \left\{ x \in \partial^e E \;:\; \lim_{r \to 0^+} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))} = -\nu_E(x) \in \mathbb{S}^{n-1} \right\}.$$

Then,

(i) $\partial^* E$ is countably $\mathcal{H}^{n-1}$-rectifiable;

(ii) *for all* $x \in \partial^* E$, $\chi_{E_{x,r}} \to \chi_{H_{\nu_E(x)}}$ *in* $L^1_{loc}(\mathbb{R}^n)$ *as* $r \to 0^+$, *where* $E_{x,r} := (E-x)/r$ *and* $H_{\nu_E(x)}$ *denotes the half-space through* 0 *whose exterior normal is* $\nu_E(x)$*;*

(iii) *for any Borel set* A, $P(E;A) = \mathcal{H}^{n-1}(A \cap \partial^* E)$*, thus in particular* $P(E) = \mathcal{H}^{n-1}(\partial^* E)$*;*

(iv) $\int_E \operatorname{div} g = \int_{\partial^* E} g \cdot \nu_E \, d\mathcal{H}^{n-1}$ *for any* $g \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$.

Another notion of boundary is the one introduced by Federer, i.e. the *essential boundary*, defined below.

Definition 1.37. We define the essential boundary of E

$$\partial^e E := \mathbb{R}^n \setminus (E^{(0)} \cup E^{(1)}) \,,$$

where $E^{(\alpha)}$ has been defined in Definition 1.20.

Remark 1.38. Notice that the inner normal $-\nu_E$ is the Radon-Nikodým derivative of the Radon measure $D\chi_E$.

The next theorem clarifies the link between the reduced boundary and the essential boundary of sets of (locally) finite perimeter, stating that they are the same up to null-measure sets.

Theorem 1.39 (Federer Structure Theorem)**.** *Let* E *be a set of finite perimeter. Then,* $\partial^* E \subset E^{(1/2)} \subset \partial^e E$ *and one has*

$$\mathcal{H}^{n-1}\left(\partial^e E \setminus \partial^* E\right) = 0 \,.$$

1.3.4. P-indecomposability of sets of finite perimeter

We here recall the notion of *P-indecomposability* defined below.

Definition 1.40. A set E is said to be P-indecomposable if for every disjoint sets $E_1, E_2 \subseteq E$ such that $P(E) = P(E_1) + P(E_2)$ either $|E_1|$ or $|E_2|$ has null measure.

Analogously to connectedness, one can define the *P-indecomposable components* of E by means of the following theorem.

Theorem 1.41 (Theorem 1 of [ACMM01])**.** *Let* E *be a set of finite perimeter. Then, there exists a unique finite or countably finite family of pairwise disjoint P-indecomposable sets* $\{E_i\}_i$ *of positive Lebesgue measure such that* $P(E) = \sum_i P(E_i)$*. Moreover*

$$\mathcal{H}^{n-1}\left(E^{(1)} \setminus \bigcup E_i^{(1)}\right) = 0 \,,$$

and the set E_i *are maximal P-indecomposable i.e. any P-indecomposable set* $F \subset E$ *is contained in only one of the* E_i*, up to null measure sets.*

1.3.5. The Minkowski content

We here introduce a third possible definition for the perimeter, known as the $(n-1)$-dimensional Minkowski content. Since in this thesis we will deal only with the $(n-1)$-dimensional (bilateral, inner and outer) content, for sake of brevity we shall suppress the indication of the dimension and only speak of (bilateral, inner and outer) Minkowski content. For a given set E, we shall use the notation $[E]_t$, for a positive t, to indicate the Minkowski enlargement of E, i.e.

$$[E]_t := \{x \in \mathbb{R}^n \, : \, \mathrm{dist}(x, E) \leq t\} \, ,$$

while $[E]^t$ for the inner parallel set of E at distance t, which is

$$[E]^t := \{x \in \mathbb{R}^n \, : \, \mathrm{dist}(x, \mathbb{R}^n \setminus E) \geq t\} \, ,$$

Definition 1.42. For any set $E \subset \mathbb{R}^n$ we define its (bilateral) Minkowski content $\mathcal{M}(E)$ as

$$\mathcal{M}(E) := \lim_{\varepsilon \to 0^+} \frac{|[E]_\varepsilon \setminus [E]^\varepsilon|}{\varepsilon} \, ,$$

whenever the limit exists.

Whenever a set E displays some regularity of the boundary, the Minkowski content exists and it agrees with the $\mathcal{H}^{n-1}$-measure of the topological boundary and, due to Proposition 1.31 with the De Giorgi's perimeter, as stated in the next proposition.

Proposition 1.43 (Paragraph 3.2.39 of [Fed69])**.** *A set $E \subset \mathbb{R}^n$ whose boundary is $(n-1)$-rectifiable admits Minkoswki content and $\mathcal{M}(E) = \mathcal{H}^{n-1}(\partial E)$.*

We shall now define the outer (inner) Minkoswki content, which take in account only the difference between the set and the enlargement (inner parallel set).

Definition 1.44. For any set $E \subset \mathbb{R}^n$ we define its outer Minkowski content $\mathcal{M}_+(E)$ as

$$\mathcal{M}_+(E) := \lim_{\varepsilon \to 0^+} \frac{|[E]_\varepsilon \setminus E|}{\varepsilon} \, ,$$

whenever the limit exists. Analogously we define its inner Minkowski content $\mathcal{M}_-(E)$ as

$$\mathcal{M}_-(E) := \lim_{\varepsilon \to 0^+} \frac{|E \setminus [E]^\varepsilon|}{\varepsilon} \, ,$$

whenever the limit exists.

The following theorems are collected from [ACV08] where are stated in a slightly more general way. We here report those in a more specific case which is the one we shall use later on. Particularly in Theorem 1.46, we swapped the outer content with the inner content, which can be done painlessly since the inner content of a set E coincides with the outer content of its complement set $\mathbb{R}^n \setminus E$.

Theorem 1.45 (Theorem 2 of [ACV08])**.** *Let $A \subset \mathbb{R}^{n-1}$ be a countably $\mathcal{H}^{n-1}$-rectifiable compact set and suppose there exist $\gamma > 0$ and a Radon measure η in $\mathbb{R}^{n-1}$ which is absolutely continuous with respect to $\mathcal{H}^{n-1}$ such that for all $x \in A$ and all $r \in (0,1)$ one has*

$$\eta(B_r(x)) \geq \gamma r^{n-1}\,.$$

Then, $\mathcal{M}(A) = \mathcal{H}^{n-1}(A)$.

Theorem 1.46 (Theorem 5 of [ACV08])**.** *Let E be a set of finite perimeter in $\mathbb{R}^n$ and assume that its Minkowski content $\mathcal{M}(E)$ exists and coincide with $P(E)$. Then, its inner Minkowski content exists and $\mathcal{M}_-(E) = P(E)$.*

We shall use Theorem 1.45 and Theorem 1.46 combined in the following way.

Theorem 1.47. *Let E be a set of finite perimeter such that there exist $\gamma > 0$ and a Radon measure η in $\mathbb{R}^{n-1}$ which is absolutely continuous with respect to $\mathcal{H}^{n-1}$ such that for all $x \in \partial^* E$ and all $r \in (0,1)$ one has*

$$\eta(B_r(x)) \geq \gamma r^{n-1}\,.$$

Then, $\mathcal{M}_-(E) = P(E)$.

1.3.6. Sets with positive reach and Steiner's formulas

For a bounded, convex set $A \subset \mathbb{R}^n$, the volume of the Minkowski enlargement $[A]_r$ can be expressed as a degree-n polynomial in r with coefficients depending on A for any $r > 0$. This was originally shown in [Ste40], cf. [Sch14b]. For $n = 2$, this polynomial takes the form

$$|[A]_r| = |A| + rP(A) + \pi r^2. \tag{1.6}$$

Polynomial expansions of the same type were shown for C^2 sets in [Wey39] for $r > 0$ sufficiently small. If $A \subset \mathbb{R}^2$ is simply connected[2] and of class C^2 (actually, $C^{1,1}$ will suffice), then the expansion holds in the same form (1.6). In [Fed59], Federer gave a unified treatment of this theory with the introduction of sets of positive reach. He defined the reach of a set A to be

$$\mathrm{reach}(A) = \sup\{r : \text{ if } x \in [A]_r, \text{ then } x \text{ has a unique projection onto } A\}$$

and showed a polynomial expansion for $|[A]_r|$ for $0 < r < \mathrm{Reach}(A)$.

If $A \subset \mathbb{R}^2$ is simply connected with positive reach, the proof of this polynomial expansion is fairly simple. For any $0 < t < \mathrm{reach}(A)$, $[A]_t$ is a simply connected set of class $C^{1,1}$. Hence, (1.6) holds for $[A]_t$, that is, for $t < r$,

$$|[A]_r| - |[A]_t| = (r-t)P([A]_t) + \pi(r-t)^2. \tag{1.7}$$

[2] The general formula takes into account the genus g of the set, see for instance [Fed59]. For simply connected sets one has $g = 0$.

Since $\lim_{t\to 0^+} |[A]_t| = |A|$, it follows that $\lim_{t\to 0^+} P([A]_t) = c_A$ exists. Hence, taking $t \to 0^+$ in (1.7), we see that (1.6) holds for A with c_A replacing $P(A)$. And actually, dividing (1.7) by $r-t$, then letting $t \to 0^+$ and $r \to 0^+$ respectively, it follows that $c_A = \mathcal{M}_+(A)$. So, if $A \subset \mathbb{R}^2$ is a simply connected set with positive reach, then

$$|[A]_r| = |A| + \mathcal{M}_+(A)r + \pi r^2 \qquad 0 < r < \mathrm{reach}(A). \tag{1.8}$$

Differentiating this identity, we also find that

$$P([A]_r) = \mathcal{M}_+(A) + 2\pi r \qquad 0 < r < \mathrm{reach}(A). \tag{1.9}$$

1.3.7. Approximation of sets

Rather than dealing with sets with finite perimeter but wild topological boundary, one prefers to work on inner approximations with smooth boundary and then (hopefully) be able to pass to the limit. This, though, can be done under some regularity hypotheses on the boundary. The first result, which for long time has been the only known one, is somewhat useless in this regard since it requires Lipschitz regularity which is already a strong request.

Theorem 1.48. *Let Ω be a Lipschitz of finite perimeter. Then, there exists a sequence of open smooth sets E_i compactly contained in Ω such that $E_i \xrightarrow{L^1} \Omega$ and $P(E_i) \to P(\Omega)$.*

Recently a much weaker approximation theorem was proved and the request made on Ω is that its perimeter coincides with the $\mathcal{H}^{n-1}$ measure of its topological boundary. The result is essentially due to Schmidt [Sch14a]. A minor tweak to the statement is proved in [LS16]. Namely, we add to the original statement that whenever Ω is connected the sequence of approximating sets can be chosen to be connected.

Theorem 1.49 (Theorem 1.1 of [Sch14a])**.** *Suppose that Ω is a bounded open set in $\mathbb{R}^n$ such that $P(\Omega) = \mathcal{H}^{n-1}(\partial\Omega) < +\infty$. Then, for every $\delta > 0$ there exist an open set Ω_δ with smooth boundary in $\mathbb{R}^n$ such that*

$$\Omega_\delta \subset\subset \Omega, \qquad \Omega \setminus \Omega_\delta \subset ([\partial\Omega]_\delta \cap [\partial\Omega_\delta]_\delta), \qquad |\Omega \setminus \Omega_\delta| < \delta, \qquad P(\Omega_\delta) \le P(\Omega) + \delta\,. \tag{1.10}$$

Moreover, Ω_δ can be chosen connected as soon as Ω is connected.

Proof. The first part of the statement is due to [Sch14a]. For sake of completeness we here give a sketch of it. By a standard mollification argument, it is enough to prove the existence of measurable Ω_δ satysfying (1.10). Moreover, by a contradiction argument one sees that it is enough to show $\Omega \setminus \Omega_\delta \subset [\partial\Omega]_\delta$ rather than the second request in (1.10). Thus, if $P(\Omega) = \infty$, the choice $\Omega_\delta = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \delta/2\}$ satisfies the claim. If, on the other hand, the perimeter is finite one proceeds as follows. Consider the boundary split in $\partial^*\Omega$ and the remaining part

$\partial\Omega \setminus \partial^*\Omega$. The latter can be covered with balls B_k of radii $r_k \leq \delta/2$ such that $\sum_k r_k^{n-1} < \delta$. On the other hand, one can provide a cover for $\partial^*\Omega$ by the means of [Sch14a, Lemma 3.1] which allows to cover the reduced boundary $\partial^*\Omega$ in countably many disjoint Borel sets F_i such that if $x \in F_i$, the volume of $\Omega \cap \{y \in B_r(x) \,:\, y \cdot \nu(x) \geq 0\}$ is bound from above by $\delta^2 r^n$ and such that $F_i \cap B_r(x)$ is contained in the tangent hyperplane to Ω at x. One can then furtherly refine this covering in few additional steps: firstly, one takes a spherical covering $B^i_j = B_{r^i_j}$ of these F_i such that $\sum_j \omega_{n-1}(r^i_j)^{n-1} \leq \mathcal{H}^{n-1}(\partial^* F_i) + 2^{-i}\delta$; secondly one slightly enlarges it in cylinders C^i_j contained in balls B_δ centered on points of $\partial^*\Omega$, such that

$$\mathcal{H}^{n-1}(\Omega \cap \partial C^i_j) \leq [\omega_{n-1} + 3^n\delta + 6\delta(n-1)\omega_{n-1}](r^i_j)^{n-1} .$$

Clearly the union of C^i_j and B_k provides a covering of $\partial\Omega$. Thanks to boundedness, we can extract a finite subcover of $\partial\Omega$ given by

$$S = C^{i_1}_{j_1} \cup \cdots \cup C^{i_n}_{j_n} \cup B_1 \cup \cdots \cup B_m .$$

Then, $\Omega_\delta := \Omega \setminus S$ provides a set satisfying (1.10). Clearly $\Omega_\delta \subset\subset \Omega$ and $\Omega \setminus \Omega_\delta \subset S \subset [\partial\Omega]_\delta$ since the C_i and B_i are each contained in a B_δ. It remains to be seen the perimeter bound. Since has

$$\begin{aligned} P(\Omega_\delta) \leq \mathcal{H}^{n-1}(\partial\Omega_\delta) &\leq \sum_{i,j} \mathcal{H}^{n-1}(\Omega \cap \partial C^i_j) + \sum_k \mathcal{H}^{n-1}(\partial B_k) \\ &\leq (1 + c_n\delta) \sum_{i,j} \omega_n (r^i_j)^{n-1} + c_n \sum_k r_k^{n-1} \\ &\leq (1 + c_n\delta) \sum_i [\mathcal{H}^{n-1}(F_i) + 2^{-i}\delta] + c_n\delta \\ &\leq (1 + c_n\delta)[\mathcal{H}^{n-1}(\partial^*\Omega) + \delta] + c_n\delta , \end{aligned}$$

which yields the claim.

We here report the full proof of the second part, as shown in [LS16]. Fix a compact set $K \subset \Omega$ such that $|\Omega \setminus K| < \delta$, then setting $d = \min\{\mathrm{dist}(x, \partial\Omega) \,:\; x \in K\}$ we take a finite covering of K by balls of radius $d/2$ and let $x_1, \ldots, x_N$ denote their centers. By connectedness, for any $h, k \in \{1, \ldots, N\}$ there exists a path $\Gamma_{hk} \subset \Omega$ connecting x_h to x_k, so that the set

$$\widetilde{K} = \bigcup_{h=1}^{N} B_{d/2}(x_h) \cup \bigcup_{h,k=1}^{N} \Gamma_{hk} ,$$

is contained in Ω, connected, compact, and such that $|\Omega \setminus \widetilde{K}| < \delta$. Let now $\tilde{\delta} = \min(\min\{\mathrm{dist}(x, \partial\Omega) : x \in \widetilde{K}\}, \delta) > 0$, then by (1.10) with $\tilde{\delta}$ replacing δ we get an open set $\Omega_{\tilde{\delta}}$ which necessarily has a connected component A containing $\widetilde{K}$, so that (1.10) and the last part of the statement are satisfied by setting $\Omega_\delta = A$. $\square$

We remark that whenever the inner Minkowski content of Ω exists and coincides with $P(\Omega)$ a similar result to Theorem 1.49 holds. Namely, one has the following proposition.

Proposition 1.50. *Let Ω be an open bounded set of finite perimeter, such that $P(\Omega) = \mathcal{M}_-(\Omega)$. Then, there exists a sequence $\{\Omega_j\}_j$ of relatively compact open sets with smooth boundary, satisfying the same properties as in Theorem 1.49.*

Proof. It is enough to observe that $\mathcal{M}_-(\Omega) = P(\Omega)$ coupled with coarea formula gives

$$\lim_{r \to 0^+} \frac{1}{r} \int_0^r \mathcal{H}^{n-1}(\partial[\Omega]^t)\, dt = 0\,,$$

where $[\Omega]^t$. Hence there exists a strictly decreasing, infinitesimal sequence $\{r_j\}_j$ such that the sequence $\{\Omega_j\}_j$ defined by $\Omega_j = [\Omega]^{r_j}$ satisfies the required properties, with a possible exception of the smoothness. Finally, in order to enforce $\partial\Omega_j$ smooth one can apply standard approximation by smooth sets. □

1.4 Isoperimetric inequality

The classic isoperimetric inequality states that for every set E of finite perimeter and finite volume, the following holds

$$|E|^{n-1} \leq n\omega_n P(E)^n\,, \tag{1.11}$$

and the equality is attained if and only if E is a ball i.e. one has

$$P(E) \geq P(B_E)\,, \tag{1.12}$$

where B_E is the ball such that $|E| = |B_E|$.

A set $\Omega \subset \mathbb{R}^n$ is said to be a domain of isoperimetry if it supports a relative isoperimetric inequality, i.e. if there exists a constant $k = k(n, \Omega)$ such that for all $E \subset \Omega$ one has

$$\min\{|E|; |\Omega \setminus E|\}^{n-1} \leq kP(E;\Omega)^n\,. \tag{1.13}$$

As we shall recall in Section 1.5, if a set Ω enjoys (1.13) then it admits the classic Sobolev and BV embeddings into the appropriate L^p spaces. We shall remind that being Lipschitz guarantees the relative isoperimetric inequality as stated below.

Theorem 1.51. *If E is a Lipschitz set, then it supports a relative isoperimetric inequality.*

1.5 Embedding and trace theorems

We here recall the embedding and trace theorems with sharp conditions as stated in [Maz11]. We here report only their statements for the BV setting but we recall they hold as well for any Sobolev space into the appropriate L^p space.

Theorem 1.52. *Let Ω be a bounded, connected open set and set $p^* = np/(n-p)$. Then, there exist a constant $K = K(n, \Omega) > 0$ such that, for all $u \in BV(\Omega)$*

$$\|u\|_{L^{1^*}} \leq K\|u\|_{BV} ,$$

and a constant $K_p = K(n, p, \Omega)$ such that, for all $u \in W^{1,p}(\Omega)$

$$\|u\|_{L^{p^*}} \leq K_p\|u\|_{W^{1,p}} ,$$

if and only if Ω supports a relative isoperimetric inequality (1.13) for some k.

Theorem 1.53. *Let Ω be a bounded, connected open set such that one has $P(\Omega) = \mathcal{H}^{n-1}(\partial\Omega)$. Then, there exist linear continuous operators $T : BV(\Omega) \to L^1(\partial\Omega)$ and $T_p : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, if and only if there exists a constant $k = k(n, \Omega)$ such that for all $E \subset \Omega$ one has*

$$\min\{P(E; \partial\Omega); P(\Omega \setminus E; \partial\Omega)\} \leq kP(E; \Omega) . \tag{1.14}$$

In particular thanks to [AG78, Theorem 10 (a)] we have the following.

Theorem 1.54. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected open set with $P(\Omega) = \mathcal{H}^{n-1}(\partial\Omega) < +\infty$. Then the following are equivalent:*

(i) there exists $k = k(\Omega)$ such that for all $E \subset \Omega$

$$\min\{P(E; \Omega^c), P(\Omega \setminus E; \Omega^c)\} \leq kP(E; \Omega);$$

(ii) there exists a continuous trace operator from $BV(\Omega)$ to $L^1(\partial\Omega)$ with the following property: for any $\varphi \in L^1(\partial\Omega)$ there exists $\Psi \in W^{1,1}(\mathbb{R}^n)$ such that φ is the trace of Ψ on $\partial\Omega$.

1.6 The mean curvature

The derivation that follows can be found in more generality in [Sim84]. Let M be a manifold of dimension n embedded in $\mathbb{R}^{n+1}$ of class C^2, and let T_yM be its tangent space at $y \in M$. We shall denote by $\{\tau_i\}_{i=1,\dots,n}$ any orthonormal basis of the tangent space. Given any element $\tau \in T_yM$ one can define the directional derivative of a regular enough function f on M as

$$D_\tau f = \frac{d}{dt} f(\gamma(t))\Big|_{t=0} ,$$

where γ is a regular curve onto M whose tangent vector at y is τ. For a real-valued (and regular enough) function f one can define the tangential (to M) gradient as

$$\nabla^M f(y) = \sum_{i=1}^{n} \tau_j \cdot D_{\tau_j} f \,.$$

If f is the restriction of a C^1 curve defined on $\mathbb{R}^{n+1}$, then the tangential gradient is the projection onto T_yM of the usual gradient, i.e.

$$\nabla^M f(y) = \nabla f - (\nabla f \cdot \nu)\nu \,,$$

where ν is the versor generating the one-dimensional normal space $(T_yM)^\perp$. Then, given a C^1 vector field $X = (X_1, \ldots, X_{n+1}) : M \to \mathbb{R}^{n+1}$ we define its tangential (to M) divergence as

$$\mathrm{div_M}\, X = \sum_{j=1}^{n+1} e_j \cdot \nabla^M X_j \ = \sum_{i=1}^{n} \tau_i \cdot D_{\tau_i} X,$$

where $\{e_j\}_{j=1,\ldots,n+1}$ is the standard orthonormal basis of $\mathbb{R}^{n+1}$.

The *second fundamental form* of M is the bilinear form $B_y : T_yM \times T_yM \to (T_yM)^\perp$ acting as

$$B_y(\tau, \eta) = -(\eta \cdot D_\tau \nu)\nu \,,$$

The mean curvature H of M is defined as

$$H = \mathbf{H} \cdot \nu = \left(\sum_{i=1}^{n} B_y(\tau_i, \tau_i) \right) \cdot \nu = \left(-\sum_{i=1}^{n} (\tau_i \cdot D_{\tau_i} \nu) \right) \nu \cdot \nu \ = -\mathrm{div}_{\,M} \nu \,, \tag{1.15}$$

while $\mathbf{H}$ is usually referred to as the mean curvature vector. It can be proved that H is an invariant and it does not depend on the choice of the local representative, nor on the basis of the tangent. We recall the divergence formula for smooth manifolds M with empty boundary which states that

$$\int_M \mathrm{div_M}\, X \, d\,\mathcal{H}^n(x) = \int_M X \cdot \mathbf{H} \, d\,\mathcal{H}^n(x) \,. \tag{1.16}$$

We shall now compute the mean curvature H for a submanifold M that is the graph of a function u defined on $\Omega \subset \mathbb{R}^n$. For such a M, one has

$$\nu = \frac{(-\nabla u, 1)}{\sqrt{1 + |\nabla u|^2}} \,. \tag{1.17}$$

From now on, vectors of $\mathbb{R}^n$ shall be considered embedded into $\mathbb{R}^{n+1}$. From (1.15) and (1.17) we find

$$\begin{aligned} H = -\mathrm{div_M}\, \nu = -\sum_{j=1}^{n+1} e_j \nabla^M \nu_j &= -\sum_{j=1}^{n+1} e_j \cdot (\nabla \nu_j - (\nabla \nu_j \cdot \nu)\nu) \qquad (1.18) \\ = -\sum_{j=1}^{n} e_j \cdot (\nabla \nu_j - (\nabla \nu_j \cdot \nu)\nu) &- e_{n+1} \cdot (\nabla \nu_{n+1} - (\nabla \nu_{n+1} \cdot \nu)\nu) \,. \end{aligned}$$

On one hand, one has

$$
\begin{aligned}
e_{n+1} \cdot (\nabla \nu_{n+1} - (\nabla \nu_{n+1} \cdot \nu)\nu) = & \\
&= e_{n+1} \cdot \nabla \left(\frac{1}{\sqrt{1+|\nabla u|^2}} \right) - \frac{1}{\sqrt{1+|\nabla u|^2}} \left(\nabla \left(\frac{1}{\sqrt{1+|\nabla u|^2}} \right) \cdot \frac{(-\nabla u, 1)}{\sqrt{1+|\nabla u|^2}} \right) \\
&= \frac{1}{1+|\nabla u|^2} \left(\nabla \left(\frac{1}{\sqrt{1+|\nabla u|^2}} \right) \cdot \nabla u \right)
\end{aligned} \tag{1.19}
$$

On the other hand

$$
\begin{aligned}
\sum_{j=1}^{n} e_j \cdot (\nabla \nu_j - (\nabla \nu_j \cdot \nu)\nu) = & \\
&= \sum_{j=1}^{n} e_j \cdot \left(\nabla \frac{-\partial_j u}{\sqrt{1+|\nabla u|^2}} \right) + \left(\nabla \left(- \frac{\partial_j u}{\sqrt{1+|\nabla u|^2}} \right) \cdot \frac{(-\nabla u, 1)}{\sqrt{1+|\nabla u|^2}} \right) \frac{\partial_j u}{\sqrt{1+|\nabla u|^2}} \\
&= -\mathrm{div} \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) + \sum_{j=1}^{n} \left(\nabla \left(\frac{\partial_j u}{\sqrt{1+|\nabla u|^2}} \right) \cdot \nabla u \right) \frac{\partial_j u}{1+|\nabla u|^2} \\
&= -\mathrm{div} \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) + \sum_{j=1}^{n} \frac{(\partial_j u)^2}{1+|\nabla u|^2} \left(\nabla u \cdot \nabla \left(\frac{1}{\sqrt{1+|\nabla u|^2}} \right) \right) + \frac{\partial_j u (\nabla(\partial_j u) \cdot \nabla u)}{(1+|\nabla u|^2)^{3/2}} \\
&= -\mathrm{div} \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) + \frac{|\nabla u|^2}{1+|\nabla u|^2} \left(\nabla \left(\frac{1}{\sqrt{1+|\nabla u|^2}} \right) \cdot \nabla u \right) + \sum_{j=1}^{n} \frac{\partial_j u (\nabla(\partial_j u) \cdot \nabla u)}{(1+|\nabla u|^2)^{3/2}}
\end{aligned}
$$

Finally, since $\partial_j |\nabla u|^2 = 2\partial_j(\nabla u) \cdot \nabla u$, combining these two in (1.19), all terms but the divergence one cancel out, leaving

$$
H = \mathrm{div} \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) .
$$

Then, one can state the Prescribed Mean Curvature problem as follows. Given an open domain $\Omega \subset \mathbb{R}^n$ and a prescribed scalar function H defined on Ω, does it exist a function u such that its graph has mean curvature H? Equivalently, does the nonlinear PDE

$$
\mathrm{div} \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) = H \, , \tag{1.20}
$$

admit solution? We shall then refer to (1.20) as to the Prescribed Mean Curvature equation. Furthermore, we shall define the vector field Tu as

$$
Tu := \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} .
$$

1.7 The first variation of the perimeter

We here recall the first variation of the perimeter. Full computations and details are available, for instance, in [Mag12]. A family $\{f_t\}_t$, for $|t| \leq \varepsilon$ is said to be a local variation in Ω if it defines a one-parameter family of diffeomorphisms such that

$$f_0(x) = x\,, \qquad \forall x \in \mathbb{R}^n \tag{1.21}$$

$$\{x \in \mathbb{R}^n \,:\, f_t(x) \neq x\} \subset\subset \Omega\,, \qquad \forall |t| \leq \varepsilon \tag{1.22}$$

Clearly, if the family defines a local variation in Ω, then the sets $f_t(E)$ and E agree outside Ω. Moreover, one has the validity of the Taylor expansion, uniformly on $\mathbb{R}^n$

$$f_t(x) = x + tT(x) + o(t) \tag{1.23}$$

$$\nabla f_t(x) = \mathrm{Id} + t\nabla T(x) + o(t) \tag{1.24}$$

where the vector field $T \in C_c^\infty(\Omega;\mathbb{R}^n)$, called the initial velocity of $\{f_t\}$ is

$$T(x) = \frac{\partial}{\partial t} f_t(x,0)\,.$$

Since, conversely, given a vector field T one can construct a local variation, we shall say that $\{f_t\}$ is a local variation associated with T. Thus, for a given T, one has as first variation of the perimeter

$$P(f_t(E);\Omega) = P(E;\Omega) + t\int_{\partial^* E} \mathrm{div}_{\,\partial E} T(x)\, d\mathcal{H}^{n-1}(x) + o(t)\,.$$

For smooth enough sets, applying (1.16) to the manifold ∂E gives

$$\left.\frac{d}{dt}\right|_{t=0} P(f_t(E);\Omega) = \int_{\partial^* E} H(T\cdot\nu_E)\, d\mathcal{H}^{n-1}(x)\,. \tag{1.25}$$

Thus, any perimeter minimizer must have zero first order variation, hence its boundary must have zero mean curvature.

1.8 Perimeter (Λ, r_0)-minimizers

Roughly speaking a set E is said to be a (Λ, r_0)-minimizer of the perimeter in an open set Ω, whenever its perimeter is locally (to a scale smaller than r_0) less than the one of any other competitor up to a volume term times Λ. The formal definition is presented below, along with some results and comments. For further details one can refer to [Mag12].

Definition 1.55. Given an open set $\Omega \subset \mathbb{R}^n$ and a set of locally finite perimeter E, we say that E is a (Λ, r_0)-perimeter minimizer in Ω if there exist two constants $\Lambda \in [0,\infty)$ and $r_0 > 0$, such that

$$P(E; B_r(x)) \leq P(F; B_r(x)) + \Lambda|E\Delta F|\,,$$

whenever $E\Delta F$ is compactly contained in $B_r(x) \cap \Omega$ and $r < r_0$.

Trivially, a perimeter local minimizer (at a given scale r_0) is a (Λ, r_0)-minimizer for $\Lambda = 0$. On the other hand, notice that for $\Lambda > 0$ the term $\Lambda|E\Delta F|$ appears as a higher order perturbation of local perimeter minimality. Indeed at small scales r the perimeter $P(E; B_r(x))$ behaves like r^{n-1} while the term $\Lambda|E\Delta F|$ scales like r^n. Thus, a (Λ, r_0)-perimeter minimizer is "almost" a local perimeter minimizer with an increasing precision at increasingly smaller scales. Therefore, these minimizers have similar properties to local perimeter minimizers such as regularity of the boundary (see Section 2.3), density estimates and compactness properties.

An important example is given by the minimizers of the so-called prescribed mean curvature functional

$$\mathcal{F}[E] = P(E; \Omega) + \int_E g(x)\, dx\,, \tag{1.26}$$

defined on $E \subset \Omega$ for some bounded Borel function $g : \mathbb{R}^n \to \mathbb{R}$. Any minimizer E of $\mathcal{F}$ is a (Λ, r_0)-perimeter minimizer in Ω with $\Lambda = \|g\|_\infty$ and arbitrary r_0. Indeed, take any competitor F such that $E\Delta F \subset\subset B_r(x) \cap \Omega$, then

$$P(E; \Omega) \le P(F; \Omega) + \int_F g - \int_E g \le P(F; \Omega) + \|g\|_\infty |E\Delta F|\,.$$

Since F and E agree on $\Omega \setminus B_r(x)$ the claim follows. Now, notice that any minimizer of (1.26) have prescribed mean curvature H equal to $-g$. Indeed, expanding to the first order the volume term $\int_E g$ with respect to a one-parameter family of diffeomorphisms gives

$$\int_{f_t(E)} g = \int_E g + t \int_{\partial^* E} g(T \cdot \nu_E)\, d\mathcal{H}^{n-1}(x) + o(t)\,,$$

thus

$$\frac{d}{dt}\Big|_{t=0} \mathcal{F} = \int_{\partial^* E} H(T \cdot \nu_E)\, d\mathcal{H}^{n-1}(x) + \int_{\partial^* E} g(T \cdot \nu_E)\, d\mathcal{H}^{n-1}(x)\,.$$

Hence, any minimizer of (1.26) must be such that $H = -g$.

CHAPTER 2

Technical tools

2.1 The super reduced boundary

In this section we introduce a new notion in the context of Geometric Measure Theory: the *super-reduced boundary*. Roughly speaking, it is the subset of the reduced boundary of those points $\bar{x}$ such that the signed distance associated with subsequent scalings centered in $\bar{x}$ converge to the signed distance associated with the tangent half-space in $\bar{x}$. The following definition and theorems characterizing this new set have been firstly proved in [LS16].

From now on, we shall denote by $d_E(x)$ the signed distance function of E evaluated at x, that is

$$d_E(x) = \mathrm{dist}(x, E) - \mathrm{dist}(x, \mathbb{R}^n \setminus E)\,.$$

Definition 2.1 (Super-reduced boundary)**.** Let E be a set of locally finite perimeter in $\mathbb{R}^n$. We say that x belongs to the super-reduced boundary of E, and write $x \in \partial^{**}E$, if $x \in \partial^{*}E$ and, setting $\nu = \nu_E(x)$, one has that the signed distance function $d_{E_{x,r}}$ converges to the signed distance function d_{H_ν} associated with the tangent half-space to E at x in $C^0_{loc}(H_\nu)$, as $r \to 0$.

A useful and more geometric characterization of the super-reduced boundary is given in the next proposition.

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set with locally finite perimeter. Given $x \in \partial^*\Omega$ we denote by $H_x = H_{\nu_\Omega(x)}$ the tangent half-space to Ω at x. Then $x \in \partial^{**}\Omega$ if and only if for all $\varepsilon, R > 0$ there exists $r_0 > 0$ such that for all $0 < r < r_0$ and all $y \in \partial\Omega_{x,r} \cap H_x$ with $|y| \leq R$ we have that $-d_{H_x}(y) \leq \varepsilon|y|$.*

Proof. First we prove the "only-if" part. For notational convenience we write d_r instead of $d_{E_{x,r}}$. Take $x \in \partial^{**}\Omega$ and assume by contradiction that there exist $\varepsilon, R > 0$, an infinitesimal sequence of radii $\{r_k\}_{k\in\mathbb{N}}$ and a sequence of points $\{y_k\}_{k\in\mathbb{N}}$ such that $y_k \in \partial\Omega_{x,r_k} \cap H_x$ with

$|y_k| \le R$, for which $-d_{H_x}(y_k) > \varepsilon|y_k|$ for all k. We set $z_k = \frac{R}{|y_k|}y_k = \lambda_k y_k$ so that $|z_k| = R$ and $-d_{H_x}(z_k) > \varepsilon R$. Since $z_k \in \partial\Omega_{x,\lambda_k^{-1}r_k}$ we have $d_{\lambda_k^{-1}r_k}(z_k) = 0$ for all k. Up to subsequences we can assume that z_k converges to $\bar z$ as $k \to \infty$, with $|\bar z| = R$ and $-d_{H_x}(\bar z) \ge \varepsilon R$. Since $\lambda_k \ge 1$ by definition, the sequence $\lambda_k^{-1}r_k$ is infinitesimal and thus by hypothesis we have

$$\lim_{k\to\infty} -d_{\lambda_k^{-1}r_k}(\bar z) = -d_{H_x}(\bar z) \ge \varepsilon R\,. \tag{2.1}$$

On the other hand, since $d_{\lambda_k^{-1}r_k}$ is 1-Lipschitz for all k, we have

$$-d_{\lambda_k^{-1}r_k}(\bar z) \le -d_{\lambda_k^{-1}r_k}(z_k) + |\bar z - z_k| = |\bar z - z_k| \underset{k\to\infty}{\longrightarrow} 0\,,$$

which contradicts (2.1).

Then we prove the "if" part. We fix $R > 0$ and z such that $d_{H_x}(z) < 0$ and $|z| \le R$. Given $\varepsilon > 0$ there exists $r_0 > 0$ such that for all $0 < r < r_0$ and all $y \in \partial\Omega_{x,r} \cap H_x$ with $|y| \le 2R$ we have

$$-\,d_{H_x}(y) \le \varepsilon|y|\,. \tag{2.2}$$

Up to possibly reducing ε, we can assume that $-d_{H_x}(z) > \varepsilon|z|$. Since by Theorem 1.36(ii) we have $\chi_{\Omega_{x,r}} \to \chi_{H_x}$ in $L^1_{loc}(\mathbb{R}^n)$ as $r \to 0$, we deduce that $z \in \Omega_{x,r}$ for r small enough, otherwise there would exist points of $\partial\Omega_{x,r} \cap H_x$ getting closer and closer to z, in contradiction with (2.2). To prove that $x \in \partial^{**}\Omega$ we only need to show that the following inequalities hold:

(i) $d_r(z) \le d_{H_x}(z) + \varepsilon$;

(ii) $d_r(z) \ge d_{H_x}(z) - \varepsilon$.

Let us start by proving (i). We fix $\varepsilon' = \frac{2\varepsilon}{3R}$ and consider the points z_c and z_h defined as the projections of z, respectively, on the closed cone

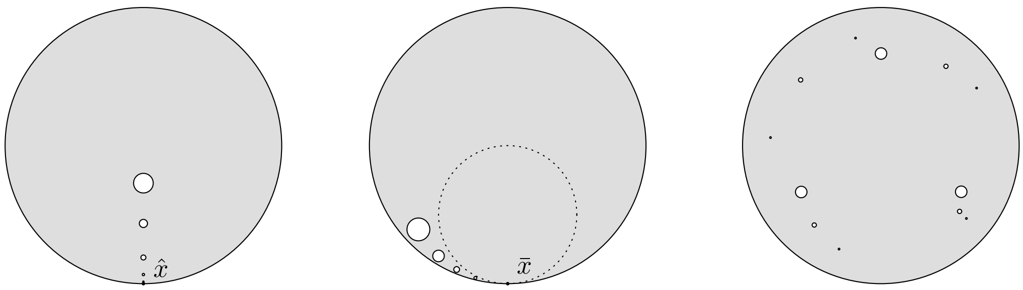
$$C_{\varepsilon'} = \{y \in \mathbb{R}^n :\ y\cdot\nu \ge -\varepsilon'\,|y|\}$$

and on the tangent half-space H_x (more precisely, z_c might be non-uniquely defined, however the following argument does not depend on the choice of this projection). We have then $-d_r(z) \ge |z - z_c|$ and $-d_{H_x}(z) = |z - z_h|$. Let us denote by c the intersection of the segment from z to z_h with the boundary of the cone $C_{\varepsilon'}$. The triangles of vertices z, z_c, c and $0, z_h, c$ are similar, therefore the following chain of inequalities holds true:

$$\begin{aligned}
-d_r(z) &\ge |z - z_c| = (1-{\varepsilon'}^2)^{\frac12}\big(|z - z_h| - \varepsilon'|c|\big)\\
&\ge (1-{\varepsilon'}^2)^{\frac12}\big(-d_{H_x}(z) - \varepsilon'|z|\big)\\
&\ge (1-{\varepsilon'}^2)^{\frac12}\big(-d_{H_x}(z) - R\varepsilon'\big) = -d_{H_x}(z) - R\varepsilon' + O({\varepsilon'}^2),
\end{aligned}$$

hence for $\varepsilon << 1$ one has

$$-d_r(z) \ge -d_{H_x}(z) - \frac{3}{2}R\varepsilon' = -d_{H_x}(z) - \varepsilon\,,$$



(a) The point $\hat{x} \in \partial B_1$ belongs to $\partial^*\Omega \setminus \partial^{**}\Omega$ since none of the cones with vertex $\hat{x}$ is "clean".

(b) The point $\bar{x} \in \partial B_1$ belongs to $\partial^{**}\Omega$ but $\partial\Omega$ is not Lipschitz in its neighborhood.

(c) This set is such that $\partial^*\Omega \setminus \partial^{**}\Omega = \partial B_1$ thus it has positive $\mathcal{H}^1$ measure.

Figure 2.1: The above figures display the different situations that might happen according to Remark 2.3

which yields (i).
Let us now prove (ii). We set $\rho_\varepsilon = -d_{H_x}(z) + \varepsilon$ and consider the ball $B_{\rho_\varepsilon}(z)$, for which one necessarily has

$$|(B_{\rho_\varepsilon}(z) \setminus H_x) \cap B_R| = m_{\varepsilon,z} > 0\,. \tag{2.3}$$

Since $\chi_{\Omega_{x,r}} \to \chi_{H_x}$ in L^1_{loc} for r small enough we have

$$|(\Omega_{x,r} \setminus H_x) \cap B_R| < m_{\varepsilon,z}. \tag{2.4}$$

Then, the fact that $z \in \Omega_{x,r}$ combined with (2.3) and (2.4) implies $B_{\rho_\varepsilon}(z) \nsubseteq \Omega_{x,r}$, so that there exists $y \in \partial\Omega_{x,r} \cap B_{\rho_\varepsilon}(z)$. This shows (ii) and concludes the proof of the proposition. □

Remark 2.3. By definition one has $\partial^{**}\Omega \subset \partial^*\Omega$. We remark that this inclusion might be strict, see Figure 2.1(a) showing a point $\hat{x} \in \partial^*\Omega \setminus \partial^{**}\Omega$. It can be easily proved that if $\partial\Omega$ is Lipschitz then $\partial^{**}\Omega = \partial^*\Omega$, while the converse is not true in general like in Figure 2.1(b). Indeed one can consider the unit, open disk $B_1 \subset \mathbb{R}^2$ and a sequence $\{B_i = B_{r_i}(c_i)\}_{i\in\mathbb{N}}$ of mutually disjoint, open disks with the following properties: $B_i \subset B_1 \setminus B_{1/2}((0,-1/2))$, $\sum_{i=1}^\infty r_i < +\infty$ and $c_i \to (0,-1)$ as $i \to \infty$. Then setting

$$\Omega = B_1 \setminus \bigcup_{i=1}^\infty \overline{B_i}$$

one can check that Ω is open, bounded, with finite perimeter, and such that $\partial^{**}\Omega = \partial^*\Omega$. However Ω is not globally Lipschitz: in particular, its boundary cannot be represented as the graph of a function in any neighborhood of the point $\bar{x} = (0,-1)$. On the other hand, one can easily construct examples of domains Ω such that $\mathcal{H}^{n-1}(\partial^*\Omega \setminus \partial^{**}\Omega) > 0$. For instance, for every $n \geq 2$ one can take a regular n-gon inscribed in the disk $B_{1-1/n}$ and remove from B_1 all closed balls of radius $r_n = \frac{1}{n2^n}$ centered at each vertex of the n-gon (see Figure 2.1(c)). The resulting set Ω has finite perimeter since

$$P(\Omega) \leq 2\pi \left(1 + \sum_{n=3}^{\infty} \frac{1}{2^n}\right) < \infty\,.$$

Moreover, $\partial B_1 \subset \partial^*\Omega$ as for any fixed $x \in \partial B_1$ and any fixed r one has the estimates

$$|B_r(x)|/2 \geq |B_r(x) \cap \Omega| \geq |B_r(x)|/2 - \sum_{n \geq 1/r+1} n|B_{(n2^n)^{-1}}| = |B_r(x)|/2 - o(r^2)\,,$$

where the last equality follows because

$$\sum_{n \geq 1/r+1} n|B_{(n2^n)^{-1}}| \leq \pi \sum_{n \geq 1/r} \frac{1}{n2^{2n}} \leq \pi r \sum_{n \geq 1/r} \frac{1}{2^n}\frac{1}{2^n} \leq \pi r^3 \sum_{n \geq 1/r} \frac{1}{2^n}$$

Finally, $\partial B_1 \cap \partial^{**}\Omega = \emptyset$. Fix $\bar{x} \in \partial B_1$. Up to a rotation and translation we can suppose $\bar{x}$ is the origin and the center of the ball is $c = (0,1)$. A generic cone then is given by $(x, \lambda|x|)$, and we localize in B_r. For the given r, for any n such that $1/n \leq r$ the ball $B_{1-1/n}(c)$ intersects the cone in two points p_-, p_+ symmetric with respect to the y-axis. The chord theorem ensures that the angle θ at the origin formed by these two points is such that

$$|p_+ - p_-| = 2\left(1 - \frac{1}{n}\right) \sin\left(\frac{1}{2}\theta\right) \geq \left(1 - \frac{1}{n}\right)\frac{\theta}{2}$$

On the other hand, recalling that $p_+ = (x, \lambda x)$ for some $x > 0$ and that $|p_+ - c| = 1 - 1/n$ a straightforward computation gives

$$|p_+ - p_-| = \frac{2}{1+\lambda^2}\frac{1}{n} + o\left(\frac{1}{n}\right)\,.$$

Thus, the n-polygon inscribed in $B_{1-1/n}(c)$ has for sure one vertex in B_r if the following inequality holds

$$\theta \leq \frac{2n}{n-1}|p_+ - p_-| = \frac{4}{1+\lambda^2} \cdot \frac{1}{n-1} + o\left(\frac{1}{n}\right) \leq \frac{2\pi}{n}\,,$$

which is true for $n >> 1$.

2.2 A Gauss-Green theorem for vector fields

Very general forms of the Gauss-Green Theorem have been obtained, see for instance [DG61, Fed45, Fed58], [BM69, Vol67], [Anz83, Zie83], and [Pfe05, DPP04]. We recall in particular the extensions of the divergence theorem for bounded, divergence-measure vector fields on sets with finite perimeter [CF99, CF03, CT05, CTZ09]. These last results rely on a notion of *weak normal trace* of a bounded, divergence-measure vector field ξ on the reduced boundary of E, where $E \subset\subset \Omega$ is a set of finite perimeter and Ω is the domain of the vector field. This notion of trace already appears in [Anz83], in the special case of E being an open bounded set with Lipschitz boundary. A crucial tool used in [CTZ09] is the approximation of E by smooth sets which are "mostly" contained in the measure-theoretic interior of E with respect to the measure $\mu = \operatorname{div} \xi$. Actually, this is the main reason why E needs to be compactly contained in the domain of the vector field ξ. On the other hand, if such a domain Ω has finite perimeter and $P(\Omega) = \mathcal{H}^{n-1}(\partial\Omega)$ then one can consider the vector field $\hat{\xi}$ defined as $\hat{\xi} = \xi$ on Ω and $\hat{\xi} = 0$ on $\mathbb{R}^n \setminus \Omega$, so that by relying on Theorem 1.49 it is possible to show that $\operatorname{div} \hat{\xi}$ is a finite measure on $\mathbb{R}^n$. Then by applying [CTZ09, Theorem 25.1] one might show the validity of the divergence theorem for the field ξ on $E = \Omega$, which in turn leads to the generalized Gauss-Green formula

$$\int_\Omega \varphi \operatorname{div} \xi + \int_\Omega \nabla\varphi \cdot \xi = \int_{\partial\Omega} \varphi\, [\xi \cdot \nu]\, d\mathcal{H}^{n-1}\,, \tag{2.5}$$

where ν is the exterior weak normal to $\partial^*\Omega$, $[\xi \cdot \nu]$ denotes the weak normal trace of ξ on $\partial^*\Omega$, and $\varphi \in C_c^\infty(\mathbb{R}^n)$. In this section, we give a very direct proof of (2.5) when $\Omega \subset \mathbb{R}^n$ is *weakly regular*, accordingly to the next defintion.

Definition 2.4. An open, bounded set of finite perimeter $\Omega \subset \mathbb{R}^n$ is *weakly regular* if

$$P(\Omega) = \mathcal{H}^{n-1}(\partial\Omega) \tag{2.6}$$

and there exists a constant $k = k(\Omega) > 0$ such that

$$\min\{P(E; \Omega^c), P(\Omega \setminus E; \Omega^c)\} \le k\, P(E; \Omega)\,. \tag{2.7}$$

The task of proving (2.5) will be accomplished by adapting the construction proposed by Anzellotti in [Anz83] (see also the recent papers [BS15, SS16]). More precisely, we will show in Theorem 2.7 that for every vector field $\xi \in L^\infty(\Omega;\mathbb{R}^n) \cap C^0(\Omega;\mathbb{R}^n)$ with divergence in $L^\infty(\Omega)$ there exists a function $[\xi\cdot\nu] \in L^\infty(\partial\Omega)$ such that (2.5) holds for any $\varphi \in BV(\Omega)$. The assumptions made on Ω (i.e. (2.6) and (2.7)) and on the vector field ξ are tuned to reflect those satisfied by Tu for a solution u in a weakly regular set Ω of (PMC), the Prescribed Mean Curvature problem which has been introduced in Section 1.6 and will be extensively studied in Chapter 4. The results of this section have been firstly proved in [LS16], a paper whose main interest was the Prescribed Mean Curvature equation.

In this section, on top of proving the validity of (2.5), we shall prove that the weak normal trace is a proper extension of the normal component of the usual trace of ξ on $\partial\Omega$ whenever such a trace is defined in a measure theoretic sense, as shown in Proposition 2.8. Notice that in general, the value of $[\xi \cdot \nu](z)$ cannot be understood as the measure-theoretic limit of the normal component of ξ, as shown by Example 2.9. We are therefore legitimate to ask whether some conditions on the weak normal trace and on the boundary of Ω are enough to ensure that ξ admits a classical trace at $\mathcal{H}^{n-1}$-almost every point of $\partial\Omega$. We answer this question in Theorem 2.11, which in turn relies on a key result (Theorem 2.10) and on the notion of *super-reduced boundary* of Ω introduced in Section 2.1.

Finally, we stress that all open, bounded and connected Lipschitz sets, as well as some sets with inner cusps or with some controlled porosity (see for instance Section 4.5), are weakly regular and therefore (2.5) holds on them.

2.2.1. The weak normal trace

Before delving into the proof of the Gauss-Green formula, we recall the following theorem which is the characterization of $W_0^{1,1}(\Omega)$ as the space of functions in $W^{1,1}(\Omega)$ having zero trace at $\partial\Omega$ which holds independently from the regularity of the boundary.

Theorem 2.5 ([Swa07, Theorem 5.2]). *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in W^{1,1}(\Omega)$. Then, $u \in W_0^{1,1}(\Omega)$ if and only if*

$$\lim_{r\to 0} \frac{1}{r^n} \int_{B_r(x)\cap\Omega} |u(y)|\, dy = 0$$

for $\mathcal{H}^{n-1}$-almost all $x \in \partial\Omega$.

Let $\Omega \subset \mathbb{R}^n$ be a weakly regular connected set, i.e. an open, bounded set with finite perimeter satisfying (2.6) and (2.7). We denote by $X(\Omega)$ the collection of vector fields $\xi \in L^\infty(\Omega;\mathbb{R}^n) \cap C^0(\Omega;\mathbb{R}^n)$ such that $\operatorname{div}\xi \in L^\infty(\Omega)$. For every $u \in BV(\Omega)$ we define the pairing

$$\langle \xi, u\rangle_{\partial\Omega} = \int_\Omega u \operatorname{div}\xi + \int_\Omega \xi \cdot Du\,. \tag{2.8}$$

The map $\langle\cdot,\cdot\rangle_{\partial\Omega} : X(\Omega)\times BV(\Omega) \to \mathbb{R}$ is bilinear. Under the assumption Ω connected and weakly regular, Theorem 1.53 holds, thus if $u, v \in W^{1,1}(\Omega)$ have the same trace on $\partial\Omega$ then by Theorem 2.5 we infer that $u - v$ is in $W_0^{1,1}$, hence there exists a sequence $\{g_j\}$ of functions in $C_c^\infty(\Omega)$ such that $g_j \to u - v$ weakly in $W^{1,1}(\Omega)$, so that we have

$$\begin{aligned}
\langle \xi, u - v\rangle_{\partial\Omega} &= \int_\Omega (u-v)\operatorname{div}\xi + \int_\Omega \xi\cdot D(u-v) \\
&= \lim_j \int_\Omega g_j \operatorname{div}\xi + \int_\Omega \xi\cdot\nabla g_j = \lim_j \int_{\partial\Omega_j} g\xi\cdot\nu_j = 0\,,
\end{aligned}$$

where $\Omega_j \subset\subset \Omega$ is a smooth set containing the support of g_j. This shows that the pairing defined in (2.8) only depends on the trace of u on $\partial\Omega$. Then by Theorem 1.54 (ii) (originally proven in [AG78]) we infer that $\langle \xi, u\rangle_{\partial\Omega} = \langle \xi, v\rangle_{\partial\Omega}$ whenever $u, v \in BV(\Omega)$ have the same trace on $\partial\Omega$.

At this point we can show the continuity of the pairing (2.8) in the topology of $L^\infty(\Omega;\mathbb{R}^n)\times L^1(\partial\Omega)$. The following, key lemma extends [Anz83, Lemma 5.5]).

Lemma 2.6. *Let Ω be connected and weakly regular. Then for every $u \in L^1(\partial\Omega)$ and $\varepsilon > 0$ there exists $w_\varepsilon \in BV(\Omega)\cap C^\infty(\Omega)$ such that*

(i) the trace of w_ε on $\partial\Omega$ equals u $\mathcal{H}^{n-1}$-almost everywhere on $\partial\Omega$,

(ii) $\int_\Omega |\nabla w_\varepsilon| \le \int_{\partial\Omega} |u| + \varepsilon$,

(iii) $w_\varepsilon(x) = 0$ *whenever* $\operatorname{dist}(x, \partial\Omega) > \varepsilon$,

(iv) $\int_\Omega |w_\varepsilon| \leq \varepsilon$,

(v) $\|w_\varepsilon\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\partial\Omega)}$.

Proof. Let us fix $\varepsilon > 0$. By Theorem 1.54 (ii) there exists $\Psi \in W^{1,1}(\mathbb{R}^n)$ such that its trace on $\partial\Omega$ coincides with u. Up to an application of Meyer-Serrin's approximation theorem, we can additionally assume that $\Psi \in C^\infty(\Omega)$. Moreover we fix a sequence $\{\Psi_j\}_j$ of smooth functions such that $\|\Psi - \Psi_j\|_{W^{1,1}(\mathbb{R}^n)} \to 0$ as $j \to \infty$. Again by Theorem 1.54 (ii) we have that the trace operator from $BV(\Omega)$ to $L^1(\partial\Omega)$ is continuous, hence

$$\int_{\partial\Omega} |\Psi_j| \, d\mathcal{H}^{n-1} \to \int_{\partial\Omega} |\Psi| \, d\mathcal{H}^{n-1} = \int_{\partial\Omega} |u| \, d\mathcal{H}^{n-1} \qquad \text{as } j \to \infty.$$

Given $\delta, \eta > 0$ we define $\chi_{\delta,\eta}(x) = \chi_{\Omega_\delta} * \rho_\eta(x)$, where ρ_η is a standard symmetric mollifier with support in $B_\eta(0)$, while $\Omega_\delta \subset\subset \Omega$ is obtained in virtue of Theorem 1.49, so that the Hausdorff distance between $\partial\Omega_\delta$ and $\partial\Omega$ is smaller than δ and $|P(\Omega_\delta) - P(\Omega)| \leq \delta$. We note that up to choosing δ and η small enough we get $\operatorname{spt}(\chi_{\delta,\eta}) \subset\subset \Omega$, $\chi_{\delta,\eta} = 1$ on the set $\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$, and $\left| \int_\Omega |\nabla \chi_{\delta,\eta}| - P(\Omega_\delta) \right| \leq \delta$. Then we define $w_{\delta,\eta}(x) = \Psi(x)(1 - \chi_{\delta,\eta}(x))$ and, for any fixed vector field $g \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ with $\|g\|_\infty \leq 1$ and compact support in Ω, up to choosing δ and η small enough as well as j sufficiently large we obtain

$$\begin{aligned}
\int_\Omega \nabla w_{\delta,\eta} \cdot g \, dx &= \int_\Omega (1 - \chi_{\delta,\eta}) \, \nabla \Psi \cdot g \, dx - \int_\Omega \Psi \, \nabla \chi_{\delta,\eta} \cdot g \, dx \\
&\leq \int_\Omega (1 - \chi_{\delta,\eta}) \, |\nabla \Psi| - \int_\Omega \Psi_j \, \nabla \chi_{\delta,\eta} \cdot g \, dx - \int_\Omega (\Psi - \Psi_j) \, \nabla \chi_{\delta,\eta} \cdot g \, dx \\
&\leq \frac{\varepsilon}{4} + \int_\Omega |\Psi_j| \, |\nabla \chi_{\delta,\eta}| \, dx + \int_\Omega \chi_{\delta,\eta} \Big(\nabla(\Psi - \Psi_j) \cdot g + (\Psi - \Psi_j) \operatorname{div} g \Big) \, dx \\
&\leq \frac{\varepsilon}{4} + \int_\Omega |\Psi_j| \, |\nabla \chi_{\delta,\eta}| \, dx + (1 + \|\operatorname{div} g\|_\infty) \int_\Omega \Big(|D(\Psi - \Psi_j)| + |\Psi - \Psi_j| \Big) \, dx \\
&\leq \int_\Omega |\Psi_j| \, |\nabla \chi_{\delta,\eta}| \, dx + \frac{\varepsilon}{2} \leq \int |\Psi_j| \, d|D\chi_\Omega| + \frac{3}{4}\varepsilon \leq \int_{\partial\Omega} |u| \, d\mathcal{H}^{n-1} + \varepsilon \,.
\end{aligned}$$

We finally set $w_\varepsilon = w_{\delta,\eta}$ and, by taking the supremum over g, we find

$$\int_\Omega |\nabla w_\varepsilon| \, dx \leq \int_{\partial\Omega} |u| \, d\mathcal{H}^{n-1} + \varepsilon \,,$$

which proves (ii). Finally, (i), (iii) and (v) are immediate from the construction, while (iv) is easily shown to hold up to possibly taking smaller δ and η. □

Now, given $\varepsilon > 0$ and $\varphi \in BV(\Omega) \cap L^\infty(\Omega)$, taking w_ε as in Lemma 2.6 (with $u = \varphi$ on $\partial\Omega$), and setting $\Omega_\varepsilon = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \geq \varepsilon\}$ we obtain

$$\begin{aligned}
|\langle \xi, \varphi \rangle_{\partial\Omega}| &= |\langle \xi, w_\varepsilon \rangle_{\partial\Omega}| \\
&\leq \|\varphi\|_{L^\infty(\Omega)} \int_{\Omega\setminus\Omega_\varepsilon} |\mathrm{div}\,\xi| \; + \|\xi\|_{L^\infty(\Omega)} \int_\Omega |\nabla w_\varepsilon| \\
&\leq \|\varphi\|_{L^\infty(\Omega)} \int_{\Omega\setminus\Omega_\varepsilon} |\mathrm{div}\,\xi| \; + \|\xi\|_{L^\infty(\Omega)} \left(\int_{\partial\Omega} |\varphi| \; + \varepsilon \right),
\end{aligned}$$

which by the arbitrary choice of ε leads to

$$|\langle \xi, \varphi \rangle_{\partial\Omega}| \leq \|\xi\|_{L^\infty(\Omega)} \int_{\partial\Omega} |\varphi| \,. \tag{2.9}$$

One can check by a truncation argument that (2.9) holds for each $\varphi \in BV(\Omega)$. An immediate consequence of (2.9) is the fact that the linear functional $N_\xi : L^1(\partial\Omega) \to \mathbb{R}$ defined as $N_\xi(u) = \langle \xi, u \rangle_{\partial\Omega}$ is continuous on $L^1(\partial\Omega)$, thus it can be represented by a function in $L^\infty(\partial\Omega)$, hereafter denoted by $[\xi \cdot \nu]$. This function is the so-called *weak normal trace* of the vector field $\xi \in X(\Omega)$ on $\partial\Omega$. Another immediate consequence of (2.9) is the following L^∞-estimate of the weak normal trace:

$$\|[\xi \cdot \nu]\|_{L^\infty(\partial\Omega)} \leq \|\xi\|_{L^\infty(\Omega)} \,. \tag{2.10}$$

Summing up, we have proved that (2.8) can be rewritten in the form of the generalized Gauss-Green formula stated in the next theorem.

Theorem 2.7. *Let $\Omega \subset \mathbb{R}^n$ be connected and weakly regular. Let $\xi \in X(\Omega)$ and $\varphi \in BV(\Omega)$, then*

$$\int_\Omega \varphi \,\mathrm{div}\,\xi \; + \int_\Omega \xi \cdot D\varphi = \int_{\partial\Omega} \varphi \,[\xi \cdot \nu] \, d\mathcal{H}^{n-1} \,. \tag{2.11}$$

The next proposition shows that the weak normal trace is a proper extension of the normal component of the usual trace of ξ on $\partial\Omega$, whenever such a trace exists in measure-theoretic sense.

Proposition 2.8. *Let $\Omega \subset \mathbb{R}^n$ be connected and weakly regular. Let $\xi \in X(\Omega)$ and let $z \in \partial^*\Omega$ be a Lebesgue point for the weak normal trace $[\xi \cdot \nu]$. Assume*

$$\operatorname*{ap-lim}_{x \to z} \xi(x) = w \,, \tag{2.12}$$

then

$$[\xi \cdot \nu](z) = w \cdot \nu(z) \,. \tag{2.13}$$

Proof. We can assume that $z = 0$ up to a translation. We fix a sequence $r_i \downarrow 0$ as $i \to \infty$. Given any function (or vector field) f defined in Ω, we set

$$\Omega_i = r_i^{-1}\Omega, \qquad f_i(y) = f(r_i y) \,.$$

We note that $Df_i(y) = r_i Df(r_i y)$ in the sense of distributions. By (2.12) we infer that for all $\alpha > 0$ the set

$$N_i(\alpha) = r_i^{-1} N(\alpha) = \{y \in \Omega_i : \; |\xi_i(y) - w| \geq \alpha\}$$

satisfies

$$\lim_{i\to\infty} |N_i(\alpha)\cap B_1| = 0\,. \tag{2.14}$$

On the other hand, the fact that $z=0$ is by assumption a Lebesgue point for $[\xi\cdot\nu]$ implies that

$$[\xi\cdot\nu](0) = \lim_{i\to\infty} \mu_i^{-1}\int_{\partial\Omega_i\cap B_1} [\xi\cdot\nu]_i(y)\,d\mathcal{H}^{n-1}(y)\,, \tag{2.15}$$

where $\mu_i = \mathcal{H}^{n-1}(\partial\Omega_i\cap B_1)$. Now we take $\delta\in(0,1)$ and set $\alpha=\delta^2$ and

$$\varphi(y) = \max(0,\min(1,(1-|y|)/\delta))\,.$$

By Theorem 1.36(ii), setting $H = H_{\nu(0)}$ for brevity, we obtain

$$\left|\int_{\Omega_i\cap B_1} D\varphi(x)\,dx - \int_{H\cap B_1} D\varphi(x)\,dx\right| \le \delta^{-1}|(\Omega_i\Delta H)\cap B_1| = m_i(\delta)\to 0 \qquad \text{as } i\to\infty\,. \tag{2.16}$$

Moreover by Theorem 2.7 we get for a suitable constant $C>0$

$$\begin{aligned}\left|\int_{H\cap B_1} D\varphi(x)\,dx - \omega_{n-1}\nu(0)\right| &= \left|\int_{\partial H\cap B_1} \varphi(x)\,d\mathcal{H}^{n-1}(x) - \omega_{n-1}\right|\\ &= \omega_{n-1}\int_0^1 [1-(1-\delta t)^{n-1}]\,dt\\ &\le C\delta\,.\end{aligned} \tag{2.17}$$

Then by (2.15), (2.16), (2.17), and Theorem 1.36(ii), we find

$$\begin{aligned}\omega_{n-1}\Big|[\xi\cdot\nu](0) - w\cdot\nu(0)\Big| &\le \left|\lim_{i\to\infty}\int_{\partial\Omega_i\cap B_1}[\xi\cdot\nu]_i\,d\mathcal{H}^{n-1} - w\cdot\int_{H\cap B_1} D\varphi(x)\,dx\right| + C\delta\\ &\le \limsup_{i\to\infty}\left|\int_{\Omega_i\cap B_1}\varphi\,\mathrm{div}\,\xi_i\right| + \left|\int_{\Omega_i\cap B_1}(\xi_i - w)\cdot D\varphi\right| + m_i(\delta) + C\delta\\ &= \limsup_{i\to\infty}\big(A_i + B_i + m_i(\delta)\big) + C\delta\,.\end{aligned} \tag{2.18}$$

Then we notice that $A_i + m_i(\delta)\to 0$ as $i\to\infty$, while

$$\begin{aligned}B_i &= \left|\int_{(\Omega_i\cap B_1)\setminus N_i(\alpha)}(\xi_i - w)\cdot D\varphi + \int_{N_i(\alpha)\cap B_1}(\xi_i - w)\cdot D\varphi\right|\\ &\le \frac{\omega_n\alpha}{\delta} + \frac{2\|\xi\|_\infty}{\delta}|N_i(\alpha)\cap B_1|\\ &\le \omega_n\delta + \frac{2\|\xi\|_\infty}{\delta}|N_i(\alpha)\cap B_1|\,.\end{aligned}$$

Therefore by passing to the limit as $i\to\infty$ in (2.18) and using (2.14) we finally get

$$\omega_{n-1}\Big|[\xi\cdot\nu](0) - w\cdot\nu(0)\Big| \le (\omega_n + C)\delta\,,$$

which implies (2.13) at once by the arbitrary choice of $\delta\in(0,1)$. □

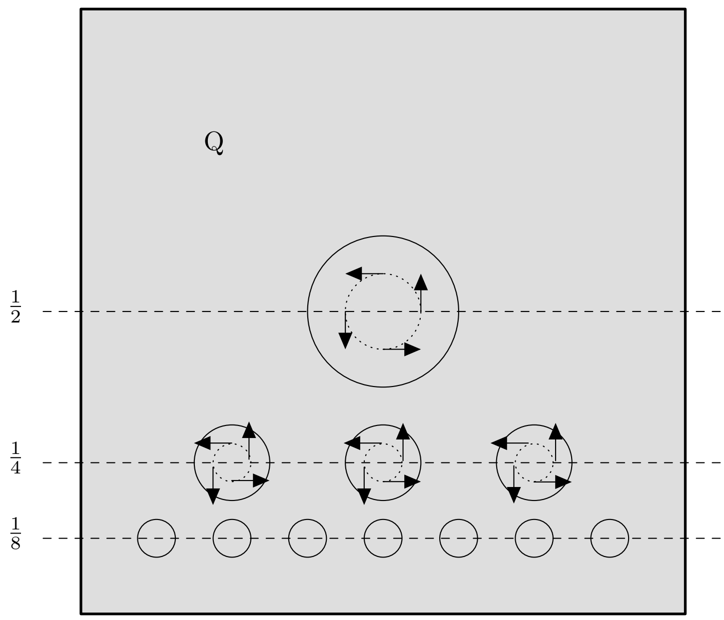


Figure 2.2: The twisting vector field ξ constructed in Example 2.9

We stress that the weak normal trace $[\xi \cdot \nu]$ of a vector field $\xi \in X(\Omega)$ at $x \in \partial\Omega$ does not necessarily coincides with any pointwise, almost-everywhere, or measure-theoretic limit of the scalar product $\xi(y) \cdot \nu(x)$, as $y \to x$. To see this one can consider the following example.

Example 2.9. Take the unit square $Q = (0,1)^2 \subset \mathbb{R}^2$ and for $i \geq 1$ and $j = 1, \ldots, 2^i - 1$ set $x_{ij} = \frac{j}{2^i}$, $y_i = \frac{1}{2^i}$, $r_i = \frac{1}{2^{i+2}}$. Then, for such i and j we take $\varphi_i \in C_c^\infty(\mathbb{R})$ with compact support in $(0, r_i)$, so that in particular $\varphi_i(0) = \varphi_i(r_i) = 0$, and define $p_{ij} = (x_{ij}, y_i) \in Q$. Notice that by our choice of parameters, the balls $\{B_{ij} = B_{r_i}(p_{ij})\}_{i,j}$ are pairwise disjoint. Whenever $p \in B_{ij}$ we set

$$\xi(p) = \varphi_i(|p - p_{ij}|)(p - p_{ij})^\perp \,,$$

while $\xi(p) = 0$ otherwise (see Figure 2.2). One can suitably choose φ_i so that $\xi \in C^\infty(Q) \cap L^\infty(Q)$ and $\|\xi\|_{L^\infty(B_{ij})} = 1$ for all i, j. Moreover $\operatorname{div} \xi = 0$ on Q and thus for any $f \in W^{1,1}(Q)$, by Theorem 2.7 and owing to the definition of ξ, one has

$$\int_{\partial Q} f\,[\xi \cdot \nu]\, d\mathcal{H}^1 = \int_Q \xi \cdot Df = \sum_{i,j} \int_{B_{ij}} \xi \cdot Df = \sum_{i,j} \int_{\partial B_{ij}} f\,\xi \cdot \nu_{ij} = 0\,,$$

so that $[\xi \cdot \nu] = 0$ on ∂Q. At the same time, ξ twists in any neighborhood of any point $p_0 = (x_0, 0)$,

$x_0 \in (0,1)$. In conclusion, the scalar product $\xi(p)\cdot\nu(p_0)$ does not converge to 0 in any pointwise or measure-theoretic sense.

In general one should expect weak-type convergence of the normal component of the trace, as this corresponds to the weak-star convergence of the measures associated to the (classical) normal trace of ξ on the boundaries of subdomains of Ω that converge to Ω in measure and perimeter (in the sense of Theorem 1.49).

Nevertheless, in Theorem 2.10 below we characterize the behavior of a field $\xi \in X(\Omega)$ near $\partial\Omega$ in the case when its weak normal trace $[\xi\cdot\nu]$ attains the maximal value $\|\xi\|_{L^\infty(\Omega)}$ at $\mathcal{H}^{n-1}$-almost all $x \in \partial\Omega$. Indeed, in this case we expect that ξ cannot oscillate too much near $\partial\Omega$. For technical reasons we need to make a further, even though still mild, assumption on Ω. Indeed we require that $P(\Omega)$ coincides with the inner Minkowski content $\mathcal{M}_-(\Omega)$ (see Definition 1.44). Under this extra assumption, Theorem 2.10 establishes a close correlation between vector fields in $X(\Omega)$ admitting a maximal weak normal trace on $\partial\Omega$ and the gradient of the signed distance function d_Ω.

Theorem 2.10. *Let Ω be connected and weakly regular, with $P(\Omega) = \mathcal{M}_-(\Omega)$. Let $\xi \in X(\Omega)$ be such that $[\xi\cdot\nu](z) = \|\xi\|_{L^\infty(\Omega)}$ for $\mathcal{H}^{n-1}$-almost all $z \in \partial\Omega$. Then, denoting by $d_\Omega(x)$ the signed distance function of Ω, for every $t > 0$ and for $\mathcal{H}^{n-1}$-almost all $z \in \partial\Omega$ the set*

$$N(t) = \{x \in \Omega : \ \xi(x)\cdot\nabla d_\Omega(x) < \|\xi\|_{L^\infty(\Omega)} - t\}$$

has zero density at z, i.e., it satisfies

$$\lim_{r\to 0} \frac{|N(t)\cap B(z,r)|}{r^n} = 0. \tag{2.19}$$

Proof of Theorem 2.10. Up to a renormalization we assume that $|\xi| \le 1$ on Ω and that $[\xi\cdot\nu] = 1$ almost everywhere on $\partial\Omega$. We fix $\varepsilon > 0$ and set $\varphi_\varepsilon(x) = \big(1 + \frac{d_\Omega(x)}{\varepsilon}\big)_+$. Owing to the hypotheses, the Gauss-Green Theorem 2.7 holds, thus

$$\int_\Omega \varphi_\varepsilon \operatorname{div}\xi + \int_\Omega \xi\cdot D\varphi_\varepsilon = \int_{\partial\Omega} [\xi\cdot\nu]\, d\mathcal{H}^{n-1} = P(\Omega)\,.$$

We observe that $\varphi_\varepsilon = 0$ on $[\Omega]^\varepsilon = \{x \in \Omega : \ \operatorname{dist}(x,\partial\Omega) > \varepsilon\}$ and that $\nabla\varphi_\varepsilon(x) = \frac{\nabla d_\Omega(x)}{\varepsilon}$ for almost all $x \in \Omega \setminus [\Omega]^\varepsilon$, so that we get

$$
\begin{aligned}
P(\Omega) &= \frac{1}{\varepsilon}\left(\int_{\Omega\setminus[\Omega]^\varepsilon} \xi \cdot \nabla d_\Omega\right) + \eta_\varepsilon \\
&= \frac{1}{\varepsilon}\left(\int_{N(t)\setminus[\Omega]^\varepsilon} \xi \cdot \nabla d_\Omega + \int_{\Omega\setminus(N(t)\cup[\Omega]^\varepsilon)} \xi \cdot \nabla d_\Omega\right) + \eta_\varepsilon \\
&\le \frac{(1-t)|N(t)\setminus[\Omega]^\varepsilon| + |\Omega\setminus(N(t)\cup[\Omega]^\varepsilon)|}{\varepsilon} + \eta_\varepsilon \\
&= \frac{|\Omega\setminus[\Omega]^\varepsilon|}{\varepsilon} - t\frac{|N(t)\setminus[\Omega]^\varepsilon|}{\varepsilon} + \eta_\varepsilon \\
&\le P(\Omega) - t\frac{|N(t)\setminus[\Omega]^\varepsilon|}{\varepsilon} + 2\eta_\varepsilon\,,
\end{aligned} \tag{2.20}
$$

where η_ε is infinitesimal as $\varepsilon \to 0$. Clearly (2.20) implies that

$$
\frac{|N(t)\setminus[\Omega]^\varepsilon|}{\varepsilon} \le \frac{2\eta_\varepsilon}{t}\,. \tag{2.21}
$$

Now we argue by contradiction assuming that there exist $t, \tau > 0$, a decreasing sequence $(r_i)_i$, and a sequence of compact sets $M_i \subset \partial\Omega$ with $\mathcal{H}^{n-1}(M_i) > \tau$ for all i, such that

$$
|N(t)\cap B_{r_i}(z)| \ge \tau\, r_i^n \qquad \forall\, z \in M_i,\ \forall\, i \in \mathbb{N}\,.
$$

We notice that

$$
\begin{aligned}
\tau^2 r_i^n &\le \int_{M_i} |N(t)\cap B_{r_i}(z)|\, d\mathcal{H}^{n-1}(z) \\
&\le \int_{\partial\Omega}\int_{\mathbb{R}^n} \chi_{N(t)}(x)\cdot\chi_{B_{r_i}(z)}(x)\, dx\, d\mathcal{H}^{n-1}(z) \\
&= \int_{\{x:\ d(x)\ge -r_i\}} \chi_{N(t)}(x)\mathcal{H}^{n-1}(\partial\Omega\cap B_{r_i}(z))\, dx\,.
\end{aligned} \tag{2.22}
$$

By writing (2.7) with $E = \Omega\cap B_r(x)$, and by noticing that the minimum in (2.7) is realized by $P(E;\mathbb{R}^n\setminus\Omega)$ whenever $0 < r < r_\Omega$, with

$$
r_\Omega = \left(\frac{|\Omega|}{\omega_n\Big((k+1)^{\frac{n}{n-1}}+1\Big)}\right)^{\frac{1}{n}},
$$

we obtain the upper density estimate

$$
P(\Omega; B_r(x)) \le nk\omega_n r^{n-1} \tag{2.23}
$$

valid for all $x \in \mathbb{R}^n$ and $0 < r < r_\Omega$. Now, by (2.22) combined with (2.23) and (2.6) we get

$$
\tau^2 r_i^n \le C r_i^{n-1}\int_{\{x:\ d(x)\ge -r_i\}} \chi_{N(t)}(x)\, dx \le C r_i^{n-1}|N(t)\setminus[\Omega]^{r_i}|
$$

for a suitable constant $C > 0$, and thus by (2.21) we find

$$\frac{\tau^2}{C} \le \frac{|N(t) \setminus [\Omega]^{r_i}|}{r_i} \le \frac{2\eta_{r_i}}{t}\,,$$

that is, a contradiction with the infinitesimality of η_{r_i} as $i \to \infty$. This proves (2.19), as wanted. $\square$

The next theorem shows that, under the extra assumptions $P(\Omega) = \mathcal{M}_-(\Omega)$ and $[\xi \cdot \nu](x_0) = \|\xi\|_{L^\infty(\Omega)}$ with $x_0 \in \partial^{**}\Omega$, the reverse of Proposition 2.8 holds.

Theorem 2.11. *Let $\Omega \subset \mathbb{R}^n$ be connected and weakly regular. We additionally assume that $P(\Omega) = \mathcal{M}_-(\Omega)$. Then for any $\xi \in X(\Omega)$ and for $\mathcal{H}^{n-1}$-almost every $x_0 \in \partial^{**}\Omega$ such that $[\xi \cdot \nu](x_0) = \|\xi\|_{L^\infty(\Omega)}$, we have*

$$\operatorname*{ap-lim}_{x \to x_0} \xi(x) = \|\xi\|_{L^\infty(\Omega)} \nu(x_0). \tag{2.24}$$

Proof. Let $x_0 \in \partial^{**}\Omega$. For any $r > 0$ we take as usual $\Omega_{x_0,r} = r^{-1}(\Omega - x_0)$ and, for notational simplicity, we denote by $d_r(z)$ the signed distance function associated with $\Omega_{x_0,r}$. As $r \to 0$, and since $x_0 \in \partial^*\Omega$, by Theorem 1.36 (ii) we know that $\chi_{\Omega_{x_0,r}} \to \chi_{H_\nu}$ in L^1_{loc}, where $H_\nu = \{y \in \mathbb{R}^n :\ y \cdot \nu < 0\}$, with $\nu = \nu_\Omega(x_0)$. Let $d_{H_\nu}(z)$ be the signed distance associated with H_ν. By our assumption on x_0 we have in particular that $d_r \to d_{H_\nu}$ in $C^0(H_\nu \cap B_1)$ as $r \to 0$. Let us fix a decreasing infinitesimal sequence $\{r_k\}_{k \in \mathbb{N}}$, then for almost every $z \in H_\nu \cap B_1$ we have that the gradients $\nabla d_{r_k}(z)$ and $\nabla d_{H_\nu}(z)$ are defined for every k. By a well-known property of distance functions, for every k there exists a unique $y_k \in \partial\Omega_{x,r_k}$ such that $d_{r_k}(z) = -|y_k - z|$ and in particular one has $\nabla d_{r_k}(z) = \frac{y_k - z}{|y_k - z|}$. Since $d_{r_k}(z) \to d_{H_\nu}(z) \ge -1$ as $k \to \infty$ we can assume without loss of generality that $|y_k| \le 2$ for all k. Let $p \in \partial H_\nu \cap B_1$ be such that $d_{H_\nu}(z) = -|p - z|$. Similarly we take $p_k \in \partial H_\nu \cap B_2$ as the projection of y_k onto ∂H_ν. Now we show that

$$\lim_{k \to \infty} |y_k - p| = 0\,. \tag{2.25}$$

We split the proof of (2.25) in two cases. In the first case we assume $d_{H_\nu}(y_k) \ge 0$ and notice that

$$\begin{aligned}
|y_k - p|^2 &= |p_k - y_k|^2 + |p - p_k|^2 \\
&\le |p_k - y_k|^2 + |p - p_k|^2 + 2|p_k - y_k||z - p| \\
&= |y_k - z|^2 - |z - p|^2 = d_{r_k}(z)^2 - d_{H_\nu}(z)^2 \to 0 \qquad \text{as } k \to \infty.
\end{aligned}$$

In the second case, we assume $d_{H_\nu}(y_k) < 0$ and find

$$\begin{aligned}
|p_k - p|^2 &= |z - p_k|^2 - |z - p|^2 \\
&\le (|z - y_k| + |y_k - p_k|)^2 - |z - p|^2 \\
&= d_{r_k}(z)^2 - d_{H_\nu}(z)^2 + d_{H_\nu}(y_k)^2 + 2|d_{r_k}(z)|\,|d_{H_\nu}(y_k)| \\
&\le d_{r_k}(z)^2 - d_{H_\nu}(z)^2 + 4\varepsilon_k^2 + 8\varepsilon_k\,,
\end{aligned}$$

where $\varepsilon_k = d_{H_\nu}(y_k)$ is infinitesimal in virtue of Proposition 2.2. By recalling that $d_{r_k}(z) \to d_{H_\nu}(z)$ we conclude that $|p_k - p| \to 0$ as $k \to \infty$. Then

$$|y_k - p|^2 = |y_k - p_k|^2 + |p_k - p|^2 \leq 4\varepsilon_k^2 + |p_k - p|^2 \to 0\,, \qquad \text{as } k \to \infty,$$

which completes the proof of (2.25). Now, owing to (2.25) we obtain

$$\nabla d_{r_k}(z) = \frac{y_k - z}{|y_k - z|} \underset{k\to\infty}{\longrightarrow} \frac{p-z}{|p-z|} = \nabla d_{H_\nu}(z) = \nu$$

for almost every $z \in B_1 \cap H_\nu$. Consequently, for all $\tau > 0$ the set

$$M_\tau(x_0) = \{x \in \Omega :\ |\nabla d_\Omega(x) - \nu(x_0)| > \tau\}$$

satisfies

$$\begin{aligned}\frac{|M_\tau(x_0) \cap B_r(x_0)|}{r^n} &\leq |\{z \in \Omega_{x_0,r} \cap B_1 \cap H_\nu :\ |\nabla d_r(z) - \nabla d_{H_\nu}(z)| > \tau\}| \\ &\quad + |(\Omega_{x_0,r} \setminus H_\nu) \cap B_1| \\ &\underset{r\to 0^+}{\longrightarrow} 0\,.\end{aligned} \tag{2.26}$$

We now fix $t > 0$. Setting

$$N_t(x_0) = \{x \in \Omega :\ \xi(x)\cdot\nu < \|\xi\|_\infty - t\} \quad \text{and} \quad S_{\tau,r}(x_0) = M_\tau(x_0) \cap B_r(x_0)\,,$$

we have the following inclusions:

$$\begin{aligned}N_t(x_0) \cap B_r(x_0) &= \{x \in \Omega \cap B_r(x_0) :\ \xi(x)\cdot\nu < \|\xi\|_\infty - t\} \\ &\subset \big[\{x \in \Omega \cap B_r(x_0) :\ \xi(x)\cdot\nu < \|\xi\|_\infty - t\} \setminus M_\tau(x_0)\big] \cup S_{\tau,r}(x_0) \\ &\subset \{x \in \Omega \cap B_r(x_0) :\ \xi(x)\cdot\nabla d(x) < \|\xi\|_\infty - t + \|\xi\|_\infty \tau\} \cup S_{\tau,r}(x_0)\end{aligned} \tag{2.27}$$

Choosing $\tau = \frac{t}{2\|\xi\|_\infty}$ we deduce by (2.26), (2.27) and Theorem 2.10 that

$$\lim_{r\to 0} \frac{|N_t(x_0) \cap B_r(x_0)|}{r^n} = 0\,,$$

which shows that

$$\operatorname*{ap-lim}_{x\to x_0} \xi(x)\cdot\nu(x_0) = \|\xi\|_\infty\,. \tag{2.28}$$

We finally observe that (2.28) is equivalent to (2.24) since for every $t > 0$ one has the inclusion

$$\{x \in \Omega :\ \big|\xi(x) - \|\xi\|_\infty \nu(x_0)\big| \geq t\} \ \subset\ \left\{x \in \Omega :\ \xi(x)\cdot\nu(x_0) \leq \|\xi\|_\infty \left(1 - \frac{t^2}{2\|\xi\|_\infty^2}\right)\right\}.$$

$\square$

2.3 A weak Young law for (Λ, r_0)-perimeter minimizers

In this section we prove a weak Young law for (Λ, r_0)-perimeter minimizers, i.e. our claim is that such minimizers in Ω meets $\partial\Omega$ in a tangential way whenever the meeting point $x \in \partial\Omega$ is in $\partial^*\Omega$. This property is widely known for local perimeter minimizers whenever $\partial\Omega$ is regular in a neighborhood of x; in [LP16] has been proved in the more general case when $x \in \partial^*\Omega$. We here present a proof of this fact for the larger class of (Λ, r_0)-perimeter minimizers; it follows the one of [LP16] up to some minor technical tweaks in order to account for the higher order term given by $\Lambda|E\Delta F|$.

Theorem 2.12 (Weak Young's Law)**.** *Let Ω be an open set with locally finite perimeter and let E be a (Λ, r_0)-minimizer in $\overline{\Omega}$. Then $\partial E \cap \Omega$ meets $\partial^*\Omega$ in a tangential way, i.e., for any $x \in \partial^*\Omega \cap \overline{(\partial E \cap \Omega)}$ one has that $x \in \partial^* E$ and $\nu_E(x) = \nu_\Omega(x)$.*

Proof. Let us fix a point $x \in \partial^*\Omega \cap \partial E$ and let $x + H$ be the half space obtained by blowing up Ω around x. We divide the proof in three steps. In the first one we prove that E and Ω have the same tangential space at x, while in the third one we prove that x is in $\partial^* E$ and that the outward normal is equal to the one outward Ω. Step 2 provides a tool to prove Step 3.

Step 1. Let us prove that E has the same tangent space $x + H$ at x. In order to do so, we need to prove perimeter and volume density estimates for $E \subset \Omega$ at x. Fix $m(r) := |E \cap B_r(x)|$ so that one has $m'(r) = P(E \cap B_r(x), \partial B_r(x))$. By our assumptions, one has $m(r) > 0$ for all $r > 0$. Being E a (Λ, r_0)-minimum, for any $r < r_0$ and any competitor F, such that $F\Delta E \subset\subset B_r(x) \cap \overline{\Omega}$, one has

$$P(E; B_r(x)) \leq P(F; B_r(x)) + \Lambda|F\Delta E|.$$

Fix two radii, $r_2 < r_1 < r_0$ and consider as competitor in $B_{r_1}(x) \cap \overline{\Omega}$ the set $F := E \setminus B_{r_2}$. Therefore, exploiting the Λ-minimality one has

$$P(E; B_{r_1}(x)) \leq P(F; B_{r_1}(x)) + \Lambda|E\Delta F| \leq P(E; B_{r_1}(x) \setminus B_{r_2}(x)) + m'(r_2) + \Lambda|E \cap B_{r_2}(x)|.$$

Thus

$$P(E; B_{r_2}) \leq P(E; B_{r_1}(x)) \leq c_2 \Lambda m(r_2) + m'(r_2). \tag{2.29}$$

Due to the latter and to the isoperimetric inequality, it follows

$$\begin{aligned} c_3 m(r_2)^{\frac{n-1}{n}} &= c_3 |E \cap B_{r_2}(x)|^{\frac{n-1}{n}} \leq P(E \cap B_{r_2}(x)) \\ &= P(E; B_{r_2}(x)) + P(E \cap B_{r_2}(x); \partial B_{r_2}(x)) \leq c_2 \Lambda m(r_2) + 2m'(r_2). \end{aligned} \tag{2.30}$$

Hence for r_2 small enough we have

$$\frac{m'(r_2)}{m(r_2)^{\frac{n-1}{n}}} \geq c_4.$$

By integrating on $(\rho/2, \rho)$ the latter we have as lower bound on $m(\rho)$ the quantity $c\rho^n$ which in turn yields the volume density estimate.

Regarding the perimeter, directly from (2.29) one can infer that $P(E; B_{r_2}) \leq \Lambda\omega_n r_2^n + m'(r_2)$, which, for r_2 small enough implies

$$P(E; B_{r_2}) \leq c_5 r_2^{n-1},$$

which then yields the perimeter density estimate.

Now blowing up E at x we find a limit set E_∞ contained in the half-space $x + H$ with $x \in \partial E_\infty$. It can be shown that E_∞ is not empty and minimizes the perimeter without volume constraint with respect to any compact variation contained in $x + H$. By convexity of H and by a maximum principle argument [Sim87] one infers that E admits $x + H$ as unique blow up at the point x.

Step 2. Let us prove that

$$\lim_{r\to 0} \frac{P(E; B_r(x))}{r^{n-1}} = \omega_{n-1} \tag{2.31}$$

holds. Let E_r be $r^{-1}(E - x)$. Since the blow up of E at x is the half space $x + H$ one has the L^1_{loc}-convergence $\chi_{E_r} \to \chi_H$ as r goes to 0. By the lower semi-continuity of the perimeter we have

$$\liminf_{r\to 0} \frac{P(E; B_r(x))}{r^{n-1}} = \liminf_{r\to 0} P(E_r; B_1) \geq P(H; B_1) \geq \omega_{n-1},$$

therefore to prove (2.31) it is enough to show that

$$\limsup P(E_r; B_1) \leq \omega_{n-1}. \tag{2.32}$$

Argue by contradiction and suppose there exists a sequence of radii r_i going to 0 such that

$$P(E_{r_i}; B_1) \geq \omega_{n-1} + \varepsilon. \tag{2.33}$$

Recall that $x \in \partial^*\Omega$, therefore for r_i small enough one has

$$P(\Omega_{r_i}; B_s) \leq s^{n-1}\omega_{n-1} + \varepsilon/3, \qquad \text{for all } 1 < s < 2, \tag{2.34}$$

where Ω_{r_i} is defined in the same manner of E_{r_i}. Due to the L^1-convergence in $B_2(0)$ of $\chi_{E_{r_i}}$ to χ_H and by coarea formula one can find a suitable

$$t \in \left(1, \left(\frac{\omega_{n-1} + \varepsilon/2}{\omega_{n-1} + \varepsilon/3}\right)^{\frac{1}{n-1}}\right)$$

such that

$$P(\Omega_i; \partial B_t) = P(E_i; \partial B_t) = 0 \tag{2.35}$$

$$\mathcal{H}^{n-1}(E_i \Delta \Omega_i \cap \partial B_t) < \frac{\varepsilon}{4} \tag{2.36}$$

hold. Consider now the sets $F_i := (E \cup B_{tr_i}(x)) \cap \Omega$, for which, due to the previous, one has

$$P(F_i, B_{r_0}(x)) = P(E; (\Omega \cap B_{r_0}(x)) \setminus B_{tr_i}(x)) + P(\Omega; B_{tr_i}(x)) + r_i^{n-1}\mathcal{H}^{n-1}(E_i \Delta \Omega_i \cap \partial B_t).$$

For r_i small enough that $tr_i < r_0$, the set F_i is a competitor to E in B_{r_0}, therefore

$$\begin{aligned} r^{n-1}(\omega_{n-1} + \varepsilon) &\le P(E; B_{r_i}(x)) \le P(E; B_{r_0}(x)) - P(E; (\Omega \cap B_{r_0}(x)) \setminus B_{tr_i}(x)) \\ &\le P(F; B_{r_0}(x)) - P(E; (\Omega \cap B_{r_0}(x)) \setminus B_{tr_i}(x)) + \Lambda|F \Delta E| \\ &\le P(F; B_{r_0}(x)) - P(E; (\Omega \cap B_{r_0}(x)) \setminus B_{tr_i}(x)) + \Lambda|E \cap B_{tr_i}(x)| \\ &\le P(\Omega; B_{tr_i}(x)) + r_i^{n-1}\frac{\varepsilon}{4} + \Lambda\omega_n(tr_i)^n \qquad (2.37) \\ &\le (tr_i)^{n-1}(\omega_{n-1} + \varepsilon/3) + r_i^{n-1}\frac{\varepsilon}{4} + \Lambda\omega_n(tr_i)^n \\ &< r_i^{n-1}(\omega_{n-1} + \varepsilon/2) + r_i^{n-1}\frac{\varepsilon}{2} \le r^{n-1}(\omega_{n-1} + \varepsilon), \end{aligned}$$

which leads to a contradiction.
Step 3. Owing to (2.31), in order to show that $x \in \partial^* E$ and that $\nu_E(x) = \nu_\Omega(x)$ it is enough to prove that

$$\lim_{r \to 0} -\frac{D\chi_E(B - r(x)) \cdot \nu_\Omega(x)}{\omega_{n-1} r^{n-1}} = 1\,. \qquad (2.38)$$

In virtue of Theorem 1.36 (iv), for almost every $r > 0$ one has

$$\begin{aligned} D\chi_E(B - r(x)) \cdot v &= \int_{E \cap \partial B_r(x)} v \cdot N \, d\mathcal{H}^{n-1} = \int_{H \cap \partial B_r(0)} v \cdot N \, d\mathcal{H}^{n-1} + A(x, r) \qquad (2.39) \\ &= \omega_{n-1} r^{n-1} + A(x, r)\,, \end{aligned}$$

where N is the outward normal to $\partial B_r(x)$ and

$$\begin{aligned} |A(x, r)| &= \left| v \cdot \int_{\partial B_r(x)} (\chi_E(y) - \chi_{x+H}(y)) N(y) \, d\mathcal{H}^{n-1}(y) \right| \\ &\le \int_{\partial B_r(x)} |\chi_E(y) - \chi_{x+H}(y)| \, d\mathcal{H}^{n-1}(y)\,. \end{aligned}$$

Now for any fixed $\delta > 0$, define the set $\Sigma(x, \delta) \subseteq (0, +\infty)$ of radii $r > 0$ such that $A(x, r) > \delta r^{n-1}$. Hence, by the L^1_{loc}-convergence of $r^{-1}(E - x)$ to the half-space H we infer that

$$\lim_{\rho \to 0^+} \frac{\mathcal{H}^1(\Sigma(x, \delta) \cap (0, \rho))}{\rho} = 0\,.$$

Therefore, for any decreasing infinitesimal sequence of radii $\{r_i\}_i$ we can find another sequence $\{\rho_i\}_i$ such that $\rho_i \notin \Sigma(x, \delta)$ for all i and $\rho_i = r_i + o(r_i)$ as $i \to \infty$. Suppose by contradiction that (2.38) does not hold. Then, there exist $\alpha > 0$ and a decreasing infinitesimal sequence $\{r_i\}_i$ such that

$$\left| \frac{D_{\chi_E}(B_{r_i}(x) \cdot v}{\omega_{n-1} r_i^{n-1}} \right| \ge \alpha\,, \qquad (2.40)$$

for all $i \in \mathbb{N}$. By suitably choosing δ as $\omega_{n-1}\alpha/2$ and considering the sequence ρ_i defined above, one gets in (2.39) with the substitution $r = \rho_i$

$$\left|D_{\chi_E}(B_{\rho_i}(x)) \cdot v - \omega_{n-1}\rho_i^{n-1}\right| = |A(x, \rho_i)| \leq \frac{\alpha}{2}\omega_{n-1}\rho_i^{n-1} .$$

On the other hand, by (2.31), we also have

$$|D_{\chi_E}(B_{\rho_i}(x)) - D_{\chi_E}(B_{r_i}(x))| \leq P(E; B_{\rho_i}(x)\Delta B_{r_i}(x)) \leq \omega_{n-1}|\rho_i^{n-1} - r_i^{n-1}| + o(r_i^{n-1}) = o(r_i^{n-1})$$

as $i \to \infty$. Combining these two latter inequalities yields to

$$\left|D_{\chi_E}(B_{r_i}(x)) \cdot v - \omega_{n-1}r_i^{n-1}\right| \leq \frac{\alpha}{2}\omega_{n-1}\rho_i^{n-1} + o(r_i^{n-1}) = \frac{\alpha}{2}\omega_{n-1}r_i^{n-1} + o(r_i^{n-1}) ,$$

which contradicts (2.40) for i large enough. □

2.4 The largest ball enclosed in a Jordan domain

Given a closed planar curve one might wonder which is the largest ball one can fit in the region enclosed by the curve. Besides this question being interesting in its own merit, it was one of the steps to be addressed during the proof of Theorem 3.8 as, to apply the rolling ball property (see Theorem 3.1 (viii)) we had to ensure at least a ball of a given radius was contained in the initial set.

First of all, one needs to satisfy the hypotheses of the Jordan theorem which states that a Jordan curve divides the plane in two connected components whose boundary is the curve itself (the well-known Jordan theorem). Moreover, one is bounded (the interior or *Jordan domain*) and one is unbounded (the exterior). In 1959 Pestov and Ionin proved that in the region enclosed by a C^2-regular Jordan curve with curvature bounded from above by $1/r$ one can fit a ball of radius r (see [IP59, HT95]). We here report the definition of Jordan curve, a slightly more general result than Jordan theorem and Pestov and Ionin theorem before delving into a generalization of this latter result.

Definition 2.13 (Jordan curve)**.** A *Jordan curve* in the plane $\mathbb{R}^2$ is the image of a homeomorphism γ defined on $\mathbb{S}^1$.

Theorem 2.14 (Jordan-Schoenflies theorem)**.** *Given a Jordan curve $\gamma(\mathbb{S}^1)$ there exists a homeomorphism $\Gamma : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Gamma_{|\mathbb{S}^1} = \gamma$.*

To introduce the next theorem we recall the definition of signed curvature. Let γ be a C^2 regular curve parametrized with respect arc length, i.e. such that $\|\gamma'(s)\| = 1$. Then the signed curvature κ of γ is defined as

$$\kappa(s) = \gamma''(s) \cdot \gamma'(s) .$$

Theorem 2.15 (Ionin and Pestov theorem)**.** *Let $\gamma(\mathbb{S}^1)$ be a Jordan curve of class C^2 with curvature bounded from above by $1/r$. Then, the Jordan domain enclosed by $\gamma(\mathbb{S}^1)$ contains a ball of radius r.*

We here provide a generalization of Theorem 2.15 for curves whose curvature is only known to be bounded in the viscosity sense. The general idea follows [HT95], yet many adjustments have to be made in order to take into account the lack of regularity of the boundary. The notion of cut locus of E had to be changed, and the deduction of its structural properties presents many tricky points to overcome the loss of regularity which in [HT95] allowed to infer that the cut locus is a strong deformation of the boundary, which of course is no more true in this general setting. Roughly speaking, the cut locus of a set E, is the subset of points x such that if y is a point of projection of x onto ∂E, then on the ray from y to x, any point after x do not project onto y (for a more precise definition see Definition 2.22, where also a subset of these cut points is defined, namely the focal points). The proof here presented has been firstly discussed in [LNS17]. Before stating the theorem we aim to prove, we introduce few useful definitions.

Definition 2.16 (Curvature bounded from above in the viscosity sense)**.** Given a constant $h \in \mathbb{R}$, a set $E \subset \mathbb{R}^2$ has curvature bounded above by h at $x \in \partial E$ in the viscosity sense if the following holds. Suppose $A \subset \partial B_r(y)$ is a circular arc that locally touches E from outside at x, i.e. x is in the relative interior of A and there exists a radius ε such that $E \cap B_\varepsilon(x) \subset \overline{B_r(y)} \cap B_\varepsilon(x)$. Then $r \geq 1/h$.

Definition 2.17. Let $\Omega \subset \mathbb{R}^2$ be an open bounded set. We say that $\partial\Omega$ is locally homeomorphic to an interval if there exist $r_0 > 0$ and a modulus of continuity ω_0, such that for any $y_1, y_2 \in \partial\Omega$ with $|y_1 - y_2| < r_0$, one can find an open set U containing y_1, y_2, with $\operatorname{diam}(U) \leq \omega_0(|y_1 - y_2|)$, and such that $\partial\Omega \cap U$ is homeomorphic to an open interval.

The following proposition shows that the boundary of every Jordan domain is locally homeomorphic to an interval, in the sense of Definition 2.17.

Proposition 2.18. *If $\gamma(\mathbb{S}^1)$ is a Jordan curve then there exist $r_0 > 0$ and a modulus of continuity ω_0, with the following property: for any $y_1, y_2 \in \gamma(\mathbb{S}^1)$, with $|y_1 - y_2| < r_0$, there exists an open set U containing y_1, y_2, with* $\operatorname{diam}(U) \leq \omega_0(|y_1 - y_2|)$, *such that $\gamma(\mathbb{S}^1) \cap U$ is homeomorphic to an open interval.*

Proof. By definition of Jordan curve γ is a homeomorphism from $\mathbb{S}^1$ onto $\gamma(\mathbb{S}^1)$. By the Jordan-Schoenflies theorem this can be extended to a homeomorphism $\Gamma : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Gamma(\mathbb{S}^1) = \gamma(\mathbb{S}^1)$ and $\Gamma(B_1)$ equals the bounded Jordan domain enclosed by $\gamma(\mathbb{S}^1)$. The restriction of Γ to B_2 is uniformly continuous, hence there exists a modulus of continuity η_0 such that the diameter of $\Gamma(B_t(x))$ is bounded by $\eta_0(t/4)$, for all $B_t(x) \subset B_2$. One can then choose $t_0 > 0$ and a finite covering $\{B_{t_0}(x_1), \ldots, B_{t_0}(x_N)\}$ of $\mathbb{S}^1$, with $x_j \in \mathbb{S}^1$ for all j, such that $B_t(x) \cap \mathbb{S}^1$ is homeomorphic to an open interval for all $x \in \mathbb{S}^1$ and $0 < t < 4t_0$. Then we set r_0 to be the supremum of $r > 0$ such that whenever $y_1, y_2 \in \gamma(\mathbb{S}^1)$ with $|y_1 - y_2| < r$, then there exists $j \in \{1, \ldots, N\}$ such that $y_1, y_2 \in \Gamma(B_{t_0}(x_j))$. Since Γ^{-1} is uniformly continuous when restricted to $\Gamma(B_2)$, the set of r satisfying the above property is nonempty, thus $r_0 > 0$. Then, setting

$$\omega_0(d) = \sup\{\eta_0(t) : \ t = |\Gamma^{-1}(y_1) - \Gamma^{-1}(y_2)|, \ y_i \in \gamma(\mathbb{S}^1), \ d = |y_1 - y_2|\},$$
$$t_{1,2} = |\Gamma^{-1}(y_1) - \Gamma^{-1}(y_2)|, \qquad x_1 = \Gamma^{-1}(y_1), \qquad U = \Gamma(B_{2t_{1,2}}(x_1)),$$

one has that $y_1, y_2 \in U$, $\operatorname{diam}(U) \leq \omega_0(|y_1 - y_2|)$, and that for $d < r_0$, the set $\gamma(\mathbb{S}^1) \cap U = \Gamma(B_{2|y_1-y_2|}(x_1) \cap \mathbb{S}^1)$ is homeomorphic to an open interval, as wanted. □

Remark 2.19. We observe that the converse of Proposition 2.18 is also true, namely that every bounded open set whose boundary is locally homeomorphic to an open interval is enclosed by a finite family of pairwise disjoint Jordan curves.

We can now state the main theorem of this section which generalizes Theorem 2.15.

Theorem 2.20. *Suppose $E \subset \mathbb{R}^2$ is simply connected and such that ∂E is locally homeomorphic to an interval. If the curvature of E is bounded from above by $h > 0$ in the viscosity sense, then E contains a ball of radius $1/h$.*

Clearly, the viscosity hypothesis replaces the stronger C^2 and bounded curvature κ requests. Furthermore, on one hand we do not require ∂E to be a single Jordan curve, but we allow it to be a union of pairwise disjoint curves in virtue of Remark 2.19. Finally, the simple connectedness hypothesis is used in the proof of Theorem 2.20 to ensure that the cut locus of ∂E contains no loops.

2.4.1. The cut locus

Throughout this section, we suppose that $E \subset \mathbb{R}^2$ is a bounded, simply connected set with curvature bounded from above by 1 in the viscosity sense. We assume further that ∂E is locally homeomorphic to an interval and set r_0 as in Definition 2.17. We assume $\omega_0(r_0) < 1$ and $r_0 < 1/4$, without loss of generality.

Observe that $B_d(x) \subset E$ where $d = \operatorname{dist}(x, \partial E)$. To prove Theorem 2.20, we must show that E contains at least one point x such that $\operatorname{dist}(x, \partial E) \geq 1$. For any $x \in E$, we let

$$P_x = \{y \in \partial E : \operatorname{dist}(x, \partial E) = |x - y|\} \tag{2.41}$$

be the set of projections of x onto ∂E. Given $y \in P_x$, we let

$$z_{x,y}(t) := x + t(x - y). \tag{2.42}$$

When no confusion arises, we use the shorthand z_t in place of $z_{x,y}(t)$.

The following lemma will play a key role in what follows.

Lemma 2.21. *Suppose $E \subset \mathbb{R}^2$ is an open set with ∂E locally homeomorphic to an interval with curvature bounded from above by 1 in the viscosity sense. Assume that for some $x \in E$ and $y_1, y_2 \in P_x$ one has $0 < |y_1 - y_2| < r_0$. Then $\mathrm{dist}(x, \partial E) \geq 1$.*

Proof. We denote by $\gamma \subset \partial E$ a curve homeomorphic to a closed interval that connects y_1 and y_2, as obtained according to Definition 2.17. We assume by contradiction that $\bar{r} = \mathrm{dist}(x, \partial E) < 1$. Fix $1 > r > \max(\bar{r}, \omega_0(r_0))$, then denote by m and ℓ the midpoint and the line orthogonal to the vector $y_1 - y_2$ passing through m, respectively. Choose a point $z \in \ell$ in such a way that $|z - y_i| = r$ for $i = 1, 2$ and $m + t(m - z) \in \gamma$ for some $t > 0$ (we remark that, since $m \in E$, there is at least one possible choice of such a point z). Let A be the (uniquely defined) half of $\partial B_r(z)$ which contains y_i for $i = 1, 2$ and is symmetric with respect to the line ℓ. We consider all translations $A + sv$ of A, with $s \geq 0$ and $v = m - z$, and observe that $A + sv \cap \gamma = \emptyset$ for s sufficiently large. Moreover, the maximum distance of γ from ℓ is bounded by $\omega_0(r_0)$, hence the distance of each endpoint of $A + sv$ from γ is bounded below by $r - \omega_0(r_0) > 0$, for all $s \geq 0$. Then, we let

$$\bar{s} = \inf\{s : s > 0,\ (A + sv) \cap \gamma = \emptyset\}.$$

As both A and γ are closed, the set $F = \gamma \cap (A + \bar{s}v)$ is nonempty and made of points belonging to the relative interior of A. If $\bar{s} = 0$, then $y_1, y_2 \in F$ and $A \cap \ell \in \gamma$. Therefore γ coincides with the closure of $A \cap B_{\bar{r}}(x)$, but this is not possible as the intersection $\gamma \cap B_{\bar{r}}(x)$ is empty. Hence we must have $\bar{s} > 0$. Since now F only contains points in the relative interior of γ, there exists $w \in F$ such that the arc $A + \bar{s}v$ touches E locally from outside at w. As the curvature of ∂E is bounded above by 1 in the viscosity sense, we obtain $r \geq 1$, that is, a contradiction. This concludes the proof of the lemma. $\square$

We introduce the following definitions:

Definition 2.22. Let $x \in E$.

1. We call x a cut point if there exists $y \in P_x$ such that

$$\sup\{t : |z_{x,y}(t) - y| = \mathrm{dist}(z_{x,y}(t), \partial E)\} = 0.$$

2. We call x a focal point if there exists $y \in P_x$ such that

$$\sup\{t : y \text{ is a local minimizer of } \mathrm{dist}(z_{x,y}(t), \cdot\,) \text{ among points in } \partial E\} = 0.$$

   We call such a $y \in P_x$ a *focal projection* of x.

We let $\mathcal{C}$ denote the set of cut points of E. Note that any focal point is also a cut point. Furthermore, if $\#P_x > 1$, then x is a cut point, and if $\#P_x = \infty$, then x is a focal point. In Figure 2.3 an example of cut locus for a set is given.

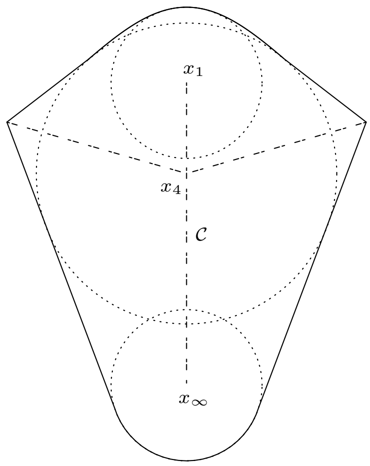


Figure 2.3: The 4 dashed lines form the cut locus $\mathcal{C}$ of the depicted set. The point x_1 is a focal point with unique projection, the point x_∞ is a focal point with infinite projections and the point x_4 is a cut point with 4 projections. All the other points in $\mathcal{C}$ have 2 projections

Remark 2.23 (Geometric interpretation of focal points). Let x be a focal point and let $y \in P_x$ be a focal projection. Let $d = |x - y|$ and $d_t = |z_t - \bar{y}|$, so that $y \in \partial B_d(x) \cap \partial B_{d_t}(z_t) \cap \partial E$. Given $\varepsilon > 0$ we denote by $\Sigma_\varepsilon(y)$ a portion of ∂E with diameter less than ε and homeomorphic to an open interval, such that $y \in \Sigma_\varepsilon(y)$ (see Definition 2.17). By the definition of focal point we have

$$\Sigma_\varepsilon(y) \cap B_d(x) = \emptyset, \qquad \Sigma_\varepsilon(y) \cap B_{d_t}(z_t) \neq \emptyset.$$

Remark 2.24 (Nesting property). With the geometric interpretation of focal point in mind, we observe the following nesting property. If there exists $\bar{t} > 0$ such that $\bar{y} \in \partial E$ minimizes $\operatorname{dist}(z_{x,\bar{y}}(\bar{t}), \cdot\,)$ among all $y \in \Sigma_\varepsilon(\bar{y})$, then $\bar{y}$ also minimizes $\operatorname{dist}(z_{x,y}(t), \cdot\,)$ among all $y \in \Sigma_\varepsilon(\bar{y})$ for all $0 < t < \bar{t}$. Indeed, $\Sigma_\varepsilon(\bar{y})$ does not intersect $B_d(z_{x,\bar{y}}(\bar{t}))$ with $d = |z_{x,\bar{y}}(\bar{t}) - \bar{y}|$, so it also does not intersect the ball of smaller radius $|z_{x,\bar{y}}(t) - \bar{y}|$ centered in $z_{x,\bar{y}}(t)$, which is obviously contained in $B_d(z_{x,\bar{y}}(\bar{t}))$.

We have the following corollary of Lemma 2.21.

Corollary 2.25. *Suppose $E \subset \mathbb{R}^2$ is an open set whose boundary ∂E is locally homeomorphic to an interval, such that the curvature of E is bounded from above by 1 in the viscosity sense. Suppose $x \in E$ is a cut point with $\operatorname{dist}(x, \partial E) < 1$ and $y \in P_x$. For any $0 < \varepsilon < r_0$, there exists $\delta_\varepsilon > 0$ such that every $z \in B_{\delta_\varepsilon}(x)$ has a unique projection onto $\Sigma_\varepsilon(y)$.*

Proof. Fix $\delta > 0$ small enough so that $d_z = \operatorname{dist}(z, \Sigma_\varepsilon(y)) < 1$ for all $z \in B_\delta(x)$. Assume that $y_1, y_2 \in \Sigma_\varepsilon(y) \cap \partial B_{d_z}(z)$ for some $z \in B_\delta(x)$. This implies that $|y_1 - y_2| \le \varepsilon < r_0$. Hence, $y_1 = y_2$ by Lemma 2.21. $\square$

Next, we show that focal points lie at distance at least 1 from ∂E.

Proposition 2.26. *Suppose $E \subset \mathbb{R}^2$ is an open set with ∂E locally homeomorphic to an interval, and with curvature bounded from above by 1 in the viscosity sense. Let r_0 be as in Definition 2.17 and assume that $r_0 < 1$ without loss of generality. If $x \in E$ is a focal point, then* $\operatorname{dist}(x, \partial E) \geq 1$.

Proof. Let $\bar{y} \in P_x$ be a focal projection for x. Let $d = |x - \bar{y}|$ and $d_t = |z_t - \bar{y}|$. We show that $d_t \geq 1$ for all $t > 0$.

First, suppose that $\Sigma_\varepsilon(\bar{y})$ is contained in $\overline{B}_{d_t}(z_t)$ for $\varepsilon > 0$ sufficiently small. In this case, $\partial B_{d_t}(z_t) \cap B_\varepsilon(\bar{y})$ is a circular arc touching E from outside at $\bar{y}$, so $d_t \geq 1$ as the curvature of E is bounded above by 1 in the viscosity sense.

Next, suppose that $\Sigma_\varepsilon(\bar{y}) \setminus \overline{B}_{d_t}(z_t)$ is nonempty for every $\varepsilon > 0$. Then, since x is a focal point, for $\varepsilon < r_0$ there are infinitely many pairs of points $y_1, y_2 \in \Sigma_\varepsilon(\bar{y}) \cap \partial B_{d_t}(z_t)$ with $y_1 \neq y_2$ such that

$$\gamma \subset \Sigma_\varepsilon(\bar{y}) \setminus B_{d_t}(z_t), \tag{2.43}$$

letting γ denote the subset of $\Sigma_\varepsilon(\bar{y})$ that is homeomorphic to an interval and has endpoints y_1 and y_2. Fix one such pair. Let $r = \max\{d_t, r_0\}$. Arguing by contradiction as in the proof of Lemma 2.21, we finally show that $r \geq 1$. As $r_0 < 1$, we determine that $d_t \geq 1$, concluding the proof. $\square$

Proposition 2.27. *Suppose $E \subset \mathbb{R}^2$ is an open set with ∂E locally homeomorphic to an interval, and with curvature bounded from above by 1 in the viscosity sense. Let $x \in E$ be a cut point. Then at least one of the following holds:*

1. *$\#P_x > 1$,*

2. *x is a focal point.*

Proof. Suppose x is a cut point with $\#P_x = 1$, and let $\bar{y}$ denote the unique point in P_x.
Step 1. We start proving the following statement: *for all $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that*

$$P_{z_t} \subset \Sigma_\varepsilon(\bar{y}) \qquad \forall 0 \leq t < t_\varepsilon$$

(here we still adopt the notation introduced in Remark 2.23). If not, then we may find $\varepsilon > 0$ $t_n \to 0$ and y_n, where $y_n \in P_{z_{t_n}}$ and $|y_n - \bar{y}| \geq \varepsilon$. Since ∂E is compact, up to a subsequence, $y_n \to y_1 \in \partial E$ with $y_1 \notin \Sigma_\varepsilon(\bar{y})$. In particular, $y_1 \neq \bar{y}$. On the other hand, we deduce that $y_1 \in P_x$, since

$$\operatorname{dist}(z_{t_n}, \partial E) \to \operatorname{dist}(x, \partial E) = |x - \bar{y}|$$

and

$$\operatorname{dist}(z_{t_n}, \partial E) = |z_{t_n} - y_n| \to |x - y_1|,$$

thus $|x - y_1| = |x - \bar{y}|$, but this contradicts $\#P_x = 1$.

Step 2: x is a focal point. If x is not a focal point, then the supremum in Definition 2.22 (2) is strictly positive. By the nesting property (Remark 2.24), there exist $\varepsilon > 0$ and $\bar{t} > 0$ such that $\bar{y}$ minimizes $\operatorname{dist}(z_t, \cdot\,)$ among all $y \in \Sigma_\varepsilon(\bar{y})$ for all $0 < t < \bar{t}$. On the other hand, x is a cut point, so $\bar{y} \notin P_{z_t}$ for all $t > 0$, and therefore $y \notin P_{z_t}$ for any other $y \in \Sigma_\varepsilon(\bar{y})$. This contradicts Step 1. $\square$

Proposition 2.28. *Suppose $E \subset \mathbb{R}^2$ is an open set with ∂E locally homeomorphic to an interval, and with curvature bounded from above by 1 in the viscosity sense. Suppose $x \in E$ is a cut point with $\operatorname{dist}(x, \partial E) = d < 1$ and $\#P_x = k \geq 2$. Then, there exists $\delta > 0$ such that $\mathcal{C} \cap B_\delta(x)$ is the union of k curves of class C^1 meeting at x.*

Proof. Let $P_x = \{y_1, \dots, y_k\}$ and $d = \operatorname{dist}(x, \partial E)$. Fix $\varepsilon < r_0/4$ so small that $\Sigma_i := \Sigma_\varepsilon(y_i)$ and $\Sigma_j := \Sigma_\varepsilon(y_j)$ have positive distance for all $1 \leq i, j \leq k$ with $i \neq j$.

Observe that there exists $r > d$ such that $B_r(x) \cap \partial E \subset \bigcup_{i=1}^k \Sigma_i$. Hence, for $\delta > 0$ sufficiently small,

$$P_z \subset \bigcup_{i=1}^{k} \Sigma_i$$

for all $z \in B_\delta(x)$. Up to further decreasing δ, we may also assume that $0 < \delta < \delta_\varepsilon$, where δ_ε is as in Corollary 2.25. Consequently, each $z \in B_\delta(x)$ has a unique projection y_z^i onto Σ_i for all $i \in \{1, \dots, k\}$. The functions $\rho_i : B_\delta(x) \to \mathbb{R}$ defined by

$$\rho_i(z) = \operatorname{dist}(z, \Sigma_\varepsilon(y_i))$$

are therefore continuously differentiable at every $z \in B_\delta(x)$ with

$$\nabla \rho_i(z) = \frac{z - y_z^i}{|z - y_z^i|} \qquad \forall z \in B_\delta(x)$$

(the continuity of the differential depends on the continuity of the map $z \mapsto y_z^i$, which in turns follows from the uniqueness of the projection of z onto $\Sigma_\varepsilon(y_i)$).

Case 1: $k = 2$. Consider the differentiable function $f : B_\delta(x) \to \mathbb{R}$ defined by $f(z) = \rho_1(z) - \rho_2(z)$. There exist F_1 and F_2 connected subsets of S^1 such that $\nabla \rho_1(B_\delta(x)) \subset F_1$ and $\nabla \rho_2(B_\delta(x)) \subset F_2$. Since Σ_1 and Σ_2 have positive distance, up to decreasing δ, F_1 and F_2 also have positive distance. Up to a rotation, assume that e_2 bisects the angle between F_1 and F_2. Then, for every $v = (v_1, v_2) \in F_i$, we have $|v_2| < c < 1$, and so $|v_1| > C > 0$. Therefore, we find that $\partial_{\mathrm{e}_1} f > 0$ for all $z \in B_\delta(x)$. Applying the implicit function theorem, we see that the set $\{f = 0\}$ is the graph of a continuous function of x_2. Since $\{f = 0\} = \mathcal{C} \cap B_\delta(x)$, the proof is complete in this case. Finally, we note that ∇f is orthogonal to $\nabla \rho_1 + \nabla \rho_2$, and $\nabla \rho_1 + \nabla \rho_2$ bisects the angle between $\nabla \rho_1$ and $\nabla \rho_2$.

Case 2: $k > 2$. The union of the segments $\{[y_i, x]\}$ divide $B_d(x)$ into k circular sectors $S_1, \dots, S_k$. We possibly relabel the points y_i in such a way that y_i and y_{i+1} are associated with the sector S_i for $i = 1, \dots k$ (where $k+1$ is identified with 1). We apply Case 1 and find that the set of points that are equidistant from Σ_i and Σ_{i+1} form a continuous curve γ_i. Next, we claim that if $z \in B_\delta(x) \cap S_i \setminus \{x\}$, then $P_z \subset \Sigma_i \cup \Sigma_{i+1}$. Indeed, we already know that $P_z \subset \bigcup_{j=1}^k \Sigma_j$, thus we only have to show that whenever $j \neq i, i+1$ we have $P_z \cap \Sigma_j = \emptyset$. By contradiction, suppose that there exists $y \in P_z \cap \Sigma_j$ for some $j \neq i, i+1$. In this case the segment $[y, z]$ intersects either $[y_i, x]$ or $[y_{i+1}, x]$ at some $\bar{z}$. Without loss of generality, let us suppose that it is $[y_i, x]$. First we notice that $\bar{z} \neq x$, otherwise we would have that $P_x = \{y\}$, i.e., a contradiction with the assumption on x. On the other hand, for $\bar{z} \neq x$ the ball $B_{|\bar{z}-y_i|}(\bar{z})$ is contained in $B_d(x)$ and the boundaries of these balls touch only at y_i, hence we obtain

$$|z - y_i| \leq |z - \bar{z}| + |\bar{z} - y_i| < |z - \bar{z}| + |\bar{z} - y| = |z - y|\,,$$

that is, a contradiction with the fact that $y \in P_z$. This shows our claim and thus the inclusion

$$\gamma_i \cap B_\delta(x) \cap S_i \subset \mathcal{C} \cap B_\delta(x) \cap S_i\,. \tag{2.44}$$

We now show that all points $z \in B_\delta(x) \cap S_i \setminus \gamma_i$ do not belong to $\mathcal{C}$. Indeed the curve γ_i splits the set $B_\delta(x) \cap S_i$ in two regions A_i, B_i made of points whose projection onto ∂E is unique and belongs to, respectively, Σ_i and Σ_{i+1}. This is a consequence of the fact that $\rho_i(z) = \rho_{i+1}(z)$ only if $z \in \gamma_i$. This implies that z has a unique projection onto ∂E at a distance strictly less than 1, hence it cannot be a focal point by Proposition 2.26. Moreover, by Proposition 2.27 it cannot be a cut point.

By repeating the previous argument for each sector S_i we finally obtain that $\mathcal{C} \cap B_\delta(x)$ is the union of exactly k curves $\gamma_1, \dots, \gamma_k$ of class C^1 meeting at x. We also notice that the half-tangent to any such curve γ_i at x bisects the corresponding sector S_i. □

2.4.2. Proof of Theorem 2.20

This section is devoted to the proof of Theorem 2.20, whose statement is recalled below.

Theorem. *Suppose $E \subset \mathbb{R}^2$ is simply connected and such that ∂E is locally homeomorphic to an interval. If the curvature of ∂E is bounded from above by h in the viscosity sense, then E contains a ball of radius $1/h$.*

Proof. Without loss of generality, we set $h = 1$. Up to rescaling, we may assume that $h = 1$. Suppose for contradiction that $\operatorname{dist}(x, \partial E) < 1$ for all $x \in \mathcal{C}$. In particular, by Proposition 2.26, $\mathcal{C}$ contains no focal points, and so by Proposition 2.27, $1 < \#P_x < \infty$ for all $x \in \mathcal{C}$. Then, Proposition 2.28 implies that $\mathcal{C}$ is the union of continuous curves. In particular, $\mathcal{C}$ is a graph where each vertex has finite valence.

We claim that $\mathcal{C}$ is simply connected. If $\mathcal{C}$ contains a loop, then, since E is simply connected, there is some $x \in E \setminus \mathcal{C}$, in the interior of the loop, in the sense that, given any continuous path ψ from x to ∂E, ψ intersects $\mathcal{C}$. However, as $x \notin \mathcal{C}$, it has a unique projection on ∂E: $P_x = \{y\}$. Let ψ be the line segment between x and y, and let z be some point in the intersection of ψ and $\mathcal{C}$. Since $z \in \mathcal{C}$, there exists some $y_z \in P_z$, $y_z \neq y$. However, this implies that

$$|x - y_z| \leq |x - z| + |z - y_z| \leq |x - z| + |z - y| = |x - y| = \operatorname{dist}(x, \partial E).$$

It follows that $y_z \in P_x$, so $x \in \mathcal{C}$. This yields a contradiction, and we determine that $\mathcal{C}$ is simply connected. Therefore, $\mathcal{C}$ is a union of disjoint trees. We now claim that $\mathcal{C}$ is compact, which will complete the proof. Indeed, if $\mathcal{C}$ is compact, then it is a *finite* union of trees. Therefore, it has at least one vertex v of valence 1, and by Proposition 2.28, $\operatorname{dist}(v, \partial\Omega) \geq 1$. Since $\mathcal{C}$ is bounded, we argue by contradiction and suppose $\mathcal{C}$ is not closed. Hence, there is a sequence of non-focal cut points $\{x_i\}$ such that $\operatorname{dist}(x_i, \partial E) < 1$, converging to some $x \notin \mathcal{C}$. As $\{x_i\} \subset \mathcal{C}$, we can take two sequences $\{y_i\}$ and $\{w_i\}$ such that $y_i \neq w_i$ and $|x_i - y_i| = |x_i - w_i| = \operatorname{dist}(x_i, \partial E)$ for all i. By compactness, $y_i \to y \in \partial E$ and $w_i \to w \in \partial E$ up to subsequences, and $\{y, w\} \subset P_x$ by the continuity of the distance function. Since $x \notin \mathcal{C}$, the two points must coincide, i.e. $w = y$. Hence, $|y_i - w_i| \to 0$. So, for i large enough, $0 < |y_i - w_i| < r_0$ and so $\operatorname{dist}(x_i, \partial E) \geq 1$ by Lemma 2.21. This contradicts our assumptions and we deduce that $\mathcal{C}$ is compact, completing the proof. $\square$

CHAPTER 3

The Cheeger problem

For a bounded, open set $\Omega \subset \mathbb{R}^n$, with $n \geq 2$ the *Cheeger constant* is defined as

$$h_1(\Omega) := \inf_{S \subseteq \Omega} \frac{P(S, \mathbb{R}^n)}{|S|}, \tag{3.1}$$

where the infimum is sought among all open subsets of finite perimeter and positive volume. The constant $h_1(\Omega)$ is trivially greater or equal than zero. The task of finding sets $E \subset \Omega$ attaining the infimum in (3.1) is known as the *Cheeger problem* and any set E that minimizes (3.1) is said to be a *Cheeger set* for Ω and we shall denote by $\mathcal{C}_1(\Omega)$ the collection of sets that are Cheeger sets for Ω. If $\Omega \in \mathcal{C}_1(\Omega)$ we will say it is a *self-Cheeger set*. Any set $E \in \mathcal{C}_1(\Omega)$ is a (Λ, r_0)-perimeter minimizer inside Ω. Indeed, if $E \in \mathcal{C}_1(\Omega)$, then it is a volume-constrained (to $V = |E|$) perimeter-minimizer in Ω. It is known then that a volume-constrained perimeter-minimizer in a set Ω is as well a (Λ, r_0)-perimeter minimizer in Ω (see for instance [Mag12, Example 21.3]). Therefore, the classic regularity results of minimal surfaces of [Giu84, GMT81, GMT83] can be applied. We recall some properties in the following theorem and, for more details and many more properties, we recommend the introductory surveys by Leonardi and Parini [Leo15, Par11].

Theorem 3.1. *Let $E \in \mathcal{C}_1(\Omega)$. Then, the following properties hold:*

(i) The constrained boundary of E, i.e. $\partial E \cap \partial \Omega$, contains at least two points;

(ii) The free boundary of E, i.e. $\partial E \cap \Omega$, is analytic possibly except for a closed singular set whose Hausdorff dimension does not exceed $n - 8$;

(iii) The mean curvature of the free boundary is constant at every regular point and it is equal to $h_1(\Omega)/(n-1)$;

(iv) In dimension $n = 2$, any connected component of the free boundary can not be longer than $\pi h_1(\Omega)^{-1}$;

(v) *The free boundary of E meets $\partial\Omega$ tangentially at the regular points of $\partial\Omega$: more precisely, if the boundary of E cointains a point $x \in \partial\Omega$ at which the normal vector to $\partial\Omega$ is defined, then also the normal vector to ∂E is defined at x, and the two vectors coincide;*

(vi) *Any P-indecomposable component of E is itself in $\mathcal{C}_1(\Omega)$;*

(vii) *The volume of E is bounded from below as follows*

$$|E| \geq \omega_n \left(\frac{n}{h_1(\Omega)}\right)^n ,$$

(viii) *In dimension $n = 2$ the rolling ball property holds, i.e. if the maximal Cheeger set E in Ω contains two balls of radius $r = 1/h_1$ and there exists a $C^{1,1}$ curve $\gamma : [0,1] \to \Omega$ with curvature bounded by $h_1(\Omega)$ connecting the two centers, then $B_r(\gamma(t)) \subset E$ for all $t \in [0,1]$ (see [LP16, Lemma 2.12]).*

Statement (iii) of the previous proposition gains a special meaning when the ambient space Ω is in $\mathbb{R}^2$ as the only planar curves with constant mean curvature are unions of arcs of circle of same radius. This trivial fact is really important as it allows to prove many other properties for the planar case that either require much more complex proofs or have no higher dimensional counterpart. Among these we recall the next theorem which was first partly proved in [SZ97] and partly in [KLR06] and states that, if Ω is planar and convex, then its Cheeger set is unique, it is the union of all balls in Ω of radius $h_1(\Omega)^{-1}$ and Steiner's formula holds.

Theorem 3.2. *Let Ω be a convex, planar set. Then, there exists a unique Cheeger set for Ω, E which is convex and*[1]

$$E = [\Omega]^r \oplus B_r \,, \tag{3.2}$$

where $[\Omega]^r$ is called the inner Cheeger set, *and denotes the inner parallel set to Ω at distance $r = h_1(\Omega)^{-1}$, i.e. $[\Omega]^r := \{x \in \Omega \,:\, \mathrm{dist}(x;\partial\Omega) > h_1(\Omega)^{-1}\}$. Moreveor, the* inner Cheeger formula *holds, i.e.*

$$|[\Omega]^r| = \pi r^2 \,.$$

As a corollary one obtains the following characterization for planar self-Cheeger sets.

Corollary 3.3. *Let Ω be a convex, planar set. Then, Ω is the only element in $\mathcal{C}_1(\Omega)$ if and only if*

$$\bar{\kappa} \leq \frac{P(\Omega)}{|\Omega|} \,, \tag{3.3}$$

where $\bar{\kappa}$ is the maximum value of the curvature of $\partial\Omega$.

A result in the same spirit of Corollary 3.3 has been proved in [Che80], namely

[1] The Minkowski sum $A \oplus B$ is defined as $A \oplus B = \{a + b \,:\, a \in A \,,\, b \in B\}$.

Theorem 3.4. *Let Ω be a bounded, planar set with piecewise smooth boundary. Let Ω satisfy the* interior rolling ball property *with radius $r = |\Omega|/P(\Omega)$, i.e.*

- *for any regular point $x \in \partial\Omega$, one can fit a tangent to x ball of radius r inside Ω such that no pair of antipodal points is in $\partial\Omega$;*
- *for any non regular point $x \in \partial\Omega$, one can fit a ball centered on any direction between the inner left-hand and right-hand normals to x such that no pair of antipodal points is in $\partial\Omega$*

Then, Ω is the unique element in $\mathcal{C}_1(\Omega)$.

The convexity hypothesis on Ω of Theorem 3.2 is quite strong. Up to our knowledge only one similar result has been proved in [KP11, LP16] and deals with the special case of strips and annuli. In Section 3.1 we prove an analogue theorem for simply connected domains, with boundary a Jordan curve of zero 2-Lebesgue measure, that are "neckless". This second property is somewhat analogue to the interior rolling ball property defined in Theorem 3.4, and indeed from our result one can derive the previous one, at least in the simply connected case. To this aim the generalization of a result of Ionin and Pestov proved in Section 2.4 is used.

The Cheeger problem has sprung in many different contexts since its introduction. In the original paper [Che70], it arose in connection with estimates, on a compact n-dimensional Riemannian manifold M without boundary, of $\lambda_1(M)$, the first eigenvalue of the Laplace-Beltrami operator. In there, he showed that

$$\lambda_1(M) \geq \frac{1}{4}\frac{P(A)^2}{\min\{V(A), V(M\setminus A)\}^2}\,,$$

where $P(A)$ (and respectively $V(A)$) denotes the Riemannian perimeter (volume). For the analogous problem for the Laplacian with Dirichlet boundary conditions on a bounded open set $\Omega \subset \mathbb{R}^n$, the following holds

$$\lambda_1(\Omega) \geq \left(\frac{h_1(\Omega)}{2}\right)^2 .$$

Later on in [KF03], this inequality was generalized to $\lambda_1(p;\Omega)$, the first eigenvalue of the p-Laplacian with Dirichlet boundary conditions and was proved that

$$\lambda_1(p;\Omega) \geq \left(\frac{h_1(\Omega)}{p}\right)^p , \qquad \forall p > 1\,.$$

Moreover, in the same paper the authors show that $\lambda_1(p;\Omega) \xrightarrow{p\to 1} h_1(\Omega)$.

Another application of the classic Cheeger problem is the study of plate failure under stress as studied in [Kel80]. Following the notation of that paper and denoting by p the constant uniform pressure to which is subject the planar plate Ω, one wants to determine the minimal value of p for which the plate breaks down (without considering bending or buckling phenomena). The author shows that failure does not occur if and only if for every $E \subset \Omega$ one has

$$p|E| \leq \sigma P(E)\,, \tag{3.4}$$

where $\sigma > 0$ is a constant. Clearly, equation (3.4) is equivalent to ask $p \leq \sigma h_1(\Omega)$.

Another research area in which the Cheeger constant plays a big role is the one of maximal flow-minimal cut problems. As studied in [Gri06, Str83, Str10], given a planar and bounded set Ω and given two functions $f, c : \Omega \to \mathbb{R}$, one wants to find the maximal value of a constant λ such that there exists a vector field $v : \Omega \to \mathbb{R}^2$ satisfying

$$\begin{cases} \operatorname{div} v & = \lambda f \,, \\ |v| & \leq c \,. \end{cases} \tag{3.5}$$

The scenario depicted is the following: given a source or sink term, f, one wants to find the maximal flow in Ω under a capacity constraint, c. When the given functions are identically equal to 1, then the maximal value λ is exactly the Cheeger constant and the boundary of an associated Cheeger set is the corresponding minimal cut. Notice that the task of finding a vector field satisfying (3.5) while $f, c \equiv 1$ also came up in [Fin79] in which the author was dealing with existence of capillary surface. More on this will be said further on. These maximal flow-minimal cut techniques have found application in image processing and reconstruction in particular in the medical field (see [AT06]). Another different approach to image reconstruction, using the now-called ROF model (see [CL97, ROF92]) was proved to be strictly linked to the Cheeger problem. In those papers the proposed model aimed to regularize noisy images by preserving the essential contours of the objects while removing the noise. The associated functional is proved to be uniquely minimized by the characteristic function of the ambient space Ω (times a multiplicative constant where the Cheeger ratio of Ω is involved) if and only if the set is *calibrable*. We here do not delve in further details but we just recall that calibrability for a bounded mean-convex sets is equivalent to being a Cheeger set. One implication can be found in [ACC05a] while the opposite one in [Leo15]. Image processing and calibrability have been dealt to as well in [BCN02, ACC05b, CCN11].

Finally, I want to recall that the classic problem (3.1) has then been tweaked in many different directions: it has been proposed in gaussian settings (see [CMJN10]), or modified to the non-local fractional perimeter (see [BLP14]), or considered with weights to the perimeter and volume as in [CC07, CCMN08, CFM09] or dealt with suitable powers of the volume [PS17]. Specifically, the weighted version proved to have interesting applications both in landslide modelling [HILRR02, HILR05, ILR05] and the existence of surfaces of prescribed mean curvature, see [Gia74, Giu78]. This last is tightly related to the capillarity problem [CF74] which will be discussed in depth in Chapter 4. Therefore, in Section 3.2 we introduce a further generalization of problem (3.1) which encloses both the weighted version and the "volume-power"'s one. Namely, we will deal with

$$\inf_{E \subseteq \Omega} \frac{\int_{\partial^* E} g(x, \nu_E(x)) \, d\mathcal{H}^{n-1}(x)}{\left(\int_E f \, dx\right)^{1/\alpha}} \,,$$

where $\alpha \in [1, 1^*)$, $g(y, v)$ is a scalar function, lower semi-continuous in $(y, v) \in \mathbb{R}^n \times \mathbb{R}^n$, convex and positively 1-homogeneous in v and such that for some $C > 0$ one has

$$\frac{1}{C}|v| \leq g(y,v) \leq C|v|\,.$$

We will first show that the problem admits minimizers and that those minimizers display nice isoperimetric properties, i.e. they allow the classic Sobolev embeddings and, under the further assumption $P(E) = \mathcal{H}^{n-1}(\partial E)$, the trace theorem. This result is not surprising and it is somewhat expected for minimizers to behave quite well. In [DS98] the authors prove this kind of properties to hold for quasi-minimizers with respect to any variation, while we here allow only inner variations.
The results presented in Section 3.1 have been proved in [LNS17], while the one in Section 3.2 in [Sar16]. Finally the example shown in Subsection 3.3 was explicitly built in [LS17].

3.1 The inner Cheeger formula for simply connected domains

Finding Cheeger sets of a given domain Ω explicitly is a generally difficult problem. Some numerical methods based on duality theory were employed in [LRO05, BCC07, CCP09] to approximate the maximal Cheeger set of Ω. However, apart from a few specific examples, Cheeger sets had been precisely characterized for only two classes of domains: convex planar sets and planar strips. We here provide an extension to Theorem 3.2 to a wider class of sets that covers as well the extension to planar strips given in [KP11] and [LP16] (but for the generalized annuli which are not covered in our result).

Accordingly to [KLR06, KP11, LP16], for convex sets and strips Ω in $\mathbb{R}^2$, there is a unique Cheeger set E characterized as

$$E = \bigcup_{x\in[\Omega]^r} B_r(x) = [\Omega]^r \oplus B_r\,, \tag{3.6}$$

where $r = h_1(\Omega)^{-1}$ and $[\Omega]^r$ is the inner parallel set to Ω at distance r, as in Theorem 3.2. Moreover, in both cases the inner Cheeger formula holds, which we recall to be

$$|[\Omega]^r| = \pi r^2\,. \tag{3.7}$$

One should not expect a characterization of the type (3.6) to hold in general. Since the constant mean curvature condition forces $\partial E \cap \Omega$ to comprise spherical caps only when $n = 2$, it is unsurprising that counterexamples are easily found for $n \geq 3$. But, even in $\mathbb{R}^2$, we see that (3.6) can fail to hold for all Cheeger sets of certain domain. For instance, we recall the bow-tie domain $\mathcal{W}$ constructed in [LP16, Example 4.2] and depicted in Figure 3.1. The Cheeger set $E_{\mathcal{W}}$ of $\mathcal{W}$ is unique, but $E_{\mathcal{W}}$ includes the "bottleneck" of $\mathcal{W}$, which is not contained in the right-hand side of (3.6).

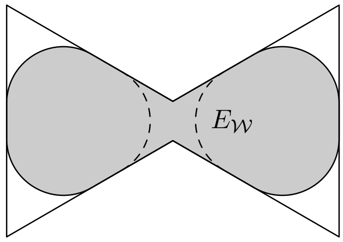


Figure 3.1: The Cheeger set $E_{\mathcal{W}}$ of a bow-tie domain, which does not satisfy (3.6)

In general, given a Cheeger set E of Ω, it is not even true that every connected component of $\partial E \cap \Omega$ is contained in the boundary of a ball of radius $r = h(\Omega)^{-1}$ fully contained in Ω, as illustrated in the following example.

Example 3.5 (The heart domain)**.** We define the set Ω_θ via the following construction. Let B be the ball of radius 1 centered at $(x_1, x_2) = (0, 0)$. For each $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$, let x_θ be the point in $\partial B \cap \{x_2 \le 0\}$ that forms the angle 2θ with $x_0 = (1, 0)$. Let ℓ_θ be the tangent line to B at x_θ, and let E_θ be the region enclosed be the arc of ∂B with angle $2\pi - 2\theta \ge \pi$, ℓ, and $\{x_1 = 1\}$. Finally, let Ω_θ be the union of E_θ with its reflection across $\{x_1 = 1\}$ (see Figure 3.2).

The edges of Ω formed by ℓ_θ and its reflection meet at an angle, forming an outer corner of Ω. The Cheeger set therefore cannot adhere to this corner, and thus the boundary of the Cheeger set will have a circular arc of radius $\frac{1}{h_1(\Omega_\theta)}$ as shown in Figure 3.2.

We compute

$$|\Omega_\theta| = 2[\pi - \theta + \tan\theta],$$
$$P(\Omega_\theta) = 2[2(\pi - \theta) + \tan\theta].$$

We therefore have

$$r = \frac{1}{h_1(\Omega_\theta)} \ge \frac{|\Omega_\theta|}{P(\Omega_\theta)} = 1 - \frac{\pi - \theta}{2(\pi - \theta) + \tan\theta}.$$

If $\frac{1}{2}\tan\theta < 1 - \frac{\pi-\theta}{2(\pi-\theta)+\tan\theta}$, then it is impossible to fit a ball of radius r in Ω along the bottom corner. Since

$$1 - \frac{\pi - \theta}{2(\pi - \theta) + \tan\theta} > \frac{1}{2}\tan\theta$$

for $\theta = \frac{\pi}{4}$, then by continuity, one can choose $\frac{\pi}{4} < \theta$ sufficiently close to $\frac{\pi}{4}$ so that the strict inequality still holds.

In this section we show that for a broad class of domains in $\mathbb{R}^2$ including both convex sets and strips, the maximal Cheeger set is given by (3.6). To this end, in the following definition we introduce the concept of *bottleneck of radius* r.

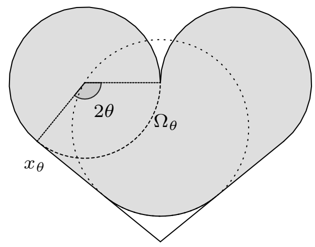


Figure 3.2: The Heart domain Ω_θ of Example 3.5

Definition 3.6. A domain Ω *has no bottlenecks of radius* r if the following condition holds. Suppose $B_r(x_0)$ and $B_r(x_1)$ are two balls of radius r contained in Ω, then there exists a $C^{1,1}$ curve with curvature bounded from above by $1/r$ parametrized by $\gamma : [0,1] \to \Omega$ such that

$$\gamma(0) = x_0, \qquad \gamma(1) = x_1, \qquad \text{and} \qquad B_r(\gamma(t)) \subset \Omega \text{ for all } t \in [0,1].$$

Remark 3.7. We notice for future reference that the property of having no bottlenecks of size r implies the path-connectedness of $[\Omega]^r$.

The main result is the following.

Theorem 3.8. *Suppose $\Omega \subset \mathbb{R}^2$ is an open bounded set such that*

$$\partial\Omega \text{ is locally homeomorphic to an interval and } |\partial\Omega| = 0, \tag{T}$$

$$\Omega \text{ is simply connected,} \tag{SC}$$

$$\Omega \text{ has no bottlenecks of radius } r = 1/h_1(\Omega). \tag{NB}$$

Then the maximal Cheeger set E of Ω is given by (3.6) *and the inner Cheeger formula* (3.7) *holds.*

Let us make some comments on the assumptions we make on Ω. On one hand, the assumptions (SC) and (NB) are essentially necessary. This can be understood by considering two simple examples: one is the bow-tie domain $\mathcal{W}$ shown in Figure 3.1 (here the problem is that the domain does not satisfy (NB)); the other is a domain Ω given by a disc D minus a sufficiently small disc H placed inside D and near ∂D, for which one can show that, if the radius of H is small enough, then the whole domain Ω is the unique Cheeger set of itself, even though it does not satisfy (3.6) (here the problem is that Ω is not simply connected, i.e., it does not satisfy (SC)). Hypothesis (NB) can be probably relaxed, by taking a weaker definition of bottleneck. Namely, in Definition 3.6 one might drop the regularity hypotheses on γ to the lone continuity. The stronger requirements we imposed are tuned to use Theorem 3.1 (viii), which can possibly hold as well for less regular γs. On the other hand, it is not clear whether the topological assumption (T) is necessary. We strongly use the locally homeomorphic structure in the proof of Lemma

2.21, which is a key proposition for proving the theorem and the hypothesis $|\partial\Omega| = 0$ for the proof of Proposition 3.15. We remark however that, at the same time, (T) is not too stringent, as it is satisfied by every domain bounded by a finite number of closed disjoint Jordan curves that are not Osgood-type curves, i.e. homeomorphisms of $\mathbb{S}^1$ with positive 2-Lebsegue measure (for explicit examples one can refer to [Sag94]).

An apparent drawback of Theorem 3.8 is the fact that it requires the a priori knowledge of the value of $h_1(\Omega)$ to verify hypothesis (NB). We thus prove the following corollary, whose assumptions are totally independent of $h_1(\Omega)$ and, therefore, much easier to verify. The idea is simple: instead of requiring the no-bottleneck property exactly for the (unknown) radius $r = 1/h_1(\Omega)$, we can ask a more restrictive but easier-to-check assumption (NB') involving a lower and an upper bound on r.

Corollary 3.9. *Suppose $\Omega \subset \mathbb{R}^2$ is such that (T) and (SC) hold. Suppose further that*

$$\Omega \text{ has no bottlenecks of radius } s, \text{ for all } \frac{\operatorname{inr}(\Omega)}{2} \leq s \leq \frac{1}{2}\left(\frac{|\Omega|}{\pi}\right)^{\frac{1}{2}}, \tag{NB'}$$

where $\operatorname{inr}(\Omega)$ *denotes the inradius of* Ω*. Then the maximal Cheeger set E of Ω is given by* (3.6) *and the inner Cheeger formula* (3.7) *holds.*

Remark 3.10. We observe that the assumptions (T), (SC), and (NB) (or (NB')) also imply the uniqueness of the *minimal Cheeger set*. This can be seen reasoning by contradiction as follows. Suppose that there exist two disjoint minimal Cheeger sets $E_1, E_2 \subset \Omega$, then by Theorem 2.20 there exist two balls $B_r(x_1) \subset E_1$ and $B_r(x_2) \subset E_2$. By (NB) there exists a path γ such that we can roll, say, $B_r(x_1)$ towards $B_r(x_2)$ along γ, without exiting Ω. This rolling process would then build a one-parameter family P_t of Cheeger sets such that $P_0 = E_1 \cup E_2$ and, for some $t > 0$, $\partial P_t \cap \Omega$ would necessarily exhibit a non-admissible singularity at some point, whence a contradiction.

Let us briefly discuss the method of proof for our main theorems. We prove Theorem 3.8 in two steps. First, we show that a maximal Cheeger set E of Ω contains the union of all balls of radius $r = 1/h_1(\Omega)$ contained in Ω. Second, we show that this containment is, in fact, an equality. For the first step, we have at our disposal the rolling ball property of Theorem 3.1 (viii): if E contains $B_r(x_0)$ for some x_0, then it contains any ball of the same radius that can be reached by rolling $B_r(x_0)$ inside of Ω. Coupling the rolling ball property with the no-bottlenecks assumption (NB), we reduce the problem to show that E contains at least *one* ball of radius r. Since the curvature of E is bounded from above by $h_1(\Omega)$ in the viscosity sense (see Definition 2.16 and Lemma 3.13), this follows from Theorem 2.20.

The structural properties of the cut locus introduced in Section 2.4 of ∂E are also the key to proving the second step of the proof of Theorem 3.8, that is, showing that the union of balls of radius $1/h_1(\Omega)$ contained in Ω is the entirety of E. Essentially, if $G = E \setminus \cup B_r$ is nonempty, then the portion of the cut locus that is contained in G contradicts the established properties. Then, to prove the inner Cheeger formula (3.7), we make use of Steiner's formula (1.8).

The rest of the Section is divided as follows. In Subsection 3.1.1 we show that any Cheeger set E of Ω has curvature bounded in the viscosity sense by $h_1(\Omega)$. In Subsection 3.1.2 we show that if a set Ω is simply connected and its boundary is a finite union of disjoint Jordan curves, the same holds for any of its Cheeger sets. Finally, in Subsection 3.1.3 we prove Theorem 3.8.

3.1.1. Curvature bounds in the variational and viscosity sense

Definition 3.11 (Curvature bounded from above in the variational sense)**.** Given $h \in L^1_{loc}$, a set of finite perimeter $E \subset \mathbb{R}^2$ has variational curvature bounded above by h if the following holds. There exists r_0 such for all $F \subset E$ with $F \Delta E \subset\subset B_r(x_0)$, with $0 < r < r_0$ and $x_0 \in \mathbb{R}^2$,

$$P(E; B_r(x_0) \leq P(F; B_r(x_0)) + \int_{E\setminus F} h(x)\, dx.$$

Lemma 3.12. *A Cheeger set E of Ω has variational curvature bounded from above by $h_1(\Omega)$.*

Proof. A Cheeger set E minimizes the energy

$$I(E) = P(E) - h_1(\Omega)|E|$$

among all sets $F \subset E$. Rearranging the inequality $I(E) \leq I(F)$, we find

$$P(E) \leq P(F) + h_1(\Omega)(|E| - |F|) = P(F) + \int_{E\setminus F} h_1(\Omega)\, dx. \qquad \square$$

Lemma 3.13. *If E has variational mean curvature bounded from above by a constant $h \in \mathbb{R}$, then E has curvature bounded from above by h in the viscosity sense at every point $x \in \partial E$.*

Proof. Up to rescaling we may assume that $h = 1$, i.e.,

$$P(E) - |E| \leq P(F) - |F| \qquad \text{for all } F \subset E. \tag{3.8}$$

Arguing by contradiction, we assume that there exist $x_0, y_0 \in \mathbb{R}^2$, $\varepsilon > 0$ and $0 < r < 1$, such that $x_0 \in \partial E \cap \partial B_r(y_0)$ and $\partial E \cap B_\varepsilon(x_0) \setminus \{x_0\} \subset B_r(y_0)$. Up to an isometry, we have $y_0 = (0, 0)$ and $x_0 = (0, r)$. In the infinite strip $S = (-1, 1) \times \mathbb{R}$ we consider the unit vector field $g(x_1, x_2) = \left(x_1, \sqrt{1 - x_1^2}\right)$ with divergence constantly equal to one. Consider the one-parameter family of unit half-circles

$$A_t = \left\{(x_1, x_2) : \ |x_1| < 1,\ x_2 = t + \sqrt{1 - x_1^2}\right\}, \quad t \in \mathbb{R},$$

which foliates S (notice that g is normal to A_t for all t). Let t_0 be such that $x_0 \in A_{t_0}$. For any $t \in (t_0 - \varepsilon, t_0)$ we set

$$E_t = \left(E \setminus B_\varepsilon(x_0)\right) \cup \left(E \cap B_\varepsilon(x_0) \cap \{x \in S : \ x \text{ lies below } A_t\}\right).$$

Note that $E_{t_0} = E$ because A_{t_0} is outside B_r, while in general $E_t \subset E$. By the divergence theorem, for almost every $t \in (t_0 - \varepsilon, t_0)$ we get

$$|E \setminus E_t| = \int_{E \setminus E_t} \operatorname{div} g = \int_{\partial_{t,+} E \cap B_\varepsilon(x_0)} g \cdot \nu_E - \mathcal{H}^1(E \cap A_t \cap B_\varepsilon(x_0))\,,$$

where $\partial_{t,+}E$ denotes the set of points in ∂E that belong to S and stay below A_t. Since $|g| = 1$, this last computation shows in particular that

$$|E \setminus E_t| < \mathcal{H}^1(\partial_{t,+}E \cap B_\varepsilon(x_0)) - \mathcal{H}^1(E \cap A_t \cap B_\varepsilon(x_0))\,. \tag{3.9}$$

Notice that the previous inequality is always strict since otherwise each connected component of $\partial_{t,+}E$ is contained in some arc $A_{t'}$, but then necessarily $t' = t$ and thus x_0 cannot be a point of ∂E, that is, a contradiction. Now we set

$$a = \mathcal{H}^1(E \cap A_t \cap B_\varepsilon(x_0)), \qquad b = \mathcal{H}^1(\partial_{t,+}E \cap B_\varepsilon(x_0))\,,$$

and estimate $P(E_t)$ and $|E_t|$ using (3.9):

$$|E_t| = |E| - |E \setminus E_t| > |E| - (b - a)$$

and

$$P(E_t) = P(E) - (b - a)\,.$$

In conclusion we find

$$P(E_t) - |E_t| < P(E) - (b - a) - |E| + (b - a) = P(E) - |E|\,,$$

which contradicts (3.8). □

3.1.2. Properties inherited by a Cheeger set

Proposition 3.14. *Suppose $\Omega \subset \mathbb{R}^2$ is an open bounded set such that $\partial\Omega$ is a finite union of pairwise disjoint Jordan curves. Then, for any Cheeger set E of Ω, ∂E is a finite union of pairwise disjoint Jordan curves.*

Proof. Let E be a Cheeger set in Ω and let $x \in \partial E \cap \partial\Omega$. By Proposition 2.18 there are an open neighborhood U of x with arbitrarily small diameter and a simple parametric curve $\sigma : [-1, 1] \to \mathbb{R}^2$ such that $\sigma(-1, 1) = \partial\Omega \cap U$ and $\sigma(0) = x$. Since $\partial E \cap \Omega$ consists of the union of disjoint circular arcs, to prove the proposition it suffices to show that $\partial E \cap U$ is the image of a continuous injective parametrization. We now split the proof in two steps.

Step 1. Suppose additionally that $x \in \overline{\partial E \cap \Omega}$. Then, up to choosing a smaller neighborhood U of x, the connected components of $\partial E \cap \Omega \cap U$ are circular arcs with radius $r = 1/h_1(\Omega)$ that satisfy the following properties:

(i) both endpoints of an arc $\alpha \subset \partial E \cap \Omega \cap U$ belong to $\sigma(-1,1)$, up to a possible exception of a finite number of arcs with only one endpoint on $\sigma(-1,1)$;

(ii) if $\alpha \subset \partial E \cap \Omega \cap U$ has both endpoints on $\sigma(-1,1)$, and if $a_i = \sigma(t_i)$ for $i = 1,2$ denote the endpoints of α, then either $t_1, t_2 \le 0$ or $t_1, t_2 \ge 0$;

(iii) if $\alpha, \beta \subset \partial E \cap \Omega \cap U$ denote two distinct arcs having all endpoints on $\sigma(-1,1)$, and if $a_i = \sigma(t_i)$ and $b_i = \sigma(s_i)$, $i = 1,2$, respectively denote the endpoints of α and β, then assuming w.l.o.g. that $t_1 < t_2$ and $s_1 < s_2$, we either have $t_2 < s_1$ or $s_2 < t_1$.

If an endpoint of an arc $\alpha \subset \partial E \cap \Omega \cap U$ does not lie in $\sigma(-1,1)$, then it must lie in U^c. Hence, if α has one or more endpoints lying in U^c and $B_r(x) \cap \alpha$ is nonempty, where $r = \mathrm{dist}(x, U^c)/2$ then the length of α is at least r. Therefore, as E has finite perimeter, we deduce that there acn only be finitely many such arcs. Up to decreasing the diameter of U, this proves (i).

The proof of (ii) goes as follows: assume by contradiction that there exists $\alpha \subset \partial E \cap U \cap \Omega$ with endpoints $a_1 = \sigma(t_1)$ and $a_2 = \sigma(t_2)$ such that, for instance, $t_1 < 0 < t_2$. Then concatenating the curves α and $\sigma([t_1, t_2])$ we obtain a Jordan curve γ that encloses a domain D. Since x belongs to the closure of $\partial E \cap \Omega$, but does not belong to α, it must be that $D \cap E$ has positive measure. On the other hand, $D \cap E$ is also a P-indecomposable component of E, thus it is a Cheeger set of Ω. Since the Jordan curve γ is contained in U, and U has an arbitrarily small diameter, we reach a contradiction with the uniform volume lower bound of Theorem 3.1 (vii). Finally, the proof of (iii) is obtained by arguing as in the proof of (ii).

Step 2. Let now $x \in \partial E \cap \partial\Omega$. We have two possibilities: either $x \in \overline{\partial E \cap \Omega}$, or not. In the latter case the conclusion is immediately achieved, as there would exist a neighborhood U of x such that $\partial E \cap U = \partial\Omega \cap U$. In the former case we can apply Step 1 and discuss the following 3 cases. The first case is when x is the endpoint of m arcs $\alpha_1, \dots, \alpha_m$ of $\partial E \cap \Omega \cap U$ with $m \ge 3$: this case can be excluded by the following argument. Fix $t > 0$ such that $B_t(x) \subset U$, then denote by q_j the intersection of α_j with $\partial B_t(x)$, for $j = 1, \dots, m$. We also assume without loss of generality that the q_j's are ordered with respect to the standard, positive orientation of $\partial B_t(x)$, and that the arc of $\partial B_t(x)$ between q_1 and q_2 is contained in E (this can be always guaranteed up to reversing the orientation and consistently relabeling the points). Let θ_{12} be the angle spanned by the half-tangents to α_1 and α_2. Observe that $\theta_{12} \ge \pi$ otherwise we could "cut the angle" producing a variation of E that would improve the quotient $P(E)/|E|$, that is, a contradiction. Then, we consider the arc α_3 and observe that it must be tangent to α_2 at x, otherwise we could perturb the Cheeger set by cutting the angle θ_{31} formed by the half-tangents to α_3 and α_1. But then we could "shortcut" $\alpha_2 \cup \alpha_3$ near x as depicted in Figure 3.3, reducing the perimeter and increasing the area, a contradiction with the minimality of E.

The second case is when x is the endpoint of exactly two arcs: here we simply deduce that $\partial E \cap U \cap \Omega$ coincides with the union of these two arcs intersected with $U \cap \Omega$.

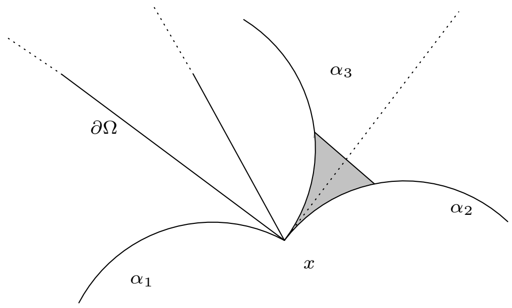


Figure 3.3: The cut from α_2 to α_3 adds the grayed out area to the competitor producing a better Cheeger ratio

The third case is when x is the endpoint of at most one arc in $\partial E \cap U \cap \Omega$. Then, owing to Step 1, the set $\partial E \cap \Omega \cap U$ is given by the union of, at most, two sequences $\{\alpha_j\}_j$ and $\{\beta_j\}_j$ of arcs, whose endpoints are, respectively, $a_{j,1} = \sigma(t_{j,1})$, $a_{j,2} = \sigma(t_{j,2})$, $b_{j,1} = \sigma(s_{j,1})$, $b_{j,2} = \sigma(s_{j,2})$. Of course, one of the two sequences could be undefined (this happens precisely when x is an endpoint of some arc), but the subsequent argument is not affected by this possibility. We thus assume without loss of generality that both sequences are defined. By properties (ii) and (iii) proved in Step 1, setting $A_j = (t_{j,1}, t_{j,2})$ and $B_j = (s_{j,1}, s_{j,2})$, we deduce that

$$t_{j,i} \le 0 \le s_{j,i} \qquad \text{and} \qquad A_j \cap A_k = B_j \cap B_k = \emptyset\,,$$

for all $j \ne k \in \mathbb{N}$ and $i = 1, 2$. It is then clear how to define a parameterization of $\partial E \cap U$ by concatenating circular arcs with pieces of σ. To do this we may assume that every arc α_j (β_j) is parametrized over the interval A_j (B_j), so that we can define for $t \in (-1, 1)$

$$\gamma(t) = \begin{cases} \alpha_j(t) & \text{if } t \in A_j, \\ \beta_j(t) & \text{if } t \in B_j, \\ \sigma(t) & \text{if } t \notin \bigcup_{j\in\mathbb{N}}(A_j \cup B_j). \end{cases}$$

It is easy to check that γ is well-defined in $(-1, 1)$ and provides a continuous, injective parametrization of $\partial E \cap U$. This completes the proof of the proposition. $\square$

Proposition 3.15. *Suppose $\Omega \subset \mathbb{R}^2$ is open, bounded, and simply connected, such that $|\partial\Omega| = 0$. Then any Cheeger set E of Ω is Lebesgue-equivalent to a simply connected open set.*

Proof. Step one. We show that E is Lebesgue-equivalent to the set E° of its interior points. It is enough to check that, given a sequence $\{\Omega_j\}_{j\in\mathbb{N}}$ of relatively compact open subsets of Ω such that $\Omega = \bigcup_{j\in\mathbb{N}} \Omega_j$, one has

$$|(E \setminus E^\circ) \cap \Omega_j| = 0 \qquad \text{for all } j\,. \tag{3.10}$$

Indeed, if we combine (3.10) with the assumption $|\partial\Omega| = 0$ we get

$$|E \setminus E^\circ| \le |\partial E \cap \partial\Omega| + \sum_{j\in\mathbb{N}} |(E \setminus E^\circ) \cap \Omega_j| = 0\,.$$

On the other hand, (3.10) holds since $\partial E \cap \Omega_j$ coincides with the intersection of a finite union of circular arcs with Ω_j, thus in particular $|\partial E \cap \Omega_j| = 0$ for all j.

Step two. Let $G = \Omega \setminus E$. We show the following fact: if G' is a P-indecomposable component of G then

$$P(G';\Omega) < P(G')\,.$$

Indeed, if $P(G';\Omega) = P(G')$, then

$$\begin{aligned}
P(E \cup G') &= P(E \cup G'; \partial\Omega) + P(E \cup G'; \Omega) \\
&\le P(E;\partial\Omega) + P(E;\Omega) + P(G';\Omega) - 2\mathcal{H}^1(\partial^* E \cap \partial^* G' \cap \Omega) \\
&= P(E) - \mathcal{H}^1(\partial^* E \cap \partial^* G' \cap \Omega) \\
&\le P(E)\,,
\end{aligned}$$

which would in turn contradict the fact that E is a Cheeger set, as $|E \cup G'| > |E|$.

Step three. Assuming E open according to Step one, we let γ be a nontrivial simple closed curve contained in E and let F be the bounded subset of $\mathbb{R}^2$ with $\partial F = \gamma$. Note that F is compactly contained in Ω since its closure is a compact set with boundary $\partial F = \gamma$ at a positive distance from ∂E. We see that $F \cap G$ is empty; otherwise $F \cap G$ contains a P-indecomposable component G' of G. As $G' \subset F \subset\subset \Omega$, this contradicts Step Two. We conclude that F is compactly contained in E, hence, E is simply connected. □

3.1.3. Proof of Theorem 3.8

The following lemma will be needed in the proof.

Lemma 3.16. *Suppose that $\Omega \subset \mathbb{R}^2$ is simply connected and has no bottlenecks of radius r. Then $[\Omega]^r$ is path-connected and* $\mathrm{reach}([\Omega]^r) \ge r$.

Proof. Recall that the condition that Ω has no bottlenecks of size r is equivalent to $[\Omega]^r$ being path-connected (see Remark 3.7). Suppose $\mathrm{reach}([\Omega]^r) < r$, so there exists $x_0 \in \Omega \setminus [\Omega]^r$ such that $\mathrm{dist}(x_0, [\Omega]^r) = t < r$ and x_0 has a non-unique projection onto $[\Omega]^r$. Take y_1, y_2 to be two distinct points such that $y_i \in \partial B_t(x_0) \cap \partial[\Omega]^r$ for $i = 1, 2$. Let ℓ be the line passing through y_1 and y_2, splitting $\mathbb{R}^2$ into two open half-planes H^+ and H^-, and let $\partial^\pm\Omega = \partial\Omega \cap H^\pm$.

Since $x_0 \notin [\Omega]^r$, there exists $y_0 \in \partial\Omega$ such that $x_0 \in B_r(y_0)$. Geometrically, we see that y_0 cannot be contained in ℓ, so without loss of generality, say $y_0 \in \partial^+\Omega$. We now construct a path from y_0 to a point $\bar{y} \in \partial^-\Omega$ that disconnects $[\Omega]^r$.

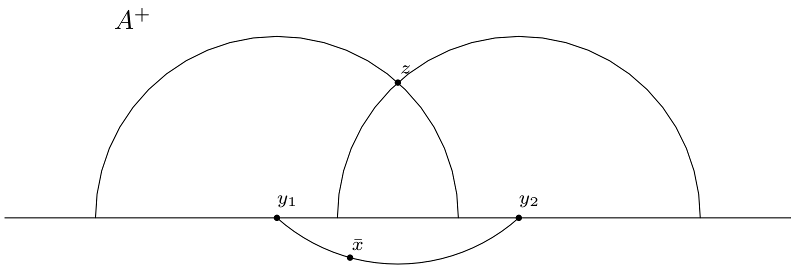


Figure 3.4: A geometric configuration in the proof of Lemma 3.16

Take z to be the unique point in H^+ such that $y_1, y_2 \in \partial B_r(z)$. Being $t < r$, the circular arc $\sigma = \partial B_r(z) \cap H^-$ is contained in $B_t(x_0)$; moreover one has $\sigma \cap \Omega \neq \emptyset$ and $\sigma \cap [\Omega]^r = \emptyset$. Let us fix $\bar{x} \in \sigma \cap \Omega$ and observe that its distance from $\partial^+\Omega$ is necessarily larger or equal to $|\bar{x} - z| = r$. To see this we can consider the set $A^+ = H^+ \setminus \big(B_r(y_1) \cup B_r(y_2)\big)$ and notice that $\mathrm{dist}(\bar{x}, A^+) = |\bar{x} - z| = r$, which implies $\mathrm{dist}(\bar{x}, \partial^+\Omega) \geq r$ (see Figure 3.4). On the other hand $\mathrm{dist}(\bar{x}, \partial\Omega) < r$, therefore we must have $\mathrm{dist}(\bar{x}, \partial\Omega) = \mathrm{dist}(\bar{x}, \partial^-\Omega)$, so that there exists $\bar{y} \in \partial^-\Omega$ satisfying $|\bar{x} - \bar{y}| = \mathrm{dist}(\bar{x}, \partial\Omega)$. Now we consider the piecewise-linear path

$$\Gamma = [\bar{y}, \bar{x}] \cup [\bar{x}, x_0] \cup [x_0, y_0]\,.$$

By construction one has $\Gamma \setminus \{y_0, \bar{y}\} \subset \Omega$ and $\Gamma \cap [\Omega]^r = \emptyset$, thus Γ disconnects $[\Omega]^r$ in two nonempty components, one containing y_1 and the other containing y_2 (notice indeed that the segment $[\bar{x}, x_0]$ necessarily cuts $[y_1, y_2]$ into two nontrivial subsegments). We thus reach a contradiction and the proof is complete. $\square$

Proof of Theorem 3.8. Up to rescaling, we may assume that $h_1(\Omega) = 1$. Let E be the maximal Cheeger set of Ω. By Section 3.1.2 and by Lemma 3.12 the hypotheses of Theorem 2.20 are met by E with curvature bounded by one. Thus, E contains a ball of radius one. By the rolling ball property (see Theorem 3.1(viii)) and the assumption that Ω has no bottlenecks of radius one, we have

$$\bigcup_{B_1 \subset \Omega} B_1 \subset E.$$

We now aim to show the opposite inclusion. Let

$$G = E \setminus \bigcup_{B_1 \subset \Omega} B_1.$$

If G were empty, then the theorem would follow at once. Suppose by contradiction that G is nonempty, then choose $x \in G$ and notice that, necessarily, $\mathrm{dist}(x, \partial E) < 1$. Let P_x denote the set of projections of x onto ∂E as in (2.41). Take $y \in P_x$ and consider the ray $z_t = y + t(x-y)$ as in (2.42). For some $t > 1$, $z = z_t$ is a cut point with $y \in P_z$. Furthermore, $\operatorname{dist}(z, \partial E) < 1$, otherwise x would belong to the union of balls of radius 1 contained in Ω, which is not possible as $x \in G$. Hence, by Proposition 2.26, z is not a focal point, so by Proposition 2.27, we have $\#P_z > 1$. Let γ be a maximal path (with respect to inclusion) in $\mathcal{C} \cap G$ containing z; such a path exists by Zorn's Lemma and is defined, say, on a bounded open interval (a, b). By Proposition 2.28, γ is not the singleton $\{z\}$. We now split the proof in two cases.

Case one: the endpoints $\gamma(a)$ and $\gamma(b)$ of the curve γ are well-defined in the limit sense. Arguing as in the proof of Theorem 2.20, we determine that the closure of γ is not a loop. Moreover, as already observed, $\operatorname{dist}(\gamma(a), \partial E) \geq 1$ and $\operatorname{dist}(\gamma(b), \partial E) \geq 1$. Since Ω has no bottlenecks of radius one, there exists a path $\tilde{\gamma}$ with endpoints $\gamma(a)$ and $\gamma(b)$ such that $B_1(z) \subset \Omega$ for all $z \in \tilde{\gamma}$. The rolling ball property ensures that these balls are contained in E as well. Now, consider the closed loop σ obtained by concatenation of the two paths γ and $\tilde{\gamma}$. Notice that σ is a simple loop as γ and $\tilde{\gamma}$ do not intersect. Since E is simply connected, the domain D_σ bounded by σ is compactly contained in E. Furthermore, since γ is piece-wise C^1 and of positive length, almost all points $z \in \gamma$ are such that $\#P_z = 2$ (and the set of such points is nonempty). Let us fix such a point $x \in \gamma$ with $\#P_x = 2$ and $\operatorname{dist}(x, \partial E) < 1$. Then, the segments $[y_i, x]$ for $y_i \in P_x$, $i = 1, 2$, are transversal to the tangent to γ at x, and lie on opposite sides of γ. Hence, one of the segments has nonempty intersection with D_σ. Suppose it is $[y_1, x]$. Since $y_1 \in \partial E$, the segment $[y_1, x]$ must intersect σ at some $x' \neq x$. Furthermore, we find that $x' \notin \tilde{\gamma}$, as this would imply that $\operatorname{dist}(x, \partial E) \geq 1$. This is impossible by the properties of γ. Hence, $x' \in \gamma$. However, this contradicts the fact that $\#P_{x'} = 1$. This concludes the proof of(3.6) in the first case.

Now we show the validity of (3.7). It remains to be showed the inner Cheeger formula, which we would like to apply to $E = [[\Omega]^r]_r$. To do so, we need to check that $[\Omega]^r$ has positive reach and it is simply connected. The former is granted by Lemma 3.16, and moreover the lemma gives us $\operatorname{reach}([\Omega]^r) \geq r$. Therefore, there exists a projection operator from E into $[\Omega]^r$, say P_E. Take any loop $\gamma \subset [\Omega]^r$, we aim to find a retraction in $[\Omega]^r$ of it into a point. Since E is simply connected by Theorem 3.15, there is a retraction ψ of γ into a point. The composition of $P_E \circ \psi$ provides the wanted retraction. Thus, $[\Omega]^r$ is simply connected, and then we can apply the Steiner formula (1.8). Since

$$\frac{1}{r} = h_1(\Omega) = \frac{P(E)}{|E|},$$

we deduce that

$$r\mathcal{M}_+([\Omega]^r) + 2\pi r^2 = rP(E) = |E| = |[\Omega]^r| + r\mathcal{M}_+([\Omega]^r) + \pi r^2.$$

That is,

$$|[\Omega]^r| = \pi r^2.$$

Case two: the endpoints of γ are not well-defined. In this case we might replace γ with another curve $\hat{\gamma}$ obtained in the following way. First, we restrict γ to a sequence of compact subintervals

$[\alpha_j, \beta_j] \subset (a, b)$ such that $\alpha_j \to a$ and $\beta_j \to b$ as $j \to \infty$. Second, assuming in addition that $\gamma(\alpha_j) \to z_a$ and $\gamma(\beta_j) \to z_b$, we obtain by maximality of γ (and owing to Propositon 2.28) that $\mathrm{dist}(z_a, \partial E) \geq 1$ and $\mathrm{dist}(z_b, \partial E) \geq 1$, so that it is not restrictive to assume the existence of $j \in \mathbb{N}$ and of $t \in (\alpha_j, \beta_j)$ such that, setting $x = \gamma(t)$, we have $\#P_x = 2$ and

$$\mathrm{dist}(x, \partial E) < \min\{\mathrm{dist}(\gamma(\alpha_j), \partial E),\ \mathrm{dist}(\gamma(\beta_j), \partial E)\}\,.$$

Third, by connecting $\gamma(\alpha_j)$ and $\gamma(\beta_j)$ to, respectively, z_a and z_b with two segments, we would obtain $\hat{\gamma}$ as a replacement of γ, having z_a and z_b as endpoints. Up to choosing j large enough, we can also assume that $\mathrm{dist}(x, \partial E) < \mathrm{dist}(y, \partial E)$ for every y belonging to each straight segment. Then we can repeat the same proof as in Case one, with $\hat{\gamma}$ in place of γ. □

We now prove Corollary 3.9.

Proof of Corollary 3.9. We obtain the following simple bounds above and below on the Cheeger constant of Ω:

$$2\left(\frac{\pi}{\Omega}\right)^{\frac{1}{2}} \leq h_1(\Omega) \leq \frac{2}{\mathrm{inr}(\Omega)}.$$

Indeed, the bound below comes from applying the isoperimetric inequality to any $E \subset \Omega$ and using $|E| \leq |\Omega|$, while the upper bound follows simply by taking the competitor $E =$ a ball contained in Ω with the largest possible radius in the minimization problem. Hence, the assumption (NB') on Ω in particular implies that Ω has no bottlenecks of radius $r = 1/h_1(\Omega)$. By applying Theorem 3.8 we conclude the proof. □

3.2 Generalized weighted Cheeger sets

The Cheeger problem has naturally arisen in many different contexts and several generalization have sprung up. In this section we describe the results contained in [Sar16], where a new tweak of the problem has been considered. Fixed an ambient space Ω and denoted by E its subsets of finite perimeter and non zero volume, recall that in [PS17] the authors studied

$$h^\alpha(\Omega) := \inf\left\{\frac{P(E)}{|E|^{1/\alpha}}\right\}\,, \tag{3.11}$$

for values of $\alpha \in [1, 1^*)$, where 1^* equals $n/(n-1)$, while in [ILR05] the authors dealt with

$$h_{f,g}(\Omega) := \inf\left\{\frac{\int_{\partial * E} g\, d\,\mathcal{H}^{n-1}(x)}{\int_E f\, dx}\right\}\,, \tag{3.12}$$

where $f \geq 0, f \not\equiv 0$ and $g \geq g_0 > 0$ are bounded and continuous functions on $\overline{\Omega}$. The generalization we propose here encloses both (3.11) and (3.12) and goes further beyond. We now collect few useful definitions and the notation we will use throughout the whole section, starting by the *weighted volume*.

Definition 3.17 (Weighted volume)**.** Let E be a Lebesgue measurable set in $\mathbb{R}^n$ and $f \in L^\infty(\mathbb{R}^n)$ a positive not identically vanishing weight. We define the weighted Lebesgue measure of E as

$$|E|_f := \int_E f\,dx\,. \tag{3.13}$$

The next proposition, even if somewhat trivial, states an analogue of the classic isoperimetric inequality where the role of the volume is taken up by the weighted volume just defined.

Proposition 3.18. *Let $f \in L^\infty(\mathbb{R}^n)$ a positive not identically vanishing weight. There exists a constant $c = c(f,n)$ such that*

$$|E|_f^{n-1} \le c(f,n)P(E)^n\,,$$

for all Lebesgue measurable subsets $E \subset \mathbb{R}^n$ of finite volume.

Proof. Since f is bounded, we have $|E|_f \le \|f\|_\infty |E|$. Thus, by this and the classic isoperimetric inequality (1.11) we have

$$|E|_f^{n-1} \le \|f\|_\infty^{n-1}|E|^{n-1} \le n\omega_n\|f\|_\infty^{n-1}P(E)^n\,. \qquad \square$$

Once given the definition of weighted volume, we proceed on giving the one of *weighted perimeter*. There are several possibilities for this which end up being equivalent whenever the weight is smooth enough (usually Lipschitz) and the boundary of the ambient space is smooth enough (again Lipschitz). These kind of perimeter are either defined as the total variation of the characteristic functions of sets in weighted BV spaces, as in [CC07] or with a weight inside the integral over the reduced boundary of the set [ILR05]. We here choose this latter one. A twist on the definition of [ILR05] on top of the lower regularity, it is in that we consider a weight depending on both the points of $\partial^* E$ and the outer normal to the set in those points. Basically, what we consider to be our weighted perimeter is a Finsler-type surface energy.

Definition 3.19 (Weighted perimeter)**.** Let E be a Borel set in $\mathbb{R}^n$ and let $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a lower semicontinuous function, convex and positively 1-homogeneous in the second variable for which it exists $C > 0$ such that

$$\frac{1}{C}|v| \le g(x,v) \le C|v|\,, \tag{3.14}$$

for all $(x,v) \in \mathbb{R}^n \times \mathbb{R}^n$. We define the perimeter of E weighted through g in a Borel set $\Omega \subset \mathbb{R}^n$ as

$$P_g(E;\Omega) := \int_{(\partial^* E)\cap\Omega} g(x,\nu_E(x))\,d\mathcal{H}^{n-1}(x). \tag{3.15}$$

We set $P_g(E) = P_g(E;\mathbb{R}^n)$.

Thanks to (3.15), we are able to establish both an upper and a lower bound to the weighted perimeter in terms of the classic perimeter. Hence, the sets of finite perimeter are all and only the sets of weighted finite perimeter in the sense of Definition 3.19.

Proposition 3.20. *The weighted perimeter $P_g(E;\Omega)$ of a set E in Ω has both a lower bound and an upper bound in terms of the classical perimeter $P(E;\Omega)$ given by*

$$\frac{1}{C}P(E;\Omega) \leq P_g(E;\Omega) \leq CP(E;\Omega)\,.$$

Proof. Due to to (3.15) and De Giorgi's characterization of the perimeter of Theorem 1.36 (iii), the claim is immediate. □

We now show that the weighted perimeter functional (3.15) is lower semicontinuous with respect to the L^1 convergence of the characteristic functions. This fact is important since it allows to use the Direct Method of Calculus of Variations to prove the existence of minimizers for the generalized weighted Cheeger problem (3.16), which will be stated shortly after.

Theorem 3.21. *Let Ω be an open set and let $\{E_k\}_k$ be a sequence of bounded sets such that $\sup_k P_g(E_k) < +\infty$. Then, there exists a set E such that $E_k \xrightarrow{L^1} E$ and $P_g(E;\Omega) \leq \liminf_k P_g(E_k;\Omega)$.*

Proof. By Proposition 3.20, the sequence $\{E_k\}_k$ is such that $\sup_k P(E_k) < +\infty$. Then, by Theorem 1.33, there exists E such that $E_k \xrightarrow{L^1} E$ and $D\chi_{E_k}$ weakly star converges to $D\chi_E$. Since the outer normal to a set $-\nu_E$ is the Radon-Nikodým derivative of $D\chi_E/|D\chi_E|$, we can apply Reshetnyak's Theorem 1.19 and obtain

$$\begin{aligned}
P_g(E;\Omega) &= \int_{(\partial^* E)\cap\Omega} g(x,\nu_E(x))\,d\,\mathcal{H}^{n-1}(x) = \int_\Omega g(x,\nu_E(x))\,d\,|D\chi_E| \\
&\leq \liminf_k \int_\Omega g(x,\nu_{E_k}(x))\,d\,|D\chi_{E_k}| = \liminf_k \int_{(\partial^* E)\cap\Omega} g(x,\nu_{E_k}(x))\,d\,\mathcal{H}^{n-1}(x) \\
&= \liminf_k P_g(E_k;\Omega)
\end{aligned}$$

□

We are now ready to define the problem we are interested in, and in an analog way to the classic problem we shall talk of (f,g,α)-*Cheeger constant, set or problem.*

Definition 3.22 (Generalized weighted Cheeger set)**.** Let Ω be an open, bounded and connected set in $\mathbb{R}^n$. Let $\alpha \in [1, 1^*)$, where $1^* := n/(n-1)$. We define the (f,g,α)-Cheeger constant of Ω as

$$h^\alpha_{f,g}(\Omega) := \inf\left\{\frac{P_g(E)}{|E|_f^{1/\alpha}}\right\}\,, \tag{3.16}$$

where the infimum is sought amongst all non empty subsets of Ω with finite perimeter. We shall denote by $\mathcal{C}^\alpha_{f,g}(\Omega)$ the family of minimizers of (3.16).

It is understood that whenever no particular interest lies in the choice of the triplet (f,g,α) we shall drop it and simply refer to the Cheeger problem, constant or set.

Note that the triplet $(1,|v|,1)$ corresponds to the classic Cheeger problem (3.1), while the triplet $(f(x), g(x)|v|, 1)$ corresponds to (3.12) (up to choosing more regular g in some of those papers), and finally the triplet $(1,|v|,\alpha)$ corresponds to (3.11). In all these cases, existence is known and proved in the related papers.

3.2.1. Existence

Existence of minimizers for the infimum in (3.16) is quite easy to prove. The proof is fairly standard and exploits the Direct Method of Calculus of Variations. In order to show it, we will use Proposition 3.20 and Reshetnyak's Theorem 1.19 on the functional P_g.

Theorem 3.23. *For an open bounded set Ω the family $\mathcal{C}^\alpha_{f,g}(\Omega)$ is not empty, i.e. there exists at least one set $E \subset \Omega$ such that*

$$h^\alpha_{f,g}(\Omega) = \frac{P_g(E)}{|E|_f^{1/\alpha}}\,.$$

Proof. Being Ω bounded, it is clear that $h(\Omega) < \infty$. Let $\{E_k\}_k$ be a minimizing sequence for (3.16). Since Ω is bounded, we have that $\{E_k\}_k$ is an equibounded family in L^1. Let now be $\varepsilon > 0$: it has to exist an index $\bar{k}$ such that

$$\left| h(\Omega) - \frac{P_g(E_k)}{|E_k|_f^{1/\alpha}} \right| \leq \varepsilon\,,$$

for all $k \geq \bar{k}$.Therefore,

$$P_g(E_k) \leq (\varepsilon + h(\Omega))\,(\|f\|_\infty |\Omega|)^{1/\alpha}\,,$$

thus $\{E_k\}_k$ is an equibounded family in the BV norm. Thus, up to subsequences, it converges in the L^1 topology and pointwise almost everywhere to a function u. Hence, it is a characteristic function of a set $E \subset \Omega$. Using the lower semicontinuity of P_g of Theorem 3.21 and the L^1 convergence of $f\chi_{E_k}$ to $f\chi_E$, we infer that $E \in \mathcal{C}^\alpha_{f,g}(\Omega)$, as soon as we prove $|E|_f > 0$. Argue by contradiction and suppose it equals zero. Hence, $|E_k|_f \to 0$. Fix a ball B_{r_k} of the same volume of E_k, then by the lower bound of Proposition 3.20 and by the isoperimetric inequality (1.12) we get

$$\frac{P_g(E_k)}{|E_k|_f^{1/\alpha}} \geq \frac{1}{C\|f\|_\infty^{1/\alpha}} \frac{P(E_k)}{|E_k|^{1/\alpha}} \geq \frac{1}{C\|f\|_\infty^{1/\alpha}} \frac{P(B_{r_k})}{|B_{r_k}|^{1/\alpha}} = \frac{n\omega_n^{1-\frac{1}{\alpha}} r_k^{n-1-\frac{n}{\alpha}}}{C\|f\|_\infty^{1/\alpha}} \to \infty\,,$$

against the fact that E_k is a minimizing sequence. □

3.2.2. Isoperimetric properties

We shall now focus on the isoperimetric properties of sets in $\mathcal{C}^\alpha_{f,g}(\Omega)$. We assume from now on that Ω is connected and belongs to $\mathcal{C}^\alpha_{f,g}(\Omega)$ and we prove that it admits the classic Sobolev embeddings and, under the further hypothesis $P(\Omega) = \mathcal{H}^{n-1}(\partial\Omega)$, the classic trace theorem.

Proposition 3.24. *Let $\Omega \in \mathcal{C}^\alpha_{f,g}(\Omega)$ be connected. Then there exists $k = k(n,\Omega) > 0$ such that for every $E \subset \Omega$ one has*

$$\min\{P(E;\Omega^c), P(\Omega\setminus E;\Omega^c)\} \leq k\,P(E;\Omega). \tag{3.17}$$

Proof. Being Ω connected, by [Maz11, comments to Chapter 9] one has that (3.17) is equivalent to

$$\sup_{x\in\partial\Omega}\lim_{\rho\to 0^+}\sup\left\{\frac{P(E;\Omega^c)}{P(E;\Omega)}\middle| E\subset\Omega\cap B_\rho(x)\right\}<+\infty. \tag{3.18}$$

In order to prove (3.18) we assume by contradiction that for some $x\in\partial\Omega$ there exists a decreasing sequence of radii $\rho_j\to 0$ and a sequence of sets $E_j\subset\Omega\cap B_{\rho_j}(x)$, such that

$$\frac{P(E_j;\Omega^c)}{P(E_j;\Omega)}\xrightarrow[j\to\infty]{}\infty. \tag{3.19}$$

We then have

$$0\le\frac{P(E_j;\Omega)}{P(E_j)}\le\frac{P(E_j;\Omega)}{P(E_j;\Omega^c)}\xrightarrow[j\to\infty]{}0, \tag{3.20}$$

which tells us that $P(E_j;\Omega)=o(P(E_j))$. Let now $\Omega_j=\Omega\setminus E_j$. We aim to find a contradiction to the minimality of Ω. By using that for any A

$$P_g(A;\Omega^c)=P_g(A)-P_g(A;\Omega)\,,$$

by Proposition 3.20 and (3.20), we have

$$\begin{aligned}
h^\alpha_{f,g}(\Omega)&\le\frac{P_g(\Omega_j)}{|\Omega_j|_f^{1/\alpha}}=\frac{P_g(\Omega)-P_g(E_j;\Omega^c)+P_g(E_j;\Omega)}{(|\Omega|_f-|E_j|_f)^{1/\alpha}}=\frac{P_g(\Omega)-P_g(E_j)+2P_g(E_j;\Omega)}{(|\Omega|_f-|E_j|_f)^{1/\alpha}}\\
&\le\frac{P_g(\Omega)-C^{-1}P(E_j)+2CP(E_j;\Omega)}{(|\Omega|_f-|E_j|_f)^{1/\alpha}}\le\frac{P_g(\Omega)-C^{-1}P(E_j)+o(P(E_j))}{(|\Omega|_f-|E_j|_f)^{1/\alpha}}
\end{aligned}$$

In order to proceed, notice that $1-x^\beta\le(1-x)^\beta$ for $x\in[0,1]$ and $\beta\in(0,1)$. Thus, by using the isoperimetric inequality stated in Proposition 3.18, the chain of inequalities goes on as

$$h^\alpha_{f,g}(\Omega)\le\frac{P_g(\Omega)-k|E_j|_f^{\frac{n-1}{n}}+o\left(|E_j|_f^{\frac{n-1}{n}}\right)}{|\Omega|^{1/\alpha}-|E_j|_f^{1/\alpha}}<\frac{P_g(\Omega)}{|\Omega|_f^{1/\alpha}}=h^\alpha_{f,g}(\Omega) \tag{3.21}$$

for $j>>1$ and $k>0$, since $|E_j|_f\to 0$ as $f\in L^1(\Omega)$ and $|E_j|\to 0$ and $\alpha<1^*$. Hence, a contradiction. This shows (3.18), thus (3.17). $\square$

By Proposition 3.24 and Theorem 1.53 we immediately deduce the following corollary.

Corollary 3.25. *Let $\Omega\in\mathcal{C}^\alpha_{f,g}(\Omega)$ be connected. If $P(\Omega)=\mathcal{H}^{n-1}(\partial\Omega)$ then there exists a linear continuous operator (the trace) $T:BV(\Omega)\to L^1(\partial\Omega)$.*

In the following lemma we show how inequality (3.17) implies the relative isoperimetric inequality.

Lemma 3.26. *Let Ω be an open, bounded and connected set. If there exists $k=k(\Omega)>0$ such that*

$$\min\{P(E;\Omega^c),P(\Omega\setminus E;\Omega^c)\}\le k\,P(E;\Omega)\qquad\forall\,E\subset\Omega, \tag{3.22}$$

then a relative isoperimetric inequality holds on Ω, i.e. there exists $K=K(\Omega)>0$ such that

$$\min\{|E|;|\Omega\setminus E|\}^{\frac{n-1}{n}}\le K\,P(E;\Omega),\qquad\forall E\subset\Omega. \tag{3.23}$$

Proof. Since $E \subset \Omega$ we have

$$P(E; \Omega^c) = P(E) - P(E; \Omega), \qquad P(\Omega \setminus E; \Omega^c) = P(\Omega \setminus E) - P(E, \Omega).$$

Plugging these identities in (3.22) and then exploiting the isoperimetric inequality give

$$\begin{aligned} \frac{k+1}{n\omega_n^{1/n}} P(E; \Omega) &\geq \frac{1}{n\omega_n^{1/n}} \min\{P(E), P(\Omega \setminus E)\} \\ &\geq \min\left\{ |E|^{\frac{n-1}{n}}, |\Omega \setminus E|^{\frac{n-1}{n}} \right\}, \end{aligned} \tag{3.24}$$

which is the claim for $K(\Omega) = (k(\Omega)+1)^{-1} n\omega_n^{1/n}$. □

By combining Proposition 3.24, Lemma 3.26 and Theorem 1.52 one obtains the following result.

Corollary 3.27. *Let $\Omega \in \mathcal{C}^{\alpha}_{f,g}(\Omega)$ be connected and let $p^* = np/(n-p)$. Then there exists a constant $K = K(\Omega) > 0$ such that*

$$\|u\|_{L^{p^*}} \leq K \|u\|_{W^{1,p}}$$

for all $u \in W^{1,p}(\Omega)$, thus the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous for all $1 \leq q \leq p^*$ and compact for $1 \leq q < p^*$.

The assumption $\Omega = \Omega^{(1)}$, i.e. that it coincides with its points of density 1, implies $P(\Omega) = \mathcal{H}^{n-1}(\partial\Omega)$ whenever Ω is a set admitting a relative isoperimetric inequality. This is indeed an easy consequence of the following lemma and of Federer's Theorem 1.39.

Lemma 3.28. *Let Ω be an open set such that a relative isoperimetric inequality holds on it. Then, $\Omega^{(0)} \cap \partial\Omega = \emptyset$.*

Proof. Argue by contradiction and fix a point $x_0 \in \Omega^{(0)} \cap \partial\Omega$. For any r set

$$m(r) := |\Omega \cap B_r(x_0)|.$$

By Proposition 1.32 (iv), for almost every r, one has

$$m'(r) = P(B_r(x_0); \Omega).$$

By the relative isoperimetric inequality on Ω, for r small enough, it follows

$$\frac{m'(r)}{m(r)^{\frac{n-1}{n}}} \geq C. \tag{3.25}$$

By integrating (3.25) between ρ and 2ρ one obtains

$$m^{\frac{1}{n}}(2\rho) - \frac{n}{c}\rho \geq m^{\frac{1}{n}}(\rho) \geq 0,$$

hence

$$m(2\rho) \geq \left(\frac{n}{2c}\right)^n 2^n \rho^n.$$

However, this contradicts the assumption $x_0 \in \Omega^{(0)}$. □

Proposition 3.29. *Let Ω be an open, bounded set of finite perimeter such that $\Omega = \Omega^{(1)}$ and such that it supports a relative isoperimetric inequality. Then, Ω is weakly-regular, i.e. satisfies as well $P(\Omega) = \mathcal{H}^{n-1}(\partial\Omega)$.*

Proof. By Federer's theorem $\mathcal{H}^{n-1}(\partial^e\Omega) = P(\Omega)$. Therefore it suffices to show that both $\Omega^{(0)} \cap \partial\Omega$ and $\Omega^{(1)} \cap \partial\Omega$ have null measure. By hypotheses the latter is empty, while by the previous lemma the former is empty as well. □

Corollary 3.30. *Let $\Omega \in \mathcal{C}^{\alpha}_{f,g}(\Omega)$ be connected and such that $\Omega = \Omega^{(1)}$. Then, there exists a linear continuous operator (the trace) $T : BV(\Omega) \to L^1(\partial\Omega)$.*

Proof. By hypothesis $\Omega = \Omega^{(1)}$ and by Proposition 3.29 we have $P(\Omega) = \mathcal{H}^{n-1}(\partial\Omega)$. Therefore Corollary 3.25 holds. □

3.3 A Cheeger set with a fat Cantor set in its boundary

In this section we provide an example, firstly described in [LS17], of connected set $\Omega \subset \mathbb{R}^2$ that is a minimal Cheeger set, whose reduced boundary $\partial^*\Omega$ coincides with the super-reduced boundary $\partial^{**}\Omega$ while the perimeter $P(\Omega)$ is strictly smaller than both $\mathcal{H}^1(\partial\Omega)$ and $\mathcal{M}_-(\Omega)$. We also note that, as a consequence of the construction, it is not possible to find a Lebesgue-equivalent open set Ω' for which the perimeter coincides either with $\mathcal{H}^1(\partial\Omega)$ or $\mathcal{M}_-(\Omega)$. Such an example is interesting for two reasons. On one hand, strikingly enough, being a minimizer of an isoperimetric-like problem is not enough to ensure that the hypotheses of Theorem 1.49 are met, thus it is not possible to approximate it from within the interior. On the other hand, it provides a connected Cheeger set that does not satisfy the hypotheses of Corollary 3.30, thus showing that the request $\Omega = \Omega^{(1)}$ in there can not be relaxed.

In virtue of Proposition 3.24 and Proposition 3.29, in order to build a minimal Cheeger set Ω such that its perimeter is strictly smaller than $\mathcal{H}^1(\partial\Omega)$ we must ensure that the set of points of density 1 for Ω that are also contained in $\partial\Omega$ has positive $\mathcal{H}^1$ measure.

Consider the concentric balls $B_1, B_\varepsilon \subset \mathbb{R}^2$, where the radius $\varepsilon < 1$ will be fixed later on. We now define a set $F^\varepsilon \subset B_\varepsilon$ whose topological boundary contains a "fat" Cantor set with positive $\mathcal{H}^1$ measure. We shall show that the open set $\Omega := B_1 \setminus \overline{F^\varepsilon}$ satisfies the requests we made at the beginning of the section.

We consider the segment $C^\varepsilon_0 = [-\varepsilon, \varepsilon] \times \{0\} \subset \overline{B_\varepsilon}$ and iteratively construct a decreasing sequence C^ε_i, $i \in \mathbb{N}$, of compact subsets of C^ε_0, obtained at each step i of the construction by removing 2^{i-1} open segments S^i_j, $j = 1, \dots, 2^{i-1}$, of length

$$\mathcal{H}^1(S^i_j) = 2^{1-2i}\mathcal{H}^1(C^\varepsilon_{i-1}), \qquad \text{for all } j,$$

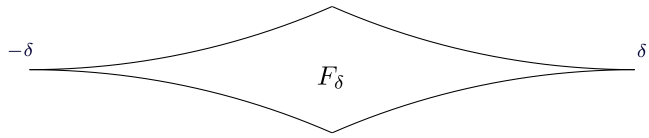


Figure 3.5: The shape of the planar set F_δ

and placed in the middle of each closed segment of C^ε_{i-1}, so that the total loss of length at step i equals $2^{-i}\mathcal{H}^1(C^\varepsilon_{i-1})$. Consequently, the set $C^\varepsilon = \lim_{i\to\infty} C^\varepsilon_i$ satisfies

$$\mathcal{H}^1(C^\varepsilon) = 2\varepsilon \prod_{k=1}^{\infty} \left(1 - 2^{-k}\right) > 0$$

(the strict positivity of the infinite product can be easily inferred by the fact that the series $\sum_{k=1}^{\infty} \log(1-2^{-k})$ is convergent). C^ε is a so-called "fat" Cantor set.

Let now $\delta > 0$ be fixed. We set

$$f_\delta(x) = \begin{cases} 1 - \sqrt{1 - (|x| - \delta)^2} & \text{if } x \in (-\delta, \delta), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$F_\delta = \left\{(x,y) \in \mathbb{R}^2 \,:\, |x| \le \delta,\ |y| \le f_\delta(x)\right\},$$

which is depicted in Figure 3.5. Notice that ∂F_δ is a union of four circular arcs of radius 1. For $i \in \mathbb{N}$ we set $\delta_i = 2^{-2i}\mathcal{H}^1(C^\varepsilon_{i-1})$ and let m^i_j denote the midpoint of S^i_j, then define

$$F^\varepsilon = \bigcup_{i\in\mathbb{N}} \bigcup_{j=1}^{2^{i-1}} F^i_j\,,$$

where $F^i_j = m^i_j + F_{\delta_i}$. Before going on we need the following lemma.

Lemma 3.31. *The set F^ε is contained in $\overline{B_\varepsilon}$.*

Proof. As soon as $\varepsilon < 1$ we have $\delta_i < 1/2$ for all $i \in \mathbb{N}$. Then F^ε is contained in the region bounded by the graphs of f and $-f$, where f is 1-Lipschitz and defined as

$$f(x) = \sum_{i=1}^{\infty} \sum_{j=1}^{2^{i-1}} f_{\delta_i}(x - \mu^i_j)\,,$$

with $(\mu^i_j, 0) = m^i_j$. Since the support of f is contained in $[-\varepsilon, \varepsilon]$ and the Lipschitz constant of f is smaller than 1, we easily conclude that the graph of $\pm f$ restricted to $[-\varepsilon,\varepsilon]$ is entirely contained in $\overline{B_\varepsilon}$, as wanted. $\square$

We finally let

$$\Omega = B_1 \setminus \overline{F^\varepsilon}\,. \tag{3.26}$$

Proposition 3.32. *The open set Ω defined in (3.26) satisfies $P(\Omega) < \mathcal{H}^1(\partial\Omega)$.*

Proof. In general we have $P(F^\varepsilon) \le \mathcal{H}^1(\partial F^\varepsilon)$, therefore $P(F^\varepsilon)$ is finite because $\mathcal{H}^1(\partial F^\varepsilon)$ is finite by construction. According to Theorem 1.36 we only need to show that $P(F^\varepsilon) = \mathcal{H}^1(\partial^* F^\varepsilon) < \mathcal{H}^1(\partial F^\varepsilon)$. Clearly $\partial F^\varepsilon = C^\varepsilon \cup (F^\varepsilon)^{(1/2)} \cup \hat{F}^\varepsilon$, where $\hat{F}^\varepsilon$ is the set of corner points of ∂F^ε that do not belong to the segment C_0^ε. Since $\hat{F}^\varepsilon$ is at most countable, it has null $\mathcal{H}^1$-measure and therefore

$$\mathcal{H}^1(\partial F^\varepsilon) = \mathcal{H}^1(C^\varepsilon) + \mathcal{H}^1\left((F^\varepsilon)^{(1/2)}\right) = \mathcal{H}^1(C^\varepsilon) + \mathcal{H}^1(\partial^* F^\varepsilon)\,,$$

also owing to Theorem 1.39. The claim follows at once by recalling that $\mathcal{H}^1(C^\varepsilon) > 0$. □

Proposition 3.33. *The set Ω is such that $\partial^*\Omega = \partial^{**}\Omega$ set-wise. At the same time $P(\Omega) < \mathcal{M}_-(\Omega)$.*

Proof. Since at any point $x \in \partial^*\Omega$ the set Ω defined in (3.26) is locally Lipschitz, $\partial^*\Omega$ coincides with the super-reduced boundary $\partial^{**}\Omega$ (see Remark 2.3). On the other hand, one has

$$\mathcal{M}_-(\Omega) \ge \mathcal{H}^1(C^\varepsilon) + P(\Omega) > P(\Omega)\,,$$

which completes the proof of the claim. □

Now we show that Ω is a minimal Cheeger set (or, in other words, that it is the unique Cheeger set of itself). This proof will be obtained through some intermediate steps. First of all, by the boundedness of Ω and by Theorem 3.23 we know that Ω admits at least a Cheeger set, from now on generically denoted as E. Then we have the following result.

Proposition 3.34. *Let $\varepsilon < 1/4$ and let Ω be as in (3.26). Then*

(i) $h_1(\Omega) \in \left(2, \frac{2}{1-\varepsilon}\right]$;

(ii) if E is a Cheeger set of Ω then any connected component of $\partial E \cap \Omega$ is a circular arc with curvature equal to $h_1(\Omega)$ and length less or equal than $\pi h_1(\Omega)^{-1}$;

(iii) any Cheeger set of Ω is P-indecomposable;

(iv) the minimal Cheeger set E_0 of Ω is unique and 2-symmetric;

(v) if $\varepsilon < 1/24$ then the minimal Cheeger set E_0 has only one connected component.

Proof. By Lemma 3.31 we have the inclusions $B_1 \setminus \overline{B_\varepsilon} \subset \Omega \subset B_1$. Then (i) follows directly from noting that the Cheeger problem (3.1) has a monotonicity property. On the other hand

(ii) follows from Theorem 3.1 (ii)-(iv). The proof of (iii) is a bit more involved. By Theorem 3.1 (vii) we have the following lower bound for the volume of any Cheeger set E:

$$|E| \geq \omega_2 \left(\frac{2}{h_1(\Omega)}\right)^2 \geq \omega_2(1-\varepsilon)^2 = \pi(1-\varepsilon)^2. \tag{3.27}$$

We now argue by contradiction supposing that E is P-decomposable, so that there exist S and T, both with positive measure, and such that $E = S \cup T$ and $P(E) = P(S) + P(T)$. Then S and T are both Cheeger sets of Ω (see for instance [Par11]), hence they must satisfy (3.27). Since $\varepsilon < 1/4$ we obtain $|E| = |S| + |T| > 18\pi/16 > \pi = |B_1|$, which is clearly not possible. In order to prove (iv) we notice that, thanks to the symmetry of Ω, the reflection $\tilde{E}_0$ of E_0 with respect to one of the two coordinate axes is a Cheeger set of Ω, too. By the lower bound on the volume one has $|E_0 \cap \tilde{E}_0| > 0$, then by well-known properties of Cheeger sets, such intersection is also a Cheeger set of Ω. Therefore by minimality of E_0 we infer $E_0 = E_0 \cap \tilde{E}_0 = \tilde{E}_0$ up to null sets, which shows the claimed symmetry of E_0. Notice moreover that, by the same argument, E_0 is unique. Finally, for the proof of (v) we can suppose without loss of generality that there are just two connected components E_1, E_2 of E_0, one symmetric to the other with respect to the line containing the segment C_0^ε. By (iii) we must have $P(E_0) < P(E_1) + P(E_2) = 2P(E_1)$. Moreover the strict inequality implies that $\mathcal{H}^1(\partial^* E_1 \cap C_0^\varepsilon) > 0$, so that we obtain

$$2P(E_1) \leq P(E_0) + 2\mathcal{H}^1(C_0^\varepsilon) = P(E_0) + 4\varepsilon\,. \tag{3.28}$$

Hence by (3.28) and the isoperimetric inequality we infer

$$\begin{aligned}
\frac{4}{1-\varepsilon}|E_1| &\geq 2h_1(\Omega)|E_1| = h_1(\Omega)|E_0| = P(E_0) \\
&\geq 2P(E_1) - 4\varepsilon \geq 4\sqrt{\pi}|E_1|^{\frac{1}{2}} - 4\varepsilon = 4\sqrt{\frac{\pi}{2}}|E_0|^{1/2} - 4\varepsilon \\
&\geq \frac{4\pi}{\sqrt{2}}(1-\varepsilon) - 4\varepsilon.
\end{aligned}$$

Then if $\varepsilon < 1/24$ we find

$$|E_0| = 2|E_1| \geq \sqrt{2}\pi(1-\varepsilon)^2 - 2\varepsilon(1-\varepsilon) > \pi,$$

that is, a contradiction. □

Owing to Lemma 3.31 we infer that $\partial\Omega = \partial B_1 \cup \partial F^\varepsilon$. Hereafter we adopt the same notation introduced in the proof of Proposition 3.32, i.e., we denote by $\hat{F}^\varepsilon$ the set of corner points of ∂F^ε that do not belong to C_0^ε.

Theorem 3.35. *Let $\varepsilon < 1/24$. Then Ω defined in (3.26) is a minimal Cheeger set.*

Proof. Let E_0 be a minimal Cheeger set of Ω. By Proposition 3.34 (iv) we know that E_0 is 2-symmetric and unique. Assume now by contradiction that E_0 does not coincide with Ω up to null sets. This implies that $\partial E_0 \cap \Omega \neq \emptyset$, thus there exists at least one, by the 2-symmetry of E_0) connected component of $\partial E_0 \cap \Omega$ consisting of a circular arc α of radius $r = h_1(\Omega)^{-1}$, whose endpoints p, q necessarily belong to $\partial\Omega$. We now rule out all possibilities depending on where the endpoints p and q are located. This will be accomplished by the discussion of the following four cases.

Case 1: one of the endpoints of α belongs to ∂B_1. Let us assume without loss of generality that $p \in \partial B_1$. In this case we have to distinguish two subcases. First, if $q \in \partial B_1$ then α must touch ∂B_1 in a tangential way at both p and q, however the radius r is smaller than $1/2$, so that necessarily $p = q$, that is, α is a full circle, which is in contrast with Proposition 3.34 (ii). Second, if $q \in \partial F^\varepsilon$, the arc α can be symmetric neither with respect to the x-axis nor with respect to the y-axis. Therefore, by symmetry, $\partial E_0 \cap \Omega$ has at least three more other connected components. None of these can touch, but in the endpoints. Then there exist at least two connected components of E_0, which yields a contradiction according to Proposition 3.34 (v).

Case 2: one of the endpoints of α belongs to $\partial^* F^\varepsilon$. Again we can assume that $p \in \partial^* F^\varepsilon$. In this case the arc α is contained in the closure of the ball of radius 1 that is tangent to $\partial^* F^\varepsilon$ at p and does not intersect F^ε (by construction of F^ε there is exactly one such ball for any $p \in \partial^* F^\varepsilon$). Consequently the only possibilities for q are either to coincide with p (which is not possible as discussed in the first subcase of Case 1) or to belong to ∂B_1 (which has been excluded in the second subcase of Case 1). This completes the discussion of this case.

Case 3: p and q belong to the fat Cantor set C^ε. By the assumption on ε coupled with Proposition 3.34 (i) we infer that $r = h_1(\Omega)^{-1} > 2\varepsilon$. Since $|p - q| \leq 2\varepsilon$ we deduce that α is the smaller arc cut by the chord $\overline{pq}$ on one of the two possible circles of radius r passing through both p and q. Without loss of generality we may assume that α is contained in the upper half-plane delimited by the x-axis, so that we have necessarily that $\alpha \subset B_\varepsilon$ and that E_0 has a connected component E_0' bounded by α and by a piece of the graph of the function f over $[-\varepsilon, \varepsilon]$. Taking into account (3.27) and Proposition 3.34 (v), this leads to

$$\pi\varepsilon^2 \geq |E_0'| \geq \pi \left(\frac{2}{h(\Omega)} \right)^2 \geq \pi(1-\varepsilon)^2,$$

in contradiction with the fact that $\varepsilon < 1/24$.

Case 4: one endpoint belongs to $\hat{F}^\varepsilon$, the other to $\hat{F}^\varepsilon \cup C^\varepsilon$. As before we assume that p is a corner point on the graph of f and that $q \in \hat{F}^\varepsilon \cup C^\varepsilon$, without loss of generality. We preliminarily notice that q belongs to the upper half-plane: indeed arguing by contradiction, and via the same argument involving the chord $\overline{pq}$ of Case 3, one would infer that α necessarily crosses the segment C_0^ε, which is not possible. This means that q belongs to the graph of f over $[-\varepsilon, \varepsilon]$. Moreover, the curvature vector associated with α at p must have a positive component with respect to the y-axis, otherwise we would fall into the same situation of Case 3 (i.e., the presence of a too small connected component of E_0). Consequently, by comparing the graph of f (whose generalized curvature is bounded from above by 1) with the arc α (whose curvature is $h_1(\Omega) \geq 2$) we deduce by the maximum principle that their intersection can only contain p, which contradicts the fact that q belongs to that intersection. This concludes the discussion of Case 4, and thus the proof of the theorem. $\square$

CHAPTER 4

The Prescribed Mean Curvature problem

Let Ω be an open bounded set in $\mathbb{R}^n$ and let $H : \Omega \to \mathbb{R}$ be a Lipschitz continuous function. A classical solution to the Prescribed Mean Curvature equation is a function $u : \Omega \to \mathbb{R}$ of class C^2 satisfying

$$\operatorname{div}\left(\frac{\nabla u(x)}{\sqrt{1+|\nabla u(x)|^2}}\right) = H(x) \qquad \forall\, x \in \Omega\,. \tag{PMC}$$

The left-hand side corresponds to the mean curvature of the graph of u at the point $(x, u(x))$ (refer to Section 1.6).

The existence and the properties of solutions to (PMC), possibly satisfying some given boundary conditions, have been the object of extensive studies in the past, also due to the close connection between (PMC) and the physical phenomenon of capillarity whose link is described in Section 4.1. After the pioneering works by Young [You05], Laplace [Lap06], and Gauss [Gau30], it is nowadays a well-known fact that the mean curvature of a capillary surface in a cylindrical container with cross-section Ω is determined by the surface tension, by the wetting properties of the fluid with respect to the container, and by the presence of external forces such as gravity. The modern theory of capillarity has its roots in a series of fundamental papers by Finn [Fin65], Concus-Finn [CF69, CF74], Emmer [Emm73, Emm76], Gerhardt [Ger74, Ger75, Ger76], Giaquinta [Gia74], Giusti [Giu76, Giu78], and many others (see [Fin86] and the references therein). Other contributions to the theory have been obtained in various directions, see for instance Tam [Tam86b, Tam86a], Finn [Fin88], Concus-Finn [CF91], Caffarelli-Friedman [CF85], as well as more recent works by De Philippis-Maggi [DPM15], Caffarelli-Mellet [CM07] and Lancaster [Lan10]. However the above list is far from being complete.

A necessary condition on the pair (Ω, H) for the existence of a solution to (PMC) can be easily found by integrating (PMC) on any relatively compact set $A \subset \Omega$ with smooth boundary. Indeed, by applying the divergence theorem we get

$$\left| \int_A H\, dx \right| \le \int_{\partial A} |\langle Tu, \nu\rangle|\, d\mathcal{H}^{n-1}\,,$$

where ν is the exterior normal to ∂A and $\mathcal{H}^{n-1}$ is the Hausdorff $(n-1)$-dimensional measure in $\mathbb{R}^n$. Then using the fact that the vector field

$$Tu(x) := \frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}}$$

has modulus less than 1 on Ω, we obtain for every such A the strict inequality

$$\left| \int_A H \; dx \right| < P(A), \tag{4.1}$$

where $P(A)$ denotes the perimeter of A (when ∂A is smooth, $P(A) = \mathcal{H}^{n-1}(\partial A)$; more generally, $P(A)$ has to be understood in the sense of Definition 1.29).

Notice that whenever H is a non-vanishing positive L^∞ weight on Ω the necessary condition (4.1) says that the Cheeger ratio $P(A)/|A|_H \geq 1$, for all relatively compact subsets $A \subset \Omega$ with positive volume. Hence, the existence of solutions to (PMC) is closely related to the generalized Cheeger problem discussed in Section 3.2.

In the fundamental paper [Giu78], Giusti proved the necessary condition (4.1) to be also sufficient for the existence of solutions to (PMC) in any bounded connected open set Ω with Lipschitz boundary. More specifically, he showed that if (4.1) holds together with the strict inequality

$$\left| \int_\Omega H \, dx \right| < P(\Omega) \tag{4.2}$$

then one can find many variational solutions (see [Gia74]) attaining any given Dirichlet $L^1(\partial\Omega)$ boundary datum in a weak sense. We shall call any pair (Ω, H) satisfying (4.1) an *admissible* pair. On the other hand, a much more subtle situation occurs when the equality

$$\left| \int_\Omega H \, dx \right| = P(\Omega) \tag{4.3}$$

holds, as it corresponds to the so-called *extremal case*. Whenever an admissible pair (Ω, H) is such that (4.3) holds, we will call the pair *extremal*.

It is worth noting that in the non-extremal case the existence of solutions is genuinely variational, while in the extremal case, one can essentially consider a suitably translated sequence of variational (non-extremal) solutions u_i of (PMC), defined on subsets Ω_i that converge to Ω both in volume and in perimeter, as $i \to \infty$. Then, one obtains a so-called *generalized solution* u defined on Ω as the limit of u_i (in the sense of the L^1-convergence of the subgraphs, see [Mir77]). The extremal case is particularly relevant because it corresponds to capillarity for a perfectly wetting fluid under zero-gravity conditions. It is well known that the fluid-gas interface meets the (smooth) boundary of the cylindrical container with a constant contact angle γ depending only on the fluids and the material of the cylinder; in other words one expects that any solution u in the extremal case automatically satisfies the boundary condition of Neumann type

$$Tu \cdot \nu = \cos\gamma \qquad \text{on } \partial\Omega\,. \tag{4.4}$$

In [Giu78], it was considered the perfectly-wetting situation i.e. $\gamma = 0$. At the same time, one also experimentally observes that the solution u is unique up to additive constants. This is what Giusti showed to be a consequence of a more general equivalence result (see Theorem 2.1 in [Giu78]) that he proved for the extremal case under the strong regularity assumption $\partial\Omega \in C^2$. Later, Finn observed that the regularity requirements on $\partial\Omega$ can be reduced to piecewise Lipschitz (see [Fin86, Chapter 6]) if one is interested in the existence of solutions to (PMC) in the 2-dimensional case, and to "C^1 up to a $\mathcal{H}^{n-1}$-negligible set" if uniqueness up to vertical translations has to be shown in the extremal case.

In the recent paper [LS16] we prove Giusti's characterization of existence and uniqueness of solutions to (PMC) under the very mild regularity hypothesis on Ω to be weakly regular (see Definition 2.4). In Section 4.2, we discuss how these can be regarded as minimal assumptions for the problem. In Section 4.3 we prove the existence of solutions in both the non-extremal and extremal cases. Then, in Section 4.4 we prove a result analogous to [Giu78, Theorem 2.1] stating a series of facts equivalent to the extremality, among which, most notably, the uniqueness up to vertical translations. Moreover, some remarks on stability are given in Proposition 4.13. Finally, in Section 4.5 we build a family of non-smooth "porous sets" that satisfy the hypotheses of the stability proposition. In particular, those sets provide examples of sets that were not covered in the previous existence theorems.

4.1 A capillarity-type functional

The capillary problem was firstly derived in a modern way by Gauss in [Gau30] where he uses the principle of the virtual work according to which the energy of a mechanical system in equilibrium is unvaried under virtual displacements consistent with the constraints. Suppose that the equilibrium interface between two fluids Γ_u in a cylinder of cross-section Ω is a surface given by the graph of a function u. In a gravity-free environment the energy of u is given by the sum of three terms, i.e. the *free surface energy*, the *wetting energy* and a *volume constraint*. Physically, the surface energy quantifies the work necessary to build the surface separating the two phases and it is proportional to the surface area (times the so-called surface tension). The wetting energy represents the adhesion energy between the two fluids and the rigid walls of the cylinder. In this term as well the surface tension plays a role times the cosine of the angle that is created by the surface and the walls which is constant and dependant only on the materials and not the shape of $\partial\Omega$. Finally, the volume constraint represents the finiteness of mass of the fluid we are dealing with. The energy functional therefore (up to the multiplicative factor of the surface tension parameter) is

$$\mathcal{F}[u] := \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx - \cos\gamma \int_{\partial\Omega} u \, d\mathcal{H}^{n-1}(x) + \int_\Omega \lambda u \, dx \,, \tag{4.5}$$

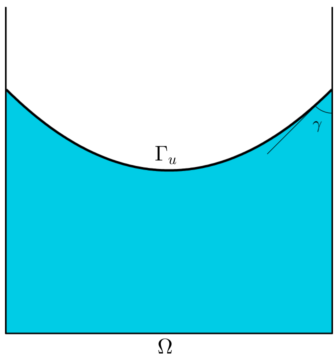


Figure 4.1: The contact angle for a capillary surface

where the first term represents the free surface energy, the second term the wetting energy considering a contact angle of width $\gamma \in [0, \pi/2]$ as in Figure 4.1, while the last is the volume constraint times λ, a Lagrange parameter to be determined. Now, any minimizer of (4.5) must solve the Euler-Lagrange equation for $\mathcal{F}$ which we can find by suitably computing the first order expansion of the functional. Let us compute $\mathcal{F}[u+\varepsilon\eta]$ for a smooth function η defined on $\overline{\Omega}$ and $\varepsilon << 1$.

$$\begin{aligned}\mathcal{F}[u+\varepsilon\eta] &= \int_\Omega \sqrt{1+|\nabla(u+\varepsilon\eta)|^2}\,dx - \cos\gamma\int_{\partial\Omega}(u+\varepsilon\eta)\,d\mathcal{H}^{n-1}(x) + \int_\Omega \lambda(u+\varepsilon\eta)\,dx\\ &= \varepsilon\int_\Omega \frac{\nabla u\cdot\nabla\eta}{\sqrt{1+|\nabla u|^2}}\,dx + o(\varepsilon) + \varepsilon\int_\Omega \lambda\eta\,dx - \varepsilon\cos\gamma\int_{\partial\Omega}\eta\,d\mathcal{H}^{n-1} + \mathcal{F}[u]\,,\end{aligned}$$

thus, one has

$$\begin{aligned}\frac{\mathcal{F}[u+\varepsilon\eta]-\mathcal{F}[u]}{\varepsilon} &= \int_\Omega\left(\frac{\nabla u\cdot\nabla\eta}{\sqrt{1+|\nabla u|^2}}+\lambda\eta\right)dx - \cos\gamma\int_{\partial\Omega}\eta\,d\mathcal{H}^{n-1} + o(1)\\ &= \int_\Omega \eta\,(-\mathrm{div}\,Tu+\lambda)\,dx + \int_{\partial\Omega}\eta\,(Tu\cdot\nu-\cos\gamma)\,d\mathcal{H}^{n-1} + o(1)\,.\end{aligned}$$

Hence, by (PMC), a minimizer u has necessarily mean curvature H equal to the Lagrange parameter, λ, and it is such that $Tu\cdot\nu = \cos\gamma$ on the boundary. We are thus led to the following PDE with prescribed boundary condition

$$\begin{cases}\mathrm{div}\,(Tu) = H\,, & \text{in } \Omega\\ Tu\cdot\nu = \cos\gamma\,, & \text{on } \partial\Omega\end{cases} \tag{4.6}$$

Notably H can not be chosen but it is implicitily determined by (4.6). Indeed by integrating the PDE on Ω, using Gauss-Green theorem (we are here supposing to deal with smooth sets) and exploiting the boundary condition we get

$$\left|\int_\Omega H\,dx\right| = \left|\int_\Omega \mathrm{div}\,(Tu)\,dx\right| = \left|\int_{\partial\Omega} Tu\cdot\nu\,d\,\mathcal{H}^{n-1}(x)\right| = \cos\gamma P(\Omega)\,. \tag{4.7}$$

We here remark that computing the same steps but on a smooth compact subset E of Ω and using the fact that the vector field Tu has norm strictly less than 1 in these subsets we get

$$\left| \int_E H \, dx \right| < \cos\gamma P(E) \,. \tag{4.8}$$

Notice that whenever the function H is chosen to be positive and not identically vanishing, the LHS of (4.7) and (4.8) amounts, respectively, to $|\Omega|_H$ and to $|E|_H$, i.e. the volume of Ω and E weighted through H as defined in Chapter 3. It is worth noting that whenever H is chosen to be as above (4.7) coupled with (4.8) reads as

$$\Omega \text{ is the unique element in } \mathcal{C}^1_{H,g}(\Omega) \,, \tag{4.9}$$

where $g(x, v) = |v| \cos\gamma$ and $\mathcal{C}^1_{H,g}(\Omega)$ has been defined in Chapter 3.

4.2 Weakly regularity hypothesis

In the previous section we have already noted that the capillarity problem is strongly linked to the solvability of the (PMC) equation. Therefore in order to deal with it, we shall here investigate under which conditions the existence of solutions to the prescribed mean curvature equation is ensured and when the solution is unique.

Our main interest is to identify the minimal regularity assumptions on Ω under which existence (and possibly uniqueness) of solutions can be guaranteed. The Lipschitz regularity assumption in [Giu78] on Ω was made as it is enough to guarantee the validity of a Gauss-Green theorem, the existence of a continuous trace operator from $BV(\Omega)$ to $L^1(\partial\Omega)$ as well as of an extension operator from $L^1(\partial\Omega)$ to $W^{1,1}(\mathbb{R}^n)$, according to classical results holding for BV and Sobolev spaces. We are going to make the minimal assumption to ensure the validity of the Gauss-Green Theorem 2.7 proved in Chapter 2, the existence of a continuous trace operator from $BV(\Omega)$ to $L^1(\partial\Omega)$ as well as an extension operator from $L^1(\partial\Omega)$ to $W^{1,1}(\mathbb{R}^n)$ and show that these are enough to yield existence and uniqueness.

Specifically, we assume that $\Omega \subset \mathbb{R}^n$ is a weakly regular open, bounded and connected set with finite perimeter that, additionaly, coincides with its measure-theoretic interior i.e.

$$x \in \Omega \qquad \text{iff } \exists r > 0 \,:\, |\mathbb{R}^n \setminus (B_r(x) \cap \Omega)| = 0 \tag{4.10}$$

which, roughly speaking, means we do not allow Ω to have "measure-zero holes". From now on, we shall say that Ω is a domain if it is an open, bounded and connected set with finite perimeter. For sake of convenience we shall recall here explicitly the defintion of weak-regularity as given in Definition 2.4. An open, bounded set Ω is said to be weakly regular, if there exists $k = k(\Omega) > 0$ such that

$$\min\{P(E; \Omega^c), P(\Omega \setminus E; \Omega^c)\} \leq k \, P(E; \Omega) \tag{4.11}$$

for all $E \subset \Omega$ and it is such that

$$P(\Omega) = \mathcal{H}^{n-1}(\partial\Omega). \tag{4.12}$$

Whenever an open, bounded set Ω satisfies (4.11) and (4.12), we say that Ω is weakly regular. We shall show that whenever Ω is a weakly regular domain, any admissible couple (Ω, H) admits solution of (PMC) and any extremal couple has a unique (up to translation) "weakly vertical" solution. In particular uniqueness and verticality shall be shown to be equivalent conditions to the maximality of Ω with respect to H and to an integral boundary condition as of Theorem 4.8. We stress that hypotheses (4.11) and (4.12) are valid for domains with inner cusps or with some porosity (see Section 4.5), which of course fall outside of the Lipschitz class.

The weak regularity can be regarded as a minimal assumption in the following sense. On one hand, if one assumes (4.12), then by Theorem 1.53, (4.11) is equivalent to the existence of a continuous and surjective trace operator from $BV(\Omega)$ to $L^1(\partial\Omega)$. Analogously by Theorem 1.54, the requirement (4.11) is equivalent the extension operator $L^1(\partial\Omega)$ to $W^{1,1}(\mathbb{R}^n)$. Moreover, (4.12) allows to use the interior approximation Theorem 1.49. Finally, the two previous hypotheses allow a Gauss-Green formula on Ω as proved in Theorem 2.7

Firstly, the interior approximation theorem is needed to ensure the derivation of the necessary condition (4.1) and the Gauss-Green formula to compute whether or not the couple is extremal. Secondly, the trace operator and extension operator are needed in order to deal with the minimization of the functional $\mathcal{J}$ defined in the next section, whose Euler-Lagrange equation is exactly (PMC). Lastly, the Gauss-Green formula is used in Thorem 4.8 dealing with the equivalency of uniqueness, maximality and the attaining of boundary data.

On the other hand, by Federer's Structure Theorem 1.39, the request (4.12) amounts to ask that the set of points of $\partial\Omega$ that are of density 0 or 1 for Ω is $\mathcal{H}^{n-1}$-negligible, which can be considered as a very mild regularity assumption on $\partial\Omega$. Moreover, in virtue of (4.9), in the extremal case Ω satisfies the hypotheses of Proposition 3.24 thus only (4.12) needs to be assumed. Clearly, whenever $\Omega = \Omega^{(1)}$, which is stronger than (4.10), Proposition 3.29 holds as well, yielding (4.12). Thus, for the extremal case one only needs to ask to Ω to coincide with its points of density 1.

We here show that for a weakly regular domain Ω condition (4.1) can be extended from $A \subset\subset \Omega$ to any proper subset of Ω.

Proposition 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying condition (4.12). Assume that the necessary condition (4.1) holds for every $A \subset\subset \Omega$, then it also holds for every $A \subset \Omega$ such that $0 < |A| < |\Omega|$.*

Proof. Let us fix a measurable set $A \subset \Omega$ with $0 < |A| < |\Omega|$ and finite perimeter. By Theorem 1.49 there exists a sequence $\{\Omega_j\}_{j\in\mathbb{N}}$ of relatively compact, smooth open subsets of Ω, such that $|\Omega \setminus \Omega_j| \to 0$ and $P(\Omega_j) \to P(\Omega)$ as $j \to \infty$. Now take $A_j = A \cap \Omega_j$ and notice that $A_j \subset\subset \Omega$,

$P(A_j) < +\infty$, and $A_j \to A$ in L^1 as $j \to \infty$. Since

$$P(A_j) + P(A \cup \Omega_j) \le P(A) + P(\Omega_j),$$

and owing to the fact that $A \cup \Omega_j \to \Omega$ in L^1 as $j \to \infty$, we deduce that

$$\begin{aligned} P(A) &\le \liminf_j P(A_j) \le \limsup_j P(A_j) \le \limsup_j \Big(P(A) + P(\Omega_j) - P(A \cup \Omega_j) \Big) \\ &= P(A) + P(\Omega) - \liminf_j P(A \cup \Omega_j) \le P(A) + P(\Omega) - P(\Omega) = P(A), \end{aligned}$$

which proves that

$$\lim_j P(A_j) = P(A). \tag{4.13}$$

Now we observe that $P(A;\Omega) > 0$, which follows from the connectedness of Ω coupled with the fact that $0 < |A| < |\Omega|$. Therefore owing to (1.10) we can assume that $P(A_j;\Omega_{j_0}) \ge c > 0$ for a suitably large j_0 and for all $j \ge j_0$, which means that

$$\begin{aligned} \left| \int_{A_j} H\, dx \right| = \left| \int_{\partial^* A_j} \langle Tu, \nu \rangle \, d\mathcal{H}^{n-1} \right| &\le P(A_j; \mathbb{R}^n \setminus \Omega_{j_0}) + \int_{\partial^* A_j \cap \Omega_{j_0}} |\langle Tu, \nu\rangle| \, d\mathcal{H}^{n-1} \\ &\le P(A_j; \mathbb{R}^n \setminus \Omega_{j_0}) + \alpha P(A_j; \Omega_{j_0}) = P(A_j) - (1-\alpha)c, \end{aligned}$$

where $\alpha < 1$ is the supremum of $|\langle Tu, \nu\rangle|$ on Ω_{j_0}. Since $|A_j| \to |A|$ as $j \to \infty$, by the necessary condition written for A_j, and passing to the limit as $j \to \infty$, we get by (4.13)

$$\left| \int_A H\, dx \right| \le P(A) - (1-\alpha)c < P(A), \tag{4.14}$$

whence the conclusion follows. □

4.3 Existence theorems

In this section we prove that whenever our hypotheses are met, (PMC) has solutions. In the existence proof we will have first to discuss the easier *non-extremal case*, in which the pair (Ω, H) is non extremal, and then the more involved *extremal case*, that is when the pair (Ω, H) satisfies (4.3). It is worth noting that through the proofs we provide in the non-extremal case the existence of solutions is genuinely variational, while in the extremal case one recovers a solution as a limit of variational solutions defined on subdomains.

To deal with the non-extremal case, we will follow the argument of [Gia74, Giu78], which is based on the minimization of the functional

$$\mathcal{J}[u] = \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx + \int_\Omega Hu\, dx + \int_{\partial\Omega} |u - \varphi|\, d\mathcal{H}^{n-1}, \tag{4.15}$$

defined on $BV(\Omega)$, for a given $\varphi \in L^1(\partial\Omega)$. Note that the Euler-Lagrange equation of $\mathcal{J}$, obtained by perturbations with compact support in Ω, is precisely the prescribed mean curvature equation (PMC). The weakly regularity hypothesis on Ω, by Theorem 1.53, implies that the last term in (4.15) is well-defined.

Before the proof we report the next lemma which corresponds to [Giu78, Lemma 1.1], and thus we omit its proof. It says that if the couple (Ω, H) is not extremal, then in (4.1) there is a uniform detachment from the perimeter.

Lemma 4.2. *Let Ω be a weakly regular domain. If (Ω, H) is an admissible non-extremal pair, then there exists $\varepsilon_0 > 0$ such that the stronger inequality*

$$\left| \int_A H \, dx \right| \leq (1 - \varepsilon_0) P(A)$$

holds for all such A.

Theorem 4.3 (Existence, non-extremal case)**.** *Let Ω be a weakly regular domain. If the pair (Ω, H) is an admissible non-extremal pair, i.e. (4.1) holds along with*

$$\left| \int_\Omega H \, dx \right| < P(\Omega),$$

then the functional $\mathcal{J}$ defined in (4.15) is minimized in $BV(\Omega)$.

Proof. Fix a ball B containing Ω and extend the function H to 0 in $B \setminus \Omega$. Fix a function $\Phi \in W_0^{1,1}(B)$ such that $\Phi = \varphi$ on $\partial\Omega$ (this can be done according to Theorem 1.54). Then minimizing $\mathcal{J}$ on $BV(\Omega)$ is equivalent to minimizing $\widetilde{\mathcal{J}}$ defined as

$$\widetilde{\mathcal{J}} : u \mapsto \int_B \sqrt{1 + |\nabla u|^2} \, dx + \int_B Hu \, dx,$$

in $K = \{u \in BV(B) | u = \Phi \quad \text{in } B \setminus \Omega\}$, which is a closed subset of $BV(B)$. Owing to Proposition 4.1 and by the assumption on Ω we can apply Lemma 4.2 and get the lower bound

$$\int_\Omega Hu \, dx \geq -(1 - \varepsilon_0) \int_B |Du| - c \int_{\partial\Omega} |\varphi| \, d\mathcal{H}^{n-1}$$

for some $\varepsilon_0 > 0$, whence

$$\widetilde{\mathcal{J}}[u] \geq \varepsilon_0 \int_B |Du| \, dx - c \int_{\partial\Omega} |\varphi| \, d\mathcal{H}^{n-1}. \tag{4.16}$$

Exploiting Poincaré's inequality on the ball B one finally shows the coercivity of $\widetilde{\mathcal{J}}$ in $L^1(\Omega)$. Since it is also lower semi-continuous within respect to the L^1-norm we infer the existence of a minimizer of $\widetilde{\mathcal{J}}$ in K, hence of a minimizer of $\mathcal{J}$ in $BV(\Omega)$. □

In order to prove the existence of minimizers for an extremal pair, following [Mir77] we introduce the notion of *generalized solution* of (PMC). For technical reasons, we consider the epigraph of u instead of its subgraph, therefore the definition is slighty offset from the one in [Mir77] (but of course equivalent up to changing the minus sign in (4.17)).

Definition 4.4. A function $u : \Omega \to [-\infty, +\infty]$ is said to be a generalized solution to (PMC) if the epigraph of u

$$U = \{(x, y) \in \Omega \times \mathbb{R} : \ y > u(x)\},$$

minimizes the functional

$$P(U) - \int_U H \ dx\, dy, \tag{4.17}$$

locally in $\Omega \times \mathbb{R}$.

It is clear that any classical solution to (PMC) is also a generalized solution. Moreover, any generalized solution of (PMC) can be shown to satisfy some key properties, that we collect in the following proposition (see [Giu78] and [Mir64, Mir77] for the proof).

Proposition 4.5. *Let u be a generalized solution of* (PMC) *and define $N_\pm = \{x \in \Omega : \ u(x) = \pm\infty\}$. Then the following properties hold.*

(i) If $x \in N_\pm$ then $|N_\pm \cap B_r(x)| > 0$ for all $r > 0$.

(ii) The set $N_\pm$ minimizes the functional

$$E \mapsto P(E) \pm \int_E H\, dx$$

locally in Ω.

(iii) The function u is smooth on $\Omega \setminus (N_+ \cup N_-)$.

(iv) Given a sequence $\{u_k\}$ of generalized solutions of (PMC)*, then up to subsequences the epigraphs U_k of u_k converge to an epigraph U of a function u locally in $L^1(\Omega\times\mathbb{R})$, moreover u is a generalized solution of* (PMC)*.*

(v) If u is locally bounded, then u a classical solution of (PMC)*.*

The next lemma is a straightforward adaptation of [Giu78, Lemma 1.2]. The proof is the same up to choosing a sequence $\{\Omega_j\}_j$ as provided by Theorem 1.49 with $\varepsilon = 1/j$.

Lemma 4.6. *Let Ω be a weakly regular domain. Let the pair (Ω, H) be an extremal pair. Let $E \subset \Omega$ be a set of finite perimeter minimizing the functional*

$$P(E) - \int_E H\, dx$$

locally in Ω. Then either $E = \emptyset$ or $E = \Omega$, up to null sets.

We now come to the existence of solutions of (PMC) in the extremal case.

Theorem 4.7 (Existence, extremal case)**.** *Let Ω be a weakly regular domain. Assume that the pair (Ω, H) is extremal. Then there exists a solution u of* (PMC)*.*

Proof. We closely follow the argument in the proof of [Giu78, Theorem 1.1]. By Theorem 1.49 we find a sequence of smooth, connected sets $\Omega_j \subset\subset \Omega$, such that $|\Omega \setminus \Omega_j| \to 0$ and $P(\Omega_j) \to P(\Omega)$ as $j \to +\infty$. Since (4.1) holds for any $A \subset \Omega_j$ (and in particular for $A = \Omega_j$), in virtue of Theorem 4.3 (existence in the non-extremal case) we find a minimizer $u_j \in BV(\Omega_j)$ of $\mathcal{J}$ restricted to $BV(\Omega_j)$, as every Ω_j satisfies (4.11). Setting

$$t_j = \inf\Big\{t : \ |\{x \in \Omega_j : \ u_j(x) \geq t\}| \leq |\Omega_j|/2\Big\}$$

we obtain

$$\min(|\{x \in \Omega_j : \ u_j(x) \geq t_j\}|, |\{x \in \Omega_j : \ u_j(x) \leq t_j\}|) \ \geq \ |\Omega_j|/2 \ \geq \ |\Omega|/4$$

for all j large enough. Therefore, we can consider the sequence of vertically translated functions $\{u_j(x) - t_j\}_j$ defined for $x \in \Omega_j$, and relabel it as $\{u_j\}_j$, so that

$$\min\Big(|\{x \in \Omega_j : \ u_j(x) \geq 0\}|, \ |\{x \in \Omega_j : \ u_j(x) \leq 0\}|\Big) \ \geq \ |\Omega|/4 \tag{4.18}$$

for all j large enough. Then, by applying Proposition 4.5 (iv) on Ω_{j_0} for any fixed $j_0 \in \mathbb{N}$, and by a diagonal argument, we infer that u_j locally converges up to subsequences to a generalized solution u as $j \to \infty$, in the sense that the epigraph U_j locally converges to the epigraph of u in $L^1_{loc}(\Omega \times \mathbb{R})$ as $j \to \infty$. Let us set $N_\pm = \{x \in \Omega : \ u(x) = \pm\infty\}$ as in Proposition 4.5. We claim that $N_\pm$ are both empty, which in turn implies by Proposition 4.5 (v) that u is a classical solution of (PMC). Indeed by Proposition 4.5 (ii) the set N_- minimizes the functional $P(E) - \int_E H\,dx$ defined for $E \subset \Omega$, thus by Lemma 4.6 we have either $N_- = \emptyset$ or $N_- = \Omega$. Similarly, the set $\Omega \setminus N_+$ minimizes $P(E) - \int_E H\,dx$ (this follows from the fact that N_+ minimizes $P(E) + \int_E H\,dx$), hence either $N_+ = \Omega$ or $N_+ = \emptyset$. By (4.18) we conclude that $N_\pm = \emptyset$, which proves our claim. $\square$

4.4 Characterization of extremality

Extremality arises in physical models of capillarity for perfectly wetting fluids as (4.6), the uniqueness and the stability of solutions with respect to suitable perturbations of the domain are of special interest in this case.

In [Giu76] Giusti showed that, assuming C^2 regularity of $\partial\Omega$ and (4.1), the extremality condition (4.3) is equivalent to a series of facts, and in particular to the uniqueness of the solution of (PMC) up to vertical translations.

Here we obtain essentially the same result only assuming that Ω is weakly regular. Before stating our main result, we present a list of properties using the same labels as those appearing in [Giu78].

(E) *(Extremality)* The pair (Ω, H) is extremal, i.e., $\left|\int_\Omega H\,dx\right| = P(\Omega)$.

(U) *(Uniqueness)* The solution of (PMC) is unique up to vertical translations.

(M) *(Maximality)* Ω is maximal, i.e. no solution of (PMC) can exist in any domain strictly containing Ω.

(V) *(weak Verticality)* There exists a solution u of (PMC) which is weakly vertical at $\partial\Omega$, i.e.

$$[Tu \cdot \nu] = 1 \qquad \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega\,,$$

where $[Tu \cdot \nu]$ is the weak normal trace of Tu on $\partial\Omega$.

(V') *(integral Verticality)* There exists a solution u of (PMC) and a sequence $\{\Omega_i\}_i$ of smooth subdomains, such that $\Omega_i \subset\subset \Omega$, $|\Omega \setminus \Omega_i| \to 0$, $P(\Omega_i) \to P(\Omega)$, and

$$\lim_{i\to\infty} \int_{\partial\Omega_i} Tu(x) \cdot \nu \, d\mathcal{H}^{n-1} = P(\Omega),$$

as $i \to \infty$.

Then we come to the main result of this section.

Theorem 4.8. *Let Ω and H be given, such that Ω is weakly regular domain and the pair (Ω, H) is an admissible pair. Then the properties (E), (U), (M), (V) and (V') are equivalent.*

Before proving Theorem 4.8 some further comments about properties (V) and (V') above are in order. In [Giu78] the property (V) is stated in the stronger, pointwise form $Tu(x) = \nu(x)$ for all $x \in \partial\Omega$ (moreover $\partial\Omega$ is assumed of class C^2, hence Tu can be continuously extended on $\partial\Omega$ owing to well-known regularity results, see [Emm76]) while (V') is stated by using the one-parameter family of inner parallel sets (which is again well-defined owing to the C^2-smoothness of $\partial\Omega$).

In our more general case we replace the original properties as stated in [Giu78] with the weaker ones (V) and (V'). Nevertheless, by relying on the results of Section 2.2.1 we can easily prove the following fact.

Proposition 4.9. *Let Ω be a weakly regular domain, such that $P(\Omega) = \mathcal{M}_-(\Omega)$ and $\mathcal{H}^{n-1}(\partial^*\Omega \setminus \partial^{**}\Omega) = 0$. Let u be a solution of (PMC) on Ω. Then (V) is equivalent to require that*

$$\operatorname*{ap-lim}_{x\to x_0} Tu(x) = \nu(x_0) \tag{4.19}$$

for $\mathcal{H}^{n-1}$-almost every $x_0 \in \partial\Omega$.

Proof. It is an immediate consequence of Theorem 2.11, Proposition 2.8 and property (4.11). □

Remark 4.10. We notice that the assumption (4.11) in Proposition 4.9 is not necessary as soon as we aim at showing that (V) implies (4.19). Indeed, by Theorem 4.8 we obtain in particular that (V) implies (E). On the other hand, it is not difficult to show that (E) implies (4.11) (see Section 3.2.2).

$$\begin{tikzcd}
 & (E) \arrow[r, Leftrightarrow] \arrow[dd, Rightarrow] & (M) \\
(U) \arrow[ur, Rightarrow] & & \\
 & (V') \arrow[r, Leftrightarrow] \arrow[ul, Rightarrow] & (V)
\end{tikzcd}$$

Figure 4.2: The implication scheme for the proof of Theorem 4.8

The Maximum Principle Lemma that we state hereafter has been originally proved in [Fin65] and then in [Giu78]. We remark that it remains valid under the weaker assumptions guaranteeing the interior smooth approximation property, in the sense of Theorem 1.49.

Lemma 4.11 (Maximum Principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be weakly regular. Let u and v be two functions of class $C^2(\Omega)$, such that* $\operatorname{div}(Tu) \le \operatorname{div}(Tv)$ *in Ω. Assume that $\partial\Omega = \Gamma_1 \cup \Gamma_2$ with Γ_1 relatively open in $\partial\Omega$ and $u, v \in C^0(\Omega \cup \Gamma_1)$. Assume further that*

$$\lim_{i\to\infty} \int_{\partial\Omega_i \setminus A} (1 - Tu \cdot \nu)\, d\mathcal{H}^{n-1} = 0$$

for every open set $A \supset \Gamma_1$, where $\{\Omega_i\}_{i\in\mathbb{N}}$ is a sequence of smooth and relatively compact open subsets of Ω, such that $|\Omega \setminus \Omega_i| \to 0$ and $P(\Omega_i) \to P(\Omega)$ as $i \to \infty$. Then

(a) if $\Gamma_1 \ne \emptyset$ then $u \ge v$ in Ω;

(b) if $\Gamma_1 = \emptyset$ then $u = v + c$.

We finally come to the proof of Theorem 4.8.

Proof of Theorem 4.8. We shall split the proof in five steps, according to the scheme displayed in Figure 4.2.
Step one: (E) $\Rightarrow$ (V'). Owing to (E) we have

$$P(\Omega) \overset{(E)}{=} \int_\Omega H\, dx = \lim_{i\to\infty} \int_{\Omega_i} H\, dx = \lim_{i\to\infty} \int_{\Omega_i} \operatorname{div}(Tu)\, dx = \lim_{i\to\infty} \int_{\partial\Omega_i} Tu(x) \cdot \nu\, d\mathcal{H}^{n-1},$$

which implies (V').

Step two: (E) $\Leftrightarrow$ (M). Let us start by showing (E) $\Longrightarrow$ (M). We argue by contradiction and suppose there exists a solution u of (PMC) defined on $\widetilde{\Omega} \supsetneq \Omega$. Then Proposition 4.1 gives

$$\left| \int_\Omega H\, dx \right| < P(\Omega),$$

which immediately contradicts (E). Let us now show the implication (M) $\Longrightarrow$ (E). Again by contradiction we assume that

$$\left| \int_A H\, dx \right| < P(A)$$

for all $A \subset \Omega$. By Lemma 4.2 there exists $\varepsilon_0 > 0$ such that

$$\left| \int_A H\,dx \right| < (1-\varepsilon_0) P(A) \tag{4.20}$$

for all $A \subset \Omega$. Now we claim that (compare with Lemma 2.1 in [Giu78]) given a ball B such that $\Omega \subset\subset B$, for all $0 < \varepsilon < \varepsilon_0$ one can find an open set $\Omega_\varepsilon \subset B$ with smooth boundary, such that $\Omega \subset\subset \Omega_\varepsilon$ and

$$\left| \int_A H\,dx \right| < (1-\varepsilon) P(A), \qquad \forall\, A \subset \Omega_\varepsilon. \tag{4.21}$$

Of course, the validity of (4.21) would allow us to apply Theorem 4.3 on Ω_ε, which in turn would contradict our assumption (M). In order to show (4.21) we argue again by contradiction, i.e., we assume that there exists $\varepsilon \in (0, \varepsilon_0)$ such that, for every U with smooth boundary satisfying $\Omega \subset\subset U$, one can find $A \subset U$ for which (4.21) fails. In particular, for every $k \in \mathbb{N}$ we may choose a suitable U_k as specified below, such that $\Omega \subset\subset U_k$, $|U_k \setminus \Omega| < 1/k$, ∂U_k is smooth and there exists $A_k \subset U_k$ for which

$$\left| \int_{A_k} H\,dx \right| \geq (1-\varepsilon) P(A_k) \tag{4.22}$$

holds. By (4.22) we have that

$$P(A_k) \leq \frac{|B| \sup_B |H|}{1-\varepsilon} \qquad \forall\, k \in \mathbb{N},$$

hence we can extract a not relabeled subsequence A_k converging to some $A \subset B$ in L^1. On the other hand, since $|A_k \setminus \Omega| \leq |U_k \setminus \Omega| \to 0$ as $k \to \infty$, we infer that $A \subset \Omega$ up to null sets. By (4.22), by the lower semi-continuity of the perimeter and by the continuity of the term $\int_{A_k} H\,dx$ with respect to L^1-convergence, we conclude that

$$\left| \int_A H\,dx \right| \geq (1-\varepsilon) P(A)$$

which is in contrast with (4.20). We are left to prove that such a sequence U_k exists. To this aim we consider the open set $V = B \setminus \overline{\Omega}$ and notice that $P(V) = P(B) + P(\Omega) = \mathcal{H}^{n-1}(\partial B) + \mathcal{H}^{n-1}(\partial\Omega) = \mathcal{H}^{n-1}(\partial V)$ owing to the assumption on Ω. We can now apply Theorem 1.49 to V with $\delta_k = \min(\mathrm{dist}(\partial B, \partial\Omega)/3, 1/k)$ and set $U_k = B \setminus (V_{\delta_k} \cup \mathcal{N}_{2\delta_k}(\partial B))$. Thanks to (1.10) we find that ∂U_k is smooth, $\Omega \subset\subset U_k$ and $|U_k \setminus \Omega| < \delta_k \leq 1/k$, as wanted.

Step three: (V') $\Longrightarrow$ (U). We consider two solutions u, v of (PMC), then if we take $\Gamma_1 = \emptyset$ and thanks to the property $P(\Omega_i) \to P(\Omega)$ as $i \to \infty$, we infer that the assumptions of Lemma 4.11(b) are satisfied. Consequently there exists a constant $c \in \mathbb{R}$ such that $u = v + c$.

Step four: (U) $\Longrightarrow$ (E). Let u be the unique solution of $\mathrm{div}\,(Tu) = H$ on Ω, up to vertical translations. By contradiction we suppose that

$$\int_\Omega H\;dx < P(\Omega)\,.$$

Arguing as in Step two we find a bounded and smooth domain $\tilde{\Omega} \supsetneq \Omega$ for which (4.1) holds. By Theorems 4.3 and 4.7 there exists a solution $\tilde{u}$ of $\operatorname{div}(T\tilde{u}) = H$ on $\tilde{\Omega}$. Then (U) implies the existence of $t \in \mathbb{R}$ such that $u = \tilde{u} + t$ on Ω. By internal regularity of $\tilde{u}$, we infer that $u \in C^1(\overline{\Omega})$. Fix now a function $\varphi \in C^2(\mathbb{R}^n)$ such that

$$\mathcal{H}^{n-1}(\{x \in \partial\Omega : \ \varphi(x) - u(x) \neq s\}) > 0 \qquad \forall\, s \in \mathbb{R}. \tag{4.23}$$

The choice of φ satisfying (4.23) can be easily made as follows: if u is constant on $\partial\Omega$, then one can choose any smooth function φ taking different values on two distinct points of $\partial\Omega$; conversely, if u is not constant on $\partial\Omega$ then one can take $\varphi = 0$. Now we consider a minimizer w of the functional

$$\int_\Omega \sqrt{1 + |Dw|^2} + \int_\Omega Hw + \int_{\partial\Omega} |w - \varphi| \, d\mathcal{H}^{n-1},$$

then w necessarily satisfies (PMC). By the assumed uniqueness up to translations one has that $w = u + s$ for some $s \in \mathbb{R}$. Then it follows that

$$|Tu(x_0)| = |Tw(x_0)| < 1. \tag{4.24}$$

Moreover by (4.23) we have that $w \neq \varphi$ on some set $K \subset \partial^*\Omega$ with $\mathcal{H}^{n-1}(K) > 0$. Fix now a point $x_0 \in K$ and assume without loss of generality that $\varphi(x_0) > w(x_0)$. Set now $\mathcal{C} = \Omega \times \mathbb{R}$, $p_0 = (x_0, w(x_0)) \in \partial\mathcal{C}$, and notice that by the continuity of w and φ on $\partial\Omega$ there exists $R > 0$ such that the subgraph of φ contains the ball $B_R(p_0) \subset \mathbb{R}^{n+1}$. Owing to the choice of $B_R(p_0)$, the epigraph

$$W := \{p = (x, y) \in \mathcal{C} : \ y > w(x)\}$$

necessarily minimizes the functional

$$P(W; B_R(p_0)) - \int_{W \cap B_R(p_0)} H$$

with obstacle $\mathbb{R}^{n+1} \setminus \mathcal{C}$ inside $B_R(p_0)$. In other words, for any set U that coincides with W outside the set $A := B_R(p_0) \cap \overline{\mathcal{C}}$, one has

$$P(W; B_R(p_0)) - \int_{W \cap B_R(p_0)} H \leq P(U; B_R(p_0)) - \int_{U \cap B_R(p_0)} H. \tag{4.25}$$

It is then easy to show that W is a (Λ, R)-perimeter minimizer in $\overline{\mathcal{C}}$ (see Definition 1.55), where R is the radius of the ball defined above and $\Lambda = \sup_\Omega |H|$. Indeed for any ball $B_r \subset B_R(p_0)$ and any set U such that $U \Delta W \subset\subset B_r \cap \overline{\mathcal{C}}$, by (4.25) one has that

$$\begin{aligned}
P(W; B_r) &= P(W; B_R(p_0)) - P(W; B_R(p_0) \setminus \overline{B_r}) \\
&\leq P(U; B_R(p_0)) - P(U; B_R(p_0) \setminus \overline{B_r}) - \int_{B_R(p_0)} H(\chi_U - \chi_W) \\
&\leq P(U; B_r) + \sup_\Omega |H| \, |U \Delta W|,
\end{aligned}$$

which proves the (Λ, R)-minimality of W in $\overline{\mathcal{C}}$. Then by Theorem 2.12 we infer that $\nu_W(p_0) = \nu_{\mathcal{C}}(p_0)$, which contradicts (4.24).

Step five: (V) and (V') are equivalent. We can consider the sequence Ω_j of Theorem 1.49 and apply Theorem 2.7 to get

$$\int_{\Omega\setminus\Omega_j} H(x)\,dx = \int_{\Omega\setminus\Omega_j} \operatorname{div} Tu(x)\,dx = \int_{\partial\Omega} [Tu\cdot\nu]\,d\mathcal{H}^{n-1} - \int_{\partial\Omega_j} Tu\cdot\nu_j\,d\mathcal{H}^{n-1}\,. \tag{4.26}$$

Now, observing that the left-hand side of (4.26) is infinitesimal as $j\to\infty$ the equivalence between (V) and (V') is immediate.

The proof is finally completed by combining the previous five steps. $\square$

We now show a well-known consequence of Lemma 4.11, which can be obtained by arguing as in Step two of the proof of Theorem 4.8.

Proposition 4.12. *Assume that u is a solution of* (PMC) *on Ω and that either (V) or (V') holds. Then u is bounded from below.*

Proof. Let B denote a ball compactly contained in Ω and consider the open set $S = \Omega\setminus\overline{B}$. By Lemma 4.2 and arguing as in Step two of the proof of Theorem 4.8 we find a solution w of (PMC) which is of class $C^1(\overline{S})$. Since in particular $u\in C^2(\overline{B})$ we can assume that $w\le u$ on ∂B up to a vertical translation, hence by Lemma 4.11(a) we deduce that $w\le u$ on S, which gives the conclusion at once. $\square$

We conclude the section with some remarks about the stability of solutions of (PMC) in the extremal case. One might ask whether or not there exists some perturbation $(\Omega_\varepsilon, H_\varepsilon)$ of an extremal pair (Ω, H), such that $(\Omega_\varepsilon, H_\varepsilon)$ satisfies the necessary condition (4.1) and the solution u_ε of (PMC) on Ω_ε is in a suitable sense a small perturbation of u up to translations, as soon as ε is small. The following proposition contains a result in this direction.

Proposition 4.13 (Stability)**.** *Let $\{\Omega_j\}_j$ be a sequence of domains and $\{H_j\}_j$ a sequence of Lipschitz functions, such that Ω_j is weakly regular and the pair (Ω_j, H_j) is extremal. Assume moreover that $\Omega_j\to\Omega_\infty$ in L^1 and $P(\Omega_j)\to P(\Omega_\infty)$, as $j\to\infty$, with Ω_∞ a weakly regular domain, and that H_j uniformly converges to H_∞ such that the pair $(\Omega_\infty, H_\infty)$ is extremal as well. Then the sequence of unique (up to translations) solutions $\{u_j\}_j$ to the* (PMC) *problem for the pair (Ω_j, H_j) converges to a solution u_∞ of* (PMC) *for the pair $(\Omega_\infty, H_\infty)$, in the sense of the $L^1_{loc}(\mathbb{R}^{n+1})$-convergence of the epigraphs.*

Proof. Due to our hypotheses, the existence of a solution u_j to (PMC) for the pair (Ω_j, H_j) (also for $j=\infty$) is guaranteed by Theorem 4.7. Arguing as in Theorem 4.7, for any j large enough we can find a suitable t_j such that the translated solution $u_j + t_j$ which we just rename u_j satisfies

$$\min\Big(|\{x\in\Omega_j:\ u_j(x)\ge 0\}|,\ |\{x\in\Omega_j:\ u_j(x)\le 0\}|\Big)\ \ge\ |\Omega|/4\,.$$

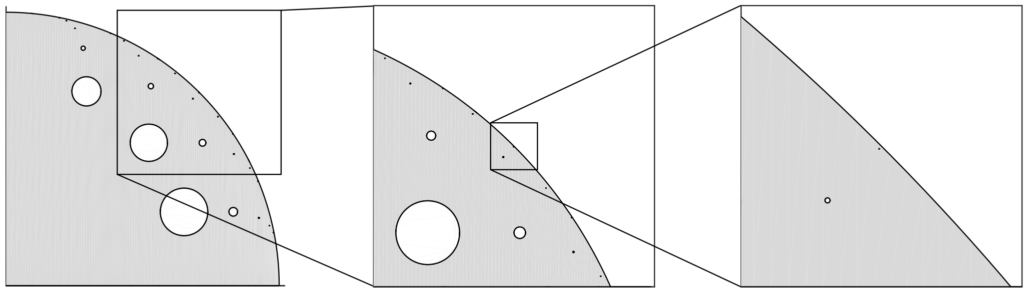


Figure 4.3: The “Swiss cheese” set constructed in Section 4.5

Then we find that the epigraphs U_j of u_j converge in $L^1_{loc}(\mathbb{R}^{n+1})$ to a set U^*_∞ which is the epigraph of a classical solution u^*_∞ defined on Ω_∞. By Theorem 4.8 we have that $u^*_\infty = u_\infty$ up to a translation, thus the thesis follows. □

4.5 An example of non-smooth extremal pair

In a forthcoming paper [LS17], an explicit example of an extremal pair (Ω, H) and of a sequence of extremal pairs (Ω_j, H_j) satisfying the hypotheses of Proposition 4.13 is constructed, for the special case $H_j = P(\Omega_j)/|\Omega_j|$. The family is a non-smooth perturbation of the unit disk, built by removing a sequence of smaller and smaller disks from the unit disk in $\mathbb{R}^2$, in such a way that it looks like a sort of Swiss cheese with holes accumulating towards a portion of its boundary (see Figure 4.3). This shows the following, remarkable fact: while a generic small and smooth perturbation of the unit disk may produce a dramatic change in the capillary solution (and even end up with non-existence of a solution), there exist some non-smooth perturbations that, instead, preserve both existence and stability.

We consider the set J of pairs $\mathbf{j} = (j_1, j_2)$ such that $j_1, j_2 \in \mathbb{N}$ and $j_2 \le j_1$, then for any $\mathbf{j} \in J$ we set

$$\mathbf{j} + 1 = \begin{cases} (j_1 + 1, 1) & \text{if } j_2 = j_1, \\ (j_1, j_2 + 1) & \text{if } j_2 < j_1. \end{cases}$$

We fix two sequences $(\varepsilon_{\mathbf{j}})_{\mathbf{j}\in J}$ and $(r_{\mathbf{j}})_{\mathbf{j}\in J}$ of positive real numbers between 0 and $\frac{1}{2}$, that will be specified later, and define

$$\begin{aligned}
\rho_{\mathbf{j}} &= 1 - \varepsilon_{\mathbf{j}}, \\
\theta_{\mathbf{j}} &= j_2 \cdot \frac{\pi}{2(j_1+1)}, \\
x_{\mathbf{j}} &= \rho_{\mathbf{j}} \left(\cos(\theta_{\mathbf{j}}), \sin(\theta_{\mathbf{j}})\right), \\
B_{\mathbf{j}} &= B_{r_{\mathbf{j}}}(x_{\mathbf{j}}),
\end{aligned}$$

so that in particular $x_{\mathbf{j}}$ is a point of $B_1 = B_1(0)$ contained in the first quadrant, for all $\mathbf{j} \in J$. We write $\mathbf{j} \preceq \mathbf{j}'$ (or equivalently $\mathbf{j}' \succeq \mathbf{j}$) if $\mathbf{j}$ precedes or is equal to $\mathbf{j}'$ with respect to the standard lexicographic order on J. The notion of "limit as $\mathbf{j} \to \infty$" is the obvious one associated with this order relation. We require the following properties on the sequences introduced above:

(i) $\varepsilon_{\mathbf{j}}$ and $r_{\mathbf{j}}$ are non-increasing with respect to $\mathbf{j}$, and $r_{\mathbf{j}} < \varepsilon_{\mathbf{j}}/4$;

(ii) $\sum_{k=1}^{\infty} k\, r_{(k,1)} < +\infty$;

(iii) $\overline{B_{\mathbf{j}}} \subset B_1$ and $\overline{B_{\mathbf{j}}} \cap \overline{B_{\mathbf{k}}} = \emptyset$ for all $\mathbf{j}, \mathbf{k} \in J$ with $\mathbf{j} \neq \mathbf{k}$;

Notice that the above requirements imply that $\varepsilon_{\mathbf{j}}$ and $r_{\mathbf{j}}$ are infinitesimal, moreover one has from (i) and (ii) that $\sum_{\mathbf{j} \in J} r_{\mathbf{j}} < +\infty$. We set

$$\Omega := B_1 \setminus \bigcup_{\mathbf{j} \in J} \overline{B_{\mathbf{j}}}, \tag{4.27}$$

which is an open set since the only accumulation points of the sequence of "holes" $B_{\mathbf{j}}$ are contained in ∂B_1. Up to a further refinement on the requests of the parameters $\varepsilon_{\mathbf{j}}$ and $r_{\mathbf{j}}$, the porous family satisfying the hypotheses of Proposition 4.13 is given by $\{\Omega_{\mathbf{k}}\}$, where

$$\Omega_{\mathbf{k}} := B_1 \setminus \bigcup_{\mathbf{j} \geq \mathbf{k}} \overline{B_{\mathbf{j}}}. \tag{4.28}$$

We shall see that the set Ω defined above in (4.27), for a suitable choice of parameters, is a minimal Cheeger set.

We shall start by proving some facts on the set Ω, using only the very generic assumptions (i)-(iii). The first one, Proposition 4.14 below, shows that the topological boundary $\partial\Omega$ coincides with the reduced boundary $\partial^*\Omega$. Moreover, if we have

(iv) $\displaystyle\sum_{\mathbf{j} \in J} \varepsilon_{\mathbf{j}} < +\infty$

then $\partial^*\Omega$ coincides with $\partial^{**}\Omega$ up to a $\mathcal{H}^1$-negligible set.

Proposition 4.14. *Under the above assumptions (i)-(iii) one has $\partial\Omega = \partial^*\Omega$. If we additionally assume (iv), then $\partial^*\Omega = \partial^{**}\Omega$ up to a $\mathcal{H}^1$-negligible set.*

Proof. Of course $\partial^*\Omega \subseteq \partial\Omega$. In order to prove the opposite inclusion we fix $y \in \partial\Omega$ and argue as follows. If $y \in \partial B_{\mathbf{j}}$ for some $\mathbf{j} \in J$, or $y \in \partial B_1 \setminus \{z = (z_1, z_2) \in \mathbb{R}^2 \,:\, z_1 \geq 0, z_2 \geq 0\}$, then there exists a neighborhood U_y of y such that $\partial\Omega \cap U_y$ is an arc of ∂B_1 or $\partial B_{\mathbf{j}}$, hence trivially $y \in \partial^*\Omega$. Assume now that $y \in \partial B_1$ with non-negative coordinates y_1, y_2. It is standard to check that, in this case, $y \in \partial^*\Omega$ if and only if

$$P(\Omega; B_s(y)) \leq 2s + o(s), \quad s \to 0\,. \tag{4.29}$$

In order to show (4.29) we first set

$$J_2(j_1, s) = \Big\{ j_2 \in \{1, \dots, j_1\} :\ |x_{\mathbf{j}} - y| < s + r_{\mathbf{j}} < 2s \Big\}\,.$$

Then there exists a least index $j_1(s) \in \mathbb{N}$ such that $J_2(j_1, s)$ is empty whenever $j_1 < j_1(s)$, while in general we obtain

$$\# J_2(j_1, s) \leq 1 + \frac{32(j_1+1)s}{\pi} \quad \text{when } j_1 \geq j_1(s). \tag{4.30}$$

To prove this estimate on the cardinality of $J_2(j_1, s)$ we observe that for $\mathbf{j} = (j_1, j_2)$ and $\mathbf{j}' = (j_1, j_2')$ belonging to $J_2(j_1, s)$ we have

$$\frac{1}{2}\Big|(\cos\theta_{\mathbf{j}} - \cos\theta_{\mathbf{j}'}, \sin\theta_{\mathbf{j}} - \sin\theta_{\mathbf{j}'})\Big| \leq |x_{\mathbf{j}} - x_{\mathbf{j}'}| \leq |x_{\mathbf{j}} - y| + |x_{\mathbf{j}'} - y| < 4s\,, \tag{4.31}$$

where for the first inequality we have also used the fact that $|x_{\mathbf{j}}| > \frac{1}{2}$ for all $\mathbf{j}$. Then, setting

$$h = |\theta_{\mathbf{j}} - \theta_{\mathbf{j}'}| = \frac{|j_2' - j_2|\pi}{2(j_1+1)}$$

one easily obtains from (4.31) that

$$\sin h \leq |(\cos\theta_{\mathbf{j}} - \cos\theta_{\mathbf{j}'}, \sin\theta_{\mathbf{j}} - \sin\theta_{\mathbf{j}'})| < 8s\,,$$

whence assuming $s < \frac{1}{16}$ one deduces

$$h \leq 16s\,,$$

which implies $|j_2 - j_2'| \leq 32(j_1+1)s/\pi$. Then (4.30) follows at once. In conclusion we find

$$\begin{aligned}
P(\Omega; B_s(y)) &= 2s + o(s) + P\left(\bigcup_{\mathbf{j}\in J} B_{\mathbf{j}}; B_s(y)\right) \\
&\leq 2s + o(s) + \sum_{j_1=1}^{\infty} \sum_{j_2 \in J_2(j_1,s)} 2\pi r_{\mathbf{j}} \\
&\leq 2s + o(s) + s \sum_{j_1=j_1(s)}^{\infty} [2\pi + 64(j_1+1)] r_{(j_1,1)} \\
&= 2s + o(s)
\end{aligned}$$

where the last equality relies on the assumption (ii) and the fact that $j_1(s) \to +\infty$ as $s \to 0$.

Let us now prove that (iv) implies $\mathcal{H}^1(\partial^*\Omega \setminus \partial^{**}\Omega) = 0$. By Proposition 2.2 the set $\partial^*\Omega \setminus \partial^{**}\Omega$ is contained in the countable union $\bigcup_{n\in\mathbb{N}} S_n$, where S_n is the set of points $w \in \partial B_1$ for which there exists an increasing sequence $(\mathbf{j}^k)_{k\in\mathbb{N}}$ of double-indices such that

$$(w - x_{\mathbf{j}^k}) \cdot w \geq \frac{|w - x_{\mathbf{j}^k}|}{n} \qquad \text{for all } k \in \mathbb{N}. \tag{4.32}$$

Now the goal is to show that, for each $n \in \mathbb{N}$, $\mathcal{H}^1(S_n) = 0$ which is equivalent to show that the Hausdorff premeasure $\mathcal{H}^1_\eta(S_n) = 0$ for all $\eta > 0$. To this aim we notice that, by (4.32), any $w \in S_n$ is contained in $B_{C_n\varepsilon_{\mathbf{j}}}(x_{\mathbf{j}})$ for infinitely many pairs $\mathbf{j}$ and for a suitable constant $C_n > 0$ depending only on n. We can thus fix $\delta > 0$ and find $\mathbf{j^0} \in J$ depending on δ and η, such that $C_n\varepsilon_{\mathbf{j}} < \eta$ for all $\mathbf{j} \succeq \mathbf{j^0}$ and moreover $\sum_{\mathbf{j}\succeq\mathbf{j^0}} \varepsilon_{\mathbf{j}} < \delta$, so that we obtain

$$S_n \subset \bigcup_{\mathbf{j}\succeq\mathbf{j^0}} B_{C_n\varepsilon_{\mathbf{j}}}(x_{\mathbf{j}})$$

and consequently

$$\mathcal{H}^1_\eta(S_n) \leq C_n\delta\,.$$

Since $\delta > 0$ is arbitrary we infer that $\mathcal{H}^1_\eta(S_n) = 0$ for all $\eta > 0$, which concludes the proof. $\square$

Proposition 4.15. *Under the assumptions (i)-(iv) we have $P(\Omega) = \mathcal{M}_-(\Omega)$.*

Proof. By Theorem 1.47, and since $\partial\Omega = \partial^*\Omega$ by Proposition 4.14, it is enough to show that there exists a finite measure η absolutely continuous with respect to the perimeter measure $\mu = \mathcal{H}^1_{|\partial\Omega}$ and a constant $\gamma > 0$, such that for all $y \in \partial\Omega$ and $r \in (0,1)$ one has

$$\eta(B_r(y)) \geq \gamma r\,. \tag{4.33}$$

Let us set

$$\eta = \mu_{|\partial B} + \sum_{\mathbf{j}\in J} \frac{\varepsilon_{\mathbf{j}}}{r_{\mathbf{j}}} \mu_{|\partial B_{\mathbf{j}}}\,,$$

Clearly the measure η is finite since $\mu_{|\partial B_{\mathbf{j}}} \leq 2\pi r_{\mathbf{j}}$ for all $\mathbf{j} \in J$ and the convergence of $\sum_{\mathbf{j}\in J} \varepsilon_{\mathbf{j}}$. Given $y \in \partial\Omega \cap \partial B_1$ one trivially has $\eta(B_r(y)) \geq r$ for all $r \in (0,1)$. Hence we only have to show (4.33) when $y \in \partial\Omega \setminus \partial B_1$. We thus fix $y \in \partial B_{\mathbf{j}}$ for $\mathbf{j} \in J$ and distinguish three cases depending on the value of r.

Case 1: $0 < r \leq 2r_{\mathbf{j}}$. In this case we have

$$\eta(B_r(y)) \geq \frac{\varepsilon_{\mathbf{j}}}{r_{\mathbf{j}}} \mu_{|\partial B_{\mathbf{j}}}(B_r(x)) \geq \frac{\varepsilon_{\mathbf{j}} r}{r_{\mathbf{j}}} \geq r\,,$$

where the last inequality follows from $r_{\mathbf{j}} \leq \varepsilon_{\mathbf{j}}$, a consequence of assumption (i).

Case 2: $r \in (2r_{\mathbf{j}}, 2\varepsilon_{\mathbf{j}}]$. Since $r \geq 2r_{\mathbf{j}}$ one has $B_r(x) \supset B_{\mathbf{j}}$, hence using the other inequality $r \leq 2\varepsilon_{\mathbf{j}}$ one obtains

$$\eta(B_r(x)) \geq \frac{\varepsilon_{\mathbf{j}}}{r_{\mathbf{j}}} \mu(\partial B_{\mathbf{j}}) = 2\pi\varepsilon_{\mathbf{j}} \geq \pi r.$$

Case 3: $2\varepsilon_{\mathbf{j}} < r < 1$. In this last case, $B_r(x)$ intersects a portion of the boundary of B_1 whose length is comparable to r, hence

$$\eta(B_r(x)) \geq \mu(\partial B_1 \cap B_r(x)) \geq cr$$

for a universal constant $c > 0$. The previous three cases provide a complete proof of (4.33), and thus of the proposition, as soon as we choose $\gamma = \min\{1, \pi, c\}$. □

In the rest of the section we focus on an open set Ω constructed as before under the assumptions (i)-(iii) and $\sum_{\mathbf{j}\in J} \varepsilon_{\mathbf{j}} < +\infty$. By Theorem 3.23, Ω admits at least one Cheeger set. We will denote by E a Cheeger set of Ω. The main goal now is to show that, necessarily, $E = \Omega$ up to null sets. This result will be obtained through some intermediate steps and under some slightly more specific choice of the parameters $\varepsilon_{\mathbf{j}}$ and $r_{\mathbf{j}}$.

Proposition 4.16. *Let Ω be defined as in (4.27) and let E be a Cheeger set of Ω. Assume that (i)-(iv) hold. Then setting*

$$\delta = \frac{1 + \sum_{\mathbf{j}} r_{\mathbf{j}}}{1 - \sum_{\mathbf{j}} r_{\mathbf{j}}^2} - 1$$

one has

$$2 \leq h_1(\Omega) \leq 2(1 + \delta), \tag{4.34}$$

$$|E| \geq \frac{\pi}{(1 + \delta)^2}\,. \tag{4.35}$$

Proof. The first inequality in (4.34) follows directly from the inclusion $\Omega \subset B$ and from the monotonicity of the Cheeger constant (with respect to inclusions), while the second is a consequence of $h_1(\Omega) \leq \frac{P(\Omega)}{|\Omega|}$. Then (4.35) follows from (3.27) at once. □

Let us make the following assumptions on $\varepsilon_{\mathbf{j}}$ and $r_{\mathbf{j}}$.

(v) $\sum_{\mathbf{j}} r_{\mathbf{j}} \leq 1/(2^8 + 1)$;

(vi) $\varepsilon_{\mathbf{1}} < 1/4$;

(vii) $\varepsilon_{\mathbf{j}+1} \leq \frac{3}{10}\varepsilon_{\mathbf{j}}$;

(viii) $r_{\mathbf{j}} \leq 2^{-18}\varepsilon_{\mathbf{j}}^3$.

It is easy to check that the set of requests (v)-(viii) implies the set of requests (i)-(iv). Indeed, the growth condition (vii) implies the convergence of $\sum \varepsilon_{\mathbf{j}}$, i.e. (iv). The growth conditions (vii) and (viii) ensure that $\varepsilon_{\mathbf{j}}$ and $r_{\mathbf{j}}$ are strictly decreasing. This coupled with (vi) implies (i). Then (vii), (viii) and (vi) imply that

$$kr_{(k,1)} \leq k\varepsilon_{(k,1)}^3 \leq k\varepsilon_{\mathbf{1}}^3 \left(\frac{3}{10}\right)^{3(k^2-k)/2}$$

thus (ii) holds. Finally (iii) follows from (vii) and (viii), as it is equivalent to ask that $\varepsilon_{\mathbf{j}} - \varepsilon_{\mathbf{j}+1} \geq r_{\mathbf{j}} + r_{\mathbf{j}+1}$. We remark that the stronger inequality $\varepsilon_{\mathbf{j}} - 2\varepsilon_{\mathbf{j}+1} \geq r_{\mathbf{j}} + 2r_{\mathbf{j}+1}$ holds, which we shall use later on.

Notice that (v) implies $\delta < 1/2^7$. Indeed let $\eta = \sum_{\mathbf{j}} r_{\mathbf{j}}$. Then, since $\eta > \sum_{\mathbf{j}} r_{\mathbf{j}}^2$ one has

$$\delta = \frac{1 + \sum_{\mathbf{j}} r_{\mathbf{j}}}{1 - \sum_{\mathbf{j}} r_{\mathbf{j}}^2} - 1 \leq \frac{1+\eta}{1-\eta} - 1 \leq \frac{1}{2^7}\,. \tag{4.36}$$

Thus, by Proposition 4.16 we have

$$2 \leq h_1(\Omega) \leq 2(1+\delta) < 3\,. \tag{4.37}$$

Theorem 4.17. *Let $\varepsilon_{\mathbf{j}}$ and $r_{\mathbf{j}}$ be such that (v)-(viii) hold. Then, Ω is a minimal Cheeger set.*

The proof of Theorem 4.17 will require some preliminary lemmas.

Lemma 4.18. *Let Γ be an arc swept by a disk of radius $r < 1/2$ contained in an annulus of inner and outer radii equal to, respectively, $1/2$ and 1. Denote by o the center of the annulus and by a, b the endpoints of Γ. If the region R enclosed by (the vectors) a, b and Γ is convex then*

$$|p| \geq \min\{|a|, |b|\} \qquad \forall\, p \in \Gamma\,.$$

Proof. The configuration described in the statement is depicted in Figure 4.4. To prove the lemma we argue by contradiction and suppose that there exists $p_0 \in \Gamma \setminus \{a, b\}$ such that

$$|p_0| = \min_{p\in\Gamma} |p| < \min\{|a|, |b|\}\,.$$

If we denote by c the center of the disk sweeping the arc Γ, by minimality of p_0 we have that p_0, c, o lie on the same line. Moreover, being the region R convex by our assumption, we infer that c and o lie on the same half-plane cut by the tangent in p_0 to Γ. We now claim that c lies in between o and p_0. If this were not the case one would have $|p_0 - c| > |p_0|$ which in turn implies $r > 1/2$ against our hypotheses. Therefore we have $|p_0 - c| + |c| = |p_0|$ and by the triangular inequality

$$|a| \leq |c| + |c - a| = |c| + |p_0 - c| = |p_0|,$$

against our initial assumption. □

Lemma 4.19 (Density estimate)**.** *Let E be a Cheeger set of $A \subset \mathbb{R}^2$. Fix $z \in A$ and $r > 0$ such that $B_r(z) \subset A$. Then*

$$|B_r(z) \setminus E| \leq \pi r^2/36 \quad \Rightarrow \quad B_{2r/3}(z) \subset E\,, \tag{4.38}$$

where the inclusion must be understood up to null sets.

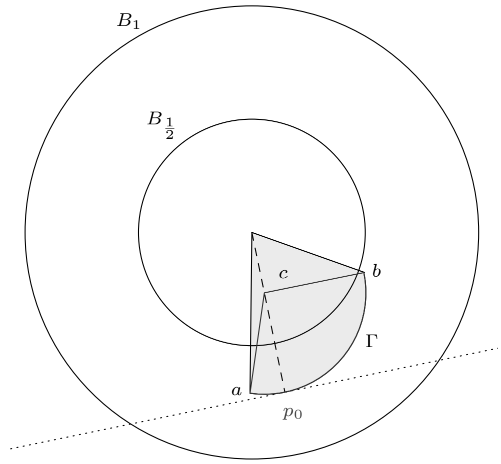


Figure 4.4: The configuration of Lemma 4.18

Proof. Let us set $m(r) = |B_r(z) \setminus E|$ and define $F = E \cup B_r(z)$ as a competitor. The minimality of E implies that

$$\frac{P(E)}{|E|} \le \frac{P(F)}{|F|} = \frac{P(E, \mathbb{R}^2 \setminus \overline{B_r(z)}) + m'(r)}{|E| + m(r)}$$
$$= \frac{P(E) - P(B_r(z) \setminus E) + 2m'(r)}{|E| + m(r)}$$

for almost all $r > 0$, hence

$$\frac{P(E)}{|E|} m(r) + P(B_r(z) \setminus E) \le 2m'(r)\,.$$

In particular we find that $P(B_r(z) \setminus E) \le 2m'(r)$, therefore by the isoperimetric inequality in $\mathbb{R}^2$ we obtain

$$m'(r) \ge \sqrt{\pi}\, m(r)^{\frac{1}{2}}\,. \tag{4.39}$$

If we now assume by contradiction that $m(2r/3) > 0$ then we can integrate the differential inequality

$$\frac{m'(t)}{m(t)^{\frac{1}{2}}} \ge \sqrt{\pi}$$

between $2r/3$ and r, thus obtaining

$$0 < m(2r/3)^{\frac{1}{2}} \le m(r)^{\frac{1}{2}} - \frac{\sqrt{\pi r^2}}{6} \le 0\,,$$

that is, a contradiction. □

Lemma 4.20. *Let Ω be constructed as before, and let δ be as in Proposition 4.16. If (v)-(viii) hold, then the disk $B_{1/2}$ is contained in any Cheeger set E of Ω.*

Proof. By (4.35) and (4.36) we have that

$$|B_{3/4} \setminus E| \le |B_1| - |E| \le \pi - \frac{\pi}{(1+\delta)^2} = \frac{2+\delta}{(1+\delta)^2}\pi\delta \le \frac{2+\delta}{1+\delta}\pi\delta \le 2\pi\delta \le \frac{\pi(3/4)^2}{36},$$

hence we can apply Lemma 4.19 and obtain that $B_{1/2} = B_{\frac{2}{3}\cdot\frac{3}{4}} \subset \Omega$ is also contained in E. □

Let us fix a Cheeger set E of Ω and assume that $\partial E \cap \Omega \neq \emptyset$. Then we consider the (at most countable) collection $\{\Gamma_k\}_{k\in\mathbb{N}}$ of the closures of the connected components of $\partial E \cap \Omega$. Notice that Γ_k is a closed circular arc of radius $r = h_1(\Omega)^{-1}$.

We observe that $\cup_k \Gamma_k$ is locally compact in B_1, as only a finite number of arcs can have a nonempty intersection with B_t, for all $0 < t < 1$. Then we have the following result.

Lemma 4.21. *Assume (v)-(viii) and that $\partial E \cap \Omega \neq \emptyset$. Denote by p_0 a point of $\cup_k \Gamma_k$ minimizing the distance from the origin. Then there exists k_0 such that p_0 is one of the endpoints of Γ_{k_0}.*

Proof. Since $\cup_k \Gamma_k \cap B_1$ is nonempty and locally compact in B_1, there exists $k_0 \in \mathbb{N}$ such that $p_0 \in \Gamma_{k_0}$. Assume now by contradiction that p_0 is not one of the endpoints a_0, b_0 of Γ_{k_0}, then owing to Lemma 4.20, $B_{1/2} \subset E$. Thus by Lemma 4.18, the region enclosed by Γ_{k_0} and the segments connecting a_0 and b_0 to the origin cannot be convex. Therefore, since $B_{1/2} \subset E$, the segment σ_0 connecting p_0 to the origin must intersect the boundary of E at some first point q_0 strictly closer than p_0 to the origin. Indeed, the Cheeger set locally lies on the convex side of Γ_{k_0} near p_0. To conclude we need to exclude the possibility that $q_0 \in \partial\Omega \setminus \partial B_1$, which means that $q_0 \in \partial B_{\mathbf{j}}$ for some $\mathbf{j}$. Let now consider the shortest of the two closed arcs of $\partial B_{\mathbf{j}}$ cut by σ_0 (note that the arc could degenerate to a single point), and call it γ. Notice that all the points of γ have a distance from the origin which is strictly less than $|p_0|$. Then γ must contain at least an endpoint of some Γ_k, otherwise there would exist an open neighbourhood U of γ such that $U \cap \partial E \cap \Omega = \emptyset$, but this cannot hold as U must contain points of E (this comes from the fact that $q_0 \in \gamma$) as well as points of $\Omega \setminus E$ (this is a consequence of the fact that the connected component of $\sigma_0 \cap \Omega$ having an endpoint on $\partial B_{\mathbf{j}}$, and being the closest to p_0, is made of points of $\Omega \setminus E$). Therefore $q_0 \in \Omega$, hence $q_0 \in \Gamma_k$ for some k, which contradicts the minimality of p_0. This concludes the proof. □

Lemma 4.22. *Assume (v)-(viii) and let p_0 be as in Lemma 4.21. Then letting α be the angle spanned by the half-tangent to Γ_{k_0} in p_0 and the segment connecting p_0 to the origin, one has*

$$\alpha > \frac{\pi}{2} + \frac{d_0}{2}, \tag{4.40}$$

where $d_0 = \operatorname{dist}(p_0, \partial B_1)$.

Proof. Let $B_{\mathbf{j}}$ be the ball whose boundary contains p_0. Let p_1 be the second endpoint of Γ_{k_0} and denote by p_* the point of Γ_{k_0} minimizing the distance from ∂B_1. Since $p_1 \in \partial\Omega$, by construction of Ω we infer that either $p_1 \in \partial B_1$, or $p_1 \in \partial B_{\mathbf{j}'}$ with $\mathbf{j} \prec \mathbf{j}'$, therefore the distance

$d_* = \mathrm{dist}(p_*, \partial B_1)$ must satisfy $d_* < d_0/2$. Indeed this holds true if $\varepsilon_{\mathbf{j}} - 2\varepsilon_{\mathbf{j}+1} \geq r_{\mathbf{j}} + 2r_{\mathbf{j}+1}$, which follows from conditions (vii) and (viii). Let c be the center of the arc Γ_{k_0} and consider the triangle T with vertices p_0, c and the origin. Notice that $|p_0 - c| = r < 1/2$ and $|p_0| = 1 - d_0$ while by the triangular inequality applied to the triangle T^* of vertices p^*, c, o we have

$$|c| \geq |p^*| - r = 1 - r - d^* \geq 1 - r - d_0/2\,.$$

Moreover if we assume that $\alpha < \pi$ (otherwise the estimate would be trivial) then the internal angles of T at p_0 and at the origin (respectively, γ and β) are smaller than $\pi/2$. Indeed for $\alpha < \pi$ we find that

$$\langle p_0, \nu_{\mathbf{j}}(p_0)\rangle < 0\,,$$

where $\nu_{\mathbf{j}}(p_0)$ denotes the outer normal to $\partial B_{\mathbf{j}}$ at p_0, thus $\alpha > \pi/2$. Then, $\gamma = \alpha - \pi/2 \in [0, \pi/2)$. Finally, $|p_0| > r$, whence $\beta < \pi/2$ as claimed. Consequently the orthogonal projection z of c onto the line through the opposite side of T must lie between the origin and p_0, that is, $|p_0| = |z| + |p_0 - z|$. Then we have

$$|c|^2 - |z|^2 = r^2 - |p_0 - z|^2\,,$$

whence by rearranging terms

$$\begin{aligned}
|c|^2 - r^2 &= |z|^2 - |p_0 - z|^2 \\
&= |p_0| \cdot (|z| - |p_0 - z|) \\
&= |p_0| \cdot (|p_0| - 2|p_0 - z|) \\
&= (1 - d_0)(1 - d_0 - 2|p_0 - z|)\,.
\end{aligned}$$

On the other hand

$$|c|^2 - r^2 \geq (1 - r - d_0/2)^2 - r^2 = 1 + d_0^2/4 - 2r - d_0 + d_0 r\,,$$

thus we find

$$2|p_0 - z| \leq 1 - d_0 - \frac{1 + d_0^2/4 - (2 - d_0)r - d_0}{1 - d_0}\,.$$

Consequently we have

$$\begin{aligned}
\cos\gamma &= \frac{|p_0 - z|}{r} \\
&\leq \frac{2r(1 - d_0) + rd_0 - d_0 + 3d_0^2/4}{2r(1 - d_0)} \\
&= 1 - \frac{d_0(1 - r) - 3d_0^2/4}{2r(1 - d_0)} \\
&< 1 - d_0/4\,,
\end{aligned}$$

where the last inequality follows as soon as $d_0 < 1/3$. Being $d_0 \leq \varepsilon_{\mathbf{1}} + r_{\mathbf{1}}$, this condition is met thanks to (vi) and (vii). Then, we have

$$\sin^2\gamma = 1 - \cos^2\gamma > 1 - (1 - d_0/4)^2 = d_0/2 - d_0^2/4 > d_0^2/4$$

and thus we conclude that

$$\gamma > \sin\gamma > d_0/2\,.$$

Since $\alpha = \pi/2 + \gamma$, we get (4.40). □

Lemma 4.23. *Assume (v)-(viii) and let p_0, Γ_{k_0}, d_0 and α be as in Lemma 4.22. Let $p \in \Gamma_{k_0}$ be a point such that $0 < |p_0 - p| < d_0/12$. Then, denoting by η the angle in p_0 spanned by the half-tangent to Γ_{k_0} at p_0 and the segment from p_0 to p, one has*

$$\xi := \alpha - \eta > \frac{\pi}{2} + \frac{d_0}{4}\,. \tag{4.41}$$

Proof. Let c be the center of the disk sweeping Γ_{k_0} and let h be the projection of p onto the half-tangent to Γ_{k_0} at p_0. Since $\xi = \alpha - \eta$, by Lemma 4.22, it is enough to provide an upper bound for η.

To this aim we consider the triangles T of vertices p_0, p_h and h and S of vertices p_0, c and m, where m is the midpoint of the segment $p - p_0$, as in Figure 4.5. It is easy to see they are similar with angles $\pi/2, \eta$, and $\pi/2 - \eta$. Therefore we have the proportionality relation

$$\frac{|p-h|}{|p-p_0|} = \frac{|p-p_0|}{2r}\,,$$

whence by recalling that $0 < \eta < \pi/2$ and that $r > 1/3$ by (4.34) and the condition on δ one obtains

$$\frac{\eta}{2} \leq \sin(\eta) = \frac{|p-h|}{|p-p_0|} = \frac{|p-p_0|}{2r} < \frac{d_0}{24r} < \frac{d_0}{8}\,. \tag{4.42}$$

This upper bound on η combined with (4.40) yields the claim. □

Remark 4.24. Note that Lemmas 4.22 and 4.23 hold whenever p_0 is the endpoint of an arc Γ such that p_0 minimizes $|p|$ among $p \in \Gamma$.

It is not difficult to show via a compactness argument that, for a suitable choice of parameters, the set $\Omega_{\mathbf{j}}$ defined as

$$\Omega_{\mathbf{j}} := B_1 \setminus \bigcup_{\mathbf{i} \leq \mathbf{j}} \overline{B_{\mathbf{i}}}\,,$$

is a minimal Cheeger set for all $\mathbf{j}$. Then, by passing to the limit as $\mathbf{j} \to \infty$ and by exploiting Theorem 2.7 of [LP16], we would infer that Ω is a Cheeger set as well. However, this simple argument tells us nothing about the uniqueness of the Cheeger set of Ω. In other words, there seems to be no way of deducing that $\Omega = \lim_{\mathbf{j}} \Omega_{\mathbf{j}}$ is a minimal Cheeger set from the minimality of $\Omega_{\mathbf{j}}$. This is due to the lack of uniform a-priori estimates in the spirit of the quantitative

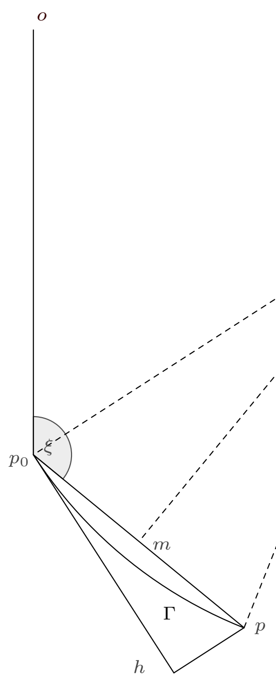


Figure 4.5: The configuration of Lemma 4.23

isoperimetric inequality (see in particular [CL12, CL13]). In this specific case, the existence of a modulus of continuity φ independent of $\mathbf{j}$, such that

$$P(E)/|E| - h(\Omega_{\mathbf{j}}) \geq \varphi(|\Omega_{\mathbf{j}} \setminus E|)$$

for all $\mathbf{j}$ and all measurable $E \subset \Omega_{\mathbf{j}}$, would be needed. By an application of the selection principle introduced in [CL12] we can obtain $\varphi = \varphi_{\mathbf{j}}$, however it is not clear how to exclude a possible degeneracy of the sequence $\{\varphi_{\mathbf{j}}\}_{\mathbf{j}}$, as $\mathbf{j} \to \infty$. Therefore we are forced to directly prove uniqueness by showing that, for any Cheeger set E of Ω, the intersection $\partial E \cap \Omega$ is necessarily empty. Owing to the connectedness of Ω and the fact that $B_{1/2} \subset E$, this is sufficient to conclude that $E = \Omega$ up to null sets. Before delving into the proof, we remark that there are four different kinds of arcs inside $\partial E \cap \Omega$, depending on where their endpoints lie:

(a) arcs Γ with both endpoints on ∂B_1;

(b) arcs Γ with both endpoints on $\partial B_{\mathbf{j}}$ for some $\mathbf{j}$;

(c) arcs Γ with an endpoint of ∂B_1 and one of $\partial B_{\mathbf{j}}$ for some $\mathbf{j}$;

(d) arcs Γ with an endpoint on $\partial B_{\mathbf{j}}$ and one on $\partial B_{\mathbf{i}}$ with $\mathbf{j} \neq \mathbf{i}$.

While cases (a) and (b) can be easily excluded by Theorem 3.1 (iv) and (v), cases (c) and (d) are much trickier. For these latter two cases the argument is actually the same: we will build a competitor that has a smaller Cheeger ratio, thus contradicting the minimality of E. In order to do so, we will also employ Lemma 4.23.

Proof of Theorem 4.17. Argue by contradiction and suppose $\partial E \cap \Omega \neq \emptyset$.

Step 1. We start by showing that cases (a) and (b) cannot happen. Let Γ be the arc with endpoints $p, q \in \partial B_{\mathbf{j}}$. Being these points regular, by Theorem 3.1 (v) the arc Γ must be tangent to $B_{\mathbf{j}}$ in both points. By Proposition 4.16 and the choice of $r_{\mathbf{j}}$ the curvature of $B_{\mathbf{j}}$ is strictly greater than the curvature of Γ. Therefore one necessarily has that points p and q coincide which implies that Γ is a full circle which contradicts Theorem 3.1 (iv). An analogue reasoning holds for an arc Γ with endpoints $p, q \in \partial B_1$.

Step 2. We now show that cases (c) and (d) cannot happen. We will exhibit a competitor to E that has a better Cheeger ratio against the minimality of E. Pick the point p_0 provided by Lemma 4.21 and consider the arc Γ_{p_0} with endpoint p_0. There exists a pair $\mathbf{j}$ such that $p_0 \in \partial B_{\mathbf{j}}$. Trivially there exists at least another point q_0 on the boundary of $B_{\mathbf{j}}$ from which another arc of $\partial E \cap \Omega$ departs. Let $z \in \partial B_{\mathbf{j}}$ be the "north pole", i.e. the closest point to the origin. Note that there is only a finite number of arcs of $\partial E \cap \Omega$ touching $\partial B_{\mathbf{j}}$. Moreover, since $|p_0| > r$ we find that $|p_0| > |z|$ (otherwise we would have $p_0 = z$ and this would contradict the fact that p_0 minimizes the distance of points of Γ_{p_0} from the origin). This shows that z is contained in a connected component φ of $\partial B_{\mathbf{j}} \setminus \mathcal{E}_{\mathbf{j}}$, where $\mathcal{E}_{\mathbf{j}}$ denotes the (finite) set of endpoints of arcs of $\partial E \cap \Omega$ that lie on $\partial B_{\mathbf{j}}$. One of the endpoints of φ is, of course, p_0. Let q_0 denote the other endpoint belonging to the arc Γ_{q_0}.

From now on we shall assume that φ is smaller than a half-circle, otherwise the construction of the competitor would be even easier.

Since p_0 minimizes the distance of $\partial E \cap \Omega$ from the origin we have that

$$d_{q_0} := \operatorname{dist}(q_0, \partial B_1) \leq \operatorname{dist}(p_0, \partial B_1) =: d_{p_0}.$$

We now fix two points $q \in \Gamma_{q_0}$ and $p \in \Gamma_{p_0}$ such that

$$|p - p_0| = |q - q_0| = \frac{d_{q_0}}{16}. \tag{4.43}$$

We can apply Lemma 4.23 to the couples of points p, p_0 and q, q_0 obtaining the estimate from below of the angles ξ_q and ξ_p (that correspond to ξ in Lemma 4.23):

$$\xi_q, \xi_p > \frac{\pi}{2} + \frac{d_{q_0}}{4}. \tag{4.44}$$

We now modify the Cheeger set E into $\widetilde{E}$ by adding the region delimited by $\partial B_{\mathbf{j}}, \Gamma_{q_0}, \Gamma_{p_0}$ and the segment $p{-}q$. To contradict the minimality of E it is enough to show that $\delta P = P(\widetilde{E}) - P(E) < 0$

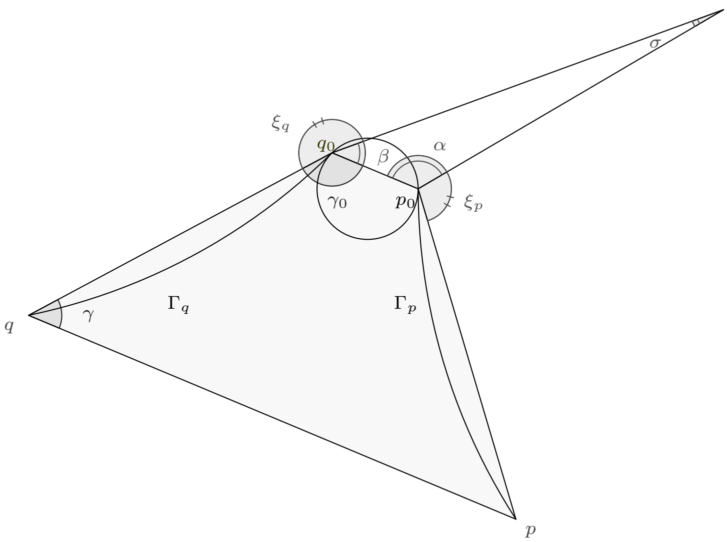


Figure 4.6: The competitor of Theorem 4.17 step 2

for ε small enough. It is straightforward that

$$\delta P \leq 2\pi r_{\mathbf{j}} - |p-p_0| - |q-q_0| + |p-q| = 2\pi r_{\mathbf{j}} - 2|p-p_0| + |p-q|. \tag{4.45}$$

Therefore we need to estimate $|p-q|$ from above. In order to do so, we will employ the angles of the isosceles trapezoid with vertices p_0, q_0, q, p (and, respectively, angles γ_0 and γ) and the triangle T of vertices o, p_0, q_0 (and, respectively, angles σ, α, β), denoted as in Figure 4.6. We then have

$$\begin{cases} \gamma_0 + \gamma = \pi & (4.46a) \\ \alpha + \beta + \xi_q + \xi_p + 2\gamma_0 = 4\pi & (4.46b) \\ \alpha + \beta + \sigma = \pi & (4.46c) \end{cases}$$

where (4.46a) denotes the (half of the) sum of interior angles of the trapezoid, (4.46b) the sum of the angles in p_0 and in q_0, and (4.46c) the sum of the interior angles of the triangle T.

Subtracting (4.46c) to (4.46b), and combining the resulting equality with (4.44) we find

$$2\gamma_0 < 2\pi + \sigma - \frac{d_{q_0}}{2}$$

which coupled with (4.46a) gives

$$\gamma > \frac{d_{q_0}}{4} - \frac{\sigma}{2}\,.$$

We now estimate σ from above as follows. First notice that its sine is small

$$\sin(\sigma) = \frac{|p_0 - q_0|}{1 - d_{q_0}} \sin(\alpha) \leq 4r_{\mathbf{j}} \leq 2^{-4}\varepsilon_{\mathbf{j}}\,,$$

where the last inequality is guaranteed by (viii). Thus σ itself is small, i.e.

$$\frac{\sigma}{2} \le \sigma - \frac{\sigma^3}{6} \le \sin\sigma \le 2^{-4}\varepsilon_{\mathbf{j}} \le 2^{-3}d_{q_0}\,,$$

eventually getting the lower bound

$$\gamma > \frac{d_{q_0}}{8}\,.$$

Since $|p-q| > |p_0 - q_0|$, the angle γ is smaller than $\pi/2$, thus

$$0 \le \cos\gamma \le \cos\left(\frac{d_{q_0}}{8}\right) \le 1 - \frac{d_{q_0}^2}{2^7} + \frac{d_{q_0}^4}{3\cdot 2^{15}} \le 1 - \frac{d_{q_0}^2}{2^8}\,. \tag{4.47}$$

From (4.43), (4.45) and (4.47) it follows that

$$\begin{aligned}\delta P &\le 2\pi r_{\mathbf{j}} - 2|p-p_0| + |p-q|\\ &\le 2\pi r_{\mathbf{j}} - 2|p-p_0| + 2|p-p_0|\cos\gamma + 2r_{\mathbf{j}}\\ &\le 2r_{\mathbf{j}}(\pi+1) + \frac{d_{q_0}}{2^3}(\cos(\gamma)-1) \le 2r_{\mathbf{j}}(\pi+1) - \frac{d_{q_0}^3}{2^{11}}\end{aligned}$$

Since by (viii) we have $r_{\mathbf{j}} \le 2^{-18}\varepsilon_{\mathbf{j}}^3$ and $d_{q_0} \ge \varepsilon_{\mathbf{j}}/2$, we obtain

$$\frac{d_{q_0}^3}{2^{11}} \ge \frac{\varepsilon_{\mathbf{j}}^3}{2^{14}} \ge 16r_{\mathbf{j}} > 2r_{\mathbf{j}}(\pi+1)\,,$$

thus $\delta P < 0$, a contradiction. This concludes the proof of the theorem. □

Remark 4.25. We proved that if (v)-(viii) hold, then the set Ω is a minimal Cheeger set. One can wonder if by assuming only (i)-(iii), and a sequence $\varepsilon_{\mathbf{j}}$ not satisfying (iv) the same can hold for a suitable choice of $r_{\mathbf{j}}$. It is reasonable to think the resulting set will still be a minimal Cheeger by letting the radii decay fast enough. The main difference in the proof would be the construction of the competitor in Theorem 4.17 step 2. This one would be more subtle and would require different cuts depending on the positions of the points p_0 and q_0. Were this true, recalling Proposition 4.14 one might end up with a minimal Cheeger set Ω (and, accordingly to (4.28) a family $\{\Omega_{\mathbf{k}}\}$) such that

$$\mathcal{H}^1(\partial^*\Omega \setminus \partial^{**}\Omega) > 0 \tag{4.48}$$

and

$$P(\Omega) < \mathcal{M}_-(\Omega)\,. \tag{4.49}$$

We here show that for the specific choice $(\varepsilon_{\mathbf{j}})_{\mathbf{j}\in J}$ defined by

$$\varepsilon_{\mathbf{j}} = \varepsilon_{(j_1,j_2)} = \frac{5}{j_1+10} \tag{4.50}$$

for all $\mathbf{j} = (j_1, j_2) \in J$ (and for any choice of $r_{\mathbf{j}}$ such that (i)-(iii) are satisfied) one indeed has (4.49) and (4.50). We first show that any $y \in \partial B_1 \cap \{z \in \mathbb{R}^2 :\ z_1, z_2 \ge 0\}$ does not belong to $\partial^{**}\Omega$.

Let $\theta \in [0, \pi/2]$ be such that $y = (\cos\theta, \sin\theta)$. For any $j_1 \in \mathbb{N}$ we can choose $j_2 \in \{1, \dots, j_1\}$ to be such that

$$|x_{\mathbf{j}} - y| \qquad \text{is minimum for } \mathbf{j} = (j_1, j_2) \tag{4.51}$$

among pairs (j_1, h). This allows us to define a subsequence $(\mathbf{j}^h)_h$ such that $x_{\mathbf{h}} \to y$ as $h \to \infty$, where we have set $x_{\mathbf{h}} = x_{\mathbf{j}^h}$. Moreover by (4.51) and by the choice of $\varepsilon_{\mathbf{j}}$ we have

$$y \cdot (y - x_{\mathbf{h}}) \geq c|x_{\mathbf{h}} - y|$$

for all $\mathbf{h}$ and for a suitable constant $c > 0$, which implies $y \notin \partial^{**}\Omega$ by Proposition 2.2, as claimed. This shows (4.48).

We now prove (4.49). To this aim we set $A = \partial B_1 \cap \{z \in \mathbb{R}^2 : \; z_1, z_2 \geq 0\}$ and fix $0 < t < 1$. At least for t small enough we prove that

$$\begin{aligned} V_t &:= \Omega \cap \{z : \; z_1, z_2 \geq 0\} \cap [A]_{11t/10} \\ &\subset \Omega \cap [\partial\Omega]_t \,. \end{aligned} \tag{4.52}$$

Indeed, let us fix $z = r(\cos\theta, \sin\theta) \in V_t \setminus [\partial\Omega]_t$ and take the least index $j_1 = j_1(t)$ such that $\varepsilon_{\mathbf{j}} < t$. Clearly we also have $\varepsilon_{\mathbf{j}} \geq 9t/10$, otherwise we would contradict the minimality of j_1. At the same time, for a suitable choice of $j_2 \in \{1, \dots, j_1\}$ we have as before that $|z - x_{\mathbf{j}}|$ is minimized by $\mathbf{j} = (j_1, j_2)$ among indices of the form (j_1, h). On the other hand, $1 - 11t/10 \leq r \leq 1 - t$ and $9t/10 \leq \varepsilon_{\mathbf{j}} \leq t$, so that in particular $j_1 \geq \frac{5}{t} - 10$. Then by assuming $t < 1/4$ we obtain $\frac{j_1+10}{j_1+1} < 2$ and then by applying the triangular inequality we get

$$|z - x_{\mathbf{j}}| \leq \frac{t}{5} + \frac{\pi}{2(j_1+1)} = \frac{t}{5} + \frac{\pi}{10} \cdot \frac{j_1+10}{j_1+1} \cdot \varepsilon_{\mathbf{j}} < t \,,$$

which proves (4.52). We finally conclude that

$$\liminf_{t \to 0^+} \frac{|\Omega \cap [\partial\Omega]_t|}{t} \geq P(\Omega) + \frac{1}{10}\mathcal{H}^1(A) > P(\Omega) \,,$$

as wanted.

List of Figures

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