

# REAL SUBSET SUMS AND POSETS WITH AN INVOLUTION

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ABSTRACT. In this paper we carry out in an abstract order context some real subset combinatorial problems. Specifically, let  $(X, \leq, c)$  be a finite poset, where  $c : X \rightarrow X$  is an order-reversing and involutive map such that  $c(x) \neq x$  for each  $x \in X$ . Let  $B_2 = \{N < P\}$  be the Boolean lattice with two elements and  $\mathcal{W}_+(X, B_2)$  the family of all the order-preserving 2-valued maps  $A : X \rightarrow B_2$  such that  $A(c(x)) = P$  if  $A(x) = N$  for all  $x \in X$ . In this paper we build a family  $\mathcal{B}_{w_+}(X)$  of particular subsets of  $X$ , that we call  $w_+$ -bases on  $X$ , and we determine a bijection between the family  $\mathcal{B}_{w_+}(X)$  and the family  $\mathcal{W}_+(X, B_2)$ . In such a bijection, a  $w_+$ -basis  $\Omega$  on  $X$  corresponds to a map  $A \in \mathcal{W}_+(X, B_2)$  whose restriction of  $A$  to  $\Omega$  is the smallest 2-valued partial map on  $X$  which has  $A$  as its unique extension in  $\mathcal{W}_+(X, B_2)$ . Next we show how each  $w_+$ -basis on  $X$  becomes, in a particular context, a sub-system of a larger system of linear inequalities, whose compatibility implies the compatibility of the whole system.

## 1. INTRODUCTION

Involution partially ordered sets (briefly *involution posets*) are order structures classically used to describe specific combinatorial aspects in Ockam algebras and de Morgan algebras (see [9] for more details). These posets generalize the classical Boolean algebras [23, 38], and in literature they are studied in several different mathematical contexts (see for example [1, 11, 16, 19, 25, 26, 30, 31, 36, 37]).

In this paper we use involution posets in a set operatorial perspective, in analogy with what has been recently done for integral domains [14] and monoid actions [15].

Our basic motivation arises from an extremal combinatorial sum problem studied in [12] and in [13]. In such a problem one has two fixed integer parameters  $n \geq r \geq 0$  and a numerical map

$$f : \{r, \dots, 1, -1, \dots, -(n-r)\} := I(n, r) \rightarrow \mathbb{R}$$

such that

$$f(r) \geq \dots \geq f(1) \geq 0 > f(-1) \geq \dots \geq f(-(n-r)),$$

which in the present paper we will call  $(n, r)$ -function. Let  $F(n, r)$  be set of all  $(n, r)$ -functions. If  $f \in F(n, r)$  and  $C \subseteq I(n, r)$ , we call  $\sum_{k \in C} f(k)$  a *partial sum* of  $f$ . In particular, the *total sum* of  $f$  is  $\sum_{k \in I(n, r)} f(k)$ .

Now, in [13] the problem of computing the number

$$\gamma(n, r) := \min |\{C \subseteq I(n, r) : \sum_{k \in C} f(k) \geq 0\}|,$$

where the minimum is taken over all the functions  $f \in F(n, r)$  such that  $\sum_{k \in I(n, r)} f(k) \geq 0$ , has been solved by using a particular involution lattice  $(S(n, r), \sqsubseteq, c)$  introduced in [6] (in relation with a famous conjecture of Manickam and Singhi [29]), whose elements can be identified with the subsets of  $I(n, r)$  (for detail see [6]). In analogy with what has been done in [2, 3, 5, 17, 32, 33, 34, 39] for several types of discrete structures, in [7] the lattice  $S(n, r)$  is also described as a discrete dynamical system of signed integer partitions by means of some evolution rules. The signed integer partitions were introduced by Andrews in [4] and further studied in [27]. Recently, having as reference the classical integer partition lattices introduced in [10], several types of lattice structures of signed integer partitions have been investigated in terms discrete dynamical systems in [7, 8, 24]. In the specific case of  $S(n, r)$ , the partial order  $\sqsubseteq$  is naturally related (see [6]) to a particular class of linear system inequalities on  $n$  real variables

$$u_r \geq \dots \geq u_1 \geq 0 > u_{-1} \geq \dots \geq u_{-(n-r)},$$

that we call  $(n, r)$ -systems. An  $(n, r)$ -system  $\mathcal{S}$  has exactly one inequality of the type  $\sum_{i \in B} u_i \geq 0$  (or  $< 0$ ) for each subset  $B$  of the indexes set  $I(n, r)$ . Each  $(n, r)$ -system  $\mathcal{S}$  can be identified with an order-preserving 2-valued map  $A_{\mathcal{S}} : S(n, r) \rightarrow B_2$ , where  $B_2 := \{N < P\}$  is the 2-chain. If we assume that the

*Date:* September 4, 2021.

*1991 Mathematics Subject Classification.* Primary: 06A06, 06A07; Secondary: 05A05, 90C09.

*Key words and phrases.* Involution Posets, 2-Valued Maps, Systems of Linear Inequalities, Valuations, Lattices.

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inequality

$$u_r + \cdots + u_1 + u_{-1} + \cdots + u_{-(n-r)} \geq 0$$

appears in the  $(n, r)$ -system  $\mathcal{S}$ , then the 2-valued map  $A_{\mathcal{S}}$  is such that  $A_{\mathcal{S}}(B^c) = P$  if  $A_{\mathcal{S}}(B) = N$ , where  $B^c$  is the complementary subset of  $B$  in  $I(n, r)$ . For further details on this type of problems see also [6]. As mentioned above, the present study has as its underlying motivation the research of some particular types of  $(n, r)$ -functions. Moreover, our most relevant applications are linked to the involution map  $c : S(n, r) \rightarrow S(n, r)$  introduced in [6].

To this regard, we often must consider the compatibility of a correspondent  $(n, r)$ -system  $\mathcal{S}$ . It is therefore natural to study the 2-valued map  $A_{\mathcal{S}}$  to obtain some information on the system  $\mathcal{S}$ .

However, from a more theoretical perspective, if one investigates in detail the above involution map  $c$  on  $S(n, r)$  in connection with any 2-valued map  $A_{\mathcal{S}}$ , he realizes that there two basic properties having a fundamental role. These two properties (that we investigated in a more general context) are the monotonicity of any 2-valued map  $A_{\mathcal{S}}$  and the condition  $A_{\mathcal{S}}(B^c) = P$  whenever  $A_{\mathcal{S}}(B) = N$ .

In the present paper we investigate such properties in some more general order structures, which are the specific types of involution posets. More in detail, we start with an arbitrary finite *involution poset*  $(X, \leq, c)$ , that is a partially ordered set  $(X, \leq)$  equipped with an involutive and order-reversing map  $c : X \rightarrow X$ , having, in our case, the further property that  $c(x) \neq x$  for all  $x \in X$ . On  $X$  we examine then the family  $\mathcal{W}_+(X, B_2)$  of all the 2-valued maps  $B : X \rightarrow \{N < P\}$  which are order-preserving and such that  $B(c(x)) = P$  if  $B(x) = N$ .

In the study of the  $(n, r)$ -systems, it is important to know (see [6] for details) when a 2-valued partial map  $A$  on  $S(n, r)$  determines a unique total map  $B \in \mathcal{W}_+(S(n, r), B_2)$  which extends  $A$ . Moreover, given any total map  $B \in \mathcal{W}_+(S(n, r), B_2)$ , it is also important to know how it is made the “minimum” sub-partial map in the set of all the partial maps having  $B$  as unique extension.

Similar extension problems already arise in classical Boolean algebra literature (see [38]), and in our context they naturally conduct us to introduce the notion of *core* for any fixed family  $\mathcal{H}$  of 2-valued maps on  $X$ .

More specifically, let  $\mathcal{H}$  be a family of 2-valued global maps on  $X$ . We say that a 2-valued partial map  $A$  on  $X$  is a  $\mathcal{H}$ -*core* if there is a unique map  $B_A \in \mathcal{H}$  which extends  $A$ , and, in this case, we also say that  $B_A$  is the total map in  $\mathcal{H}$  generated by  $A$  and that  $A$  is a  $\mathcal{H}$ -core of  $B_A$ . Moreover, we say that a  $\mathcal{H}$ -core  $A$  is a *fundamental*  $\mathcal{H}$ -core if for each other  $\mathcal{H}$ -core  $C$  such that  $B_A = B_C$ , the map  $C$  extends  $A$ . In this paper, we first build the fundamental core of a generic 2-valued map  $B \in \mathcal{W}_+(X, B_2)$ . Next, we show which must be the particular form of a finite subset  $V$  of  $X$  so that it is the fundamental core of some  $B \in \mathcal{W}_+(X, B_2)$ , and, in this case, we also determine the corresponding form of the map  $B$ .

Finally, we translate these abstract results in the terminology of the  $(n, r)$ -systems, when  $X = S(n, r)$ , showing as the concept of core is related to the study of the compatibility of such types of linear inequalities systems.

The paper is organized as follows.

Section 2 contains an explanatory discussion of the basic notions. In this section we establish the definitions and the general results that we use in the remaining part of the paper.

In Section 3 we determine the fundamental core of an arbitrary 2-valued map  $B \in \mathcal{W}_+(X)$ , where  $X$  is a finite SIP. The last part of the same section is devoted to compute the specific  $\mathcal{W}_+(X)$ -core of a 2-valued total map in a particular case, when  $X$  is a given infinite SIP.

Section 4 provides the specific form that must have a subset of  $X$  in order to be the fundamental core of some 2-valued map  $B \in \mathcal{W}_+(X)$ .

In Section 5 we take  $X = S(n, r)$  and we show that its partial order is characterized in a natural way from the partial sums  $\sum_{k \in C} f(k)$ , where  $C \subseteq I(n, r)$  and  $f \in F(n, r)$ . In this section we also remark the fact that each  $f \in F(n, r)$  induces a sum function  $\sum_f : S(n, r) \rightarrow \mathbb{R}$  which is a valuation on the lattice  $S(n, r)$ .

Section 6 is devoted to the analysis of the  $(n, r)$ -systems. We use the notion of core in order to provide a local criterion that characterize the compatibility of such systems.

Finally, in Section 7 we give an example that shows how the family  $\mathcal{W}_+(S(n, r), B_2)$  is not sufficiently small in order to be in bijective correspondence with the family of all the  $(n, r)$ -systems having non-negative total sum. In this section we define a subfamily  $\mathcal{FC}_+(n, r)$  of  $\mathcal{W}_+(S(n, r), B_2)$  which for us is a good candidate to be a family of 2-valued maps that capture all the combinatorial properties necessary and sufficient to characterize the family of the  $(n, r)$ -consistent systems. We leave this as an open problem for a future research.

## 2. DEFINITIONS, NOTATIONS AND SOME GENERAL RESULTS.

In what follows we use the same terminology of [18].

Let  $(X, \leq)$  be a poset. If  $Z \subseteq X$ , we will set  $\downarrow Z = \{x \in X : \exists z \in Z \text{ such that } z \geq x\}$ ,  $\uparrow Z = \{x \in X : \exists z \in Z \text{ such that } z \leq x\}$ . In particular, if  $z \in X$ , we will set  $\downarrow z = \downarrow\{z\} = \{x \in X : z \geq x\}$ ,  $\uparrow z = \uparrow\{z\} = \{x \in X : z \leq x\}$ .  $Z$  is called a *down-set* of  $X$  if for each  $z \in Z$  and  $x \in X$  with  $z \geq x$ , then  $x \in Z$ . Analogously,  $Z$  is called an *up-set* of  $X$  if for each  $z \in Z$  and  $x \in X$  with  $z \leq x$ , then  $x \in Z$ .  $\downarrow Z$  is the smallest down-set of  $X$  which contains  $Z$  and  $Z$  is a down-set in  $X$  if and only if  $Z = \downarrow Z$ . Similarly  $\uparrow Z$  is the smallest up-set of  $X$  which contains  $Z$  and  $Z$  is an up-set in  $X$  if and only if  $Z = \uparrow Z$ .

If  $X$  and  $Y$  are two arbitrary posets, a partial map from  $X$  to  $Y$  is a map  $A : Z \rightarrow Y$ , where  $Z$  is a subset of  $X$ . A map  $A : X \rightarrow Y$  is called a total map. If  $A : Z \subseteq X \rightarrow Y$  is a partial map, we will say that  $A$  is *order-preserving* (briefly OP) if for every  $x_1, x_2 \in Z$  such that  $x_1 \leq x_2$  in  $X$ , then  $A(x_1) \leq A(x_2)$  in  $Y$ . We denote by  $\mathcal{OP}(X, Y)$  the family of all total maps from  $X$  to  $Y$  which are order-preserving.

Denote with  $B_2$  the Boolean lattice composed of a chain with 2 elements that we will denote by  $N$  (the minimal element) and  $P$  (the maximal element). The set of all the partial maps from  $X$  to  $B_2$ , here denoted by  $\mathcal{BPM}(X)$ , is a poset with the natural order: if  $A, B \in \mathcal{BPM}(X)$ , then  $A \leq B$  if and only if  $B$  extends  $A$ .

A *2-valued partial map* (briefly BPM) on  $X$  is an element  $A$  of  $\mathcal{BPM}(X)$ . If  $\text{dom}(A) = X$ , we will say that  $A$  is a *2-valued total map* (BTM) on  $X$ . We say that a 2-valued partial map  $A$  on  $X$  is *up-positive* if  $A^{-1}(P)$  is an up-set of  $X$ ; we say that it is *down-negative* if  $A^{-1}(N)$  is a down-set of  $X$ .

Given a 2-valued partial map  $A$  on  $X$ , a minimal element in  $A^{-1}(P)$  is called *minimal positive* of  $A$ ; a maximal element in  $A^{-1}(N)$  is called *maximal negative* of  $A$ . If  $Z \subseteq X$  we denote by  $\text{Min}(Z)$  the set of minimal elements of  $Z$  and with  $\text{Max}(Z)$  the set of maximal elements of  $Z$ . If  $A$  is a 2-valued partial map on  $X$  and if  $Z$  is a subset of  $X$ , we set:

$$\begin{aligned} Z_P^A &= A^{-1}(P) \cap Z = \{x \in Z \cap \text{dom}(A) : A(x) = P\}, \\ Z_N^A &= A^{-1}(N) \cap Z = \{x \in Z \cap \text{dom}(A) : A(x) = N\}. \end{aligned}$$

As usual, we denote by  $\sqcup$  the disjoint union between two sets.

The following proposition shows that the concepts of up-positivity, down-negativity and of order-preserving are equivalent for 2-valued total maps.

**Proposition 2.1.** *Let  $X$  be an arbitrary poset and  $A$  a BTM on  $X$ . Then the following are equivalent:*

- i)  $A$  is order-preserving (OP);
- ii)  $A$  is up-positive (UP);
- iii)  $A$  is down-negative (DN).

*Proof.* Straightforward. □

Our next question is the following: let  $X$  be a poset and let  $A$  be a 2-valued partial map on  $X$ . Which properties has  $A$  to have in order that there exists a unique particular type of BTM  $B$  on  $X$  which extends  $A$ ?

**Definition 2.2.** *Let  $X$  be an arbitrary poset and let  $\mathcal{H}$  be a family of BTM's on  $X$ . A BPM  $A$  on  $X$  is a  $\mathcal{H}$ -core if there exists a unique  $B \in \mathcal{H}$  such that  $A \leq B$ . If  $A$  is a  $\mathcal{H}$ -core, the unique map  $B \in \mathcal{H}$  which extends  $A$  is called the  $\mathcal{H}$ -map spanned by the core  $A$ . We also say that  $A$  spans  $B$  or that  $A$  is a  $\mathcal{H}$ -core of  $B$  and sometime we write  $B = B_A$  to mean that  $B$  is spanned by  $A$ . If the family  $\mathcal{H}$  is clear from the context, we say simply core instead of  $\mathcal{H}$ -core.*

**Definition 2.3.** *We say that  $A$  is a  $\mathcal{H}$ -fundamental core of  $B$  if it is a  $\mathcal{H}$ -core of  $B$  and if, for each  $\mathcal{H}$ -core  $A'$  of  $B$ ,  $A \leq A'$ .*

Obviously, if there exists a  $\mathcal{H}$ -fundamental core of  $B$ , then it is unique, therefore we can speak of the  $\mathcal{H}$ -fundamental core of  $B$ .

We will denote by  $\text{Core}_{\mathcal{H}}(X)$  the family of all the  $\mathcal{H}$ -cores on  $X$ . Let us define a function  $f : \text{Core}_{\mathcal{H}}(X) \rightarrow \mathcal{H}$  by setting  $f(A) = B_A$ . Naturally  $f$  is surjective, since for all  $B \in \mathcal{H}$  we have  $B = B_B = f(B)$ .

We define on  $\text{Core}_{\mathcal{H}}(X)$  the following relation

$$A \sim A' \Leftrightarrow B_A = B_{A'}.$$

Then  $\sim$  is an equivalence relation and by the universal property of the quotient there exists a unique injective map  $f^*$  from  $Core_{\mathcal{H}}(X)/\sim$  into  $\mathcal{H}$  induced by  $f$ . Since  $f$  is surjective, it follows that  $f^*$  is bijective.

In this paper we study the cores for two particular families of BTM's and we determine explicitly the set  $Core_{\mathcal{H}}(X)/\sim$  for these families of 2-valued maps. The two families of BTM's that we will examine are defined on a particular class of posets, which are the involution posets. An *involution poset* (IP) is a poset  $(X, \leq)$  with a unary map  $c : X \rightarrow X$  such that:

- I1)  $c(c(x)) = x$ , for all  $x \in X$ ;
- I2) if  $x, y \in X$  and if  $x \leq y$ , then  $c(y) \leq c(x)$ .

In what follows, if  $x \in X$ , we usually write  $x^c$  instead of  $c(x)$ .

In this paper, we consider involution posets  $(X, \leq, c)$  having the following further property:

- I3)  $x^c \neq x$  for all  $x \in X$ .

**Definition 2.4.** We call *strong involution poset (SIP)* an involution poset  $(X, \leq, c)$  which satisfies I3). Let us note that, by property I3), a SIP must have at least two elements.

**Remark 2.5.** Let us observe that if  $X$  is an involution poset, by I1) it follows that  $c$  is bijective and by I1) and I2) it holds that if  $x, y \in X$  are such that  $x < y$ , then  $y^c < x^c$ . Moreover it holds that  $(\downarrow Z)^c = \uparrow(Z^c)$  and  $(\uparrow Z)^c = \downarrow(Z^c)$ .

If  $(X, \leq, c)$  is an involution poset and if  $Z \subseteq X$ , we will set  $Z^c = \{z^c : z \in Z\}$ . If  $X$  is a SIP, we will say that a 2-valued partial map  $A$  on  $X$  is *complemented-positive* [*complemented-negative*] if  $A^{-1}(N)^c \subseteq A^{-1}(P)$  [ $A^{-1}(P)^c \subseteq A^{-1}(N)$ ].

**Definition 2.6.** If  $X$  is a SIP, a BPM  $A$  on  $X$  is called *positively weighted 2-valued partial map* (briefly *+WBPM*) if it is up-positive, down-negative and complemented-positive; in particular, if  $A$  is also total on  $X$ , it is called *positively weighted 2-valued total map* (briefly *+WBTM*).

Similar definitions are given when complemented-positive is replaced with complemented-negative: +WBPM with -WBPM and +WBTM with -WBPM.

Moreover, we use term WBPM to mean a +WBPM or a -WBPM, and WBTM to mean a +WBTM or a -WBTM.

**Example 2.7.** Let  $X$  be a set. Then the power set  $\mathcal{P}(X)$  with the usual set inclusion relation  $\subseteq$  and the complement operation  $c$  is a SIP.

Let  $X = \mathbb{R}$ . On  $(\mathcal{P}(X), \subseteq, c)$  let us define a BTM  $B$  by setting:

$$B(Y) = \begin{cases} P & \text{if } Y \text{ is unbounded from above,} \\ N & \text{otherwise.} \end{cases}$$

This map  $B$  is up-positive, down-negative and complemented positive, but not complemented negative, so it is a +WBTM but not a -WBTM.

If  $X$  is a SIP, we denote by  $\mathcal{W}_+(X)$  the family of all the +WBTMs on  $X$  and by  $\mathcal{W}_-(X)$  that of all the -WBTMs on  $X$ . Then  $\mathcal{W}_+(X)$  and  $\mathcal{W}_-(X)$  are the two families that we will study in this paper. Obviously, if  $X$  is a SIP, by virtue of Proposition 2.1, it follows that  $\mathcal{W}_+(X)$  [ $\mathcal{W}_-(X)$ ] is the sub-family of all the maps in  $\mathcal{OP}(X, B_2)$  which are also complemented positive [negative].

By Proposition 2.1 it follows that if  $A$  is a BTM on  $X$ , then  $A$  is a +WBTM [-WBTM] if and only if  $A$  is up-positive and complemented positive [negative]. The following proposition shows that each Boolean lattice is also a SIP and that a Boolean lattices morphism is a +WBTM and a -WBTM.

**Proposition 2.8.** Let  $(X, \wedge, \vee, 0, 1, ')$  be a Boolean lattice, then  $X$  is a SIP. Moreover, if  $A : X \rightarrow B_2$  is a Boolean lattice morphism, then  $A$  is both a +WBTM and a -WBTM.

*Proof.* Let  $c : X \rightarrow X$  be such that  $c(x) = x'$  where  $x'$  is the complement of  $x$  in  $X$ , i.e. the unique element of  $X$  such that  $x \wedge x' = 0$  and  $x \vee x' = 1$ . By the well-known properties of the function  $x \mapsto x'$ , it follows that  $c$  satisfies the properties I1) I2) I3).

By definition of morphism of Boolean lattices,  $A$  is such that

$$\begin{aligned} A(a \vee b) &= A(a) \vee A(b), & A(a \wedge b) &= A(a) \wedge A(b), \\ A(0) &= N, & A(1) &= P, & A(a') &= (A(a))'. \end{aligned}$$

It is well-known by the general theory that  $A$  is order-preserving (hence also up-positive and down-negative). Finally, if  $x \in X$  is such that  $A(x) = N$ , then  $A(x^c) = A(x') = A(x)' = N' = P$  because in  $B_2$  the complement of  $N$  is  $P$ . Hence  $A$  is complemented positive. Similarly we prove that  $A$  is also complemented negative.  $\square$

**Proposition 2.9.** *Let  $X$  be a SIP and  $A$  a +WBPM on  $X$ , then:*

- i) if  $w$  is a minimal positive of  $A$  and  $A(w^c) = N$ , then  $w^c$  is a maximal negative of  $A$ ;*
- ii) if  $x, x^c \in \text{dom}(A)$  and  $x^c \leq x$ , then  $A(x) = P$ .*

*Moreover, if  $A$  is a -WBPM on  $X$ , then:*

- i') if  $w$  is a maximal negative of  $A$  and  $A(w^c) = P$ , then  $w^c$  is a minimal positive of  $A$ ;*
- ii') if  $x, x^c \in \text{dom}(A)$  and  $x^c \leq x$ , then  $A(x^c) = N$ .*

*Proof.* *i)* Let  $y \in X$  such that  $A(y) = N$  and  $w^c \leq y$ . Then  $y^c \leq w$ . Moreover, since  $A$  is complemented positive, we have that  $A(y^c) = P$ . By minimality of  $w$  in  $A^{-1}(P)$ ,  $y^c = w$ , so  $w^c = y$  and  $w^c$  is a maximal negative of  $A$ .

*ii)* Suppose by contradiction that  $A(x) = N$ . Since  $A$  is complemented positive, we have that  $x^c \in \text{dom}(A)$  and  $A(x^c) = P$ . Since  $x^c \leq x$  and  $A$  is up-positive, we have that  $A(x) = P$  and this is a contradiction.

The proof of *i')* and *ii')* is similar. □

By Proposition 2.9-*ii)* and Proposition 2.9-*ii')* the elements  $w \in X$  such that  $w^c \leq w$  are called *complemented*.

### 3. THE $\mathcal{H}$ -CORES WHEN $\mathcal{H} = \mathcal{W}_+(X)$ OR $\mathcal{H} = \mathcal{W}_-(X)$

In this section we assume that  $X$  is a finite strong involution poset and we determine the  $\mathcal{W}_+(X)$ -fundamental core of an arbitrary +WBTM in  $\mathcal{W}_+(X)$  and the  $\mathcal{W}_-(X)$ -fundamental core of an arbitrary -WBTM in  $\mathcal{W}_-(X)$ .

**Remark 3.1.** *Although almost all of the following results still hold when  $X$  is infinite, we will consider only finite SIPs. The reason of that is that, in the finite case, each subset  $S$  of  $X$  has some minimal and maximal elements, so if  $A$  is a BPM on a  $X$ , then  $\text{Min}(A^{-1}(P))$  and  $\text{Max}(A^{-1}(N))$  are not empty. Propositions 3.14, 3.15, 3.16 and 3.17 analyze, through an example (see Example 2.7), the infinite case. In particular in Proposition 3.17 we will show that in the infinite case we can have situations that, as far as our analysis is concerned, appear to have a marginal importance.*

**Proposition 3.2.** *i) Let  $B \in \mathcal{W}_+(X)$  and let  $A$  be a  $\mathcal{W}_+(X)$ -core of  $B$ . Then*

$$B^{-1}(N) \subseteq \downarrow A^{-1}(N).$$

*ii) Let  $B \in \mathcal{W}_-(X)$  and let  $A$  be a  $\mathcal{W}_-(X)$ -core of  $B$ . Then*

$$B^{-1}(P) \subseteq \uparrow A^{-1}(P).$$

*Proof.* *i)* Let us suppose, by contradiction, that  $W := B^{-1}(N) \setminus \downarrow A^{-1}(N)$  is not empty and let  $\bar{w}$  be a maximal element in  $W$ . Define a 2-valued total map  $B'$  on  $X$  by setting

$$B'(x) := \begin{cases} B(x) & \text{if } x \neq \bar{w}, \\ P & \text{if } x = \bar{w}. \end{cases}$$

We first show that  $B'$  is up-positive.

Let us prove first that  $B'$  is up-positive. For this, let  $z \in B'^{-1}(P)$  and let  $x \in X$  such that  $z < x$ . If both  $x$  and  $z$  are different from  $\bar{w}$ , then  $B'(z) = B(z)$ ,  $B'(x) = B(x)$  and the thesis follows directly by the assumption made on  $B$ . If  $z < x = \bar{w}$ , then  $B'(x) = B'(\bar{w}) = P$ . Finally let  $z = \bar{w} < x$ . By maximality of  $\bar{w}$  in  $W$ ,  $x \notin W$ , so  $x \notin B^{-1}(N)$  or  $x \in B^{-1}(N) \cap \downarrow A^{-1}(N)$ . In the first case, it holds trivially  $B'(x) = B(x) = P$ . In the second case, there exists  $y \in A^{-1}(N)$  such that  $z < x < y$ . Thus  $z = \bar{w} \in \downarrow A^{-1}(N)$ , by contradicting the fact that  $\bar{w} \in W$ .

We prove that  $B'$  is complemented positive, i.e. that  $(B'^{-1}(N))^c \subseteq B'^{-1}(P)$ . For this, let  $x \in X$  such that  $B'(x) = N$ . Let us prove that  $B'(x^c) = P$ .

Since  $B'(\bar{w}) = P$ ,  $x \neq \bar{w}$ . If  $x^c = \bar{w}$ , then clearly  $B'(x^c) = B'(\bar{w}) = P$ . If  $x^c \neq \bar{w}$ , then  $B'(x^c) = B(x^c) = P$  because  $x \in B^{-1}(N)$  and  $B$  is complemented positive.

*ii)* Similar arguments apply. □

**Corollary 3.3.** *i) Let  $B \in \mathcal{W}_+(X)$  and let  $A$  be a  $\mathcal{W}_+(X)$ -core of  $B$ . Then*

$$B^{-1}(N) = \downarrow A^{-1}(N).$$

*ii) Let  $B \in \mathcal{W}_-(X)$  and let  $A$  be a  $\mathcal{W}_-(X)$ -core of  $B$ . Then*

$$B^{-1}(P) = \uparrow A^{-1}(P).$$

*Proof.* *i)* In view of Proposition 3.1-*i)*, it suffices to show that  $\downarrow A^{-1}(N) \subseteq B^{-1}(N)$ . Let  $x \in \downarrow A^{-1}(N)$ . Then there exist  $y \in X$  such that  $A(y) = N$  and  $x \leq y$ . Since  $B$  extends  $A$  and it is down-negative,  $B(x) = N$ , i.e.  $x \in B^{-1}(N)$ .

*ii)* This follows by the same reasoning of part *i)* and using then Proposition 3.2-*ii)*.  $\square$

**Corollary 3.4.** *i)* Let  $B \in \mathcal{W}_+(X)$  and let  $A$  be a  $\mathcal{W}_+(X)$ -core of  $B$ . Then

$$\text{Max}(B^{-1}(N)) \subseteq A^{-1}(N)$$

*ii)* Let  $B \in \mathcal{W}_-(X)$  and let  $A$  be a  $\mathcal{W}_-(X)$ -core of  $B$ . Then

$$\text{Min}(B^{-1}(P)) \subseteq A^{-1}(N)$$

*Proof.* *i)* Let  $x \in \text{Max}(B^{-1}(N))$ , by Proposition 3.2-*i)* it follows that  $x \in \downarrow A^{-1}(N)$ . This implies that there exists an element  $y \in A^{-1}(N) \subseteq B^{-1}(N)$  such that  $x \leq y$ . By the maximality of  $x$  in  $B^{-1}(N)$ , it holds that  $x = y \in A^{-1}(N)$ .

*ii)* Analogously using Proposition 3.2-*ii)*.  $\square$

**Corollary 3.5.** *i)* Let  $B \in \mathcal{W}_+(X)$  and let  $A$  be a  $\mathcal{W}_+(X)$ -core of  $B$  such that  $A^{-1}(N)$  is an anti-chain on  $X$ . Then

$$\text{Max}(B^{-1}(N)) = A^{-1}(N).$$

*ii)* Let  $B \in \mathcal{W}_-(X)$  and let  $A$  be a  $\mathcal{W}_-(X)$ -core of  $B$  such that  $A^{-1}(P)$  is an anti-chain on  $X$ . Then

$$\text{Min}(B^{-1}(P)) = A^{-1}(P).$$

*Proof.* *i)* By Corollary 3.4-*i)*, we need to prove the inclusion  $\text{Max}(B^{-1}(N)) \supseteq A^{-1}(N)$ . Suppose that  $x \in A^{-1}(N)$  and that (by contradiction)  $x \notin \text{Max}(B^{-1}(N))$ . Then there exists an element  $y \in B^{-1}(N)$  such that  $x < y$ . By Proposition 3.2-*i)*, we have that  $y \in \downarrow A^{-1}(N)$ . Then there exists an element  $z \in A^{-1}(N)$  such that  $z \geq y$ , and hence we have that  $z \geq y > x$ , with  $z, x \in A^{-1}(N)$ , and this contradicts the hypothesis that  $A^{-1}(N)$  is an anti-chain on  $X$ .

*ii)* Similar proof by using Corollary 3.4-*ii)* and Proposition 3.2-*ii)*.  $\square$

**Proposition 3.6.** *i)* Let  $B \in \mathcal{W}_+(X)$  and  $A$  a  $\mathcal{W}_+(X)$ -core of  $B$ . Then

$$B^{-1}(P) \subseteq \uparrow(A^{-1}(P) \cup (A^{-1}(N))^c).$$

*ii)* Let  $B \in \mathcal{W}_-(X)$  and  $A$  a  $\mathcal{W}_-(X)$ -core of  $B$ . Then

$$B^{-1}(N) \subseteq \downarrow(A^{-1}(N) \cup (A^{-1}(P))^c).$$

*Proof.* *i)* Let us assume, by contradiction, that  $B^{-1}(P) \setminus \uparrow(A^{-1}(P) \cup (A^{-1}(N))^c)$  is not empty and let  $\bar{w}$  be a minimal element in this set. Let us define a 2-valued total map  $B'$  on  $X$  by setting

$$B'(x) = \begin{cases} B(x) & \text{if } x \neq \bar{w}, \\ N & \text{if } x = \bar{w}. \end{cases}$$

We prove now that  $B'$  is a +WBTM on  $X$ .

Let us prove first that  $B'$  is up-positive. For this, let  $z \in X$  such that  $B'(z) = P$  and let  $x \in X$  with  $z < x$ . Note that  $z \neq \bar{w}$  because  $B'(z) = P \neq N = B'(\bar{w})$ . If both  $x$  and  $z$  are different from  $\bar{w}$ , then  $B'(z) = B(z)$ ,  $B'(x) = B(x)$  and the thesis follows directly by the assumption made on  $B$ . If  $z < x = \bar{w}$ , then, by minimality of  $\bar{w}$  in  $W$ ,  $z \notin B^{-1}(P) \setminus \uparrow(A^{-1}(P) \cup (A^{-1}(N))^c)$  and since  $B(z) = P$ ,  $z \in \uparrow(A^{-1}(P) \cup (A^{-1}(N))^c)$ . But in this case we would have that  $x = \bar{w} \in \uparrow(A^{-1}(P) \cup (A^{-1}(N))^c)$ , because this set is an up set, by contradicting our assumptions on  $\bar{w}$ .

We show now that  $B'$  is complemented positive, i.e. that  $(B'^{-1}(N))^c \subseteq B'^{-1}(P)$ . For this, let  $x \in X$  such that  $B'(x) = N$ . We have to prove that  $B'(x^c) = P$ . We claim that  $B'(\bar{w}^c) = P$ . Indeed, by contradiction, suppose that  $B'(\bar{w}^c) = N$ ; since  $\bar{w}^c \neq \bar{w}$  we will have that  $A(\bar{w}^c) = A'(\bar{w}^c) = N$ , and hence  $\bar{w}^c \in B^{-1}(N)$ . By Corollary 3.3-*i)*, it follows that  $\bar{w}^c \in \downarrow A^{-1}(N)$ . Hence there exists an element  $y \in A^{-1}(N)$  such that  $\bar{w}^c \leq y$ , and hence  $y^c \leq (\bar{w}^c)^c = \bar{w}$ , with  $y^c \in A^{-1}(N)^c$ . Therefore  $z \in \uparrow(A^{-1}(N))^c$ , by contradicting the hypothesis made on  $\bar{w}$ . This proves our claim.

If  $x \neq \bar{w}$  and  $x \neq \bar{w}^c$ , then  $N = B'(x) = B(x)$ . Since  $B$  is complemented positive, we have that  $B(x^c) = P$ , and hence, since  $x^c \neq \bar{w}$ , by definition of  $B'$  we have that  $B'(x^c) = B(x^c) = P$ . If  $x = \bar{w}$ , then  $B'(x^c) = B'(\bar{w}^c) = P$ . Finally, if  $x = \bar{w}^c$ , then the hypothesis  $B'(x) = N$  is empty because we have

proved that  $B'(\bar{w}^c) = P$ . Hence  $A'$  is complemented positive.

In the same way we can prove *ii*), using Corollary 3.3-*ii*).  $\square$

**Corollary 3.7.** *i*) Let  $B \in \mathcal{W}_+(X)$  and  $A$  be a  $\mathcal{W}_+(X)$ -core of  $B$ . Then

$$B^{-1}(P) = \uparrow(A^{-1}(P)) \cup (A^{-1}(N))^c.$$

*ii*) Let  $B \in \mathcal{W}_-(X)$  and  $A$  be a  $\mathcal{W}_-(X)$ -core of  $B$ . Then

$$B^{-1}(N) = \downarrow(A^{-1}(N)) \cup (A^{-1}(N))^c.$$

*Proof.* *i*) Let  $z \in \uparrow(A^{-1}(P)) \cup (A^{-1}(N))^c$ . Then there exists an element  $z' \in A^{-1}(P) \cup A^{-1}(N)^c$  such that  $z' \leq z$ . Since  $B(z') = P$  and  $B$  is up-positive, we have that  $B(z) = P$ , i.e.  $z \in B^{-1}(P)$ . This implies that  $\uparrow(A^{-1}(P)) \cup (A^{-1}(N))^c \subseteq B^{-1}(P)$ . The assertion follows by Proposition 3.6-*i*).  $\square$

*ii*) Similarly, by using Proposition 3.6-*ii*).  $\square$

**Theorem 3.8.** *i*) Let  $B \in \mathcal{W}_+(X)$  and  $A$  a  $\mathcal{W}_+(X)$ -core of  $B$ , then

$$X = (\uparrow(A^{-1}(P)) \cup (A^{-1}(N))^c) \sqcup \downarrow A^{-1}(N).$$

*ii*) Let  $B \in \mathcal{W}_-(X)$  and  $A$  a  $\mathcal{W}_-(X)$ -core of  $B$ , then

$$X = (\downarrow(A^{-1}(N)) \cup (A^{-1}(N))^c) \sqcup \uparrow A^{-1}(P).$$

*Proof.* Since  $X = B^{-1}(P) \sqcup B^{-1}(N)$ , the assertion is a direct consequence of Corollaries 3.3 and 3.7.  $\square$

**Proposition 3.9.** *i*) Let  $B \in \mathcal{W}_+(X)$  and  $A$  a  $\mathcal{W}_+(X)$ -core of  $B$ . Then

$$\text{Min}(B^{-1}(P)) \subseteq A^{-1}(P) \cup (A^{-1}(N))^c.$$

*ii*) Let  $B \in \mathcal{W}_-(X)$  and  $A$  a  $\mathcal{W}_-(X)$ -core of  $B$ . Then

$$\text{Max}(B^{-1}(N)) \subseteq A^{-1}(N) \cup (A^{-1}(P))^c.$$

*Proof.* *i*) Suppose that  $x \in \text{Min}(B^{-1}(P))$ . By Corollary 3.7-*i*), since  $\uparrow(A^{-1}(P)) \cup (A^{-1}(N))^c = \uparrow A^{-1}(P) \cup \uparrow(A^{-1}(N))^c$ , it follows that  $x \in \uparrow A^{-1}(P)$  or  $x \in \uparrow(A^{-1}(N))^c$ . If  $x \in \uparrow A^{-1}(P)$ , then there exists  $y \in A^{-1}(P)$  such that  $y \leq x$ . By the minimality of  $x$  in  $B^{-1}(P)$ , we will have that  $x = y \in A^{-1}(P)$ . If  $x \in \uparrow(A^{-1}(N))^c$ , then there exists  $z \in (A^{-1}(N))^c$  such that  $z \leq x$ . By the minimality of  $x$  in  $B^{-1}(P)$ , it follows also that  $x = z \in (A^{-1}(N))^c$ . Hence  $\text{Min}(B^{-1}(P)) \subseteq A^{-1}(P) \cup (A^{-1}(N))^c$ .  $\square$

*ii*) Likewise, by using Corollary 3.7-*ii*).  $\square$

**Proposition 3.10.** *i*) Let  $B \in \mathcal{W}_+(X)$  and  $A$  a  $\mathcal{W}_+(X)$ -core of  $B$ , then

$$\text{Min}(B^{-1}(P)) = \text{Min}(A^{-1}(P) \cup (A^{-1}(N))^c).$$

*ii*) Let  $B \in \mathcal{W}_-(X)$  and  $A$  a  $\mathcal{W}_-(X)$ -core of  $B$ , then

$$\text{Max}(B^{-1}(N)) = \text{Max}(A^{-1}(N) \cup (A^{-1}(P))^c).$$

*Proof.* *i*) We start proving the inclusion  $\text{Min}(B^{-1}(P)) \subseteq \text{Min}(A^{-1}(P) \cup (A^{-1}(N))^c)$ . Suppose that  $z \in \text{Min}(B^{-1}(P))$ . By Proposition 3.9-*i*), it follows that  $z \in A^{-1}(P) \cup (A^{-1}(N))^c$ . Suppose by contradiction that  $z \notin \text{Min}(A^{-1}(P) \cup (A^{-1}(N))^c)$ , then there exists  $\bar{z} \in A^{-1}(P) \cup (A^{-1}(N))^c$  such that  $\bar{z} < z$ . Since  $B(\bar{z}) = P$ , this contradicts the minimality of  $z$  in  $B^{-1}(P)$ . Hence  $\text{Min}(B^{-1}(P)) \subseteq \text{Min}(A^{-1}(P) \cup (A^{-1}(N))^c)$ .

Now we prove the other inclusion  $\text{Min}(B^{-1}(P)) \supseteq \text{Min}(A^{-1}(P) \cup (A^{-1}(N))^c)$ . Suppose that  $w \in \text{Min}(A^{-1}(P) \cup (A^{-1}(N))^c)$ . Obviously  $w \in B^{-1}(P)$ . If, by contradiction,  $w \notin \text{Min}(B^{-1}(P))$ , then there exists an element  $\bar{w} \in B^{-1}(P)$  such that  $\bar{w} < w$ . By Corollary 3.7-*i*) it follows that either  $\bar{w} \in \uparrow A^{-1}(P)$  or  $\bar{w} \in \uparrow(A^{-1}(N))^c$ .

1) if  $\bar{w} \in \uparrow A^{-1}(P)$ , there exists an element  $\tilde{w} \in A^{-1}(P)$  such that  $\tilde{w} \leq \bar{w}$ , therefore  $\tilde{w} \leq \bar{w} < w$ , and this contradicts that  $w \in \text{Min}(A^{-1}(P) \cup (A^{-1}(N))^c)$ .

2) if  $\bar{w} \in \uparrow(A^{-1}(N))^c$ , then there exists an element  $\tilde{w} \in (A^{-1}(N))^c$  such that  $\tilde{w} \leq \bar{w}$ , hence  $\tilde{w} \leq \bar{w} < w$ , and this contradicts that  $w \in \text{Min}(A^{-1}(P) \cup (A^{-1}(N))^c)$ .

*ii*) Analogously by using Proposition 3.9-*ii*) and Corollary 3.7-*ii*).  $\square$

**Proposition 3.11.** *Let  $B$  be a +WBTM or a -WBTM on  $X$ . Then, setting*

$$N(B) := \text{Min}(B^{-1}(P)) \cup \text{Max}(B^{-1}(N)),$$

*it follows that  $B|_{N(B)}$  is a  $\mathcal{W}_+(X)$ -core of  $B$  on  $X$ .*

*Proof.* Let  $\text{Min}(B^{-1}(P)) = \{w_1, \dots, w_k\}$  and  $\text{Max}(B^{-1}(N)) = \{v_1, \dots, v_q\}$ . Then

$$N(B) = \{w_1, \dots, w_k, v_1, \dots, v_q\}$$

We start observing that  $B^{-1}(P)$  and  $B^{-1}(N)$  are two anti-chains in  $X$ , because they are, respectively, the minimal elements of  $B^{-1}(P)$  and the maximal elements of  $B^{-1}(N)$ . Let  $B'$  be another +WBTM on  $X$  such that  $B'|_{N(B)} = B|_{N(B)}$  i.e. such that  $B'(w_1) = \dots = B'(w_k) = P$  and  $B'(v_1) = \dots = B'(v_q) = N$ . Suppose, by contradiction, that  $B' \neq B$ . Then there exists an element  $w \in X$  such that  $B(w) \neq B'(w)$ . Then there are two possibilities:

- i)*  $B(w) = P$  and  $B'(w) = N$ . In this case,  $w \in B^{-1}(P)$  and, since  $B^{-1}(P) = \cup_{i=1}^k (\uparrow w_i)$ , there exists  $i \in \{1, \dots, k\}$  such that  $w \in (\uparrow w_i)$ , i.e. such that  $w_i \leq w$ . Since  $B'(w_i) = P$ , we will have that  $w_i \in (B')^{-1}(P)$  and since  $B'$  is up-positive and  $w_i \leq w$ , it follows that  $B'(w) = P$ , and this is a contradiction.
- ii)*  $B(w) = N$  and  $B'(w) = P$ . In this case,  $w \in B^{-1}(N)$ , and hence, by  $B^{-1}(N) = \cup_{j=1}^q (\downarrow v_j)$ , there exists  $j \in \{1, \dots, q\}$  such that  $w \in (\downarrow v_j)$ , i.e. such that  $w \leq v_j$ . Since  $v_j \in (B')^{-1}(N)$  and since  $B'$  is down-negative, it follows that  $B'(w) = N$  which is a contradiction.

Therefore  $B = B'$ , and hence  $B|_{N(B)}$  is a  $\mathcal{W}_+(X)$ -core of  $B$ .  $\square$

In the next theorem we show that each +WBTM and each -WBTM have a unique fundamental core and we also describe such a core.

**Theorem 3.12.** *i) Let  $B \in \mathcal{W}_+(X)$ . Then, setting*

$$C_+(B) := N(B) \setminus (\text{Max}(B^{-1}(N)))^c,$$

*it holds that  $B|_{C_+(B)}$  is the  $\mathcal{W}_+(X)$ -fundamental core of  $B$ .*

*ii) Let  $B \in \mathcal{W}_-(X)$ . Then, setting*

$$C_-(B) := N(B) \setminus (\text{Min}(B^{-1}(P)))^c,$$

*it holds that  $B|_{C_-(B)}$  is the  $\mathcal{W}_-(X)$ -fundamental core of  $B$ .*

*Proof.* *i)* Let  $\text{Min}(B^{-1}(P)) = \{w_1, \dots, w_k\}$  and  $\text{Max}(B^{-1}(N)) = \{v_1, \dots, v_q\}$ . Then  $\{v_1^c, \dots, v_q^c\} \subseteq B^{-1}(P)$ . If  $\{v_1^c, \dots, v_q^c\} \cap \{w_1, \dots, w_k\} = \emptyset$  then  $C_+(B)$  coincides with  $N(B)$  and hence the assertion holds by Proposition 3.11.

If  $\{v_1^c, \dots, v_q^c\} \cap \{w_1, \dots, w_k\} \neq \emptyset$  we assume, without loss of generality, that

$\{v_1^c, \dots, v_q^c\} \cap \{w_1, \dots, w_k\} = \{w_1, \dots, w_p\}$ , for some  $p$  such that  $1 \leq p \leq \min(k, q)$ .

Re-ordering the indexes, we can assume that  $w_1 = v_1^c, \dots, w_p = v_p^c$ . Then we have that:

$$C_+(B) = \{w_{p+1}, \dots, w_k, v_1, \dots, v_p, v_{p+1}, \dots, v_q\}.$$

Observe that  $C_+(B) \cap B^{-1}(P) = \{w_{p+1}, \dots, w_k\}$  is an anti-chain in  $X$ , because it is a subset of the anti-chain  $\text{Min}(B^{-1}(P))$ ;  $C_+(B) \cap B^{-1}(N) = \{v_1, \dots, v_q\}$  is an anti-chain in  $X$  because it coincides with the anti-chain  $\text{Max}(B^{-1}(N))$ . Let  $B'$  be another +WBTM on  $X$  and suppose that  $B'(w_{p+1}) = \dots = B'(w_k) = P$ , and  $B'(v_1) = \dots = B'(v_q) = N$ . We need to prove that  $B' = B$  on all  $X$ . Suppose by contradiction that  $B \neq B'$  on  $X$ , then there exists an element  $w \in X$  such that  $B(w) \neq B'(w)$ . First suppose that  $B'(w) = P$  and  $B(w) = N$ . In this case,  $w \in B^{-1}(N)$  and hence by  $B^{-1}(N) = \cup_{j=1}^q (\downarrow v_j)$  there exists  $j \in \{1, \dots, q\}$  such that  $w \in \downarrow v_j$  i.e. such that  $w \leq v_j$ . Since  $v_j \in (B')^{-1}(N)$  and since  $B'$  is down-negative, by  $w \leq v_j$  it follows that  $w \in (B')^{-1}(N)$  i.e.  $B'(w) = N$ , which is a contradiction.

Finally suppose that  $B'(w) = N$  and  $B(w) = P$ . In this case, since  $w \in B^{-1}(P)$  and since  $B^{-1}(P) = \cup_{i=1}^k (\uparrow w_i)$  there exists  $i \in \{1, \dots, k\}$  such that  $w \in \uparrow w_i$ , i.e. such that  $w_i \leq w$ . We distinguish two cases:

- j<sub>1</sub>)* if  $i \in \{p+1, \dots, k\}$ , then  $w_i \in (B')^{-1}(P)$  and since  $B'$  is up-positive, by  $w_i \leq w$  it follows that  $w \in (B')^{-1}(P)$  i.e.  $B'(w) = P$ , and this is a contradiction.
- j<sub>2</sub>)* if  $i \in \{1, \dots, p\}$ , then we will have  $w_i = v_i^c$ . Since  $w_i \leq w$ , we will have that  $w^c \leq w_i^c = (v_i^c)^c = v_i$ ; since  $v_i \in (B')^{-1}(N)$  and  $B'$  is down-negative, it follows that  $w^c \in (B')^{-1}(N)$ . Therefore since  $B'$  is complemented positive,  $w = (w^c)^c \in (B')^{-1}(P)$ , i.e.  $B'(w) = P$ , which is a contradiction.



This shows that  $B|_{C_+(B)}$  is a  $\mathcal{W}_+(X)$ -core of  $B$  on  $X$ .

Let now  $A$  be a  $\mathcal{W}_+(X)$ -core of  $B$  on  $X$ . Notice first that

$$C_+(B) = (\text{Min}(B^{-1}(P)) \setminus (\text{Max}(B^{-1}(N)))^c) \sqcup \text{Max}(B^{-1}(N)).$$

Moreover, by Proposition 3.9-i) we have that

$$(1) \quad \text{Min}(B^{-1}(P)) \subseteq A^{-1}(P) \cup (A^{-1}(N))^c,$$

and, by Corollary 3.4-i) we have that

$$(2) \quad \text{Max}(B^{-1}(N)) \subseteq A^{-1}(N).$$

Therefore, in order to show that  $C_+(B) \subseteq \text{dom}(A)$ , by (2), it is sufficient to prove that

$$H = \text{Min}(B^{-1}(P)) \setminus (\text{Max}(B^{-1}(N)))^c \subseteq A^{-1}(P).$$

Since  $H \subseteq \text{Min}(B^{-1}(P))$ , by (1) it is sufficient to prove that  $H \cap (A^{-1}(N))^c = \emptyset$ . Suppose on the contrary that there exists  $w \in H$  such that  $w \in (A^{-1}(N))^c$ . In this case there exists  $\tilde{w} \in A^{-1}(N)$  such that  $\tilde{w}^c = w$ . It follows that  $w$  is a minimal positive of  $A$  ( $w \in H$ ) such that  $w^c = \tilde{w}$  is negative for  $A$ , hence by Proposition 2.9-i) it follows that  $\tilde{w} \in \text{Max}(B^{-1}(N))$  and therefore  $w = \tilde{w}^c \in \text{Max}(B^{-1}(N))^c$ , which is a contradiction since  $w \in H$ .

ii) Similarly by using Proposition 3.11, Proposition 3.9-ii), Corollary 3.4-ii) and Proposition 2.9-ii).  $\square$

**Remark 3.13.** *Let us note that all the previous results can be given in an equivalent form, a ‘negative’ one, by using the properties of the involution map  $c$  with respect to the order relation and thus with respect to the operators  $\uparrow, \downarrow, \text{Min}$  and  $\text{Max}$ .*

In the remaining part of the present section we consider the specific SIP  $X = (\mathcal{P}(\mathbb{R}), \subseteq, c)$  and the +WBTM  $B$  defined in Example 2.7. In such cases we characterize all the  $\mathcal{W}_+(X)$ -cores of  $B$ .

**Proposition 3.14.** *Let  $A$  be a BPM such that  $B$  extends  $A$ . Then, if  $Z$  is a bounded from above real subset which is not in  $\text{dom}(A)$  such that, for any bounded from above real subset  $Y$  such that  $Z \subseteq Y$  it holds that  $Y \notin \text{dom}(A)$ , then  $A$  is not a  $\mathcal{W}_+(X)$ -core of  $B$ .*

*Proof.* Let us define a +WBTM  $B'$  different from  $B$  such that  $B'|_{\text{dom}(A)} = A$ , by setting:

$$B'(Y) = \begin{cases} P & \text{if } Y \text{ is bounded from above and } Z \subseteq Y, \\ B(Y) & \text{otherwise.} \end{cases}$$

Prove that  $B'$  is up-positive. For this let  $Y \subseteq Y'$  such that  $B'(Y) = P$ . Then  $Y$  is bounded from above and  $Z \subseteq Y$  or  $Y$  is unbounded from above. Thus also  $Y'$  is bounded from above and  $Z \subseteq Y'$  or  $Y'$  is unbounded from above, so  $B'(Y') = P$  and  $B'$  is up-positive.

Let us prove now that  $B'$  is complemented positive, i.e. that if  $Y \subseteq \mathbb{R}$  is such that  $B'(Y) = N$ , then  $Y$  is bounded from above and  $Z \not\subseteq Y$ . So  $Y^c$  is unbounded from above and thus  $B'(Y^c) = P$ . This proves our assert.  $\square$

**Proposition 3.15.** *Let  $A$  be a BPM such that  $B$  extends  $A$ . Then, if  $Z$  is an unbounded from above real subset which is not in  $\text{dom}(A)$  such that, for any unbounded from above real subset  $Y$  such that  $Y \subseteq Z$  it holds that  $Y \notin \text{dom}(A)$ , then  $A$  is not a  $\mathcal{W}_+(X)$ -core of  $B$ .*

*Proof.* Let  $B'$  be the total map on  $X$  defined by setting:

$$B'(Y) = \begin{cases} N & \text{if } Y \text{ is unbounded from above and } Y \subseteq Z, \\ B(Y) & \text{otherwise.} \end{cases}$$

The proof that this map is a +WBTM is similar to that described in the previous proposition. Moreover it is different from  $B$ , so the proposition holds.  $\square$

By the previous propositions it follows the following result.

**Proposition 3.16.** *Let  $A$  be a BPM on  $X$ . Then  $A$  is a  $\mathcal{W}_+(X)$ -core of  $B$  if and only if the following conditions hold:*

i)  $B$  extends  $A$ .

ii) if  $Z$  is a bounded from above real subset which is not in  $\text{dom}(A)$ , then there exists a bounded from above real subset  $Y$  in  $\text{dom}(A)$  such that  $Z \subseteq Y$ .

iii) if  $Z$  is an unbounded from above real subset which is not in  $\text{dom}(A)$ , then there exists an unbounded from above real subset  $Y$  in  $\text{dom}(A)$  such that  $Y \subseteq Z$ .

*Proof.* It follows directly by Proposition 3.14, Proposition 3.15 and because  $B$  is order preserving.  $\square$

The following proposition is easy to prove.

**Proposition 3.17.** *Let  $X$  and  $B$  as in the previous propositions. Then the following conditions hold:*

i) *there is not a  $\mathcal{W}_+(X)$ -fundamental core of  $B$ ;*

ii)  *$\text{Max}(B^{-1}(N)) = \text{Min}(B^{-1}(P)) = \emptyset$ ;*

iii) *there is not a  $\mathcal{W}_+(X)$ -core  $A$  of  $B$  such that  $A^{-1}(N)$  is an anti-chain on  $X$ .*

#### 4. ESSENTIAL PROPERTIES OF A $\mathcal{W}_\pm(X, B_2)$ -FUNDAMENTAL CORE

In this section we determine the properties characterizing the fundamental core of a WBTM. This will lead us to define the concepts of  $w_+$ -basis and  $w_-$ -basis for  $X$ . At the end of the section we will show that each  $w_+$ -basis identifies uniquely the fundamental core of a +WBTM on  $X$  and each  $w_-$ -basis identifies uniquely the fundamental core of a -WBTM on  $X$ .

In all this section,  $X$  will denote a finite SIP.

**Definition 4.1.** i) *A  $w_+$ -basis for  $X$  is an ordered pair  $\langle W_+ | W_- \rangle$ , where  $W_+$  and  $W_-$  are two disjoint anti-chains on  $X$  such that:*

$$\text{B1+)} \quad (\downarrow W_+) \cap (W_-^c) = \emptyset;$$

$$\text{B2+)} \quad \uparrow((W_+ \cup W_-)^c) \cap \downarrow W_- = \emptyset;$$

$$\text{B3+)} \quad X = \uparrow((W_+ \cup W_-)^c) \cup \downarrow W_-.$$

ii) *A  $w_-$ -basis for  $X$  is an ordered pair  $\langle W_+ | W_- \rangle$ , where  $W_+$  and  $W_-$  are two disjoint anti-chains of  $X$  such that:*

$$\text{B1-)} \quad (\uparrow W_-) \cap (W_+^c) = \emptyset;$$

$$\text{B2-)} \quad \downarrow((W_- \cup W_+)^c) \cap \uparrow W_+ = \emptyset;$$

$$\text{B3-)} \quad X = \downarrow((W_- \cup W_+)^c) \cup \uparrow W_+.$$

Two  $w_+$ -bases [ $w_-$ -bases]  $\langle W_+ | W_- \rangle$  and  $\langle W'_+ | W'_- \rangle$  are considered equal if  $W_+ = W'_+$  and  $W_- = W'_-$ .

**Proposition 4.2.** i) *If  $B \in \mathcal{W}_+(X)$  and if  $A = B|_{C_+(B)}$ , then  $\langle A^{-1}(P) | A^{-1}(N) \rangle$  is a  $w_+$ -basis for  $X$ .*

ii) *If  $B \in \mathcal{W}_-(X)$  and if  $A = B|_{C_-(B)}$ , then  $\langle A^{-1}(P) | A^{-1}(N) \rangle$  is a  $w_-$ -basis for  $X$ .*

*Proof.* i) By the definition of  $C_+(B)$ , we have that  $A^{-1}(P) = \text{Min}(B^{-1}(P)) \setminus (\text{Max}(B^{-1}(N)))^c$  and  $A^{-1}(N) = \text{Max}(B^{-1}(N))$ . By Theorem 3.12-i) we know that  $A$  is a  $\mathcal{W}_+(X)$ -core of  $B$ , therefore, by Theorem 3.8-i), we have

$$X = (\uparrow(A^{-1}(P)) \cup (A^{-1}(N))^c) \sqcup \downarrow A^{-1}(N)$$

Moreover, since the elements of  $A^{-1}(P)$  are a part of the minimal positives of  $B$  and the elements of  $A^{-1}(N)$  are all the maximal negatives of  $B$ , it follows that  $A^{-1}(P)$  and  $A^{-1}(N)$  are two disjoint anti-chains of  $X$ . It will remain to prove that  $(\downarrow A^{-1}(P)) \cap (A^{-1}(N))^c = \emptyset$ . Since

$$A^{-1}(P) = \text{Min}(B^{-1}(P)) \setminus (A^{-1}(N))^c,$$

then  $A^{-1}(P) \cap (A^{-1}(N))^c = \emptyset$ . Let us suppose now by contradiction that there exists an element  $z \in \downarrow A^{-1}(P) \cap (A^{-1}(N))^c$ . This implies the existence of an element  $x \in A^{-1}(P)$  such that  $z \leq x$ . Since  $A^{-1}(P) \cap (A^{-1}(N))^c = \emptyset$ , we have that  $z < x$  (if  $z = x$ , then  $x \in A^{-1}(P) \cap (A^{-1}(N))^c = \emptyset$ ). Since  $x$  is a minimal positive of  $A$  and  $A(z) = P$  (because  $z \in (A^{-1}(N))^c$ ), this is a contradiction.

ii) Similarly, by using Theorem 3.12-ii) and Theorem 3.8-ii).  $\square$

The following proposition is essential in [13]:

**Proposition 4.3.** i) *Let  $\langle W_+ | W_- \rangle$  be a  $w_+$ -basis for  $X$ . If we set  $W := W_+ \sqcup W_-$  and*

$$B(x) := \begin{cases} P & \text{if } x \in \uparrow(W_+ \cup (W_-)^c) \\ N & \text{if } x \in \downarrow W_- \end{cases}$$

*then  $B$  is a +WBTM on  $X$  and  $W = C_+(B)$ .*

ii) Let  $\langle W_+ | W_- \rangle$  be a  $w_-$ -basis for  $X$ . If we set  $W := W_+ \sqcup W_-$  and

$$B(x) := \begin{cases} P & \text{if } x \in \uparrow W_+ \\ N & \text{if } x \in \downarrow(W_- \cup (W_+)^c), \end{cases}$$

then  $B$  is a  $-$ WBTM on  $X$  and  $W = C_-(B)$ .

*Proof.* i) Let us observe that  $B$  is well defined because  $\uparrow(W_+ \cup (W_-)^c) \cap \downarrow W_- = \emptyset$ ; moreover, by setting  $A := B|_W$ , we have  $W_+ = A^{-1}(P)$  and  $W_- = A^{-1}(N)$ .

Let us prove now that  $B$  is up-positive and complemented positive. Trivially  $B^{-1}(P) = \uparrow(W_+ \cup (W_-)^c)$  is an up-set. Furthermore, since  $(B^{-1}(N))^c = (\downarrow W_-)^c = \uparrow(W_-^c) \subseteq B^{-1}(P)$ , it follows that  $B$  is complemented positive. Hence  $B$  is a  $+$ WBTM.

Suppose now that  $B'$  is another  $+$ WBTM on  $X$  such that  $B'|_W = A$ . We need to prove that  $B = B'$  on all  $X$ . Assume, by contradiction, that there exists  $w \in X$  such that  $B(w) \neq B'(w)$ .

- 1) Suppose that  $w \in \uparrow W_+$ . In this case,  $B(w) = P$ , hence it holds  $B'(w) = N$ . Since  $w \in \uparrow W_+$ , there exists  $\tilde{w} \in W_+$  such that  $\tilde{w} \leq w$ ; by hypothesis  $B$  and  $A$  coincides on  $W$  and hence  $B(\tilde{w}) = A(\tilde{w}) = P$ . Since  $B$  is up-positive and  $\tilde{w} \leq w$ , we have that  $B(w) = P$  and this is a contradiction.
- 2) Suppose that  $w \in \downarrow W_-$ . In this case,  $B(w) = N$ , and hence  $B'(w) = P$ . Since  $w \in \downarrow W_-$ , there exists  $\tilde{w} \in W_-$  such that  $w \leq \tilde{w}$ ; by hypothesis  $B(\tilde{w}) = A(\tilde{w}) = N$ . Since  $B$  is down-negative, with  $w \leq \tilde{w}$ , we have that  $B(w) = N$ . This is a contradiction.
- 3) Suppose that  $w \in \uparrow(W_-^c)$ . In this case,  $B(w) = P$ , and hence  $B'(w) = N$ . Since  $w \in \uparrow(W_-^c)$ , there exists  $\tilde{w} \in W_-^c$  such that  $\tilde{w} \leq w$ . Since  $\tilde{w} \in W_-^c$ , there exists  $\bar{w} \in W_-$  such that  $\tilde{w} = \bar{w}^c$ , and hence  $B(\bar{w}) = A(\bar{w}) = N$ . Since  $A$  and  $B$  are complemented positive, it follows that  $B(\bar{w}^c) = A(\bar{w}^c) = P$ , i.e.  $B(\tilde{w}) = P$ . Now since  $\tilde{w} \leq w$  and  $B$  is up-positive, it follows that  $B(w) = P$ , and this is a contradiction.

Hence  $A$  is a  $W_+(X)$ -core of  $B$ . By Theorem 3.12-i) it follows then that  $C_+(B) \subseteq W$ .

Let us prove now that

$$(3) \quad A^{-1}(P) \subseteq \text{Min}(B^{-1}(P)).$$

Suppose that  $w \in A^{-1}(P)$  and that by contradiction  $w \notin \text{Min}(B^{-1}(P))$ .

In this case there exists  $\bar{w} \in B^{-1}(P)$  such that  $\bar{w} < w$ . By Corollary 3.7 i) we have that  $\bar{w} \in \uparrow A^{-1}(P)$  or  $\bar{w} \in \uparrow(A^{-1}(N))^c$ .

1) if  $\bar{w} \in \uparrow A^{-1}(P)$ , there exists  $\tilde{w} \in A^{-1}(P)$  such that  $\tilde{w} \leq \bar{w}$  and hence we will have that  $\tilde{w} < \bar{w} \leq w$ , with  $\tilde{w}, w \in A^{-1}(P)$ , by contradicting the hypothesis that  $A^{-1}(P)$  is an anti-chain.

2) if  $\bar{w} \in \uparrow(A^{-1}(N))^c$ , there exists  $\tilde{w} \in (W_N^A)^c$  such that  $\tilde{w} \leq \bar{w}$  and hence  $\tilde{w} \leq \bar{w} < w$ , with  $\tilde{w} \in (A^{-1}(N))^c$  and  $w \in A^{-1}(P)$ , against the  $B1+$ ) and the hypothesis that  $\langle A^{-1}(P) | A^{-1}(N) \rangle$  is a  $w_+$ -basis for  $X$ . This proves (3).

Let us suppose now that  $C_+(B) \neq W$ . Since  $C_+(B) \subseteq W$ , this implies that  $C_+(B) \subsetneq W$ , and hence that  $|C_+(B)| < |W|$ . Let  $\tilde{W} = C_+(B)$  and set  $\tilde{A} = B|_{\tilde{W}}$ .

Since  $A$  and  $\tilde{A}$  are both two  $W_+(X)$ -cores for  $B$ , by Proposition 3.10-i) we know that

$$\text{Min}(A^{-1}(P) \sqcup (A^{-1}(N))^c) = \text{Min}(B^{-1}(P)) = \text{Min}(\tilde{A}^{-1}(P) \cup (\tilde{A}^{-1}(N))^c).$$

Since  $\langle A^{-1}(P) | A^{-1}(N) \rangle$  is a  $w_+$ -basis for  $X$ , we have that  $A^{-1}(N)$  is an anti-chain, moreover, by definition of  $C_+(B)$ , also  $\tilde{A}^{-1}(N)$  is an anti-chain; by Corollary 3.5 i), then it follows that  $A^{-1}(N) = \text{Max}(B^{-1}(N)) = \tilde{A}^{-1}(N)$ , and hence  $(A^{-1}(N))^c = (\tilde{A}^{-1}(N))^c$ . Since  $W = A^{-1}(P) \sqcup A^{-1}(N)$  and  $\tilde{W} = \tilde{A}^{-1}(P) \sqcup \tilde{A}^{-1}(N)$ , by the equality  $A^{-1}(N) = \tilde{A}^{-1}(N)$  and by the inequality  $|\tilde{W}| < |W|$ , it follows that

$$(4) \quad |\tilde{A}^{-1}(P)| < |A^{-1}(P)|.$$

By (3) we have that

$$(5) \quad A^{-1}(P) \subseteq \text{Min}(B^{-1}(P)) = \text{Min}(A^{-1}(P) \sqcup (A^{-1}(N))^c) = \text{Min}(\tilde{A}^{-1}(P) \cup (\tilde{A}^{-1}(N))^c).$$

Since  $A^{-1}(P) \cap (\tilde{A}^{-1}(N))^c = A^{-1}(P) \cap (A^{-1}(N))^c = \emptyset$  in view of the fact that  $\langle A^{-1}(P) | A^{-1}(N) \rangle$  is a  $w_+$ -basis for  $X$ , by (5) it follows that  $A^{-1}(P) \subseteq \tilde{A}^{-1}(P)$ , and hence  $|A^{-1}(P)| \leq |\tilde{A}^{-1}(P)|$ , that is in contradiction with (4). This proves that  $W = C_+(B)$ .

ii) The same reasoning applies by using Theorem 3.12-ii), Corollary 3.7-ii) and Proposition 3.10-ii).  $\square$

Now, we denote by  $\mathcal{B}_{w_+}(X)$  the family of all  $w_+$ -bases on  $X$  and by  $\mathcal{B}_{w_-}(X)$  the family of all  $w_-$ -bases on  $X$ . If  $B \in \mathcal{W}_+(X)$ , by Proposition 4.2-*i*), it follows that  $\langle A^{-1}(P)|A^{-1}(N) \rangle \in \mathcal{B}_{w_+}(X)$ , where  $A = B|_{C_+(B)}$ . This defines an application  $h_+ : \mathcal{W}_+(X) \rightarrow \mathcal{B}_{w_+}(X)$  such that  $h_+(A) = \langle A^{-1}(P)|A^{-1}(N) \rangle$ , where  $W = C_+(B)$ .

If  $B \in \mathcal{W}_-(X)$ , by Proposition 4.2-*ii*) we can define a similar map  $h_- : \mathcal{W}_-(X) \rightarrow \mathcal{B}_{w_-}(X)$ . It holds then the following result.

**Theorem 4.4.** *The maps  $h_+$  and  $h_-$  are bijective.*

*Proof.* The map  $h_+$  is onto by virtue of Proposition 4.3-*i*). We prove now that  $h_+$  is a one-to-one map. Let  $B$  and  $\tilde{B}$  be two  $+$ WBMTs on  $X$  such that  $\langle A^{-1}(P)|A^{-1}(N) \rangle = \langle \tilde{A}^{-1}(P)|\tilde{A}^{-1}(N) \rangle$ , where  $A = B|_{C_+(B)}$  and  $\tilde{A} = \tilde{B}|_{C_+(\tilde{B})}$ .

Then, if  $w \in \tilde{A}^{-1}(P)$ , we have that  $\tilde{B}(w) = P$  and also  $B(w) = P$ ; similarly, if  $w \in \tilde{A}^{-1}(N)$ , we have that  $\tilde{B}(w) = N$  and  $B(w) = N$ . Therefore  $B|_{\tilde{W}} = \tilde{B}|_{\tilde{W}}$ . Since  $\tilde{A}$  is a  $\mathcal{W}_+(X)$ -core of  $\tilde{B}$ , it follows that  $B = \tilde{B}$ .

The case of  $h_-$  is analogue. □

## 5. THE INVOLUTION LATTICE $S(n, r)$

In what follows  $n$  and  $r$  are two integer numbers such that  $n \geq r \geq 0$ .

We set  $I(n, r) := \{r, \dots, 1, -1, \dots, -(n-r)\}$  and  $A(n, r) := I(n, r) \cup \{0\}$ . Now we briefly recall the definition of the lattice  $S(n, r)$ . Let  $A(n, r) := I(n, r) \cup \{0\} = \{r, \dots, 1, 0, -1, \dots, -(n-r)\}$ . We call  $(n, r)$ -string a  $n$ -pla of integers

$$(6) \quad i_1 \dots i_r | j_1 \dots j_{n-r},$$

such that:

*i*)  $r \geq i_1 \geq \dots \geq i_r \geq 0 \geq j_1 \geq \dots \geq j_{n-r} \geq -(n-r)$ ;

*ii*) the unique element in (6) which can be repeated is 0.

We denote by  $S(n, r)$  the set of all  $(n, r)$ -strings. On  $S(n, r)$  we consider the partial order on the components, that we denote by  $\sqsubseteq$ . To simplify the notations, in all the numerical examples the integers on the right of the vertical bar  $|$  will be written without minus sign. Since  $(S(n, r), \sqsubseteq)$  is a finite distributive lattice it is also graded, with minimum element  $0 \dots 0 | 2 \dots (n-r)$  and maximum element  $r(r-1) \dots 21 | 0 \dots 0$ . In [6] it has been shown that if we define

$$(i_1 \dots i_k \ 0 \dots 0 | 0 \dots 0 \ j_1 \dots j_l)^c := i'_1 \dots i'_{r-k} \ 0 \dots 0 | 0 \dots 0 \ j'_1 \dots j'_{n-r-l},$$

where  $\{i'_1, \dots, i'_{r-k}\}$  is the usual complement of  $\{i_1, \dots, i_k\}$  in  $\{1, \dots, r\}$  and  $\{j'_1, \dots, j'_{n-r-l}\}$  is the usual complement of  $\{j_1, \dots, j_l\}$  in  $\{-1, \dots, -(n-r)\}$ , then  $S(n, r)$  becomes a SIP.

It is straightforward to prove the following result.

**Proposition 5.1.** *The lattice  $S(n, r)$  is isomorphic to the direct product of the two lattice  $S(r, r)$  and the dual lattice  $(S(n-r, n-r))^\delta$ .*

We call  $(n, r)$ -function a function  $f : A(n, r) \rightarrow \mathbb{R}$  such that

$$(7) \quad f(r) \geq \dots \geq f(1) \geq f(0) = 0 > f(-1) \geq \dots \geq f(-(n-r)).$$

We denote by  $F(n, r)$  the set of the  $(n, r)$ -functions. The  $(n, r)$ -function  $f$  is a  $(n, r)$ -positive weight function [negative weight function] if

$$(8) \quad f(r) + \dots + f(1) + f(-1) + \dots + f(-(n-r)) \geq 0 \ [< 0].$$

We denote by  $WF_+(n, r)$  the set of the  $(n, r)$ -positive weight functions and with  $WF_-(n, r)$  the set of the  $(n, r)$ -negative weight functions. If  $w = i_1 \dots i_r | j_1 \dots j_{n-r} \in S(n, r)$  and  $f \in F(n, r)$ , we set  $\sum_f(w) := f(i_1) + \dots + f(i_r) + f(j_1) + \dots + f(j_{n-r})$ .

The next result shows as the order structure in  $S(n, r)$  is strictly related to the properties of the family of maps  $\{\sum_f : f \in F(n, r)\}$ .

We recall at first the definition of valuation on an arbitrary lattice  $X$ . If  $X$  is a lattice, a map  $\nu : X \rightarrow \mathbb{R}$  is called a *valuation* on  $X$  if for all  $a, b \in X$ :  $\nu(a \wedge b) + \nu(a \vee b) = \nu(a) + \nu(b)$ . For classical and more recent studies concerning the valuations on distributive lattices see [20, 21, 22, 28].

**Proposition 5.2.** *i) If  $f \in F(n, r)$ , the map  $\sum_f$  is a valuation on  $S(n, r)$ .*

*ii) If  $w, w' \in S(n, r)$ , then  $w \sqsubseteq w'$  if and only if  $\sum_f(w) \leq \sum_f(w')$  for each  $f \in F(n, r)$ .*

*Proof.* *i)* Follows directly from the definition of  $\sqsubseteq$ .

*ii)* Let  $w = a_r \dots a_1 | b_1 \dots, b_{n-r}$  and  $w' = a'_r \dots a'_1 | b'_1 \dots, b'_{n-r}$  two elements in  $S(n, r)$ . If  $w \sqsubseteq w'$ , it is immediate that  $\sum_f(w) \leq \sum_f(w')$  for each  $f \in F(n, r)$ . We assume now that  $\sum_f(w) \leq \sum_f(w')$  for each  $f \in F(n, r)$  and that the condition  $w \sqsubseteq w'$  is false. This means that there exists some  $i \in \{1, \dots, r\}$  such that  $a_i > a'_i$  or some  $j \in \{1, \dots, n-r\}$  such that  $b_j > b'_j$ . Let us suppose at first that there exists  $i \in \{1, \dots, r\}$  such that  $a_i > a'_i$  and we assume that  $i$  is maximal among all the positive integers  $l \in \{1, \dots, r\}$  such that  $a_l > a'_l$ , therefore

$$(9) \quad a_r \geq \dots \geq a_{i+1} \geq a_i > a'_i \geq a'_{i-1} \geq \dots \geq a'_1 \quad \text{and} \quad a'_r \geq a_r, \dots, a'_{i+1} \geq a_{i+1}.$$

We consider now the following function:

$$f(\alpha) := \begin{cases} -1 & \text{if } \alpha \in \{-1, \dots, -(n-r)\} \\ 0 & \text{if } \alpha \in \{0, 1, \dots, a_i - 1\} \\ +1 & \text{if } \alpha \in \{a_i, \dots, r\} \end{cases}$$

Then  $f \in F(n, r)$  and by (9) it follows that  $\sum_f(w) \geq (r-i+1) + \sum_{1 \leq j \leq n-r} f(b_j) > (r-i) + \sum_{1 \leq j \leq n-r} f(b_j) = (r-i) + \sum_{1 \leq j \leq n-r} f(b'_j) = \sum_f(w')$ , that is a contradiction.

We can suppose then  $a_i \leq a'_i$  for all  $i = 1, \dots, r$ , so there exists  $j \in \{1, \dots, n-r\}$  such that  $b_j > b'_j$  and we assume that  $j$  is minimal among all the positive integers  $l \in \{1, \dots, n-r\}$  such that  $b_l > b'_l$ , therefore

$$(10) \quad b_1 \geq \dots \geq b_{j-1} \geq b_j > b'_j \geq b'_{j+1} \geq \dots \geq b'_{n-r} \quad \text{and} \quad b'_1 \geq b_1, \dots, b'_{j-1} \geq b_{j-1}$$

We must now distinguish two cases. First we suppose that  $b_j = 0$ . In this case we consider the following function:

$$h(\alpha) := \begin{cases} 0 & \text{if } \alpha \in \{0, 1, \dots, r\} \\ -1 & \text{if } \alpha \in \{-1, \dots, -(n-r)\} \end{cases}$$

Then  $h \in F(n, r)$  and by (10) it follows that  $\sum_h(w) \geq (-1)(n-r-j) > (-1)(n-r-j+1) = \sum_h(w')$ , that is a contradiction. We assume now that  $b_j < 0$ . In this case we consider the following function:

$$g(\alpha) := \begin{cases} 0 & \text{if } \alpha = 0 \\ +1 & \text{if } \alpha \in \{1, \dots, r\} \\ -1 & \text{if } \alpha \in \{-1, \dots, b_j\} \\ -2 & \text{if } \alpha \in \{b_j - 1, \dots, -(n-r)\} \end{cases}$$

Then  $g \in F(n, r)$  and by (10) we have:

$$\begin{aligned} \sum_g(w) &\geq \sum_{1 \leq l \leq j-1} g(b_l) + (-1) + \sum_{j+1 \leq l \leq n-r} g(b_l) > \sum_{1 \leq l \leq j-1} g(b_l) + (-2) + \sum_{j+1 \leq l \leq n-r} g(b_l) \\ &= \sum_{1 \leq l \leq j-1} g(b'_l) + g(b'_j) + \sum_{j+1 \leq l \leq n-r} g(b_l) \geq \sum_{1 \leq l \leq j-1} g(b'_l) + (-2) + (-2)(n-r-j) = \\ &\sum_g(w') \end{aligned}$$

that is a contradiction. This completes the proof of *ii)*.  $\square$

In particular, if  $\sum_f$  is one-to-one, then  $\sum_f$  is also a linear extension of  $S(n, r)$ . In [35] Rota showed that a valuation on a finite distributive lattice is uniquely determined by the values that it takes on the join-irreducible elements of the lattice, therefore, in our case, this means that  $\sum_f$  is uniquely determined by the values that it takes on the join-irreducible elements of the distributive lattice  $S(n, r)$ .

For each  $f \in F(n, r)$ , we can define a 2-valued map  $B_f : S(n, r) \rightarrow \{N, P\}$  as follows:

$$B_f(w) := \begin{cases} P & \text{if } \sum_f(w) \geq 0 \\ N & \text{if } \sum_f(w) < 0. \end{cases}$$

Then it is clear that  $B_f$  is a +WBTM if  $f \in WF_+(n, r)$  and  $B_f$  is a -WBTM if  $f \in WF_-(n, r)$ .

## 6. $(n, r)$ -SYSTEMS

In the present section, let us suppose that we have  $r$  variables  $x_r, \dots, x_1$  and other  $(n-r)$  variables  $y_1, \dots, y_{(n-r)}$ .

We call  $(n, r)$ -system of size  $p$  a system  $\mathcal{S}$  of linear inequalities having the following form:

$$(11) \quad \mathcal{S} : \begin{cases} x_r \geq \cdots \geq x_1 \geq 0 \geq y_1 \geq \cdots \geq y_{n-r} \\ \sum_{i \in A_1} t_i \geq 0 \text{ (or } \leq 0) \\ \sum_{i \in A_2} t_i \geq 0 \text{ (or } \leq 0) \\ \cdots \\ \cdots \\ \sum_{i \in A_p} t_i \geq 0 \text{ (or } \leq 0) \end{cases}$$

where  $A_1, \dots, A_p$  are different subsets of  $I(n, r)$  of cardinality strictly greater than 1; moreover  $t_i = x_i$  if  $i \in \{r, \dots, 1\}$  and  $t_i = y_i$  if  $i \in \{-1, \dots, -(n-r)\}$ . When the subsets  $A_1, \dots, A_p$  coincide with all the possible subsets of  $I(n, r)$  different from the singletons and from the empty set, we say that the  $(n, r)$ -system (11) is *total*. Furthermore, when in (11) the following inequality appears:

$$(12) \quad x_r + \cdots + x_1 + y_1 + \cdots + y_{n-r} \geq 0 \text{ (or } \leq 0)$$

we say that it is an  $(n, r)$ -*positively weighted* system (or an  $(n, r)$ -*negatively weighted* system).

Let  $\mathcal{S}, \mathcal{S}'$  be two  $(n, r)$ -systems: we say that they are *equal* (in symbols  $\mathcal{S} = \mathcal{S}'$ ) if they have exactly the same inequalities, otherwise we say that they are *different* (in symbols  $\mathcal{S} \neq \mathcal{S}'$ ).

If they are both consistent (i.e. they have solutions) and equivalent (i.e. they have the same solutions) we shall write  $\mathcal{S} \equiv \mathcal{S}'$ .

We denote by

- $Syst(n, r)$  the set of all  $(n, r)$ -systems;
- $TSyst(n, r)$  the set of all  $(n, r)$ -total systems;
- $CSyst(n, r)$  the set of all  $(n, r)$ -systems which are also consistent;
- $CTSyst(n, r) := CSyst(n, r) \cap TSyst(n, r)$ ;
- $W_+Syst(n, r)$  the set of all  $(n, r)$ -positively weighted systems;
- $W_+CSyst(n, r)$  the set of all  $(n, r)$ -consistent positively weighted systems;
- $W_+TSyst(n, r)$  the set of all  $(n, r)$ -total positively weighted systems;
- $W_+CTSyst(n, r) := W_+CSyst(n, r) \cap W_+TSyst(n, r)$ ;
- $W_-Syst(n, r), W_-CSyst(n, r), W_-TSyst(n, r), W_-CTSyst(n, r)$  the analogues of the previous four sets but negatively weighted.

Notice that if  $\mathcal{S}, \mathcal{S}' \in CTSyst(n, r)$  and  $\mathcal{S} \neq \mathcal{S}'$ , then  $\mathcal{S}$  and  $\mathcal{S}'$  can not be equivalent.

Let us consider an  $(n, r)$ -system  $\mathcal{S}$  as in (11). Since there is an obvious bijection between the power set  $\mathcal{P}(I(n, r))$  and  $S(n, r)$ , all the subsets  $A_1, \dots, A_p$  in (11) can be identified with strings of  $S(n, r)$ , that we denote by  $w_1, \dots, w_p$  (for example, if  $n = 7, r = 4$ , we identify the subset  $\{1, 3, 4, -1\}$  with the string 4310|001, or the subset  $\{2, -2, -3\}$  with 2000|023).

Let us note that  $0 \dots 0|0 \dots 0$  will be identified always with the empty subset of  $I(n, r)$ . It is immediate to see that if  $w_k \sqsubseteq w_j$  for some  $k, j$ , then  $\sum_{i \in A_k} t_i \leq \sum_{i \in A_j} t_i$ .

We denote by  $(S(n, r) \rightsquigarrow B_2)$  the poset of the 2-valued partial maps on  $S(n, r)$  (see [18] for the definition of a poset of partial maps).

Moreover, we also set

- $\xi_r = r0 \dots 0|0 \dots 0, \dots, \xi_1 = 10 \dots 0|0 \dots 0, \xi_0 = 00 \dots 0|0 \dots 0,$
- $\eta_1 = 0 \dots 0|0 \dots 01, \dots, \eta_{n-r} = 0 \dots 0|0 \dots 0(n-r),$
- $\Omega_{\mathcal{S}} = \{w_1, \dots, w_p, \xi_r, \dots, \xi_1, \xi_0, \eta_1, \dots, \eta_{n-r}\}.$

**Definition 6.1.** Let  $\mathcal{S} \in Syst(n, r)$ . The induced  $\mathcal{S}$ -2-valued partial map (briefly  $\mathcal{S}$ -BPM)  $A_{\mathcal{S}} : \Omega_{\mathcal{S}} \subseteq S(n, r) \rightarrow B_2$  is defined as follows:

for  $j \in \{1, \dots, p\}$ ,

$$A_{\mathcal{S}}(w_j) = \begin{cases} P & \text{if } \sum_{i \in A_j} t_i \geq 0 \\ N & \text{if } \sum_{i \in A_j} t_i < 0 \end{cases}$$

$A_{\mathcal{S}}(\xi_0) = A_{\mathcal{S}}(\xi_1) = \dots = A_{\mathcal{S}}(\xi_r) = P$  and  $A_{\mathcal{S}}(\eta_1) = \dots = A_{\mathcal{S}}(\eta_{n-r}) = N$ .

**Definition 6.2.** If  $\mathcal{S}, \mathcal{S}' \in Syst(n, r)$ , we set  $\mathcal{S} \lesssim \mathcal{S}'$  if  $\mathcal{S}$  is a subsystem of  $\mathcal{S}'$ .

This obviously defines a partial order  $\lesssim$  on  $Syst(n, r)$ . We denote by  $\mathcal{B}(n, r)$  the sub-poset of all the 2-valued partial maps  $A \in (S(n, r) \rightsquigarrow B_2)$  such that  $\xi_r, \dots, \xi_1, \xi_0, \eta_1, \dots, \eta_{n-r} \in \text{dom}(A)$  and  $A(\xi_0) = A(\xi_1) = \dots = A(\xi_r) = P, A(\eta_1) = \dots = A(\eta_{n-r}) = N$  and with  $\mathcal{BT}(n, r)$  the subset of all the total maps of  $\mathcal{B}(n, r)$ . Then the map  $\chi : Syst(n, r) \rightarrow \mathcal{B}(n, r)$  such that  $\chi(\mathcal{S}) = A_{\mathcal{S}}$ , for each  $\mathcal{S} \in Syst(n, r)$ , is an isomorphism of posets. We denote by  $\tau : \mathcal{B}(n, r) \rightarrow Syst(n, r)$  the inverse of  $\chi$  and we set  $\tau(b) = \mathcal{S}_B$  if  $B \in \mathcal{B}(n, r)$ . Obviously the restriction of  $\chi$  to  $\mathcal{BT}(n, r)$  defines an isomorphism between  $\mathcal{BT}(n, r)$  and

$TSyst(n, r)$ , and we continue respectively to denote by  $\chi$  and  $\tau$  this isomorphism and its inverse. A very difficult question is then:

$Q_1$ ) What are the maps in  $\chi(W_+CTSyst(n, r))$  and in  $\chi(W_-CTSyst(n, r))$ ?

Roughly speaking, what are the order-properties that characterize a 2-valued map  $B \in \mathcal{BT}(n, r)$  in such a way that  $\mathcal{S}_B$  is a consistent system?

The following two results are very simple but they provide a possible strategy in order to approach the problem raised in  $Q_1$ ).

**Positive local criterion (p.l.c.)** *Let  $\mathcal{H}$  be a family of maps of  $\mathcal{BT}(n, r)$  such that*

$$\chi(W_+CTSyst(n, r)) \subseteq \mathcal{H} \text{ and } \mathcal{H} \cap \chi(W_-CTSyst(n, r)) = \emptyset,$$

*and let  $B \in \mathcal{H}$ . Let  $A$  denote a  $\mathcal{H}$ -core of  $B$ . Then,  $\mathcal{S}_B$  is consistent if and only if  $\mathcal{S}_A \in W_+CSyst(n, r)$  and, in this case, if  $f \in WF_+(n, r)$  is a solution of  $\mathcal{S}_A$ , it is also a solution of  $\mathcal{S}_B$ .*

*Proof.* If  $\mathcal{S}_B$  is consistent, then necessarily it holds that  $\mathcal{S}_B \in W_+CTSyst(n, r)$  because

$$\mathcal{H} \cap \chi(W_-CTSyst(n, r)) = \emptyset \text{ and } B = \chi(\mathcal{S}_B)$$

Hence  $\mathcal{S}_A \in W_+CSyst(n, r)$ . On the other side, if  $\mathcal{S}_A \in W_+CSyst(n, r)$ , it has a solution  $f \in WF_+(n, r)$ . Then  $\mathcal{S}_f \in W_+CTSyst(n, r)$  and hence, by hypothesis,  $\chi(\mathcal{S}_f) \in \mathcal{H}$ . It is easy to observe that  $\chi(\mathcal{S}_f) = B_f$ . Therefore  $B_f \in \mathcal{H}$ . If we denote by  $W$  the domain of  $A$ , we have  $(B_f)|_W = A$  since  $f$  is a solution of  $\mathcal{S}_A$ ; therefore  $(B_f)|_W = B|_W$ . Since  $B|_W$  is a  $\mathcal{H}$ -core of  $B$ , we have that  $B = B_f$ ; hence  $\mathcal{S}_B$  is consistent and  $f$  is one of its solution.  $\square$

**Negative local criterion (n.l.c.)** *Let  $\mathcal{H}$  be a family of maps of  $\mathcal{BT}(n, r)$  such that*

$$\chi(W_-CTSyst(n, r)) \subseteq \mathcal{H} \text{ and } \mathcal{H} \cap \chi(W_+CTSyst(n, r)) = \emptyset,$$

*and let  $B \in \mathcal{H}$ . Let  $A$  denote a  $\mathcal{H}$ -core of  $B$ . Then,  $\mathcal{S}_B$  is consistent if and only if  $\mathcal{S}_A \in W_-CSyst(n, r)$  and, in this case, if  $f \in WF_-(n, r)$  is a solution of  $\mathcal{S}_A$ , it is also a solution of  $\mathcal{S}_B$ .*

*Proof.* Similar to that of p.l.c.  $\square$

The previous results give us some ‘‘local’’ criteria that are useful in two directions: ‘‘from global to local’’ and ‘‘from local to global’’. In the direction ‘‘from global to local’’, to decide if a map  $B$  that we choose in a special family  $\mathcal{H}$  of 2-valued total maps of  $\mathcal{BT}(n, r)$  determines an  $(n, r)$ -consistent total system. In this case the previous criteria are useful if we know, for each given map  $B \in \mathcal{H}$  a  $\mathcal{H}$ -core that is ‘‘sufficiently’’ small. In the direction ‘‘from local to global’’, we can ask if a given system  $\mathcal{S}$  of  $W_+CSyst(n, r)$  (or of  $W_-CSyst(n, r)$ ) is equivalent to some  $\mathcal{S}' \in W_+CTSyst(n, r)$  (or to some  $\mathcal{S}' \in W_-CTSyst(n, r)$ ) and if  $\mathcal{S}$  has a minimal cardinality between all the  $(n, r)$ -subsystems having the same solutions of  $\mathcal{S}'$ .

We set now

$$\mathcal{OP}(n, r) = \mathcal{OP}(S(n, r), B_2) \cap \mathcal{BT}(n, r),$$

$$\mathcal{W}_+(n, r) = \{B \in \mathcal{W}_+(S(n, r), B_2) \cap \mathcal{BT}(n, r) \mid B(r \cdots 21|12 \cdots (n-r)) = P\},$$

$$\mathcal{W}_-(n, r) = \{B \in \mathcal{W}_-(S(n, r), B_2) \cap \mathcal{BT}(n, r) \mid B(r \cdots 21|12 \cdots (n-r)) = N\}.$$

The family  $\mathcal{W}_+(n, r)$  satisfies the hypotheses of p.l.c. and the family  $\mathcal{W}_-(n, r)$  satisfies the hypotheses of n.l.c., i.e:

$$\chi(W_+CTSyst(n, r)) \subseteq \mathcal{W}_+(n, r) \text{ and } \mathcal{W}_+(n, r) \cap \chi(W_-CTSyst(n, r)) = \emptyset,$$

$$\chi(W_-CTSyst(n, r)) \subseteq \mathcal{W}_-(n, r) \text{ and } \mathcal{W}_-(n, r) \cap \chi(W_+CTSyst(n, r)) = \emptyset.$$

We can then apply the local criteria to the previous two families of 2-valued total maps on  $S(n, r)$ . If we apply the p.l.c. to a map  $B \in \mathcal{W}_+(n, r)$ , we take the  $\mathcal{W}_+(S(n, r), B_2)$ -fundamental core, that is also a  $\mathcal{W}_+(n, r)$ -core. Similarly, if we apply the n.l.c. to a map  $B \in \mathcal{W}_-(n, r)$ , we take the  $\mathcal{W}_-(S(n, r), B_2)$ -fundamental core, that is also a  $\mathcal{W}_-(n, r)$ -core. In these cases we say simply ‘‘the fundamental core’’ of  $B$ .

In what follows, for simplicity, we will write a partial map as the set of the strings of its domain followed by an  $N$  if they are negative or by a  $P$  if they are positive.

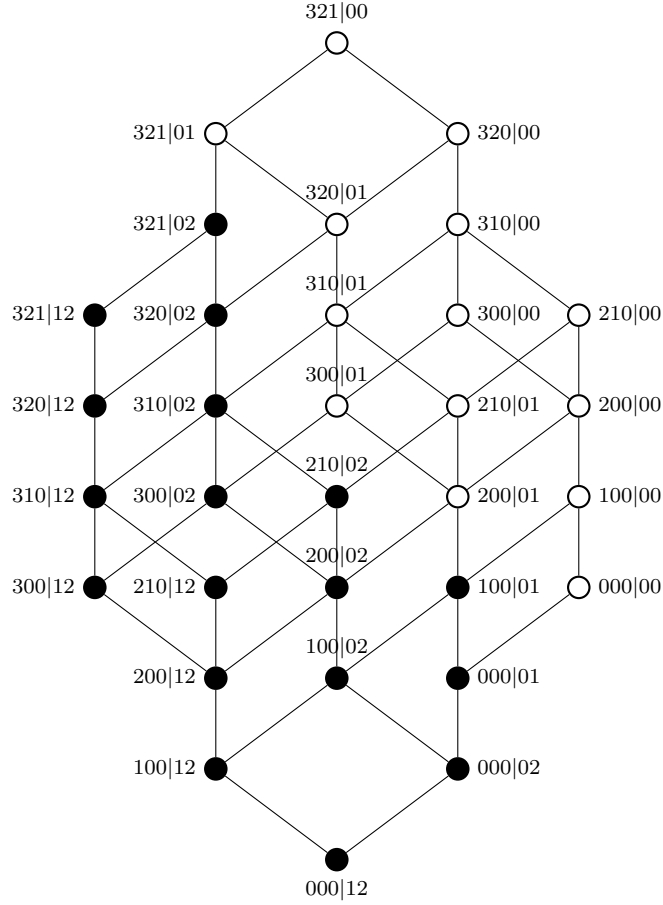


FIGURE 1. The 2-valued map discussed in Example 6.3

**Example 6.3.** Let us consider the case  $n = 5$  and  $r = 3$ . We take the system  $\mathcal{S} \in W\_TSyst(5, 3)$  such that the relative 2-valued map  $B_{\mathcal{S}}$  associated to it is represented in Figure 1 (the white nodes are P and the black nodes are N).

It results that  $B_{\mathcal{S}} \in \mathcal{W}\_-(5, 3)$ . It is easy to verify that the fundamental core of  $B_{\mathcal{S}}$  is the partial map

$$A = \{321|02N, 100|01N, 000|00P, 200|01P\}$$

The  $(5, 3)$ -system  $\mathcal{S}_A$  is then the following:

$$\mathcal{S}_B : \begin{cases} x_3 \geq x_2 \geq x_1 \geq 0 > y_1 \geq y_2 \\ x_1 + x_2 + x_3 + y_2 < 0 \\ x_1 + y_1 < 0 \\ x_2 + y_1 \geq 0 \end{cases}$$

A solution of this system is easily given by:

$$x_3 = \frac{1}{2}, x_2 = \frac{1}{3}, x_1 = \frac{1}{6}, y_1 = -\frac{1}{5}, y_2 = -\frac{6}{5}.$$

Then, by n.l.c. it follows that  $\mathcal{S} \in W\_CTSyst(5, 3)$ , i.e. it is consistent and has the same solutions of  $\mathcal{S}_B$ .

**Example 6.4.** Let us consider again the case  $n = 5$  and  $r = 3$ . We take the system  $\mathcal{S} \in W\_+TSyst(5, 3)$  such that the relative 2-valued map  $B_{\mathcal{S}}$  associated to it is represented in Figure 2 (as before, the white nodes are P and the black nodes are N).

It is easy to verify that the fundamental core of  $B_{\mathcal{S}}$  is the partial map

$$A = \{320|02N, 321|12P, 000|00P, 000|01N\}$$



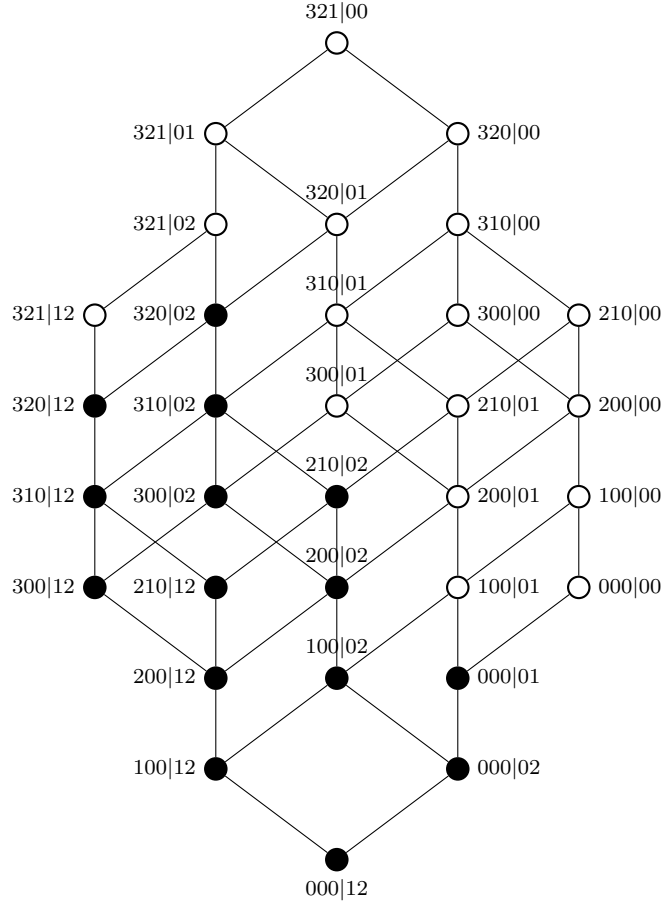


FIGURE 2. The 2-valued map discussed in Example 6.4

Therefore the system  $\mathcal{S}_A$  is the following:

$$\begin{cases} x_3 \geq x_2 \geq x_1 \geq 0 > y_1 \geq y_2 \\ x_1 + x_2 + x_3 + y_1 + y_2 \geq 0 \\ x_3 + x_2 + y_2 < 0 \end{cases}$$

A solution of this system is easily given by

$$x_3 = 1, x_2 = 1, x_1 = 0.9, y_1 = -0.8, y_2 = -2.1.$$

By p.l.c. it follows then that  $\mathcal{S} \in W_+CTSys(5, 3)$ , i.e.  $\mathcal{S}$  is consistent and equivalent to  $\mathcal{S}_B$ .

In the previous example, we showed two different  $(5, 3)$ -total systems both consistent. In the next example, we show a  $(6, 3)$ -total system  $\mathcal{S}$  that is not consistent but it is such that  $B_{\mathcal{S}} \in \mathcal{W}_+(6, 3)$ . The next example shows that the inclusion  $\chi(W_+CTSys(n, r)) \subseteq \mathcal{W}_+(n, r)$  is strict, i.e. there exist maps in  $\mathcal{W}_+(n, r)$  whose associated  $(n, r)$ -system has no solutions.

**Example 6.5.** Let us consider the map  $B \in \mathcal{BT}(6, 3)$  represented in Figure 3 (the white nodes are P and the black nodes are N).

Then it is easy to observe that  $B \in \mathcal{W}_+(6, 3)$  and that the fundamental core of  $B$  is the following partial map:

$$B = \{321|123P, 300|003N, 210|003N, 200|002N, 100|001N, 000|000P\}$$

Hence  $\mathcal{S}_B$  is the following  $(6, 3)$ -positively weighted system:

$$\begin{cases} x_3 \geq x_2 \geq x_1 \geq 0 > y_1 \geq y_2 \geq y_3 \\ x_1 + x_2 + x_3 + y_1 + y_2 + y_3 \geq 0 \\ x_3 + y_3 < 0 \\ x_2 + y_2 < 0 \\ x_1 + y_1 < 0 \\ x_2 + x_1 + y_3 < 0. \end{cases}$$

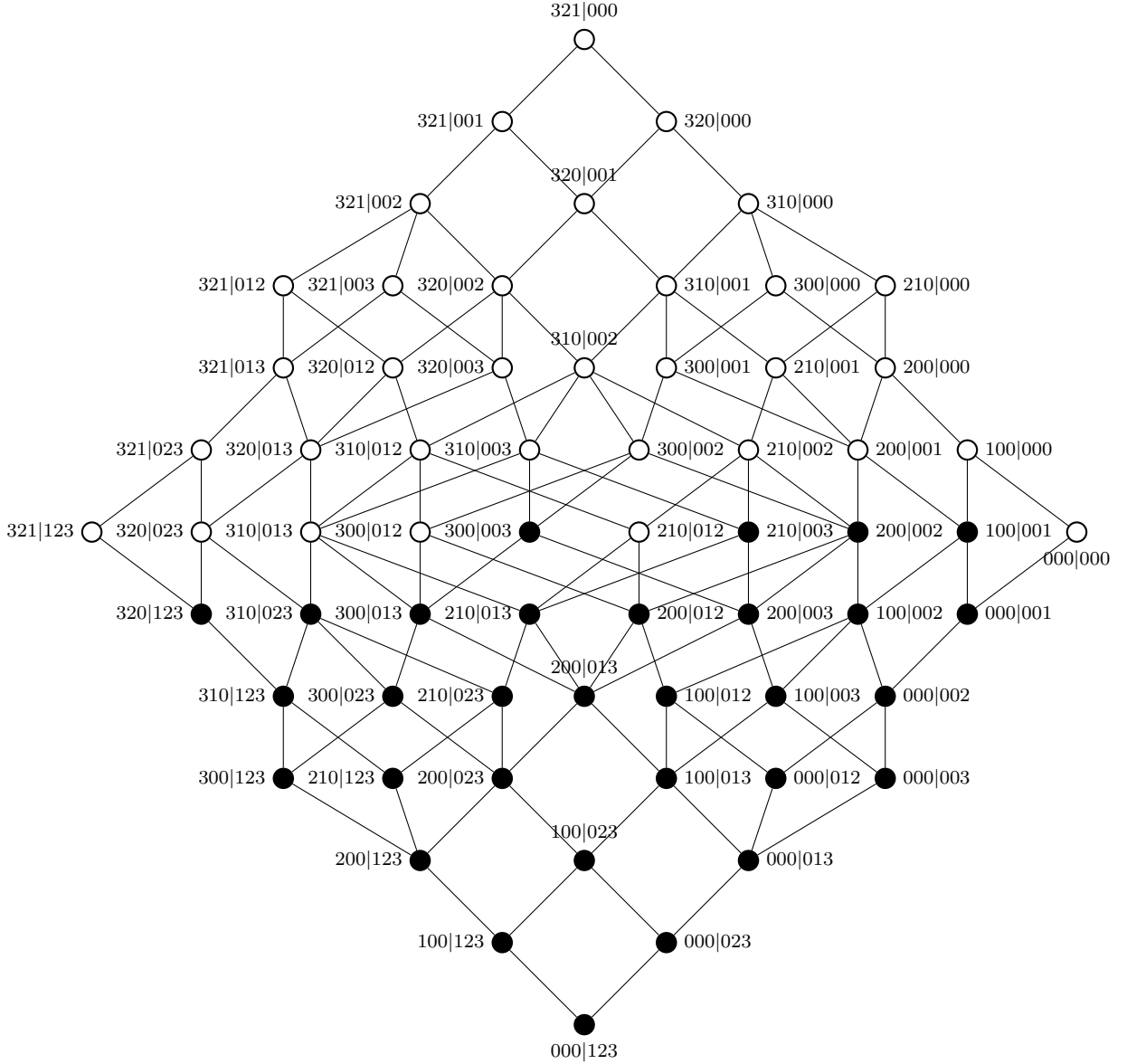


FIGURE 3. The 2-valued map discussed in Example 6.5

Obviously the previous system  $\mathcal{S}_A$  is not consistent, therefore also  $\mathcal{S}_B$  is not consistent. Hence  $BA \in \mathcal{W}_+(6, 3)$ , but  $B \notin \chi(\mathcal{W}_+CTSys(6, 3))$ .

Let us note that, for the previous map  $B$ , there does not exist an  $f \in \mathcal{W}_+F(n, r)$  such that  $B = B_f$ . Therefore this example shows that the answer to the open problems raised in [6] is negative.

## 7. OPEN PROBLEMS

The last example of the previous section tell us that the family  $\mathcal{W}_+(n, r)$  does not capture all the properties of the systems in  $\mathcal{W}_+CTSys(n, r)$ , therefore we give now a more restrictive condition on a family of 2-valued maps in order to catch all the properties that characterize a system in  $\mathcal{W}_+CTSys(n, r)$ . If  $w$  is a string in  $S(n, r)$  in the form (6) with  $i_1 > \dots > i_p > 0$ ,  $i_{p+1} = \dots = i_r = 0$  and  $j_1 = \dots = j_{q-1} = 0$ ,  $0 > j_q > \dots > j_{n-r}$ , for some indexes  $p$  and  $q$ , we set

$$w^* = \{i_1, \dots, i_p, j_q, \dots, j_{n-r}\}.$$

For example, if  $w = 4310|013 \in S(7, 4)$ , then  $w^* = \{1, 3, 4, -1, -3\}$ . In particular, if  $w = 0 \dots 0|0 \dots 0$  then  $w^* = \emptyset$ . It stays therefore defined a bijective map

$$* : w \in S(n, r) \mapsto w^* \in \mathcal{P}(I(n, r)).$$

**Definition 7.1.** Let  $w \in S(n, r)$ ; a partition of  $w$  is a subset  $\{w_1, \dots, w_k\}$  of  $S(n, r)$  such that  $\{w_1^*, \dots, w_k^*\}$  is a set-partition of  $w^*$ . If  $\{w_1, \dots, w_k\}$  is a partition of  $w$ , we write  $w : w_1 \wr \dots \wr w_k$ .

**Example 7.2.** If  $w = 7543100|0013 \in S(11, 7)$ , then  $w : 7000000|0000 \wr 5430000|0001 \wr 1000000|0003$ .

**Definition 7.3.** If  $A$  is a BPM on  $S(n, r)$ , we say that  $A$  is complemented pointwise if for each  $w \in \text{dom}(A)$  such that  $A(w) = T$ , where  $T = P$  or  $T = N$ , and for each partition  $w : w_1 \wr \dots \wr w_k$ , with  $\{w_1, \dots, w_k\} \subseteq \text{dom}(A)$ , we have  $A(w_i) = T$  for some  $i \in \{1, \dots, k\}$ .

**Definition 7.4.** We say that a map  $A$  is +formally consistent [-formally consistent] on  $S(n, r)$  if:

- $A \in \mathcal{OP}(n, r)$ ;
- $A$  is complemented pointwise;
- $A(r \dots 21|12 \dots (n-r)) = P$  [ $A(r \dots 21|12 \dots (n-r)) = N$ ].

We denote by  $\mathcal{FC}_+(n, r)$  [ $\mathcal{FC}_-(n, r)$ ] the family of all the maps +formally consistent [-formally consistent] on  $S(n, r)$ . It is immediate to observe that  $\mathcal{FC}_+(n, r) \subseteq \mathcal{W}_+(n, r)$  [ $\mathcal{FC}_-(n, r) \subseteq \mathcal{W}_-(n, r)$ ] and  $\chi(W_+CT\text{Syst}(n, r)) \subseteq \mathcal{FC}_+(n, r)$  [ $\chi(W_-CT\text{Syst}(n, r)) \subseteq \mathcal{FC}_-(n, r)$ ].

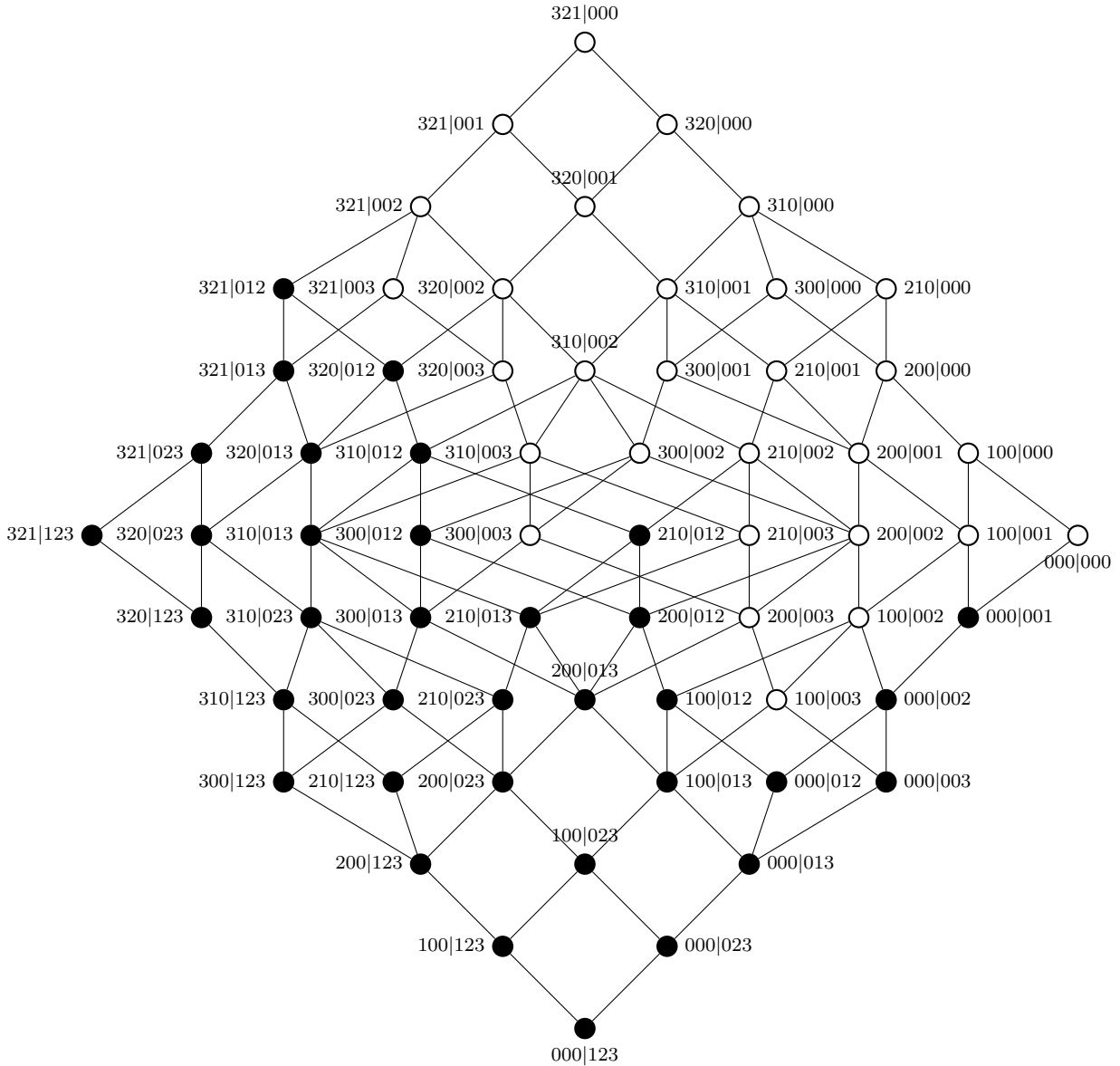


FIGURE 4. The 2-valued map discussed in Example 7.5

**Open Problems:**

$Q_2$ )  $\chi(W_+CTSys(n, r)) = \mathcal{FC}_+(n, r)$ ?

$Q_3$ )  $\chi(W_-CTSys(n, r)) = \mathcal{FC}_-(n, r)$ ?

The next example shows a map  $B \in \mathcal{W}_-(n, r)$  such that the restriction of  $B$  to its  $\mathcal{W}_-(n, r)$ -fundamental core is complemented pointwise but  $B$  is not -formally consistent: this proves that the open problem  $Q_3$ ) is false if the system and the 2-valued map are not total. We can provide an analogue example for the case  $B \in \mathcal{W}_+(n, r)$ , hence also the problem  $Q_2$ ) is false if the system and the 2-valued map are not total.

**Example 7.5.** Let us consider the 2-valued map  $B \in \mathcal{BT}(6, 3)$  represented in Figure 4 (as in the above cases, the white nodes are P and the black nodes are N).

It is easy to prove that  $B \in \mathcal{W}_-(6, 3)$  and that the  $\mathcal{W}_-(6, 3)$ -fundamental core of  $B$  is the following partial map:

$$A = \{321|012N, 000|001N, 100|003P, 000|000P\}$$

Then  $A$  is a BPM on  $S(6, 3)$  that is complemented pointwise, but  $A \notin \mathcal{FC}_-(6, 3)$ . In fact, if we take the string  $w = 321|123$ , we have  $w : 300|003 \wr 100|002 \wr 200|001$ , with  $B(w) = N$  and  $B(300|003) = A(100|002) = A(200|001) = P$ .

The system  $\mathcal{S}_A$  determined from  $A$  is the following  $(6, 3)$ -weighted system:

$$\begin{cases} x_3 \geq x_2 \geq x_1 \geq 0 > y_1 \geq y_2 \geq y_3 \\ x_3 + x_2 + x_1 + y_1 + y_2 + y_3 < 0 \\ x_1 + y_3 \geq 0 \end{cases}$$

The system  $\mathcal{S}_A$  is not consistent, because if it were then by n.l.c. also  $\mathcal{S}_B$  would be consistent and hence  $B \in \chi(\mathcal{W}_-CTSys(6, 3)) \subseteq \mathcal{FC}_-(6, 3)$ , which contradicts our assumptions.

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