# SEPARABLE FUNCTORS AND FORMAL SMOOTHNESS

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ABSTRACT. The natural problem we approach in the present paper is to show how the notion of formally smooth (co)algebra inside monoidal categories can substitute that of (co)separable (co)algebra in the study of splitting bialgebra homomorphisms. This is performed investigating the relation between formal smoothness and separability of certain functors and led to other results related to Hopf algebra theory. Between them we prove that the existence of ad-(co)invariant integrals for a Hopf algebra H is equivalent to the separability of some forgetful functors. In the finite dimensional case, this is also equivalent to the separability of the Drinfeld Double D(H) over H. Hopf algebras which are formally smooth as (co)algebras are characterized. We prove that given a bialgebra surjection  $\pi : E \to H$  with nilpotent kernel such that H is a Hopf algebra which is formally smooth as a K-algebra, then  $\pi$  has a section which is a right H-colinear algebra homomorphism. Moreover, if H is also endowed with an ad-invariant integral, then this section can be chosen to be H-bicolinear. We also deal with the dual case.

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## INTRODUCTION

Separable functors were introduced by C. Năstăsescu, M. Van den Bergh and F. Van Oystaeyen in [NVV]. As highlighted in [CMZ], the relevance of these functors lies in a functorial version of Maschke's theorem they satisfy, namely they reflect split exact sequences. In [AMS1, Corollary 2.31], this property was applied to the following situation. Let H be a semisimple and cosemisimple Hopf algebra over a field K and denote by  ${}^{H}\mathfrak{M}^{H}$  the category of H-bicomodules. Then the forgetful functor  $U : {}^{H}_{A}\mathfrak{M}^{H}_{A} \to {}_{A}\mathfrak{M}_{A}$ , from the category of A-bimodules in  ${}^{H}\mathfrak{M}^{H}$  to the category of ordinary A-bimodules, is a separable functor and hence the multiplication of A splits as a morphism of A-bimodules and H-bicomodules (i.e. A is separable as an algebra in the monoidal category ( ${}^{H}\mathfrak{M}^{H}, \otimes_{K}, K$ )) if and only if it splits as a morphism of A-bimodules (i.e. A is separable as an ordinary K-algebra). The proof of separability of the functor U relies on the existence of an adinvariant integral (introduced by D. Ştefan and F. Van Oystaeyen in [SVO, Definition 1.11]) for

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any semisimple and cosemisimple Hopf algebra over a field K. The characterization of separable and formally smooth algebras in the framework of monoidal categories was developed in [AMS2]. The notion of formal smoothness (or quasi-freeness) for algebras over a field K was introduced by J. Cuntz and D. Quillen in [CQ] to provide a natural setting for non-commutative version of certain aspects of manifolds. A formally smooth algebra A in monoidal categories behaves like a free algebra with respect to nilpotent extensions in the sense that, under natural conditions, any algebra homomorphism  $A \to R/I$ , where I is a nilpotent ideal of an algebra R, can be lifted to an algebra homomorphism  $A \to R$ . This gives a natural way to produce algebra sections in  $\mathcal{M}$  for algebra homomorphisms  $E \to A$  which are epimorphisms with nilpotent kernel in  $\mathcal{M}$ . Like in the classical case any separable algebra in a monoidal category is in particular formally smooth. As a consequence, in [AMS1] it was shown that if E is a bialgebra such that H = E/J is a quotient Hopf algebra of E which is semisimple, J denoting the Jacobson radical of E, then the canonical Hopf projection  $\pi: E \to H$  admits a left H-colinear algebra section  $\sigma: H \to E$ . Furthermore this section can be chosen to be H-bicolinear, whenever H is also cosemisimple. In [AMS1] also the dual situation of a bialgebra E whose coradical, say H, is a Hopf subalgebra is described. In this case there is a retraction  $\pi$  of the canonical injection  $\sigma$  which is a left H-linear (bilinear if H is also semisimple) coalgebra map.

These results fit in the classification of finite dimensional Hopf algebras problem as follows.

A bialgebra with a projection is a bialgebra E over a field K endowed with a Hopf algebra Hand two bialgebra maps  $\sigma : H \to E$  and  $\pi : E \to H$  such that  $\pi \circ \sigma = \mathrm{Id}_H$ . In [Rad], M. D. Radford describes the structure of bialgebras with a projection: E can be decomposed as the smash product of H with the (right) H-coinvariant part of E which actually comes out to be a braided bialgebra in the monoidal category  ${}^{H}_{H}\mathcal{YD}$  of Yetter-Drinfeld modules over H. It is meaningful that, even relaxing some assumption on  $\pi$  (as was done by P. Schauenburg in [Scha1]) or on  $\sigma$  (see [AMS1]), it is possible to reconstruct E by means of a suitable bosonization type procedure. An occurrence of this situation is given by the results in [AMS1] described above.

The natural problem we approach in the present paper is to show how the notion of formally smooth (co)algebra inside monoidal categories can substitute that of (co)separable (co)algebra in the study of splitting bialgebra homomorphisms. This is performed investigating the relation between formal smoothness and separability of certain functors and led to other results related to Hopf algebra theory. Between them we prove that the existence of ad-(co)invariant integrals for a Hopf algebra H is equivalent to the separability of suitable forgetful functors (Theorem 3.12). In the finite dimensional case, this is also equivalent to the separability of the Drinfeld Double D(H)as an extension of H (Theorem 3.14).

Hopf algebras which are formally smooth as (co)algebras are characterized in Propositions 5.4, 5.5, 9.4 and 9.5 (see also [MO, Theorem 1.2]). In particular we obtain that the the underline (co)algebra structures of a Hopf algebra is formally smooth if and only if it is hereditary.

As a result we prove that given a bialgebra surjection  $\pi : E \to H$  with nilpotent kernel such that H is a Hopf algebra which is formally smooth as a K-algebra, then  $\pi$  has a section which is a right H-colinear algebra homomorphism (Theorem 5.10). Moreover, if H is also endowed with an ad-invariant integral, then this section can be chosen to be H-bicolinear (Theorem 4.8). Dually, we prove that, if H is a Hopf subalgebra of a bialgebra E which is formally smooth as a K-coalgebra and such that  $Corad(E) \subseteq H$ , then E has a weak projection onto H (Theorem 9.16). Furthermore, if H is also endowed with an ad-coinvariant integral, then this retraction can be chosen to be H-bilinear (Theorem 8.11). As an application, in Proposition 9.18 we prove that every connected Hopf algebra E over a field K with char (K) = 0 has a weak projection  $\pi : E \to K[x]$ , for every  $x \in P(E) \setminus \{0\}$ .

The paper is organized as follows. We begin in Section 1 by recalling the definition of monoidal category and by listing the most important examples for this paper. We recall the notion of projectivity (respectively injectivity) of objects in a category  $\mathfrak{C}$  with respect to a class of homomorphisms in  $\mathfrak{C}$  and some general facts about separable functors. We obtain the main result of this section,

Theorem 1.13, providing a diagrammatic method to establish when a separable functor F preserves or reflects relative projective (resp. injective) objects.

This technique is applied, in Section 2, in the case when F is the forgetful functor  ${}_{A}\mathcal{M}_{A} \to {}_{A}\mathfrak{M}_{A}$ , where  $\mathcal{M}$  denotes one of the categories  $\mathfrak{M}^{H}$ ,  ${}^{H}\mathfrak{M}^{H}$  of right, two-sided comodules over a Hopf algebra H respectively, and A is an algebra in  $\mathcal{M}$ . In Theorem 2.12 we prove that, if F is separable, then A is formally smooth as an algebra in  $\mathcal{M}$  if and only if it is formally smooth as an algebra in  $\mathfrak{M}_{K}$  (i.e. regardless the H-comodule structure of A). A remarkable fact is that the functor F is separable whenever H has an ad-invariant integral (see Lemma 3.11). In Proposition 2.6, a characterization of separable algebras in a monoidal category by means of separable functors is given. We also deal with the dual results.

In Section 3 the existence of *ad*-invariant integrals is related to separability of suitable functors. In particular H has an *ad*-invariant integral if and only if the forgetful functor  ${}^{H}_{H}\mathcal{YD} \to {}^{H}\mathfrak{M}$  is separable (see Theorem 3.12). In the finite dimensional case, this is equivalent to say that the Drinfeld Double D(H) is a separable extension of H (see Theorem 3.14).

Section 4 is devoted to the study of splitting properties of surjective algebra homomorphisms by means of the characterization of formally smooth algebras in monoidal categories given in [AMS2]. Using the results of Section 3, we prove Theorem 4.7 that can be applied to the case A = H where H itself is a formally smooth algebra in  $\mathfrak{M}_K$  which is endowed with an *ad*-invariant integral (Theorem 4.8). Theorem 4.5 deals with the case when H needs not to have an *ad*-invariant integral but it is formally smooth as an algebra either in  $\mathfrak{M}^H$  or in  ${}^H\mathfrak{M}^H$ .

The main results of Section 5 are contained in Propositions 5.4 and 5.5, were we characterize when H fulfills these properties by means of a suitable map  $\tau: H^+ \to H \otimes H^+$ , where  $H^+$  is the augmentation ideal. Moreover a Hopf algebra H comes out to be formally smooth as a K-algebra if and only if it is formally smooth as an algebra in  $\mathfrak{M}^H$  if and only if it is a hereditary K-algebra (note that a hereditary algebra needs not to be formally smooth as an algebra in general, while the converse is always true). In Theorem 5.8, we apply these facts to the particular case when H is the group algebra KG (compare with [LB, Theorem 2]). The main application is Theorem 5.10 where we prove that given a bialgebra surjection  $\pi: E \to H$  with nilpotent kernel such that H is a Hopf algebra which is formally smooth as a K-algebra, then  $\pi$  has a section which is a right H-colinear algebra homomorphism. The results of this section are used in Section 6 to handle some particular case related to group algebras.

Sections 7, 8 and 9 are devoted to the proof of all dual results.

**Preliminaries and Notation.** In a category  $\mathcal{M}$  the set of morphisms from X to Y will be denoted by  $\mathcal{M}(X, Y)$ . If X is an object in  $\mathcal{M}$ , then the functor  $\mathcal{M}(X, -)$  from  $\mathcal{M}$  to  $\mathfrak{Sets}$  associates to any morphism  $u: U \to V$  in  $\mathcal{M}$  the map that will be denoted by  $\mathcal{M}(X, u)$ . We say that a morphism  $f: X \to Y$  in  $\mathcal{M}$  splits (respectively *cosplits*) or has a section (resp. retraction) in  $\mathcal{M}$  whenever there is a morphism  $g: Y \to X$  such that  $f \circ g = \mathrm{Id}_Y$  (resp.  $g \circ f = \mathrm{Id}_X$ ). In this case we also say that f is a splitting (resp. cospliting) morphism.

Throughout, K is a field and, when working in the category  $\mathfrak{M} = \mathfrak{M}_K$  of vector spaces, we write  $\otimes$  for tensor product over K. We use Sweedler's notation for comultiplications  $\Delta(c) = c_{(1)} \otimes c_{(2)} = c_1 \otimes c_2$ , and the versions  ${}^C\rho(x) = x_{<-1>} \otimes x_{<0>} = x_{-1} \otimes x_0$  and  $\rho^C(x) = x_{<0>} \otimes x_{<1>} = x_0 \otimes x_1$  for left and right comodules respectively (we omit the summation symbol for the sake of brevity).

### 1. Preliminary results

1.1. Monoidal Categories. Throughout this paper, the symbol  $(\mathcal{M}, \otimes, \mathbf{1})$  denotes a strict monoidal category with *unit*  $\mathbf{1} \in \mathcal{M}$  and *tensor product*  $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ . See [Ka, Chap. XI]) for a general reference.

The notions of algebra, module over an algebra, coalgebra and comodule over a coalgebra can be introduced in the general setting of monoidal categories. Given an algebra A in  $\mathcal{M}$  one can define the categories  ${}_{A}\mathcal{M}$ ,  $\mathcal{M}_{A}$  and  ${}_{A}\mathcal{M}_{A}$  of left, right and two-sided modules over A respectively. Similarly, given a coalgebra C in  $\mathcal{M}$ , one can define the categories of C-comodules  ${}^{C}\mathcal{M}, \mathcal{M}^{C}, {}^{C}\mathcal{M}^{C}$ . For more details, the reader is referred to [AMS2].

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The relative tensor and cotensor functors. Let  $(\mathcal{M}, \otimes, \mathbf{1})$  be a monoidal category. Assume that  $\mathcal{M}$  is abelian and let A be an algebra in  $\mathcal{M}$ . It can be proved (see e.g. [Ar]) that  ${}_{A}\mathcal{M}$  is an abelian category, whenever the functor  $A \otimes (-) : \mathcal{M} \to \mathcal{M}$  is additive and right exact. In the case when both the functors  $A \otimes (-) : \mathcal{M} \to \mathcal{M}$  and  $(-) \otimes A : \mathcal{M} \to \mathcal{M}$  are additive and right exact, then the category  ${}_{A}\mathcal{M}_{A}$  is abelian too.

Since, sometimes, we have to work with more than one algebra in  $\mathcal{M}$  and its bimodules, it is convenient to assume that  $X \otimes (-) : \mathcal{M} \to \mathcal{M}$  and  $(-) \otimes X : \mathcal{M} \to \mathcal{M}$  are additive and right exact, for any  $X \in \mathcal{M}$ . Hence we are led to the following definitions.

DEFINITIONS 1.2. Let  $\mathcal{M}$  be a monoidal category.

We say that  $\mathcal{M}$  is an **abelian monoidal category** if  $\mathcal{M}$  is abelian and both the functors  $X \otimes (-)$ :  $\mathcal{M} \to \mathcal{M}$  and  $(-) \otimes X : \mathcal{M} \to \mathcal{M}$  are additive and right exact, for any  $X \in \mathcal{M}$ .

We say that  $\mathcal{M}$  is a **coabelian monoidal category** if  $\mathcal{M}^o$  is an abelian monoidal category, where  $\mathcal{M}^o$  denotes the dual monoidal category of  $\mathcal{M}$ . Recall that  $\mathcal{M}^o$  and  $\mathcal{M}$  have the same objects but  $\mathcal{M}^o(X,Y) = \mathcal{M}(Y,X)$  for any X,Y in  $\mathcal{M}$ .

Given an algebra A in  $\mathcal{M}$ , there exists a suitable functor  $\otimes_A : {}_A\mathcal{M}_A \times {}_A\mathcal{M}_A \to {}_A\mathcal{M}_A$  that makes the category  $({}_A\mathcal{M}_A, \otimes_A, A)$  monoidal (an algebra in this category will be called an A-algebra): see [AMS2, 1.11].

The tensor product over A in  $\mathcal{M}$  of a right A-module V and a left A-module W is defined to be the coequalizer:

Note that, since  $\otimes$  preserves coequalizers, then  $V \otimes_A W$  is also an A-bimodule, whenever V and W are A-bimodules.

Dually, let  $\mathcal{M}$  be a coabelian monoidal category.

Given a coalgebra  $(C, \Delta, \varepsilon)$  in  $\mathcal{M}$ , there exists of a suitable functor  $\Box_C : {}^C \mathcal{M}^C \times {}^C \mathcal{M}^C \to {}^C \mathcal{M}^C$ that makes the category  $({}^C \mathcal{M}^C, \Box_C, C)$  monoidal (a coalgebra in this category will be called a C-coalgebra).

The cotensor product over C in  $\mathcal{M}$  of a right C-bicomodule V and a left C-comodule W is defined to be the equalizer:

 $0 \longrightarrow V \square_C W \xrightarrow{C \leq V, W} V \otimes W \xrightarrow{C \leq V, W} V \otimes C \otimes W$ 

Note that, since  $\otimes$  preserves equalizers, then  $V \square_C W$  is also a *C*-bicomodule, whenever *V* and *W* are *C*-bicomodules.

What follows is a list of the most important monoidal categories meeting our requirements.

**Examples of "good" monoidal categories**. We provide a list of the monoidal categories we need in this paper. They are "good" in the sense that they are (co)abelian monoidal categories.

• The category  $(\mathfrak{M}_K, \otimes_K, K)$  of all vector spaces over a field K.

Let  $(H, m_H, u_H, \Delta_H, \varepsilon_H, S)$  be a Hopf algebra over field K. Then we have the following categories (see [Scha2] for more details).

• The category  ${}_{H}\mathfrak{M} = ({}_{H}\mathfrak{M}, \otimes_{K}, K)$ , of all left modules over H: the unit K is a left H-module via  $\varepsilon_{H}$  and the tensor  $V \otimes W$  of two left H-modules can be regarded as an object in  ${}_{H}\mathfrak{M}$  via the diagonal action. Analogously the category  $\mathfrak{M}_{H}$  can be introduced.

• The category  $_{H}\mathfrak{M}_{H} = (_{H}\mathfrak{M}_{H}, \otimes_{K}, K)$ , of all two-sided modules over H: the unit K is a Hbimodule via  $\varepsilon_{H}$  and the tensor  $V \otimes W$  of two H-bimodules carries, on both sides, the diagonal action.

We can dualize all the structures given for modules in order to obtain categories of comodules.

• The category  ${}^{H}\mathfrak{M} = ({}^{H}\mathfrak{M}, \otimes_{K}, K)$ , of all left comodules over H: the unit K is a left H-comodule via the map  $k \mapsto 1_{H} \otimes k$  and the tensor product  $V \otimes W$  of two left H-comodules can be regarded as an object in  ${}^{H}\mathfrak{M}$  via the codiagonal coaction. Analogously the category  $\mathfrak{M}^{H}$  can be introduced. • The category  ${}^{H}\mathfrak{M}^{H} = ({}^{H}\mathfrak{M}^{H}, \otimes_{K}, K)$  of all two-sided comodules over H: the unit K is a H-bicomodule via the maps  $k \mapsto 1_{H} \otimes k$  and  $k \mapsto k \otimes 1_{H}$ ; the tensor  $V \otimes W$  of two H-bicomodules carries, on both sides, the codiagonal coaction.

As observed, given an algebra A in an abelian monoidal category  $(\mathcal{M}, \otimes, \mathbf{1})$ , we can construct the monoidal category of A-bimodules  $({}_{A}\mathcal{M}_{A}, \otimes_{A}, A)$ . Applying this (in particular for A := H) to the categories  $(\mathfrak{M}_{K}, \otimes_{K}, K), (\mathfrak{M}^{H}, \otimes_{K}, K), ({}^{H}\mathfrak{M}, \otimes_{K}, K)$  and  $({}^{H}\mathfrak{M}^{H}, \otimes_{K}, K)$ , we obtain respectively: •  ${}_{A}\mathfrak{M}_{A} = ({}_{A}\mathfrak{M}_{A}, \otimes_{A}, A), {}_{A}\mathfrak{M}^{H}_{A} = ({}_{A}\mathfrak{M}^{H}, \otimes_{A}, A), {}_{A}^{H}\mathfrak{M}_{A} = ({}_{A}^{H}\mathfrak{M}_{A}, \otimes_{A}, A), {}_{A}^{H}\mathfrak{M}^{H}_{A} = ({}_{A}^{H}\mathfrak{M}^{H}, \otimes_{A}, A).$ 

Given a coalgebra C in a coabelian monoidal category  $(\mathcal{M}, \otimes, \mathbf{1})$ , we can construct the monoidal category of C-bicomodules  $({}^{C}\mathcal{M}^{C}, \Box_{C}, C)$ . Applying this (in particular for C := H) to the categories  $(\mathfrak{M}_{K}, \otimes_{K}, K), (\mathfrak{M}_{H}, \otimes_{K}, K), ({}_{H}\mathfrak{M}, \otimes_{K}, K)$  and  $({}_{H}\mathfrak{M}_{H}, \otimes_{K}, K)$ , we obtain respectively: •  ${}^{C}\mathfrak{M}^{C} = ({}^{C}\mathfrak{M}^{C}, \Box_{C}, C), {}^{C}\mathfrak{M}^{C}_{H} = ({}^{C}\mathfrak{M}^{C}_{H}, \Box_{C}, C), {}^{C}_{H}\mathfrak{M}^{C} = ({}^{C}_{H}\mathfrak{M}^{C}, \Box_{C}, C), {}^{C}_{H}\mathfrak{M}^{C}_{H} = ({}^{C}_{H}\mathfrak{M}^{C}, \Box_{C}, C).$ 

It is well known that  $({}^{H}_{H}\mathfrak{M}^{H}_{H}, \otimes_{H}, H)$  and  $({}^{H}_{H}\mathfrak{M}^{H}_{H}, \Box_{H}, H)$  are equivalent monoidal categories (see [Scha2, Theorem 5.7]).

We now consider the categories of Yetter-Drinfeld modules over H. Recall that a *twisted antipode* for H is an antipode  $\overline{S}$  for  $H^{op}$  (and hence also for  $H^{cop}$ ). One can check that  $S^{-1}$  is a twisted antipode whenever S is bijective. If H is commutative or cocommutative then  $S^2 = S \circ S = \text{Id}_H$  and consequently  $\overline{S} = S$ .

• The category  ${}^{H}_{H}\mathcal{YD} = ({}^{H}_{H}\mathcal{YD}, \otimes_{K}, K)$ , of all left-left Yetter-Drinfeld modules over H: the unit K is a left H-comodule via the map  $k \mapsto 1_{H} \otimes k$  and a left H-module via  $\varepsilon_{H}$ ; the tensor product  $V \otimes W$  of two left-left Yetter-Drinfeld modules can be regarded as an object in  ${}^{H}_{H}\mathcal{YD}$  via the diagonal action and the codiagonal coaction.

Recall that an object V in  ${}^{H}_{H}\mathcal{YD}$  is a left H-module and a left H-comodule satisfying, for any  $h \in H, v \in V$ , the compatibility condition:

 $(h_1v)_{<-1>}h_2 \otimes (h_1v)_{<0>} = h_1v_{<-1>} \otimes h_2v_{<0>}$  or  $(hv)_{<-1>} \otimes (hv)_{<0>} = h_1v_{<-1>}S(h_3) \otimes h_2v_{<0>}$ . Analogously the categories  $\mathcal{YD}_H^H$ ,  $_H\mathcal{YD}^H$  and  $^H\mathcal{YD}_H$  can be defined. The compatibility conditions are respectively:

 $(vh_2)_{<0>} \otimes h_1(vh_2)_{<1>} = v_{<0>}h_1 \otimes v_{<1>}h_2 \text{ or } (vh)_{<0>} \otimes (vh)_{<1>} = v_{<0>}h_2 \otimes S(h_1)v_{<1>}h_3,$ 

 $(h_2v)_{<0>} \otimes (h_2v)_{<1>}h_1 = h_1v_{<0>} \otimes h_2v_{<1>} \text{ or } (hv)_{<0>} \otimes (hv)_{<1>} = h_2v_{<0>} \otimes h_3v_{<1>}\overline{S}(h_1),$ 

 $h_2(vh_1)_{<-1>} \otimes (vh_1)_{<0>} = v_{<-1>}h_1 \otimes v_{<0>}h_2 \text{ or } (vh)_{<-1>} \otimes (vh)_{<0>} = \overline{S}(h_3)v_{<-1>}h_1 \otimes v_{<0>}h_2,$ 

for all  $h \in H, v \in V$  and where in the last two cases the right conditions are available when H has a twisted antipode  $\overline{S}$ .

1.3. Relative Projectivity and Injectivity. A main tool for studying (co)separable and formally smooth (co)algebras is relative projectivity (respectively injectivity). Most of the material introduced below can be found in [HS, Chap. IX, page 307-312] and [We, Chap. 8, page 279-281].

Let  $\mathfrak{C}$  be an arbitrary category and let  $\mathcal{H}$  be a class of homomorphisms in  $\mathfrak{C}$ . An object  $P \in \mathfrak{C}$  is called *f*-projective where  $f: C_1 \to C_2$  is a morphism, if  $\mathfrak{C}(P, f): \mathfrak{C}(P, C_1) \to \mathfrak{C}(P, C_2)$  is surjective. P is  $\mathcal{H}$ -projective if it is *f*-projective for every  $f \in \mathcal{H}$ . Dually, an object  $I \in \mathfrak{C}$  is called *f*-injective, where  $f: C_1 \to C_2$  is a morphism, if and only if, considered as an object in the opposite category  $\mathfrak{C}^{op}$ , it is  $f^{op}$ -projective, where  $f^{op}: C_2 \to C_1$  is in  $\mathfrak{C}^{op}$ . I is called  $\mathcal{H}$ -injective if it is *f*-injective for every  $f \in \mathcal{H}$ .

All the results we will obtain for projectivity, can be dualized to get their analogues for injectivity.

THEOREM 1.4. Let  $\mathbb{H} : \mathfrak{B} \to \mathfrak{A}$  be a covariant functor and consider:

 $\mathcal{E}_{\mathbb{H}} := \{ f \in \mathfrak{B} \mid \mathbb{H}(f) \text{ splits in } \mathfrak{A} \}.$ 

Let  $\mathbb{T} : \mathfrak{A} \to \mathfrak{B}$  be a left adjoint of  $\mathbb{H}$  and let  $\varepsilon : \mathbb{TH} \to \mathrm{Id}_{\mathfrak{B}}$  be the counit of the adjunction. Then, for any object  $P \in \mathfrak{B}$ , the following assertions are equivalent:

(a) P is  $\mathcal{E}_{\mathbb{H}}$ -projective.

(b) Every morphism  $f: B \to P$  in  $\mathcal{E}_{\mathbb{H}}$  has a section.

(c)  $\varepsilon_P : \mathbb{TH}P \to P$  has a section.

(d) There is a splitting morphism  $\pi : \mathbb{T}X \to P$  for a suitable object  $X \in \mathfrak{A}$ . In particular all objects of the form  $\mathbb{T}X$ ,  $X \in \mathfrak{A}$ , are  $\mathcal{E}_{\mathbb{H}}$ -projective.

*Proof.* Let  $\eta : \mathrm{Id}_{\mathfrak{A}} \to \mathbb{HT}$  be the unit of the adjunction.

 $(a) \Rightarrow (b)$ . Assume that  $P \in \mathfrak{B}$  is  $\mathcal{E}_{\mathbb{H}}$ -projective i.e. that for every  $f: B \to B_2$  in  $\mathcal{E}_{\mathbb{H}}$  and for every morphism  $\gamma: P \to B_2$ , there exists a morphism  $\beta: P \to B$  such that  $\gamma = f \circ \beta$ . In particular, for  $B_2 := P$  and  $\gamma := \mathrm{Id}_P$ , there exists a morphism  $\beta: P \to B$  such that  $\mathrm{Id}_P = f \circ \beta$ .

 $(b) \Rightarrow (c)$ . Since  $\mathbb{H}(\varepsilon_B) \circ \eta_{\mathbb{H}B} = \mathrm{Id}_{\mathbb{H}B}$ , we infer that  $\mathbb{H}(\varepsilon_B)$  splits and hence the counit  $\varepsilon_B : \mathbb{T}\mathbb{H}B \to B$  belongs to  $\mathcal{E}_{\mathbb{H}}$  for any  $B \in \mathfrak{B}$ .

 $(c) \Rightarrow (d)$ . Obvious.

 $(d) \Rightarrow (a)$ . Let  $f: B_1 \to B_2$  be in  $\mathcal{E}_{\mathbb{H}}$  and denote by  $g: \mathbb{H}B_2 \to \mathbb{H}B_1$  the section of  $\mathbb{H}(f)$ . Let  $\gamma: P \to B_2$ . Assume that  $\pi: \mathbb{T}X \to P$  is a split morphism for a suitable object  $X \in \mathfrak{A}$ . Let  $\sigma: P \to \mathbb{T}X$  be a section of  $\pi$  and  $\tau: P \to B_1$  be defined by

$$P \xrightarrow{\sigma} \mathbb{T}X \xrightarrow{\mathbb{T}(\eta_X)} \mathbb{T}\mathbb{H}\mathbb{T}X \xrightarrow{\mathbb{T}\mathbb{H}(\pi)} \mathbb{T}\mathbb{H}P \xrightarrow{\mathbb{T}\mathbb{H}(\gamma)} \mathbb{T}\mathbb{H}B_2 \xrightarrow{\mathbb{T}(g)} \mathbb{T}\mathbb{H}B_1 \xrightarrow{\varepsilon_{B_1}} B_1.$$

We have

$$\begin{split} f \circ \tau &= f \circ \varepsilon_{B_1} \circ \mathbb{T}(g) \circ \mathbb{TH}(\gamma) \circ \mathbb{TH}(\pi) \circ \mathbb{T}(\eta_X) \circ \sigma \\ &= \varepsilon_{B_2} \circ \mathbb{T}\left[\mathbb{H}(f) \circ (g)\right] \circ \mathbb{TH}(\gamma \circ \pi) \circ \mathbb{T}(\eta_X) \circ \sigma \\ &= \varepsilon_{B_2} \circ \mathbb{TH}(\gamma \circ \pi) \circ \mathbb{T}(\eta_X) \circ \sigma = \gamma \circ \pi \circ \varepsilon_{\mathbb{T}X} \circ \mathbb{T}(\eta_X) \circ \sigma = \gamma \circ \pi \circ \sigma = \gamma \end{split}$$

and hence P is  $\mathcal{E}_{\mathbb{H}}$ -projective.

Since  $\mathrm{Id}_{\mathbb{T}X} : \mathbb{T}X \to \mathbb{T}X$  is an isomorphism, by  $(d) \Rightarrow (a)$ , we have that  $\mathbb{T}X$  is  $\mathcal{E}_{\mathbb{H}}$ -projective.  $\Box$ 

For completeness we include the dual statement of Theorem 1.4.

THEOREM 1.5. Let  $\mathbb{T}: \mathfrak{A} \to \mathfrak{B}$  be a covariant functor and consider:

 $\mathcal{I}_{\mathbb{T}} := \{ g \in \mathfrak{A} \mid \mathbb{T}(g) \text{ cosplits in } \mathfrak{B} \}.$ 

Let  $\mathbb{H} : \mathfrak{B} \to \mathfrak{A}$  be a right adjoint of  $\mathbb{T}$  and let  $\eta : \mathrm{Id}_{\mathfrak{A}} \to \mathbb{HT}$  be the unit of the adjunction. Then, for any object  $I \in \mathfrak{A}$ , the following assertions are equivalent:

(a) I is  $\mathcal{I}_{\mathbb{T}}$ -injective.

- (b) Every morphism  $f: I \to A$  in  $\mathcal{I}_{\mathbb{T}}$  has a retraction.
- (c)  $\eta_I : I \to \mathbb{HT}I$  has a retraction.
- (d) There is a cosplitting morphism  $i: I \to \mathbb{H}Y$  for a suitable object  $Y \in \mathfrak{B}$ .

In particular all objects of the form  $\mathbb{H}Y, Y \in \mathfrak{B}$ , are  $\mathcal{I}_{\mathbb{T}}$ -injective.

1.6. Separable Functors. Let  $\mathbb{U}: \mathfrak{B} \to \mathfrak{A}$  be a covariant functor. We have functors

$$Hom_{\mathfrak{B}}(\bullet, \bullet), Hom_{\mathfrak{A}}(\mathbb{U}(\bullet), \mathbb{U}(\bullet)) : \mathfrak{B}^{op} \times \mathfrak{B} \to \mathfrak{Sets}$$

and a natural transformation

$$\mathcal{U}: Hom_{\mathfrak{B}}(\bullet, \bullet) \to Hom_{\mathfrak{A}}(\mathbb{U}(\bullet), \mathbb{U}(\bullet)), \quad \mathcal{U}_{B_1, B_2}(f) := \mathbb{U}(f) \text{ for all objects } B_1, B_2 \in \mathfrak{B}.$$

The functor  $\mathbb{U}$  is called *separable* if  $\mathcal{U}$  cosplits, that is there is a natural transformation

$$\mathcal{P}: Hom_{\mathfrak{A}}(\mathbb{U}(\bullet), \mathbb{U}(\bullet)) \to Hom_{\mathfrak{B}}(\bullet, \bullet)$$

such that  $\mathcal{P} \circ \mathcal{U} = \mathbf{1}_{Hom_{\mathfrak{B}}(\bullet, \bullet)}$ , the identity natural transformation on  $Hom_{\mathfrak{B}}(\bullet, \bullet)$ . It is proved in [Raf, page 1446] that this definition is consistent with the one given in [NVV].

REMARK 1.7. Let  $\alpha : X \to Y$  be a morphism in  $\mathfrak{B}$ . If  $\mathbb{U}$  is a faithful functor, then,  $\alpha$  is an epimorphism (resp. monomorphism) whenever  $\mathbb{U}(\alpha)$  is.

Let us recall some well known property on separable functors.

LEMMA 1.8. [NVV, Proposition 1.2] Let  $\mathbb{U} : \mathfrak{B} \to \mathfrak{A}$  be a covariant separable functor and let  $\alpha : X \to Y$  be a morphism in  $\mathfrak{B}$ . If  $\mathbb{U}(\alpha)$  has a section h (resp. a retraction l) in  $\mathfrak{A}$ , then  $\alpha$  has a section (retraction) in  $\mathfrak{B}$ .

LEMMA 1.9. Let  $F : \mathfrak{A} \to \mathfrak{B}$  and  $G : \mathfrak{B} \to \mathfrak{C}$  be covariant functors. Then  $\mathcal{E}_F \subseteq \mathcal{E}_{GF}$  and  $\mathcal{I}_F \subseteq \mathcal{I}_{GF}$ . Moreover the equalities hold whenever G is separable.

THEOREM 1.10. Consider functors  $\mathbb{T}: \mathfrak{A} \to \mathfrak{B}$  and  $\mathbb{H}: \mathfrak{B} \to \mathfrak{C}$ . Then, we have that:

- 1) If  $\mathbb{T}$  and  $\mathbb{H}$  are separable, then  $\mathbb{H} \circ \mathbb{T}$  is also separable.
- 2) If  $\mathbb{H} \circ \mathbb{T}$  is separable, then  $\mathbb{T}$  is separable.
- 3) If  $\mathfrak{C} = \mathfrak{A}$  and that  $(\mathbb{T}, \mathbb{H})$  is a category equivalence, then  $\mathbb{T}$  and  $\mathbb{H}$  are both separable.

*Proof.* See [CMZ, Proposition 46 and Corollary 9].

We quote from [Raf] the so-called Rafael Theorem:

THEOREM 1.11. [Raf, Theorem 1.2] Let  $(\mathbb{T}, \mathbb{H})$  be an adjunction, where  $\mathbb{T} : \mathfrak{A} \to \mathfrak{B}$  and  $\mathbb{H} : \mathfrak{B} \to \mathfrak{A}$ . Then we have:

1)  $\mathbb{T}$  is separable if and only if the unit  $\eta : \mathrm{Id}_{\mathfrak{A}} \to \mathbb{HT}$  of the adjunction cosplits, i.e. there exists a natural transformation  $\mu : \mathbb{HT} \to \mathrm{Id}_{\mathfrak{A}}$  such that  $\mu \circ \eta = \mathrm{Id}_{\mathrm{Id}_{\mathfrak{A}}}$ , the identity natural transformation on  $\mathrm{Id}_{\mathfrak{A}}$ .

2)  $\mathbb{H}$  is separable if and only if the counit  $\varepsilon : \mathbb{TH} \to \mathrm{Id}_{\mathfrak{B}}$  of the adjunction splits, i.e. there exists a natural transformation  $\sigma : \mathrm{Id}_{\mathfrak{B}} \to \mathbb{TH}$  such that  $\varepsilon \circ \sigma = \mathrm{Id}_{\mathrm{Id}_{\mathfrak{B}}}$ , the identity natural transformation on Id<sub>B</sub>.

COROLLARY 1.12. Let  $(\mathbb{T}, \mathbb{H})$  be an adjunction, where  $\mathbb{T} : \mathfrak{A} \to \mathfrak{B}$  and  $\mathbb{H} : \mathfrak{B} \to \mathfrak{A}$ . Then we have: 1)  $\mathbb{H}$  separable  $\Rightarrow$  any object in  $\mathfrak{B}$  is  $\mathcal{E}_{\mathbb{H}}$ -projective.

2)  $\mathbb{T}$  separable  $\Rightarrow$  any object in  $\mathfrak{A}$  is  $\mathcal{I}_{\mathbb{T}}$ -injective.

*Proof.* 1) Let B be an object in  $\mathfrak{B}$ . Since  $\mathbb{H}(\varepsilon_B) \circ \eta_{\mathbb{H}B} = \mathrm{Id}_{\mathbb{H}B}$  and  $\mathbb{H}$  is separable, by Lemma 1.8,  $\varepsilon_B$  has a section in  $\mathfrak{B}$ . By Theorem 1.4, B is  $\mathcal{E}_{\mathbb{H}}$ -projective.

2) It follows analogously by Lemma 1.8 and Theorem 1.5 once observed that  $\varepsilon_{\mathbb{T}A} \circ \mathbb{T}(\eta_A) = \mathrm{Id}_{\mathbb{T}A}$ for any  $A \in \mathfrak{A}$ .

We are now ready to prove the main theorem of this section, that investigates whether a functor F (resp. F') preserves and reflects relative projective (resp. injective) objects.

THEOREM 1.13. Let  $(\mathbb{T}, \mathbb{H})$  and  $(\mathbb{T}', \mathbb{H}')$  be adjunctions and assume that, in the following diagrams,  $\mathbb{T}' \circ F'$  and  $F \circ \mathbb{T}$  (and also  $F' \circ \mathbb{H}$  and  $\mathbb{H}' \circ F$ ) are naturally equivalent:



Let P be an object in  $\mathfrak{B}$  and let I be an object in  $\mathfrak{A}$ . We have:

a) P is  $\mathcal{E}_{\mathbb{H}}$ -projective  $\Longrightarrow$  F(P) is  $\mathcal{E}_{\mathbb{H}'}$ -projective; the converse is true whenever F is separable.  $a^{op}$ ) I is  $\mathcal{I}_{\mathbb{T}}$ -injective  $\Longrightarrow$  F'(I) is  $\mathcal{I}_{\mathbb{T}'}$ -injective; the converse is true whenever F' is separable.

*Proof.* a) Let  $\varepsilon : \mathbb{TH} \to \mathrm{Id}_{\mathfrak{B}}$  be the counit of the adjunction  $(\mathbb{T}, \mathbb{H})$ .

Assume that P is  $\mathcal{E}_{\mathbb{H}}$ -projective. Then, by Theorem 1.4,  $\varepsilon_P : \mathbb{TH}P \to P$  has a section  $\beta : P \to \mathbb{TH}P$ , i.e.  $\varepsilon_P \circ \beta = \mathrm{Id}_P$ . Since  $F(\beta)$  is a section of  $F(\varepsilon_P) : \mathbb{T}'\mathbb{H}'FP \sim F\mathbb{T}\mathbb{H}P \to FP$ , by applying Theorem 1.4 to the adjunction  $(\mathbb{T}', \mathbb{H}')$  in the case when  $X = \mathbb{H}' F P$  and to the split morphism  $F(\varepsilon_P)$ , we conclude that FP is  $\mathcal{E}_{\mathbb{H}'}$ -projective.

Conversely, assume that FP is  $\mathcal{E}_{\mathbb{H}'}$ -projective and that F is separable. Let  $\eta : \mathrm{Id}_{\mathfrak{B}} \to \mathbb{HT}$  be the unit of the adjunction  $(\mathbb{T},\mathbb{H})$ . Thus  $\mathbb{H}(\varepsilon_P) \circ \eta_{\mathbb{H}P} = \mathrm{Id}_{\mathbb{H}P}$  and hence  $F'(\eta_{\mathbb{H}P})$  is a section of  $F'\mathbb{H}(\varepsilon_P)$ . Then also  $\mathbb{H}'F(\varepsilon_P)$  has a section, so that  $F(\varepsilon_P): F\mathbb{T}\mathbb{H}P \to FP$  belongs to  $\mathcal{E}_{\mathbb{H}'}$ . As FP is  $\mathcal{E}_{\mathbb{H}'}$ -projective, by Theorem 1.4, we get a section in  $\mathfrak{B}'$  of  $F(\varepsilon_P)$ . Since F is separable, by Lemma 1.8, we conclude that  $\varepsilon_P$  splits in  $\mathfrak{B}$ : hence P is  $\mathcal{E}_{\mathbb{H}}$ -projective.  $a^{op}$ ) It follows by duality. 

# 2. (CO)SEPARABLE AND FORMALLY SMOOTH (CO)ALGEBRAS

2.1. Let (A, m, u) be an algebra in a monoidal category  $(\mathcal{M}, \otimes, \mathbf{1})$ . We have the functors

$${}_{A}\mathbb{T}: \mathcal{M} \to {}_{A}\mathcal{M} \text{ where } {}_{A}\mathbb{T}(X) := A \otimes X \text{ and } {}_{A}\mathbb{T}(f) := A \otimes f_{A}$$

$$\mathbb{T}_A: \mathcal{M} \to \mathcal{M}_A \text{ where } \mathbb{T}_A(X) := X \otimes A \text{ and } \mathbb{T}_A(f) := f \otimes A,$$

 ${}_{A}\mathbb{T}_{A}: \mathcal{M} \to {}_{A}\mathcal{M}_{A} \text{ where } {}_{A}\mathbb{T}_{A}(X) := A \otimes (X \otimes A) \text{ and } {}_{A}\mathbb{T}_{A}(f) := A \otimes (f \otimes A),$ 

with their right adjoint (see [AMS2, Proposition 1.6])  ${}_{A}\mathbb{H}, \mathbb{H}_{A}, {}_{A}\mathbb{H}_{A}$ , respectively, that forget the module structures. Then the adjunctions  $(\mathbb{T}_{A}, \mathbb{H}_{A})$ ,  $({}_{A}\mathbb{T}, {}_{A}\mathbb{H})$  and  $({}_{A}\mathbb{T}_{A}, {}_{A}\mathbb{H}_{A})$ , give rise to the following classes:

$$\mathcal{E}_{A} := \mathcal{E}_{\mathbb{H}_{A}} = \{g \in \mathcal{M}_{A} \mid g \text{ splits in } \mathcal{M}\},\$$
$$_{A}\mathcal{E} := \mathcal{E}_{A}\mathbb{H} = \{g \in {}_{A}\mathcal{M} \mid g \text{ splits in } \mathcal{M}\},\$$
$$_{A}\mathcal{E}_{A} := \mathcal{E}_{A}\mathbb{H}_{A} = \{g \in {}_{A}\mathcal{M}_{A} \mid g \text{ splits in } \mathcal{M}\}$$

Recall that an algebra (A, m, u) is called *separable* in  $\mathcal{M}$  whenever the multiplication m admits a section  $A \to A \otimes A$  in  ${}_{A}\mathcal{M}_{A}$ .

Assume that  $\mathcal{M}$  is an abelian monoidal category. Then  $(\Omega^1 A, j) := \ker m$  carries a natural Abimodule structure that makes it the kernel of m in the category  ${}_A\mathcal{M}_A$ . We say that A is formally smooth in  $\mathcal{M}$  (see [AMS2, Corollary 3.12]) if and only if  $\Omega^1 A$  is an  ${}_A\mathcal{E}_A$ -projective A-bimodule.

Let us recall the following result that holds true for unitary rings.

PROPOSITION 2.2. [NVV, Proposition 1.3] For any ring homomorphism  $i: S \to R$ , the following are equivalent:

- (1) R is separable in  $({}_{S}\mathfrak{M}_{S}, \otimes_{S}, S)$ , i.e. R/S is separable.
- (2) The restriction of scalars functor  ${}_{R}\mathfrak{M} \to {}_{S}\mathfrak{M}$  is separable.
- (3) The restriction of scalars functor  $\mathfrak{M}_R \to \mathfrak{M}_S$  is separable.

As we will explain in Remark 2.5, the previous result, in general, can not be extended to algebras in a monoidal category.

LEMMA 2.3. Let A be a separable algebra in a monoidal category  $\mathcal{M}$ . The following assertions hold true:

1) The forgetful functor  ${}_{A}\mathbb{H} : {}_{A}\mathcal{M} \to \mathcal{M}$  is separable. In particular, any left A-module  $(M, {}^{A}\mu_{M})$  is  ${}_{A}\mathcal{E}$ -projective. Moreover if M is an A-bimodule, the multiplication  ${}^{A}\mu_{M} : A \otimes M \to M$  has a section  ${}^{A}\sigma_{M}$  which is A-bilinear and natural in M.

2) The forgetful functor  $\mathbb{H}_A : \mathcal{M}_A \to \mathcal{M}$  is separable. In particular, any right A-module  $(M, \mu_M^A)$  is  $\mathcal{E}_A$ -projective. Moreover if M is an A-bimodule, the multiplication  $\mu_M^A : M \otimes A \to M$  has a section  $\sigma_M^A$  which is A-bilinear and natural in M.

Proof. 1) By assumption, the multiplication m of A admits a section  $\nu : A \to A \otimes A$  in  ${}_{A}\mathcal{M}_{A}$ . Let  $(M, {}^{A}\mu_{M})$  be a left A-module and consider the morphism  ${}^{A}\sigma_{M} : M \to A \otimes M$  defined by  ${}^{A}\sigma_{M} := (A \otimes {}^{A}\mu_{M}) \circ (\nu u \otimes M) \circ l_{M}^{-1}$ , where  $u : \mathbf{1} \to A$  is the unit of A. It is straightforward to check that  ${}^{A}\sigma_{M}$  is a left A-linear section of  ${}^{A}\mu_{M}$  which is A-bilinear whenever  $M \in {}_{A}\mathcal{M}_{A}$  (see the left handed version of [AMS2, Lemma 1.29]). Since  ${}^{A}\mu$  is the counit of the adjunction  $({}_{A}\mathbb{T}, {}_{A}\mathbb{H})$ , and  ${}^{A}\sigma_{M}$  defines a natural transformation  ${}^{A}\sigma : \mathrm{Id}_{A}\mathcal{M} \to ({}_{A}\mathbb{T}) ({}_{A}\mathbb{H})$ , we get, by Theorem 1.11, that  ${}_{A}\mathbb{H}$  is separable. Note that, by Corollary 1.12, if the forgetful functor  ${}_{A}\mathbb{H} : {}_{A}\mathcal{M} \to \mathcal{M}$  is separable, then any left A-module is  ${}_{A}\mathcal{E}$ -projective.

2) It follows analogously.

PROPOSITION 2.4. Let H be a Hopf algebra over a field K. The forgetful functors  $\mathfrak{M}_{H}^{H} \to \mathfrak{M}^{H}$  and  ${}^{H}\mathfrak{M}_{H}^{H} \to {}^{H}\mathfrak{M}^{H}$  are separable.

*Proof.* Composing the functor  $(-)^{coH} : \mathfrak{M}^H \to \mathfrak{M}_K$  with the forgetful functor  $\mathfrak{M}_H^H \to \mathfrak{M}^H$ , one gets the Sweedler's equivalence of categories  $(-)^{coH} : \mathfrak{M}_H^H \to \mathfrak{M}_K$ . Since, by Theorem 1.11, this functor is separable, by Theorem 1.10, the forgetful functor  $\mathfrak{M}_H^H \to \mathfrak{M}^H$  is separable too.

Composing the functor  $(-)^{coH} : {}^{H}\mathfrak{M}^{H} \to {}^{H}\mathfrak{M}$  with the forgetful functor  ${}^{H}\mathfrak{M}^{H}_{H} \to {}^{H}\mathfrak{M}^{H}$ , one gets the Sweedler's equivalence of categories  $(-)^{coH} : {}^{H}\mathfrak{M}^{H}_{H} \to {}^{H}\mathfrak{M}$ . As in the first part, we conclude that the forgetful functor  ${}^{H}\mathfrak{M}^{H}_{H} \to {}^{H}\mathfrak{M}^{H}$  is separable.

REMARK 2.5. By Lemma 2.3, the forgetful functor  $\mathbb{H}_A : \mathcal{M}_A \to \mathcal{M}$  is separable for any separable algebra A in a monoidal category  $\mathcal{M}$ . The converse does not hold true. In fact, when  $\mathcal{M} = \mathfrak{M}^H$  and A = H, the functor  $\mathbb{H}_A$  is always separable (Proposition 2.4), but A is separable in  $\mathcal{M}$  if and only if H is a semisimple algebra ([AMS1, Proposition 2.11]).

**PROPOSITION 2.6.** Let A be an algebra in a monoidal category  $\mathcal{M}$ . The following assertions are equivalent:

- (a) A is separable in  $\mathcal{M}$ .
- (b) The forgetful functor  ${}_{A}\mathbb{H}_{A} : {}_{A}\mathcal{M}_{A} \to \mathcal{M}$  is separable.
- (c) Any A-bimodule is  ${}_{A}\mathcal{E}_{A}$ -projective.
- (d) The A-bimodule A is  ${}_{A}\mathcal{E}_{A}$ -projective.

Proof. (a)  $\Rightarrow$  (b) If  $(M, {}^{A}\mu_{M}, \mu_{M}^{A})$  is an A-bimodule, by Lemma 2.3, there are A-bilinear natural sections  ${}^{A}\sigma_{M}$  and  $\sigma_{M}^{A}$ , respectively of  ${}^{A}\mu_{M}$  and  $\mu_{M}^{A}$ . The morphism  $\sigma_{M} := ({}^{A}\sigma_{M} \otimes A) \circ \sigma_{M}^{A}$ :  $M \to A \otimes M \otimes A$  is a section in  ${}_{A}\mathcal{M}_{A}$  of the counit  $\varepsilon_{M} := \mu_{M}^{A} \circ ({}^{A}\mu_{M} \otimes A) : A \otimes M \otimes A \to M$  of the adjunction  $({}_{A}\mathbb{T}_{A,A}\mathbb{H}_{A})$ . Since  $\sigma_{M}$  is natural in M, we get a natural transformation  $\sigma : \mathrm{Id}_{\mathcal{M}} \to {}_{A}\mathbb{T}_{AA}\mathbb{H}_{A}$  such that  $\varepsilon \circ \sigma = \mathrm{Id}_{\mathrm{Id}_{M}}$ . We conclude by Theorem 1.11.

 $(b) \Rightarrow (c)$  It follows by Corollary 1.12.

 $(c) \Rightarrow (d)$  Obvious.

 $(d) \Rightarrow (a)$  Since A is  ${}_{A}\mathcal{E}_{A}$ -projective, the multiplication  $m : A \otimes A \to A$ , that is a morphism in  ${}_{A}\mathcal{E}_{A}$ , admits a section  $\sigma : A \to A \otimes A$  in  ${}_{A}\mathcal{M}_{A}$ .

COROLLARY 2.7. Any separable algebra in an abelian monoidal category  $\mathcal{M}$  is formally smooth.

COROLLARY 2.8. Let A be a separable algebra in  $\mathfrak{M}_K$ . Then any left A-module is projective in  ${}_A\mathfrak{M}$ . Hence any left A-module is also injective in  ${}_A\mathfrak{M}$  and A is semisimple. Moreover any A-bimodule is projective in  ${}_A\mathfrak{M}_A$  and hence any A-bimodule is injective in  ${}_A\mathfrak{M}_A$ .

*Proof.* Since  $\mathcal{M} = \mathfrak{M}_K$ , any epimorphism in  $\mathcal{M}$  splits. So a left A-module is  ${}_{A}\mathcal{E}$ -projective if and only if it is projective in  ${}_{A}\mathfrak{M}$  in the usual sense. The right and two-sided cases follow analogously.

2.9. Let  $(F', \phi_0, \phi_2) : (\mathcal{M}, \otimes, \mathbf{1}) \to (\mathcal{M}', \otimes', \mathbf{1}')$  be a monoidal functor between two monoidal categories, where  $\phi_2(U, V) : F'(U \otimes V) \to F'(U) \otimes' F'(V)$ , for any  $U, V \in \mathcal{M}$ , and  $\phi_0 : \mathbf{1}' \to F'(\mathbf{1})$ . Let (A, m, u) be an algebra in  $\mathcal{M}$ . It is well known that  $(A', m_{A'}, u_{A'}) := (F'(A), m_{F'(A)}, u_{F'(A)})$  is an algebra in  $\mathcal{M}'$ , where

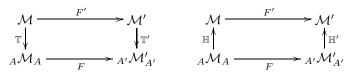
$$m_{F'(A)} := F'(A) \otimes' F'(A) \xrightarrow{\phi_2(A,A)} F'(A \otimes A) \xrightarrow{F'(m)} F'(A)$$
$$u_{F'(A)} := \mathbf{1}' \xrightarrow{\phi_0} F'(\mathbf{1}) \xrightarrow{F'(u)} F'(A).$$

Consider the functor  $F: {}_{A}\mathcal{M}_{A} \to {}_{A'}\mathcal{M}'_{A'}$  defined by  $F((M, {}^{A}\mu_{M}, \mu_{M}^{A})) = (F'(M), {}^{A'}\mu_{F'(M)}, \mu_{F'(M)}^{A'})$ , where

$${}^{A'}\mu_{F'(M)} := F'(A) \otimes' F'(M) \xrightarrow{\phi_2(A,M)} F'(A \otimes M) \xrightarrow{F'(^A\mu_M)} F'(M)$$
  
$$\mu_{F'(M)}^{A'} := F'(M) \otimes' F'(A) \xrightarrow{\phi_2(M,A)} F'(M \otimes A) \xrightarrow{F'(\mu_M^A)} F'(M).$$

Let us study a particular case of Theorem 1.13.

PROPOSITION 2.10. Let  $\mathcal{M}$  and  $\mathcal{M}'$  be abelian monoidal categories. Let A, A', F' and F as in 2.9. Then, in the following diagrams,  $\mathbb{T}' \circ F'$  and  $F \circ \mathbb{T}$  are naturally equivalent and  $F' \circ \mathbb{H} = \mathbb{H}' \circ F$ :



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where  $(\mathbb{T}, \mathbb{H})$  is the adjunction  $({}_{A}\mathbb{T}_{A}, {}_{A}\mathbb{H}_{A})$  defined in 2.1, and  $(\mathbb{T}', \mathbb{H}')$  is analogously defined. We have that:

 $P \in {}_{A}\mathcal{M}_{A}$  is  $\mathcal{E}_{\mathbb{H}}$ -projective  $\Longrightarrow$   $F(P) \in {}_{A'}\mathcal{M}'_{A'}$  is  $\mathcal{E}_{\mathbb{H}'}$ -projective; the converse is true whenever F is separable.

In particular we obtain that:

i) A is separable in  $\mathcal{M} \Longrightarrow \mathcal{A}'$  is separable in  $\mathcal{M}'$  (i.e.  $\mathbb{H}$  is separable  $\Longrightarrow \mathbb{H}'$  is separable); the converse is true whenever F is separable.

ii) If F' preserves kernels, then: A is formally smooth in  $\mathcal{M} \Longrightarrow A'$  is formally smooth in  $\mathcal{M}'$ ; the converse is true whenever F is separable.

Proof. Define  $\alpha_M : F'(A) \otimes' F'(M) \otimes' F'(A) \to F'(A \otimes M \otimes A)$  by  $\alpha_M = \phi_2(A \otimes M, A) [\phi_2(A, M) \otimes' F'(A)]$ , for any  $M \in \mathcal{M}$ . Then  $(\alpha_M)_{M \in \mathcal{M}}$  defines a natural equivalence  $\alpha : \mathbb{T}'\mathbb{F}' \to \mathbb{FT}$ . The first assertion holds by Theorem 1.13.

i) By Proposition 2.6, A is separable in  $\mathcal{M}$  if and only if  $A \in {}_{\mathcal{A}}\mathcal{M}_A$  is  $\mathcal{E}_{\mathbb{H}}$ -projective if and only if the functor  $\mathbb{H}$  is separable. Analogously A' is separable in  $\mathcal{M}'$  if and only if  $A' \in {}_{A'}\mathcal{M}'_{A'}$  is  $\mathcal{E}_{\mathbb{H}'}$ -projective if and only if the functor  $\mathbb{H}'$  is separable. Since A' = F(A), we conclude by the first part.

ii) Let  $(\Omega^1(A), j) = \ker(m_A)$  in  $\mathcal{M}$ . Since F' preserves kernels, we get that

$$(\Omega^{1}(A'), j') := \ker(m_{A'}) = (F'(\Omega^{1}(A), \phi_{2}(A, A)F'(j)))$$

in  $\mathcal{M}'$ . Observe that,  $\Omega^1(A') = \ker(m_{A'}) = \ker[F'(m)\phi_2(A, A)]$ . Now, if we regard regard  $\Omega^1(A)$  as an A-bimodule via the structures induced by  $m_A$  and  $\Omega^1(A')$  as an A'-bimodule via the structures induced by  $m_{A'}$ , we obtain that  $\Omega^1(A') = F(\Omega^1(A))$ .

By definition, A is formally smooth in  $\mathcal{M}$  if and only if  $\Omega^1 A \in {}_A\mathcal{M}_A$  is  $\mathcal{E}_{\mathbb{H}}$ -projective. Analogously A' is formally smooth in  $\mathcal{M}'$  if and only if  $\Omega^1(A') \in {}_{A'}\mathcal{M}'_{A'}$  is  $\mathcal{E}_{\mathbb{H}'}$ -projective. Since  $\Omega^1(A') = F(\Omega^1(A))$ , we conclude by the first part.  $\Box$ 

EXAMPLES 2.11. Let H be a Hopf algebra over a field K. With hypotheses and notations of Proposition 2.10, let  $\mathcal{M}' := \mathfrak{M}_K$ . We want to apply the previous result to the particular case when  $\mathcal{M}$  is either  $({}^H\mathfrak{M}^H, \otimes, K)$  or  $(\mathfrak{M}^H, \otimes, K)$ . Let A be an algebra in  $\mathcal{M}$ .

1)  $\mathcal{M} := {}^{H}\mathfrak{M}^{H}$ . The forgetful functor  $F_{1} : {}^{H}_{A}\mathfrak{M}_{A}^{H} \to {}_{A}\mathfrak{M}_{A}$  has a right adjoint  $G_{1} : {}_{A}\mathfrak{M}_{A} \to {}^{H}_{A}\mathfrak{M}_{A}^{H}$ ,  $G_{1}(M) = H \otimes M \otimes H$ , where  $G_{1}(M)$  is a bicomodule via  $\Delta_{H} \otimes M \otimes H$  and  $H \otimes M \otimes \Delta_{H}$ , and it is a bimodule with diagonal actions. For any  $M \in {}^{H}_{A}\mathfrak{M}_{A}^{H}$  the unit of the adjunction is the map  $\eta_{M} : M \to H \otimes M \otimes H, \eta_{M} = ({}^{H}\rho_{M} \otimes H) \circ \rho_{M}^{H}.$ 2)  $\mathcal{M} := \mathfrak{M}^{H}$ . The forgetful functor  $F_{r} : {}_{A}\mathfrak{M}_{A}^{H} \to {}_{A}\mathfrak{M}_{A}$  has a right adjoint  $G_{r} : {}_{A}\mathfrak{M}_{A} \to {}_{A}\mathfrak{M}_{A}^{H}$ ,

2)  $\mathcal{M} := \mathfrak{M}^H$ . The forgetful functor  $F_r : {}_A\mathfrak{M}_A^H \to {}_A\mathfrak{M}_A$  has a right adjoint  $G_r : {}_A\mathfrak{M}_A \to {}_A\mathfrak{M}_A^H$ ,  $G_r(M) = M \otimes H$ , where  $G_r(M)$  is a comodule via  $M \otimes \Delta_H$ , and it is a bimodule with diagonal actions. For any  $M \in {}_A\mathfrak{M}_A^H$  the unit of the adjunction is the map  $\eta_M : M \to M \otimes H$ ,  $\eta_M = \rho_M^H$ . In the case A = H we set  $(F_2, G_2) := (F_1, G_1)$ .

The forgetful functor  $F_b: {}^{H}_{H}\mathfrak{M}_{H}^{H} \to {}^{H}_{H}\mathfrak{M}_{H}^{H}$  has a right adjoint  $G_b: {}^{H}_{H}\mathfrak{M}_{H}^{H} \to {}^{H}_{H}\mathfrak{M}_{H}^{H}, G_b(M) = H \otimes M$ , where  $G_b(M)$  is a bicomodule via  $\Delta_H \otimes M$  and  $M \otimes \rho_M^H$ , and it is a bimodule with diagonal action. The forgetful functor  $F_a: {}^{H}_{H}\mathfrak{M}_{H}^{H} \to {}^{H}_{H}\mathfrak{M}_{H}$ , is nothing but  $F_r$  in the case A = H. Then it has a right adjoint  $G_a: {}^{H}_{H}\mathfrak{M}_{H} \to {}^{H}_{H}\mathfrak{M}_{H}^{H}$ , which is  $G_r$  for A = H.

Note that the forgetful functor  $F_2: {}^H_H\mathfrak{M}^H_H \to {}^H\mathfrak{M}_H$  can be decomposed as  $F_2 = F_a \circ F_b$ .

In view of Examples 2.11, we obtain the following important result:

THEOREM 2.12. Let H be a Hopf algebra over a field K and let  $\mathcal{M}$  denote either  ${}^{H}\mathfrak{M}{}^{H}$  or  $\mathfrak{M}{}^{H}$ . Let A be an algebra in  $\mathcal{M}$  and consider the forgetful functors  $\mathbb{H} : {}_{A}\mathcal{M}_{A} \to \mathcal{M}, \ \mathbb{H}' : {}_{A}\mathfrak{M}_{A} \to \mathfrak{M}_{K}$ and  $F : {}_{A}\mathcal{M}_{A} \to {}_{A}\mathfrak{M}_{A}$ .

We have that:

 $P \in {}_{A}\mathcal{M}_{A}$  is  $\mathcal{E}_{\mathbb{H}}$ -projective  $\Longrightarrow$  P is  $\mathcal{E}_{\mathbb{H}'}$ -projective as an object in  ${}_{A}\mathfrak{M}_{A}$ ; the converse is true whenever F is separable.

In particular we obtain that:

i) A is separable as an algebra in  $\mathcal{M} \Longrightarrow A$  is separable as an algebra in  $\mathfrak{M}_K$ ; the converse is true whenever F is separable.

ii) A is formally smooth as an algebra in  $\mathcal{M} \Longrightarrow A$  is formally smooth as an algebra in  $\mathfrak{M}_K$ ; the converse is true whenever F is separable.

*Proof.* Apply Proposition 2.10 in the case when  $\mathcal{M}' = \mathfrak{M}_K$ , and  $F' : \mathcal{M} \to \mathfrak{M}_K$  is the forgetful functor.

Dually we have.

2.13. Let  $(C, \Delta, \varepsilon)$  be a coalgebra in a monoidal category  $(\mathcal{M}, \otimes, \mathbf{1})$ . Like in the dual case, we have the functors

$${}^{C}\mathbb{H}: \mathcal{M} \to {}^{C}\mathcal{M} \text{ where } {}^{C}\mathbb{H}(X) := C \otimes X \text{ and } {}^{C}\mathbb{H}(f) := C \otimes f,$$
$$\mathbb{H}^{C}: \mathcal{M} \to \mathcal{M}^{C} \text{ where } \mathbb{H}^{C}(X) := X \otimes C \text{ and } \mathbb{H}^{C}(f) := f \otimes C,$$
$${}^{C}\mathbb{H}^{C}: \mathcal{M} \to {}^{C}\mathcal{M}^{C} \text{ where } {}^{C}\mathbb{H}^{C}(X) := C \otimes (X \otimes C) \text{ and } {}^{C}\mathbb{H}^{C}(f) := C \otimes (f \otimes C),$$

with their left adjoint  ${}^{C}\mathbb{T}, \mathbb{T}^{C}, {}^{C}\mathbb{T}^{C}$ , respectively, that forget the comodule structures. Then the adjunctions  $({}^{C}\mathbb{T}, {}^{C}\mathbb{H}), (\mathbb{T}^{C}, \mathbb{H}^{C})$  and  $({}^{C}\mathbb{T}^{C}, {}^{C}\mathbb{H}^{C})$  gives rise to the following classes:

$${}^{C}\mathcal{I} := \mathcal{I}_{{}^{C}\mathbb{T}} = \{g \in {}^{C}\mathcal{M} \mid g \text{ is a cosplits in } \mathcal{M}\},\ \mathcal{I}^{C} := \mathcal{I}_{\mathbb{T}^{C}} = \{g \in \mathcal{M}^{C} \mid g \text{ is a cosplits in } \mathcal{M}\},\ {}^{C}\mathcal{I}^{C} := \mathcal{I}_{{}^{C}\mathbb{T}^{C}} = \{g \in {}^{C}\mathcal{M}^{C} \mid g \text{ is a cosplits in } \mathcal{M}\}.$$

By duality we can obtain the definition of coseparability and formal smoothness for a coalgebra  $(C, \Delta, \varepsilon)$  in a monoidal category  $\mathcal{M}$ . We say that C is *coseparable* whenever the comultiplication  $\Delta$  cosplits in  ${}^{C}\mathcal{M}^{C}$ .

Assume that  $\mathcal{M}$  is a coabelian monoidal category. Then  $\mathfrak{V}_1C := \operatorname{Coker}\Delta_C$  carries a natural C-bicomodule structure that makes it the cokernel of  $\Delta$  in the category  ${}^C\mathcal{M}^C$ . We say that C is formally smooth in  $\mathcal{M}$  if  $\mathfrak{V}_1C$  is  ${}^C\mathcal{I}^C$ -injective. By duality, from Lemma 2.3 and Proposition 2.4, we obtain the following two results.

LEMMA 2.14. Let C be a coseparable coalgebra in a monoidal category  $\mathcal{M}$ . The following assertions hold true:

1) The forgetful functor  ${}^{C}\mathbb{T} : {}^{C}\mathcal{M} \to \mathcal{M}$  is separable. In particular, any left C-comodule  $(M, {}^{C}\rho_{M})$  is  ${}^{C}\mathcal{I}$ -injective and if M is a C-bicomodule, the comultiplication  ${}^{C}\rho_{M} : M \to C \otimes M$  has a retraction  ${}^{C}\mu$  which is C-bicolinear and natural. 2) The forgetful functor  $\mathbb{T}^{C} : \mathcal{M}^{C} \to \mathcal{M}$  is separable. In particular, any right C-comodule

2) The forgetful functor  $\mathbb{T}^C : \mathcal{M}^C \to \mathcal{M}$  is separable. In particular, any right C-comodule  $(M, \rho_M^R)$  is  $\mathcal{I}^C$ -injective and if M is a C-bicomodule, the comultiplication  $\rho_M^C : M \to M \otimes C$  has a retraction  $\mu^R$  which is C-bicolinear and natural.

PROPOSITION 2.15. Let H be a Hopf algebra with antipode S over a field K. The forgetful functors  $\mathfrak{M}_{H}^{H} \to \mathfrak{M}_{H}$  and  ${}_{H}\mathfrak{M}_{H}^{H} \to {}_{H}\mathfrak{M}_{H}$  are separable.

*Proof.* It is dual to Proposition 2.4.

REMARK 2.16. By Lemma 2.14, the forgetful functor  $\mathbb{T}^C : \mathcal{M}^C \to \mathcal{M}$  is separable for any coseparable coalgebra C in a monoidal category  $\mathcal{M}$ . The converse does not hold true. In fact, in the case when  $\mathcal{M} = \mathfrak{M}_H$  and C = H, the functor  $\mathbb{T}^C$  is always separable (Lemma 2.15), but C is coseparable in  $\mathcal{M}$  if and only if H is a cosemisimple coalgebra ([AMS1, Proposition 2.11]).

**PROPOSITION 2.17.** Let C be a coalgebra in a monoidal category  $\mathcal{M}$ . The following assertions are equivalent:

- (a) C is coseparable in  $\mathcal{M}$ .
- (b) The forgetful functor  ${}^{C}\mathbb{T}^{C}: {}^{C}\mathcal{M}^{C} \to \mathcal{M}$  is separable.
- (c) Any C-bicomodule is  ${}^{C}\mathcal{I}^{C}$ -injective.
- (d) The C-bicomodule C is  ${}^{C}\mathcal{I}^{C}$ -injective.

COROLLARY 2.18. Any coseparable coalgebra in a coabelian monoidal category  $\mathcal{M}$  is formally smooth.

COROLLARY 2.19. Let C be a coseparable coalgebra in  $\mathfrak{M}_K$ . Then any left C-comodule is injective in  ${}^{C}\mathfrak{M}$ . Hence any left C-comodule is also projective in  ${}^{C}\mathfrak{M}$  and C is cosemisimple. Moreover any C-bicomodule is injective in  ${}^{C}\mathfrak{M}^{C}$  and hence any C-bicomodule is projective in  ${}^{C}\mathfrak{M}^{C}$ .

2.20. Let  $(F', \phi_0, \phi_2) : (\mathcal{M}, \otimes, \mathbf{1}) \to (\mathcal{M}', \otimes', \mathbf{1}')$  be a monoidal functor between two monoidal categories, where  $\phi_2(U, V) : F'(U \otimes V) \to F'(U) \otimes F'(V)$ , for any  $U, V \in \mathcal{M}$ , and  $\phi_0 : \mathbf{1}' \to \mathbf{F}'(\mathbf{1})$ . Let  $(C, \Delta, \varepsilon)$  is a coalgebra in  $\mathcal{M}$ . It is well known that  $(F'(C), \Delta_{F'(C)}, \varepsilon_{F'(C)})$  is a coalgebra in  $\mathcal{M}'$ , where

$$\Delta_{F'(C)} := F'(C) \xrightarrow{F'(\Delta)} F'(C \otimes C) \xrightarrow{\phi_2^{-1}(C,C)} F'(C) \otimes' F'(C)$$
$$\varepsilon_{F'(C)} := F'(C) \xrightarrow{F'(\varepsilon)} F'(\mathbf{1}) \xrightarrow{\phi_0^{-1}} \mathbf{1}'.$$

Consider the functor  $F: {}^{C}\mathcal{M}^{C} \to {}^{C'}\mathcal{M}^{\prime C'}$  defined by  $F((M, {}^{C}\rho_{M}, \rho_{M}^{C})) = (F'(M), {}^{C'}\rho_{F'(M)}, \rho_{F'(M)}^{C'}),$ where

**PROPOSITION 2.21.** Let  $\mathcal{M}$  and  $\mathcal{M}'$  be coabelian monoidal categories. Let C, C', F' and F as in Example 2.20. Then, in the following diagrams,  $\mathbb{H}' \circ G'$  and  $G \circ \mathbb{H}$  are naturally equivalent and  $G' \circ \mathbb{T} = \mathbb{T}' \circ G$ :

$$\begin{array}{cccc} {}^{C}\mathcal{M}^{C} & \xrightarrow{G} {}^{C'}\mathcal{M'}^{C'} & {}^{C}\mathcal{M}^{C} & \xrightarrow{G} {}^{C'}\mathcal{M'}^{C'} \\ \\ \mathbb{T} & & & & \\ \mathcal{M} & \xrightarrow{G'} {}^{\mathcal{M'}} & \mathcal{M'} & & \\ \end{array}$$

where  $(\mathbb{T}, \mathbb{H})$  is the adjunction  $({}^{C}\mathbb{T}^{C}, {}^{C}\mathbb{H}^{C})$  defined in 2.13, and  $(\mathbb{T}', \mathbb{H}')$  is analogously defined. We have that:

 $I \in {}^{C}\mathcal{M}^{C}$  is  $\mathcal{I}_{\mathbb{T}}$ -injective  $\Longrightarrow G(I) \in {}^{C'}\mathcal{M'}^{C'}$  is  $\mathcal{I}_{\mathbb{T}'}$ -injective; the converse is true whenever G is separable.

In particular we obtain that:

i) C is coseparable in  $\mathcal{M} \Longrightarrow C'$  is coseparable in  $\mathcal{M}'$  (i.e.  $\mathbb{T}$  is separable  $\Longrightarrow \mathbb{T}'$  is separable); the converse is true whenever G is separable.

ii) If G' preserves cokernels, then: C is formally smooth in  $\mathcal{M} \Longrightarrow C'$  is formally smooth in  $\mathcal{M}'$ ; the converse is true whenever G is separable.

*Proof.* It is dual to Proposition 2.10.

EXAMPLES 2.22. Let H be a Hopf algebra over a field K. With hypotheses and notations of Proposition 2.21, let  $\mathcal{M}' := \mathfrak{M}_K$ . We want to apply the previous result to the particular case when  $\mathcal{M}$  is ether  $(_H\mathfrak{M}_H, \otimes, K)$  or  $(\mathfrak{M}_H, \otimes, K)$ . Let C be a coalgebra in  $\mathcal{M}$ .

1)  $\mathcal{M} := {}_{H}\mathfrak{M}_{H}$ . The forgetful functor  $G^{1} : {}_{H}^{C}\mathfrak{M}_{H}^{C} \to {}^{C}\mathfrak{M}^{C}$  has a left adjoint  $F^{1} : {}^{C}\mathfrak{M}^{C} \to {}^{C}_{H}\mathfrak{M}_{H}^{C}$ ,  $F^{1}(M) = H \otimes M \otimes H$ , where  $F^{1}(M)$  is a bimodule via  $m_{H} \otimes M \otimes H$  and  $H \otimes M \otimes m_{H}$ , and it is a bicomodule with codiagonal coactions. For any  $M \in {}^{C}_{H}\mathfrak{M}_{H}^{C}$  the counit of the adjunction is the

map  $\varepsilon_M : H \otimes M \otimes H \to M, \varepsilon_M = \mu_M^H \circ ({}^H \mu_M \otimes H).$ 2)  $\mathcal{M} := \mathfrak{M}_H$ . The forgetful functor  $G^r : {}^C \mathfrak{M}_H^C \to {}^C \mathfrak{M}^C$  has a left adjoint  $F^r : {}^C \mathfrak{M}^C \to {}^C \mathfrak{M}_H^C$ ,  $F^r(M) = M \otimes H$ , where  $F^r(M)$  is a module via  $M \otimes m_H$ , and it is a bicomodule with codiagonal coactions. For any  $M \in {}^C\mathfrak{M}_H^C$  the counit of the adjunction is the map  $\varepsilon_M : M \otimes H \to M, \varepsilon_M = \mu_M^H$ . In the case C = H we set  $(F^2, G^2) := (F^1, G^1)$ .

The forgetful functor  $G^a: {}^{H}\mathfrak{M}_{H}^{H} \to {}^{H}\mathfrak{M}_{H}^{H}$  is nothing but  $G^r$  in the case C = H. Then it has a left

adjoint  $F^a : {}^{H}\mathfrak{M}^{H} \to {}^{H}\mathfrak{M}^{H}_{H}$ , which is  $F^r$  for C = H. The forgetful functor  $G^b : {}^{H}_{H}\mathfrak{M}^{H}_{H} \to {}^{H}\mathfrak{M}^{H}_{H}$  has a left adjoint  $F^b : {}^{H}\mathfrak{M}^{H}_{H} \to {}^{H}_{H}\mathfrak{M}^{H}_{H}$ ,  $F^b(M) = H \otimes M$ , where  $F^b(M)$  is a bimodule via  $m_H \otimes M$  and  $H \otimes \mu^H_M$ , and it is a bicomodule with codiagonal coactions.

Note that the forgetful functor  $G^2 : {}^H_H \mathfrak{M}^H_H \to {}^H \mathfrak{M}^H$  can be decomposed as  $G^2 = G^a \circ G^b$ .

In view of Examples 2.22, we obtain the following important result:

THEOREM 2.23. Let H be a Hopf algebra over a field K and let  $\mathcal{M}$  denote either  ${}_{H}\mathfrak{M}_{H}$  or  $\mathfrak{M}_{H}$ . Let C be a coalgebra in  $\mathcal{M}$  and consider the forgetful functors  $\mathbb{T} : {}^{C}\mathcal{M}^{C} \to \mathcal{M}, \ \mathbb{T}' : {}^{C}\mathfrak{M}^{C} \to \mathfrak{M}_{K}$ and  $G : {}^{C}\mathcal{M}^{C} \to {}^{C}\mathfrak{M}^{C}$ .

We have that:

 $I \in {}^{C}\mathcal{M}^{C}$  is  $\mathcal{I}_{\mathbb{T}}$ -injective  $\Longrightarrow I$  is  $\mathcal{I}_{\mathbb{T}'}$ -injective as an object in  ${}^{C}\mathfrak{M}^{C}$ ; the converse is true whenever G is separable.

In particular we obtain that:

i) C is coseparable as a coalgebra in  $\mathcal{M} \Longrightarrow C$  is coseparable as a coalgebra in  $\mathfrak{M}_K$ ; the converse is true whenever G is separable.

ii) C is formally smooth as a coalgebra in  $\mathcal{M} \Longrightarrow C$  is formally smooth as a coalgebra in  $\mathfrak{M}_K$ ; the converse is true whenever G is separable.

# 3. AD-INVARIANT INTEGRALS THROUGH SEPARABLE FUNCTORS

REMARK 3.1. Let H be a Hopf algebra over a field K. For sake of brevity many results will be stated only for the category  $\mathfrak{M}_H$ . Clearly all the results still hold true for  ${}_H\mathfrak{M}$  (as  ${}_H\mathfrak{M} \simeq \mathfrak{M}_{H^{op}}$ ). Similar arguments apply to the categories  $\mathfrak{M}^H$  and  ${}^H\mathfrak{M}$ .

3.2. Let H be a Hopf algebra with antipode S over a field K and set:

$$h \triangleright x := h_1 x S(h_2) \quad \text{and} \quad x \triangleleft h := S(h_1) x h_2$$
$${}^{H} \varrho(h) := h_1 S(h_3) \otimes h_2 \quad \text{and} \quad \varrho^{H}(h) := h_2 \otimes S(h_1) h_3$$

for all  $h, x \in H$ . It is easy to check that  $\triangleright$  defines a left module action of H on itself called *left adjoint action* and that  ${}^{H}\varrho$  defines a left comodule coaction of H on itself called *left adjoint coaction*. Analogously  $\triangleleft$  gives rise to the right adjoint action and  $\varrho^{H}$  to the right adjoint coaction. If S is bijective, we can consider the following actions and coactions of H on itself:

$$h \blacktriangleright x := h_2 x S^{-1}(h_1) \quad \text{and} \quad x \blacktriangleleft h := S^{-1}(h_2) x h_1$$
$$\overline{\varrho}^H(h) = h_2 \otimes h_3 S^{-1}(h_1) \quad \text{and} \quad {}^H \overline{\varrho}(h) := S^{-1}(h_3) h_1 \otimes h_2.$$

The structures above provide two different ways of looking at H as an object in the categories of Yetter-Drinfeld modules. In fact, if  $\Delta_H$  is the comultiplication and  $m_H$  is the multiplication of H, then H can be regarded as an object in  ${}^{H}_{H}\mathcal{YD}, \mathcal{YD}^{H}_{H}, {}^{H}\mathcal{YD}^{H}, {}^{H}\mathcal{YD}_{H}$  respectively via:

$$(\triangleright, \Delta_H), (\triangleleft, \Delta_H), (\blacktriangleright, \Delta_H), (\blacktriangleleft, \Delta_H)$$
 or  $(m_H, {}^H\varrho), (m_H, \varrho^H), (m_H, \overline{\varrho}^H), (m_H, {}^H\overline{\varrho})$ 

# 3.3. The adjunctions.

The actions recalled in 3.2 are closely linked to the categories of Yetter-Drinfeld modules. We now consider some adjunctions involving these modules that will be very useful in finding equivalent conditions to the existence of an *ad*-invariant integral.

1) The forgetful functor  $F_3: {}^{H}_{H}\mathcal{YD} \to {}^{H}_{H}\mathfrak{M}$  has a right adjoint  $G_3: {}^{H}_{H}\mathfrak{M} \to {}^{H}_{H}\mathcal{YD}, G(M) = H \otimes M$ , where G(M) is a comodule via  $\Delta_H \otimes M$  and a module via the action:  $h \cdot (l \otimes m) = h_1 lS(h_3) \otimes h_2 m$ . For any  $M \in {}^{H}_{H}\mathcal{YD}$  the unit of the adjunction is the map  $\eta_M: M \to H \otimes M, \eta_M = {}^{H}\rho_M$ .

2) The forgetful functor  $F_4: \mathcal{YD}_H^H \to \mathfrak{M}_H$  has a right adjoint  $G_4: \mathfrak{M}_H \to \mathcal{YD}_H^H, G_4(M) = M \otimes H$ , where  $G_4(M)$  is a comodule via  $M \otimes \Delta_H$  and a module via the action:  $(m \otimes l) \cdot h = mh_2 \otimes S(h_1) lh_3$ . For any  $M \in \mathcal{YD}_H^H$  the unit of the adjunction is the map  $\eta_M: M \to M \otimes H, \eta_M = \rho_M^H$ .

3) Assume *H* has bijective antipode. The forgetful functor  $F_5 : {}_H \mathcal{YD}^H \to {}_H \mathfrak{M}$  has a right adjoint  $G_5 : {}_H \mathfrak{M} \to {}_H \mathcal{YD}^H, G_5(M) = M \otimes H$ , where  $G_5(M)$  is a comodule via  $M \otimes \Delta_H$  and a module via the action:  $h \cdot (l \otimes m) = h_2 l \otimes h_3 m S^{-1}(h_1)$ . For any  $M \in {}_H \mathcal{YD}^H$  the unit of the adjunction is the map  $\eta_M : M \to M \otimes H, \eta_M = \rho_M^H$ .

4) Assume H has bijective antipode. The forgetful functor  $F_6: {}^H \mathcal{YD}_H \to \mathfrak{M}_H$  has a right adjoint  $G_6: \mathfrak{M}_H \to {}^H \mathcal{YD}_H, G_6(M) = H \otimes M$ , where  $G_6(M)$  is a comodule via  $\Delta_H \otimes M$  and a module via the action:  $(l \otimes m) \cdot h = S^{-1}(h_3) lh_1 \otimes mh_2$ . For any  $M \in {}^H \mathcal{YD}_H$  the unit of the adjunction is the map  $\eta_M: M \to H \otimes M, \eta_M = {}^H \rho_M$ .

Consider now the dual version of this functors.

1<sup>op</sup>) The forgetful functor  $G^3 : {}^H_H \mathcal{YD} \to {}^H \mathfrak{M}$  has a left adjoint  $F^3 : {}^H \mathfrak{M} \to {}^H_H \mathcal{YD}, F^3(M) = H \otimes M$ , where  $F^3(M)$  is a module via  $m_H \otimes M$  and a comodule via the coaction:  ${}^H\rho(h\otimes m) = h_1m_{-1}S(h_3)\otimes h_2 \otimes m_0$ . For any  $M \in {}^H_H \mathcal{YD}$  the counit of the adjunction is the map  $\varepsilon_M : H \otimes M \to M, \varepsilon_M = {}^H\mu_M$ . 2<sup>op</sup>) The forgetful functor  $G^4 : \mathcal{YD}_H^H \to \mathfrak{M}^H$  has a left adjoint  $F^4 : \mathfrak{M}^H \to \mathcal{YD}_H^H, F^4(M) = M \otimes H$ , where  $F^4(M)$  is a module via  $M \otimes m_H$  and a comodule via the coaction:  $\rho^H(m \otimes h) = m_0 \otimes h_2 \otimes S(h_1)m_1h_3$ . For any  $M \in \mathcal{YD}_H^H$  the counit of the adjunction is the map  $\varepsilon_M^* : M \otimes H \to M, \varepsilon_M^* = \mu_M^M$ . 3<sup>op</sup>) Assume H has bijective antipode. The forgetful functor  $G^5 : {}_H \mathcal{YD}^H \to \mathfrak{M}^H$  has a left adjoint  $F^5 : \mathfrak{M}^H \to {}_H \mathcal{YD}^H, F^5(M) = H \otimes M$ , where  $F^5(M)$  is a module via  $m_H \otimes M$  and a comodule via the coaction:  $\rho^H(h \otimes m) = h_2 \otimes m_0 \otimes h_3m_1S^{-1}(h_1)$ . For any  $M \in {}_H \mathcal{YD}^H$  the counit of the adjunction is the map  $\varepsilon_M : H \otimes M \to M, \varepsilon_M = {}^H \mu_M$ .

 $4^{op}$ ) Assume H has bijective antipode. The forgetful functor  $G^6: {}^{H}\mathcal{YD}_H \to {}^{H}\mathfrak{M}$  has a left adjoint  $F^6: {}^{H}\mathfrak{M} \to {}^{H}\mathcal{YD}_H, F^6(M) = M \otimes H$ , where  $F^6(M)$  is a module via  $M \otimes m_H$  and a comodule via the coaction:  ${}^{H}\rho(m \otimes h) = S^{-1}(h_3)m_{-1}h_1 \otimes m_0 \otimes h_2$ . For any  $M \in {}^{H}\mathcal{YD}_H$  the counit of the adjunction is the map  $\varepsilon_M: M \otimes H \to M, \varepsilon_M = \mu_M^H$ .

3.4. Let K be any field. An augmented K-algebra (A, m, u, p) is a K-algebra (A, m, u) endowed with an algebra homomorphism  $p: A \to K$  called augmentation of A. An element  $x \in A$  is a *left integral in A*, whenever  $a \cdot_A x = p(a) x$ , for every  $a \in A$ . The definition of a right integral in A is analogous. A is called *unimodular*, whenever the space of left and right integrals in A coincide. A (left or right) integral x in A is called a *total integral in A*, whenever  $p(x) = 1_K$ .

Let  $(H, m_H, u_H, \Delta_H, \varepsilon_H)$  be a bialgebra.

1)  $(H, m_H, u_H, \varepsilon_H)$  is an augmented algebra. Then a left integral in H is an element  $t \in H$  such that  $h \cdot_H t = \varepsilon_H (h) t$ , for every  $h \in H$ . Moreover t is total whenever  $\varepsilon_H (t) = 1_K$ .

2)  $(H^*, m_{H^*}, u_{H^*}, \varepsilon_{H^*})$  is an augmented algebra. Then a left integral in  $H^*$  is an element  $\lambda \in H^*$ , that is a K-linear map  $f\lambda = f(1_H)\lambda$ , for every  $f \in H^*$ . Moreover  $\lambda$  is total, whenever  $\lambda(1_H) = 1_K$ . It is clear that  $\lambda \in H^*$  is a left (resp. right) integral in  $H^*$  if and only if  $h_1\lambda(h_2) = 1_H\lambda(h)$  (resp.  $\lambda(h_1)h_2 = \lambda(h)1_H$ ) for every  $h \in H$ .

If H is finite dimensional,  $H^*$  becomes a Hopf algebra: in particular one can consider the notion of left integral in  $(H^*)^*$  in the sense of 2). By means of the isomorphism

$$H \to H^{**} : h \longmapsto \left( \begin{array}{c} H^* \to K \\ f \longmapsto f(h) \end{array} \right),$$

one can check that a left integral in  $H^{**}$  is nothing but a left integral in H in the sense of 1): thus there is no danger of confusion.

For the reader's sake, we outline the following facts.

THEOREM 3.5. Let H be a Hopf algebra with antipode S over any field K. Then we have:

- 1) There exists a total integral  $t \in H$  (i.e. H is semisimple) if and only if H is separable.
- 2) There exists a total integral  $\lambda \in H^*$  (i.e. H is cosemisimple) if and only if H is coseparable.

*Proof.* 1) "  $\Leftarrow$  " Let  $\sigma : H \to H \otimes H$  an *H*-bilinear section of the multiplication *m* and set  $t_{\sigma} := (H \otimes \varepsilon_H)\sigma(1_H) \in H$ . Then  $t_{\sigma}$  is a total integral.

"  $\Rightarrow$  " Let  $t \in H$  be a total integral. Since t is a left integral and  $\Delta_H$  is an homomorphism of algebras, we have:

(1) 
$$ht_1 \otimes S(t_2) = h_1 t_1 \otimes S(h_2 t_2) h_3 = \varepsilon_H(h_1) t_1 \otimes S(t_2) h_2 = t_1 \otimes S(t_2) h, \forall h \in H,$$

so that the map  $\sigma_t : H \to H \otimes H : h \mapsto ht_1 \otimes S(t_2)$  is *H*-bilinear. Moreover  $m_H \sigma_t(h) = ht_1 S(t_2) = h\varepsilon_H(t) = h$ , so that  $\sigma_t$  is an *H*-bilinear section of  $m_H$  and *H* is separable by definition.

2) "  $\leftarrow$  " Let  $\theta$  :  $H \otimes H \to H$  an *H*-bicolinear retraction of the comultiplication  $\Delta$  and set  $\lambda_{\theta} := \varepsilon_H \theta(- \otimes 1_H) \in H^*$ . Then  $\lambda_{\theta}$  is a total integral.

"  $\Rightarrow$  " Let integral  $\lambda \in H^*$  be a left integral such that  $\lambda(1_H) = 1$ . Since  $\lambda$  is a left integral and m is an homomorphism of coalgebras, we have:

(2) 
$$x_1\lambda(x_2S(y)) = x_1S(y_2)\lambda(x_2S(y_1))y_3 = (xS(y_1))_1\lambda((xS(y_1))_2)y_2 = \lambda(xS(y_1))y_2, \forall x, y \in H,$$

so that the map  $\theta_{\lambda} : H \otimes H \to H : x \otimes y \mapsto x_1 \lambda(x_2 S(y))$  is *H*-bicolinear. Moreover  $\theta_{\lambda} \Delta(h) = h_1 \lambda(h_2 S(h_3)) = h \lambda(1_H) = h$ , so that  $\theta_{\lambda}$  is an *H*-bicolinear retraction of the comultiplication  $\Delta$  and *H* is coseparable by definition.

Our next aim is to characterize the existence of a so-called *ad*-invariant integral. A remarkable fact is that any semisimple and cosemisimple Hopf algebra H over a field K admits such an integral (see [AMS1, Theorem 2.27]).

DEFINITION 3.6. [SVO, Definition 1.11] Let H be a Hopf algebra with antipode S over any field K and let  $\lambda \in H^*$ .  $\lambda$  will be called an *ad-invariant integral* whenever:

- a)  $h_1\lambda(h_2) = 1_H\lambda(h)$  for all  $h \in H$  (i.e.  $\lambda$  is a left integral in  $H^*$ );
- b)  $\lambda(h_1 x S(h_2)) = \varepsilon(h) \lambda(x)$ , for all  $h, x \in H$  (i.e.  $\lambda$  is left linear with respect to  $\triangleright$ );

c)  $\lambda(1_H) = 1_K$ .

LEMMA 3.7. An element  $\lambda \in H^*$  is an ad-invariant integral if and only if it is a retraction of the unit  $u_H : K \to H$  of H in  ${}^{H}_{H}\mathcal{YD}$ , where H is regarded as an object in the category via the left adjoint action  $\triangleright$  and the comultiplication  $\Delta_H$ .

EXAMPLE 3.8. 1) Let G be an arbitrary group an let KG be the group algebra associated. Let  $\lambda : KG \to K$  be defined by  $\lambda(g) = \delta_{e,g}$  (the Kronecker symbol), where e denotes the neutral element of G. Then  $\lambda$  is an ad-invariant integral for KG (see [SVO, Corollary 2.8]). 2) Every commutative cosemisimple Hopf algebra has an ad-invariant integral.

REMARK 3.9. If H is a Hopf algebra with a nonzero integral then the left and right integral spaces are both one-dimensional [DNR, Theorem 5.4.2]. If H has a total integral  $\lambda \in H^*$  (i.e. H is cosemisimple), then the left and right integral space coincide [DNR, Exercise 5.5.10], and are generated by  $\lambda$ . Hence there can be at most one ad-invariant integral, namely the unique total integral.

The following lemma shows that in the definition of *ad*-invariant integral we can choose  $\triangleleft, \triangleright$  or  $\triangleleft$  instead of  $\triangleright$ . Since  $\lambda$  is in particular a total integral, it is both a left and a right integral. Thus it is the same to have a retraction of  $u_H$  in  ${}^H_H \mathcal{YD}, \mathcal{YD}^H_H, {}^H \mathcal{YD}^H$  or  ${}^H \mathcal{YD}_H$ .

LEMMA 3.10. Let H be a Hopf algebra with antipode S over any field K and let  $\lambda \in H^*$  be a total integral. Then the following are equivalent:

- (1)  $\lambda$  is left linear with respect to  $\triangleright$ .
- (2)  $\lambda$  is right linear with respect to  $\triangleleft$ .
- (3)  $\lambda$  is left linear with respect to  $\blacktriangleright$ .
- (4)  $\lambda$  is right linear with respect to  $\blacktriangleleft$ .

*Proof.* We have that  $\lambda$  is both a left integral and a right integral for  $H^*$ . Since  $\lambda$  is a total integral S is bijective (see [DNR, Corollary 5.4.6]) and hence it makes sense to consider  $S^{-1}$ .

(1)  $\Rightarrow$  (2) Observe that:  $S(x \triangleleft h) = S(S(h_1)xh_2) = S(h)_1S(x)S[S(h)_2] = S(h) \triangleright S(x)$ . Thus, since  $\lambda = \lambda S$  and  $\lambda$  is left linear with respect to  $\triangleright$ , we get  $\lambda(x \triangleleft h) = \lambda S(x \triangleleft h) = \lambda(S(h) \triangleright S(x)) = \varepsilon S(h)\lambda(S(x)) = \varepsilon(h)\lambda S(x) = \varepsilon(h)\lambda(x)$  that is  $\lambda$  is right linear with respect to  $\triangleleft$ . (2)  $\Rightarrow$  (1) follows analogously once proved the relation  $S(h \triangleright x) = S(x) \triangleleft S(h)$ . (1)  $\Rightarrow$  (3) We have:  $S[h \blacktriangleright S^{-1}(x)] = S[h_2S^{-1}(x)S^{-1}(h_1)] = h_1xS(h_2) = h \triangleright x$ . Then, since  $\lambda = \lambda S$  and  $\lambda$  is left linear with respect to  $\triangleright$ , we have  $\lambda(h \blacktriangleright x) = \lambda S(h \blacktriangleright S^{-1}S(x)) = \lambda(h \triangleright S(x)) = \varepsilon(h)\lambda S(x) = \varepsilon(h)\lambda(x)$  i.e.  $\lambda$  is left linear with respect to  $\blacktriangleright$ . (3)  $\Rightarrow$  (1) Since  $\lambda$  is left linear with respect to  $\triangleright$  one has  $\lambda(h \triangleright x) = \lambda S[h \triangleright S^{-1}(x)] = \lambda[h \triangleright S^{-1}(x)] = \varepsilon(h)\lambda SS^{-1}(x) = \varepsilon(h)\lambda(x)$  i.e.  $\lambda$  is left linear with respect to  $\triangleright$ . (1)  $\Leftrightarrow$  (4) Analogous to (1)  $\Leftrightarrow$  (3) by means of  $S^{-1}[S(x) \triangleleft h] = x \blacktriangleleft h$ .

The following result improves [AMS1, Theorem 2.29].

LEMMA 3.11. Let H be a Hopf algebra with antipode S over a field K. Assume there exists an ad-invariant integral  $\lambda \in H^*$ . Then we have that:

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- i) The forgetful functor  ${}_{A}\mathfrak{M}_{A}^{H} \to {}_{A}\mathfrak{M}_{A}$  is separable for any algebra A in  $\mathfrak{M}^{H}$ . ii) The forgetful functor  ${}_{A}^{H}\mathfrak{M}_{A} \to {}_{A}\mathfrak{M}_{A}$  is separable for any algebra A in  ${}^{H}\mathfrak{M}$ . iii) The forgetful functor  ${}_{A}^{H}\mathfrak{M}_{A}^{H} \to {}_{A}\mathfrak{M}_{A}$  is separable for any algebra A in  ${}^{H}\mathfrak{M}^{H}$ .

*Proof.* i) By Examples 2.11, the forgetful functor  $F_r : {}_A\mathfrak{M}^H_A \to_A \mathfrak{M}_A$  has a right adjoint  $G_r :$  ${}_A\mathfrak{M}_A \to {}_A\mathfrak{M}_A^H, G_8(M) = M \otimes H.$  Thus by Theorem 1.11,  $F_r$  is separable if and only if the unit  $\eta^H$ : Id<sub>A</sub> $\mathfrak{M}^H_A \to G_r F_r$  of the adjunction cosplits, i.e. there exists a natural transformation  $\mu^H: G_r F_r \to \mathrm{Id}_{A\mathfrak{M}_A^H}^H$  such that  $\mu^H_M \circ \eta^H_M = \mathrm{Id}_M$  for any M in  ${}_A\mathfrak{M}_A^H$ . Let us define:

$$\mu_M^H: M \otimes H \to M, \quad \mu_M^H(m \otimes h) = m_0 \lambda(m_1 S(h)).$$

Obviously  $(\mu_M^H)_{M \in A} \mathfrak{M}_A^H$  is a functorial morphism.

Let us check that  $\mu_M^H$  is a morphism in  ${}_A\mathfrak{M}_A^H$ , i.e. a morphism of A-bimodules and of H-bicomodules. Since  $\mu_M^A \in \mathfrak{M}^H$ , we have:  $\mu_M^H((m \otimes h)a) = m_0 a_0 \lambda(m_1 a_1 S(a_2) S(h)) = \mu_M^H(m \otimes h)a$ . Since  ${}^A\mu_M \in \mathfrak{M}^H$  and as  $\lambda$  satisfies relation b) of Definition 3.6, we get that  $\mu_M^H$  is also left Alinear:  $\mu_M^H(a(m \otimes h)) = a_0 m_0 \lambda(a_1 \triangleright m_1 S(h)) = a \mu_M^H(m \otimes h).$ 

By (2), we have:  $\lambda(xS(y_1))y_2 = x_1\lambda(x_2S(y)), \forall x, y \in H$ . Thus we get also the right *H*-collinearity of  $\mu_M^H$ :  $(\mu_M^H \otimes H)\rho^H(m \otimes h) = m_0 \otimes \lambda(m_1S(h_1))h_2 = m_0 \otimes m_1\lambda(m_2S(h)) = \rho^H\mu_M^H(m \otimes h)$ . It remains to prove that  $\mu_M^H$  is a retraction of  $\eta_M^H$ :  $\mu_M^H\eta_M^H(m) = m_0\lambda(m_1S(m_2)) = m\lambda(1_H) = m$ .

*ii*) It is analogous to *i*) by setting  ${}^{H}\mu_{M}(h \otimes m) = \lambda(hS(m_{-1}))m_{0}$ .

*iii*) We have to construct a functorial retract of  $(\eta_M)_{M \in {}^H_{A}\mathfrak{M}^H_{A}}$ , where  $\eta_M = ({}^H_M \eta \otimes H) \circ \eta^H_M$ . By the previous part, there are a functorial retraction  $(\mu_M^H)_{M \in \mathcal{M}_A^H}$  of  $(\sigma_M^H)_{M \in \mathcal{M}_A^H}$  and a functorial retract  $({}^{H}\mu_{M})_{M \in {}^{H}_{A}\mathfrak{M}_{A}}$  of  $({}^{H}\sigma_{M})_{M \in {}^{H}_{A}\mathfrak{M}_{A}}$ . Let us define the morphism  $\mu_{M} : H \otimes M \otimes H \to M$ by  $\mu_{M} = \mu_{M}^{H} \circ ({}^{H}\mu_{M} \otimes H)$ . Obviously it is a retraction of  $\sigma_{M}$  in  ${}_{A}\mathfrak{M}_{A}^{H}$ . It is easy to prove that  $\mu_{M} = {}^{H}\mu_{M} \circ (H \otimes \mu_{M}^{H})$ : hence one gets that  $\mu_{M}$  is a morphism in  ${}^{H}_{A}\mathfrak{M}_{A}^{H}$ .  $\Box$ 

We can now consider the main result concerning ad-invariant integrals. The equivalence (1)  $\Leftrightarrow$ (3b) was proved in a different way in [AMS1, Proposition 2.11].

THEOREM 3.12. Let H be a Hopf algebra over a field K. The following assertions are equivalent:

- (1) There is an ad-invariant integral  $\lambda \in H^*$ .
- (2) The forgetful functor  ${}^{H}_{A}\mathfrak{M}^{H}_{A} \to {}^{A}\mathfrak{M}_{A}$  is separable for any algebra A in  ${}^{H}\mathfrak{M}^{H}$ . (3) The forgetful functor  ${}^{H}_{H}\mathfrak{M}^{H}_{H} \to {}^{H}\mathfrak{M}_{H}$  is separable.
- (3b) H is coseparable in  $({}_{H}\mathfrak{M}_{H}, \otimes, K)$ .
- (4) The forgetful functor  ${}^{H}_{H}\mathcal{YD} \to {}^{H}\mathfrak{M}$  is separable.
- (4b) K is  $\mathcal{I}_F$ -injective where F is the forgetful functor of (4).
- *Proof.*  $(1) \Rightarrow (2)$  It follows by Lemma 3.11.
- $(2) \Rightarrow (3)$ . Obvious.

(3)  $\Leftrightarrow$  (3b). It is just Proposition 2.17 applied to  $\mathcal{M} = ({}_H\mathfrak{M}_H, \otimes, K).$ 

(3)  $\Rightarrow$  (4). Take the notations of Examples 2.11 and 3.3. Since  $F_2: {}^H_H\mathfrak{M}^H_H \to {}^H_H\mathfrak{M}_H$  is separable and  $F_2 = F_a \circ F_b$ , where  $F_b : {}^H_H \mathfrak{M}^H_H \to {}^H_H \mathfrak{M}^H_H$  and  $F_a : {}^H_H \mathfrak{M}^H_H \to {}^H_H \mathfrak{M}^H_H$ , then, by Theorem 1.10,  $F_b$  is separable. Consider the inverses  $(F')^{-1}$  and  $F^{-1}$  respectively of the functors  $F' = (-)^{coH}$ :  ${}^H_H \mathfrak{M}^H_H \to {}^H_H \mathcal{YD}$  and  $F = (-)^{coH} : {}^H_H \mathfrak{M}^H_H \to {}^H\mathfrak{M}$  (these are category equivalences; see [Scha2, Theorem 5.7]). One can easily check that  $F^{-1} \circ F_3 = F_b \circ (F')^{-1}$ . By Theorem 1.10,  $(F')^{-1}$  is separable so that  $F^{-1} \circ F_3$ , and hence  $F_3$ , is a separable functor.

 $(4) \Rightarrow (4b)$ . By Corollary 1.12 the separability of  $F_3 : {}^H_H \mathcal{YD} \to {}^H\mathfrak{M}$  (that has  $G_3$  as a right adjoint) implies that any object in  ${}^{H}_{H}\mathcal{YD}$ , in particular K, is  $\mathcal{I}_{F_3}$ -injective.

 $(4b) \Rightarrow (1)$ . Observe that  $u_H$  can be regarded as a morphism in  ${}^H_H \mathcal{YD}$ , once H is regarded as an object in  ${}^{H}_{H}\mathcal{YD}$  via the action  $\triangleright$  (defined in 3.2) and the coaction given by the comultiplication  $\Delta$ . In particular,  $u_H$  belongs to  $\mathcal{I}_{F_3}$ : in fact the counit  $\varepsilon_H$  of H is a left linear retraction of  $F_3(u_H)$ . Hence, since K is  $\mathcal{I}_{F_3}$ -injective, there is  $\lambda : H \to K$  in  ${}^H_H \mathcal{YD}$  such that  $\lambda \circ u_H = \mathrm{Id}_K$ , i.e., by Lemma 3.7, an *ad*-invariant integral. 

REMARK 3.13. The following assertions are all equivalent to the existence of an *ad*-invariant integral  $\lambda \in H^*$ .

- (5) The forgetful functor  $\mathcal{YD}_{H}^{H} \rightarrow \mathfrak{M}_{H}$  is separable.
- (6) The forgetful functor  ${}_{H}\mathcal{YD}^{H} \to {}_{H}\mathfrak{M}$  is separable and S is bijective.
- (7) The forgetful functor  ${}^{H}\mathcal{YD}_{H} \to \mathfrak{M}_{H}$  is separable and S is bijective.
- (8) K is  $\mathcal{I}_F$ -injective where F is the forgetful functor of (5),(6) or (7).

In fact, note that  ${}^{H}_{H}\mathfrak{M}^{H}_{H} \simeq \mathcal{YD}^{H}_{H}$ . Since  $\lambda$  is in particular a total integral, the antipode S is bijective and hence, by [Scha2, Corollary 6.4], we can also assume  ${}^{H}\mathcal{YD}_{H} \simeq {}^{H}_{H}\mathfrak{M}^{H}_{H} \simeq {}_{H}\mathcal{YD}^{H}$ . Now, by means of Lemma 3.10, one can proceed like in the proof of Theorem 3.12.

THEOREM 3.14. Let H be a finite dimensional Hopf algebra over a field K and let D(H) be the Drinfeld Double. The following assertions are equivalent:

- (i) There is an ad-invariant integral  $\lambda \in H^*$ .
- (ii) The forgetful functor  $_{D(H)}\mathfrak{M} \to {}_{H}\mathfrak{M}$  is separable.
- (iii) D(H) is separable in  $({}_{H}\mathfrak{M}_{H}, \otimes_{H}, H)$ , i.e. D(H)/H is separable.

*Proof.*  $(i) \Leftrightarrow (ii)$ . Since H is finite dimensional, it has bijective antipode. Hence we have  ${}^{H}_{H}\mathcal{YD} \simeq {}_{H}\mathcal{YD}^{H} \simeq {}_{D(H)}\mathfrak{M}$ . By Theorem 3.12, (i) holds if and only if the forgetful functor  ${}^{H}_{H}\mathcal{YD} \to {}_{H}\mathfrak{M}$  is separable if and only if  ${}_{D(H)}\mathfrak{M} \to {}_{H}\mathfrak{M}$  is separable.

 $(ii) \Leftrightarrow (iii)$ . It follows by Proposition 2.2 applied to the ring homomorphism  $H \to D(H) = H^{*cop} \bowtie H : h \mapsto \varepsilon_H \bowtie h$ .

PROPOSITION 3.15. Let H be a Hopf algebra with an ad-invariant integral  $\lambda \in H^*$  and let  $\mathcal{M}$  be either  $\mathfrak{M}^H$  or  ${}^H\mathfrak{M}^H$ . For any algebra A in  $\mathcal{M}$ , we have:

i) A is separable as an algebra in  $\mathcal{M}$  if and only if it is separable as an algebra in  $\mathfrak{M}_K$ .

ii) A is formally smooth as an algebra in  $\mathcal{M}$  if and only if it is formally smooth as an algebra in  $\mathfrak{M}_K$ .

*Proof.* Since H has an *ad*-invariant integral  $\lambda$ , by Lemma 3.11, the forgetful functor  $F : {}_{A}\mathcal{M}_{A} \to {}_{A}\mathfrak{M}_{A}$  is separable. By Theorem 2.12 we conclude.

## 4. Splitting algebra homomorphisms

We recall the following important result.

THEOREM 4.1. (see [AMS2, Theorem 3.13]) Let (A, m, u) be an algebra in an abelian monoidal category  $(\mathcal{M}, \otimes, \mathbf{1})$ . Then the following assertions are equivalent:

(a) A is formally smooth as an algebra in  $\mathcal{M}$ .

(b) Let  $\pi: E \to A$  be an algebra homomorphism in  $\mathcal{M}$  which is an epimorphism in  $\mathcal{M}$  and let I denote the kernel of  $\pi$ . Assume that there is  $n \in \mathbb{N}$  so that  $I^n = 0$  (I is nilpotent). If for any  $r = 1, \dots, n-1$  the canonical projection  $p_r: E/I^{r+1} \to E/I^r$  splits in  $\mathcal{M}$ , then  $\pi$  has a section which is an algebra homomorphism in  $\mathcal{M}$ .

When  $\mathcal{M}$  is  $\mathfrak{M}^H$  the previous theorem has the following application.

PROPOSITION 4.2. Let H be a Hopf algebra and let A and E be algebras in  $\mathfrak{M}^H$ . Let  $\pi : E \to A$ be an algebra homomorphism in  $\mathfrak{M}^H$  which is surjective. Assume that A is formally smooth as an algebra in  $\mathfrak{M}^H$  and that the kernel of  $\pi$  is a nilpotent ideal. Given an algebra homomorphism  $f : H \to A$  in  $\mathfrak{M}^H$ , then  $\pi$  has a section which is an algebra homomorphism in  $\mathfrak{M}^H$ .

*Proof.* It is similar to [AMS1, Theorem 2.13], where A = H = E/Rad(E) is semisimple and  $f = Id_H$ .

Let I denote the kernel of  $\pi$  and assume there is an  $n \in \mathbb{N}$  such that  $I^n = 0$ . First of all let us observe that, since  $\pi$  is a morphism in  $\mathfrak{M}^H$ , I is a subobject of E in  $\mathfrak{M}^H$ . Hence, for every r > 0,  $I^r$  is a subobject of E and the canonical maps  $E/I^{r+1} \to E/I^r$  are morphisms in  $\mathfrak{M}^H$ .

Now, the object  $I^r/I^{r+1}$  has a natural module structure over  $E/I \simeq A$ , and hence, via f, a module structure over H. With respect to this structure  $I^r/I^{r+1}$  is an object in  $\mathfrak{M}_H^H$ . Via the category equivalences  $\mathfrak{M}_H^H \simeq {}_K \mathfrak{M}$ , we get that  $I^r/I^{r+1}$  is a cofree right comodule i.e.  $I^r/I^{r+1} \simeq V \otimes H$  in  $\mathfrak{M}_H^H$ , for a suitable  $V \in {}_K \mathfrak{M}$ . In particular  $I^r/I^{r+1}$  is an injective comodule, so any canonical map  $E/I^{r+1} \to E/I^r$  has a section in  $\mathfrak{M}^H$ .

By Theorem 4.1, we conclude.

The following remark is due to the referee.

REMARK 4.3. Let H be a Hopf algebra with antipode  $S_H$  and let A be an algebra in  $\mathfrak{M}^H$ . Then the existence of an algebra homomorphism  $f: H \to A$  in  $\mathfrak{M}^H$  is equivalent to the fact that A is isomorphic as an H-comodule algebra to the smash product  $A^{co(H)} \# H$ .

In fact, using the terminology of [Mon, Definition 7.2.1, page 105], the *H*-extension  $A^{co(H)} \subseteq A$  comes out to be *H*-cleft (*f* is a right *H*-comodule map which is convolution invertible with inverse  $f \circ S_H$ ). By [Mon, Theorem 7.2.2, page 106],  $A \simeq A^{co(H)} \#_{\sigma} H$ , where  $\sigma : H \otimes H \to A^{co(H)}$  is defined by

$$\sigma(h\otimes k)=\sum f(h_1)f(k_1)fS_H(h_2k_2),$$

for every  $h, k \in H$ . Since f is an algebra homomorphism we get that  $\sigma(h \otimes k) = \epsilon_H(h)\epsilon_H(k)\mathbf{1}_A$ and hence  $A^{co(H)} \#_{\sigma} H = A^{co(H)} \# H$  is the usual smash product.

Conversely, for any algebra R, the map  $H \to R \# H$  is an algebra homomorphism in  $\mathfrak{M}^H$ .

EXAMPLE 4.4. Let H be a Hopf algebra and assume that H is formally smooth in  $\mathfrak{M}^H$ . Then, by [AMS2, Corollary 3.30], the tensor algebra  $T := T_H(\Omega^1 H)$  is formally smooth as an algebra in the monoidal category  $\mathfrak{M}^H$ . Assume that  $\pi : E \to T$  is an epimorphism that is also a morphism of algebras in  $\mathfrak{M}^H$  such that  $I := \text{Ker } \pi$  is a nilpotent coideal. By Proposition 4.2, applied to the case when  $f : H \to T$  is the canonical injection,  $\pi$  has a section which is an algebra homomorphism in  $\mathfrak{M}^H$ . (In particular the projection  $E \to T \to H$  has also a section which is an algebra homomorphism in  $\mathfrak{M}^H$ ). Observe that, in general, T is not semisimple because its dimension needs not to be finite.

THEOREM 4.5. Let H be a Hopf algebra and  $\mathcal{M}$  be either  $\mathfrak{M}^H$  or  ${}^H\mathfrak{M}^H$ . Let E be an algebra in  $\mathcal{M}$ . Let  $\pi : E \to H$  be an algebra homomorphism in  $\mathcal{M}$  which is surjective. Assume that H is formally smooth as an algebra in  $\mathcal{M}$  and that the kernel I of  $\pi$  is a nilpotent ideal. Then  $\pi$  has a section which is an algebra homomorphism in  $\mathcal{M}$  for

a) 
$$\mathcal{M} = \mathfrak{M}^{H}$$
.  
b)  $\mathcal{M} = {}^{H}\mathfrak{M}^{H}$  if any canonical map  $E/I^{r+1} \to E/I^{r}$  splits in  $\mathcal{M}$ .

*Proof.* Since  $\pi$  is a morphism in  $\mathcal{M}$ , the kernel I of  $\pi$  is a subobject of E in  $\mathcal{M}$ . Hence, for every r > 0,  $I^r$  is a subobject of E and the canonical maps  $E/I^{r+1} \to E/I^r$  are morphisms in  $\mathcal{M}$ .

a) Apply Proposition 4.2 in the case when E := H and  $f := Id_H$ .

b) It follows easily by Theorem 4.1.

Proposition 4.2 studies the existence in  $\mathfrak{M}^H$  of algebra sections of morphisms of algebras  $\pi : E \to A$  where A is a formally smooth algebra in  $\mathfrak{M}^H$  endowed with a morphism of algebras  $f : H \to A$  in  $\mathfrak{M}^H$ . The following results show that the existence of *ad*-invariant integrals provides such a section both in  $\mathfrak{M}^H$  and  ${}^H\mathfrak{M}^H$  (without f).

LEMMA 4.6. Let H be a Hopf algebra with a total integral  $\lambda \in H^*$ . Let  $\mathcal{M}$  be either  $\mathfrak{M}^H$  or  ${}^H\mathfrak{M}^H$ . Then any epimorphism in  $\mathcal{M}$  has a section in  $\mathcal{M}$ .

*Proof.* Since  $\lambda$  is a total integral in  $H^*$ , then, by Theorem 3.5, H is coseparable in  $\mathfrak{M}_K$ . Therefore any right (resp. two-sided) H-comodule is projective (see Corollary 2.19). In particular any any epimorphism in  $\mathcal{M}$  has a section in  $\mathcal{M}$ .

THEOREM 4.7. Let H be a Hopf algebra with an ad-invariant integral  $\lambda \in H^*$ . Let  $\mathcal{M}$  be either  $\mathfrak{M}^H$  or  ${}^H\mathfrak{M}^H$ . Let A and E be algebras in  $\mathcal{M}$ . Let  $\pi : E \to A$  be an algebra homomorphism in  $\mathcal{M}$  which is surjective. Assume that A is formally smooth as an algebra in  $\mathfrak{M}_K$  and that the kernel of  $\pi$  is a nilpotent ideal. Then  $\pi$  has a section which is an algebra homomorphism in  $\mathcal{M}$ .

Proof. By Proposition 3.15, A is formally smooth as an algebra in  $\mathcal{M}$ . Let  $n \geq 1$  such that  $I^n = 0$ , where  $I = \text{Ker } \pi$ . Since, in particular,  $\lambda$  is a total integral, by Lemma 4.6, any epimorphism in the category  $\mathcal{M}$  splits in  $\mathcal{M}$ . Thus, for every  $r = 1, \dots, n-1$  the canonical morphism  $\pi_r : E/I^r \to E/I^{r+1}$  has a section in the category  $\mathcal{M}$ . We can now conclude by applying Theorem 4.1 to the homomorphism of algebras  $\pi : E \to A$ . THEOREM 4.8. Let H be a Hopf algebra with an ad-invariant integral and such that H is formally smooth as an algebra in  ${}_{K}\mathfrak{M}$ . Let  $\mathcal{M}$  be either  $\mathfrak{M}^{H}$  or  ${}^{H}\mathfrak{M}^{H}$ . Let E be an algebra in  $\mathcal{M}$ . Let  $\pi : E \to H$  be a algebra homomorphism in  $\mathcal{M}$  which is surjective and with nilpotent kernel. Then  $\pi$  has a section which is an algebra homomorphism in  $\mathcal{M}$ .

REMARK 4.9. By Proposition 3.15, if H is a Hopf algebra with an *ad*-invariant integral and H is formally smooth as an algebra in  $(_K\mathfrak{M}, \otimes, K)$ , then it is formally smooth as an algebra in  $(\mathfrak{M}^H, \otimes, K)$ . Then the case  $\mathcal{M} = \mathfrak{M}^H$  of the above corollary can be also deduced by Theorem 4.5.

5. Formal Smoothness of a Hopf algebra as an algebra

In order to apply Theorem 4.5, it is useful to characterize when the algebra H is formally smooth either in  $\mathfrak{M}^H$  or in  ${}^H\mathfrak{M}^H$ .

5.1. Let H be a Hopf algebra with antipode S over a field K. We denote by  $H^+$  the augmentation *ideal*, that is the kernel of the counit  $\varepsilon : H \to K$ .

Observe that  $\varepsilon$  can be regarded as a morphism in  ${}^{H}_{H}\mathcal{YD}$ , once H is regarded as an object in  ${}^{H}_{H}\mathcal{YD}$  via the coaction  ${}^{H}\varrho$  (defined in 3.2) and the action given by the multiplication m. In this way  $H^{+} = \ker(\varepsilon)$  inherits the following structure of left-left Yetter-Drinfeld module:

$$h \cdot x = hx,$$
  ${}^{H}\rho(x) = x_{(1)}S(x_{(3)}) \otimes x_{(2)}$ 

for all  $h \in H$  and  $x \in H^+$ .

We call an *fs-section* any map  $\tau: H^+ \to H \otimes H^+$  such that:

(i) 
$$\tau(hx) = \sum_{i \in I} ha_i \otimes b_i$$

 $(ii) \sum_{i \in I} a_i \overline{b_i} = x,$ 

for all  $h \in H$  and  $x \in H^+$ , where  $\tau(x) = \sum_{i \in I} a_i \otimes b_i$ .

We say that an fs-section is *complete* whenever

$$(iii) \sum_{i \in I} a_{i(1)} b_{i(1)} S(a_{i(3)} b_{i(3)}) \otimes a_{i(2)} \otimes b_{i(2)} = x_{(1)} S(x_{(3)}) \otimes \tau(x_{(2)}).$$

LEMMA 5.2.  $\tau$  is a complete fs-section if and only if  $\tau$  is a section in  ${}^{H}_{H}\mathcal{YD}$  of the counit  $\varepsilon_{H^+}$ :  $H \otimes H^+ \to H^+$  of the adjunction  $(F^3, G^3)$  introduced in 3.3.

*Proof.* The notion of complete fs-section can be read as follows: condition (i) means that  $\tau$  is left H-linear, (iii) that  $\tau$  is left H-colinear and (ii) that  $\tau$  is a section of the counit  $\varepsilon_{H^+}$  of the adjunction  $(F^3, G^3)$ , i.e.  $\varepsilon_{H^+} \circ \tau = \mathrm{Id}_{H^+}$ .

PROPOSITION 5.3. Let H be a finite dimensional Hopf algebra over a field K and let  $H^+$  be the augmentation ideal. Let  $\tau: H^+ \to H \otimes H^+$  be a K-linear map such that

$$\tau(hx) = \sum_{i \in I} ha_i \otimes b_i$$

for all  $h \in H, x \in H^+$ , where  $\tau(x) = \sum_{i \in I} a_i \otimes b_i$ . Then  $Im(\tau) \subseteq H^+ \otimes H^+$ .

*Proof.* Since H is finite dimensional, there exists a non-zero right integral  $t \in H$ .

Let  $x \in H^+$ . Since  $Im(\tau) \subseteq H \otimes H^+$ , we can write  $\tau(x) = \sum_{i \in I} a_i \otimes b_i$ ,  $a_i \in H, b_i \in H^+$ . We have

$$\sum_{i \in I} ta_i \otimes b_i = \sum_{i \in I} t\varepsilon (a_i) \otimes b_i = t \otimes \sum_{i \in I} \varepsilon (a_i) b_i$$
  
$$\tau(tx) = \tau(t\varepsilon (x)) = 0.$$

Therefore, since  $\tau(tx) = \sum_{i \in I} ta_i \otimes b_i$ , we get  $\sum_{i \in I} \varepsilon(a_i) b_i = 0$ . Hence  $Im(\tau) \subseteq \ker(\varepsilon \otimes H^+) = H^+ \otimes H^+$ .

PROPOSITION 5.4. Let H be a Hopf algebra with antipode S over a field K and let  $H^+$  be the augmentation ideal. The following assertions are equivalent:

- (a) H is formally smooth as an algebra in  ${}^{H}\mathfrak{M}^{H}$ .
- (b)  $H^+$  is  $\mathcal{E}_G$ -projective where G is the forgetful functor  ${}^H_H \mathcal{YD} \to {}^H \mathfrak{M}$ .
- (c) There exists a complete fs-section  $\tau: H^+ \to H \otimes H^+$ .

Moreover, if H finite dimensional, the following assertion is also equivalent to the others: (d) The multiplication  $H^+ \otimes H^+ \to H^+$  has a left H-linear section  $\tau : H^+ \to H^+ \otimes H^+$ , where  $H^+ \otimes H^+$  is a left H-module via  ${}^{H}\mu_{H^+} \otimes H^+$  and such that  $\sum_{i \in I} a_{i(1)}b_{i(1)}S(a_{i(3)}b_{i(3)}) \otimes a_{i(2)} \otimes b_{i(2)} = x_{(1)}S(x_{(3)}) \otimes \tau(x_{(2)})$  for all  $x \in H^+$ , where  $\tau(x) = \sum_{i \in I} a_i \otimes b_i$ .

Proof. (b)  $\Leftrightarrow$  (c) Consider the functor  $G^3 : {}^{H}_{H}\mathcal{YD} \to {}^{H}\mathfrak{M}$  and it's left adjoint  $F^3 : {}^{H}\mathfrak{M} \to {}^{H}_{H}\mathcal{YD}$ (see 3.3). We know (see Theorem 1.4) that  $H^+$  is  $\mathcal{E}_{G^3}$ -projective if and only if the counit of the adjunction  $\varepsilon_{H^+} : F^3G^3(H^+) \to H^+$  has a section  $\tau : H^+ \to H \otimes H^+$  in  ${}^{H}_{H}\mathcal{YD}$ : thus, by Lemma 5.2,  $\tau$  is a complete fs-section.

 $(a) \Leftrightarrow (b)$  In view of Examples 2.22, consider the following diagrams:

$$\begin{array}{c} {}^{H}\mathfrak{M}_{H}^{H} \xrightarrow{F'=(-)^{coH}} {}^{H}\mathfrak{M} \\ \mathbb{T}=F^{b} \bigvee \qquad \qquad \downarrow \mathbb{T}'=F^{3} \\ {}^{H}\mathfrak{M}_{H}^{H} \xrightarrow{\sim} {}^{F=(-)^{coH}} {}^{H}_{H}\mathcal{YD} \end{array} \xrightarrow{H}\mathfrak{M}_{H}^{H} \xrightarrow{F=(-)^{coH}} {}^{H}_{H}\mathcal{YD}$$

and the forgetful functor  $G^a : {}^{H}\mathfrak{M}_{H}^{H} \to {}^{H}\mathfrak{M}^{H}$ . The second diagram is commutative. Since  $G' \circ G^3 = G^b \circ G$ , by the uniqueness of the adjoint, it is straightforward to prove that the functors  $F^3 \circ F'$  and  $F \circ F^b$  are naturally equivalent. By definition, H is formally smooth in  ${}^{H}\mathfrak{M}^{H}$ , if and only if  $\Omega^1 H$  is  $\mathcal{E}_{G^a \circ G^b}$ -projective (In fact  ${}_{H}\mathcal{E}_{H} = \mathcal{E}_{G^2}$  and  $G^2 = G^a \circ G^b$ ). By Proposition 2.4,  $G^a$  is separable, so that, by Lemma 1.9,  $\mathcal{E}_{G^b} = \mathcal{E}_{G^a \circ G^b}$ . Moreover, the functor F is separable as an equivalence of categories so that, by Theorem 1.13,  $\Omega^1 H$  is  $\mathcal{E}_{G^b}$ -projective if and only if  $H^+ \simeq F(\Omega^1 H)$  (see [Scha2, Example 5.8] for this isomorphism) is  $\mathcal{E}_{G^3}$ -projective.

 $(c) \Rightarrow (d)$  By Proposition 5.3,  $\operatorname{Im}(\tau) \subseteq H^+ \otimes H^+$  so that  $\tau$ , corestricted to  $H^+ \otimes H^+$ , is the required left *H*-linear section of the multiplication  $H^+ \otimes H^+ \to H^+$ .  $(d) \Rightarrow (c)$  Trivial.

**PROPOSITION 5.5.** Let H be a Hopf algebra over a field K and let  $H^+$  be the augmentation ideal. The following assertions are equivalent:

- (a) H is formally smooth as an algebra in  $\mathfrak{M}_K$ .
- (b) H is formally smooth as an algebra in  $\mathfrak{M}^{H}$ .
- (c)  $H^+$  is projective in  $_H\mathfrak{M}$ .
- (d) There exists an fs-section  $\tau: H^+ \to H \otimes H^+$ .
- (e) H is a hereditary K-algebra.

Moreover, if H finite dimensional, the following assertion is also equivalent to the others:

(f) The multiplication  $H^+ \otimes H^+ \to H^+$  has a left H-linear section, where  $H^+ \otimes H^+$  is a left H-module via  ${}^{H}\mu_{H^+} \otimes H^+$ .

*Proof.* The equivalences between (b), (c), (d) and (f) follow similarly to Proposition 5.4, but working with the following diagrams:

$$\mathfrak{M}_{H}^{H} \xrightarrow{F'=(-)^{coH}} {}_{K}\mathfrak{M} \qquad \mathfrak{M}_{H}^{H} \xrightarrow{F'=(-)^{coH}} {}_{K}\mathfrak{M}$$
$$\mathfrak{M}_{H}^{H} \xrightarrow{\sim} {}_{K}\mathfrak{M} \qquad \mathfrak{M}_{H}^{H} \xrightarrow{\sim} {}_{K}\mathfrak{M}$$
$$\mathfrak{M}_{H}^{H} \xrightarrow{\sim} {}_{F=(-)^{coH}} {}_{H}\mathfrak{M} \qquad \mathfrak{M}_{H}^{H} \xrightarrow{\sim} {}_{F=(-)^{coH}} {}_{H}\mathfrak{M}$$

One can check that  $(b) \Leftrightarrow (c) \Leftrightarrow H^+$  is  $\mathcal{E}_{\mathbb{H}'}$ -projective (where  $\mathbb{H}'$  is the forgetful functor  ${}_H\mathfrak{M} \to {}_K\mathfrak{M}$ ). Now, since K is a field, we have that  $\mathcal{E}_{\mathbb{H}'} = \{g \in {}_H\mathfrak{M} \mid g \text{ is a surjection}\}$ , so that  $H^+$  is  $\mathcal{E}_{\mathbb{H}'}$ -projective if and only if  $H^+$  is projective in  ${}_H\mathfrak{M}$ .

 $(b) \Rightarrow (a)$  Apply Theorem 2.12 In the case when A = H and  $\mathcal{M} = \mathfrak{M}^H$ .

 $(a) \Rightarrow (e)$  See [CQ, Proposition 5.1].

 $(e) \Rightarrow (c)$  Every left *H*-submodule of a projective left *H*-module is projective. In particular any left ideal of *H* is projective in  $_H\mathfrak{M}$ .

REMARK 5.6. Let H be a Hopf algebra with antipode S over a field K. Then H is formally smooth as an algebra in  ${}^{H}\mathfrak{M}^{H} \Rightarrow H$  is formally smooth as an algebra in  $\mathfrak{M}^{H}$ .

COROLLARY 5.7. Let H be a Hopf algebra over a field K. Assume that H has an ad-invariant integral. Let  $H^+$  be the augmentation ideal. The following assertions are equivalent:

(i) H is formally smooth as an algebra in  $\mathfrak{M}_K$ .

(ii) H is formally smooth as an algebra in  $\mathfrak{M}^{H}$ .

(iii) H is formally smooth as an algebra in  ${}^{H}\mathfrak{M}^{H}$ .

(iv)  $H^+$  is projective in  $_H\mathfrak{M}$ .

(v) There exists an fs-section  $\tau: H^+ \to H \otimes H^+$ .

(vi) H is an hereditary K-algebra.

Moreover, if H finite dimensional, the following assertion is also equivalent to the others:

(vii) The multiplication  $H^+ \otimes H^+ \to H^+$  has a left H-linear section, where  $H^+ \otimes H^+$  is a left H-module via  ${}^{H}\mu_{H^+} \otimes H^+$ .

*Proof.* Let  $\mathcal{M}$  be either  $\mathfrak{M}^H$  or  ${}^H\mathfrak{M}^H$  and observe that H is an algebra in  $\mathcal{M}$ . Then by Proposition 3.15, H is formally smooth as an algebra in  $\mathcal{M}$  if and only if it is formally smooth as an algebra in  $\mathfrak{M}_K$  that is (i), (ii) and (iii) are equivalent. By Proposition 5.5, we conclude.

By applying Corollary 5.7, we obtain the following result to be compared with [LB, Theorem 2].

THEOREM 5.8. Let G be an arbitrary group an let KG be the group algebra associated. Then the following assertions are equivalent:

(i) KG is formally smooth as an algebra in  $\mathfrak{M}_K$ .

(ii) KG is formally smooth as an algebra in  $\mathfrak{M}^{KG}$ .

(iii) KG is formally smooth as an algebra in  ${}^{KG}\mathfrak{M}^{KG}$ .

(iv) The augmentation ideal  $KG^+$  is a projective in  ${}_{KG}\mathfrak{M}$ .

(v) There exists an fs-section  $\tau : KG^+ \to KG \otimes KG^+$ .

(vi) KG is an hereditary K-algebra.

(vii) G is the fundamental group of a connected graph of finite groups whose orders are invertible in K. (see [Di, Definition 4.2, page 10]).

Moreover, if G is finite, the following assertion is also equivalent to the others:

(viii) The multiplication  $KG^+ \otimes KG^+ \to KG^+$  has a left KG-linear section, where  $KG^+ \otimes KG^+$ is a left KG-module via  ${}^{KG}\mu_{KG^+} \otimes KG^+$ .

*Proof.* By the left analogue of [Di, Theorem 2.12, page 118], (iv) and (vii) are equivalent.

By Example 3.8, the Hopf algebra KG admits an *ad*-invariant integral. The conclusion follows by Corollary 5.7.

By means of Proposition 5.5, it is now possible to rewrite Theorem 4.5 in the following form which improves Theorem 4.8 in the case  $\mathcal{M} = \mathfrak{M}^H$ .

THEOREM 5.9. Let H be a Hopf algebra and let E be an algebra in  $\mathfrak{M}^H$ . Let  $\pi : E \to H$  be an algebra homomorphism in  $\mathfrak{M}^H$  which is surjective. Assume that H is formally smooth as an algebra in  $\mathfrak{M}_K$  and that the kernel I of  $\pi$  is a nilpotent ideal. Then  $\pi$  has a section which is an algebra homomorphism in  $\mathfrak{M}^H$ .

As a consequence of Theorem 5.9, we get the following result.

THEOREM 5.10. Let H be a Hopf algebra and let E be a bialgebra. Let  $\pi : E \to H$  be a bialgebra homomorphism which is surjective. Assume that H is formally smooth as an algebra in  $\mathfrak{M}_K$  and that the kernel I of  $\pi$  is a nilpotent ideal. Then  $\pi$  has a section which is an algebra homomorphism in  $\mathfrak{M}^H$ .

REMARK 5.11. Akira Masuoka pointed out that, in the situation of Theorem 5.10, since H is a Hopf algebra so is E (see e.g. [AMS1, Lemma 3.52]).

### 6. Examples

**PROPOSITION 6.1.** Let K be any field. The group algebra  $K\mathbb{Z}$  over the set of integers admits a complete fs-section.

*Proof.* Let  $\langle g \rangle$  be the multiplicative group associated to  $\mathbb{Z}$  ( $\langle g \rangle \simeq \mathbb{Z}$ ). Let  $H = K \langle g \rangle$ . Then

$$\mathcal{B}(H) = \left(g^n - g^{n+1}\right)_{n \in \mathbb{Z}}$$

is a basis for  $H^+$ . Now define  $\tau: H^+ \to H \otimes H^+$  on generators by setting

$$\tau\left(g^{n}-g^{n+1}\right)=g^{n}\otimes\left(1-g\right),$$

for every  $n \in \mathbb{Z}$ . Clearly  $g^n \cdot (1-g) = g^n - g^{n+1}$ . Moreover

$$\tau \left[ g^a \left( g^n - g^{n+1} \right) \right] = \tau \left( g^{a+n} - g^{a+n+1} \right) = g^{a+n} \otimes (1-g) = g^a \cdot g^n \otimes (1-g) = g^a \cdot \tau \left[ \left( g^n - g^{n+1} \right) \right].$$

Since H is cocommutative, this is enough to conclude that  $\tau$  is a complete fs-section of H.

REMARKS 6.2. 1) By Proposition 6.1 and Theorem 5.8,  $K\mathbb{Z}$  is formally smooth as an algebra in  $\mathfrak{M}_K$ . Nevertheless, being not finite dimensional,  $K\mathbb{Z}$  is not separable as an algebra in  $\mathfrak{M}_K$ . More generally, the group algebra KG is formally smooth but not separable if and only if G is a free and non-trivial group (see Remark 6.5).

2) The complete fs-section  $\tau$  defined in the proof of Proposition 6.1 is such that  $\operatorname{Im}(\tau) \nsubseteq K\mathbb{Z}^+ \otimes K\mathbb{Z}^+$ . This is a counterexample for the last assertion of Proposition 5.4.

**PROPOSITION 6.3.** Let  $C_n$  be the cyclic group of order n and let  $KC_n$  be the group algebra associated. Then the following assertions are equivalent:

- (i)  $KC_n$  is formally smooth as an algebra in  $\mathfrak{M}_K$ .
- (ii)  $KC_n$  is separable as an algebra in  $\mathfrak{M}_K$ .
- (*iii*)  $n \cdot 1_K \neq 0$ .

*Proof.*  $(ii) \Leftrightarrow (iii)$  is the well known Maschke's Theorem.

 $(ii) \Rightarrow (i)$  follows by Corollary 2.7.

 $(i) \Rightarrow (iii)$  By Theorem 5.8, the multiplication  $KC_n^+ \otimes KC_n^+ \to KC_n^+$  has a section. In particular the multiplication is surjective, so that  $KC_n^+ = (KC_n^+)^2$ .

Let  $g \in C_n$  be a generator of  $C_n$ , that is o(g) = n. Then  $KC_n^+ = \sum_{i=0}^{n-1} K(1-g^i)$ . From  $1-g \in KC_n^+ = (KC_n^+)^2$ , we deduce there exists  $\alpha_{i,j} \in K$  such that

(3) 
$$1 - g = \sum_{0 \le i,j \le n-1} \alpha_{i,j} \left( 1 - g^i \right) \left( 1 - g^j \right) = \sum_{0 \le i,j \le n-1} \alpha_{i,j} \left( 1 - g^i - g^j + g^{i+j} \right).$$

Define the K-linear map  $\varphi : KC_n \to K$  by setting  $\varphi(g^i) = (1-i) \mathbf{1}_K$  for every  $0 \le i \le n-1$ . Now suppose that  $n \cdot \mathbf{1}_K = 0$ . In this case, since n = o(g), it is easy to check that  $\varphi(g^i) = (1-i) \mathbf{1}_K$ for every  $i \in \mathbb{N}$  and hence, by (3), we have

$$1 = \varphi (1 - g) = \sum_{0 \le i, j \le n - 1} \alpha_{i, j} \varphi \left( 1 - g^i - g^j + g^{i + j} \right) = 0,$$

a contradiction.

6.4. Implication  $(i) \Rightarrow (iii)$  of Proposition 6.3, can be proved in a different way. In fact (i) implies that the Hochschild cohomology  $H^2(KC_n, M)$  vanishes for every  $KC_n$ -bimodule M. By [McL, Theorem 5.5, page 292] (where the result is proved for  $\mathbb{Z}$  instead of K although the same arguments go through for any commutative ring), for every group G, one has a natural isomorphism

$$H^t(KG, M) \simeq H^t(G, {}_{\chi}M)$$

where  $_{\chi}M$  is M endowed with the left G-module structure given by  $g \cdot_{\chi}m = gmg^{-1}$  and  $H^t(G,_{\chi}M)$  denotes the group cohomology. Apply this isomorphism to the case  $G = C_n, t = 2$  and let g denote a generator of  $C_n$ .

By [McL, Theorem 7.1, page 122], for every left  $C_n$ -module L, one has

$$H^{2}(C_{n},L) = \frac{\{l \in L \mid g \cdot l = l\}}{t \cdot L},$$

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where  $t = 1 + g + g^2 + \dots + g^{n-1}$ . Now, assume that (i) holds. Then

$$\frac{\left\{x \in KC_n \mid g \cdot_{\chi} x = x\right\}}{t \cdot_{\chi} KC_n} = H^2\left(C_n, {}_{\chi}KC_n\right) \simeq H^2\left(KC_n, KC_n\right) = 0$$

Since  $C_n$  is commutative, then  $g^i \cdot_{\chi} x = g^i x g^{-i} = g^i g^{-i} x = x$  for every  $x \in KC_n$  so that

$$KC_n = \{x \in KC_n \mid g \cdot_{\chi} x = x\} = t \cdot_{\chi} KC_n = \sum_{0 \le i \le n-1} g^i \cdot KC_n \cdot g^{-i} = n \cdot KC_n \cdot g^{-i}$$

Therefore  $1 \in KC_n = n \cdot KC_n$  and hence  $n \cdot 1_K \neq 0$ .

REMARK 6.5. By Proposition 6.3,  $KC_n$  is formally smooth as an algebra in  $\mathfrak{M}_K$  if and only if it is separable as an algebra in  $\mathfrak{M}_K$ .

The groups G such that KG is **formally smooth as an algebra** in  $\mathfrak{M}_K$  **but not separable** as an algebra in  $\mathfrak{M}_K$  are precisely those having **cohomological dimension** 1. This follows in view of the isomorphism  $H^t(KG, M) \simeq H^t(G, {}_{\chi}M)$  that holds for every  $t \in \mathbb{N}$ , and for any KG-bimodule M. Note also that every left G-module N can be seen as  ${}_{\chi}({}_{KG}N_{KG})$  where  ${}_{KG}N_{KG}$  is N itself regarded as a bimodule via  $g \cdot n \cdot h := gn$ , for every  $g, h \in G, n \in N$ .

Furthermore (see [Br, Example 2, page 185]) every free group over a non-empty (possibly infinite) set has cohomological dimension 1. Conversely every group of cohomological dimension 1 is free.

# 7. Ad-coinvariant integrals through separable functors

We want now to treat the dual of all the results of the previous sections. We just state the main results that can be proved analogously.

First of all we characterize the existence of a so-called ad-coinvariant integral. A remarkable fact is that any semisimple and cosemisimple Hopf algebra H over a field K admits such an integral (see [AMS1, Theorem 2.27]).

DEFINITION 7.1. Let H be a Hopf algebra with antipode S over any field K and let  $t \in H$ . t will be called an *ad-coinvariant integral* whenever:

- a)  $ht = \varepsilon_H(h)t$  for all  $h \in H$  (i.e. t is a left integral in H);
- b)  $t_1 S(t_3) \otimes t_2 = 1_H \otimes t$ , (i.e. t is left coinvariant with respect to  ${}^H \varrho$ );

c) 
$$\varepsilon_H(t) = 1_K$$
.

Therefore we have:

LEMMA 7.2. An element  $t \in H$  is an ad-coinvariant integral if and only if the map  $\tau : K \to H : k \mapsto kt$  is a section of the counit  $\varepsilon_H : H \to K$  of H in  ${}^H_H \mathcal{YD}$ , where H is regarded as an object in the category via the left adjoint coaction  ${}^H\varrho$  and the multiplication  $m_H$ .

EXAMPLE 7.3. 1) Let G be a finite group an let  $K^G$  be the algebra of functions from G to K. Then  $K^G$  becomes a Hopf algebra which is dual to the group algebra KG. From Example 3.8, we infer that  $K^G$  has an *ad*-coinvariant integral, namely the map  $G \to K : g \mapsto \delta_{e,g}$  (the Kronecker symbol), where e denotes the neutral element of G.

2) Every cocommutative semisimple Hopf algebra has an *ad*-coinvariant integral.

REMARK 7.4. It is known that, for any Hopf algebra H with a total integral  $t \in H$ , the K-linear spaces of left and right integrals in H are both one dimensional and so both generated by t. Hence there can be only one *ad*-coinvariant integral, namely the unique total integral.

The following lemma shows that in the definition of *ad*-coinvariant integral we can choose  $\rho^H, \overline{\rho}^H$  or  ${}^{H}\overline{\rho}$  instead of  ${}^{H}\rho$ . Since *t* is in particular a total integral, it is both a left integral and a right integral. Thus it is the same to have a retraction of  $\varepsilon_H$  in  ${}^{H}_{H}\mathcal{YD}, \mathcal{YD}^{H}_{H,H}\mathcal{YD}^{H}$  or  ${}^{H}\mathcal{YD}_{H}$ .

LEMMA 7.5. Let H be a Hopf algebra with antipode S over any field K and let  $t \in H$  be a total integral. Then the following are equivalent:

(1) t is left coinvariant with respect to  ${}^{H}\varrho$ .

(2) t is right coinvariant with respect to  $\rho^{H}$ .

- (3) t is right coinvariant with respect to  $\overline{\varrho}^{H}$ .
- (4) t is left coinvariant with respect to  ${}^{H}\overline{\rho}$ .

Proof. Analogous to 3.10.

LEMMA 7.6. Let H be a Hopf algebra with antipode S over a field K. Assume there exists an ad-coinvariant integral  $t \in H$ . Then we have that:

- i) The forgetful functor  ${}^{C}\mathfrak{M}_{H}^{C} \to {}^{C}\mathfrak{M}^{C}$  is separable for any coalgebra C in  $\mathfrak{M}_{H}$ .
- ii) The forgetful functor  ${}^{C}_{H}\mathfrak{M}^{C} \to {}^{C}\mathfrak{M}^{C}$  is separable for any coalgebra C in  ${}^{H}\mathfrak{M}$ .
- iii) The forgetful functor  ${}^{C}_{H}\mathfrak{M}^{C}_{H} \to {}^{C}\mathfrak{M}^{C}$  is separable for any coalgebra C in  ${}_{H}\mathfrak{M}_{H}$ .

*Proof.* We proceed as in the proof of Lemma 3.11.

i) By Examples 2.22, the forgetful functor  $G^r: {}^C\mathfrak{M}^C_H \to {}^C\mathfrak{M}^C$  has a right adjoint  $F^r: {}^C\mathfrak{M}^C \to$  ${}^{C}\mathfrak{M}^{C}_{H}, F^{r}(M) = M \otimes H.$  Thus by Theorem 1.11,  $G^{r}$  is separable if and only if the counit  $\varepsilon^{H}$ :  $F^r G^r \to \mathrm{Id}_{\mathcal{C}\mathfrak{M}^r_G}$  of the adjunction splits, i.e. there exists a natural transformation  $\sigma^H : \mathrm{Id}_{\mathcal{C}\mathfrak{M}^r_G} \to$  $F^r G^r$  such that  $\varepsilon_M^H \circ \sigma_M^H = \operatorname{Id}_M$  for any M in  ${}^C \mathfrak{M}_H^C$ . Using (1), one can easily check that the following map works:  $\sigma_M^H : M \to M \otimes H$ ,  $\sigma_M^H(m) = mt_1 \otimes S(t_2)$ . *ii*) Analogous to *i*) by setting  ${}^H \sigma_M(m) = t_1 \otimes S(t_2)m$ . *iii*) Define  $\sigma_M := ({}^H \sigma_M \otimes H) \circ \sigma_M^H : M \to H \otimes M \otimes H$ .

We can now consider the main result concerning ad-coinvariant integrals. The equivalence  $(1) \Leftrightarrow (3b)$  was proved in a different way in [AMS1, Proposition 2.11].

THEOREM 7.7. Let H be a Hopf algebra over a field K. The following assertions are equivalent:

- (1) There is an ad-coinvariant integral  $t \in H$ .
- (2) The forgetful functor  ${}^{C}_{H}\mathfrak{M}^{C}_{H} \to {}^{C}\mathfrak{M}^{C}$  is separable for any coalgebra C in  ${}^{H}\mathfrak{M}_{H}$ . (3) The forgetful functor  ${}^{H}_{H}\mathfrak{M}^{H}_{H} \to {}^{H}\mathfrak{M}^{H}$  is separable.
- (3b) *H* is separable in  $({}^{H}\mathfrak{M}{}^{H}, \otimes, K)$ .
- (4) The forgetful functor  ${}^{H}_{H}\mathcal{YD} \to {}^{H}\mathfrak{M}$  is separable.
- (4b) K is  $\mathcal{E}_G$ -projective where G is the forgetful functor of (4).

*Proof.* Analogous to that of Theorem 3.12.

REMARK 7.8. The following assertions are all equivalent to the existence of an ad-coinvariant integral  $t \in H$ :

(5) The forgetful functor  $\mathcal{YD}_H^H \to \mathfrak{M}^H$  is separable.

- (6) The forgetful functor  ${}_{H}\mathcal{YD}^{H} \to \mathfrak{M}^{H}$  is separable and S is bijective. (7) The forgetful functor  ${}^{H}\mathcal{YD}_{H} \to {}^{H}\mathfrak{M}$  is separable and S is bijective.
- (8) K is  $\mathcal{E}_G$ -projective where G is the forgetful functor of (5),(6) or (7).

In fact, note that  ${}^{H}_{H}\mathfrak{M}^{H}_{H} \simeq \mathcal{YD}^{H}_{H}$ . Since t is in particular a total integral, the antipode S is bijective and hence, by [Scha2, Corollary 6.4], we can also assume  ${}^{H}\mathcal{YD}_{H} \simeq {}^{H}_{H}\mathfrak{M}_{H}^{H} \simeq {}_{H}\mathcal{YD}^{H}$ . Now, by means of Lemma 7.5. one prove the above equivalences.

THEOREM 7.9. Let H be a finite dimensional Hopf algebra over a field K and let D(H) be the Drinfeld Double. The following assertions are equivalent:

- (i) There is an ad-coinvariant integral  $t \in H$ .
- (ii) The forgetful functor  $\mathfrak{M}^{D(H)^*} \to \mathfrak{M}^H$  (equiv.  $_{D(H)}\mathfrak{M} \to {}_{H^*}\mathfrak{M}$ ) is separable.
- (iii)  $D(H)^*$  is coseparable in  $({}^H\mathfrak{M}^H, \Box_H, H)$  (equiv.  $D(H)/H^*$  is separable).

*Proof.* It is dual to Theorem 3.14.

**PROPOSITION 7.10.** Let H be a Hopf algebra with an ad-coinvariant integral t and let  $\mathcal{M}$  be either  $\mathfrak{M}_H$  or  $_H\mathfrak{M}_H$ . For any coalgebra C in  $\mathcal{M}$ , we have:

i) C is coseparable as a coalgebra in  $\mathcal{M}$  if and only if it is coseparable as a coalgebra in  $\mathfrak{M}_K$ .

ii) C is formally smooth as a coalgebra in  $\mathcal{M}$  if and only if it is formally smooth in  $\mathfrak{M}_{K}$ .

*Proof.* Since H has an ad-coinvariant integral t, by Lemma 7.6, the forgetful functor  $G: {}^{C}\mathcal{M}^{C} \to \mathcal{M}^{C}$  ${}^{C}\mathfrak{M}^{C}$  is separable. By Theorem 2.23 we conclude.  $\Box$ 

### 8. Splitting coalgebra homomorphisms

8.1. Let *E* be a coalgebra in an abelian monoidal category  $\mathcal{M}$ . Let us recall, (see [Mon, §5.2]), the definition of wedge of two subobject *X*, *Y* of *E* in  $\mathcal{M}$ :

$$X \wedge_E Y := Ker[(\pi_X \otimes \pi_Y) \circ \Delta_E],$$

where  $\pi_X : E \to E/X$  and  $\pi_Y : E \to E/Y$  are the canonical quotient maps.

8.2. Let now C be a subcoalgebra of E in an abelian monoidal category  $\mathcal{M}$ . Define  $(C^{\wedge_E^n})_{n\in\mathbb{N}}$  by

$$C^{\wedge_E^0} := 0, \qquad C^{\wedge_E^1} := C, \qquad \text{and} \qquad C^{\wedge_E^n} := C^{\wedge_E^{n-1}} \wedge_E C \text{ for any } n \ge 2.$$

Note that  $C^{\wedge_E^1} \hookrightarrow \cdots \hookrightarrow C^{\wedge_E^n} \hookrightarrow C^{\wedge_E^{n+1}} \hookrightarrow \cdots \hookrightarrow E$  as coalgebras.

In the case when  $\mathcal{M}$  is one of the monoidal categories  $\mathfrak{M}_K, \mathfrak{M}_H$  or  ${}_H\mathfrak{M}_H$ , then the wedge product has the following properties:

- $X \wedge_E Y = \Delta^{-1}(E \otimes Y + X \otimes E);$
- $(X \wedge_E Y) \wedge_E Z = X \wedge_E (Y \wedge_E Z);$
- $X \wedge_E Y$  is a subcoalgebra of E whenever both X and Y are subcoalgebras of E.

REMARK 8.3. Let  $\mathcal{M}$  be one of the monoidal categories  $\mathfrak{M}_K, \mathfrak{M}_H$  or  ${}_H\mathfrak{M}_H$ . Let C be a subcoalgebra of a coalgebra E in  $\mathcal{M}$ . Then  $C^{\wedge_E^1} \subseteq \cdots \subseteq C^{\wedge_E^n} \subseteq C^{\wedge_E^{n+1}} \subseteq \cdots \subseteq E$ .

Moreover, by [Sw, Remark and Proposition, page 226], one has that  $\bigcup_{n \in \mathbb{N}} C^{\wedge_E^n} = E$  if and only if  $Corad(E) \subseteq C$ . Note that  $\bigcup_{n \in \mathbb{N}} C^{\wedge_E^n} = \lim_{n \in \mathbb{N}} C^{\wedge_E^n}$ .

We recall the following important result.

THEOREM 8.4. (see [AMS2, Theorem 4.22]) Let  $(C, \Delta, \varepsilon)$  be a coalgebra in an abelian monoidal category  $(\mathcal{M}, \otimes, \mathbf{1})$  with direct limits. Then the following assertions are equivalent:

(a) C is formally smooth as a coalgebra in  $\mathcal{M}$ .

(b) Let  $\sigma : D \to E$  be a coalgebra homomorphism in  $\mathcal{M}$  which is a monomorphism in  $\mathcal{M}$ . Assume that  $E = \varinjlim C^{\wedge_E^i}$ . If for every  $r \in \mathbb{N}$  the canonical injection  $i_r : C^{\wedge_E^r} \to C^{\wedge_E^{r+1}}$  cosplits in  $\mathcal{M}$ , then  $\sigma$  has a retraction which is a coalgebra homomorphism in  $\mathcal{M}$ .

Then the previous theorem has the following application.

THEOREM 8.5. Let H be a Hopf algebra. Let  $\mathcal{M}$  be one of the monoidal categories  $\mathfrak{M}_K, \mathfrak{M}_H$  or  ${}_H\mathfrak{M}_H$ . Let C be a subcoalgebra of a coalgebra E in  $\mathcal{M}$ . Assume that C is formally smooth as a coalgebra in  $\mathcal{M}$  and that  $Corad(E) \subseteq C$ . If any inclusion map  $i_r : C^{\wedge_E^r} \to C^{\wedge_E^{r+1}}$  cosplits in  $\mathcal{M}$ , then there exists a coalgebra homomorphism  $\pi : E \to C$  in  $\mathcal{M}$  such that  $\pi_{|C} = \mathrm{Id}_C$ .

*Proof.* As observed in Remark 8.3, we have  $E = \bigcup_{n \in \mathbb{N}} C^{\wedge_E^n} = \varinjlim_{e \in \mathbb{N}} C^{\wedge_E^i}$ . The conclusion follows by applying Theorem 8.4.

PROPOSITION 8.6. Let H be a Hopf algebra. Let C be a subcoalgebra of a coalgebra E in  $\mathfrak{M}_H$ . Assume that C is formally smooth as a coalgebra in  $\mathfrak{M}_H$  and that  $Corad(E) \subseteq C$ . Given a coalgebra homomorphism  $g: C \to H$  in  $\mathfrak{M}_H$ , then there exists a coalgebra homomorphism  $\pi: E \to C$  in  $\mathfrak{M}_H$  such that  $\pi_{|C} = \mathrm{Id}_C$ .

Proof. It is similar to [AMS1, Theorem 2.17], where C = H = Corad(E) is cosemisimple and  $g = \mathrm{Id}_H$ . In order to apply Theorem 8.5, we have only to prove that any inclusion map  $C^{\wedge_E^n} \hookrightarrow C^{\wedge_E^{n+1}}$  cosplits in  $\mathfrak{M}_H$ . Since  $C^{\wedge_E^{n+1}} = C^{\wedge_E^n} \wedge_E C = C \wedge_E C^{\wedge_E^n} = \Delta_E^{-1}(E \otimes C + C^{\wedge_E^n} \otimes E)$ , the quotient  $C^{\wedge_E^{n+1}}/C^{\wedge_E^n}$  becomes a right C-comodule in  $\mathfrak{M}_H$  via the map  $\rho_L^C$ , given by  $x + C^{\wedge_E^n} \mapsto (x_1 + C^{\wedge_E^n}) \otimes x_2$ . Since  $g : C \to H$  is a morphism of coalgebras in  $\mathfrak{M}_H$ , then  $(\mathrm{Id} \otimes g) \circ \rho_L^C$  is a right H-comodule structure map for  $C^{\wedge_E^{n+1}}/C^{\wedge_E^n}$  that is right H-linear. Thus  $C^{\wedge_E^{n+1}}/C^{\wedge_E^n}$  becomes an object in  $\mathfrak{M}_H^H$ : by the fundamental theorem for Hopf modules  $(\mathfrak{M}_H^H \simeq K \mathfrak{M})$ , we get that  $C^{\wedge_E^{n+1}}/C^{\wedge_E^n} \simeq V \otimes H$  in  $\mathfrak{M}_H^H$ , for a suitable  $V \in K\mathfrak{M}$ , i.e.  $C^{\wedge_E^{n+1}}/C^{\wedge_E^n}$  is a free right H-module. In particular  $C^{\wedge_E^{n+1}}/C^{\wedge_E^n}$  is a projective right H-module, so that the inclusion map  $i : C^{\wedge_E^n} \hookrightarrow C^{\wedge_E^{n+1}}$  has a retraction in  $\mathfrak{M}_H$ .

THEOREM 8.7. Let H be a Hopf algebra and let  $\mathcal{M}$  be either  $\mathfrak{M}_H$  or  ${}_H\mathfrak{M}_H$ . Assume that H is a subcoalgebra of a coalgebra E in  $\mathcal{M}$ , that H is formally smooth as a coalgebra in  $\mathcal{M}$  and that  $Corad(E) \subseteq H$ . Then there exists a coalgebra homomorphism  $\pi : E \to H$  in  $\mathcal{M}$  such that  $\pi_{|H} = \mathrm{Id}_H$  for

a) 
$$\mathcal{M} = \mathfrak{M}_H$$

b)  $\mathcal{M} = {}_{H}\mathfrak{M}_{H}$  if any inclusion map  $H^{\wedge_{E}^{n}} \hookrightarrow H^{\wedge_{E}^{n+1}}$  cosplits in  $\mathcal{M}$ .

*Proof.*  $H^{\wedge_E^n}$  is a subcoalgebra of E in  $\mathcal{M}$  and the inclusion map  $H^{\wedge_E^n} \hookrightarrow H^{\wedge_E^{n+1}}$  is obviously a morphism in  $\mathcal{M}$ .

a) Apply Proposition 8.6 in the case when C := H and  $g := Id_H$ .

b) Apply Theorem 8.5 in the case when C = H.

EXAMPLES 8.8. Let E be a coalgebra in the category of vector spaces. Let C = Corad(E). In this case, the sequence  $(C^{\wedge_E^n})_{n \in \mathbb{N}}$  is simply denoted by  $(E_n)_{n \in \mathbb{N}}$  and it is the so-called coradical filtration of E.

Let H be a Hopf algebra and let  $\mathcal{M}$  be either  $\mathfrak{M}_H$  or  ${}_H\mathfrak{M}_H$ . Assume that E is a coalgebra in  $\mathcal{M}$  and that H = C = Corad(E). We have two cases.

 $\mathcal{M} = {}_{H}\mathfrak{M}_{H}$ ) If any inclusion  $E_n \hookrightarrow E_{n+1}$  cosplits in  ${}_{H}\mathfrak{M}_{H}$  and H is formally smooth as a coalgebra in  ${}_{H}\mathfrak{M}_{H}$ , then, by Theorem 8.7, there is an homomorphisms of coalgebras  $\pi : E \to H$  in  ${}_{H}\mathfrak{M}_{H}$  such that  $\pi_{|H} = \mathrm{Id}_{H}$ .

 $\mathcal{M} = \mathfrak{M}_H$ ) By [AMS1, Theorem 2.11], since H is cosemisimple in  $\mathfrak{M}_K$ , then H is coseparable in  $\mathfrak{M}_H$ . In particular H is formally smooth as a coalgebra in  $\mathfrak{M}_H$ . Again, by Theorem 8.7, there is an homomorphisms of coalgebras  $\pi : E \to H$  in  $\mathfrak{M}_H$  such that  $\pi_{|H} = \mathrm{Id}_H$  (see also [AMS1, Theorem 2.17]).

Proposition 8.6 studies the existence in  $\mathfrak{M}_H$  of coalgebra retractions of coalgebras inclusion  $C \hookrightarrow E$  where C is a formally smooth coalgebras in  $\mathfrak{M}_H$  endowed with a morphism of coalgebras  $g: C \to H$  in  $\mathfrak{M}_H$ . The following results show that the existence of *ad*-coinvariant integrals provides such a section both in  $\mathfrak{M}_H$  and in  $_H\mathfrak{M}_H$  (without g).

LEMMA 8.9. Let H be a Hopf algebra with a total integral  $t \in H$ . Let  $\mathcal{M}$  be either  $\mathfrak{M}_H$  or  ${}_H\mathfrak{M}_H$ . Then any monomorphism in  $\mathcal{M}$  has a retraction in  $\mathcal{M}$ .

*Proof.* Since t is a total integral in H, then H is separable by Theorem 3.5-2). Therefore any right (resp. left, two-sided) H-module is injective (see Corollary 2.8). In particular any monomorphism in  $\mathcal{M}$  has a retraction in  $\mathcal{M}$ .

THEOREM 8.10. Let H be a Hopf algebra with an ad-coinvariant integral  $t \in H$ . Let  $\mathcal{M}$  be either  $\mathfrak{M}_H$  or  ${}_H\mathfrak{M}_H$ . Let C be a subcoalgebra of a coalgebra E in  $\mathcal{M}$ . Assume that C is formally smooth as a coalgebra in  $\mathfrak{M}_K$  and that  $Corad(E) \subseteq C$ . Then there exists a coalgebra homomorphism  $\pi: E \to C$  in  $\mathcal{M}$  such that  $\pi_{|C} = \mathrm{Id}_C$ .

*Proof.* By Proposition 7.10, C is formally smooth as a coalgebra in  $\mathcal{M}$ . Since t is in particular a total integral in H, by Lemma 8.9, any monomorphism in  $\mathcal{M}$ , in particular the inclusion map  $C^{\wedge_E^n} \hookrightarrow C^{\wedge_E^{n+1}}$  for any  $n \in \mathbb{N}$ , has a retraction in  $\mathcal{M}$ . Now apply Theorem 8.5.

THEOREM 8.11. Let H be a Hopf algebra with an ad-coinvariant integral and such that H is formally smooth as a coalgebra in  $\mathfrak{M}_K$ . Let  $\mathcal{M}$  be either  $\mathfrak{M}_H$  or  ${}_H\mathfrak{M}_H$ . If H is a subcoalgebra of a coalgebra E in  $\mathcal{M}$  and  $Corad(E) \subseteq H$ , then there exists a coalgebra homomorphism  $\pi : E \to H$ in  $\mathcal{M}$  such that  $\pi_{|H} = \mathrm{Id}_H$ .

REMARK 8.12. By Proposition 7.10, if H is a Hopf algebra with an *ad*-coinvariant integral and H is formally smooth as a coalgebra in  $(\mathfrak{M}_K, \otimes, K)$ , then it is formally smooth as a coalgebra in  $(\mathfrak{M}_H, \otimes, K)$ . Then the case  $\mathcal{M} = \mathfrak{M}_H$  of the above corollary can be also deduced by Theorem 8.7.

## 9. Formal Smoothness of a Hopf algebra as a coalgebra

In order to apply Theorem 8.7, it is useful to characterize when the coalgebra H is formally smooth either in  $\mathfrak{M}_H$  or in  ${}_H\mathfrak{M}_H$ .

9.1. Let H be a Hopf algebra with antipode S over a field K. We denote by  $\overline{H}$  the cokernel of the unit  $u: K \to H$ .

Observe that u can be regarded as a morphism in  ${}^{H}_{H}\mathcal{YD}$ , once H is regarded as an object in  ${}^{H}_{H}\mathcal{YD}$ via the action  $\triangleright$  (defined in 3.2) and the coaction given by the comultiplication  $\Delta$ . In this way  $\overline{H} = \operatorname{Coker}(u)$  inherits the following structure of left-left Yetter-Drinfeld module:

$$h \cdot \overline{x} = \overline{h_1 x S(h_2)}, \qquad {}^{H} \rho(\overline{x}) = x_1 \otimes \overline{x_2}$$

for all  $h \in H$  and  $x \in H$  (by  $\overline{x}$  we denote the image of x in  $\overline{H}$ ). We call an *fs-retraction* any map  $\chi : H \otimes \overline{H} \to \overline{H}$  such that:

- (i)  $a_1 \otimes \overline{a_2} = x_1 \otimes \chi(x_2 \otimes \overline{y}),$ (*ii*)  $\chi(x_1 \otimes \overline{x_2}) = \overline{x}$ ,
- for all  $x, y \in H$ , where  $\chi(x \otimes \overline{y}) = \overline{a}$ .

We say that an fs-retraction is *complete* whenever

 $(iii) \ \chi[h_1 x S(h_4) \otimes \overline{h_2 y S(h_3)}] = \overline{h_1 a S(h_2)},$ 

for all  $h, x, y \in H$ , where  $\chi(x \otimes \overline{y}) = \overline{a}$ .

LEMMA 9.2.  $\chi$  is a complete fs-retraction if and only  $\chi$  is a retraction in  ${}^{H}_{H}\mathcal{YD}$  of the unit  $\eta_{\overline{H}} = {}^{H}\rho_{\overline{H}} : \overline{H} \to H \otimes \overline{H}$  of the adjunction  $(F_3, G_3)$  introduced in 3.3

*Proof.* The notion of complete fs-retraction can be read as follows: condition (i) means that  $\chi$  is left H-colinear, (iii) that  $\chi$  is left H-linear and (ii) that  $\chi$  is a retraction of the unit  $\eta_{\overline{H}}$  of the adjunction  $(F_3, G_3)$ , i.e.  $\chi \circ \eta_{\overline{H}} = \mathrm{Id}_{\overline{H}}$ .

**PROPOSITION 9.3.** Let H be a finite dimensional Hopf algebra over a field K and let  $\overline{H}$  be the cokernel of the unit  $u_H: K \to H$ . Let  $\chi: H \otimes \overline{H} \to \overline{H}$  be a K-linear map such that

$$a_1 \otimes \overline{a_2} = x_1 \otimes \chi (x_2 \otimes \overline{y})$$

for all  $x, y \in H$ , where  $\chi(x \otimes \overline{y}) = \overline{a}$ . Then  $\chi: H \otimes \overline{H} \to \overline{H}$  quotients to a map  $\overline{\chi}: \overline{H} \otimes \overline{H} \to \overline{H}$ .

**PROPOSITION 9.4.** Let H be a Hopf algebra with antipode S over a field K and let  $\overline{H}$  be the cokernel of the unit  $u: K \to H$ . The following assertions are equivalent:

- (a) H is formally smooth as a coalgebra in  $_{H}\mathfrak{M}_{H}$ .
- (b)  $\overline{H}$  is  $\mathcal{I}_F$ -injective where F is the forgetful functor  ${}^H_H \mathcal{YD} \to {}_H \mathfrak{M}$ .
- (c) There exists a complete fs-retraction  $\chi: H \otimes \overline{H} \to \overline{H}$ .

Moreover, if H finite dimensional, the following assertion is also equivalent to the others:

(d) The comultiplication  $\overline{H} \to \overline{H} \otimes \overline{H}$  has a left H-colinear retraction  $\overline{\chi} : \overline{H} \otimes \overline{H} \to \overline{H}$ , where  $\overline{H} \otimes \overline{H}$  is a left H-comodule via  ${}^{H}\rho_{\overline{H}} \otimes \overline{H}$  and such that  $\overline{\chi}[\overline{h_1 x S(h_4)} \otimes \overline{h_2 y S(h_3)}] = \overline{h_1 a S(h_2)}$ , for every  $h, x, y \in H$ , where  $\overline{\chi}(\overline{x} \otimes \overline{y}) = \overline{a}$ .

*Proof.* Analogous to Proposition 5.4.

The referee pointed out that the equivalence  $(c) \Leftrightarrow (e)$  in the following proposition was also proved in [MO, Theorem 1.2].

**PROPOSITION 9.5.** Let H be a Hopf algebra over a field K and let  $\overline{H}$  be the cokernel of the unit  $u: K \to H$ . The following assertions are equivalent:

- (a) H is formally smooth as a coalgebra in  $\mathfrak{M}_K$ .
- (b) H is formally smooth as a coalgebra in  $\mathfrak{M}_H$ .
- (c)  $\overline{H}$  is injective in  ${}^{H}\mathfrak{M}$ .
- (d) There exists an fs-retraction  $\chi: H \otimes \overline{H} \to \overline{H}$ .
- (e) H is a hereditary K-coalgebra.

Moreover, if H finite dimensional, the following assertion is also equivalent to the others:

(f) The comultiplication  $\overline{H} \to \overline{H} \otimes \overline{H}$  has a left H-colinear retraction, where  $\overline{H} \otimes \overline{H}$  is a left H-comodule via  ${}^{H}\rho_{\overline{H}} \otimes \overline{H}$ .

REMARK 9.6. Let H be a Hopf algebra over a field K. Then H is formally smooth as an coalgebra in  ${}_{H}\mathfrak{M}_{H} \Rightarrow H$  is formally smooth as an algebra in  $\mathfrak{M}_{H}$ .

COROLLARY 9.7. Let H be a Hopf algebra over a field K. Assume that H has an ad-coinvariant integral. Let  $\overline{H}$  be the cokernel of the unit  $u_H : K \to H$ . The following assertions are equivalent:

(i) H is formally smooth as a coalgebra in  $\mathfrak{M}_K$ .

(ii) H is formally smooth as a coalgebra in  $\mathfrak{M}_H$ .

(iii) H is formally smooth as a coalgebra in  ${}_{H}\mathfrak{M}_{H}$ .

(iv)  $\overline{H}$  is injective in  ${}^{H}\mathfrak{M}$ .

- (v) There exists an fs-retraction  $\chi: H \otimes \overline{H} \to \overline{H}$ .
- (vi) H is a hereditary K-coalgebra.

(vii) The comultiplication  $\overline{H} \to \overline{H} \otimes \overline{H}$  has a left *H*-colinear retraction, where  $\overline{H} \otimes \overline{H}$  is a left *H*-comodule via  ${}^{H}\rho_{\overline{H}} \otimes \overline{H}$ .

*Proof.* It is analogous to Corollary 5.7. Note that here H is always finite dimensional since we have an ad-coinvariant (in particular total) integral in H.

THEOREM 9.8. Let G be a finite group an let  $K^G$  be the Hopf algebra of functions from G to K. Then the following assertions are equivalent:

- (i)  $K^G$  is formally smooth as a coalgebra in  $\mathfrak{M}_K$ .
- (ii)  $K^G$  is formally smooth as a coalgebra in  $\mathfrak{M}_{K^G}$ .
- (iii)  $K^G$  is formally smooth as a coalgebra in  ${}_{K^G}\mathfrak{M}_{K^G}$ .
- (iv)  $\overline{K^G}$  is injective in  ${}^{K^G}\mathfrak{M}$ .
- (v) There exists an fs-retraction  $\chi: K^G \otimes \overline{K^G} \to \overline{K^G}$ .
- (vi)  $K^G$  is a hereditary K-coalgebra.

(vii) The comultiplication  $\overline{K^G} \to \overline{K^G} \otimes \overline{K^G}$  has a left  $K^G$ -colinear retraction, where  $\overline{K^G} \otimes \overline{K^G}$  is a left  $K^G$ -comodule via  ${}^{K^G} \rho_{\overline{K^G}} \otimes \overline{K^G}$ .

*Proof.* By Example 7.3, the Hopf algebra  $K^G$  admits an *ad*-coinvariant integral. The conclusion follows by Corollary 9.7.

REMARK 9.9. Let G be a finite group. In this case both KG and  $K^G$  are finite dimensional. As observed in Example 7.3,  $K^G$  becomes a Hopf algebra which is dual to the group algebra KG. In particular,  $K^G$  is formally smooth as a coalgebra in  $\mathfrak{M}_K$  if and only if KG is formally smooth as an algebra in  $\mathfrak{M}_K$ . Hence all the assertions in Theorem 5.8 and in Theorem 9.8 are equivalent. In the particular case when G is  $C_n$ , the cyclic group of order n, then, by Proposition 6.3  $K^G$  is formally smooth as a coalgebra in  $\mathfrak{M}_K$  if and only if  $n \cdot 1_K \neq 0$ .

**PROPOSITION 9.10.** Let K[X] be the polynomial ring endowed with the unique Hopf algebra structure defined by

$$\Delta(X) = 1 \otimes X + X \otimes 1$$

Then K[X] is formally smooth as a coalgebra in  $\mathfrak{M}_K$  if and only if char (K) = 0.

*Proof.* Let A = K[X]. Assume that A is formally smooth as a coalgebra in  $\mathfrak{M}_K$ . Note that  $\overline{A} = \sum_{n>0} KX^n$ . We have

(4) 
$$\Delta(X^a) = (1 \otimes X + X \otimes 1)^a = \sum_{0 \le i \le a} \binom{a}{i} (1 \otimes X)^{a-i} (X \otimes 1)^i = \sum_{0 \le i \le a} \binom{a}{i} X^i \otimes X^{a-i}.$$

By Proposition 9.5, there exists a fs-retraction  $\chi : A \otimes \overline{A} \to \overline{A}$ . For every  $a, b \in \mathbb{N}$ ,  $\chi \left( X^a \otimes \overline{X^b} \right) \in \overline{A} = \sum_{n \geq 0} KX^n$  so that we can choose  $\alpha_u^{a,b} \in K$  such that

$$\chi\left(X^a\otimes\overline{X^b}\right)=\sum_{u\geq 1}\alpha_u^{a,b}X^u,$$

where  $\alpha_u^{a,b} = 0$ , for every  $u \ge \deg\left(\chi\left(X^a \otimes \overline{X^b}\right)\right)$ . By condition (i) of the definition of fs-retraction, we have

$$a_1 \otimes \overline{a_2} = x_1 \otimes \chi(x_2 \otimes \overline{y})$$

for every  $x, y \in H$ , where  $a \in H$  is defined by  $\overline{a} = \chi (x \otimes \overline{y})$ . We apply this, for every b > 0, to the case

$$x \otimes \overline{y} = X \otimes X^b, \qquad \overline{a} = \chi \left( X \otimes X^b \right) = \sum_{u \ge 1} \alpha_u^{1,b} X^u.$$

Since

$$\begin{cases} a_1 \otimes \overline{a_2} \stackrel{(4)}{=} \sum_{u \ge 1} \alpha_u^{1,b} \sum_{0 \le i \le u} \binom{u}{i} X^i \otimes \overline{X^{u-i}} = \sum_{u \ge 1} \alpha_u^{1,b} \sum_{0 \le i \le u-1} \binom{u}{i} X^i \otimes X^{u-i} ,\\ x_1 \otimes \chi(x_2 \otimes \overline{y}) = 1 \otimes \chi \left( \overline{X} \otimes X^b \right) + X \otimes \chi \left( 1 \otimes X^b \right), \end{cases}$$

we get

$$\sum_{u\geq 1} \alpha_u^{1,b} \sum_{0\leq i\leq u-1} \binom{u}{i} X^i \otimes X^{u-i} = 1 \otimes \chi \left( X \otimes X^b \right) + X \otimes \chi \left( 1 \otimes X^b \right).$$

Therefore

$$\begin{cases} \sum_{u\geq 2} \alpha_u^{1,b} u X^1 \otimes X^{u-1} = X \otimes \chi \left( 1 \otimes X^b \right), \\ \sum_{u\geq 3} \alpha_u^{1,b} \sum_{2\leq i\leq u-1} {u \choose i} X^i \otimes X^{u-i} = 0, \end{cases}$$

so that

$$\begin{cases} \sum_{\substack{u\geq 2\\ u_u^{1,b}\binom{u}{i}} = 0, \text{ for every } u \geq 3 \text{ and } 2 \leq i \leq u-1 \end{cases}$$

Now, from these equalities, where the last one is applied in the case when i = u - 1, we deduce

$$\chi\left(1\otimes X^{b}\right) = \sum_{u\geq 2} \alpha_{u}^{1,b} u X^{u-1} = \alpha_{2}^{1,b} 2 X^{2-1} = 2\alpha_{2}^{1,b} X.$$

If char  $(K) \neq 0$ , there is a prime p such that char (K) = p. Since  $p \mid {p \choose i}, \forall 1 \leq i \leq p-1$ , by condition (ii) of the definition of fs-retraction, we have

$$X^{p} = \chi \left( X_{1}^{p} \otimes \overline{X_{2}^{p}} \right) \stackrel{(4)}{=} \sum_{0 \le i \le p-1} \binom{p}{i} \chi \left( X^{i} \otimes X^{p-i} \right) = \chi \left( 1 \otimes X^{p} \right) = 2\alpha_{2}^{1,p} X.$$

that is a contradiction. Therefore char (K) = 0.

Conversely, if char (K) = 0. Consider the vector space C = K[X] of polynomials in one variable. C can be regarded as a Hopf algebra with the following structures

$$\Delta(X^a) = \sum_{i+j=a} X^i \otimes X^j \quad \text{and} \quad X^a X^b = \binom{a+b}{a} X^{a+b}, \text{ for every } a, b \ge 0.$$

By the universal property of the polynomial ring, there exists a unique algebra homomorphism  $\varphi : A \to C$  such that  $\varphi(X) = X$ . In fact  $\varphi(X^n) = \varphi(X)^n = n!X^n$ , for every  $n \ge 0$ , and  $\varphi$  is a Hopf algebra isomorphism (in view of the condition on the characteristic, one can construct an inverse for  $\varphi$ ). We conclude by observing that C is exactly the cotensor coalgebra  $T_K^c(K)$  which is always formally smooth as a coalgebra in  $\mathfrak{M}_K$  (see [JLMS]).

REMARK 9.11. Akira Masuoka pointed out that the "if" part of Proposition 9.10 is the same as [MO, Example 1.8], where it is proved that the polynomial ring K[X] is an hereditary coalgebra when char (K) = 0 (see also Proposition 9.5).

By means of Proposition 9.5, it is now possible to rewrite Theorem 8.7 in the following form which improves Theorem 8.11 in the case  $\mathcal{M} = \mathfrak{M}_H$ .

THEOREM 9.12. Let H be a Hopf algebra which is a subcoalgebra of a coalgebra E in  $\mathfrak{M}_H$ . Assume that H is formally smooth as a coalgebra in  $\mathfrak{M}_K$  and that  $Corad(E) \subseteq H$ . Then there exists a coalgebra homomorphism  $\pi : E \to H$  in  $\mathfrak{M}_H$  such that  $\pi_{|H} = \mathrm{Id}_H$ . REMARK 9.13. The referee pointed out to our attention [MO, Theorem 1.2]. In view of  $(i) \Rightarrow (iv)$  of this result, since any formally smooth coalgebra is also hereditary (see [JLMS, Proposition 2.2]), one gets Theorem 9.12.

DEFINITION 9.14. [Scha1, Definition 5.1] Let E be a bialgebra and let H be a Hopf subalgebra of E. Recall that a *weak projection* (onto H) is a retraction  $\pi : E \to H$  for the inclusion map which is a left H-linear coalgebra map.

9.15. Let E be a bialgebra and H a Hopf subalgebra. Given a weak projection  $\pi : E \to H$  one can construct a K-linear isomorphism  $\psi : E \to H \otimes R$ , where  $R = E/H^+E$ . The bialgebra structure that  $H \otimes R$  inherits via  $\psi$  has been described in [Scha1, Section 5] and in [Scha3, Section 5].

As a consequence of the left hand version of Theorem 9.12, we get the following result.

THEOREM 9.16. Let H be a Hopf subalgebra of a bialgebra E. Assume that H is formally smooth as a coalgebra in  $\mathfrak{M}_K$  and that  $Corad(E) \subseteq H$ . Then E has a weak projection onto H.

REMARK 9.17. Akira Masuoka pointed out that, in the situation of Theorem 9.16, by Takeuchi's lemma [Mon, Lemma 5.2.10], E is necessarily a Hopf algebra.

PROPOSITION 9.18. Let E be a connected Hopf algebra over a field K with char (K) = 0. Assume that  $E \neq K$ . Then, for every  $x \in P(E) \setminus \{0\}$ , there exists a weak projection  $\pi : E \to K[x]$ . In particular we have a K-linear isomorphism.

$$E \simeq K[x] \otimes \frac{E}{xE}.$$

*Proof.* Since  $E \neq K$ , we have  $P(E) \neq \{0\}$ . Let  $x \in P(E) \setminus \{0\}$ . Note that K[X] is isomorphic to the tensor algebra  $T_K(KX)$  as a Hopf algebra, the isomorphism being given by the assignment

$$X^n \mapsto \underbrace{X \otimes \cdots \otimes X}_n.$$

By the universal property of tensor algebra, there is a unique Hopf algebra homomorphism  $\sigma$ :  $K[X] \to E$ , such that  $\sigma(X) = x$ . Since char (K) = 0, we have that K[X] is a connected coalgebra with P(K[X]) = KX. As  $\sigma_{|KX}$  is injective, by [Mon, Lemma 5.3.3, page 65],  $\sigma$  is injective and hence  $\operatorname{Im}(\sigma) \simeq K[X]$  as Hopf algebras. Therefore, by Proposition 9.10,  $H := \operatorname{Im}(\sigma)$  is formally smooth as a coalgebra in  $\mathfrak{M}_K$ . Clearly  $Corad(E) = K \subseteq H$ . We conclude by applying Corollary 9.16 and observing that  $K[x]^+ = (x)$ , the left ideal of K[x] generated by x.

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