BORELLI'S EDITION OF BOOKS V-VII OF APOLLONIUS'S *CONICS*, AND LEMMA 12 IN NEWTON'S *PRINCIPA*

by

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Summary

To solve the direct problem of central forces when the trajectory is an ellipse and the force is directed to its centre, Newton made use of the famous Lemma 12 (1687, 47) that was later recognized equivalent to proposition 31 of book VII of Apollonius's *Conics* (Apollonius 1710). In this paper, in which we look for Newton's possible sources for Lemma 12, we compare Apollonius's original proof, as edited by Borelli (Apollonius 1661), with those of other authors, including that given by Newton himself. Moreover, after having retraced its editorial history, we evaluate the dissemination of (Apollonius 1661) before the printing of the *Principia*.

Sunto

Per risolvere il problema diretto delle forze centrali, quando la traiettoria è un'ellisse e la forza è diretta verso il centro, Newton fece uso del famoso Lemma 12 (1687, 47), successivamente riconosciuto equivalente alla Proposizione 31 del libro VII delle *Coniche* di Apollonio (Apollonius 1710). In questo articolo, cercando le possibili fonti di Newton per il Lemma 12, confrontiamo la dimostrazione originale di Apollonio, come edita da Borelli (Apollonius 1661), con quelle di altri autori, inclusa quella data da Newton stesso. Inoltre, dopo averne ripercorso la storia editoriale, valutiamo la diffusione di (Apollonius 1661) prima della stampa dei *Principia*.

1. INTRODUCTION

To solve the particular case of the direct problem of central forces when the trajectory is an ellipse and the force is directed to its centre (Newton 1687, Prop. 10, 11, 12, Lib. I, sect. II), Newton made use of the famous Lemma 12: *All parallelograms circumscribed about a given ellipse are equal among themselves. The same holds true for the parallelograms of a hyperbola circumscribed about its diameters*.[1] This lemma, which Newton referred to by simply writing "Constat utrumq; ex Conicis" (both known from Conics), is, if correctly stated,[2] equivalent to propositions 31 of book VII of Apollonius's *Conics*. The fact that Edmond Halley's edition of the first seven books of the *Conics* (Apollonius 1710) had not been published at the time Newton was composing the *Principia*, raises the question as to what sources Newton used; although difficult to answer, plausible conjectures may be made to this regard. (Guicciardini 1999, 53).

[1] "Parallelogramma omnia circa datam Ellipsin descripta esse inter se aequalia. Idem intellige de Parallogrammis in Hyperbola circum diametros ejus decriptis", (Newton 1687, 47). In the second edition of the *Principia* (1713) the statement was corrected in the form: "Parallelogramma omnia, circa datae Ellipseos vel Hyperbolae diametros quasvis conjugate descripta, esse inter se aequalia" (Newton 1687). The English translation is from (Newton 1729).

[2] The parallelograms have to be understood circumscribed to the conic section with touching points at the extremities of pairs of conjugate diameters. Although this is not clearly expressed in Newton's enunciation, in the subsequent proposition he applied the lemma in this correct form.

It is well-known that the original Greek versions of the last four books of Apollonius's *Conics* have not survived, but Arabic translations of books V to VII reached the Western Latin world during the 16th and 17th centuries.

The first to arrive was the paraphrase by Abū 'l-Fath Mahamūd al-Isfahānī,[3] which was given as a gift to Cardinal Ferdinando Medici by the Patriarch of Antioch after he came to Rome in 1578. Some fifty years later the codex was brought to Florence, where it became part of the library of the Grand Duke of Tuscany. This manuscript, which from now on we will refer to as "MS L",[4] was edited and translated into Latin by Alfonso Borelli in 1661, who was engaged in the project soon after being appointed to the chair of mathematics in the University of Pisa in 1656.

Until then, of the eight original books of Apollonius's *Conics*, only the first four were known in the West, thanks to the first Latin edition by Giovanni Battista Memo (1537). The existence of books V to VIII was known from the synopsis that Apollonius made in the general preface of book I, and from the account of it that Pappus gave in his *Collection*, edited and translated into Latin by Federico Commandino (Pappus 1588). The eighth book of the *Conic*s seems to have been lost forever.

Two other Arabic versions of books V to VII arrived in Europe around the 1629. The one brought from Constantinople by the German Orientalist Christian Raue, better known as Christianus Ravius (1613-1677), was a mutilated reworking of those books by the Persian Abū 'l-Husayn 'Abd al-Malik al-Shīrāzī. In 1669, this manuscript was published and translated into Latin by Ravius himself (Apollonius 1669). The other was brought to Leiden by the Dutchman Jacob van Gool, Latinized as Jacobus Golius (1596-1667), on his return from a prolonged travel through the East. The latter manuscript we owe to the three sons of Mūsā b. Shākir – simply known as the Banū Mūsā –, a translation from the original Greek, is the one used by Halley for (Apollonius 1710). We will refer to it as "MS B".[5]

In an attempt to answer the aforementioned question about Newton's possible sources, we have been led to compare Apollonius's proof of proposition 31, as edited by Borelli, with those of other authors that appeared before and after the printing of (Apollonius 1661), in particular that given by Newton himself in the margin of a page of Johan de Witt's *Elementa Curvarum Linearum* edited by van Schooten (1661). At the same time, we were induced to retrace the awkward editorial process of the Arabic "MS L" in order to evaluate the dissemination of Borelli's edition before the *Principia* saw the light of day, and to see which books on conics Newton had at his disposal for his studies. Finally, we have drawn some conclusions.

2. On the events involving "MS L", and the interest it aroused

The history of "MS B", and of Halley's edition of Apollonius's *Conics* (1710), is well-known.[6] Less known seems to be that of "MS L", and of (Apollonius 1661), which deserves to be briefly recalled.

Our main source is (Giovannozzi 1916); in it the editorial history is reconstructed with the help of the manuscripts of the *Collezione Galileiana* in the National Library of Florence. Soon after Paul ver Eecke's French edition of Apollonius's *Conics* (1923) appeared in print, some Italian historians wrote

[3] For the spelling of Arabic and Persian names we refer to (Toomer 1990).
[4] The letter L in our notation indicates the Laurenziana Library of Florence where the manuscript is kept, MS. Orientale 296.
[5] The letter B in our notation indicates the Bodleian Library of Oxford where the manuscript is kept, MS. Marsh 667.
[6] See (Heath 1896), and (Toomer 1990).

to correct a few mistakes made in the preface concerning the origin of "MS L". The well-documented contributions by E. Bortolotti (1924), and by A. Agostini (1930), clarify and expand the information given in the preface of (Apollonius 1661). More recently L. Guerrini (1999) completed the reconstruction with the help of thirty-four letters, regarding the editorial process, that Borelli sent to Carlo Roberto Dati. We have drawn from all of them.

In 1578, the Patriarch of Antioch Ignatius Ni'matallāh (best known as "Neama", or "Nehama", in the Western world) was forced into exile in Rome. Here he enjoyed the protection of Pope Gregorius XIII who, well aware that the Patriarch was a good astronomer and expert in chronology, wanted him to be involved in the reform of the calendar by taking part in the appointed Commission, whose principal member was the mathematician and astronomer Christoph Klau,[7] better known with the Latinized name of Christophorus Clavius (1538-1612). Exiled in Rome, the Patriarch found himself in poor economic conditions, but Cardinal Ferdinando Medici, going beyond his duties and responsibilities towards the Patriarchs of Alexandria, Antioch and the Kingdom of Ethiopia, provided him with substantial help. On leaving his country Neama took with him many oriental manuscripts, among which one contained an Arabic version of Apollonius's *Conics*. Out of gratitude, and in exchange for a life pension, Neama gifted his codices to Cardinal Ferdinando. Neama's collections became part of the endowment of the Medicean Oriental printing-house, when, in 1584, it was founded in Rome under the patronage of the Cardinal with Giovanni Battista Raimondi (1536-1614) in charge. (Maccagni, Derenzini 1985, 680). Raimondi, professor of mathematics at the University of Rome La Sapienza, was fluent in Oriental languages after having extensively travelled in the Middle-East, and so he soon established a friendship with Neama.

It is not known who first discovered that one of Neama's manuscripts contained the first seven books of the *Conics*, but it may well have been Raimondi, even if at the very beginning it was erroneously thought to contain also the eighth book.[8]

We cannot but think that Clavius was also informed of the appearance in Rome of such a precious manuscript. The English mathematician and philologist Henry Savile (1549-1622) was also soon to learn of its existence.

During his long travel through Europe, Savile was always on the lookout for codes and rare books, and in the autumn of 1581 he stopped in Rome where he was a guest in Raimondi's home. This afforded him the opportunity to see Neama's collection of oriental manuscripts. Savile was able to examine it carefully, and to draw up a catalogue.[9] If not already informed by Raimondi, Savile identified the precious manuscript of Apollonius's *Conics*, and he immediately informed his friend

[7] For the uncertainty about Clavius's German surnane see (Lattis 1994).

[8] In a catalogue found by Guglielmo Libri in the Bibliothéque Royale of Paris, under the title "Libri imprimendi in lingua arabica Romae in typographia Ducis Hetruriae cui praeest Jo. Baptista Raymundus" and belonging to Nicolas-Claude Fabri de Peiresc, there are the entries "Apollonii Pergei libri 8, de conis" and "Ejusdem liber de sectionibus", where, among those regarding Archimedes, we found "liber lemmatum, ex Thebit traditione", perhaps Archimedes's opera published by Borelli (1661) which was enclosed in the same manuscript of the seven books of Apollonius's *Conics* (Libri 1838, 236-237). The same catalogue can be found, with small differences, in (Labbe 1653, 250-258). In this work there is another list of manuscripts under the title "Libri Chaldaei et arabici in poter del Reverendissimo Patriarca di Antiochia in Roma", which lists "In Mathematica Euclide, Apollonio lib.7 Theordosio: Additione sopra Euclide di Nesir il dinel Iusi" (Ibidem, 258).

[9] The catalogue is preserved at the Bodleian Library in Oxford, MS. Add. C 296, ff. 172^v-173^v.

and colleague Andreas Dudith in Wrocław of the discovery. Savile also tried to get a copy of the manuscript, but whether he did is doubtful (Maccagni, Derenzini, 684, and note 36).[10]

The news of the presence of an Arabic version of Apollonius's *Conics* in Rome quickly spread across Europe. For instance, in 1590, Guidobaldo Dal Monte asked Clavius if there was any hope of having the last four books published in the near future (Baldini 2003, 61).[11] It is also possible that other mathematicians, such as Clavius's pupil Gregoire de Saint Vincent (1584-1667),who studied in Rome from 1605 to 1612, might have had an opportunity to see the manuscript.

Until his death in 1614, Raimondi remained the preserver and custodian of the assets of the Medicean printing-house, and of its treasured manuscripts. The same year, Federico Cesi (1585-1630), the founder of the Accademia dei Lincei, and Antonio Santini (1577-1662), a pupil of Galileo Galilei, wrote separately to the great scientist asking him to intervene with the Grand Duke Cosimo II in order to have the Arabic manuscript containing the last books of the *Conic*s translated and published. The Grand Duke sent an emissary to Rome, who, after settling the question of Raimondi's inheritance, arranged for the assets of the "Tipografia Medicea", and its collection of manuscripts, to be transferred to Florence. Unfortunately, Galileo was unable to convince the Grand Duke of the importance of the editorial project, and so it was that the precious manuscript ended up forgotten on the shelves of the Medici library (Agostini 1930, 285-285).

In 1630, the French mathematician and philosopher Pierre Gassendi (1592-1655), who had published the catalogue of Golius's manuscript collection (Gassendi 1630), brought to light the existence in Leiden of another manuscript containing an Arabic version of Apollonius's *Conics* (Toomer 1990, xxiii). Golius, who was a professor of Arabic, promised to translate it into Latin and to publish it, but seems to have prevented anyone from having access to his manuscript.

In the 1630s, several important geometrical works appeared in print. René Descartes (1596-1650) published *La Géométrie* (1637), Claude Mydorge (1585-1647) the *Prodromi catopticorum etc.* (1639), and Girard Desargues (1591-1661) his *Brouillon projet* (1639). These works, not only simplified the proofs of some of Apollonius's propositions contained in the first four books of the *Conics*, but also extended the horizon of the theory with new results.

In 1644, Marin Mersenne (1588-1648) informed the scientific community that Golius intended to translate and publish the manuscript in his possession. In the "Praefatio in Apollonii Pergaei Conica" included in (Mersenne 1644, 274-275), he gave the Latin translation of the introductions of books VI and VII, the first proposition of book VI, and the number of the propositions of books VI (77) and VII (51). Mersenne added that at the end of the codex it was stated that the eighth book had not been translated into Arabic because the book was missing in the original Greek manuscript. All this information was evidently transmitted to him by Golius, and being unable to read Arabic he hoped to get help from someone with a knowledge of the language. This notwithstanding, Mersenne's attempt to obtain a copy of the codex was not successful (Toomer 1996, 34, 50).

In the same "Praefatio", Mersenne reported on Mydorge's doubts that the last three books could be attributed to Apollonius, because the first proposition of book VI, mentioned above, was stated only

[10] Two copies, beside the original manuscript, are preserved at the Biblioteca Laurenzia in Florence, MS. Orientale 275 and MS. Orientale 270.

[11] The knowledge of the presence in Rome of the Arabic version of Apollonius's *Conics* is recorded in Clavius' correspondence. See (Clavius 1992, letters 65, 256).

for right cones, even if it was valid for any oblique cone (Mersenne 1644, 274-275). This means that Mydorge was privately informed by Mersenne, or Golius himself, before 1644.

Meanwhile, during the years 1645 to 1647, having received permission from Ravius, the well-known mathematician John Pell (1611-1685), professor of mathematics at Amsterdam and Breda, made a remarkable attempt to publish a Latin translation of his manuscript of Apollonius's Conics. In May 1645, Pell had finished the translation and Ravius offered to pay for the woodcut engraving of the figures for Pell's edition. This seems to have alarmed Golius and possibly the Republic of Letters, too. Pell never made his own version public and it appears to have vanished without further trace. (Toomer 1996, 183-186).

In 1645, the Maronite Catholic philosopher and linguist Ibrāhīm al-Ḥāqilānī, Latinized Abhraham Ecchellensis (1605-1664), with the help of the mathematicians Michelangelo Ricci (1619-1682) and Evangelista Torricelli (1608-1647), tried to obtain permission from the Grand Duke Cosimo to translate the manuscript (Bortolotti 1924). The request was likely provoked by the news given in (Mersenne 1644) and perhaps also by the news of Pell's translation. But, notwithstanding the requests of such leading scientists, once again the Grand Duke paid no heed to the request. Only several years later did favourable conditions present themselves.

3. On Borelli's Edition of "MS L", and Apollonius's proof of proposition 31

Around 1628, Giovanni Alfonso Borelli (1608-1679) went to Rome to study under the guidance of Benedetto Castelli (1578-1643), a professor of mathematics who taught Euclid's *Elements* and Apollonius's *Conics* at the *La Sapienza* University. In Rome Borelli was in touch with Torricelli, who was also one of Castelli's pupils. In 1637, having built up a reputation as a good mathematician, Borelli was appointed as a teacher of mathematics at the University of Messina (Sicily). Here he was involved in the edition of the posthumous work of the famous Sicilian mathematician Francesco Maurolico (1494-1575), *Emendatio et restitutio conicorum Apollonii Pergaei* (1654), that included a restoration of books V and VI. In Messina, Borelli also conceived the project of a revision of Euclid's work, which was later published with the title *Euclides restitutus* in 1658. In 1656, Borelli was appointed to the chair of mathematics at the University of Pisa. Soon after his arrival there, he turned his attention to the famous Arabic manuscript that Prince Leopold kept in the Florentine library, which was supposed to contain the "eight books" of Apollonius's *Conics*. Borelli wrote to the Prince, asking him to sponsor the translation work,[12] but he was not successful.

In 1658, however, having finally obtained the Grand Duke's authorisation, Borelli went to Rome with the manuscript to make contact with Abhraham Ecchellensis, who thirteen years earlier had already offered, without success, to translate the manuscript. In June, the translation work was already underway,[13] beginning from the first four books. The work turned out to be more difficult than expected, because of the numerous mathematical errors in the text and in the lettering of figures. Having the translation from Arabic of the first four books at their disposal helped with the translation also of the remaining three.
On the 14th of August, Borelli was able to write to Prince Leopoldo: "by the grace of God I have come almost to the end of this very stunted translation of Apollonius, which I, with good conscience, could call my own, because it was necessary to find first the demonstrations in order to be able to benefit from this so defective and incorrect manuscript".[14]

[12] See the letter that Borelli addressed to Prince Leopoldo, on 12 April 1656, in (Giovannozzi 1916, 5).
[13] See the letter of Borelli to Viviani published in (Tenca, 1956, 116)
[14] First published in Italian by Giovannozzi (1916, 7).

On completion of the Latin translation, the arrangement of the text for the press aroused some discussion. Prince Leopoldo, welcoming a suggestion of the mathematician Michelangelo Ricci (1619-1682), wished to accompany the printing of the unpublished books with that of the first four, following the redaction of Abu l-Fath, and to place the Arabic text side by side with the Latin translation. This ambitious project led to a slowdown in the editorial process that annoyed Borelli. The editorial project was even more delayed, because Prince Leopoldo also wanted a work by Viviani to be released first.

Vincenzo Viviani (1622-1703) had written a restoration of the fifth book of Apollonius's *Conics* years before, which, for family reasons and health problems, he had not yet published. When in June 1658 he got wind of Borelli's publishing project, he hastened to return to his work and print it before Borelli's, so that the many years of study he had devoted to the work would not be nullified. Viviani's reasons were accepted by the Grand Duke, who, on July 8th 1658, put his seal on the manuscript. The following year the work was printed (Viviani 1659).[15] On 20th of July 1658, Borelli wrote to Viviani approving his project to publish his "divinatio" adding that "I will be able to testify that you did not know anything from these last books". Moreover, Borelli complained that probably his epitome of conic sections ("compendiosissimo trattato de'conici disteso di mia mano") written some time before, would not be published (Tenca 1956, 116-117). Borelli was certainly referring to the work published only in the year of his death (Borelli 1679).

In the years 1658-1660, however, worries resurfaced as to what Golius might do with the manuscript in his possession. He had composed the *Lexicon Arabico-Latinum* (1653), and in Florence there were no doubts regarding his ability to translate the manuscript, and, moreover, it was known that Golius's manuscript was a copy of Apollonius and not a paraphrase. This worry led to an acceleration of the Florentine editorial project by abandoning the idea of publishing the first four books and the Arabic text (Guerrini 1999, 514). On the other hand, as we learn from Giovannozzi (1916, 17), even Golius seemed anxious to know what was happening in Florence, and in 1660, through his brother Pieter,[16] he tried to get information from Ecchellensis about the situation of the printing. Dati suggested answering that the fifth book had already been printed and that the other two were being printed, but above all what he wanted was that nothing be sent to Leiden to prevent Golius from taking advantage of it.

Thus the printing of Apollonius's work started in the winter of 1660, managed by the Florentine philosopher and scientist Carlo Dati (1619-1676). The delay caused by the printing of Viviani's work led to a worsening of the already precarious relations between Borelli and Viviani (Targioni Tozzetti 1780), and, according to Guido Grandi (1712, 61), if Borelli praised Viviani in the *Praefatio ad lectorem* (Borelli 1661), he had done it unwillingly.
The following year, in August, the joint effort between Borelli and Ecchellensis came to fruition in the publication *Apollonii Pergaei Conicorum libri V, VI, VII* (Apollonius1661). Copies of the volume were sent to M. Ricci, A. Kircher, G.D. Cassini, F. Riccioli, C. Berigardo, S. degli Angeli, and abroad, to M. Thévenot, J. Hevelke (Hevelius), and C. Huygens, among others (Giovannozzi 1916, 24).

Two years after Borelli's edition of books V--VII of Apollonius's *Conics* had been published, Golius, again through his brother in Rome, resumed contact with Ecchellensis asking him to provide him with all the Arabic terms of Apollonius. It is very likely he had serious difficulty in interpreting the geometry. When Viviani was informed of the request, he presented it to Prince Leopoldo, who, this

[15] In the preface Viviani tell us the long story, which began when he was twenty-two, and his words show how much he cared about this work.
[16] Pieter van Gool was a Carmelite, he had travelled in Syria and Lebanon and in those years he was at the Convent of San Pancrazio in Rome, see (Thomas 1939, 828-829).

time, consented to Golius's request. A noble thought, which came to nothing however, since, as we now know, Golius's frequently vowed edition of Apollonius never saw the light of day.

In 1665, Carlo Dati proposed a reprint of Borelli's edition of Apollonius's V-VII books and suggested the addition of the first four. In his reply, on 31st January 1665, Borelli wrote that the translation from Arabic of the first books had already been done, and their closeness to the Greek text would help to give credit to those already printed "because if he (the Arabic translator) does not put anything of his own into them beside the things written by Apollonius, consequently it will confirm that in the last three books there is nothing but the doctrine of the same author."[17] In the same letter Borelli returned to his compendium and suggested adding his "epitome of the elements of conic sections which are demonstrated in a different way from Apollonius, and thus V.S. would come to publish a work not only useful, but still highly reputable for the new things added, which alone usually entice the experts of our century".[18] Unfortunately, Dati's project came to nothing, so Borelli's work had to wait more than twenty years to be printed.

In 1668, Borelli returned to Messina to his old chair of mathematics, but owing to his opposition to Spanish rule of Sicily, four years later he had to flee from Sicily for Rome, where he died in 1679. Before his death he was able to publish his compendium, *Elementa conica Apollonii Pergaei et Archimedis Opera nova et breviori method dimostrata* (Borelli 1679), thus demonstrating his devotion to the work. In this work, as the title says, Borelli adopted a new method to treat the conic sections, that, in his opinion, avoided the tortuous procedure ("perplexam primariam conicarum sectionum generatione") used by Apollonius (also by Mydorge (1639) and Saint Vincent (1647)), which consisted in sectioning a cone by two planes, one through the axis and another through a straight line orthogonal to the base of the triangle through the axis. Although this method allowed one to define the principal diameter of the conic section, secondary diameters and their properties had to be studied by passing in the plane through a very laborious process ("progressu laboriosissimo"). Borelli's new method consisted of the systematic use of what he called "conteminalis analogia" (that is, harmonic ratio), which led to the simultaneous definition of all diameters.[19] We shall return to this work later on.

3.1 *Apollonius's proof of Proposition 31*

In (Apollonius 1661) proposition 31 appears on page 370, just before the end of book VII, stated in the following form:

In the ellipse, and in conjugate sections [the opposite branches of two conjugate hyperbolas] the parallelogram bounded by the axes is equal to the parallelogram bounded by any pair of conjugate diameters, if its angles are equal to the angles the conjugate diameters form at the centre.[20]

We remark, as the subsequent proof shows, that the requirement on the angles, without which the proposition is false, means that the parallelogram can be most naturally thought as circumscribed to the conic at the extremities of the conjugate diameters.

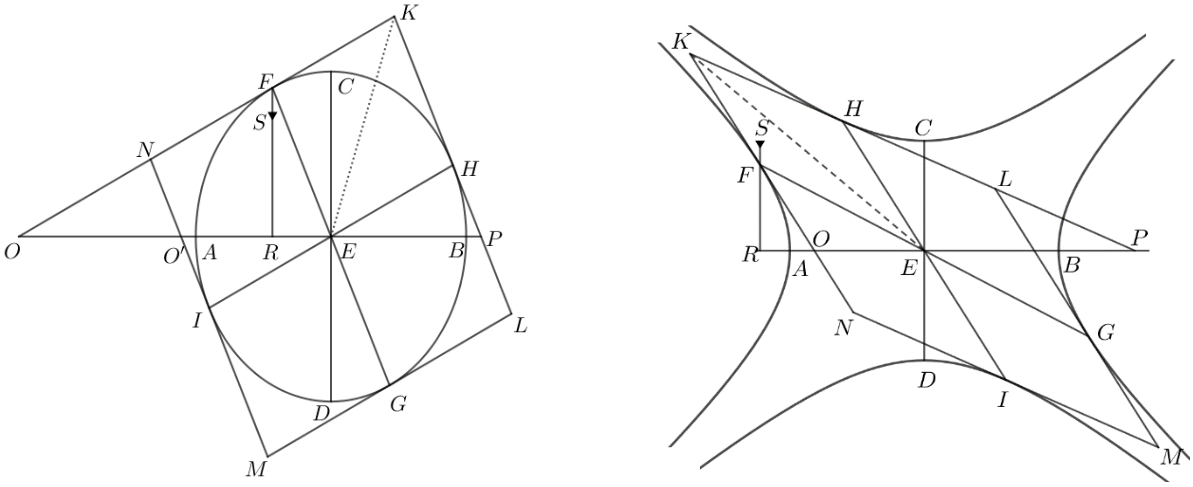
[17] See the letter of Borelli to Dati, the 31st January 1665, published in (Guerrini 1999, 513).
[18] The letter is published in (Guerrini 1999, 514).
[19] For more information on the method see (Brigaglia, Nastasi 1984, 14-15).
[20] "In ellypsi, et sectionibus conjugatis parallelogrammum sub axibus contentum aequale est parallelogrammuo à quibuscunque duabus conjugatis diametric comprehenso, si eorum anguli aequales fuerint angulis ad centrum contentis à conjugatis diametris", (Apollonius 1661, 370).

Borelli inserted the aforementioned statement into "Sectio Undecima Continens Proposit. XXXI et XXXII Apollonii".[21] The proof makes use of Propositions 37, 39 of book I of Apollonius's *Conics*, that it is useful to recall (Heath 1896, 28): *In a hyperbola, an ellipse, or a circle, if QV be an ordinate to the diameter PP', and the tangent at Q meets PP' in T, then* [*being C the centre, and p the parameter*]

$$CV \times CV = CP^2$$

$$QV^2 : CV \times VT = p : PP' \text{ [} or\ CD^2 : CP^2 \text{]}$$

That said, the proof goes as follows. Let *AB*, *CD* be the axes, *E* the centre, and *FG*, *IH* a pair of conjugate diameters (see fig. 1). Let the tangents to the conic at the points *F*, *I*, *G*, *H* be drawn, and let *K*, *L*, *M*, *N* be the points where they intersect each other. Produce *AB* until it meets the tangents at *F* and *H* in the points *O* and *P* respectively. $AB \times CD = Par(MK)$ has to be proven.[22]



(a) (b)

Fig. 1 (a) enhanced version of the diagram for the proof of the first part of proposition XXXII (Apollonius 1661, 370). In Borelli's edition the point *O* is erroneously placed in the position "*O'* ". (b) diagram for the proof of proposition XXXI as in (Apollonius 1661, 371), here the point *O* is in the correct position.

Let *FR* be perpendicular to *AB*, and *SR* mean proportional between *OR* and *RE*. Then, by virtue of prop. 37, I, we have

$$AE^2 : EC^2 = (OR \times RE) : FR^2$$

$$AE^2 : EC^2 = SR^2 : FR^2,$$

$$AE : EC = SR : FR$$

$$AE^2 : (AE \times EC) = (SR \times OE) : (FR \times OE),$$

Thus alternately we get

[21] In a note on p. 372, Borelli pointed out that since in the Arabic manuscript the statement concerning the ellipse was numbered "9", he thought it right to insert it as prop. XXXII, after prop. XXXI, the one concerning the conjugate hyperbolas.

[22] According to Borelli's notation, *Par*(*MK*) denotes the parallelogram *KLMN* (*K* and *M* are two opposite vertices).

$$AE^2 : (SR \times OE) = (AE \times EC) : (FR \times OE).$$

On the other hand, again by prop. 37, I,[23] it follows that

$$AE^2 = RE \times OE$$

hence

$$(RE \times OE) : (SR \times OE) = (AE \times EC) : (FR \times OE) \quad (*)$$

Moreover, by prop.4 of book VII, according to which $OF^2 : EH^2 = OR : RE$, and the similarity of the triangles *EOF* and *EHP*, it follows that:[24]

$$OF^2 : EH^2 = T(EOF) : T(EHP),^{25}$$

and therefore:[26]

$$Par(EK)^2 = 2T(EOF) \times 2T(EHP)$$

Remembering that $SR^2 = OR \times RE$ we get

$$2T(EOF) : Par(EK) = SR : RE = (SR \times OE) : (RE \times OE),$$

and also

$$2T(EOF) = FR \times OE$$

Thus, $(FR \times OE) : Par(EK) = (SR \times OE) : (RE \times OE)$. But, according to (*) $(RE \times OE) : (SR \times OE) = (AE \times EC) : (FR \times OE)$, hence $Par(EK) = (AE \times EC)$, and the claim is proved.

If we compare the proof given above with that in (Apollonius 1710), we see that both follow exactly the same route, but the latter includes the all details that Borelli exhibited in his notes.[27]

We feel it is appropriate to recall proposition 31 from Halley's edition: *If two conjugate diameters are taken in an ellipse, or in the opposite conjugate sections; the parallelogram bounded by them is equal to the rectangle bounded by the axes, provided its angles are equal to those formed at the centre by the conjugate diameters*.[28]

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[23] We stress that in the text it is erroneously indicated as prop. 39, I.

[24] In the text proposition 24 of book II is quoted, but it is a mistake as the latter regards the parabola.

[25] In this formula and in the following, "*T* (*ABC*)" denotes the triangle of vertices *A*, *B*, *C*.

[26] To explain this implication Borelli made a note adding:

$T(EOF) : T(EHP) = OR : RE$ (I)

Now, the triangle *EFK* is medium proportional between the two similar triangles *EOF*, and *EHP*, in fact, the two triangles *EOF*, *EFK*, having equal height, are between them as their respective basis, that is $T(EOF) : T(EFK) = OF : FK = OF : EH$. The same is *EHP* and *EHK*, which also have the same height and then they are between them as their basis, that is $T(EHP) : T(EKH) = HP : HK = HP : FE$. On the other hand *T*(*EFK*) = *T*(*EKH*), thus we have

$T(EOF) : T(EFK) = T(EFK) : T(EHP)$ (II)

Since $SR^2 = OR \times RE$, from (I) and (II) it follows that

$$T(EOF) : T(EFK) = SR : RE,$$

Therefore, we get

$$2T(EOF) : 2T(EFK) = SR : RE,$$
$$2T(EOF) : Par(EK) = SR : RE,$$

[27] In translating from the Arabic the lettering of the diagrams, Borelli adopted the Latin alphabet, while Halley used the Greek, as was natural since he was editing the first four books in the original language.

[28] "Si ducantur diametri quaevis conjugatae in Ellipsi, vel inter sectiones oppositas conjugatas; erit parallelogrammum contentum sub his diametris aequale rectangulo sub ipsis Axibus facto: modo anguli ejus aequales sint angulis ad centrum sectionis à diametris conjugatis comprehensis", (Apollonius 1710, 115); we have adopted the English translation by Heath (Apollonius 1896).

We do not find any substantial differences between the two versions.

4. BARROW, HIS LIBRARY, AND NEWTON'S EARLY READINGS

Isaac Barrow (1630-1677) matriculated at Trinity College in Cambridge in 1646, and four years later he was nominated as a fellow. In June 1655, after obtaining a three-year traveling fellowship, Barrow left England for the Continent. He first stopped in Paris, where he stayed until February 1656, then he left for Florence, where he would remain for eight months.

In Florence, Barrow established a friendship with the mathematician Carlo Renaldini (1615-1698), founding member of the Accademia del Cimento, whose algebraic work *Opus mathematicum* had been published the year before. We may argue that, through Rinaldini, Barrow was introduced to other mathematicians and scientists of the Tuscan area. Very likely he met Vincenzo Viviani, and we may also suppose he met Borelli, who, at the time of Barrow's arrival in Florence had just started lecturing at the University of Pisa, and was probably already looking for the "MS L".

Barrow spent much time in Duke's Library (today Biblioteca Laurenziana), which in addition to its magnificent collection of manuscripts and illuminated books, included a collection of thousands of ancient coins and medals, kept by an Englishman named Fitton (Feingold 1990, 48), but we do not know if he was able to take a look at the famous Arabic manuscript. Just how much Barrow may have learned in Florence is difficult to judge, but already in the 1660s he showed a profound knowledge of Italian mathematics and geometry.

In November of 1656, Barrow left Florence, and from Leghorn sailed to Smyrna for his journey through Turkey. He stayed there for two years, then he embarked in Constantinople for Venice, and finally returned to Cambridge in September 1659.

In 1663, Barrow became Lucasian professor at Cambridge. In the years 1667-1669, Newton attended his lectures on geometry and optics, and had many private conversations with him.

Barrow established a friendship with John Collins (1625-1683), a mathematical practitioner who held an extensive correspondence with many English and European scientists, including Borelli. Since the late 1660s, as M. Feingold informs us, Collins had been involved in editorial projects for publishing mathematical works, and was eager to edit whatever Barrow had written (Feingold 1990, 69). In 1670, Collins tried to convince Barrow to add to his work on the first four books of Apollonius's *Conics*, the last three, published by Borelli in 1661, but translated into Barrow's symbolic formalism.[29] The following year Collins tried to enlist Edward Bernard (1638-1697) in the project. In fact, Bernard, who was Savilian Professor of Astronomy at Oxford, was in possession of a copy of Golius's manuscript he had made in Leiden soon after Golius's death.[30] Collins proposed to Bernard that books V-VII, once translated and handled with Barrow's method, could be printed with Barrow's comments on the first four. However, the project did not go well, most likely due to Bernard's refusal to edit the last three books (Feingold 1990, 76-77), and Barrow's edition of Apollonius's *Conics* appeared in 1675 without books V-VII, see (Barrow 1675).

[29] Barrow had developed a symbolic formalism in his *Optical lectures* (1667) and *Geometrical lectures* (1670). It consisted in the use of special symbols by which statements and proofs could be make easier to read.

[30] When Golius died in 1667, his valuable collection of manuscripts remained in the hands of his heirs, but the copies he had made of the Arabic version of the *Conic*s were available in the Leiden University library.

Barrow died unexpectedly in 1677. At this time, he possessed a private library of one thousand volumes. He had formed his library during the 1660s, and for most of the scientific works he was indebted to Collins, who also traded in books. Barrow was quite liberal in lending books to friends, and it is known that Newton had had free access to Barrow's library since his undergraduate studies (Feingold 1990, 336). Among the mathematical works that Newton borrowed and read in 1664-65, there was the van Schooten edition of Descartes' *Geometria*.

A catalogue of the library was drafted following Newton's advice.[31] The volumes were in part sold directly, and in part sent up for sale in London. Newton appears to have been a major beneficiary of the dispersal of Barrow's library, of which he took advantage to form his own collection.

It was only in 1696 that, on behalf of the Archbishop of Dublin, Narcisus Marsh, Bernard was able to purchase the manuscript at the auction held in Leiden of Golius's collection. Unfortunately, Bernard died shortly after his return from Leiden, and he could not complete his own editorial project. However, the manuscript reached Edmund Halley (1656-1742), who, in about twelve years, produced the *editio princeps* of the Greek text and the Latin translation of books V-VII (Toomer 1990, xxv).

4.1 *Newton's proof of Proposition 31 for the ellipse*

In his own copy of the second part of Descartes's *Geometria*, containing De Witt's *Elementa Curvarum linearum*, edited by van Schooten (De Witt 1661),[32] Newton annotated a proof of proposition 31, for the case of the ellipse, as a corollary to a theorem by De Witt given therein.[33] To understand Newton's proof it is useful to recall the construction of the ellipse provided by De Witt (1661, 204-205).

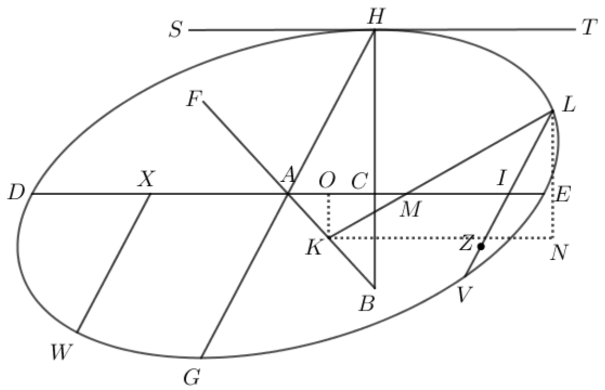


Fig. 2 reproduces diagram III in (De Witt 1661, 206)

Let two straight lines *FB* and *DE* be fixed intersecting in point *A* at any angle (*A* is called *centre* and *DE directrix*) and let two segments be fixed (called *intervals*). Then De Witt took points *B* on *FB* and *C* on *DE*, so that the segment *BC* is equal to the first *interval*. On the *directrix DE,* the points *D* and *E* were taken so that their distance from *A* is equal to the sum of the two *intervals*. Then, on the straight line *BC* De Witt took the segment *CH* equal to the second *interval*, and he called *BH* the *describing*, and *H* the *describing point* or *efficient point*. Next, he placed the *describing* in the first

[31] The catalogue is published in (Feingold 1990), and it includes two copies of (Apollonius 1661), and a copy of (Saint Vincent 1647).
[32] That volume came from Barrow's library, and years before Newton had studied on it.
[33] We stress that Newton's proof cannot be extended to the case of the hyperbola.

position, that is perpendicular to the *directrix DE* (as in fig. 2), he joined *H* with *A*, and produced it up to *G* so that *AG* = *AH* (*GH* is called *secant*). Now, if point *C* moves on the *directrix DE*, and correspondingly *B* moves on *FB* so that *BC* maintains its fixed length (as in the positions *M*, *K* in fig. 2), the *efficient point H* describes a curve.

Then De Witt stated the following theorem (Proposition 13): *For any angle, and any pair of intervals, the curve being described as above, it results that for any point on the directrix, the square of the applied at this point parallel to the secant, is to the rectangle that the same point intercepts on directrix, as the square of the secant is to the square of the directrix*.[34]

If the point on the directrix is denoted by *I* (see fig. 2), the theorem says that

$$LI^2 : DI \times IE = HG^2 : DE^2.$$

At the end of the proof, which for sake of brevity we omit here, De Witt concluded, "it is evident that the described curve is what the Ancients named ellipse, of which the directrix and the secant are conjugate diameters, that become the axes if the given angle is a right-angle".[35] Let us observe that the proposition expresses the chord theorem for conics (in the case when the chords are *HG*, *LV* cut by *DE*), and therefore the condition that the point *V* (see fig. 2) belongs to the conic passing through *D*, *H*, *L*, *E* and *G*.

In the subsequent corollary 7, De Witt showed how to construct the conic having a given pair of conjugate diameters. He took two segments *DE* and *GH* meeting in their mid point *A*, then, issuing from *H* the perpendicular *HC* to *DE*, he produced it to *B* so that *BH* = *AE*. Then, he joined *B* with *A* to determine the angle *BAC* that, together with the describing *BH* and the intervals equal to *BC* and *CH* respectively, allows us to describe the ellipse by the method above. The curve so constructed will have *DE* and *GH* as conjugate diameters.

Afterwards, De Witt stated Proposition 14: *Given an ellipse described around whatever pair of axes, draw the conjugate diameter of a given one.*[36]

In an ellipse *SYXZ*, De Witt considered a diameter *DE*, and to construct its conjugate he proceeded as follows, see fig. 3. He took the point *O* on the semi-axis *AZ* so that *DO* = *AS*, and denoted *W* the point where *DO* intersects *AS*. Then he took *P* on *AZ* so that *AP* = *AW*, and *R* on *AX* so that *AR* = *AO*. Next he drew the straight line *PR* and produced it until it intersected the ellipse at point *H*.

[34] "In quocunque *angulo*, et quibuslibet *intervalli*s, juxta definitiones hoc capite propositas, curvâ descriptâ, hoc ipsi proprium erit, ut quadratum cujuslibet *secanti* aequidistantis, à quolibet *directricis* puncto ad curvam applicatae, eandem rationem habeat ad rectangulum sub partibus *directricis* per applicatam factis, quam quadratum *secantis* ad quadratum *directricis*", (De Witt. 1661, 205).

[35] "Atque ita liquet, praedictam curvam eam ipsam esse, quae Veteribus Ellipsis dicta fuit, directricem verò ac secantesm eas ipsa, quas conjugatas diametros appellabimus, aut, si angolus rectus fuerit, conjugatos axes vocârunt", (De Witt 1661, 208). In fact, by taking the directrix and the secant as coordinate axes, and putting *x*, *y* the coordinate of *L*, $AE = a$ and $AH = b$, from the relation above we have $y^2 :(a + x)\,(a - x) = (2b)^2:(2a)^2$, which leads to the equation $x^2/a^2 + y^2/b^2 = 1$. If we denote α the angle *CAH*, and we consider orthogonal coordinates *X*, *Y* with origin at *A* and the directrix as *X*-axis, the equation of the conic is $a^{-2}(X\text{-}Y\cot\alpha)^2 + (b\cdot\sin\alpha)^{-2}Y^2 - 1 = 0$.

[36] "In Ellipsi circa quoscunque axes descriptâ, ducta quaelibet diameter transversa est, habet que secundam sibi conjugatam", (De Witt 1661, 213).

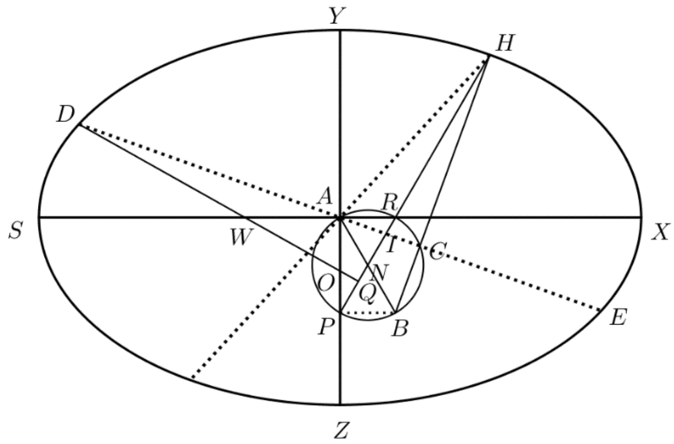


Fig. 3 reproduced the diagram I in (De Witt 1661, 214)

He showed that *HG*, *G* being the intersection between the straight line *HA* and the ellipse, is the sought-for diameter. We stress that the proof, too long to be described here in its entirety, involving De Witt's construction as described above, refers to several propositions from book I and III of the *Conics*, and finally requires the application of corollary 7. In fact, De Witt concluded the proof with the words, "By corollary 7 of proposition 13, *DE* and *HG* are two conjugate diameters of the ellipse constructed with the angle *BAC* and the intervals *BC*, *CH*. Since the two ellipses [the given one, and that constructed] coincide, *DE*, *HG* are also conjugate diameters of the starting ellipse".

We point out here that in the course of the proof, by a skilful use of similar triangles, De Witt shows that points *B* and *C*, where the first is taken on the straight line *HC* so that *HB* = *DA*, and the second is the foot of the perpendicular to *DE* issued from *H*, belong to the circle through the points *A*, *R*, and *P* (so that *PB* = *AR*, and *PR*, *AB* are diameters).

In his essay De Witt gave five corollaries of this theorem, but Newton added a sixth. On his copy of the second part of Descartes's *Geometria*, edited by van Schooten, Newton noted in his own hand:[37]

"Corollary 6: Parallelogramma omnia circa datam Ellipsin descripta sunt inter se aequalia. Nam (fig.1) [here fig 3] *RH·CH ∴BH = DA·PH = AS ergo RH (= AY)×PH = CH×DA. Per Euclid: 3 prop. 36 coroll*.".

In fact, by the chord theorem for the circle (Euclid's *Elements*, book 3, prop. 36) we have *RH* : *CH* = *BH* : *PH*; then, since *BH* = *DA* and *PH* = *AS*, it follows that *RH* : *CH* = *DA* : *AS*. Moreover, we have *RH* = *AY*, and thus *AY*:*CH* = *DA*:*AS*, which gives *AY*×*AS* = *CH*×*DA*, and the corollary is proved.

Let us note that the statement refers to "circumscribed parallelograms to an ellipse", differing from the standard enunciation in Apollonius, but, and this is quite odd, it also refers to "all" such parallelograms, which is clearly false.

Galuzzi in (1990, 401) refers to this corollary as "the ingenious corollary of Newton", but on the basis of De Witt's result and the well-known property that the tangents at the extremities of a diameter are parallel to its conjugate, the corollary is almost evident. The question then is: did Newton already know the result or did he deduce it from the proof of De Witt's theorem?

According to Galuzzi, and as Whiteside pointed out to him, this note "could reflect a reading posterior to the writing of *De motu*, and even of the *Principia*". In the light of the interest aroused in England

[37] A reproduction of Newton's original note is also given in (Galuzzi 1990, 415).

in the 1670s by the three books of Apollonius's *Conics* edited by Borelli, the presence of this result at the end of book VII, as well as in Saint Vincent's *Opus* - two works that Newton had at hand in Barrow's library - leads us to think that Newton was aware of the result long before the years 1684-1686, during which he wrote the *Principia* (see Guicciardini 1999). However, having noticed a certain similarity between the diagram for proposition XI of the *Principia* (book I, section III) and that in De Witt (see fig. 3), we may venture that, when Newton was developing his ideas in connection with the *De motu* and elliptic orbits, he was led back to De Witt's construction, and to write down the restricted form of proposition 31 as corollary 6 above.

A further proof of the simultaneity is the fact that in the corollary, and in the first edition of the *Principia*, the two sentences (both incorrect as already noticed) have the same wording.

5. Proposition 31 with Saint Vincent and La Hire

As known, Newton was reluctant to indicate his sources, but he made one exception in the *Principia*. In (1687, 69) he quoted a result from La Hire's *Sectiones Conicae* (1685), and referred to him as "Clarissimus Geometra".[38] However, as said above, he gave no reference for the Lemma 12, and this induces us to think he may have attributed it to Apollonius.

David Gregory (1661-1708) in his notes to the *Principia*, completely forgetting Apollonius, attributed Lemma 12 (i.e. proposition 31) to Saint Vincent, quoting (1647, props. 72),[39] and he did the same in the *Astronomia* (1702, 47). In those notes, Gregory also quoted de La Hire (1685, IV, prop. 43; V, prop. 21). This may be considered quite a strange thing, because Gregory might well have been familiar with the last three books of Apollonius's *Conics* edited by Borelli.

Both Saint Vincent's (1647) and La Hire's (1695) deserve our attention.

5.1 *Saint Vincent's own proof*

In his major work, *Opus geometricum etc*. (1647), Gregoire de Saint Vincent (1584-1667) proved two results that, put together, are equivalent to proposition 31; they are:

(book IV, prop. LXXII) [in an ellipse] *The rectangle constructed with the semi-axes is equal to the parallelogram constructed with* [any pair of] *conjugate semi-diameters*;[40]

(book VI, prop. XLIX). *The parallelograms whose opposite sides are tangent to two conjugate hyperbolas at the extremities of two conjugate diameters are equivalent among them*.[41]

[38] Newton was aware of other works by La Hire. Certainly he knew La Hire's *Nouvelle méthode etc*. (1673), see for instance (Guicciardini 2006, 84, note 15), and the *Nouveaux éléments des sections coniques* (1679), a copy of which he had in his library (Harrison 1978). According to Whiteside, see (MP, VI, 271), from the *Planiconiques* appended to (La Hire 1673), Newton might have found inspiration for his studies on transformation of figures.

[39] MS 210 of the Royal Society, see (Guicciardini 1999, 179-183).

[40] "Rectangulum sub dimidjis axibus aequale est parallelogrammo sub semidiametris coniugatis" (Saint Vincent 1647, 281).

[41] "Si fuerint binae hyperbolarum coniugationes *A*, *B*, *C*, *D*: ponantur autem per E centrum duae quoque diametrorum coniugationes per quarum vertices contingents actae constituant duo quadrilatera *FGHA*, *OPQR*. Dico illa esse aequalia inter se", (Saint Vincent 1647, 560).

We stress that the first statement is not correct, though in the proof Saint Vincent considered parallelograms whose angle are equal to the angles formed by the conjugate diameters at the centre.

Given an ellipse with axes *AC* and *BD* (see fig. 4), and having drawn any diameter *EG*, Saint Vincent considered the conjugate diameter *EF*, and called *H* the intersection point of the tangents at *G* and *F* to the ellipse. In proposition LXXII, he showed that the parallelogram *EGHF* and the rectangle constructed on the semi-axes *AE* and *BE* are equivalent.

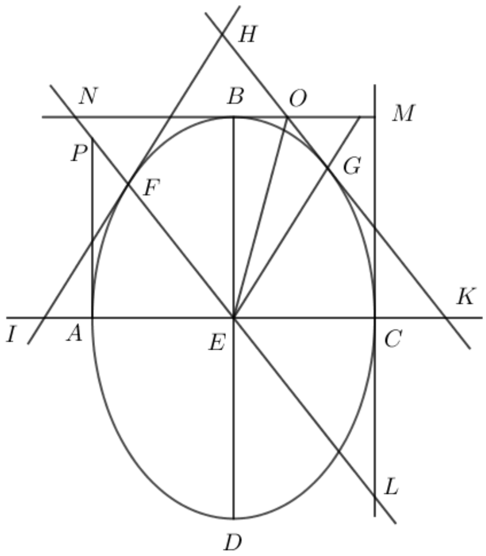


Fig. **4** reproduces the diagram in (Saint Vincent 1647, 281).

The proof given by Saint Vincent rests on two properties of the ellipses that he described in Propositions No. XXXVIII and LXX of book IV. For simplicity we state them in synthetic form and in reference to fig. 4: If *BE* is a semi-axis and *EG* is any diameter, let *O* be the intersection point between the tangents at *B* and at *G* to the ellipse, then triangles *EBO* and *EGO* are equivalent (in fact, the segment *BG* is divided in the middle by the diameter *OE*); If *AE* is a semi-axis and *EF* is a diameter, let *P* be the point where the diameter intersects the tangent at *A* to the ellipse, and *I* be the point where the tangent in *F* meets the straight line *AE*, then the triangles *APE* and *IFE* are equivalent.

To prove proposition LXXII, Saint Vincent proceeded as follows. He noticed that the parallelogram *NOKE* is divided into two equal parts by *EO*, then, since by the above *T(EOB) = T(EOG)*, it follows that *T(EBN) = T(EGK)*. Now, because *AP* and *CL* are parallel and *AE = EC*, we have that T(*EGK) = T(EBN)*. Therefore, for the second property above, we have *T(ECL) = T(EAP) = T(EIF)*. On the other hand, the triangles *EGK* and *IFE* are similar, and the same is true for the triangles *EBN* and *ECL*. Hence *KE*, *EI*, *NE*, *EL* are proportional, and the triangles *IFE*, *EGK*, *IHK* are similar, as are triangles *CLE*, *EBN*, *LMN*. So $T(IEF){:}T(LEC) = T(IHK){:}T(LMN)$ and also $T(EGK){:}T(EBN) = T(IHK){:}T(LMN)$. But $T(IEF) = T(LEC)$; $T(EGK){:}T(EBN)$, then $T(IHK) = T(LMN)$. By subtracting from the triangle *IHK* the triangles *IFE*, *EGK* and from the triangle *LMN* the triangles *ECL*, *EBN* respectively equal to the triangles *IFE*, *EGK*, we have the equivalence between the parallelogram *GEFH* and the rectangle *ECMB*.

Then Saint Vincent stated two corollaries, the second of which is the following: [In an ellipse] *All parallelograms whose (opposite) sides are a pair of conjugate diameters, are equivalent.*[42]

5.2 *La Hire's "Expositio brevis" of Apollonius*

In the years 1660-1664, Philippe de La Hire (1640-1718) travelled through Italy. He stayed mainly in Venice and Padua, where he studied with Stefano degli Angeli (1623-1697), who was a friend of Borelli. La Hire also visited Florence and Rome, see (Taton 1953).

In *Sectiones conicae* (1685), La Hire developed the theory by following a mix of the method he had adopted in (La Hire 1673), and of that of harmonic division, that, as we have seen, was already taken as the main tool for developing the theory of conic sections by Borelli in (1679).

Our attention is now drawn to the *Expositio brevis*[43] that La Hire included at the end of his main work. As he wrote in the preface (La Hire 1685, iv), he thought it right to pay a tribute to Apollonius's work by adding a summary of the seven books "which have come to us", and of the propositions proved therein, so that they could be compared with those "explained" by him. La Hire, continued by saying that he had followed a different method from that used by Apollonius, and remarked that if he had omitted some propositions that were necessary to Apollonius, he had recovered them in the summary. La Hire pointed out that he had proceeded in this way so that nothing was lost, and that his work related with the whole of Apollonius, so that it would be possible to judge what belonged to one and what to the other, and what had been added by other geometers who contributed to the last three books.[44] La Hire wrote that many geometers "all excellent for genius and doctrine" had written about conic sections, but, for fear of omitting anyone, he would not praise anyone. "Some of them", La Hire went on to say, "have limited themselves to comment on Apollonius, or to transmit the elements of conic sections [he meant the first four books] following a different route, though without going into depth". However, he quoted two authors who, according to him, had contributed to extending the boundaries of this science. They were Saint Vincent, and Viviani who restored the fifth book, and "If his deductions are compared with those contained in the fifth book,[45] you easily see how much praise he deserves because he never saw it".

Thus, La Hire did not quote Borelli's edition of books V—VII of the *Conics*, though in writing the summary he most likely had it to hand, and to which, without any doubt, he refers in the preface when speaking of the seven books "which have come to us". This is confirmed by a comparison between the propositions listed in the *Expositio brevis*, and those present in (Apollonius 1661): there is a one-to-one correspondence which preserves the numbering of the propositions.

In the *Sectiones conicae*, Apollonius's proposition 31 is stated and proved separately for the hyperbola and the ellipse, precisely in proposition XLIII of the fourth book, and in proposition XXI of the fifth book, respectively. We remark that La Hire's proofs proceed exactly as those given by Saint Vincent. However, in the statements, the parallelograms are for the first time referred to as "circumscribed" about the conic section, and, for this reason, it is worth enunciating them (see fig.

[42]"Sequitur secondo parallelogramma sub totis diametris coniugatis, inter se esse aequalia cum sint quadrupla eorum quae hac propositione ostensa sunt aequalia", (Saint Vincent 1647, 281).

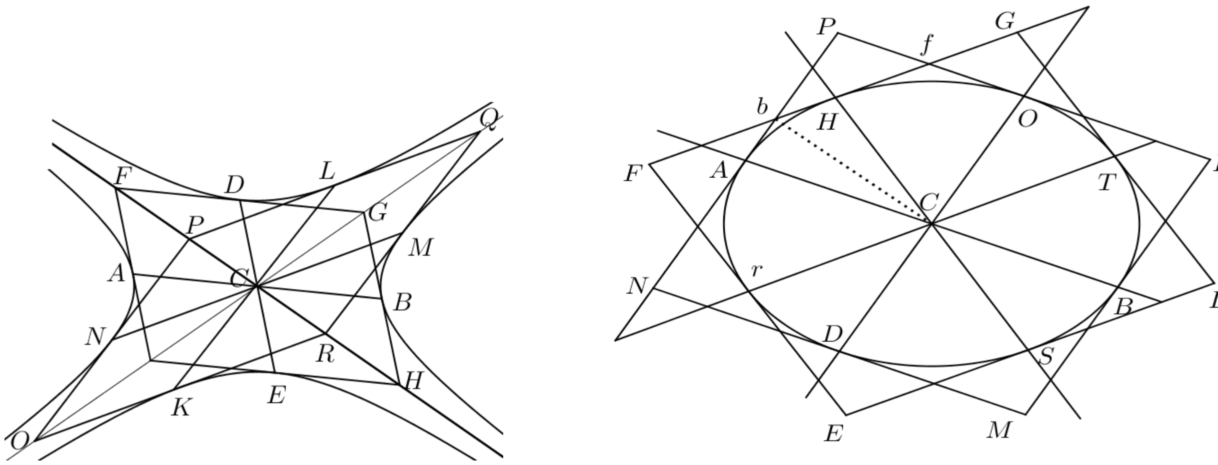
[43] "Expositio brevis, singularum propositionum Conicorum Apollonii Pergaei, cum ipsarum demonstrantionibus ex nostra methodo deductis", (La Hire, 1685, 220).

[44] La Hire implicitly referred to Viviani, see below but when specifying "the last three books" he included necessarily Borelli.

[45] He meant the fifth of Apollonius in Borelli's edition.

5a, b): *If a parallelogram FGHI is circumscribed about conjugate sections NA, DL, BM, KE whose sides are parallel to two conjugate diameters ED, BA drawn through their extremities, and with similar method another parallelogram OPQR is drawn through the extremities of other two conjugate diameters, then the parallelograms FGHI, OPQR are equal*.[46]

If a parallelogram MNPK whose parallel sides are equal two conjugate diameters AB, DO respectively, is circumscribe about an ellipse, and another parallelogram ELGF, whose sides are parallel to other two conjugate diameters HS, IT is also circumscribed, then the parallelogram MNPK and the parallelogram ELGF are equal. [47]



(a) (b)

Fig. 5 (a) reproduces the diagram in (La Hire 1685, 86). (b) is a simplified version of the diagram in (La Hire 1685, 99).

In the *Expositio brevis* La Hire gathered these two propositions under the heading “Proposit. XXXI, XXXII”, and are stated as follows: *In conjugate sections and in the ellipse, the parallelogram constructed with the axes, is equal to the parallelogram constructed with any two conjugated diameters, provided the angles are equal to those between the diameters themselves*.[48]

The reference to (Apollonius 1661) is evident, though not declared.

6. FINAL REMARKS AND CONCLUSION

The knowledge of the existence in Rome of an Arabic manuscript containing books V-VII of Apollonius’s *Conics* had spread quite rapidly since 1578. Although copies of the manuscript were made, we do not know if Savile succeeded in obtaining his own exemplar, for which it seems he had

[46] “In sectionibus conjugatis *NA, DL, BM, KE* si circumscribatur parallelogrammum *FGHI* à rectis parallelis duabus diametris inter se conjugatis *ED, BA,* et per ipsorum terminus ductis, et simili methodo circumscribatur aliud parallelogrammum *OPQR* à rectis ductis per terminos diametrorum conjugatarum, et ipsis parallelis: Dico parallelogramma *FGHI, OPQR* esse inter se aequalia”, (La Hire 1685, 85).

[47] “Si Ellipsi circumscribatur parallelogrammum *MNPK* cujus latera sint duabus diametris inter se conjugatis *AB*, *DO* aequidistantia: similiter aliud parallelogrammum *ELGF* circumscribatur, cujus latera sint duabus aliis *HS*, *IT* diametris inter se conjugatis parallela”, (La Hire 1685, 99).

[48] “In sectionibus conjugatis et Ellipsi parallelogrammum sub axibus aequale est parallelogrammo sub duabus quibuscumque diametris inter se conjugatis, in angulis ipsarum diametrorum conjugatarum”, (La Hire 1685, 243).

worked hard. Neither can we be sure that those mathematicians who did actually see it were able to get inspiration for their studies.

Raimondi aimed to translate and edit it, but we can imagine, as the events around Borelli's edition teach us, that he encountered difficulties that he was unable to overcome. Maurolico, the mathematician who, more than any other, could have been of help in this work, had died two years before the arrival of the manuscript in Rome.

In the decade 1630-1640, important progress was made in geometry, and in particular in the study of conic sections, and in the next twenty years some mathematicians tried to recover the lost books of the *Conics*. Thus, opposite interests arose around the manuscript concerning, on the one hand, those who wanted it translated and published, and, on the other, those who would have felt deprived of their work of reconstruction of the lost books.

Unfortunately, the favourable events that led Borelli to edit books V-VII occurred only seventeen years after the arrival of "MS L" in Rome.

When (Apollonius 1661) appeared in print, copies were sent to several mathematicians in Italy and abroad, and the content of books V-VII of Apollonius's *Conics* became known in its entirety. Two copies of it entered Barrow's library, where, presumably, a copy of Saint Vincent's *Opus Geometricum* was already present. Newton may have known of these works from Barrow, or may have discovered them during his early reading in Barrow's library. However, there is no evidence that Newton was aware of the content of proposition 31 before writing the "*De Motu*".

In the statement of this proposition, in the form of Newton's Lemma 12, both in the manuscript of the *De Motu* (as Lemma 4) and in the first edition of the *Principia*, Newton referred to "parallelogramma circa datam ellipsin descripta", that is, to "parallelograms circumscribed about a given ellipse". In the manuscript of *De motu*, presented to Halley in 1684, Newton added "Patet ex Conicis", [49] that is "It is clear from *Conics*". This claim may have induced some to believe that Newton divined this result (see later). In the *Principia* Newton completed the lemma by enunciating for the conjugate hyperbolas the same result he had used in the case of the ellipse, and this time he added that "Constat Utrunq; ex Conics". Newton showed he was aware that both results were known to Apollonius and he aimed to state a general result. This seems like a last-minute addition after having read La Hire (1685), as it was for the focus-directrix proprieties expounded in section 4 of the *Principia* (Whiteside 1970, 117).

According to Rowlands (2017, 52) Lemma 12 was apparently added, following Halley's advice, along with other preliminary lemmas, to help readers understand the propositions in section two of the *Principia*.

Rowlands, somehow following Whiteside (1970), asserts that the lemma probably did not derive from Apollonius, which Newton had most likely never studied in any depth. We believe that this assertion clashes with the depth of the geometrical results expounded in the *Principia* and Newton's admiration of the method of Ancient Greek geometry. For instance, he proved to be very familiar with the chord theorem (Apollonius 1566) even if it was only hastily recalled in the fifth section of the *Principia*.

Let us stress that Newton did not enunciate the Lemma 12 either in the form of Apollonius's proposition 31 or of Saint Vincent's two propositions, but to simplify the sentence, he considered circumscribed parallelograms, as La Hire had done in (1685).

[49] Page 41 of the manuscript held at the Cambridge University Library, see also (Rigaud 1838, Appendix, 2).

A few years after the printing of the *Principia*, Newton realized he had written a book which was not easy to read and, he reluctantly contemplated a revision of his work in which David Gregory and Roger Cotes (1682-1716) were involved (Whiteside 1970, 116).

In the years 1709-1713, Cotes cooperated closely with Newton on the revision of the *Principia* in the editing of the second publication of the work (Newton 1713). In 1708, at the age of 26, Cotes had been appointed the first "Plumiam" Professor of Astronomy and Experimental Philosophy at Cambridge, as proposed by Newton and William Whiston. However, Cotes's initially friendly relationship with Newton cooled toward the end of their collaboration. In the manuscript draft of the preface of the second edition of the *Principia,* Newton acknowledged Cotes's co-operation (Newton 1961, 112-113), but this did not appear in the printed version (Newton 1713). Being on very friendly terms with Whiston, Cotes most likely exchanged some opinions with him about the revision work. Many years later, writing his memoirs, in connection with Lemma 12 Whiston made his famous observation (Whiston 1749, 39):

> Sir Isaac, in Mathematicks, could sometimes see almost by Intuition, even without Demonstration; as was the Case in that famous Proposition in his *Principia,* that *All parallelograms circumscribed about the Conjugated Diameters of an Ellipsis are equal*; which he told Mr *Cotes* he used before it had ever been demonstrated by any one, as it was afterward.

To this regards Whiteside (1970, 118) remarked, "[Whiston] omitted to mention – if ever he knew– that the theorem instanced was the thirty-first of the recently rediscovered seventh book of Apollonius's Conics", clearly referring to Halley's edition (1710). "Newton's mathematical intuition whatever that was", continued Whiteside, "may indeed have let him see this Lemma is, in the case of the ellipse, all but self-evident by orthogonal projection at a fixed angle from a circle inscribed in a square, but clearly, if Whiston spoke true, his knowledge of the later books of Apollonius was far from perfect".

We are led to think that, in the case of the ellipse, the only case that really interested him, Newton's first source for Lemma 12 was his "corollary 6", the proof of which is derived from De Witt's construction. When Newton tackled the question of the motion of the planets, he was led back to De Witt's construction of the ellipse, probably because of the similarity of his diagram (book I, sec. III, proposition 11) with De Witt's (see our figure 3) and to give his own proof of the lemma for this case.

David Gregory attributed Lemma 12 to Saint Vincent and La Hire, and we do not know why Gregory did not refer directly to Apollonius *via* (Apollonius 1661).

Adriaen Verwer (1644/5-1717) in a note on the margin of his own copy of the first edition of the *Principia*,[50] finally indicated Lemma 12 as "Apollonii Pergaei, l. VII. Prop. 31, 32", clearly referring to (Apollonius 1661).

Only in the second edition of the *Principia* (1713), did Newton complete the wording in Lemma 12, which then appeared in the following correct form: *All parallelograms circumscribed about an ellipse or a hyperbola at* [the extremities of] *any conjugate diameters, are equal among them*. Constat ex Conicis.[51]

Of the three Arabic manuscripts which arrived in the West, that of Banū Mūsā was recognized as the closest to the original Greek. So Halley's edition of Apollonius (1710) was considered the most

[50] Available on line in the website of the Utrecht University Library.

[51] "Parallelogramma omnia, circa datae Ellipseos vel hyperbola diametros quasvis conjugatas descripta, esse inter se aequalia", (Newton 1713, 45).

reliable of the three. Nevertheless, by that time such interest was essentially historical, "whereas", as G. Toomer writes (1990, xxi), "if it had appeared before, instead of after, the work of the great 17th-century mathematicians such as Descartes, Fermat, Desargues and Newton, (all of whom were familiar with *Conics* I-IV), it could have influenced the development of mathematics". We cannot say the same of Borelli's edition, which – though not the most faithful to the original by Apollonius, but still retaining all its substance – seems to have played a non-marginal role in the later part of seventeenth century geometry.

Unfortunately, Borelli's edition of Books V to VII of Apollonius's *Conics* did not get the credit it deserved. The case of La Hire, as we have underlined, is exemplary.

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