On the Cesàro average of the "Linnik numbers"

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Abstract

Let Λ be the von Mangoldt function and

$$r_Q(n) = \sum_{m_1+m_2^2+m_3^2=n} \Lambda(m_1)$$

be the counting function for the numbers that can be written as sum of a prime and two squares (that we will call Linnik numbers, for brevity). Let N a sufficiently large integer. We prove that for $k > 3/2$ we have

$$\sum_{n\leq N} r_Q(n) \frac{(N-n)^k}{\Gamma(k+1)} = M(N,k) + O\left(N^{k+1}\right)$$

where $M(N,k)$ is essentially a weighted sum, over non-trivial zeros of the Riemann zeta function, of Bessel functions of complex order and real argument. We also prove that with this technique the bound $k > 3/2$ is optimal.

1 Introduction

We continue the recent work of Languasco and Zaccagnini on additive problems with prime summands. In [9] and [10] they study the Cesàro weighted

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explicit formula for the Goldbach numbers (the integers that can be written as sum of two primes) and for the Hardy-Littlewood numbers (the integers that can be written as sum of a prime and a square). In a similar manner, we will study a Cesàro weighted explicit formula for the integers that can be written as sum of a prime and two squares. We will obtain an asymptotic formula with a main term and more terms depending explicitly on the zeros of the Riemann zeta function. The study of these numbers is classical. For example Hardy and Littlewood in [7] studied the number of solutions of the equation

$$n = p + a^2 + b^2$$

and Linnik in [13] derived an asymptotic formula for the number of representations of these numbers. Similar averages of arithmetical functions are common in literature, see, e.g., Chandrasekharan - Narasimhan [2] and Berndt [1] who built on earlier classical work. For our work we will need the Bessel functions $J_v(u)$ of complex order v and real argument u. For their definition and main properties we refer to Watson [15], but we recall that they were introducted by Daniel Bernoulli and they are the canonical solution of the differential equation

$$u^2 \frac{d^2 J}{du^2} + u \frac{dJ}{du} + \left(u^2 - v^2\right) J = 0$$

for any complex number v. In particular, equation (8) on page 177 of [15] gives the Sonine representation

$$J_\nu(u) = \frac{(u/2)^\nu}{2\pi i} \int_{(a)} e^s s^{-\nu-1} e^{-u^2/(4s)} ds \tag{1.1}$$

where the notation $\int_{(a)}$ means $\int_{a-i\infty}^{a+i\infty}$. The method we will use in this additive problem is based on a formula due to Laplace [11], namely

$$\frac{1}{2\pi i} \int_{(a)} v^{-s} e^v dv = \frac{1}{\Gamma(s)} \tag{1.2}$$

with $\mathrm{Re}(s) > 0$ and $a > 0$ (see, e.g., formula 5.4 (1) on page 238 of [4]). As in [10], we combine this approach with line integrals with the classical methods dealing with infinite sum over primes and integers. Similarly as [10] the problem naturally involves the modular relation for the complex Jacobi θ_3 function; the presence of the Bessel functions in our statement strictly depends on such modularity relation.

2 Preliminary definitions and Lemmas

Let

$$r_Q(n) = \sum_{m_1+m_2^2+m_3^2=n} \Lambda(m_1)$$

and let $J_v(u)$ be the Bessel function of complex order v and real argument u. Let $z = a + iy$, $a > 0$, and

$$\theta_3(z) = \sum_{m\in\mathbb{Z}} e^{-m^2 z}, \tag{2.1}$$

$$\widetilde{S}(z) = \sum_{m\geq 1} \Lambda(m) e^{-mz}, \tag{2.2}$$

$$\omega_2(z) = \sum_{m\geq 1} e^{-m^2 z}, \tag{2.3}$$

and we can see that

$$\theta_3(z) = 1 + 2\omega_2(z). \tag{2.4}$$

Furthermore we have the functional equation (see, for example, the proposition VI.4.3 of Freitag-Busam [5] page 340)

$$\theta_3(z) = \left(\frac{\pi}{z}\right)^{1/2} \theta_3\left(\frac{\pi^2}{z}\right), \ \mathrm{Re}(z) > 0 \tag{2.5}$$

and so

(2.6)

$$\omega_2^2(z) = \left(\frac{1}{2}\left(\frac{\pi}{z}\right)^{1/2} - \frac{1}{2}\right)^2 + \frac{\pi}{z}\omega_2^2\left(\frac{\pi^2}{z}\right) + \left(\left(\frac{\pi}{z}\right)^{1/2} - 1\right)\left(\left(\frac{\pi}{z}\right)^{1/2} \omega_2\left(\frac{\pi^2}{z}\right)\right).$$

A trivial but important estimate is

$$|\omega_2(z)| \leq \omega_2(a) \leq \int_0^\infty e^{-at^2} dt = \frac{\sqrt{\pi}}{2\sqrt{a}} \ll a^{-1/2}. \tag{2.7}$$

Let us introduce the following

Lemma 2.1. *Let $z = a + iy$, $a > 0$ and $y \in \mathbb{R}$. Then*

$$\widetilde{S}(z) = \frac{1}{z} - \sum_{\rho} z^{-\rho}\Gamma(\rho) + E(a, y) \tag{2.8}$$

where $\rho = \beta + i\gamma$ runs over the non-trivial zeros of $\zeta(s)$ and

$$E(a, y) \ll |z|^{1/2} \begin{cases} 1, & |y| \leq a \\ 1 + \log^2(|y|/a), & |y| > a. \end{cases}$$

(For a proof see Lemma 1 of [9]. The bound for $E(a,y)$ has been corrected in [8]). So in particular, taking $z=\frac{1}{N}+iy$ we have

$$\left|\sum_{\rho} z^{-\rho}\Gamma(\rho)\right| = \left|\frac{1}{z}-\widetilde{S}(z)+E\left(\frac{1}{N},y\right)\right| \ll N+\frac{1}{|z|}+\left|E\left(\frac{1}{N},y\right)\right|$$

$$\ll \begin{cases} N, & |y|\leq 1/N \\ N+|z|^{1/2}\log^{2}(2N|y|), & |y|>1/N. \end{cases} \tag{2.9}$$

Now we have to recall that the Prime Number Theorem (PNT) is equivalent, via Lemma 2.1, to the statement

$$\widetilde{S}(a)\sim a^{-1}, \text{ when } a\to 0^{+}$$

(see Lemma 9 of [7]). For our purposes it is important to introduce the Stirling approximation

$$|\Gamma(x+iy)|\sim\sqrt{2\pi}e^{-\pi|y|/2}|y|^{x-1/2} \tag{2.10}$$

(see for example §4.42 of [14]) uniformly for $x\in[x_1,x_2]$, x_1 and x_2 fixed, and the identity

$$\left|z^{-w}\right| = |z|^{-\mathrm{Re}(w)}\exp\left(\mathrm{Im}(w)\arctan(y/a)\right). \tag{2.11}$$

We now quote Lemmas 2 and 3 from [9]:

Lemma 2.2. *Let $\beta+i\gamma$ run over the non-trivial zeros of the Riemann zeta function and let $\alpha>1$ be a parameter. The series*

$$\sum_{\rho,\,\gamma>0}\gamma^{\beta-1/2}\int_{1}^{\infty}\exp\left(-\gamma\arctan(1/u)\right)\frac{dy}{u^{\alpha+\beta}}$$

converges provided that $\alpha>3/2$. For $\alpha\leq 3/2$ the series does not converge. The result remains true if we insert in the integral a factor $\log^{c}(u)$, for any fixed $c\geq 0$.

Lemma 2.3. *Let $\beta+i\gamma$ run over the non-trivial zeros of the Riemann zeta function, let $z=a+iy$, $a\in(0,1)$, $y\in\mathbb{R}$ and $\alpha>1$. We have*

$$\sum_{\rho}|\gamma|^{\beta-1/2}\int_{\mathbb{Y}_1\cup\mathbb{Y}_2}\exp\left(\gamma\arctan\left(\frac{y}{a}\right)-\frac{\pi}{2}|\gamma|\right)\frac{dy}{|z|^{\alpha+\beta}}\ll_{\alpha} a^{-\alpha}$$

where $\mathbb{Y}_1=\{y\in\mathbb{R}:\ \gamma y\leq 0\}$ and $\mathbb{Y}_2=\{y\in[-a,a]:\ y\gamma>0\}$. The result remains true if we insert in the integral a factor $\log^{c}(|y|/a)$, for any fixed $c\geq 0$.

We now establish an important Lemma. We will use it to prove that there is a limitation in our technique. Essentially the lower bound of k is linked to the number of squares in the problem. We have

Lemma 2.4. *Let $\beta+i\gamma$ run over the non-trivial zeros of the Riemann zeta-function, let N, d be positive integers, $\|.\|$ the euclidean norm in $\mathbb{R}^d$ and $k>0$ be a real number. Then the series*

$$\sum_{\bar{l}\in(0,\infty)^d}\sum_{\gamma>0}\gamma^{-k-3/2}\int_0^{\gamma}e^{-N\left\|\bar{l}\right\|^2 v^2/\gamma^2}e^{-v}v^{k+\beta}dv,$$

where

$$\sum_{\bar{l}\in(0,\infty)^d}=\sum_{l_1\geq 1}\sum_{l_2\geq 1}\cdots\sum_{l_d\geq 1},$$

converges if $k>d-1/2$ and this result is optimal.

Proof. From (2.4) we have that

$$\omega_2^d(z)=\frac{1}{2^d}\sum_{m=0}^{d}\binom{d}{m}(-1)^{d-m}\,\theta_3^m(z).$$

Hence

$$\begin{aligned}I=&\sum_{\bar{l}\in(0,\infty)^d}\sum_{\gamma>0}\gamma^{-k-3/2}\int_0^{\gamma}e^{-N\left\|\bar{l}\right\|^2 v^2/\gamma^2}e^{-v}v^{k+\beta}dv\\ =&\sum_{\gamma>0}\gamma^{-k-3/2}\int_0^{\gamma}\omega_2^d\left(\frac{Nv^2}{\gamma^2}\right)e^{-v}v^{k+\beta}dv\\ =&\frac{1}{2^d}\sum_{m=0}^{d}\binom{d}{m}(-1)^{d-m}\sum_{\gamma>0}\gamma^{-k-3/2}\int_0^{\gamma}\theta_3^m\left(\frac{Nv^2}{\gamma^2}\right)e^{-v}v^{k+\beta}dv.\end{aligned}$$

Now, using the functional equation (2.5) we have that

$$\begin{aligned}I=&\frac{1}{2^d}\sum_{m=0}^{d}\binom{d}{m}(-1)^{d-m}\frac{\pi^{m/2}}{N^{m/2}}\sum_{\gamma>0}\gamma^{m-k-3/2}\int_0^{\gamma}\theta_3^m\left(\frac{\pi^2\gamma^2}{Nv^2}\right)e^{-v}v^{k+\beta-m}dv\\ =&\frac{1}{2^d}\sum_{m=0}^{d}\binom{d}{m}(-1)^{d-m}\frac{\pi^{m/2}}{N^{m/2}}\sum_{\gamma>0}\gamma^{m-k-3/2}I_{\gamma,m},\end{aligned}$$

say. Now we claim that

$$\theta_3\left(\frac{\pi^2\gamma^2}{Nv^2}\right)\asymp 1,$$

where the notation $f(x) \asymp g(x)$ means $g(x) \ll f(x) \ll g(x)$, since $\theta_3(x)$ is a continuous function in the interval $\left[\frac{\pi^2}{N}, \infty\right)$ (i.e. the range of $1/v^2$) and

$$\lim_{x\to\infty} \theta_3(x) = 1.$$

So we have

$$I_{\gamma,m} \asymp \sum_{\gamma>0} \gamma^{m-k-3/2} \int_0^{\gamma} e^{-v} v^{k+\beta-m} dv$$

and now, assuming $k+\beta-m+1>0$, we get

$$\int_0^{\gamma} e^{-v} v^{k+\beta-m} dv \asymp 1.$$

Hence

$$I_{\gamma,m} \asymp_k \sum_{\gamma>0} \gamma^{m-k-3/2}$$

and the last series converges if $k > m - 1/2$. Since $m = 0, \ldots, d$ for a global convergence we must have $k > d - 1/2$ and this result is optimal. □

Let us introduce another lemma

Lemma 2.5. *Let $\rho = \beta + i\gamma$ run over the non-trivial zeros of the Riemann zeta function, let $z = \frac{1}{N} + iy$, $N > 1$ natural number, $y \in \mathbb{R}$ and $\alpha > 3/2$. We have*

$$\sum_{\rho} |\Gamma(\rho)| \int_{(1/N)} \left|e^{Nz}\right| \left|z^{-\rho}\right| |z|^{-\alpha} |dz| \ll_{\alpha} N^{\alpha}.$$

Proof. Put $a = \frac{1}{N}$. Using the identity (2.11) and (2.10) we get that the left hand side in the statement above is

$$\sum_{\rho} |\gamma|^{\beta-1/2} \int_{\mathbb{R}} \exp\left(\gamma \arctan\left(\frac{y}{a}\right) - \frac{\pi}{2}|\gamma|\right) \frac{dy}{|z|^{\alpha+\beta}}. \tag{2.12}$$

and so by Lemma 2.3 (2.12) is $\ll_{\alpha} a^{-\alpha}$ in $\mathbb{Y}_1 \cup \mathbb{Y}_2$. For the other part we can see that

$$\sum_{\rho} \gamma^{\beta-1/2} \int_a^{\infty} \exp\left(-\gamma \arctan\left(\frac{a}{y}\right)\right) \frac{dy}{|z|^{\alpha+\beta}}$$

$$= a^{-\alpha-\beta+1} \sum_{\rho} \gamma^{\beta-1/2} \int_1^{\infty} \exp\left(-\gamma \arctan\left(\frac{1}{u}\right)\right) \frac{dy}{u^{\alpha+\beta}}$$

since

$$|z|^{-1} \asymp \begin{cases} a^{-1} & |y| \le a, \\ |y|^{-1} & |y| \ge a, \end{cases} \tag{2.13}$$

and so by Lemma 2.2 we have the convergence if $\alpha > 3/2$. □

3 Settings

Using (2.1), (2.2) and (2.3) it is not hard to see that

$$\widetilde{S}(z)\,\omega_2^2(z) = \sum_{m_1\geq 1}\sum_{m_2\geq 1}\sum_{m_3\geq 1}\Lambda(m_1)\,e^{-\left(m_1+m_2^2+m_3^2\right)z} = \sum_{n\geq 1} r_Q(n)\,e^{-nz}.$$

Let $z = a + iy$, $a > 0$ and let us consider

$$\frac{1}{2\pi i}\int_{(a)} e^{Nz}z^{-k-1}\widetilde{S}(z)\,\omega_2^2(z)\,dz = \frac{1}{2\pi i}\int_{(a)} e^{Nz}z^{-k-1}\sum_{n\geq 1} r_Q(n)\,e^{-nz}dz.$$

Now we prove that we can exchange the integral with the series. From (2.7) and the Prime Number Theorem in the form quoted above we have

$$\sum_{n\geq 1}\left|r_Q(n)\,e^{-nz}\right| = \widetilde{S}(a)\,\omega_2^2(a) \ll a^{-2}$$

hence

$$\int_{(a)}\left|e^{Nz}z^{-k-1}\right|\left|\widetilde{S}(z)\,\omega_2^2(z)\right||dz| \ll a^{-2}e^{Na}\left(\int_{-a}^{a} a^{-k-1}dy + 2\int_{a}^{\infty} y^{-k-1}dy\right) \ll_k a^{-2-k}e^{Na}$$

assuming $k > 0$. So finally we have

$$\sum_{n\leq N} r_Q(n)\,\frac{(N-n)^k}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_{(a)} e^{Nz}z^{-k-1}\widetilde{S}(z)\,\omega_2^2(z)\,dz. \tag{3.1}$$

Now, using (2.8), we can write (3.1) as

$$\sum_{n\leq N} r_Q(n)\,\frac{(N-n)^k}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_{(a)} e^{Nz}z^{-k-1}\left(\frac{1}{z}-\sum_{\rho} z^{-\rho}\Gamma(\rho)\right)\omega_2^2(z)\,dz+$$

$$+\,O\left(\int_{(a)}\left|e^{Nz}\right||z|^{-k-1}\left|\omega_2^2(z)\right||E(a,y)|\,|dz|\right) \tag{3.2}$$

and the error term can be estimated, using Lemma 2.1, (2.7) and (2.13) as

$$a^{-1}e^{Na}\left(\int_{-a}^{a} a^{-k-1}dy + \int_{a}^{\infty} y^{-k-1/2}\left(1+\log^2(y/a)\right)dy\right) \ll_k e^{Na}a^{-k-1}$$

assuming $k > 1/2$. Hereafter we will consider $a = 1/N$. We have

$$\sum_{n\leq N} r_Q(n)\,\frac{(N-n)^k}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_{(1/N)} e^{Nz}z^{-k-1}\left(\frac{1}{z}-\sum_{\rho} z^{-\rho}\Gamma(\rho)\right)\omega_2^2(z)\,dz + O\left(N^{k+1}\right)$$

and now, using the functional equation (2.6), we get

$$\begin{aligned}\sum_{n\leq N} r_Q(n)\frac{(N-n)^k}{\Gamma(k+1)} =& \frac{1}{8\pi i}\int_{(1/N)} e^{Nz}z^{-k-1}\left(\frac{1}{z}-\sum_{\rho} z^{-\rho}\Gamma(\rho)\right)\left(\left(\frac{\pi}{z}\right)^{1/2}-1\right)^2 dz\\ &+\frac{1}{2\pi i}\int_{(1/N)} e^{Nz}z^{-k-1}\left(\frac{1}{z}-\sum_{\rho} z^{-\rho}\Gamma(\rho)\right)\frac{\pi}{z}\omega_2^2\left(\frac{\pi^2}{z}\right)dz\\ &+\frac{1}{2\pi i}\int_{(1/N)} e^{Nz}z^{-k-1}\left(\frac{1}{z}-\sum_{\rho} z^{-\rho}\Gamma(\rho)\right)\left(\left(\frac{\pi}{z}\right)^{1/2}-1\right)\left(\left(\frac{\pi}{z}\right)^{1/2}\omega_2\left(\frac{\pi^2}{z}\right)\right)dz\\ &+O\left(N^{k+1}\right)\\ =&I_1+I_2+I_3+O\left(N^{k+1}\right),\end{aligned}$$

say.

4 Evaluation of I_1

From I_1 we will find the main terms $M_1(N,k)$ and $M_2(N,k)$ of our asymptotic formulae. We have

$$\begin{aligned}I_1 =& \frac{1}{8\pi i}\int_{(1/N)} e^{Nz}z^{-k-2}\left(\left(\frac{\pi}{z}\right)^{1/2}-1\right)^2 dz\\ &-\frac{1}{8\pi i}\int_{(1/N)} e^{Nz}z^{-k-1}\sum_{\rho} z^{-\rho}\Gamma(\rho)\left(\left(\frac{\pi}{z}\right)^{1/2}-1\right)^2 dz\\ =& I_{1,1}-I_{1,2},\end{aligned}$$

say. From $I_{1,1}$ we observe that

$$I_{1,1}=\frac{\pi}{8\pi i}\int_{(1/N)} e^{Nz}z^{-k-3}dz+\frac{1}{8\pi i}\int_{(1/N)} e^{Nz}z^{-k-2}dz-\frac{\pi^{1/2}}{4\pi i}\int_{(1/N)} e^{Nz}z^{-k-5/2}dz$$

so, if we put $Nz=s$, $ds=Ndz$ and use (1.2) we get immediately

$$\begin{aligned}I_{1,1} =& \frac{\pi}{4}\frac{N^{k+2}}{2\pi i}\int_{(1)} e^s s^{-k-3}ds+\frac{N^{k+1}}{4}\frac{1}{2\pi i}\int_{(1)} e^s s^{-k-2}ds-\frac{\pi}{2}\frac{N^{k+3/2}}{2\pi i}\int_{(1)} e^s s^{-k-5/2}ds\\ =& M_1(N,k).\end{aligned}$$

From $I_{1,2}$ we have

$$\begin{aligned} I_{1,2} =& \frac{\pi}{8\pi i}\int_{(1/N)} e^{Nz} z^{-k-2}\sum_{\rho} z^{-\rho}\Gamma\left(\rho\right)dz \\ &+\frac{1}{8\pi i}\int_{(1/N)} e^{Nz} z^{-k-1}\sum_{\rho} z^{-\rho}\Gamma\left(\rho\right)dz \\ &-\frac{\pi^{1/2}}{4\pi i}\int_{(1/N)} e^{Nz} z^{-k-3/2}\sum_{\rho} z^{-\rho}\Gamma\left(\rho\right)dz \\ =& \mathcal{I}_1+\mathcal{I}_2-\mathcal{I}_3, \end{aligned}$$

say. We observe that by Lemma 2.5 we have the absolute convergence of these integrals if, respectively, we have $k>-1/2$, $k>1/2$ and $k>0$. Hence for $k>1/2$ we have

$$\mathcal{I}_1=\frac{\pi}{4}\sum_{\rho}\Gamma\left(\rho\right)\frac{1}{2\pi i}\int_{(1/N)} e^{Nz} z^{-k-2-\rho}dz=\frac{\pi}{4}\sum_{\rho}\frac{\Gamma\left(\rho\right)}{\Gamma\left(k+2+\rho\right)}N^{k+1+\rho}$$

$$\mathcal{I}_2=\frac{1}{4}\sum_{\rho}\Gamma\left(\rho\right)\frac{1}{2\pi i}\int_{(1/N)} e^{Nz} z^{-k-1-\rho}dz=\frac{1}{4}\sum_{\rho}\frac{\Gamma\left(\rho\right)}{\Gamma\left(k+1+\rho\right)}N^{k+\rho}$$

$$\mathcal{I}_3=\frac{\pi^{1/2}}{2}\sum_{\rho}\Gamma\left(\rho\right)\frac{1}{2\pi i}\int_{(1/N)} e^{Nz} z^{-k-3/2-\rho}dz=\frac{\pi^{1/2}}{2}\sum_{\rho}\frac{\Gamma\left(\rho\right)}{\Gamma\left(k+3/2+\rho\right)}N^{k+1/2+\rho}.$$

5 Evaluation of I_2

We have

$$\begin{aligned} I_2 =& \frac{\pi}{2\pi i}\int_{(1/N)} e^{Nz} z^{-k-3}\omega_2^2\left(\frac{\pi^2}{z}\right)dz \\ &-\frac{\pi}{2\pi i}\int_{(1/N)} e^{Nz} z^{-k-2}\sum_{\rho} z^{-\rho}\Gamma\left(\rho\right)\omega_2^2\left(\frac{\pi^2}{z}\right)dz \\ =& I_{2,1}-I_{2,2}, \end{aligned}$$

say.

Evaluation of $\mathbf{I_{2,1}}$

We have that

$$I_{2,1}:=\frac{\pi}{2\pi i}\int_{(1/N)} e^{Nz} z^{-k-3}\omega_2^2\left(\frac{\pi^2}{z}\right)dz=\frac{\pi}{2\pi i}\int_{(1/N)} e^{Nz} z^{-k-3}\left(\sum_{l_1\geq 1} e^{-l_1^2\pi^2/z}\right)\left(\sum_{l_2\geq 1} e^{-l_2^2\pi^2/z}\right)dz;$$

so let us prove that we can exchange the integral with the series. Let us consider

$$A_1 := \sum_{l_1 \geq 1} \int_{(1/N)} \left|e^{Nz}\right| |z|^{-k-3} e^{-l_1^2 \pi^2 \mathrm{Re}(1/z)} \left| \omega_2 \left(\frac{\pi^2}{z} \right) \right| |dz| ,$$

say. From

$$\mathrm{Re}\,(1/z) = \frac{N}{1+N^2y^2} \gg \begin{cases} N & |y| \leq 1/N \\ 1/\left(Ny^2\right) & |y| > 1/N \end{cases} \tag{5.1}$$

we have

$$A_1 \ll \sum_{l_1 \geq 1} \int_0^{1/N} \frac{e^{-l_1^2 N}}{|z|^{k+3}} \omega_2 \left(N\right) dy + N^{1/2} \sum_{l_1 \geq 1} \int_{1/N}^{\infty} \frac{y e^{-l_1^2/\left(Ny^2\right)}}{|z|^{k+3}} dy = U_1 + U_2$$

hence, recalling (2.7) and (2.13),

$$U_1 \ll N^{k+2} \omega_2^2 \left(N\right) \ll N^{k+1}$$

and from (2.13) (with $a = 1/N$) we get

$$U_2 \ll N^{1/2} \sum_{l_1 \geq 1} \int_{1/N}^{\infty} \frac{e^{-l_1^2/\left(Ny^2\right)}}{y^{k+2}} dy \ll N^{k/2+1} \sum_{l_1 \geq 1} \frac{1}{l_1^{k+1}} \int_0^{l_1^2 N} u^{k/2-1/2} e^{-u} du$$

$$\leq \Gamma \left(\frac{k+1}{2} \right) N^{k/2+1} \sum_{l_1 \geq 1} \frac{1}{l_1^{k+1}} \ll_k N^{k/2+1}$$

assuming $k > 0$. Now we have to study the convergence of

$$A_2 := \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \int_{(1/N)} \left|e^{Nz}\right| |z|^{-k-3} e^{-l_1^2 \pi^2 \mathrm{Re}(1/z)} e^{-l_2^2 \pi^2 \mathrm{Re}(1/z)} |dz| ,$$

say. Again from (2.13) we have

$$A_2 \ll \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \int_0^{1/N} \frac{e^{-\left(l_1^2+l_2^2\right)N}}{|z|^{k+3}} dy + \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \int_{1/N}^{\infty} \frac{e^{-\left(l_1^2+l_2^2\right)/\left(Ny^2\right)}}{|z|^{k+3}} dy$$

$$= V_1 + V_2,$$

say. For V_1 we can repeat the same reasoning of U_1 thus getting

$$V_1 \ll N^{k+2} \omega_2^2 \left(N\right) \ll N^{k+1}$$

and for V_2, assuming $k > 1$, we have

$$V_2 \ll \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \int_{1/N}^{\infty} \frac{e^{-\left(l_1^2+l_2^2\right)/\left(Ny^2\right)}}{y^{k+3}} dy \ll_k N^{k/2+1/2}.$$

Then finally we have

$$I_{2,1}=\frac{\pi}{2\pi i}\sum_{l_1\geq 1}\sum_{l_2\geq 1}\int_{(1/N)}e^{Nz}z^{-k-3}e^{-\left(l_1^2+l_2^2\right)\pi^2/z}dz=N^{k+2}\pi\sum_{l_1\geq 1}\sum_{l_2\geq 1}\frac{1}{2\pi i}\int_{(1)}e^{s}s^{-k-3}e^{-\left(l_1^2+l_2^2\right)\pi^2 N/s}ds$$

from which, recalling the definition of the Bessel functions (1.1) we have, taking $u=2\pi\left(l_1^2+l_2^2\right)^{1/2}N^{1/2}$ and assuming $k>1$, that

$$I_{2,1}=\frac{N^{k/2+1}}{\pi^{k+1}}\sum_{l_1\geq 1}\sum_{l_2\geq 1}\frac{J_{k+2}\left(2\pi\left(l_1^2+l_2^2\right)^{1/2}N^{1/2}\right)}{\left(l_1^2+l_2^2\right)^{k/2+1}}.$$

Evaluation of $\mathbf{I_{2,2}}$

We have to calculate

$$I_{2,2}:=\frac{\pi}{2\pi i}\int_{(1/N)}e^{Nz}z^{-k-2}\sum_{\rho}z^{-\rho}\Gamma(\rho)\left(\sum_{l_1\geq 1}e^{-l_1^2\pi^2/z}\right)\left(\sum_{l_2\geq 1}e^{-l_2^2\pi^2/z}\right)dz$$

and again we have to prove that is possible to exchange the integral with the series. So let us consider

$$A_3:=\sum_{l_1\geq 1}\int_{(1/N)}\left|e^{Nz}\right|\left|z^{-k-2}\right|\left|\sum_{\rho}z^{-\rho}\Gamma(\rho)\right|e^{-l_1^2\pi^2\mathrm{Re}(1/z)}\left|\omega_2\left(\frac{\pi^2}{z}\right)\right||dz|,$$

say. Now using (2.9) and (2.7) we have

$$A_3\ll N^{1/2}\sum_{l_1\geq 1}\int_0^{1/N}\frac{e^{-l_1^2N}}{|z|^{k+2}}dy+N^{3/2}\sum_{l_1\geq 1}\int_{1/N}^{\infty}\frac{ye^{-l_1^2/\left(Ny^2\right)}}{|z|^{k+2}}dy+N^{1/2}\sum_{l_1\geq 1}\int_{1/N}^{\infty}y\log^2(2Ny)\frac{e^{-l_1^2/\left(Ny^2\right)}}{|z|^{k+3/2}}dy$$

$$=W_1+W_2+W_3,$$

say. For W_1 and W_2 we can easily see that

$$W_1\ll N^{k+3/2}\omega_2(N)\ll N^{k+1}$$

and, taking $u=l_1^2/\left(Ny^2\right)$, we obtain

$$W_2\ll N^{3/2}\sum_{l_1\geq 1}\int_{1/N}^{\infty}\frac{e^{-l_1^2/\left(Ny^2\right)}}{y^{k+1}}dy$$

$$\ll N^{k/2+3/2}\sum_{l_1\geq 1}\frac{1}{l_1^k}\int_0^{l_1^2N}e^{-u}u^{k/2-1}du\ll_k N^{k/2+3/2}$$

assuming $k>1$. We have now to check W_3. Taking again $u=l_1^2/\left(Ny^2\right)$ we have, assuming $k>3/2$, that

$$\begin{aligned} W_3 \ll & N^{k/2-1/4}\sum_{l_1\geq 1}\frac{1}{l_1^{k-1/2}}\int_0^{l_1^2 N}\log^2\left(\frac{4Nl_1^2}{u}\right)e^{-u}u^{k/2-5/4}du \\ \ll & N^{k/2-1/4}\sum_{l_1\geq 1}\frac{1}{l_1^{k-1/2}} \ll_k N^{k/2}. \end{aligned}$$

Let us consider

$$A_4 := \sum_{l_1\geq 1}\sum_{l_2\geq 2}\int_{(1/N)}\left|e^{Nz}\right|\left|z^{-k-2}\right|\left|\sum_{\rho} z^{-\rho}\Gamma\left(\rho\right)\right| e^{-l_1^2\pi^2\operatorname{Re}(1/z)}e^{-l_2^2\pi^2\operatorname{Re}(1/z)}\left|dz\right|,$$

say. By (2.9) we get

$$\begin{aligned} A_4 \ll & N\sum_{l_1\geq 1}\sum_{l_2\geq 2}\int_0^{1/N}\frac{e^{-\left(l_1^2+l_2^2\right)N}}{|z|^{k+2}}dy+\sum_{l_1\geq 1}\sum_{l_2\geq 2}\int_{1/N}^{\infty}\frac{e^{-\left(l_1^2+l_2^2\right)/\left(Ny^2\right)}}{|z|^{k+2}}dy \\ & +\sum_{l_1\geq 1}\sum_{l_2\geq 1}\int_{1/N}^{\infty}\log^2\left(2Ny\right)\frac{e^{-\left(l_1^2+l_2^2\right)/\left(Ny^2\right)}}{|z|^{k+3/2}}dy \\ = & R_1+R_2+R_3, \end{aligned}$$

say. So we have immediately

$$R_1 \ll N^{k+2}\omega^2\left(N\right) \ll N^{k+1}$$

and, if we take $u=\left(l_1^2+l_2^2\right)/\left(Ny^2\right)$, we obtain

$$R_2 \ll \sum_{l_1\geq 1}\sum_{l_2\geq 1}\int_{1/N}^{\infty}\frac{e^{-\left(l_1^2+l_2^2\right)/\left(Ny^2\right)}}{y^{k+2}}dy \ll_k N^{(k+1)/2}$$

for $k>1$. So it remains to evaluate R_3. Again we take $u=\left(l_1^2+l_2^2\right)/\left(Ny^2\right)$ and we have

$$\begin{aligned} R_3 \ll & N^{k/2+1/4}\sum_{l_1\geq 1}\sum_{l_2\geq 1}\frac{\log^2\left(4N\left(l_1^2+l_2^2\right)\right)}{\left(l_1^2+l_2^2\right)^{k/2+1/4}}\int_0^{\left(l_1^2+l_2^2\right)^{1/2}N}e^{-u}u^{k/2-3/4}du \\ & -N^{k/2+1/4}\sum_{l_1\geq 1}\sum_{l_2\geq 1}\frac{1}{\left(l_1^2+l_2^2\right)^{k/2+1/4}}\int_0^{\left(l_1^2+l_2^2\right)^{1/2}N}\log^2\left(u\right)e^{-u}u^{k/2-3/4}du \end{aligned}$$

and the convergence follows if $k>3/2$. Note that the estimation of R_3 is optimal. For proving it, take $c=\left(l_1^2+l_2^2\right)/N$, assume $k\leq 3/2$ and $y>1$. We have

$$S := \sum_{l_1\geq 1}\sum_{l_2\geq 1}\int_{1/N}^{\infty}\log^2\left(2Ny\right)\frac{e^{-c/y^2}}{y^{k+3/2}}dy \geq \sum_{l_1\geq 1}\sum_{l_2\geq 1}\int_1^{\infty}\log^2\left(2Ny\right)\frac{e^{-c/y^2}}{y^{k+3/2}}dy.$$

Now, since $y \geq 1$ we have $\log^2(2Ny) \geq \log^2(2N)$ and since $k \leq 3/2$, we have

$$S \geq \log(2N) \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \int_1^\infty \frac{e^{-c/y^2}}{y^{k+3/2}} dy \geq \log(2N) \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \int_1^\infty \frac{e^{-c/y^2}}{y^3} dy$$

$$= \log(2N) \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \frac{1}{2c} \left(1 - e^{-c}\right) \geq \frac{N \log(2N) \left(1 - e^{-2/N}\right)}{2} \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \frac{1}{l_1^2 + l_2^2}.$$

The last double series diverges since

$$\sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \frac{1}{l_1^2 + l_2^2} \geq \sum_{l_1 \geq 1} \sum_{1 \leq l_2 \leq l_1} \frac{1}{l_1^2 + l_2^2} \geq \frac{1}{2} \sum_{l_1 \geq 1} \frac{1}{l_1}.$$

Now we have to estimate

$$A_5 := \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \sum_{\rho} |\Gamma(\rho)| \int_{(1/N)} \left|e^{Nz}\right| \left|z^{-k-2}\right| \left|z^{-\rho}\right| e^{-l_1^2 \pi^2 \mathrm{Re}(1/z)} e^{-l_2^2 \pi^2 \mathrm{Re}(1/z)} |dz|,$$

say. Using (2.10) and (2.11) we have

$$A_5 \ll \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \sum_{\rho, \gamma > 0} e^{-\pi\gamma/2} \gamma^{\beta - 1/2} \int_{1/N}^\infty |z|^{-k-2} |z|^{-\beta} \exp(\gamma \arctan(Ny)) e^{-l_1^2 \pi^2 \mathrm{Re}(1/z)} e^{-l_2^2 \pi^2 \mathrm{Re}(1/z)} |dz|.$$

Let $Q_k = \sup_\beta \left\{\Gamma\left(\frac{k}{2} + \frac{\beta}{2} + \frac{1}{2}\right)\right\}$ and assume $y < 0$. Using the trivial bound $\gamma \arctan(Ny) - \gamma\frac{\pi}{2} \leq -\gamma\frac{\pi}{2}$, we have

$$\begin{aligned} A_5 \ll & N^{k+1} \sum_{l_1 \geq 1} e^{-l_1^2 N} \sum_{l_2 \geq 1} e^{-l_2^2 N} \sum_{\rho, \gamma > 0} N^\beta e^{-\pi\gamma/2} \gamma^{\beta - 1/2} \\ & + N^{(k+1)/2} Q_k \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \frac{1}{(l_1^2 + l_2^2)^{(k+1)/2}} \sum_{\rho, \gamma > 0} N^\beta \frac{e^{-\pi\gamma/2} \gamma^{\beta - 1/2}}{(l_1^2 + l_2^2)^\beta} \ll_k N^k. \end{aligned} \tag{5.2}$$

If $y > 0$ we have

$$\begin{aligned} A_5 \ll & \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \sum_{\rho: \gamma > 0} e^{-\pi\gamma/2} \gamma^{\beta - 1/2} \int_0^{1/N} N^{k+2+\beta} e^{-\left(l_1^2 + l_2^2\right) N} dy \\ & + \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \sum_{\rho: \gamma > 0} \gamma^{\beta - 1/2} \int_{1/N}^\infty \exp\left(\gamma \left(\arctan(Ny) - \frac{\pi}{2}\right)\right) \frac{e^{-\left(l_1^2 + l_2^2\right)/\left(Ny^2\right)}}{y^{k+2+\beta}} dy \end{aligned}$$

and by a well-known trigonometric identity follows that

$$\begin{aligned} A_5 \ll & N^{k+1} + \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \sum_{\rho: \gamma > 0} \gamma^{\beta - 1/2} \int_{1/N}^\infty \exp\left(-\gamma \arctan\left(\frac{1}{Ny}\right)\right) \frac{e^{-\left(l_1^2 + l_2^2\right)/\left(Ny^2\right)}}{y^{k+2+\beta}} dy \\ \ll & N^{k+1} + \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \sum_{\rho: \gamma > 0} \gamma^{\beta - 1/2} \int_{1/N}^\infty \exp\left(-\frac{\gamma}{Ny} - \frac{l_1^2 + l_2^2}{Ny^2}\right) y^{-k-2-\beta} dy \end{aligned}$$

and if we put $\frac{\gamma}{Ny}=v$ we get

$$A_5 \ll N^{k+1}+\sum_{l_1\geq1}\sum_{l_2\geq1}\sum_{\rho:\gamma>0}\gamma^{\beta-1/2}\int_0^{\gamma}e^{-v}e^{-\left(Nv^2\left(l_1^2+l_2^2\right)/\gamma^2\right)}\left(\frac{\gamma}{Nv}\right)^{-k-2-\beta}\frac{\gamma}{Nv^2}dv$$

$$\ll N^{k+1}+\sum_{l_1\geq1}\sum_{l_2\geq1}\sum_{\rho:\gamma>0}\gamma^{-k-3/2}\int_0^{\infty}e^{-v}e^{-\left(Nv^2\left(l_1^2+l_2^2\right)/\gamma^2\right)}v^{k+\beta}dv. \quad (5.3)$$

Now we can observe that we are in the situation of Lemma 2.4 with $d=2$ and so we can conclude immediately that we have the convergence for $k>3/2$ and this result is optimal.

We studied the convergence, so we finally have, using again the identity (1.1), that

$$I_{2,2}=\pi^{-k}N^{k/2+1/2}\sum_{\rho}\frac{\Gamma(\rho)}{\pi^{\rho}}N^{\rho/2}\sum_{l_1\geq1}\sum_{l_2\geq1}\frac{J_{k+1+\rho}\left(2\pi\left(l_1^2+l_2^2\right)^{1/2}N^{1/2}\right)}{\left(l_1^2+l_2^2\right)^{(k+1+\rho)/2}}.$$

6 Evaluation of I_3

We have

$$\begin{aligned} I_3 =&\frac{1}{2\pi i}\int_{(1/N)}e^{Nz}z^{-k-1}\left(\frac{\pi^{1/2}}{z^{3/2}}-\left(\frac{\pi}{z}\right)^{1/2}\sum_{\rho}z^{-\rho}\Gamma(\rho)-\frac{1}{z}+\sum_{\rho}z^{-\rho}\Gamma(\rho)\right)\left(\left(\frac{\pi}{z}\right)^{1/2}\omega_2\left(\frac{\pi^2}{z}\right)\right)dz\\ =&\frac{1}{2i}\int_{(1/N)}e^{Nz}z^{-k-3}\omega_2\left(\frac{\pi^2}{z}\right)dz-\frac{1}{2i}\int_{(1/N)}e^{Nz}z^{-k-2}\sum_{\rho}z^{-\rho}\Gamma(\rho)\,\omega_2\left(\frac{\pi^2}{z}\right)dz\\ &-\frac{1}{2\pi^{1/2}i}\int_{(1/N)}e^{Nz}z^{-k-5/2}\omega_2\left(\frac{\pi^2}{z}\right)+\frac{1}{2\pi^{1/2}i}\int_{(1/N)}e^{Nz}z^{-k-3/2}\sum_{\rho}z^{-\rho}\Gamma(\rho)\,\omega_2\left(\frac{\pi^2}{z}\right)dz\\ =&I_{3,1}-I_{3,2}-I_{3,3}+I_{3,4}. \end{aligned}$$

Evaluation of $\mathbf{I_{3,1}}$

We have

$$I_{3,1}:=\frac{1}{2i}\int_{(1/N)}e^{Nz}z^{-k-3}\omega_2\left(\frac{\pi^2}{z}\right)dz=\frac{1}{2i}\int_{(1/N)}e^{Nz}z^{-k-3}\sum_{m\geq1}e^{-m^2\pi^2/z}dz$$

hence we have to establish the convergence of

$$A_6:=\sum_{m\geq1}\int_{(1/N)}\left|e^{Nz}\right|\left|z\right|^{-k-3}e^{-m^2\mathrm{Re}(1/z)}\left|dz\right|,$$

say. Using (2.7), (2.13) and (5.1) we have

$$A_6 \ll N^{k+3/2} + \sum_{m\geq 1} \int_0^{\infty} y^{-k-3} e^{-m^2/\left(Ny^2\right)} dy \ll_k N^{k+3/2} \tag{6.1}$$

for $k > -1$. So we obtain, recalling (1.1), that

$$J_{3,1} = \frac{N^{k/2+1}}{\pi^{k+1}} \sum_{m\geq 1} \frac{J_{k+2}\left(2m\pi N^{1/2}\right)}{m^{k+2}}.$$

Evaluation of $\mathbf{I_{3,3}}$

We have

$$I_{3,3} := \frac{1}{2\pi^{1/2} i} \int_{(1/N)} e^{Nz} z^{-k-5/2} \sum_{m\geq 1} e^{-m^2\pi^2/z} dz$$

so we have to establish the convergence of

$$\sum_{m\geq 1} \int_{(1/N)} \left|e^{Nz}\right| |z|^{-k-5/2} e^{-m^2 \mathrm{Re}(1/z)} |dz| .$$

Arguing as for $I_{3,1}$, we have the convergence for $k > -1/2$. Summing up, we obtain

$$I_{3,3} = \frac{N^{k/2+3/4}}{\pi^{k+1}} \sum_{m\geq 1} \frac{J_{k+3/2}\left(2m\pi N^{1/2}\right)}{m^{k+3/2}}.$$

Evaluation of $\mathbf{I_{3,2}}$

We have to establish the convergence of

$$A_7 := \sum_{m\geq 1} \int_{(1/N)} \left|e^{Nz}\right| \left|z^{-k-2}\right| \left|\sum_{\rho} z^{-\rho}\Gamma(\rho)\right| \left|e^{-m^2\pi^2/z}\right| |dz| ,$$

say. Using (2.7), (2.13), (5.1) and (2.9) we get

$$\begin{aligned} A_7 \ll & N^{k+1/2} + N \sum_{m\geq 1} \int_{1/N}^{\infty} y^{-k-2} e^{-m^2/\left(Ny^2\right)} dy \\ & + \log^2(2N) \sum_{m\geq 1} \int_{1/N}^{\infty} y^{-k-3/2} e^{-m^2/\left(Ny^2\right)} dy \\ & + \sum_{m\geq 1} \int_{1/N}^{\infty} \log^2(y)\, y^{-k-3/2} e^{-m^2/\left(Ny^2\right)} dy. \end{aligned}$$

Now if we put $m^2/\left(Ny^2\right)=u$ we have

$$N\sum_{m\geq1}\int_{1/N}^{\infty}y^{-k-2}e^{-m^2/\left(Ny^2\right)}dy \ll N^{k/2+3/2}\Gamma\left(\frac{k+1}{2}\right)\sum_{m\geq1}m^{-k-1}$$

which converges if $k>0$. With the same substitution we get

$$\log^2\left(2N\right)\sum_{m\geq1}\int_{1/N}^{\infty}y^{-k-3/2}e^{-m^2/\left(Ny^2\right)}dy \ll \log^2\left(2N\right)N^{k/2+1/4}\Gamma\left(\frac{k}{2}+\frac{1}{4}\right)\sum_{m\geq1}m^{-k-1/2}$$

which converges for $k>1/2$. For the estimation of the last integral in the bound of A_7 we observe that if we take $\epsilon>0$ we have

$$\sum_{m\geq1}\int_{1/N}^{\infty}\log^2\left(y\right)y^{-k-3/2}e^{-m^2/\left(Ny^2\right)}dy \ll \sum_{m\geq1}\int_{1/N}^{\infty}y^{-k-3/2+\epsilon}e^{-m^2/\left(Ny^2\right)}dy$$

and so, arguing analogously as we did for (6.1), we get

$$\ll N^{k/2+1/4-\epsilon/2}\Gamma\left(\frac{k}{2}+\frac{1}{4}-\frac{\epsilon}{2}\right)\sum_{m\geq1}m^{-k-1/2+\epsilon}$$

and for the arbitrariness of ϵ we have the convergence for $k>1/2$. We have now to study

$$A_8:=\sum_{m\geq1}\sum_{\rho}\left|\Gamma\left(\rho\right)\right|\int_{(1/N)}\left|e^{Nz}\right|\left|z^{-k-2}\right|\left|z^{-\rho}\right|\left|e^{-m^2\pi^2/z}\right|\left|dz\right|,$$

say. By symmetry we may assume that $\gamma>0$. If $y\leq0$ we have $\gamma\arctan\left(y/a\right)-\frac{\pi}{2}\gamma\leq-\frac{\pi}{2}\gamma$ and so using (2.10) and (2.11) we get

$$A_8 \ll \sum_{m\geq1}\sum_{\gamma>0}\gamma^{\beta-1/2}\exp\left(-\frac{\pi}{2}\gamma\right)\left(\int_{-1/N}^{0}N^{k+2+\beta}e^{-m^2N}dy+\int_{-\infty}^{-1/N}\frac{e^{-m^2/\left(Ny^2\right)}}{\left|y\right|^{k+2+\beta}}dy\right)$$

$$\ll_k N^{k+3/2}+N^{k/2+1/2}Q_k\sum_{m\geq1}\frac{1}{m^{k+1}}\sum_{\gamma>0}N^{\beta/2}\frac{\gamma^{\beta-1/2}}{m^{\beta}}\exp\left(-\frac{\pi}{2}\gamma\right) \ll_k N^{k+3/2}$$

provided that $k>0$ and $Q_k=\sup_{\beta}\left\{\Gamma\left(\frac{k}{2}+\frac{1}{2}+\frac{\beta}{2}\right)\right\}$. Let $y>0$. We have

$$\begin{aligned}A_8 \ll &\sum_{m\geq1}\sum_{\gamma>0}\gamma^{\beta-1/2}\exp\left(-\frac{\pi}{4}\gamma\right)\int_{0}^{1/N}N^{k+2+\beta}e^{-m^2N}dy\\ &+\sum_{m\geq1}\sum_{\gamma>0}\gamma^{\beta-1/2}\int_{1/N}^{\infty}\exp\left(\gamma\arctan\left(Ny\right)-\frac{\pi}{2}\gamma\right)\frac{e^{-m^2/\left(Ny^2\right)}}{y^{k+2+\beta}}dy\\ &=L_1+L_2,\end{aligned}$$

say. From (2.7) and (2.13) we have

$$L_1 \ll N^{k+1} \sum_{m\geq 1} e^{-m^2N} \sum_{\gamma>0} N^{\beta}\gamma^{\beta-1/2} \exp\left(-\frac{\pi}{4}\gamma\right) \ll_k N^{k+3/2}$$

and again by a well-known trigonometric identity and taking $v = m/\left(N^{1/2}y\right)$ we have

$$L_2 \ll \sum_{m\geq 1}\sum_{\gamma>0} \gamma^{\beta-1/2} \int_{1/N}^{\infty} \exp\left(-\frac{\gamma}{Ny} - \frac{m^2}{Ny^2}\right) \frac{dy}{y^{k+2+\beta}}$$
$$= N^{(k+1)/2} \sum_{m\geq 1} \frac{1}{m^{k+1}} \sum_{\gamma>0} \frac{N^{\beta/2}}{m^{\beta}} \gamma^{\beta-1/2} \int_0^{m\sqrt{N}} \exp\left(-\frac{\gamma v}{N^{1/2}m} - v^2\right) v^{k+\beta} dv.$$

Using $e^{-v^2}v^k = O_k(1)$ if $k > 0$, we have, taking $s = \gamma v/\left(N^{1/2}m\right)$, that

$$\ll N^{k/2+1} \sum_{m\geq 1} \frac{1}{m^k} \sum_{\gamma>0} N^{\beta}\gamma^{-3/2} \int_0^{\infty} \exp(-s)\, s^{\beta} ds \ll_k N^{k/2+2}$$

for $k > 1$. Now we can exchange the series with the integral and so we have

$$I_{3,2} = \pi^{-k} N^{(k+1)/2} \sum_{\rho} \pi^{-\rho} N^{\rho/2} \Gamma(\rho) \sum_{m\geq 1} \frac{J_{k+1+\rho}\left(2m\pi\sqrt{N}\right)}{m^{k+1+\rho}}.$$

Evaluation of $\mathbf{I_{3,4}}$

We have to establish the convergence of

$$I_{3,4} := \frac{1}{2\pi^{1/2}i} \int_{(1/N)} e^{Nz} z^{-k-3/2} \sum_{\rho} z^{-\rho} \Gamma(\rho)\, \omega_2\left(\frac{\pi^2}{z}\right) dz.$$

Arguing analogously as we did for estimating $I_{3,2}$ we obtain the condition $k > 1$. We can exchange the series with the integral and obtain

$$I_{3,4} = \pi^{-k} N^{k/2+1/4} \sum_{\rho} \pi^{-\rho} N^{\rho} \Gamma(\rho) \sum_{m\geq 1} \frac{J_{k+1/2+\rho}\left(2m\pi\sqrt{N}\right)}{m^{k+1/2+\rho}}.$$

Defining

$$M_1(N,k)=\frac{\pi N^{k+2}}{4\Gamma(k+3)}+\frac{N^{k+1}}{4\Gamma(k+2)}-\frac{\pi^{1/2}N^{k+3/2}}{2\Gamma(k+5/2)}, \tag{6.2}$$

$$M_2(N,k)=-\frac{\pi}{4}\sum_{\rho}\frac{\Gamma(\rho)}{\Gamma(k+2+\rho)}N^{k+1+\rho}-\frac{1}{4}\sum_{\rho}\frac{\Gamma(\rho)}{\Gamma(k+1+\rho)}N^{k+\rho}$$
$$+\frac{\pi^{1/2}}{2}\sum_{\rho}\frac{\Gamma(\rho)}{\Gamma(k+3/2+\rho)}N^{k+1/2+\rho}, \tag{6.3}$$

$$M_3(N,k)=\frac{N^{k/2+1}}{\pi^{k+1}}\sum_{l_1\geq1}\sum_{l_2\geq1}\frac{J_{k+2}\left(2\pi\left(l_1^2+l_2^2\right)^{1/2}N^{1/2}\right)}{\left(l_1^2+l_2^2\right)^{k/2+1}}$$
$$-\pi^{-k}N^{k/2+1/2}\sum_{\rho}\frac{\Gamma(\rho)}{\pi^{\rho}}N^{\rho/2}\sum_{l_1\geq1}\sum_{l_2\geq1}\frac{J_{k+1+\rho}\left(2\pi\left(l_1^2+l_2^2\right)^{1/2}N^{1/2}\right)}{\left(l_1^2+l_2^2\right)^{(k+1+\rho)/2}}, \tag{6.4}$$

$$M_4(N,k)=\frac{N^{k/2+1}}{\pi^{k+1}}\sum_{m\geq1}\frac{J_{k+2}\left(2m\pi N^{1/2}\right)}{m^{k+2}}-\frac{N^{k/2+3/4}}{\pi^{k+1}}\sum_{m\geq1}\frac{J_{k+3/2}\left(2m\pi N^{1/2}\right)}{m^{k+3/2}}$$
$$-\pi^{-k}N^{(k+1)/2}\sum_{\rho}\pi^{-\rho}N^{\rho/2}\Gamma(\rho)\sum_{m\geq1}\frac{J_{k+1+\rho}\left(2m\pi\sqrt{N}\right)}{m^{k+1+\rho}}$$
$$+\pi^{-k}N^{k/2+1/4}\sum_{\rho}\pi^{-\rho}N^{\rho/2}\Gamma(\rho)\sum_{m\geq1}\frac{J_{k+1/2+\rho}\left(2m\pi\sqrt{N}\right)}{m^{k+1/2+\rho}}, \tag{6.5}$$

we have proved the following

Main Theorem 6.1. *Let N be a sufficient large integer. We have*

$$\sum_{n\leq N}r_Q(n)\frac{(N-n)^k}{\Gamma(k+1)}=M_1(N,k)+M_2(N,k)+M_3(N,k)+M_4(N,k)+O\left(N^{k+1}\right)$$

for $k>3/2$, where ρ runs over the non-trivial zeros of the Riemann zeta function $\zeta(s)$ and $J_v(u)$ is the Bessel function of complex order v and real argument u. Furthermore the bound $k>3/2$ is optimal using this technique.

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