TANGENTIAL WEAK DEFECTIVENESS AND GENERIC IDENTIFIABILITY

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Abstract. We investigate the uniqueness of decomposition of general tensors $T \in \mathbb{C}^{n_1+1} \otimes \cdots \otimes \mathbb{C}^{n_r+1}$ as a sum of tensors of rank 1. This is done extending the theory developed in [Me06] to the framework of non twd varieties. In this way we are able to prove the non generic identifiability of infinitely many partially symmetric tensors.

1. Introduction

The decomposition of tensors $T \in \mathbb{C}^{n_1+1} \otimes \cdots \otimes \mathbb{C}^{n_r+1}$ as a sum of *simple* tensors (i.e. tensors of rank 1) is a central problem for many applications from Multilinar Algebra to Algebraic Statistics, coding theory, blind signal separation and others, [DDL1],[DDL2],[DDL3],[KADL],[Si].

For statistical inference, it is meaningful to know if a probability distribution, arising from a model, uniquely determines the parameters that produced it. When this happens, the parameters are called *identifiable.* There are no useful models where all distributions are identifiable. Then the notion of *generic* identifiability for parametric models has been considered for instance in [AMR09] and in [SR12]. Conditions which guarantee the uniqueness of decomposition, for general tensors in the model, are quite important in the applications. When generic identifiability holds, the set of non-identifiable parameters has measure zero, thus parameter inference is still meaningful. Notice that many decomposition algorithms converge to *one* decomposition, hence a uniqueness result guarantees that the decomposition found is the chased one. We refer to [KB09] and its huge reference list, for more details.

From a purely theoretical point of view, the study of unique decompositions, or canonical forms in the early $\mathrm{XX}^{\mathrm{th}}$ century dictionary, has connection with both invariant theory, [Hi], and projective geometry, [Pa] [Ri]. It is already over a decade, [Me06], that generic identifiability of symmetric tensors has shown its close connection to modern birational projective geometry and especially to the maximal singularities methods. In a series of papers, [Me06] [Me09] [GM], the generic identifiability problem for symmetric tensors has been completely solved.

The present paper is devoted to extend this theory to arbitrary tensors and can be considered as a first step, similar to [Me06], in this direction. As for the symmetric case it is expected that identifiability is very rare and our results support this idea.

The main tool in [Me06] was the use, after [CC02], of non weakly defective varieties to study identifiability, see Section 2 for all the relevant definitions. Unfortunately it is very hard to determine the weak defectiveness of general tensors.

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This difficulty prevented, for many years, a straightforward application of the same techniques to them, see [Fo] for a similar approach in special cases.

In recent years the notion of tangential weak defectiveness, introduced in [CO], has gradually substituted the weak defectiveness and proved valid to study generic identifiability of subgeneric tensors, [CO] [BDdG] [BC] [BCO] [Kr] [CM]. In particular thanks to the main result in [CM] for the generic identifiability we may assume without loss of generality the non tangential weak defectiveness under mild numerical assumptions.

tangential weak defectiveness does not behave as weak defectiveness defectiveness with respect to the maximal singularities method. Therefore in this paper we develop tools to plug in maximal singularities methods for non tangentially weakly defective varieties. In this way we are able to prove the non identifiability of many partially symmetric tensors. The main technical result is a study of the nested singularities of tangential linear system for non tangentially weakly defective varieties and with this we are able to prove the following statement on identifiability of partially symmetric tensors.

Theorem 1. *The general tensor $T \in \mathrm{Sym}^{d_i}(\mathbb{C}^{n_1+1}) \otimes \cdots \otimes \mathrm{Sym}^{d_r}(\mathbb{C}^{n_r+1})$ is not identifiable when $d_i > n_i + 1$ for any $1 \leq i \leq r$ and $\lceil \frac{\prod \binom{n_i+d_i}{n_i}}{\sum n_i+1} \rceil > 2(\sum n_i)$.*

The paper is organized as follows. After recalling notation and definitions we study in detail the singular loci of tangential linear systems for non tangentially weakly defective varieties. The main technical result is Theorem 24 where we prove that, under suitable hypothesis, these linear system have not nested singularities. This result allow us to apply the standard Noether–Fano inequalities to show that some tangential projections are not birational, see Theorem 28. With this the non identifiability result is at hand following [Me06].

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2. Notation

We work over the complex field. Let $X \subset \mathbb{P}^N$ be an irreducible and reduced non-degenerate variety with $\dim X = n$ and $X^{(h)}$ be the h-th symmetric product of X. That is the variety parameterizing unordered sets of h points of X. Let $U_h^X \subset X^{(h)}$ be the subset of the smooth locus of $X^{(h)}$ parameterizing sets of h distinct smooth points.

Definition 2. A point $z \in U_h^X$ represents a set of h distinct points, say $\{z_1, \ldots, z_h\}$. We say that *a point $p \in \mathbb{P}^N$ is in the span of z, $p \in \langle z \rangle$*, if it is a linear combination of the z_i.

Definition 3. The *abstract h-Secant variety* is the irreducible and reduced variety

$$sec_h(X) := \overline{\{(z,p) \in U_h^X \times \mathbb{P}^N | p \in \langle z \rangle\}} \subset X^{(h)} \times \mathbb{P}^N.$$

Let $\pi : X^{(h)} \times \mathbb{P}^N \to \mathbb{P}^N$ be the projection onto the second factor. The *h-Secant variety* is

$$\mathbb{S}\mathrm{ec}_h(X) := \pi(sec_h(X)) \subset \mathbb{P}^N,$$

and $\pi_h^X := \pi_{|sec_h(X)} : sec_h(X) \to \mathbb{P}^N$ is the h-secant map of X.

The irreducible variety $sec_h(X)$ has dimension $hn + h - 1$. One says that X is *h-defective* if

$$\dim \mathbb{S}\mathrm{ec}_h(X) < \min\{\dim sec_h(X), N\}.$$

For simplicity we will say that X is not defective if it is not h-defective for any h.

Definition 4. Let $X \subset \mathbb{P}^N$ be a non-degenerate subvariety. We say that a point $p \in \mathbb{P}^N$ *has rank h with respect to* X if $p \in \langle z \rangle$, for some $z \in U_h^X$ and $p \notin \langle z' \rangle$ for any $z' \in U_{h'}^X$, with $h' < h$.

We call $g := g(X)$ the general rank of X (the rank of the general point of $\mathbb{P}^N$ with respect to X) and we say that $X \subset \mathbb{P}^N$ is perfect if

$$\frac{N+1}{\dim X + 1} \in \mathbb{N}$$

Note that $g(\dim X + 1) = N + 1$ if π_g^X is generically finite (or equivalently, if X is perfect and not defective).

Definition 5. A point $p \in \mathbb{P}^N$ is *h-identifiable with respect to* $X \subset \mathbb{P}^N$ if p is of rank h and $(\pi_h^X)^{-1}(p)$ is a single point. The variety X is said to be *h-identifiable* if π_h^X is a birational map, that is the general point of $\mathbb{S}\mathrm{ec}_h(X)$ is h-identifiable. For simplicity we will say that $X \subset \mathbb{P}^N$ *is generically identifiable* if the general point of $\mathbb{P}^N$ is g-identifiable.

Remark 6. Note that π_g^X is generically finite if and only if X is perfect and not defective. These are therefore necessary conditions for generic identifiability.

Definition 7. Let $X \subset \mathbb{P}^N$ be a non-degenerate variety and $\{x_1, \ldots, x_h\} \subset X$ a subset of h general points. The variety X is said *h-weakly defective* if the general hyperplane singular along h general points is singular along a positive dimensional subvariety passing through the points. Let $H \in \mathcal{H}(h) := |\mathcal{I}(1)_{x_1^2, \ldots, x_h^2}|$ be a general section, we call $\Gamma_h(H)$ its locus of tangency passing through $x_1, \ldots, x_h$, (contact locus or contact variety in [CC02]) that is the union of all the irreducible components of $Sing(H)$ passing through the points $x_1, \ldots, x_h$.

Definition 8. For a linear system $\mathcal{H}$ we set

$$\Gamma(\mathcal{H}) := \bigcap_{H \in \mathcal{H}} Sing(H)$$

the common singular locus.

Remark 9. We want to stress that, by [CC02], if $\Gamma_h(H)$ is zero dimensional in a neighborhood of $\{x_1, \ldots, x_h\}$ then $\Gamma_h(H) = \{x_1, \ldots, x_h\}$.

The notion of tangentially weakly defective varieties has been introduced in [CO]. Here we follow the notations of [BBC].

For a subset $A = \{x_1, \ldots, x_h\} \subset X$ of general points we set

$$M_A := \langle \bigcup_i \mathbb{T}_{x_i} X \rangle.$$

By Terracini Lemma (see [Ter11]) the space M_A is the tangent space to $\mathbb{S}\mathrm{ec}_h(X)$ at a general point in $\langle A \rangle$.

Definition 10. The *tangential h-contact locus* $\Gamma_h(A)$ is the closure in X of the union of all the irreducible components which contain at least one point of A, of the locus of points of X where M_A is tangent to X. We will write $\gamma_h := \dim \Gamma_h(A)$. We say that X is *h-twd (h-tangentially weakly defective)* if $\gamma_h > 0$.

Remark 11. Note that in general it is difficult to predict the behavior of $\Gamma(\mathcal{H}(h))$ for non h-twd varieties. By definition $\Gamma(\mathcal{H}(h))$ is zero dimensional in a neighborhood of the assigned singular points but not much is known about singular components away from these. Our Proposition 22 is a first attempt to study this problem, under strong hypothesis.

For what follows it is useful to introduce also the notion of tangential projection.

Definition 12. Let $X \subset \mathbb{P}^N$ be a variety and $A = \{x_1, \ldots, x_h\} \subset X$ a set of general points. The *h-tangential projection from A of X* is

$$\tau_h : X \dashrightarrow \mathbb{P}^M$$

the linear projection from M_A. That is, by Terracini Lemma, the projection from the tangent space of a general point $z \in \langle A \rangle$ of $\mathbb{S}\mathrm{ec}_h(X)$ restricted to X.

Remark 13. By Terracini's Lemma τ_h is the rational map associated to the linear system $\mathcal{H}(h) = |\mathcal{I}_{x_1^2 \ldots, x_h^2}(1)|$.

Remark 14. If X is $h-$defective the tangent space to a general point in $\mathbb{S}ec_h(X)$ is tangent along a positive dimensional variety through the h points, therefore X is $h-$twd. If X is $h-$twd the general hyperplane section tangent along h points has a positive dimensional singularity and therefore X is $h-$weakly defective. Note that the opposite implications are in general not true see for instance [BBC, Example 4.10] for the first and some scrolls for the latter.

3. Properties of contact loci for non twd varieties

In this section we study properties of the contact loci $\Gamma_{g-1}(H)$ (for a general $H \in \mathcal{H}(g-1)$) of projective varieties that are non defective and not $(g-1)$-twd. In particular in view of the applications to Noether–Fano inequalities we have in mind to study the infinitesimally near singularities of $\mathcal{H}$.

We start recalling [CC02, Proposition 3.6] and its generalization to twd. This Proposition will be useful to reduce the study of $\Gamma_{g-1}(\mathcal{H})$ to the special case of $g = 2$.

Proposition 15. *Let $X \subset \mathbb{P}^N$ be an irreducible and reduced non degenerate variety. Assume that X is not h-defective and $h(\dim X + 1) - 1 < N$. Let $X_s = \tau_s(X)$ be a general tangential projection.*

i) *X is $h-$weakly defective if and only if X_s is $(h-s)-$weakly defective.*
ii) *X is $h-$twd if and only if X_s is $(h-s)-$twd.*

Proof. Point i) is [CC02, Proposition 3.6].

ii) Here we mimic [CC02, Proposition 3.6] switching weakly defectiveness with twd. The assertion is trivially true for $h = 0$. We can use induction on h and so, in order to finish the proof, it is sufficient to check the case $s = 1$. By [CM, Lemma 16] the tangential projection $\tau_1 : X \dashrightarrow X_1$ is generically finite. Therefore also the restriction $\tau_{1|_H}$ is generically finite with H a generic hyperplane section in $\mathcal{H}(h)$. This finally implies that $\tau_{1|_{\Gamma_h(\mathcal{H})}}$ is generically finite yielding $\gamma_h(X) = \gamma_{h-1}(X_1)$. □

For future reference we observe the following fact.

Lemma 16. *Let $Z \subset \mathbb{P}^n$ be a reduced projective variety of dimension $\dim(Z) = a$. Then $\operatorname{codim} |\mathcal{I}_Z(1)| \geq a + 1$ and equality is fulfilled only by linear spaces.*

Proof. If Z is a linear space there is nothing to prove. Assume that Z is not a linear space, then $\dim\langle Z \rangle > \dim Z$. We have

$$\operatorname{codim} |\mathcal{I}_Z(1)| = \operatorname{codim} |\mathcal{I}_{\langle Z \rangle}(1)| = \dim\langle Z \rangle + 1 > \dim Z + 1.$$

□

Definition 17. Let $X \subset \mathbb{P}^N$ be an irreducible and reduced non-degenerate and non h-defective variety. Let $\{x_1, \ldots, x_h\} \subset X$ be a set of general points and $\mathcal{H}(h) = |\mathcal{I}_{x_1^2, \ldots, x_h^2}(1)|$ the linear system of hyperplane sections singular in $\{x_1, \ldots, x_h\}$. Set

$$\mathcal{W}_h := \{(H, x) | H \in \mathcal{H}(h), x \in \Gamma_h(H)\} \in \mathcal{H} \times X$$

and $\pi_1^h : \mathcal{W}_h \to \mathcal{H}(h)$, $\pi_2^h : \mathcal{W}_h \to X$ the two canonical projections. We denote with $W_h := \pi_1^h(\mathcal{W}_h) \subset \mathcal{H}(h)$.

It is clear that $W_s \subset |\mathcal{I}_{x_1^2,\dots,x_h^2}(1)|$ for any $h < s$. Then we may identify W_s as a subvariety of W_h for any $h \le s$. Our next aim is to prove, in some cases, a more precise result.

Proposition 18. *Assume that X is perfect and not defective with general rank g. Set $\mathcal{H} := \mathcal{H}(g-2) = |\mathcal{I}_{x_1^2,\dots,x_{g-2}^2}(1)|$ and assume*

$$\dim(\Gamma_{g-1}(H)) = a,$$

for $H \in \mathcal{H}(g-1)$. Then we have $\operatorname{codim}_{\mathcal{H}(g-2)}(W_{g-1}) = a+1$.

Proof. The variety X is not defective, then $\dim(\mathcal{H}(g-2)) = 2n+1$. By a parameter count we have $\dim \mathcal{W}_{g-1} = 2n$.

By definition for a general $[H] \in W_{g-1}$ we have

$$\dim((\pi_1^h)^{-1}(H)) = \dim\{x \in X | x \in \Gamma_{g-1}(H)\}) = \dim \Gamma_{g-1}(H)$$

therefore we conclude that

$$\dim(W_{g-1}) = \dim(\mathcal{W}_{g-1}) - \dim((\pi_1^h)^{-1}(H)) = 2n - a$$

yielding $\operatorname{codim}_{\mathcal{H}(g-2)}(W_{g-1}) = a+1$. □

The following result is already implicitly used in [CC02] but we state it as a Proposition for the reader convenience.

Proposition 19. *Let $X \subset \mathbb{P}^{2\dim X - 1}$ be an irreducible, reduced non-degenerate variety. Assume that X is not defective, then for a general tangent hyperplane $H \in \mathcal{H}(1)$, the tangential locus $\Gamma_1(H)$ is a linear space. In particular, under these hypotheses, also $\Gamma(\mathcal{H}(1))$ is a linear space.*

Proof. If X is not 1-weakly defective, by Remark 9, $\Gamma_1(H)$ is a point. Assume that X is 1-weakly defective and $\dim \Gamma_1(H) = a$. Let $x \in X$ be a general point and $H \in \mathcal{H}(1)$ a general tangent section in x. Let us consider the variety

$$W_1 \subset |\mathcal{O}(1)| =: \mathcal{H}$$

parameterizing singular hyperplane sections. Proposition 18 yields $\operatorname{codim}_{\mathcal{H}}(W_1) = a+1$ and so $\operatorname{codim}(\mathbb{T}_{[H]}W_1) = a+1$. On the other hand, by the infinitesimal Bertini's theorem [CC02, Thm 2.2], we have

$$\mathbb{T}_{[H]}W_1 \subset \mathcal{H}(-Sing(H))$$

and so $\operatorname{codim}_{\mathcal{H}}(\mathcal{H}(-\Gamma_1(H))) \le a+1$.

Hence we conclude by Proposition 16 that $\Gamma_1(H)$ is a linear space. □

Lemma 20. *Let $X \subset \mathbb{P}^N$ be an irreducible, reduced non-degenerate projective variety. Assume that X is $1-$weakly defective with $\dim(\Gamma_1(H)) = a$, for $H \in \mathcal{H}(1)$ a general tangent hyperplane. Then a general hyperplane section X' of X satisfies $\dim(\Gamma_1(H')) = a-1$, for H' a general tangent hyperplane to X'.*

Proof. Let $x \in X$ be a general point, $H \in |\mathcal{I}_{x^2}(1)|$ a general hyperplane section singular at x and $L \in |\mathcal{I}_x(1)|$ a general hyperplane section passing through x. The divisor L is smooth in a neighborhood of x and $\operatorname{Bs}|\mathcal{I}_x(1)| = \{x\}$. Hence, by Bertini's theorem,

$$\dim(Sing(H) \cap L)) = \dim \Gamma_1(H) - 1 = a - 1$$

To conclude observe that $H_{|L}$ is a general tangent section of L at x. □

Let $(z_1,\ldots,z_n)$ be a system of local coordinates at the smooth point $(x\in X)\cong((0,\ldots,0)\in\mathbb{C}^n)$. Every divisor $H\in|\mathcal{I}_{x^2}(1)|$ can be expressed locally as

$$H=\{Q_H(z_1,\ldots,z_n)+\sum_{d\geq 3}F_d(z_1,\ldots,z_n)=0\}$$

where $Q_H(z_1,\ldots,z_n)\in\mathbb{C}[z_1,\ldots,z_n]_2$ is a quadric and F_d is a homogeneous polynomials of degree at least 3. The rank of the double point $x\in H$ is by definition the rank of the quadric Q_H. The singular locus $\mathcal{A}=Sing(Q_H)$ is a linear space $\mathcal{A}\subset\mathbb{C}^n$ of dimension $\dim(\mathcal{A})=\dim(X)-rank(Q_H)$. It is called the asymptotic space of H at the point x. Let $\nu:X'\to X$ be the blow up of X at x with exceptional divisor E. Under the identification $E=\mathbb{P}((T_xX)^*)=\mathbb{P}^{n-1}$ we have that $\nu_*^{-1}(H)\cap E=\mathbb{P}(Q_H)$ and $Sing(\nu_*^{-1}(H))\cap E\subseteq\mathbb{P}(\mathcal{A})$. Note further that to every point $y\in E$ we can associate uniquely a line $l_y\in T_xX$ corresponding to the tangent direction represented by y.

With this notation in mind we are going to improve Proposition 19.

Proposition 21. *Let $X\subset\mathbb{P}^{2\dim X-1}$ be an irreducible, reduced non-degenerate projective variety. If X is not defective, $\mathbb{P}(Sing(Q_H))=\nu_*^{-1}(\Gamma_1(H))\cap E$.*

Proof. Let $H\in\mathcal{H}(1)$ be a general hyperplane section singular at x. If $\dim(\Gamma_1(H))=0$, by [CC02, Theorem 1.4], x is an ordinary double point of H. Thus Q_H is a quadric of maximal rank.

Assume $\dim(\Gamma_1(H))=a>0$. By Proposition 19 it is enough to prove that $rank(Q_H)=\dim X-a$. Let $\nu:X'\to X$ be the blow up of X at the general point $x\in X$, with exceptional divisor E, and $H'=\nu_*^{-1}(H)$ the strict transform of H. We have

$$\nu_*^{-1}(\Gamma_1(H))\cap E\subseteq Sing(H')$$

We already observed that $Sing(H')\cap E\subseteq\mathbb{P}(Sing(Q_H))$ hence

$$\nu_*^{-1}(\Gamma_1(H))\cap E\subseteq\mathbb{P}(Sing(Q_H)).$$

This leads to

$$rank(Q_H)\leq\dim(X)-a.$$

Let $H_1,\ldots,H_a\in\mathcal{H}(1)$ be general sections. Then Lemma 20 yields that $X^a:=H_1\cap\ldots\cap H_a$ is not 1−weakly defective. Hence, by the first part of the proof, we conclude

$$rank(Q_H)\geq\dim(X)-a$$

and finish the proof. □

We take the opportunity to stress a property of $\Gamma(\mathcal{H}(g-1))$ for non twd varieties, recall Remark 11.

Proposition 22. *Let $X\subset\mathbb{P}^N$ be a non defective, perfect, irreducible, reduced and non-degenerate variety with general rank g. Assume that X is not $(g-1)$-twd. Then $\langle\mathbb{T}_{x_1}X,\ldots,\mathbb{T}_{x_{g-1}}X\rangle$ is tangent only along a zero dimensional scheme.*

Proof. Let $W\subset\langle\mathbb{T}_{x_1}X,\ldots,\mathbb{T}_{x_{g-1}}X\rangle\cap X$ be an irreducible component where $\langle\mathbb{T}_{x_1}X,\ldots,\mathbb{T}_{x_{g-1}}X\rangle$ is tangent to X. By Proposition 15 we have that $X_{g-2}:=\tau_{g-2}(X)$ is not 1-twd and not defective, where τ_{g-2} is the linear projection from $\langle\mathbb{T}_{x_2}X,\ldots,\mathbb{T}_{x_{g-1}}X\rangle$.

Claim 1. $\tau_{g-2}(W)=\tau_{g-2}(x_1)$.

Proof. Let $y=\tau_{g-2}(x_1)$ and $H\in|\mathcal{I}_{y^2}(1)|$ a general tangent hyperplane section. By Proposition 15 X_{g-2} is not 1-twd and by Proposition 19 $\Gamma_1(H)$ is a linear space, therefore

$$\Gamma_1(H)\cap\mathbb{T}_yX_{g-2}=y.$$

On the other hand, by construction, we have

$$\tau_{g-2}(W) \subset \Gamma_1(H),$$

and this proves the claim. □

The variety X is not defective and $y = \tau_{g-2}(x_1)$ is a general point of X_{g-2}. Now if X is not $(g-2)-$defective then τ_{g-2} is generically finite, see for instance [CM, Lemma 16]. Therefore $\tau_{g-2}^{-1}(y)$ is a finite scheme and we conclude by the Claim that W is 0-dimensional. □

Remark 23. It would be very interesting to understand if the result in Proposition 22 is true for smaller values of the rank. Unfortunately our proof is based on Proposition 19 and cannot be extended in this direction.

The following is the main result of this section.

Theorem 24. *Let $X \subseteq \mathbb{P}^N$ be a projective irreducible,reduced and non-degenerate variety of general rank g. Let $\{x_1, ..., x_{g-1}\}$ be general points on X and $\mathcal{H} = \mathcal{H}(g-1)$. Assume that:*

- *X is perfect and non defective,*
- *X is not $(g-1)$-twd.*

Then there is a variety Y and a birational map $\nu : Y \to X$ with the following property: for any $\epsilon > 0$ there is a $\mathbb{Q}$-divisor D, with $D \equiv \nu_*^{-1}\mathcal{H}$ such that for any point $y \in Y$

$$\operatorname{mult}_y D < 1 + \epsilon.$$

Proof. The variety X is non defective and not $(g-1)$-twd. Then, by Proposition 22, $\Gamma(\mathcal{H})$ is zero dimensional and, by [CM, Lemma 16, i)], the tangential projection τ_{g-2}, from $\{x_2, \dots, x_{g-1}\}$ is birational. Let $X_{g-2} = \tau_{g-2}(X)$ be the image of the tangential projection and define $\mathcal{H}' := (\tau_{g-2})_*\mathcal{H}$. Then, by Proposition 15 $X_{g-2} \in \mathbb{P}^{2\dim X-1}$ is not defective and not 1-twd.

Let $\sigma : Z \to X_{g-2}$ be the blow up of the point $\tau_{g-2}(x_1)$, with exceptional divisor E and $\mathcal{H}_Z = \sigma_*^{-1}\mathcal{H}'$.

Claim 2. *$\Gamma(\mathcal{H}_Z)$ is empty.*

Proof of the Claim. By Proposition 19 the tangential locus $\Gamma_1(H)$ with respect to $\mathcal{H}'$ is a linear space. The variety X_{g-2} is not 1-twd therefore $\Gamma(\mathcal{H}') = \tau_{g-2}(x_1)$. This is enough to show that $\Gamma(\mathcal{H}_Z) \subset E$.

Assume that there is a point $z \in \Gamma(\mathcal{H}_Z) \cap E$ and denote by $l_z \subset \mathbb{P}^{2\dim X-1}$ the corresponding line in the projective space. By Proposition 21 this forces $l_z \subset \Gamma_1(H)$ and the contradiction $l_z \subset \Gamma(\mathcal{H}')$. □

Let Y_1 be the completion of the Cartesian square

$$\begin{array}{ccc} Y_1 & \overset{\eta}{\dashrightarrow} & Z \\ {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \sigma} \\ X & \underset{\tau_{g-2}}{\dashrightarrow} & X_{g-2} \end{array}$$

and $\mathcal{H}_{Y_1} = (\nu \circ \tau_{g-2})_*^{-1}(\mathcal{H}') = \eta_*^{-1}(\mathcal{H}_Z)$ the strict transform linear system. By construction and Claim 2 we have that $\Gamma(\mathcal{H}_{Y_1})$ is contained in the locus where η is not an isomorphism.

This fact has two consequences. On one hand, since the locus $\Gamma(\mathcal{H})$ is zero dimensional, the general choice of the points $\{x_1, \dots, x_{g-1}\}$ and a monodromy argument obtained via the abstract secant map π_{g-1}^X allow to conclude that

$$\Gamma(\mathcal{H}) = \{x_1, \dots, x_{g-1}\}. \tag{1}$$

On the other hand $\Gamma(\mathcal{H}_{Y_1})$ is disjoint from the exceptional divisor, say E_1, over x_1.

Let $\mu : Y \to Y_1$ be the blow up of Y along $\{x_2, \ldots, x_{g-1}\}$, with exceptional divisors E_i. Then we have

$$\mathcal{H}_Y = \mu_*^{-1}(\mathcal{H}_{Y_1}) = (\mu \circ \nu)_*^{-1}(\mathcal{H}).$$

The Equation (1) forces

$$\Gamma(\mathcal{H}_Y) \subset \cup_{i=1}^{g-1} E_i.$$

We just proved that $\Gamma(\mathcal{H}_{Y_1})$ is disjoint from E_1 therefore the same is true for $\Gamma(\mathcal{H}_Y)$ and, by a monodromy argument, $\Gamma(\mathcal{H}_Y)$ is disjoint from E_i, for $i = 1, \ldots, g-1$. This is enough to conclude that $\Gamma(\mathcal{H}_Y)$ is empty. Then, for any $y \in Y$ there are at most $\dim \mathcal{H}_Y$ linearly independent divisors in $\mathcal{H}$ singular in y. Hence, for $M \gg 0$, and a general choice of $H_i \in \mathcal{H}_Y$ we have, for any y

$$\mathrm{mult}_y \frac{1}{M} \sum_1^M H_i \leq 1 + \frac{\dim \mathcal{H}_Y}{M}.$$

Then the divisor

$$D = \frac{1}{M} \sum_1^M H_i,$$

for $H_i \in \mathcal{H}_Y$ general and $M \gg 0$, allows us to conclude the proof. □

4. Noether–Fano inequalities and generic identifiability

In this section we apply the previous results on the singular locus of linear system $\mathcal{H}(g-1)$ to produce non generic identifiability statements.

First of all let us define the notion of Mori Fiber Space.

Definition 25. Let $X \subset \mathbb{P}^N$ be a normal projective variety with at worst $\mathbb{Q}-$factorial terminal singularities. X is called a *Mori Fiber Space* if it admits a morphism $\varphi : X \to S$ with S a normal variety such that:

- $\dim(S) < \dim(X)$
- The anticanonical divisor $-K_X$ is $\varphi-$ample
- The relative Picard number is equal to one, i.e. $\rho(X/S) = 1$.

We start by recalling two results in this area.

Theorem 26 ([Me06])**.** *Let $X \subseteq \mathbb{P}^N$ be a projective, irreducible non-degenerate variety. Suppose that X is generically identifiable. Then the $(g(X)-1)$-tangential projection $\tau_{g(X)-1} : X \dashrightarrow \mathbb{P}^{\dim(X)}$ is birational.*

Theorem 27 (Noether-Fano Inequalities [Co])**.** *Let $\pi : X \to X'$ and $\rho : Y \to Y'$ be two Mori fiber spaces and $\varphi : X \dashrightarrow Y$ a birational, not biregular, map*

$$\begin{array}{ccc} X & \overset{\varphi}{\dashrightarrow} & Y \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \rho} \\ X' & & Y' \end{array}$$

Choose a very ample linear system $\mathcal{H}_Y$ in Y and let $\mathcal{H}_X = \varphi_*^{-1}(\mathcal{H}_Y)$. Let $a \in \mathbb{Q}$ such that $\mathcal{H}_X \equiv -aK_X + \pi^*(A)$ for some divisor $A \in Pic(X')$.

Then either $(X, \frac{1}{a}\mathcal{H}_X)$ has not canonical singularities or $K_X + \frac{1}{a}\mathcal{H}_X$ is not NEF.

We are ready to connect the contact loci properties and the Noether–Fano inequalities to produce a tool for non identifiability statements.

Theorem 28. *Let $X^n \subset \mathbb{P}^N$ be a projective smooth non-degenerate variety and $\tau_{g-1}: X^n \dashrightarrow \mathbb{P}^{\dim X}$ be a general tangential projection, associated to the linear system $\mathcal{H} := \mathcal{H}(g-1)$. Assume that*

- *$\pi : X \to S$ is a Mori fiber space such that*
$$\mathcal{H} \equiv_\pi -aK_X + \pi^*(A)$$
with $a > 1$ a rational number and $A \in Pic(S)$
- *The $\mathbb{Q}$-divisor $K_X + \frac{1}{a}\mathcal{H}$ is NEF*
- *X is not $(g-1)-twd$*

Then τ_{g-1} is not birational, in particular X is not generically identifiable.

Proof. If $\pi_g^X : sec_g(X) \to \mathbb{P}^N$ is of fiber type then τ_{g-1} is of fiber type, see for instance [CM, Lemma16 (i)], and we conclude, by Theorem 26, that X is not identifiable.

Then thanks to Remark 6 we may assume that X is perfect and not defective. In particular τ_{g-1} is a not biregular map onto $\mathbb{P}^{\dim X}$.

By Theorem 24 there is a variety Y and a birational map $\nu : Y \to X$ with the following property: for any $\epsilon > 0$ there is a $\mathbb{Q}$-divisor D, with $D \equiv \nu_*^{-1}\mathcal{H}(g-1)$ such that for any point $y \in Y$
$$\operatorname{mult}_y D < 1 + \epsilon.$$
In particular $(Y, \frac{1}{a}\nu_*^{-1}(\mathcal{H}(g-1)))$ and henceforth $(X, \frac{1}{a}\mathcal{H}(g-1))$ have canonical singularities. Then, by Theorem 27 applied to the diagram
$$\begin{array}{ccc} X & \overset{\tau_{g-1}}{\dashrightarrow} & \mathbb{P}^{\dim X} \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ S & & Spec(\mathbb{C}) \end{array},$$
the map τ_{g-1} cannot be birational and therefore X is not generically identifiable by Theorem 26. $\square$

We are ready to prove the non identifiability statement announced in the introduction.

Definition 29. Let $\mathbf{n} = (n_1, \ldots, n_r)$ and $\mathbf{d} = (d_1, \ldots, d_r)$ be two $r-$tuples of positive integers. The *Segre-Veronese variety* $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ is the embedding of
$$\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \subset \mathbb{P}^{\prod \binom{n_i+d_i}{n_i} - 1}$$
via the complete linear system $|\pi_1^*\mathcal{O}_{\mathbb{P}^{n_1}}(d_1) \otimes ... \otimes \pi_r^*\mathcal{O}_{\mathbb{P}^{n_r}}(d_r)|$ where
$$\pi_i : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \to \mathbb{P}^{n_i}$$
are the canonical projections.

Theorem 30. *Fix two multiindexes $\boldsymbol{n} = (n_1, \ldots, n_r)$ and $\boldsymbol{d} = (d_1, \ldots, d_r)$. Let $X = SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ the corresponding Segre-Veronese variety. Assume that $d_i > n_i + 1$, for $i = 1, \ldots, r$, and*
$$\lceil \frac{\prod \binom{n_i+d_i}{n_i}}{\sum n_i + 1} \rceil > 2(\sum n_i).$$
Then X is not generically identifiable.

Proof. If X is defective or non perfect the statement is clear, recall Remark 6. Assume that X is not defective and perfect. Then $\tau_{g-1} : X \dashrightarrow \mathbb{P}^{\dim X}$ is generically finite. The numerical assumption reads
$$g(X) = \lceil \frac{\prod \binom{n_i+d_i}{n_i}}{\sum n_i + 1} \rceil > 2(\sum n_i) = 2\dim(X)$$

and, by [CM, Corollary 22], the variety X is not $(g-1)-$twd.

After reordering the indexes we may assume that

$$\frac{n_1+1}{d_1} \geq \frac{n_i+1}{d_i}, \text{ for any } i. \tag{2}$$

Let $p : X \to Y$ be the canonical projection onto the Segre-Veronese $Y = SV^{(n_2,\ldots,n_r)}_{(d_2,\ldots,d_r)}$ and $a = \frac{d_1}{n_1+1} > 1$. Then p is a Mori fiber Space and

$$K_X + \frac{1}{a}\mathcal{H}(g-1) \equiv_p 0.$$

Further note that the Mori cone of X is spanned by the lines in the factors $\mathbb{P}^{n_i}$. Moreover the effective and nef cones of X are generated by the divisors $D_i = \pi_i^*(\mathcal{O}_{\mathbb{P}^{n_i}}(1))$ (note that X is toric and the relevant cycles are exactly the torus invariant cycles of the required dimension). By Equation (2), we have

$$(K_X + \frac{1}{a}\mathcal{H}(g-1)) \cdot l_i = -(n_i+1) + \frac{n_1+1}{d_1} d_i \geq 0,$$

This shows that $K_X + \frac{1}{a}\mathcal{H}(g-1)$ is NEF and, by Theorem 28, we prove that X is not generically identifiable. □

Remark 31. In recent years the Secant varieties of Segre-Veronese varieties have been studied intensively, see for instance [AB],[AMR],[BBC1],[BCC],[FCM20],[LMR20]. However, to the best of our knowledge, this is the first result regarding non generic identifiability for infinite classes of Segre-Veronese varieties with $r \geq 2$.

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