# PREANTIPODES FOR DUAL-QUASI BIALGEBRAS 

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#### Abstract

It is known that a dual quasi-bialgebra with antipode $H$, i.e. a dual quasi-Hopf algebra, fulfils a fundamental theorem for right dual quasi-Hopf $H$-bicomodules. The converse in general is not true. We prove that, for a dual quasi-bialgebra $H$, the structure theorem amounts to the existence of a suitable map $S: H \rightarrow H$ that we call a preantipode of $H$.


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## 1. Introduction

Let $H$ be a bialgebra. It is well known that the functor $(-) \otimes H: \mathfrak{M} \rightarrow \mathfrak{M}_{H}^{H}$ determines an equivalence between the category $\mathfrak{M}$ of vector spaces and the category $\mathfrak{M}_{H}^{H}$ of right Hopf modules if and only if $H$ has an antipode i.e. it is a Hopf algebra. The if part of this statement is the so-called structure (or fundamental) theorem for Hopf modules, which is due, in the finite-dimensional case, to Larson and Sweedler, see LS, Proposition 1, page 82].

In 1989 Drinfeld introduced the concept of quasi-bialgebra in connection with the KnizhnikZamolodchikov system of partial differential equations. The axioms defining a quasi-bialgebra are a translation of monoidality of its representation category with respect to the diagonal tensor product. In Dry , the antipode for a quasi-bialgebra (whence the concept of quasi-Hopf algebra) is introduced in order to make the category of its flat right modules rigid. Also for quasi-Hopf algebras a fundamental theorem was given first by Hausser and Nill HN and then by Bulacu and Caenepeel BC. If we draw our attention to the category of co-representations of $H$, we get the concepts of dual quasi-bialgebra and of dual quasi-Hopf algebra. This notions have been introduced in Maj2 in order to prove a Tannaka-Krein type Theorem for quasi-Hopf algebras.

A fundamental theorem for finite-dimensional dual quasi-Hopf algebras can be obtained by duality using the results in HN. For an arbitrary dual quasi-Hopf algebra, the fundamental theorem is proved in Sch2 as follows. Schauenburg proves first that a dual quasi-bialgebra satisfies the fundamental theorem if and only if the category of its finite-dimensional co-representations is rigid. On the one hand any dual quasi-Hopf algebra fulfils this property. On the other hand the converse is not true in general. Thus dual quasi-Hopf algebras do not exhaust the class of dual quasi-bialgebras satisfying the fundamental theorem.

[^0]It is remarkable that the equivalence giving the fundamental theorem in the case of ordinary Hopf algebras must be substituted, in the "quasi" case, by the equivalence between the category of left $H$-comodules ${ }^{H} \mathfrak{M}$ and the category of right dual quasi-Hopf $H$-bicomodules ${ }^{H} \mathfrak{M}_{H}^{H}$ (essentially this is due to the fact that, unlike the classical case, $H$ is not a right $H$-comodule algebra but is still an $H$-bicomodule algebra). The main result of this paper, Theorem 3.9, establishes that such an equivalence amounts to the existence of a suitable map $S: H \rightarrow H$ that we call a preantipode. By the foregoing, any dual quasi-bialgebra with antipode (i.e. a dual quasi-Hopf algebra) admits a preantipode (in Theorem 3.10 we give a direct prove of this fact) but the converse is not true in general (see Remark 3.12). Finally, in Example 3.14, we construct a preantipode for a group algebra endowed with a normalized 3-cocycle.

## 2. Preliminaries

In this section we recall the definitions and results that will be needed in the paper.
Notation 2.1. Throughout this paper $\mathbb{k}$ will denote a field. All vector spaces will be defined over $\mathbb{k}$. The unadorned tensor product $\otimes$ will denote the tensor product over $\mathbb{k}$ if not stated differently.
2.2. Monoidal Categories. Recall that (see Ka, Chap. XI]) a monoidal category is a category $\mathcal{M}$ endowed with an object $1 \in \mathcal{M}$ (called unit), a functor $\otimes: \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ (called tensor product), and functorial isomorphisms $a_{X, Y, Z}:(X \otimes Y) \otimes Z \rightarrow X \otimes(Y \otimes Z), l_{X}: \mathbf{1} \otimes X \rightarrow X$, $r_{X}: X \otimes \mathbf{1} \rightarrow X$, for every $X, Y, Z$ in $\mathcal{M}$. The functorial morphism $a$ is called the associativity constraint and satisfies the Pentagon Axiom, that is the following relation

$$
\left(U \otimes a_{V, W, X}\right) \circ a_{U, V \otimes W, X} \circ\left(a_{U, V, W} \otimes X\right)=a_{U, V, W \otimes X} \circ a_{U \otimes V, W, X}
$$

holds true, for every $U, V, W, X$ in $\mathcal{M}$. The morphisms $l$ and $r$ are called the unit constraints and they obey the Triangle Axiom, that is $\left(V \otimes l_{W}\right) \circ a_{V, \mathbf{1}, W}=r_{V} \otimes W$, for every $V, W$ in $\mathcal{M}$.

The notions of algebra, module over an algebra, coalgebra and comodule over a coalgebra can be introduced in the general setting of monoidal categories. Given an algebra $A$ in $\mathcal{M}$ one can define the categories ${ }_{A} \mathcal{M}, \mathcal{M}_{A}$ and ${ }_{A} \mathcal{M}_{A}$ of left, right and two-sided modules over $A$ respectively.

Definition 2.3. A dual quasi-bialgebra is a datum $(H, m, u, \Delta, \varepsilon, \omega)$ where

- $(H, \Delta, \varepsilon)$ is a coassociative coalgebra;
- $m: H \otimes H \rightarrow H$ and $u: \mathbb{k} \rightarrow H$ are coalgebra maps called multiplication and unit respectively; we set $1_{H}:=u\left(1_{\mathrm{k}}\right)$;
- $\omega: H \otimes H \otimes H \rightarrow \mathbb{k}$ is a unital 3-cocycle i.e. it is convolution invertible and satisfies
$\omega(H \otimes H \otimes m) * \omega(m \otimes H \otimes H)=m_{\mathbb{k}}(\varepsilon \otimes \omega) * \omega(H \otimes m \otimes H) * m_{\mathbb{k}}(\omega \otimes \varepsilon) \quad$ and

$$
\begin{equation*}
v(h \otimes k \otimes l)=\varepsilon(h) \varepsilon(k) \varepsilon(l) \quad \text { whenever } \quad 1_{H} \in\{h, k, l\} \tag{1}
\end{equation*}
$$

- $m$ is quasi-associative and unitary i.e. it satisfies

$$
\begin{align*}
m(H \otimes m) * \omega & =\omega * m(m \otimes H)  \tag{3}\\
m\left(1_{H} \otimes h\right) & =h, \text { for all } h \in H  \tag{4}\\
m\left(h \otimes 1_{H}\right) & =h, \text { for all } h \in H \tag{5}
\end{align*}
$$

$\omega$ is called the reassociator of the dual quasi-bialgebra.
2.1. The category of (bi)comodules for a dual quasi-bialgebra. Let ( $H, m, u, \Delta, \varepsilon, \omega$ ) be a dual quasi-bialgebra. It is well known that the category $\mathfrak{M}^{H}$ of right $H$-comodules becomes a monoidal category as follows. Given a right $H$-comodule $V$, we denote by $\rho=\rho_{V}^{r}: V \rightarrow$ $V \otimes H, \rho(v)=v_{0} \otimes v_{1}$, its right $H$-coaction. The tensor product of two right $H$-comodules $V$ and $W$ is a comodule via diagonal coaction i.e. $\rho(v \otimes w)=v_{0} \otimes w_{0} \otimes v_{1} w_{1}$. The unit is $\mathbb{k}$, which is regarded as a right $H$-comodule via the trivial coaction i.e. $\rho(k)=k \otimes 1_{H}$. The associativity and unit constraints are defined, for all $U, V, W \in \mathfrak{M}^{H}$ and $u \in U, v \in V, w \in W, k \in \mathbb{k}$, by

$$
\begin{gathered}
a_{U, V . W}^{H}(u \otimes v \otimes w):=u_{0} \otimes\left(v_{0} \otimes w_{0}\right) \omega\left(u_{1} \otimes v_{1} \otimes w_{1}\right), \\
l_{U}(k \otimes u):=k u \quad \text { and } \quad r_{U}(u \otimes k):=u k .
\end{gathered}
$$

The monoidal category we have just described will be denoted by $\left(\mathfrak{M}^{H}, \otimes, \mathbb{k}, a^{H}, l, r\right)$.
Similarly, the monoidal categories ( $\left.{ }^{H} \mathfrak{M}, \otimes, \mathbb{k},{ }^{H} a, l, r\right)$ and ( $\left.{ }^{H} \mathfrak{M}^{H}, \otimes, \mathbb{k},{ }^{H} a^{H}, l, r\right)$ are introduced. We just point out that

$$
\begin{gathered}
{ }^{H} a_{U, V \cdot W}(u \otimes v \otimes w):=\omega^{-1}\left(u_{-1} \otimes v_{-1} \otimes w_{-1}\right) u_{0} \otimes\left(v_{0} \otimes w_{0}\right) \\
{ }^{H} a_{U, V \cdot W}^{H}(u \otimes v \otimes w):=\omega^{-1}\left(u_{-1} \otimes v_{-1} \otimes w_{-1}\right) u_{0} \otimes\left(v_{0} \otimes w_{0}\right) \omega\left(u_{1} \otimes v_{1} \otimes w_{1}\right)
\end{gathered}
$$

REMARK 2.4. We know that, if $(H, m, u, \Delta, \varepsilon, \omega)$ is a dual quasi bialgebra, we cannot construct the category $\mathfrak{M}_{H}$, because $H$ is not an algebra. Moreover $H$ is not an algebra in $\mathfrak{M}^{H}$ or in ${ }^{H} \mathfrak{M}$. On the other hand $\left(\left(H, \rho_{H}^{l}, \rho_{H}^{r}\right), m, u\right)$ is an algebra in the monoidal category ( $\left.{ }^{H} \mathfrak{M}^{H}, \otimes, \mathbb{k},{ }^{H} a^{H}, l, r\right)$ with $\rho_{H}^{l}=\rho_{H}^{r}=\Delta$. Thus, the only way to construct the category ${ }^{H} \mathfrak{M}_{H}^{H}$ is to consider the right $H$-modules in ${ }^{H} \mathfrak{M}^{H}$. Hence, we can set

$$
{ }^{H} \mathfrak{M}_{H}^{H}:=\left({ }^{H} \mathfrak{M}^{H}\right)_{H} .
$$

The category ${ }^{H} \mathfrak{M}_{H}^{H}$ is the so called category of right dual quasi-Hopf $H$-bicomodules BG, Remark 2.3].

Remark 2.5. AMS, Example 1.5(a)] Let $(A, m, u)$ be an algebra in a given monoidal category $(\mathcal{M}, \otimes, 1, a, l, r)$. Then the assignments $M \longmapsto\left(M \otimes A,(M \otimes m) \circ a_{A, A, A}\right)$ and $f \longmapsto f \otimes A$ define a functor $T: \mathcal{M} \rightarrow \mathcal{M}_{A}$. Moreover the forgetful functor $U: \mathcal{M}_{A} \rightarrow \mathcal{M}$ is a right adjoint of $T$.
2.2. An adjunction between ${ }^{H} \mathfrak{M}_{H}^{H}$ and ${ }^{H} \mathfrak{M}$. We are going to construct an adjunction between ${ }^{H} \mathfrak{M}_{H}^{H}$ and ${ }^{H} \mathfrak{M}$ that will be crucial afterwards.
2.6. Consider the functor $L:{ }^{H} \mathfrak{M} \rightarrow{ }^{H} \mathfrak{M}^{H}$ defined on objects by $L(\bullet V):={ }^{\bullet} V^{\circ}$ where the upper empty dot denotes the trivial right coaction while the upper full dot denotes the given left $H$-coaction of $V$. The functor $L$ has a right adjoint $R:{ }^{H} \mathfrak{M}^{H} \rightarrow{ }^{H} \mathfrak{M}$ defined on objects by $R\left({ }^{\bullet} M^{\bullet}\right):={ }^{\bullet} M^{c o H}$, where $M^{c o H}:=\left\{m \in M \mid m_{0} \otimes m_{1}=m \otimes 1_{H}\right\}$ is the space of right $H$-coinvariant elements in $M$.

By Remark 2.5, the forgetful functor $U:{ }^{H} \mathfrak{M}_{H}^{H} \rightarrow{ }^{H} \mathfrak{M}^{H}, U\left({ }^{\bullet} M_{\bullet}^{\bullet}\right):={ }^{\bullet} M^{\bullet}$ has a right adjoint, namely the functor $T:{ }^{H} \mathfrak{M}^{H} \rightarrow^{H} \mathfrak{M}_{H}^{H}, T\left({ }^{\bullet} M^{\bullet}\right):={ }^{\bullet} M^{\bullet} \otimes{ }^{\bullet} H_{\bullet}^{\bullet}$. Here the upper dots indicate on which tensor factors we have a codiagonal coaction and the lower dot indicates where the action takes place. Explicitly, the structure of $T\left({ }^{\bullet} M^{\bullet}\right)$ is given as follows:

$$
\begin{aligned}
\rho_{M \otimes H}^{l}(m \otimes h) & :=m_{-1} h_{1} \otimes\left(m_{0} \otimes h_{2}\right) \\
\rho_{M \otimes H}^{r}(m \otimes h) & :=\left(m_{0} \otimes h_{1}\right) \otimes m_{1} h_{2}, \\
(m \otimes h) l & :=\omega^{-1}\left(m_{-1} \otimes h_{1} \otimes l_{1}\right) m_{0} \otimes h_{2} l_{2} \omega\left(m_{1} \otimes h_{3} \otimes l_{3}\right) .
\end{aligned}
$$

Define the functors $F:=T L:{ }^{H} \mathfrak{M} \rightarrow{ }^{H} \mathfrak{M}_{H}^{H}$ and $G:=R U:{ }^{H} \mathfrak{M}_{H}^{H} \rightarrow{ }^{H} \mathfrak{M}$. Explicitly $G\left(\bullet M_{\bullet}^{\bullet}\right)=$ ${ }^{\bullet} M^{c o H}$ and $F(\bullet V):={ }^{\bullet} V^{\circ} \otimes \bullet H_{\bullet}^{\bullet}$ so that, for every $v \in V, h, l \in H$,

$$
\begin{aligned}
\rho_{V \otimes H}^{l}(v \otimes h) & =v_{-1} h_{1} \otimes\left(v_{0} \otimes h_{2}\right) \\
\rho^{r}(v \otimes h) & =\left(v \otimes h_{1}\right) \otimes h_{2} \\
(v \otimes h) l & =\omega^{-1}\left(v_{-1} \otimes h_{1} \otimes l_{1}\right) v_{0} \otimes h_{2} l_{2}
\end{aligned}
$$

The following result is essentially the right-hand version of Sch2, Lemma 2.1]
Theorem 2.7. The functor $F:{ }^{H} \mathfrak{M} \rightarrow{ }^{H} \mathfrak{M}_{H}^{H}$ is a left adjoint of the functor $G$. Moreover, the counit and the unit of the adjunction are given respectively by $\epsilon_{M}: F G(M) \rightarrow M, \epsilon_{M}(x \otimes h):=x h$ and by $\eta_{N}: N \rightarrow G F(N), \eta_{N}(n):=n \otimes 1_{H}$, for every $M \in{ }^{H} \mathfrak{M}_{H}^{H}, N \in{ }^{H} \mathfrak{M}$. Moreover $\eta_{N}$ is an isomorphism for any $N \in{ }^{H} \mathfrak{M}$. In particular the functor $F$ is fully faithful.

## 3. The notion of preantipode

The main result of this section is Theorem 3.9, where we show that, for a dual quasi-bialgebra $H$, the adjunction $(F, G)$ is an equivalence of categories if and only if $H$ admits what will be called a preantipode.

Definition 3.1. Maj1, page 66] A dual quasi-Hopf algebra $(H, m, u, \Delta, \varepsilon, \omega, s, \alpha, \beta)$ is a dual quasi-bialgebra $(H, m, u, \Delta, \varepsilon, \omega)$ endowed with a coalgebra antimorphism

$$
s: H \rightarrow H
$$

and two maps $\alpha, \beta$ in $H^{*}$, such that, for all $h \in H$ :

$$
\begin{align*}
h_{1} \beta\left(h_{2}\right) s\left(h_{3}\right) & =\beta(h) 1_{H}  \tag{6}\\
s\left(h_{1}\right) \alpha\left(h_{2}\right) h_{3} & =\alpha(h) 1_{H}  \tag{7}\\
\omega\left(h_{1} \otimes \beta\left(h_{2}\right) s\left(h_{3}\right) \alpha\left(h_{4}\right) \otimes h_{5}\right) & =\varepsilon(h)=\omega^{-1}\left(s\left(h_{1}\right) \otimes \alpha\left(h_{2}\right) h_{3} \beta\left(h_{4}\right) \otimes s\left(h_{5}\right)\right) \tag{8}
\end{align*}
$$

REmARK 3.2. Let $(H, m, u, \Delta, \varepsilon, \omega, s, \alpha, \beta)$ be a dual quasi-Hopf algebra. In BG, Remark 2.3] it is studied the problem of finding an isomorphism between $M$ and $M^{c o H} \otimes H$, for each $M \in{ }^{H} \mathfrak{M}_{H}^{H}$. The idea is to consider the surjection $P_{M}: M \rightarrow M^{c o H}$ defined, for all $m \in M$, by $P_{M}(m)=$ $m_{0} \beta\left(m_{1}\right) s\left(m_{2}\right)$. Bulacu and Caenepeel observe that a natural candidate for the bijection could be the map $\gamma_{M}: M \rightarrow M^{c o H} \otimes H$, defined, for each $M \in{ }^{H} \mathfrak{M}_{H}^{H}$, by setting $\gamma_{M}(m)=P_{M}\left(m_{0}\right) \otimes m_{1}$. Unfortunately there is no proof of the fact that $\gamma_{M}$ is bijective.

Next result characterizes when the adjunction $(F, G)$ is an equivalence of categories in term of the existence of a suitable map $\tau$.

Proposition 3.3. Let $(H, m, u, \Delta, \varepsilon, \omega)$ be a dual quasi-bialgebra. The following assertions are equivalent.
(i) The adjunction $(F, G)$ is an equivalence.
(ii) For each $M \in{ }^{H} \mathfrak{M}_{H}^{H}$, there exists a $\mathbb{k}$-linear map $\tau: M \rightarrow M^{\text {coH }}$ such that:

$$
\begin{align*}
\tau(m h) & =\omega^{-1}\left[\tau\left(m_{0}\right)_{-1} \otimes m_{1} \otimes h\right] \tau\left(m_{0}\right)_{0}, \text { for all } h \in H, m \in M  \tag{9}\\
m_{-1} \otimes \tau\left(m_{0}\right) & =\tau\left(m_{0}\right)_{-1} m_{1} \otimes \tau\left(m_{0}\right)_{0}, \text { for all } m \in M  \tag{10}\\
\tau\left(m_{0}\right) m_{1} & =m \forall m \in M \tag{11}
\end{align*}
$$

(iii) For each $M \in{ }^{H} \mathfrak{M}_{H}^{H}$, there exists a $\mathbb{k}$-linear map $\tau: M \rightarrow M^{\text {coH }}$ such that (1才) holds and

$$
\begin{equation*}
\tau(m h)=m \varepsilon(h), \text { for all } h \in H, m \in M^{c o H} \tag{12}
\end{equation*}
$$

Proof. $(i) \Rightarrow($ ii $)$ If $(F, G)$ is an equivalence, then $\epsilon_{M}: M^{c o H} \otimes H \rightarrow M$ is an isomorphism for each $M \in{ }^{H} \mathfrak{M}_{H}^{H}$. Set, for any $m \in M$ :

$$
\epsilon_{M}^{-1}(m)=m^{0} \otimes m^{1} \in M^{c o H} \otimes H
$$

We know that $\epsilon_{M}^{-1}$ is right $H$-colinear, i.e. the following diagram commutes:


In terms of elements this means that, for all $m \in M$,

$$
\begin{equation*}
m^{0} \otimes\left(m^{1}\right)_{1} \otimes\left(m^{1}\right)_{2}=\left(m_{0}\right)^{0} \otimes\left(m_{0}\right)^{1} \otimes m_{1} \tag{13}
\end{equation*}
$$

Let us define $\tau: M \rightarrow M^{c o H}, \tau(m):=m^{0} \varepsilon\left(m^{1}\right)$. By applying $M^{c o H} \otimes \varepsilon \otimes H$ to both terms in (13), we get

$$
\epsilon_{M}^{-1}(m)=\tau\left(m_{0}\right) \otimes m_{1}
$$

We will now prove (9), (10) and (11).
From the $H$-linearity of $\epsilon_{M}^{-1}$, the following diagram commutes:

i.e.
$\omega^{-1}\left(\tau\left(m_{0}\right)_{-1} \otimes m_{1} \otimes h_{1}\right) \tau\left(m_{0}\right)_{0} \otimes m_{2} h_{2}=\tau\left(m_{0} h_{1}\right) \otimes m_{1} h_{2}$, for all $m \in M, h \in H$.
Applying now $M^{c o H} \otimes \varepsilon$ on both terms, we obtain exactly (9).
$\epsilon_{M}^{-1}$ is also left $H$-colinear, that is:

i.e.

$$
\tau\left(m_{0}\right)_{-1} m_{1} \otimes \tau\left(m_{0}\right)_{0} \otimes m_{2}=m_{-1} \otimes \tau\left(m_{0}\right) \otimes m_{1}, \text { for all } m \in M
$$

To obtain (10) we have to apply $H \otimes M^{c o H} \otimes \varepsilon$ to both terms.
Finally, recalling the fact that $\epsilon_{M} \epsilon_{M}^{-1}=\mathbb{I}_{M}$, we have the following equality, for all $m \in M$,

$$
m=\epsilon_{M} \epsilon_{M}^{-1}(m)=\epsilon_{M}\left(\tau\left(m_{0}\right) \otimes m_{1}\right)=\tau\left(m_{0}\right) m_{1}
$$

(ii) $\Rightarrow($ iii $)$ It is trivial.
$($ iii $) \Rightarrow(i)$ The only thing that we have to prove is the invertibility of $\epsilon_{M}$, for any $M \in{ }^{H} \mathfrak{M}_{H}^{H}$.
Let us define $\psi: M \rightarrow M^{c o H} \otimes H$ by setting $\psi(m):=\tau\left(m_{0}\right) \otimes m_{1}$. Then, for all $m \in M$,

$$
\epsilon_{M} \psi_{M}(m)=\tau\left(m_{0}\right) m_{1} \stackrel{\text { 111 }}{=} m
$$

and for all $m \in M^{c o H}, h \in H$,

$$
\psi_{M} \epsilon_{M}(m \otimes h)=\psi_{M}(m h)=\tau\left(m_{0} h_{1}\right) \otimes m_{1} h_{2} \stackrel{m}{\underline{M}}^{c o H} \tau\left(m h_{1}\right) \otimes h_{2} \stackrel{\text { (12) }}{=} m \otimes h .
$$

So $\psi_{M}$ is the inverse of $\epsilon_{M}$, for all $M \in{ }^{H} \mathfrak{M}_{H}^{H}$.
From Theorem 2.7, we have that $\eta_{M}$ is always an isomorphism, so we have the equivalence.
Remark 3.4. Let $\tau: M \rightarrow M^{c o H}$ be a $\mathbb{k}$-linear map such that (11) holds. Following the proof of Proposition 3.3, it is clear that a map $\tau$ fulfils (12) if and only if it fulfils (9) and (10).
3.5. Let $(H, m, u, \Delta, \varepsilon, \omega)$ be a dual quasi-bialgebra. As observed in Remark 2.4, $(H, m, u)$ is an algebra in the monoidal category $\left({ }^{H} \mathfrak{M}^{H}, \otimes, \mathbb{k},{ }^{H} a^{H}, l, r\right)$, where both comodule structures are given by $\Delta$. Consider the functor $T$ of 2.6 and set

$$
H \widehat{\otimes} H:=T\left({ }^{\circ} H^{\bullet}\right)={ }^{\circ} H^{\bullet} \otimes{ }^{\bullet} H_{\bullet}^{\bullet}
$$

where in ${ }^{\circ} H^{\bullet}$ the empty dot denotes the trivial left $H$-comodule structure and the full dot denotes the right coaction given by $\Delta$. Explicitly, for $h, k, l \in H$, the structure of $H \widehat{\otimes} H$ is given by

$$
\begin{aligned}
\rho_{H \widehat{\otimes} H}^{r}(h \otimes k) & =\left(h_{1} \otimes k_{1}\right) \otimes h_{2} k_{2}, \\
\rho_{H \widehat{\otimes} H}^{l}(h \otimes k) & =k_{1} \otimes h \otimes k_{2}, \\
(h \otimes k) l & =h_{1} \otimes k_{1} l_{1} \omega\left(h_{2} \otimes k_{2} \otimes l_{2}\right) .
\end{aligned}
$$

Set

$$
\widehat{\epsilon}_{H}:=\epsilon_{H \widehat{\otimes} H}^{F, G}:(H \widehat{\otimes} H)^{c o H} \otimes H \rightarrow H \widehat{\otimes} H
$$

so that, for $x_{i} \otimes y_{i} \in(H \widehat{\otimes} H)^{c o H}$ (summation understood) and $h \in H$,

$$
\widehat{\epsilon}_{H}\left(\left(x_{i} \otimes y_{i}\right) \otimes h\right)=\left(x_{i} \otimes y_{i}\right) \cdot h=x_{i_{1}} \otimes y_{i_{1}} h_{1} \omega\left(x_{i_{2}} \otimes y_{i_{2}} \otimes h_{2}\right)
$$

Suppose now that $\widehat{\epsilon}_{H}$ is an isomorphism. Then, from the right $H$-colinearity of $\widehat{\epsilon}_{H}$, we deduce the right $H$-colinearity of $\widehat{\epsilon}_{H}^{-1}$, i.e., if we set

$$
h^{[1]} \otimes h^{[2]} \otimes h^{[3]}:=\widehat{\epsilon}_{H}^{-1}\left(h \otimes 1_{H}\right),
$$

we have:

$$
\begin{equation*}
h^{[1]} \otimes h^{[2]} \otimes\left(h^{[3]}\right)_{1} \otimes\left(h^{[3]}\right)_{2}=\left(h_{1}\right)^{[1]} \otimes\left(h_{1}\right)^{[2]} \otimes\left(h_{1}\right)^{[3]} \otimes h_{2} . \tag{14}
\end{equation*}
$$

Define

$$
\beta(h):=h^{[1]} \otimes h^{[2]} \varepsilon\left(h^{[3]}\right) \in(H \widehat{\otimes} H)^{c o H} .
$$

Then, by applying $H \otimes H \otimes \varepsilon \otimes H$ on each side of (14), we get:

$$
\begin{equation*}
\widehat{\epsilon}_{H}^{-1}\left(h \otimes 1_{H}\right)=\beta\left(h_{1}\right) \otimes h_{2} \tag{15}
\end{equation*}
$$

Let consider a new left coaction for $H \widehat{\otimes} H$ :

$$
\rho^{l}: H \widehat{\otimes} H \rightarrow H \otimes(H \widehat{\otimes} H), \rho^{l}(h \otimes k):=h_{1} \otimes h_{2} \otimes k=(\Delta \otimes H)(h \otimes k)
$$

Let us verify that $\widehat{\epsilon}_{H}$ is $H$-left colinear respect on this new coaction i.e. that the following commutes:


In fact

$$
\begin{aligned}
(\Delta \otimes H) \circ \widehat{\epsilon_{H}}\left(x_{i} \otimes y_{i} \otimes h\right) & =(\Delta \otimes H)\left(x_{i_{1}} \otimes y_{i_{1}} h_{1} \omega\left(x_{i_{2}} \otimes y_{i_{2}} \otimes h_{2}\right)\right) \\
& =x_{i_{1}} \otimes x_{i_{2}} \otimes y_{i_{1}} h_{1} \omega\left(x_{i_{3}} \otimes y_{i_{2}} \otimes h_{2}\right) \\
& =\left(H \otimes \widehat{\epsilon}_{H}\right) \circ(\Delta \otimes H \otimes H)\left(x_{i} \otimes y_{i} \otimes h\right)
\end{aligned}
$$

Then also $\widehat{\epsilon}_{H}^{-1}$ is left $H$-colinear with respect to this structure, i.e:

$$
(\Delta \otimes H \otimes H) \circ \widehat{\epsilon}^{-1}\left(x \otimes 1_{H}\right)=\left(H \otimes \widehat{\epsilon}_{H}^{1}\right) \circ(\Delta \otimes H)\left(x \otimes 1_{H}\right)
$$

If we set $h^{1} \otimes h^{2}:=\beta(h)$, the last displayed equality means

$$
\begin{equation*}
\left(\left(x_{1}\right)^{1}\right)_{1} \otimes\left(\left(x_{1}\right)^{1}\right)_{2} \otimes\left(x_{1}\right)^{2} \otimes x_{2}=x_{1} \otimes\left(x_{2}\right)^{1} \otimes\left(x_{2}\right)^{2} \otimes x_{3} \tag{16}
\end{equation*}
$$

Let us define, for any $x \in H$,

$$
S(x):=\varepsilon\left(x^{1}\right) x^{2}
$$

By applying $H \otimes \varepsilon \otimes H \otimes \varepsilon$ on each side of (16) it results:

$$
\begin{equation*}
\beta(x)=x_{1} \otimes S\left(x_{2}\right) \tag{17}
\end{equation*}
$$

Now let us deduce the properties of $S$ from the bijectivity of $\widehat{\epsilon}_{H}$ and its colinearity.
From (17) and $\beta(x) \in(H \widehat{\otimes} H)^{c o H}$, we have:

$$
x_{1} \otimes S\left(x_{3}\right)_{1} \otimes x_{2} S\left(x_{3}\right)_{2}=x_{1} \otimes S\left(x_{2}\right) \otimes 1_{H}
$$

By applying $\varepsilon \otimes H \otimes H$ on both sides, we obtain that

$$
\begin{equation*}
S\left(x_{2}\right)_{1} \otimes x_{1} S\left(x_{2}\right)_{2}=S(x) \otimes 1_{H}, \text { for all } x \in H \tag{18}
\end{equation*}
$$

From

$$
\begin{aligned}
x \otimes 1_{H} & =\widehat{\epsilon}_{H} \widehat{\epsilon}_{H}^{1}\left(x \otimes 1_{H}\right) \\
& =\widehat{\epsilon}_{H}\left(x_{1} \otimes S\left(x_{2}\right) \otimes x_{3}\right) \\
& =x_{1_{1}} \otimes S\left(x_{2}\right)_{1} x_{3} \omega\left(x_{1_{2}} \otimes S\left(x_{2}\right)_{2} \otimes x_{4}\right) \\
& =x_{1} \otimes S\left(x_{3}\right)_{1} x_{4} \omega\left(x_{2} \otimes S\left(x_{3}\right)_{2} \otimes x_{5}\right)
\end{aligned}
$$

we get

$$
x \otimes 1_{H}=x_{1} \otimes S\left(x_{3}\right)_{1} x_{4} \omega\left(x_{2} \otimes S\left(x_{3}\right)_{2} \otimes x_{5}\right)
$$

By applying $\varepsilon \otimes \varepsilon$ on both sides, we get

$$
\begin{equation*}
\omega\left(x_{1} \otimes S\left(x_{2}\right) \otimes x_{3}\right)=\varepsilon(x), \text { for all } x \in H \tag{19}
\end{equation*}
$$

From the left $H$-colinearity with respect to the usual coaction of ${\widehat{\epsilon_{H}}}^{-1}$ we have:

$$
\rho^{l} \circ \widehat{\epsilon}_{H}^{1}\left(x \otimes 1_{H}\right)=\left(H \otimes \widehat{\epsilon}_{H}^{-1}\right)\left(1_{H} \otimes x \otimes 1_{H}\right)
$$

i.e.

$$
S\left(x_{2}\right)_{1} x_{3_{1}} \otimes x_{1} \otimes S\left(x_{2}\right)_{2} \otimes x_{3_{2}}=1_{H} \otimes x_{1} \otimes S\left(x_{2}\right) \otimes x_{3}
$$

Applying $H \otimes \varepsilon \otimes H \otimes \varepsilon$ on both sides, we obtain:

$$
\begin{equation*}
S\left(x_{1}\right)_{1} x_{2} \otimes S\left(x_{1}\right)_{2}=1_{H} \otimes S(x), \text { for all } x \in H \tag{20}
\end{equation*}
$$

Definition 3.6. A preantipode for a dual quasi-bialgebra $(H, m, u, \Delta, \varepsilon, \omega)$ is a $k$-linear map $S: H \rightarrow H$ such that (18), (19) and (20) hold.

REmARK 3.7. Let $(H, m, u, \Delta, \varepsilon, \omega, S)$ be a dual quasi-bialgebra with a preantipode. Then the following equalities hold

$$
\begin{equation*}
h_{1} \varepsilon S\left(h_{2}\right)=\varepsilon S(h) 1_{H .}=\varepsilon S\left(h_{1}\right) h_{2} \text { for all } h \in H \tag{21}
\end{equation*}
$$

Lemma 3.8. Let $(H, m, u, \Delta, \varepsilon, \omega, S)$ be a dual quasi-bialgebra with a preantipode. For any $M \in$ ${ }^{H} \mathfrak{M}_{H}^{H}$ and $m \in M$, set

$$
\begin{equation*}
\tau(m):=\omega\left[m_{-1} \otimes S\left(m_{1}\right)_{1} \otimes m_{2}\right] m_{0} S\left(m_{1}\right)_{2} \tag{22}
\end{equation*}
$$

Then (2G) defines a map $\tau: M \rightarrow M^{\text {coH }}$ which fulfills (9), (10) and (11).
Proof. To prove that $\tau$ is well-defined we have to check if $\operatorname{Im} \tau \subseteq M^{c o H}$.
Let us compute, for all $m \in M$,

$$
\begin{aligned}
\rho^{r}(\tau(m)) & =\omega\left(m_{-1} \otimes S\left(m_{2}\right)_{1} \otimes m_{3}\right) m_{0} S\left(m_{2}\right)_{2} \otimes m_{1} S\left(m_{2}\right)_{3} \\
& \stackrel{18)}{ } \omega\left(m_{-1} \otimes S\left(m_{1}\right)_{1} \otimes m_{2}\right) m_{0} S\left(m_{1}\right)_{2} \otimes 1_{H} \\
& =\tau(m) \otimes 1_{H}
\end{aligned}
$$

Now let us prove (11). For all $m \in M$,

$$
\begin{aligned}
\tau\left(m_{0}\right) m_{1} & =\omega\left(m_{-1} \otimes S\left(m_{1}\right)_{1} \otimes m_{2}\right)\left(m_{0} S\left(m_{1}\right)_{2}\right) m_{3} \\
& \stackrel{\text { B) }}{ }\left[\begin{array}{c}
\omega\left(m_{-1} \otimes S\left(m_{1}\right)_{1} \otimes m_{2}\right) \omega^{-1}\left(m_{0_{-1}} \otimes S\left(m_{1}\right)_{2} \otimes m_{3}\right) \\
m_{0_{0}}\left(S\left(m_{1}\right)_{3} m_{4}\right) \omega\left(m_{0_{1}} \otimes S\left(m_{1}\right)_{4} \otimes m_{4}\right)
\end{array}\right] \\
& =m_{0}\left(S\left(m_{2}\right)_{1} m_{3}\right) \omega\left(m_{1} \otimes S\left(m_{2}\right)_{2} \otimes m_{3}\right) \\
& =2 m_{0} 1_{H} \omega\left(m_{1} \otimes S\left(m_{2}\right) \otimes m_{3}\right) \\
& =191_{H}
\end{aligned}
$$

Let us check (12). For $m \in M^{c o H}, h \in H$,

$$
\begin{aligned}
\tau(m h) & =\omega\left(m_{-1} h_{1} \otimes S\left(m_{1} h_{3}\right)_{1} \otimes m_{2} h_{4}\right)\left(m_{0} h_{2}\right) S\left(m_{1} h_{3}\right)_{2} \\
& \stackrel{(*)}{=} \omega\left(m_{-1} h_{1} \otimes S\left(h_{3}\right)_{1} \otimes h_{4}\right)\left(m_{0} h_{2}\right) S\left(h_{3}\right)_{2} \\
& \stackrel{\text { (3) }}{=} \omega\left(m_{-1} h_{1} \otimes S\left(h_{3}\right)_{1} \otimes h_{4}\right) \omega^{-1}\left(m_{0_{-1}} \otimes h_{2_{1}} \otimes S\left(h_{3}\right)_{2}\right) m_{0_{0}}\left(h_{2_{2}} S\left(h_{3}\right)_{3}\right) \\
& =\omega\left(m_{-1_{1}} h_{1} \otimes S\left(h_{4}\right)_{1} \otimes h_{5}\right) \omega^{-1}\left(m_{-1_{2}} \otimes h_{2} \otimes S\left(h_{4}\right)_{2}\right) m_{0}\left(h_{3} S\left(h_{4}\right)_{3}\right) \\
& \text { (18) } \omega\left(m_{-1} h_{1} \otimes S\left(h_{3}\right)_{1} \otimes h_{4}\right) \omega^{-1}\left(m_{-1_{2}} \otimes h_{2} \otimes S\left(h_{3}\right)_{2}\right) m_{0} \\
& \text { (14) } \omega^{-1}\left(m_{-1_{1}} \otimes h_{1} \otimes S\left(h_{4}\right)_{1} h_{5}\right) \omega\left(h_{2} \otimes S\left(h_{4}\right)_{2} \otimes h_{6}\right) \omega\left(m_{-1_{2}} \otimes h_{3} S\left(h_{4}\right)_{3} \otimes h_{7}\right) m_{0} \\
& \text { (18) } \omega^{-1}\left(m_{-1} \otimes h_{1} \otimes S\left(h_{3}\right)_{1} h_{4}\right) \omega\left(h_{2} \otimes S\left(h_{3}\right)_{2} \otimes h_{5}\right) m_{0} \\
& \text { (20) } \omega\left(h_{2} \otimes S\left(h_{3}\right) \otimes h_{4}\right) m_{0} \\
& \text { (19) } \varepsilon(h) m,
\end{aligned}
$$

where in (*) we used that $m \in M^{c o H}$. Now, by Remark 3.4, we conclude.
We are now able to state the main theorem characterizing when $(F, G)$ is an equivalence.
Theorem 3.9. For a dual quasi-bialgebra $(H, m, u, \Delta, \varepsilon, \omega)$ the following are equivalent.
(i) The adjunction $(F, G)$ of Theorem 2.7 is an equivalence of categories.
(ii) $\widehat{\epsilon}_{H}$ is a bijection.
(iii) There exists a preantipode.

Proof. $(i) \Rightarrow(i i)$ It is trivial because $\widehat{\epsilon}_{H}=\epsilon_{H \widehat{\otimes} H}^{F, G}$.
(ii) $\Rightarrow$ (iii) See Remark 3.5.
$($ iii $) \Rightarrow(i)$ Let us define $\tau$ as in (22). Apply Lemma 3.8 and Proposition 3.3.
Theorem 3.10. Let $(H, m, u, \Delta, \varepsilon, \omega, s, \alpha, \beta)$ be a dual quasi-Hopf algebra. Then

$$
S:=\beta * s * \alpha
$$

is a preantipode. Here * denotes the convolution product.
Proof. Fix $h \in H$. Let us check (18):

$$
\begin{aligned}
S\left(h_{2}\right)_{1} \otimes h_{1} S\left(h_{2}\right)_{2} & =\beta\left(h_{2}\right) s\left(h_{3}\right)_{1} \alpha\left(h_{4}\right) \otimes h_{1} s\left(h_{3}\right)_{2} \\
& \stackrel{(*)}{=} s\left(h_{3_{2}}\right) \alpha\left(h_{4}\right) \otimes h_{1} \beta\left(h_{2}\right) s\left(h_{3_{1}}\right) \\
& \stackrel{(6)}{=} s\left(h_{2}\right) \alpha\left(h_{3}\right) \otimes \beta\left(h_{1}\right) 1_{H} \\
& =S(h) \otimes 1_{H} .
\end{aligned}
$$

where in $(*)$ we used that $s$ is a coalgebra anti-homomorphism. Let us prove (20):

$$
\begin{aligned}
S\left(h_{1}\right)_{1} h_{2} \otimes S\left(h_{1}\right)_{2} & =\beta\left(h_{1}\right) s\left(h_{2}\right)_{1} \alpha\left(h_{3}\right) h_{4} \otimes s\left(h_{2}\right)_{2} \\
& \stackrel{(*)}{=} \beta\left(h_{1}\right) s\left(h_{2_{2}}\right) \alpha\left(h_{3}\right) h_{4} \otimes s\left(h_{2_{1}}\right) \\
& \stackrel{\oiint}{=} \beta\left(h_{1}\right) \alpha\left(h_{3}\right) \otimes s\left(h_{2}\right) . \\
& =1_{H} \otimes S(h) .
\end{aligned}
$$

Finally we have to verify (19):

$$
\omega\left(h_{1} \otimes S\left(h_{2}\right) \otimes h_{3}\right)=\omega\left(h_{1} \beta\left(h_{2}\right) \otimes s\left(h_{3}\right) \otimes \alpha\left(h_{4}\right) h_{5}\right) \stackrel{(8)}{=} \varepsilon(h)
$$

Corollary 3.11. (Cf. right-hand version of Sch2, Corollary 2.7]) Let $(H, m, u, \Delta, \varepsilon, \omega, s, \alpha, \beta) a$ dual quasi-Hopf algebra. Then the adjunction $(\overline{F, G})$ of Theorem 2. 7 is an equivalence of categories.

REMARK 3.12. Let us show that the converse of Theorem 3.10 does not hold true in general. By Sch1, Example 4.5.1], there is a dual quasi-bialgebra $H$ which is not a dual quasi-Hopf algebra and such that the category ${ }^{H} \mathfrak{M}_{f}$ of finite-dimensional left $H$-comodules is left and right rigid. Then, by the right-hand version of Sch2, Theorem 3.1], the adjunction $(F, G)$ of Theorem 2.7 is an equivalence of categories. By Theorem 3.9, $H$ has a preantipode.

Nevertheless, for a finite-dimensional dual quasi-bialgebra, the existence of an antipode amounts to the existence of a preantipode. This follows by duality in view of Sch2, Theorem 3.1].

Remark 3.13. Consider a dual quasi-Hopf algebra ( $H, m, u, \Delta, \varepsilon, \omega, s, \alpha, \beta$ ) and the associated preantipode $S$. Than the map $\tau$ defined by (22) becomes:

$$
\begin{aligned}
\tau(m) & =\omega\left(m_{-1} \otimes S\left(m_{1}\right)_{1} \otimes m_{2}\right) m_{0} S\left(m_{1}\right)_{2} \\
& =\omega\left(m_{-1} \otimes \beta\left(m_{1_{1}}\right) s\left(m_{1_{2}}\right)_{1} \otimes m_{2}\right) m_{0} s\left(m_{1_{2}}\right)_{2} \alpha\left(m_{1_{3}}\right) \\
& =\omega\left(m_{-1} \otimes \beta\left(m_{1}\right) s\left(m_{3}\right) \otimes m_{5}\right) m_{0} s\left(m_{2}\right) \alpha\left(m_{4}\right) \\
& =\omega\left(m_{-1} \otimes s\left(m_{3}\right) \otimes m_{5}\right) m_{0} \beta\left(m_{1}\right) s\left(m_{2}\right) \alpha\left(m_{4}\right) \\
& =\omega\left(m_{-1} \otimes s\left(m_{1}\right) \otimes m_{3}\right) P_{M}\left(m_{0}\right) \alpha\left(m_{2}\right)
\end{aligned}
$$

where $P_{M}$ denotes the map recalled in Remark 3.2 .

Example 3.14. Let $G$ be a group. Let $\theta: G \times G \times G \rightarrow \mathbb{k}^{*}:=\mathbb{k} \backslash\{0\}$ be a normalized 3-cocycle on the group $G$ in the sense of Maj1, Example 2.3.2, page 54] i.e. a map such that, for all $g, h, k, l \in H$

$$
\begin{aligned}
\theta\left(g, 1_{G}, h\right) & =1 \\
\theta(h, k, l) \theta(g, h k, l) \theta(g, h, k) & =\theta(g, h, k l) \theta(g h, k, l) .
\end{aligned}
$$

Then $\theta$ can be extended by linearity to a reassociator $\omega: \mathbb{k} G \otimes \mathbb{k} G \otimes \mathbb{k} G \rightarrow \mathbb{k}$ making $\mathbb{k} G$ a dual quasi-bialgebra with usual underlying algebra and coalgebra structures. Moreover, by AM, page 193], $\mathbb{k} G$ is indeed a dual quasi-Hopf algebra where $\alpha, \beta: \mathbb{k} G \rightarrow \mathbb{k}$ are defined on generators by $\alpha(g):=1_{\mathbb{k}}$ and $\beta(g):=\left[\omega\left(g, g^{-1}, g\right)\right]^{-1}$ and the antipode $s: \mathbb{k} G \rightarrow \mathbb{k} G$ is given by $s(g):=g^{-1}$, for all $g \in G$. By Theorem 3.10, we have a preantipode on $\mathbb{k} G$, which is defined by $S:=\beta * s * \alpha$ so that

$$
S(g):=\left[\omega\left(g, g^{-1}, g\right)\right]^{-1} g^{-1}, \text { for all } g \in G
$$

Note that, unlike the antipode, this preantipode is not an coalgebra anti-homomorphism as

$$
S\left(g_{2}\right) \otimes S\left(g_{1}\right)=\left[\omega\left(g, g^{-1}, g\right)\right]^{-1} \Delta S(g), \text { for all } g \in G
$$

## References

[AM] H. Albuquerque, S. Majid, Quasialgebra structure of the octonions. J. Algebra 220 (1999), no. 1, $188-224$.
[AMS] A. Ardizzoni, C. Menini and D. Stefan, Hochschild Cohomology And 'Smoothness' In Monoidal Categories, J. Pure Appl. Algebra, Vol. 208 (2007), 297-330.
[BC] D. Bulacu, S. Caenepeel, Integrals for (dual) quasi-Hopf algebras. Applications. J. Algebra 266 (2003), no. 2, 552-583.
[Dr] V. G. Drinfeld, Quasi-Hopf algebras. (Russian) Algebra i Analiz 1 (1989), no. 6, 114-148; translation in Leningrad Math. J. 1 (1990), no. 6, 1419-1457.
[HN] F. Hausser and F. Nill, Integral theory for quasi Hopf algebras, preprint (arXiv:math/9904164v2)
[Ka] C. Kassel, Quantum groups, Graduate Text in Mathematics 155, Springer, 1995.
[LS] R. G. Larson, M. E. Sweedler, An associative orthogonal bilinear form for Hopf algebras. Amer. J. Math. 91 1969 75-94.
[Maj1] S. Majid, Foundations of quantum group theory, Cambridge University Press, 1995
[Maj2] S. Majid, Tannaka-Kreĭn theorem for quasi-Hopf algebras and other results. Deformation theory and quantum groups with applications to mathematical physics (Amherst, MA, 1990), 219-232, Contemp. Math., 134, Amer. Math. Soc., Providence, RI, 1992.
[Sch1] P. Schauenburg, Hopf algebra extensions and monoidal categories. New directions in Hopf algebras, 321-381, Math. Sci. Res. Inst. Publ., 43, Cambridge Univ. Press, Cambridge, 2002.
[Sch2] P. Schauenburg, Two characterizations of finite quasi-Hopf algebras. J. Algebra 273 (2004), no. 2, 538-550.

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