

$\{\sigma, \tau\}$ -Rota-Baxter operators, infinitesimal Hom-bialgebras and the associative (Bi)Hom-Yang-Baxter equation

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Abstract

We introduce the concept of $\{\sigma, \tau\}$ -Rota-Baxter operator, as a twisted version of a Rota-Baxter operator of weight zero. We show how to obtain a certain $\{\sigma, \tau\}$ -Rota-Baxter operator from a solution of the associative (Bi)Hom-Yang-Baxter equation, and, in a compatible way, a Hom-pre-Lie algebra from an infinitesimal Hom-bialgebra.

Keywords: Rota-Baxter operator, Hom-pre-Lie algebra, infinitesimal Hom-bialgebra, associative (Bi)Hom-Yang-Baxter equation.

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1 Introduction

Hom-type algebras appeared in the Physics literature related to quantum deformations of algebras of vector fields; these types of algebras satisfy a modified version of the Jacobi identity involving a homomorphism, and were called Hom-Lie algebras by Hartwig, Larsson and Silvestrov in [10], [12]. Afterwards, Hom-analogues of various classical algebraic structures have

been introduced in the literature, such as Hom-(co)associative (co)algebras, Hom-dendriform algebras, Hom-pre-Lie algebras etc. Recently, structures of a more general type have been introduced in [8], called BiHom-type algebras, for which a classical algebraic identity is twisted by two commuting homomorphisms (called structure maps).

Infinitesimal bialgebras were introduced by Joni and Rota in [11] (under the name infinitesimal coalgebra). The current name is due to Aguiar, who developed a theory for them in a series of papers ([2, 3, 4]). It turns out that infinitesimal bialgebras have connections with some other concepts such as Rota-Baxter operators, pre-Lie algebras, Lie bialgebras etc. Aguiar discovered a large class of examples of infinitesimal bialgebras, namely he showed that the path algebra of an arbitrary quiver carries a natural structure of infinitesimal bialgebra. In an analytical context, infinitesimal bialgebras have been used in [22] by Voiculescu in free probability theory.

The Hom-analogue of infinitesimal bialgebras, called infinitesimal Hom-bialgebras, was introduced and studied by Yau in [23]. He extended to the Hom-context some of Aguiar's results; however, there exist several basic results of Aguiar that do not have a Hom-analogue in Yau's paper. It is our aim here to complete the study, by proving those Hom-analogues.

The associative Yang-Baxter equation was introduced by Aguiar in [2]. Let (A, μ) be an associative algebra and $r = \sum_i x_i \otimes y_i \in A \otimes A$; then r is called a solution of the associative Yang-Baxter equation if

$$\sum_{i,j} x_i \otimes y_i x_j \otimes y_j = \sum_{i,j} x_i x_j \otimes y_j \otimes y_i + \sum_{i,j} x_i \otimes x_j \otimes y_j y_i.$$

In this situation, Aguiar noticed in [1] that the map $R : A \rightarrow A$, $R(a) = \sum_i x_i a y_i$, is a Rota-Baxter operator of weight zero. We recall (see for instance [9]) that if B is an algebra and $R : B \rightarrow B$ is a linear map, then R is called a Rota-Baxter operator of weight zero if

$$R(a)R(b) = R(R(a)b + aR(b)), \quad \forall a, b \in B.$$

Rota-Baxter operators appeared first in the work of Baxter in probability and the study of fluctuation theory, and were intensively studied by Rota in connection with combinatorics. Rota-Baxter operators occurred also in other areas of mathematics and physics, notably in the seminal work of Connes and Kreimer [6] concerning a Hopf algebraic approach to renormalization in quantum field theory.

The Hom-analogue of the associative Yang-Baxter equation was introduced by Yau in [23], but without exploring the relation between this new equation and Rota-Baxter operators. Our first aim is to obtain Hom and BiHom-analogues of Aguiar's observation mentioned above, expressing a relationship between Hom and BiHom-analogues of the associative Yang-Baxter equation and certain generalized Rota-Baxter operators. The BiHom-analogue of the associative Yang-Baxter equation is defined as follows. Let (A, μ, α, β) be a BiHom-associative algebra and $r = \sum_i x_i \otimes y_i \in A \otimes A$ such that $(\alpha \otimes \alpha)(r) = r = (\beta \otimes \beta)(r)$; we say that r is a solution of the associative BiHom-Yang-Baxter equation if

$$\sum_{i,j} \alpha(x_i) \otimes y_i x_j \otimes \beta(y_j) = \sum_{i,j} x_i x_j \otimes \beta(y_j) \otimes \beta(y_i) + \sum_{i,j} \alpha(x_i) \otimes \alpha(x_j) \otimes y_j y_i.$$

To such an element r we want to associate a certain linear map $R : A \rightarrow A$, that will turn out to be a twisted version of a Rota-Baxter operator of weight zero. More precisely, the map R is defined by

$$R : A \rightarrow A, \quad R(a) = \sum_i \alpha \beta^3(x_i)(a \alpha^3(y_i)) = \sum_i (\beta^3(x_i) a) \alpha^3 \beta(y_i), \quad \forall a \in A,$$

which in the Hom case (i.e. for $\alpha = \beta$) reduces to $R(a) = \sum_i \alpha(x_i)(ay_i) = \sum_i (x_i a)\alpha(y_i)$, for all $a \in A$, and the equation it satisfies is (see Theorem 4.4)

$$R(\alpha\beta(a))R(\alpha\beta(b)) = R(\alpha\beta(a)R(b) + R(a)\alpha\beta(b)), \quad \forall a, b \in A.$$

We call a linear map satisfying this equation an $\alpha\beta$ -Rota-Baxter operator (of weight zero). This is a particular case of the following concept we introduce and study in this paper. Let B be an algebra, $\sigma, \tau : B \rightarrow B$ algebra maps and $R : B \rightarrow B$ a linear map. We call R a $\{\sigma, \tau\}$ -Rota-Baxter operator if

$$R(\sigma(a))R(\tau(b)) = R(\sigma(a)R(b) + R(a)\tau(b)), \quad \forall a, b \in B.$$

This concept is a sort of modification of the concept of (σ, τ) -Rota-Baxter operator introduced in [21] (inspired by an example in [7]). In Section 3 we prove that certain classes of $\{\sigma, \tau\}$ -Rota-Baxter operators have similar properties to those of a usual Rota-Baxter operator of weight zero (see Theorem 3.12 and its corollaries, and Proposition 3.17).

Our second aim is to extend to infinitesimal Hom-bialgebras the following result from [4] providing a left pre-Lie algebra from a given infinitesimal bialgebra.

Theorem 1.1 (Aguiar) *Let (A, μ, Δ) be an infinitesimal bialgebra, with notation $\mu(a \otimes b) = ab$ and $\Delta(a) = a_1 \otimes a_2$, for all $a, b \in A$. If we define a new operation on A by $a \bullet b = b_1 a b_2$, then (A, \bullet) is a left pre-Lie algebra.*

Let (A, μ, Δ, α) be an infinitesimal Hom-bialgebra, with notation $\mu(a \otimes b) = ab$ and $\Delta(a) = a_1 \otimes a_2$, for all $a, b \in A$. We want to define a new multiplication \bullet on A , turning it into a left Hom-pre-Lie algebra. It is not clear what the formula for this multiplication should be (note for instance that the obvious choice $a \bullet b = \alpha(b_1)(a b_2) = (b_1 a)\alpha(b_2)$ does not work), and we need to guess it. We proceed as follows. Recall first the following old result:

Theorem 1.2 (Gel'fand-Dorfman) *Let (A, μ) be an associative and commutative algebra, with notation $\mu(a \otimes b) = ab$, and $D : A \rightarrow A$ a derivation. Define a new multiplication on A by $a \star b = aD(b)$. Then (A, \star) is a left pre-Lie algebra (it is actually even a Novikov algebra).*

We make the following observation: if the infinitesimal bialgebra in Aguier's Theorem is commutative, then his theorem is a particular case of the theorem of Gel'fand and Dorfman. Indeed, by using commutativity, the multiplication \bullet becomes $a \bullet b = b_1 a b_2 = a b_1 b_2 = aD(b)$, where we denoted by D the linear map $D : A \rightarrow A$, $D(b) = b_1 b_2$, i.e. $D = \mu \circ \Delta$, and it is well-known (see [2]) that D is a derivation.

We want to exploit this observation in order to guess the formula for the multiplication in the Hom case. There, we already have an analogue of the Gel'fand-Dorfman Theorem, due to Yau (see [24]), saying that if (A, μ, α) is a commutative Hom-associative algebra and $D : A \rightarrow A$ is a derivation (in the usual sense) commuting with α and we define a new multiplication on A by $a * b = aD(b)$, then $(A, *, \alpha)$ is a left Hom-pre-Lie algebra (it is actually even Hom-Novikov). So, we begin with a commutative infinitesimal Hom-bialgebra (A, μ, Δ, α) and we define the map $D : A \rightarrow A$ also by the formula $D = \mu \circ \Delta$. The problem is that, because of the condition from the definition of an infinitesimal Hom-bialgebra satisfied by Δ (which involves the map α), D is *not* a derivation (so we cannot use Yau's result mentioned above). Instead, it turns out that D is a so-called α^2 -derivation, that is it satisfies $D(ab) = \alpha^2(a)D(b) + D(a)\alpha^2(b)$. So what we need first is a generalization of Yau's version of the Gel'fand-Dorfman Theorem, one that would apply not only to derivations but also to α^2 -derivations. A generalization dealing with

α^k -derivations, for k an arbitrary natural number, is achieved in Proposition 5.1. The outcome is a left Hom-pre-Lie algebra (actually, a Hom-Novikov algebra) whose structure map is α^{k+1} . Coming back to the case $k = 2$, by applying this result we obtain that, for the commutative infinitesimal Hom-bialgebra we started with, we are able to obtain a left Hom-pre-Lie algebra structure on it, with structure map α^3 and multiplication $x \bullet y = \alpha^2(x)D(y) = \alpha^2(x)(y_1y_2)$, which, by using commutativity and Hom-associativity, may be written as $x \bullet y = \alpha(y_1)(\alpha(x)y_2)$.

We can consider this formula even if the infinitesimal Hom-bialgebra is *not* commutative, and it turns out that this is the formula we were trying to guess (see Proposition 5.4).

Let (A, μ, Δ_r) be a quasitriangular infinitesimal bialgebra, i.e. the comultiplication is given by the principal derivation corresponding to a solution $r = \sum_i x_i \otimes y_i$ of the associative Yang-Baxter equation. There are two left pre-Lie algebras associated to A : the first one is obtained by Theorem 1.1, the second is obtained from the fact that the Rota-Baxter operator $R : A \rightarrow A$, $R(a) = \sum_i x_i a y_i$ provides a dendriform algebra, which in turn provides a left pre-Lie algebra. Aguiar proved in [4] that these two left pre-Lie algebras coincide. Our last result shows that the Hom-analogue of this fact is also true.

In a subsequent paper ([15]) we will introduce the BiHom-analogue of infinitesimal bialgebras and prove the BiHom-analogue of Theorem 1.1. It turns out that things are more complicated than in the Hom case, and moreover the result in the Hom case is *not* a particular case of the corresponding result in the BiHom case. This comes essentially from the following phenomenon. A BiHom-associative algebra (A, μ, α, β) for which $\alpha = \beta$ is the same thing as the Hom-associative algebra (A, μ, α) . But a left BiHom-pre-Lie algebra (A, μ, α, β) (as defined in [14]) for which $\alpha = \beta$ is *not* the same thing as the left Hom-pre-Lie algebra (A, μ, α) , unless α is bijective.

2 Preliminaries

We work over a base field \mathbb{k} . All algebras, linear spaces etc. will be over \mathbb{k} ; unadorned \otimes means $\otimes_{\mathbb{k}}$. Unless otherwise specified, the (co)algebras ((co)associative or not) that will appear in what follows are *not* supposed to be (co)unital, a multiplication $\mu : V \otimes V \rightarrow V$ on a linear space V is denoted by $\mu(v \otimes v') = vv'$, and for a comultiplication $\Delta : C \rightarrow C \otimes C$ on a linear space C we use a Sweedler-type notation $\Delta(c) = c_1 \otimes c_2$, for $c \in C$. For the composition of two maps f and g , we will write either $g \circ f$ or simply gf . For the identity map on a linear space V we will use the notation id_V .

Definition 2.1 ([8]) *A BiHom-associative algebra is a 4-tuple (A, μ, α, β) , where A is a linear space, $\alpha, \beta : A \rightarrow A$ and $\mu : A \otimes A \rightarrow A$ are linear maps, such that $\alpha \circ \beta = \beta \circ \alpha$, $\alpha(xy) = \alpha(x)\alpha(y)$, $\beta(xy) = \beta(x)\beta(y)$ and the so-called BiHom-associativity condition*

$$\alpha(x)(yz) = (xy)\beta(z) \tag{2.1}$$

hold, for all $x, y, z \in A$. The maps α and β (in this order) are called the structure maps of A .

A Hom-associative algebra, as defined in [18], is a BiHom-associative algebra (A, μ, α, β) for which $\alpha = \beta$. The defining relation,

$$\alpha(x)(yz) = (xy)\alpha(z), \quad \forall x, y, z \in A, \tag{2.2}$$

is called the Hom-associativity condition and the map α is called the structure map.

If (A, μ) is an associative algebra and $\alpha, \beta : A \rightarrow A$ are two commuting algebra maps, then $A_{(\alpha, \beta)} := (A, \mu \circ (\alpha \otimes \beta), \alpha, \beta)$ is a BiHom-associative algebra, called the Yau twist of A via the maps α and β .

Definition 2.2 ([19]) A Hom-coassociative coalgebra is a triple (C, Δ, α) , in which C is a linear space, $\alpha : C \rightarrow C$ and $\Delta : C \rightarrow C \otimes C$ are linear maps, such that $(\alpha \otimes \alpha) \circ \Delta = \Delta \circ \alpha$ and

$$(\Delta \otimes \alpha) \circ \Delta = (\alpha \otimes \Delta) \circ \Delta. \quad (2.3)$$

The map α is called the structure map and (2.3) is called the Hom-coassociativity condition.

For a Hom-coassociative coalgebra (C, Δ, α) , we will use the extra notation $(id \otimes \Delta)(\Delta(c)) = c_1 \otimes c_{(2,1)} \otimes c_{(2,2)}$ and $(\Delta \otimes id)(\Delta(c)) = c_{(1,1)} \otimes c_{(1,2)} \otimes c_2$, for all $c \in C$.

Definition 2.3 A left pre-Lie algebra is a pair (A, μ) , where A is a linear space and $\mu : A \otimes A \rightarrow A$ is a linear map satisfying the condition

$$x(yz) - (xy)z = y(xz) - (yx)z, \quad \forall x, y, z \in A.$$

A morphism of left pre-Lie algebras from (A, μ) to (A', μ') is a linear map $\alpha : A \rightarrow A'$ satisfying $\alpha(xy) = \alpha(x)\alpha(y)$, for all $x, y \in A$.

Definition 2.4 ([18], [24]) A left Hom-pre-Lie algebra is a triple (A, μ, α) , where A is a linear space and $\mu : A \otimes A \rightarrow A$ and $\alpha : A \rightarrow A$ are linear maps satisfying $\alpha(xy) = \alpha(x)\alpha(y)$ and

$$\alpha(x)(yz) - (xy)\alpha(z) = \alpha(y)(xz) - (yx)\alpha(z), \quad (2.4)$$

for all $x, y, z \in A$. We call α the structure map of A . If moreover the condition

$$(xy)\alpha(z) = (xz)\alpha(y), \quad \forall x, y, z \in A, \quad (2.5)$$

is satisfied, then (A, μ, α) is called a Hom-Novikov algebra.

If (A, μ) is a left pre-Lie algebra and $\alpha : A \rightarrow A$ is a morphism of left pre-Lie algebras, then $A_\alpha := (A, \alpha \circ \mu, \alpha)$ is a left Hom-pre-Lie algebra, called the Yau twist of A via the map α .

Definition 2.5 ([2]) An infinitesimal bialgebra is a triple (A, μ, Δ) , in which (A, μ) is an associative algebra, (A, Δ) is a coassociative coalgebra and $\Delta : A \rightarrow A \otimes A$ is a derivation, that is $\Delta(ab) = ab_1 \otimes b_2 + a_1 \otimes a_2b$, for all $a, b \in A$.

A morphism of infinitesimal bialgebras from (A, μ, Δ) to (A', μ', Δ') is a linear map $\alpha : A \rightarrow A'$ that is a morphism of algebras and a morphism of coalgebras.

Definition 2.6 ([23]) An infinitesimal Hom-bialgebra is a 4-tuple (A, μ, Δ, α) , in which (A, μ, α) is a Hom-associative algebra, (A, Δ, α) is a Hom-coassociative coalgebra and

$$\Delta(ab) = \alpha(a)b_1 \otimes \alpha(b_2) + \alpha(a_1) \otimes a_2\alpha(b), \quad \forall a, b \in A. \quad (2.6)$$

Definition 2.7 ([23]) Let (A, μ, α) be a Hom-associative algebra and $r = \sum_i x_i \otimes y_i \in A \otimes A$ such that $(\alpha \otimes \alpha)(r) = r$. Define the following elements in $A \otimes A \otimes A$:

$$\begin{aligned} r_{12}r_{23} &= \sum_{i,j} \alpha(x_i) \otimes y_i x_j \otimes \alpha(y_j), & r_{13}r_{12} &= \sum_{i,j} x_i x_j \otimes \alpha(y_j) \otimes \alpha(y_i), \\ r_{23}r_{13} &= \sum_{i,j} \alpha(x_i) \otimes \alpha(x_j) \otimes y_j y_i, & A(r) &= r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13}. \end{aligned}$$

We say that r is a solution of the associative Hom-Yang-Baxter equation if $A(r) = 0$, that is

$$\sum_{i,j} \alpha(x_i) \otimes y_i x_j \otimes \alpha(y_j) = \sum_{i,j} x_i x_j \otimes \alpha(y_j) \otimes \alpha(y_i) + \sum_{i,j} \alpha(x_i) \otimes \alpha(x_j) \otimes y_j y_i. \quad (2.7)$$

We introduce the following variation of the concept introduced by Yau in [23]:

Definition 2.8 *An infinitesimal Hom-bialgebra (A, μ, Δ, α) is called quasitriangular if there exists an element $r \in A \otimes A$, $r = \sum_i x_i \otimes y_i$, such that $(\alpha \otimes \alpha)(r) = r$ and r is a solution of the associative Hom-Yang-Baxter equation, with the property that*

$$\Delta(b) = \sum_i \alpha(x_i) \otimes y_i b - \sum_i b x_i \otimes \alpha(y_i), \quad \forall b \in A.$$

In this situation, we denote Δ by Δ_r .

Yau's definition requires $\Delta(b) = \sum_i b x_i \otimes \alpha(y_i) - \sum_i \alpha(x_i) \otimes y_i b$, for all $b \in A$. This is consistent with Aguiar's convention in [2]; our choice is consistent with the convention in [4].

Definition 2.9 ([13]) *A BiHom-dendriform algebra is a 5-tuple $(A, \prec, \succ, \alpha, \beta)$ consisting of a linear space A , linear maps $\prec, \succ: A \otimes A \rightarrow A$ and commuting linear maps $\alpha, \beta: A \rightarrow A$ such that α and β are multiplicative with respect to \prec and \succ and satisfying the conditions*

$$(x \prec y) \prec \beta(z) = \alpha(x) \prec (y \prec z + y \succ z), \quad (2.8)$$

$$(x \succ y) \prec \beta(z) = \alpha(x) \succ (y \prec z), \quad (2.9)$$

$$\alpha(x) \succ (y \succ z) = (x \prec y + x \succ y) \succ \beta(z), \quad (2.10)$$

for all $x, y, z \in A$. We call α and β (in this order) the structure maps of A .

A dendriform algebra, as introduced by Loday in [16], is just a BiHom-dendriform algebra $(A, \prec, \succ, \alpha, \beta)$ for which $\alpha = \beta = id_A$. A Hom-dendriform algebra, as introduced in [17], is a BiHom-dendriform algebra $(A, \prec, \succ, \alpha, \beta)$ for which $\alpha = \beta$.

Let (A, \prec, \succ) be a dendriform algebra and $\alpha, \beta: A \rightarrow A$ two commuting linear maps that are multiplicative with respect to \prec and \succ . Define two new operations on A by $x \prec_{(\alpha, \beta)} y = \alpha(x) \prec \beta(y)$ and $x \succ_{(\alpha, \beta)} y = \alpha(x) \succ \beta(y)$, for all $x, y \in A$. Then $A_{(\alpha, \beta)} := (A, \prec_{(\alpha, \beta)}, \succ_{(\alpha, \beta)}, \alpha, \beta)$ is a BiHom-dendriform algebra, called the Yau twist of A via the maps α and β .

Proposition 2.10 ([17], [20], [13]) *Let $(A, \prec, \succ, \alpha, \beta)$ be a BiHom-dendriform algebra and define a new multiplication on A by $x * y = x \prec y + x \succ y$. Then $(A, *, \alpha, \beta)$ is a BiHom-associative algebra. Moreover, if $\alpha = \beta$ and we define a new operation on A by $x \circ y = x \succ y - y \prec x$, then (A, \circ, α) is a left Hom-pre-Lie algebra.*

3 $\{\sigma, \tau\}$ -Rota-Baxter operators

In this section we introduce and study some classes of modified Rota-Baxter operators which are twisted by algebra maps. We recall first the following well-known concept:

Definition 3.1 *Let A be an algebra, $\sigma, \tau: A \rightarrow A$ algebra maps and $D: A \rightarrow A$ a linear map. We call D a (τ, σ) -derivation if $D(ab) = D(a)\tau(b) + \sigma(a)D(b)$, for all $a, b \in A$.*

The following concept is a variation of the one introduced in [21] for associative algebras.

Definition 3.2 *Let A be an algebra, $\sigma, \tau: A \rightarrow A$ algebra maps and $R: A \rightarrow A$ a linear map. We call R a (σ, τ) -Rota-Baxter operator (of weight zero) if*

$$R(a)R(b) = R(\sigma(R(a))b + a\tau(R(b))), \quad \forall a, b \in A.$$

Remark 3.3 For associative algebras, an (id, τ) -Rota-Baxter operator is the same thing as a τ -twisted Rota-Baxter operator, a concept introduced in [5].

Remark 3.4 Let R be a (σ, τ) -Rota-Baxter operator on an associative algebra A . One can easily check that the triple $(A, \sigma \circ R, \tau \circ R)$ is a Rota-Baxter system, as defined by Brzeziński in [5] (the case $\sigma = id_A$ may be found in [5]). Consequently, by [5], if we define two operations on A by $a \prec b = a\tau(R(b))$ and $a \succ b = \sigma(R(a))b$, then (A, \prec, \succ) is a dendriform algebra.

It is well-know that, if A is an algebra and $D : A \rightarrow A$ is a bijective linear map, then D is a derivation (in the usual sense) if and only D^{-1} is a Rota-Baxter operator of weight zero. This fact may be easily generalized, as follows:

Proposition 3.5 Let A be an algebra, $\sigma, \tau : A \rightarrow A$ algebra maps and $D : A \rightarrow A$ a bijective linear map with inverse $R =: D^{-1}$. Then D is a (τ, σ) -derivation if and only if R is a (σ, τ) -Rota-Baxter operator.

We are interested in the following modification of the concept of (σ, τ) -Rota-Baxter operator.

Definition 3.6 Let A be an algebra, $\sigma, \tau : A \rightarrow A$ algebra maps and $R : A \rightarrow A$ a linear map. We call R a $\{\sigma, \tau\}$ -Rota-Baxter operator (of weight zero) if

$$R(\sigma(a))R(\tau(b)) = R(\sigma(a)R(b) + R(a)\tau(b)), \quad \forall a, b \in A. \quad (3.1)$$

Remark 3.7 Let A be an algebra, $\sigma, \tau : A \rightarrow A$ bijective algebra maps and $R : A \rightarrow A$ a linear map commuting with σ and τ . Then one can easily see that R is a (σ, τ) -Rota-Baxter operator if and only if R is a $\{\sigma^{-1}, \tau^{-1}\}$ -Rota-Baxter operator.

Remark 3.8 Let (A, μ) be an algebra, $\sigma : A \rightarrow A$ an algebra map and $R : A \rightarrow A$ a Rota-Baxter operator of weight zero commuting with σ . Then one can easily see that $R \circ \sigma$ is a $\{\sigma, \sigma\}$ -Rota-Baxter operator both for (A, μ) and for $(A, \sigma \circ \mu)$.

We will be particularly interested in the following two classes of $\{\sigma, \tau\}$ -Rota-Baxter operators.

Definition 3.9 Let A be an algebra, $\alpha : A \rightarrow A$ an algebra map, $R : A \rightarrow A$ a linear map commuting with α and n a natural number. We call R an α^n -Rota-Baxter operator if it is an $\{\alpha^n, \alpha^n\}$ -Rota-Baxter operator, i.e.

$$R(\alpha^n(a))R(\alpha^n(b)) = R(\alpha^n(a)R(b) + R(a)\alpha^n(b)), \quad \forall a, b \in A. \quad (3.2)$$

Obviously, an α^0 -Rota-Baxter operator is just a usual Rota-Baxter operator of weight zero commuting with α .

Remark 3.10 From previous remarks it follows that, if A is an algebra, $\alpha : A \rightarrow A$ a bijective algebra map, $D : A \rightarrow A$ a bijective linear map commuting with α and n a natural number, then $R := D^{-1}$ is an α^n -Rota-Baxter operator if and only if D is an $(\alpha^{-n}, \alpha^{-n})$ -derivation.

Definition 3.11 Let (A, μ, α, β) be a BiHom-associative algebra and $R : A \rightarrow A$ a linear map commuting with α and β . We call R an $\alpha\beta$ -Rota-Baxter operator if it is an $\{\alpha\beta, \alpha\beta\}$ -Rota-Baxter operator, that is

$$R(\alpha\beta(a))R(\alpha\beta(b)) = R(\alpha\beta(a)R(b) + R(a)\alpha\beta(b)), \quad \forall a, b \in A. \quad (3.3)$$

Theorem 3.12 *Let (A, μ, α, β) be a BiHom-associative algebra and $\sigma, \tau, \eta, R : A \rightarrow A$ linear maps such that σ, τ, η are algebra maps, R is a $\{\sigma, \tau\}$ -Rota-Baxter operator and any two of the maps $\alpha, \beta, \sigma, \tau, \eta, R$ commute. Define new operations on A by*

$$x \prec y = \sigma(x)R\eta(y) \quad \text{and} \quad x \succ y = R(x)\tau\eta(y),$$

for all $x, y \in A$. Then $(A, \prec, \succ, \alpha\sigma, \beta\tau\eta)$ is a BiHom-dendriform algebra.

Proof. One can see that $\alpha\sigma$ and $\beta\tau\eta$ are multiplicative with respect to \prec and \succ . We compute:

$$\begin{aligned} (x \prec y) \prec \beta\tau\eta(z) &= (\sigma(x)R\eta(y)) \prec \beta\tau\eta(z) = \sigma(\sigma(x)R\eta(y))R\beta\tau\eta^2(z) \\ &= (\sigma^2(x)\sigma R\eta(y))\beta R\tau\eta^2(z) \\ &\stackrel{(2.1)}{=} \alpha\sigma^2(x)(R\sigma\eta(y)R\tau\eta^2(z)) \\ &\stackrel{(3.1)}{=} \alpha\sigma^2(x)R(\sigma\eta(y)R\eta^2(z) + R\eta(y)\tau\eta^2(z)) \\ &= \alpha\sigma^2(x)R\eta(\sigma(y)R\eta(z) + R(y)\tau\eta(z)) \\ &= \alpha\sigma(x) \prec (\sigma(y)R\eta(z) + R(y)\tau\eta(z)) \\ &= \alpha\sigma(x) \prec (y \prec z + y \succ z). \end{aligned}$$

Then we compute:

$$\begin{aligned} (x \succ y) \prec \beta\tau\eta(z) &= (R(x)\tau\eta(y)) \prec \beta\tau\eta(z) = \sigma(R(x)\tau\eta(y))R\beta\tau\eta^2(z) \\ &= (\sigma R(x)\sigma\tau\eta(y))\beta R\tau\eta^2(z) \\ &\stackrel{(2.1)}{=} \alpha\sigma R(x)(\sigma\tau\eta(y)R\tau\eta^2(z)) = R\alpha\sigma(x)\tau\eta(\sigma(y)R\eta(z)) \\ &= \alpha\sigma(x) \succ (\sigma(y)R\eta(z)) = \alpha\sigma(x) \succ (y \prec z). \end{aligned}$$

Also, by using again (3.1), one proves that $\alpha\sigma(x) \succ (y \succ z) = (x \prec y + x \succ y) \succ \beta\tau\eta(z)$, finishing the proof. \square

We have some particular cases of this theorem.

Corollary 3.13 *Let A be an associative algebra, $\sigma, \tau : A \rightarrow A$ two commuting algebra maps and $R : A \rightarrow A$ a $\{\sigma, \tau\}$ -Rota-Baxter operator commuting with σ and τ . Define new operations on A by $x \prec y = \sigma(x)R(y)$ and $x \succ y = R(x)\tau(y)$, for $x, y \in A$. Then $(A, \prec, \succ, \sigma, \tau)$ is a BiHom-dendriform algebra. Moreover, if we consider $(A, *, \sigma, \tau)$ the BiHom-associative algebra associated to it as in Proposition 2.10, then R is a morphism of BiHom-associative algebras from $(A, *, \sigma, \tau)$ to $A_{(\sigma, \tau)}$, the Yau twist of the associative algebra A via the maps σ and τ .*

Proof. Take in Theorem 3.12 $\alpha = \beta = \eta = id_A$. The second statement is obvious. \square

Remark 3.14 *Assume that we are in the hypotheses of Corollary 3.13 and moreover σ and τ are bijective; denote $\alpha = \sigma^{-1}$, $\beta = \tau^{-1}$. By Remark 3.7, R is an (α, β) -Rota-Baxter operator, so, by Remark 3.4, A becomes a dendriform algebra with operations $a \prec b = \alpha\tau^{-1}(R(b))$ and $a \succ b = \sigma^{-1}(R(a))b$. One can check that the Yau twist of this dendriform algebra via the maps σ and τ is exactly the BiHom-dendriform algebra obtained in Corollary 3.13.*

Corollary 3.15 *Let (A, μ, α) be a Hom-associative algebra, n a natural number and $R : A \rightarrow A$ an α^n -Rota-Baxter operator. Define new operations on A by $x \prec y = \alpha^n(x)R(y)$ and $x \succ y =$*

$R(x)\alpha^n(y)$, for all $x, y \in A$. Then $(A, \prec, \succ, \alpha^{n+1})$ is a Hom-dendriform algebra. Consequently, by Proposition 2.10, if we define new operations on A by

$$\begin{aligned} x * y &= x \prec y + x \succ y = \alpha^n(x)R(y) + R(x)\alpha^n(y), \\ x \circ y &= x \succ y - y \prec x = R(x)\alpha^n(y) - \alpha^n(y)R(x), \end{aligned}$$

then $(A, *, \alpha^{n+1})$ is a Hom-associative algebra and (A, \circ, α^{n+1}) is a left Hom-pre-Lie algebra.

Proof. Take in Theorem 3.12 $\alpha = \beta$, $\sigma = \tau = \alpha^n$, $\eta = id_A$. \square

Corollary 3.16 *Let (A, μ, α, β) be a BiHom-associative algebra and $R : A \rightarrow A$ an $\alpha\beta$ -Rota-Baxter operator. Let $\eta : A \rightarrow A$ be an algebra map commuting with α , β and R . Define new operations on A by $x \prec y = \alpha\beta(x)R\eta(y)$ and $x \succ y = R(x)\alpha\beta\eta(y)$, for all $x, y \in A$. Then $(A, \prec, \succ, \alpha^2\beta, \alpha\beta^2\eta)$ is a BiHom-dendriform algebra.*

Proof. Take in Theorem 3.12 $\sigma = \tau = \alpha\beta$. \square

We recall from [10] that a Hom-Lie algebra is a triple $(L, [\cdot, \cdot], \alpha)$ in which L is a linear space, $\alpha : L \rightarrow L$ is a linear map and $[\cdot, \cdot] : L \times L \rightarrow L$ is a bilinear map, such that, for all $x, y, z \in L$:

$$\begin{aligned} \alpha([x, y]) &= [\alpha(x), \alpha(y)], \\ [x, y] &= -[y, x], \quad (\text{skew-symmetry}) \\ [\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] &= 0. \quad (\text{Hom-Jacobi condition}) \end{aligned}$$

Proposition 3.17 *Let $(L, [\cdot, \cdot], \alpha)$ be a Hom-Lie algebra and $R : L \rightarrow L$ an α^n -Rota-Baxter operator, i.e. R commutes with α and*

$$[R(\alpha^n(a)), R(\alpha^n(b))] = R([\alpha^n(a), R(b)] + [R(a), \alpha^n(b)]), \quad \forall a, b \in L. \quad (3.4)$$

Then (L, \cdot, α^{n+1}) is a left Hom-pre-Lie algebra, where $a \cdot b = [R(a), \alpha^n(b)]$, for all $a, b \in L$.

Proof. Obviously, we have $\alpha^{n+1}(a \cdot b) = \alpha^{n+1}(a) \cdot \alpha^{n+1}(b)$, for all $a, b \in A$. Note that the Hom-Jacobi identity together with the skew-symmetry of the bracket $[\cdot, \cdot]$ imply

$$[\alpha(a), [b, c]] = [[a, b], \alpha(c)] + [\alpha(b), [a, c]], \quad \forall a, b, c \in A. \quad (3.5)$$

Now for $x, y, z \in A$ we compute:

$$\begin{aligned} &\alpha^{n+1}(x) \cdot (y \cdot z) - (x \cdot y) \cdot \alpha^{n+1}(z) \\ &= \alpha^{n+1}(x) \cdot [R(y), \alpha^n(z)] - [R(x), \alpha^n(y)] \cdot \alpha^{n+1}(z) \\ &= [R(\alpha^{n+1}(x)), [\alpha^n(R(y)), \alpha^{2n}(z)]] - [R([R(x), \alpha^n(y)]), \alpha^{2n+1}(z)] \\ (3.4) \quad &= [R(\alpha^{n+1}(x)), [\alpha^n(R(y)), \alpha^{2n}(z)]] - [[R(\alpha^n(x)), R(\alpha^n(y))], \alpha^{2n+1}(z)] \\ &\quad + [R([\alpha^n(x), R(y)]), \alpha^{2n+1}(z)] \\ &= [R(\alpha^{n+1}(x)), [R(\alpha^n(y)), \alpha^{2n}(z)]] - [[\alpha^n(R(x)), R(\alpha^n(y))], \alpha^{2n+1}(z)] \\ &\quad + [R([\alpha^n(x), R(y)]), \alpha^{2n+1}(z)] \\ (3.5) \quad &= [R(\alpha^{n+1}(x)), [R(\alpha^n(y)), \alpha^{2n}(z)]] - [\alpha^{n+1}(R(x)), [R(\alpha^n(y)), \alpha^{2n}(z)]] \end{aligned}$$

$$\begin{aligned}
& +[\alpha^{n+1}(R(y)), [\alpha^n(R(x)), \alpha^{2n}(z)]] + [R([\alpha^n(x), R(y)]), \alpha^{2n+1}(z)] \\
\stackrel{\text{skew-symmetry}}{=} & [\alpha^{n+1}(R(y)), [\alpha^n(R(x)), \alpha^{2n}(z)]] - [R([R(y), \alpha^n(x)]), \alpha^{2n+1}(z)] \\
= & \alpha^{n+1}(y) \cdot (x \cdot z) - (y \cdot x) \cdot \alpha^{n+1}(z),
\end{aligned}$$

finishing the proof. \square

4 The associative BiHom-Yang-Baxter equation

In this section we introduce the associative BiHom-Yang-Baxter equation, generalizing the associative Yang-Baxter equation introduced by Aguiar as well as the associative Hom-Yang-Baxter equation introduced by Yau. Moreover, we discuss its connection with the generalized Rota-Baxter operators introduced in Section 3.

Definition 4.1 *Let (A, μ, α, β) be a BiHom-associative algebra and $r = \sum_i x_i \otimes y_i \in A \otimes A$ such that $(\alpha \otimes \alpha)(r) = r = (\beta \otimes \beta)(r)$. We define the following elements in $A \otimes A \otimes A$:*

$$\begin{aligned}
r_{12}r_{23} &= \sum_{i,j} \alpha(x_i) \otimes y_i x_j \otimes \beta(y_j), & r_{13}r_{12} &= \sum_{i,j} x_i x_j \otimes \beta(y_j) \otimes \beta(y_i), \\
r_{23}r_{13} &= \sum_{i,j} \alpha(x_i) \otimes \alpha(x_j) \otimes y_j y_i, & A(r) &= r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13}.
\end{aligned}$$

We say that r is a solution of the associative BiHom-Yang-Baxter equation if $A(r) = 0$, i.e.

$$\sum_{i,j} \alpha(x_i) \otimes y_i x_j \otimes \beta(y_j) = \sum_{i,j} x_i x_j \otimes \beta(y_j) \otimes \beta(y_i) + \sum_{i,j} \alpha(x_i) \otimes \alpha(x_j) \otimes y_j y_i. \quad (4.1)$$

Remark 4.2 *Obviously, for $\alpha = \beta$ the associative BiHom-Yang-Baxter equation reduces to the associative Hom-Yang-Baxter equation (2.7).*

Remark 4.3 *Assume that the BiHom-associative algebra A in the previous definition has a unit, that is (see [8]) an element $1_A \in A$ satisfying the conditions $\alpha(1_A) = \beta(1_A) = 1_A$, $a1_A = \alpha(a)$ and $1_A a = \beta(a)$, for all $a \in A$. Then, by using the unit 1_A , one can define the elements $r_{12}, r_{13}, r_{23} \in A \otimes A \otimes A$ by $r_{12} = \sum_i x_i \otimes y_i \otimes 1_A$, $r_{13} = \sum_i x_i \otimes 1_A \otimes y_i$ and $r_{23} = \sum_i 1_A \otimes x_i \otimes y_i$. Then, the element $r_{12}r_{23}$ defined above is just the product between r_{12} and r_{23} in $A \otimes A \otimes A$, but the element $r_{13}r_{12}$ is **not** the product between r_{13} and r_{12} (which is $\sum_{i,j} x_i x_j \otimes \beta(y_j) \otimes \alpha(y_i)$), and similarly the element $r_{23}r_{13}$ is **not** the product between r_{23} and r_{13} .*

Theorem 4.4 *Let (A, μ, α, β) be a BiHom-associative algebra and $r = \sum_i x_i \otimes y_i \in A \otimes A$ such that $(\alpha \otimes \alpha)(r) = r = (\beta \otimes \beta)(r)$ and r is a solution of the associative BiHom-Yang-Baxter equation. Define the linear map*

$$R : A \rightarrow A, \quad R(a) = \sum_i \alpha\beta^3(x_i)(a\alpha^3(y_i)) = \sum_i (\beta^3(x_i)a)\alpha^3\beta(y_i), \quad \forall a \in A. \quad (4.2)$$

Then R is an $\alpha\beta$ -Rota-Baxter operator.

Proof. The fact that R commutes with α and β follows immediately from the fact that $(\alpha \otimes \alpha)(r) = r = (\beta \otimes \beta)(r)$. Now we compute, for $a, b \in A$:

$$R(\alpha\beta(a))R(\alpha\beta(b)) = \left\{ \sum_i (\beta^3(x_i)\alpha\beta(a))\alpha^3\beta(y_i) \right\} \left\{ \sum_j \alpha\beta^3(x_j)(\alpha\beta(b)\alpha^3(y_j)) \right\}$$

$$\begin{aligned}
& \stackrel{(\alpha \otimes \alpha)(r)=r}{=} \left\{ \sum_i (\alpha \beta^3(x_i) \alpha \beta(a)) \alpha^4 \beta(y_i) \right\} \left\{ \sum_j \alpha \beta^3(x_j) (\alpha \beta(b) \alpha^3(y_j)) \right\} \\
& \stackrel{(2.1)}{=} \sum_{i,j} \{ (\beta^3(x_i) \beta(a)) \alpha^3 \beta(y_i) \} \alpha \beta^3(x_j) \{ \alpha \beta^2(b) \alpha^3 \beta(y_j) \} \\
& \stackrel{(2.1)}{=} \sum_{i,j} \{ (\alpha \beta^3(x_i) \alpha \beta(a)) (\alpha^3 \beta(y_i) \alpha \beta^2(x_j)) \} \{ \alpha \beta^2(b) \alpha^3 \beta(y_j) \} \\
& \stackrel{(\beta \otimes \beta)(r)=r}{=} \sum_{i,j} \{ (\alpha \beta^4(x_i) \alpha \beta(a)) (\alpha^3 \beta^2(y_i) \alpha \beta^2(x_j)) \} \{ \alpha \beta^2(b) \alpha^3 \beta(y_j) \} \\
& \stackrel{(\alpha^2 \otimes \alpha^2)(r)=r}{=} \sum_{i,j} \{ (\alpha \beta^4(x_i) \alpha \beta(a)) \alpha^3 \beta^2(y_i x_j) \} \{ \alpha \beta^2(b) \alpha^5 \beta(y_j) \} \\
& \stackrel{(4.1)}{=} \sum_{i,j} \{ (\beta^4(x_i x_j) \alpha \beta(a)) \alpha^3 \beta^3(y_j) \} \{ \alpha \beta^2(b) \alpha^5 \beta(y_i) \} \\
& \quad + \sum_{i,j} \{ (\alpha \beta^4(x_i) \alpha \beta(a)) \alpha^4 \beta^2(x_j) \} \{ \alpha \beta^2(b) \alpha^5 (y_j y_i) \} \\
& \stackrel{(2.1)}{=} \sum_{i,j} \{ \alpha \beta^4(x_i x_j) (\alpha \beta(a) \alpha^3 \beta^2(y_j)) \} \{ \alpha \beta^2(b) \alpha^5 \beta(y_i) \} \\
& \quad + \sum_{i,j} \{ (\alpha \beta^4(x_i) \alpha \beta(a)) \alpha^4 \beta^2(x_j) \} \{ \alpha \beta^2(b) \alpha^5 (y_j y_i) \} \\
& = \sum_{i,j} \{ (\beta^3(x_i) \beta^2(x_j)) (\alpha \beta(a) \alpha^2(y_j)) \} \{ \alpha \beta^2(b) \alpha^4(y_i) \} \\
& \quad + \sum_{i,j} \{ (\beta^4(x_i) \alpha \beta(a)) \beta^2(x_j) \} \{ \alpha \beta^2(b) (\alpha(y_j) \alpha^4(y_i)) \},
\end{aligned}$$

where for the last equality we used the identities $\sum_i \alpha \beta^4(x_i) \otimes \alpha^5 \beta(y_i) = \sum_i \beta^3(x_i) \otimes \alpha^4(y_i)$ and $\sum_j \alpha \beta^4(x_j) \otimes \alpha^3 \beta^2(y_j) = \sum_j \beta^2(x_j) \otimes \alpha^2(y_j)$, for the first term, and $\sum_i \alpha \beta^4(x_i) \otimes \alpha^5(y_i) = \sum_i \beta^4(x_i) \otimes \alpha^4(y_i)$ and $\sum_j \alpha^4 \beta^2(x_j) \otimes \alpha^5(y_j) = \sum_j \beta^2(x_j) \otimes \alpha(y_j)$, for the second term, identities that are consequences of the relation $(\alpha \otimes \alpha)(r) = r = (\beta \otimes \beta)(r)$. On the other hand, we have:

$$\begin{aligned}
& R(R(a) \alpha \beta(b) + \alpha \beta(a) R(b)) \\
& = R\left(\sum_j \{ \alpha \beta^3(x_j) (\alpha \alpha^3(y_j)) \} \alpha \beta(b) + R(\alpha \beta(a) \{ \sum_j \alpha \beta^3(x_j) (b \alpha^3(y_j)) \})\right) \\
& = \sum_{i,j} \alpha \beta^3(x_i) \{ (\alpha \beta^3(x_j) (\alpha \alpha^3(y_j))) \alpha \beta(b) \} \alpha^3(y_i) \\
& \quad + \sum_{i,j} \alpha \beta^3(x_i) \{ \{ \alpha \beta(a) (\alpha \beta^3(x_j) (b \alpha^3(y_j))) \} \} \alpha^3(y_i) \\
& \stackrel{(\beta \otimes \beta)(r)=r}{=} \sum_{i,j} \alpha \beta^4(x_i) \{ (\alpha \beta^3(x_j) (\alpha \alpha^3(y_j))) \alpha \beta(b) \} \alpha^3 \beta(y_i) \\
& \quad + \sum_{i,j} \alpha \beta^3(x_i) \{ \{ \alpha \beta(a) (\alpha \beta^3(x_j) (b \alpha^3(y_j))) \} \} \alpha^3(y_i) \\
& \stackrel{(2.1)}{=} \sum_{i,j} \alpha \beta^4(x_i) \{ \{ \alpha^2 \beta^3(x_j) (\alpha(a) \alpha^4(y_j)) \} \} \{ \alpha \beta(b) \alpha^3(y_i) \}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{i,j} \alpha \beta^3(x_i) \{ \{ (\beta(a) \alpha \beta^3(x_j)) (\beta(b) \alpha^3 \beta(y_j)) \} \alpha^3(y_i) \} \\
\stackrel{(2.1)}{=} & \sum_{i,j} \{ \beta^4(x_i) \{ \alpha^2 \beta^3(x_j) (\alpha(a) \alpha^4(y_j)) \} \} \{ \alpha \beta^2(b) \alpha^3 \beta(y_i) \} \\
& + \sum_{i,j} \alpha \beta^3(x_i) \{ \{ (\beta(a) \alpha \beta^3(x_j)) (\beta(b) \alpha^3 \beta(y_j)) \} \alpha^3(y_i) \} \\
\stackrel{(\beta \otimes \beta)(r)=r}{=} & \sum_{i,j} \{ \beta^4(x_i) \{ \alpha^2 \beta^3(x_j) (\alpha(a) \alpha^4(y_j)) \} \} \{ \alpha \beta^2(b) \alpha^3 \beta(y_i) \} \\
& + \sum_{i,j} \alpha \beta^4(x_i) \{ \{ (\beta(a) \alpha \beta^3(x_j)) (\beta(b) \alpha^3 \beta(y_j)) \} \alpha^3 \beta(y_i) \} \\
\stackrel{(2.1)}{=} & \sum_{i,j} \{ \beta^4(x_i) \{ \alpha^2 \beta^3(x_j) (\alpha(a) \alpha^4(y_j)) \} \} \{ \alpha \beta^2(b) \alpha^3 \beta(y_i) \} \\
& + \sum_{i,j} \alpha \beta^4(x_i) \{ (\alpha \beta(a) \alpha^2 \beta^3(x_j)) \{ (\beta(b) \alpha^3 \beta(y_j)) \alpha^3(y_i) \} \} \\
\stackrel{(\alpha \otimes \alpha)(r)=r}{=} & \sum_{i,j} \{ \alpha \beta^4(x_i) \{ \alpha^2 \beta^3(x_j) (\alpha(a) \alpha^4(y_j)) \} \} \{ \alpha \beta^2(b) \alpha^4 \beta(y_i) \} \\
& + \sum_{i,j} \alpha \beta^4(x_i) \{ (\alpha \beta(a) \alpha^2 \beta^3(x_j)) \{ (\beta(b) \alpha^3 \beta(y_j)) \alpha^3(y_i) \} \} \\
\stackrel{(2.1)}{=} & \sum_{i,j} \{ (\beta^4(x_i) \alpha^2 \beta^3(x_j)) (\alpha \beta(a) \alpha^4 \beta(y_j)) \} \{ \alpha \beta^2(b) \alpha^4 \beta(y_i) \} \\
& + \sum_{i,j} \{ \beta^4(x_i) (\alpha \beta(a) \alpha^2 \beta^3(x_j)) \} \{ (\beta^2(b) \alpha^3 \beta^2(y_j)) \alpha^3 \beta(y_i) \} \\
\stackrel{(\alpha \otimes \alpha)(r)=r}{=} & \sum_{i,j} \{ (\beta^4(x_i) \alpha^2 \beta^3(x_j)) (\alpha \beta(a) \alpha^4 \beta(y_j)) \} \{ \alpha \beta^2(b) \alpha^4 \beta(y_i) \} \\
& + \sum_{i,j} \{ \alpha \beta^4(x_i) (\alpha \beta(a) \alpha^2 \beta^3(x_j)) \} \{ (\beta^2(b) \alpha^3 \beta^2(y_j)) \alpha^4 \beta(y_i) \} \\
\stackrel{(2.1)}{=} & \sum_{i,j} \{ (\beta^4(x_i) \alpha^2 \beta^3(x_j)) (\alpha \beta(a) \alpha^4 \beta(y_j)) \} \{ \alpha \beta^2(b) \alpha^4 \beta(y_i) \} \\
& + \sum_{i,j} \{ (\beta^4(x_i) \alpha \beta(a)) \alpha^2 \beta^4(x_j) \} \{ \alpha \beta^2(b) (\alpha^3 \beta^2(y_j) \alpha^4(y_i)) \}.
\end{aligned}$$

By using the identities $\sum_i \beta^4(x_i) \otimes \alpha^4 \beta(y_i) = \sum_i \beta^3(x_i) \otimes \alpha^4(y_i)$ and $\sum_j \alpha^2 \beta^3(x_j) \otimes \alpha^4 \beta(y_j) = \sum_j \beta^2(x_j) \otimes \alpha^2(y_j)$, for the first term, and $\sum_j \alpha^2 \beta^4(x_j) \otimes \alpha^3 \beta^2(y_j) = \sum_j \beta^2(x_j) \otimes \alpha(y_j)$, for the second term, identities that are consequences of the relation $(\alpha \otimes \alpha)(r) = r = (\beta \otimes \beta)(r)$, the final expression becomes

$$\begin{aligned}
& \sum_{i,j} \{ (\beta^3(x_i) \beta^2(x_j)) (\alpha \beta(a) \alpha^2(y_j)) \} \{ \alpha \beta^2(b) \alpha^4(y_i) \} \\
& + \sum_{i,j} \{ (\beta^4(x_i) \alpha \beta(a)) \beta^2(x_j) \} \{ \alpha \beta^2(b) (\alpha(y_j) \alpha^4(y_i)) \},
\end{aligned}$$

and this coincides with the expression we obtained for $R(\alpha\beta(a))R(\alpha\beta(b))$. \square

Corollary 4.5 *Let (A, μ, α) be a Hom-associative algebra and $r = \sum_i x_i \otimes y_i \in A \otimes A$ such that $(\alpha \otimes \alpha)(r) = r$ and r is a solution of the associative Hom-Yang-Baxter equation. Define $R : A \rightarrow A$, $R(a) = \sum_i \alpha(x_i)(ay_i) = \sum_i (x_i a)\alpha(y_i)$. Then R is an α^2 -Rota-Baxter operator.*

Proof. Take $\alpha = \beta$ in the previous theorem and note that, for $\alpha = \beta$, since $(\alpha \otimes \alpha)(r) = r$, the formula (4.2) becomes $R(a) = \sum_i \alpha(x_i)(ay_i) = \sum_i (x_i a)\alpha(y_i)$. \square

5 Hom-pre-Lie algebras from infinitesimal Hom-bialgebras

In this section we derive Hom-pre-Lie algebras from infinitesimal Hom-bialgebras, generalizing Aguiar's result in the classical case.

Proposition 5.1 *Let (A, μ, α) be a commutative Hom-associative algebra, k a natural number and $D : A \rightarrow A$ an α^k -derivation, that is D is a linear map commuting with α and*

$$D(ab) = D(a)\alpha^k(b) + \alpha^k(a)D(b), \quad \forall a, b \in A. \quad (5.1)$$

Define a new operation on A by

$$x \bullet y = \alpha^k(x)D(y), \quad \forall x, y \in A. \quad (5.2)$$

Then $(A, \bullet, \alpha^{k+1})$ is a Hom-Novikov algebra.

Proof. Since D commutes with α , it is obvious that $\alpha^{k+1}(x \bullet y) = \alpha^{k+1}(x) \bullet \alpha^{k+1}(y)$, for all $x, y \in A$. Now we compute:

$$\begin{aligned} & \alpha^{k+1}(x) \bullet (y \bullet z) - (x \bullet y) \bullet \alpha^{k+1}(z) \\ &= \alpha^{k+1}(x) \bullet (\alpha^k(y)D(z)) - (\alpha^k(x)D(y)) \bullet \alpha^{k+1}(z) \\ &= \alpha^{2k+1}(x)D(\alpha^k(y)D(z)) - \alpha^k(\alpha^k(x)D(y))D(\alpha^{k+1}(z)) \\ &\stackrel{(5.1)}{=} \alpha^{2k+1}(x)(D(\alpha^k(y))\alpha^k(D(z)) + \alpha^{2k}(y)D^2(z)) \\ &\quad - (\alpha^{2k}(x)\alpha^k(D(y)))\alpha^{k+1}(D(z)) \\ &\stackrel{(2.2)}{=} \alpha^{2k+1}(x)(\alpha^k(D(y))\alpha^k(D(z))) + \alpha^{2k+1}(x)(\alpha^{2k}(y)D^2(z)) \\ &\quad - \alpha^{2k+1}(x)(\alpha^k(D(y))\alpha^k(D(z))) \\ &\stackrel{(2.2)}{=} (\alpha^{2k}(x)\alpha^{2k}(y))\alpha(D^2(z)) = \alpha^{2k}(xy)\alpha(D^2(z)), \end{aligned}$$

and since $xy = yx$, this expression is obviously symmetric in x and y , so $(A, \bullet, \alpha^{k+1})$ is a left Hom-pre-Lie algebra. Now we compute:

$$\begin{aligned} (x \bullet y) \bullet \alpha^{k+1}(z) &= (\alpha^k(x)D(y)) \bullet \alpha^{k+1}(z) = \alpha^k(\alpha^k(x)D(y))D(\alpha^{k+1}(z)) \\ &= (\alpha^{2k}(x)\alpha^k(D(y)))\alpha^{k+1}(D(z)) \\ &\stackrel{(2.2)}{=} \alpha^{2k+1}(x)(\alpha^k(D(y))\alpha^k(D(z))) = \alpha^{2k+1}(x)\alpha^k(D(y)D(z)) \\ &\stackrel{\text{commutativity}}{=} \alpha^{2k+1}(x)\alpha^k(D(z)D(y)) = (x \bullet z) \bullet \alpha^{k+1}(y). \end{aligned}$$

So indeed $(A, \bullet, \alpha^{k+1})$ is a Hom-Novikov algebra. \square

By taking $k = 0$ in the Proposition, we obtain:

Corollary 5.2 ([24]) *Let (A, μ, α) be a commutative Hom-associative algebra and $D : A \rightarrow A$ a derivation (in the usual sense) commuting with α . Define a new operation on A by $x \bullet y = xD(y)$, for all $x, y \in A$. Then (A, \bullet, α) is a Hom-Novikov algebra.*

Proposition 5.3 *Let (A, μ, Δ, α) be an infinitesimal Hom-bialgebra. Define the linear map $D : A \rightarrow A$, $D(a) = a_1 a_2$ for all $a \in A$, i.e. $D = \mu \circ \Delta$. Then D is an α^2 -derivation.*

Proof. Obviously D commutes with α and, for all $a, b \in A$, we have

$$\begin{aligned} D(ab) &\stackrel{(2.6)}{=} (\alpha(a)b_1)\alpha(b_2) + \alpha(a_1)(a_2\alpha(b)) \\ &\stackrel{(2.2)}{=} \alpha^2(a)(b_1 b_2) + (a_1 a_2)\alpha^2(b) = \alpha^2(a)D(b) + D(a)\alpha^2(b), \end{aligned}$$

finishing the proof. \square

Let now (A, μ, Δ, α) be a commutative infinitesimal Hom-bialgebra. By using Propositions 5.3 and 5.1, we obtain a Hom-Novikov algebra (A, \bullet, α^3) , where

$$\begin{aligned} x \bullet y &\stackrel{(5.2)}{=} \alpha^2(x)D(y) = \alpha^2(x)(y_1 y_2) \\ &\stackrel{(2.2)}{=} (\alpha(x)y_1)\alpha(y_2) \\ &\stackrel{\text{commutativity}}{=} (y_1\alpha(x))\alpha(y_2) \\ &\stackrel{(2.2)}{=} \alpha(y_1)(\alpha(x)y_2). \end{aligned}$$

Inspired by this, now we have:

Proposition 5.4 *Let (A, μ, Δ, α) be an infinitesimal Hom-bialgebra, and define a new multiplication on A by*

$$x \bullet y = \alpha(y_1)(\alpha(x)y_2) = (y_1\alpha(x))\alpha(y_2), \quad \forall x, y \in A. \quad (5.3)$$

Then (A, \bullet, α^3) is a left Hom-pre-Lie algebra.

Proof. Since $(\alpha \otimes \alpha) \circ \Delta = \Delta \circ \alpha$, it is easy to see that $\alpha^3(x \bullet y) = \alpha^3(x) \bullet \alpha^3(y)$, for all $x, y \in A$. Now, for all $x, y, z \in A$ we compute:

$$\begin{aligned} &\alpha^3(x) \bullet (y \bullet z) - (x \bullet y) \bullet \alpha^3(z) \\ &= \alpha^3(x) \bullet (\alpha(z_1)(\alpha(y)z_2)) - (\alpha(y_1)(\alpha(x)y_2)) \bullet \alpha^3(z) \\ &= \alpha([\alpha(z_1)(\alpha(y)z_2)]_1)(\alpha^4(x)[\alpha(z_1)(\alpha(y)z_2)]_2) \\ &\quad - \alpha^4(z_1)\{[\alpha^2(y_1)(\alpha^2(x)\alpha(y_2))]\alpha^3(z_2)\} \\ &\stackrel{(2.6)}{=} \text{twice} \alpha(\alpha^2(z_1)(\alpha^2(y)z_{(2,1)}))[\alpha^4(x)\alpha^2(z_{(2,2)})] + \alpha(\alpha^2(z_1)\alpha^2(y_1))[\alpha^4(x)(\alpha^2(y_2)\alpha^2(z_2))] \\ &\quad + \alpha^3(z_{(1,1)})[\alpha^4(x)(\alpha(z_{(1,2)})\alpha(\alpha(y)z_2))] - \alpha^4(z_1)\{[\alpha^2(y_1)(\alpha^2(x)\alpha(y_2))]\alpha^3(z_2)\} \\ &= \alpha([\alpha^2(z_1)(\alpha^2(y)z_{(2,1)})]\alpha(\alpha^2(x)z_{(2,2)})) + \alpha^2(\alpha(z_1 y_1)[\alpha^2(x)(y_2 z_2)]) \\ &\quad + \alpha(\alpha^2(z_{(1,1)})[\alpha^3(x)(z_{(1,2)}(\alpha(y)z_2)])) - \alpha(\alpha^3(z_1)\{[\alpha(y_1)(\alpha(x)y_2)]\alpha^2(z_2)\}). \end{aligned}$$

We claim that the second and fourth terms in this expression cancel each other. To show this, it is enough to prove that $\alpha^2(z_1 y_1)[\alpha^3(x)(\alpha(y_2)\alpha(z_2))] = \alpha^3(z_1)\{[\alpha(y_1)(\alpha(x)y_2)]\alpha^2(z_2)\}$. We compute, by applying repeatedly the Hom-associativity condition:

$$\alpha^3(z_1)\{[\alpha(y_1)(\alpha(x)y_2)]\alpha^2(z_2)\} = \alpha^3(z_1)\{\alpha^2(y_1)[(\alpha(x)y_2)\alpha(z_2)]\}$$

$$\begin{aligned}
&= [\alpha^2(z_1)\alpha^2(y_1)][(\alpha^2(x)\alpha(y_2))\alpha^2(z_2)] \\
&= \alpha^2(z_1y_1)[\alpha^3(x)(\alpha(y_2)\alpha(z_2))], \quad q.e.d.
\end{aligned}$$

So, we can now write (by using both Hom-associativity and Hom-coassociativity):

$$\begin{aligned}
&\alpha^3(x) \bullet (y \bullet z) - (x \bullet y) \bullet \alpha^3(z) \\
&= \alpha([\alpha^2(z_1)(\alpha^2(y)z_{(2,1)})]\alpha(\alpha^2(x)z_{(2,2)})) + \alpha(\alpha^2(z_{(1,1)})[\alpha^3(x)(z_{(1,2)}(\alpha(y)z_2))]) \\
&= \alpha([\alpha(z_1)\alpha^2(y)]\alpha(z_{(2,1)})]\alpha(\alpha^2(x)z_{(2,2)}) + \alpha([\alpha(z_{(1,1)})\alpha^3(x)]\alpha(z_{(1,2)}(\alpha(y)z_2))) \\
&= \alpha([\alpha^2(z_1)\alpha^3(y)][\alpha(z_{(2,1)})(\alpha^2(x)z_{(2,2)})] + [\alpha(z_{(1,1)})\alpha^3(x)][\alpha(z_{(1,2)})(\alpha^2(y)\alpha(z_2))]) \\
&= \alpha([\alpha(z_{(1,1)})\alpha^3(y)][\alpha(z_{(1,2)})(\alpha^2(x)\alpha(z_2))] + [\alpha(z_{(1,1)})\alpha^3(x)][\alpha(z_{(1,2)})(\alpha^2(y)\alpha(z_2))]),
\end{aligned}$$

and this expression is obviously symmetric in x and y . \square

Remark 5.5 *The construction introduced in Proposition 5.4 is compatible with the Yau twist, in the following sense. Let (A, μ, Δ) be an infinitesimal bialgebra and $\alpha : A \rightarrow A$ a morphism of infinitesimal bialgebras. Consider the Yau twist $A_\alpha = (A, \mu_\alpha = \alpha \circ \mu, \Delta_\alpha = \Delta \circ \alpha, \alpha)$ (with notation $\mu_\alpha(x \otimes y) = x * y = \alpha(xy)$ and $\Delta_\alpha(x) = x_{[1]} \otimes x_{[2]} = \alpha(x_1) \otimes \alpha(x_2)$), which is an infinitesimal Hom-bialgebra, to which we can apply Proposition 5.4 and obtain a left Hom-pre-Lie algebra with structure map α^3 and multiplication*

$$x \bullet y = \alpha(y_{[1]}) * (\alpha(x) * y_{[2]}) = \alpha^2(y_1) * \alpha^2(xy_2) = \alpha^3(y_1xy_2).$$

This is exactly the Yau twist via the map α^3 of the left pre-Lie algebra obtained from (A, μ, Δ) by Theorem 1.1.

Assume now that we have a quasitriangular infinitesimal Hom-bialgebra $(A, \mu, \Delta_r, \alpha)$ as in Definition 2.8; there are two left Hom-pre-Lie algebras that may be associated to A , and we want to show that they coincide.

The first one is (A, \bullet, α^3) obtained from A by using Proposition 5.4, with multiplication

$$\begin{aligned}
a \bullet b &= \alpha(b_1)(\alpha(a)b_2) = \sum_i \alpha^2(x_i)(\alpha(a)(y_i b)) - \sum_i \alpha(bx_i)(\alpha(a)\alpha(y_i)) \\
&\stackrel{(2.2)}{=} \sum_i \alpha^2(x_i)[(\alpha y_i)\alpha(b)] - \sum_i [\alpha(b)\alpha(x_i)]\alpha(ay_i) \\
&\stackrel{(2.2)}{=} \sum_i [\alpha(x_i)(ay_i)]\alpha^2(b) - \sum_i \alpha^2(b)[\alpha(x_i)(ay_i)].
\end{aligned}$$

The second is obtained by applying Corollary 3.15 (for $n = 2$) to the α^2 -Rota-Baxter operator R defined in Corollary 4.5. So, its structure map is α^3 and the multiplication is

$$a \circ b = R(a)\alpha^2(b) - \alpha^2(b)R(a) = \sum_i [\alpha(x_i)(ay_i)]\alpha^2(b) - \sum_i \alpha^2(b)[\alpha(x_i)(ay_i)],$$

so indeed \bullet and \circ coincide.

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