Magnetoconvection of a micropolar fluid in a vertical channel

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Abstract

This work examines the effects of the magnetic field and of the temperature on the steady mixed convection in the fully developed flow of a micropolar fluid filling a vertical channel under the Oberbeck-Boussinesq approximation. The two boundaries are considered as isothermal and kept either at different or at equal temperatures. The velocity, the microrotation, the temperature and the induced magnetic field are analytically obtained. A selected set of pictures illustrating the influence of various parameters involved in the problem (the coupling parameter, the micropolar parameter, the Hartmann number and the buoyancy coefficient) is presented and discussed. Moreover, the results obtained for the micropolar flow are compared with the corresponding for the Newtonian fluid.

Keywords: Micropolar fluids, MHD fully developed flow, mixed convection, natural convection, Boussinesq approximation, vertical channel.

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Nomenclature

b	thermal diffusivity
C	constant such that $P = -Cx_1 + p_0$
c_0, c_d, c_a	angular viscosity coefficients
$2d$	channel width
$\mathbf{E}$	electric field
$\mathbf{g} = -g\mathbf{e_1}$	gravity acceleration
Gr	Grashof number defined by $(6)_9$
$\mathbf{H}$	total magnetic field
$h(y)$	dimensionless function desribing the induced magnetic field defined by $(6)_{15}$
$H_0\mathbf{e_2}$	external uniform magnetic field ($H_0 > 0$)
$H_1(x_2)$	induced magnetic field component in the x_1-direction
I	microinertia coefficient
k	fluid thermal conductivity
$k_{1,2}$	heat transfer coefficients evaluated at $\Pi_{1,2}$
l	characteristic length defined by $(6)_2$
L	dimensionless constant defined by $(6)_3$
M^2	Hartmann number defined by $(6)_5$
M_p^2	micropolar parameter defined by $(6)_4$
N^2	coupling number defined by $(6)_1$ $(0 < N^2 < 1)$
Nu	Nusselt number
p	pressure
$P = p + \mu_e \frac{H_1^2}{2} + \rho_0 g x_1$	difference between the hydromagnetic pressure and the hydrostatic pressure
p_0	arbitrary constant
$\mathbf{q}$	heat flux vector
Re	Reynolds number defined by $(6)_8$
$T = T(x_2)$	temperature
T_0	reference temperature
T_1, T_2	uniform temperatures ($T_2 \geq T_1$)
$\mathbf{v}$	velocity field
$v(y)$	dimensionless function describing the velocity defined by $(6)_{12}$
V_0	characteristic velocity defined by $(6)_6$
$v_1(x_2)$	velocity component in the x_1-direction
$\mathbf{w}$	microrotation field
$w(y)$	dimensionless function describing the microrotation defined by $(6)_{13}$
$w_3(x_2)$	microrotation component in the x_3-direction
y	dimensionless transverse coordinate defined by $(6)_{11}$

Greek symbols

α, β, γ	dimensionless constants defined by (11)
α_T	thermal expansion coefficient
η_e	electrical permettivity $\left(\eta_e = \dfrac{1}{\mu_e \sigma_e}\right)$
$\vartheta(y)$	dimensionless temperature defined by $(6)_{14}$
λ	buoyancy coefficient defined by $(6)_{10}$
μ	Newtonian viscosity coefficient
μ_e	magnetic permeability
μ_r	dynamic microrotation viscosity coefficient
ν_0	constant defined by $(6)_7$
ρ_0	mass density at the temperature T_0
σ_e	electrical conductivity
$\boldsymbol{\tau}_{1,2}$	skin friction at the plates $\Pi_{1,2}$
$\boldsymbol{\tau}_{p1,2}$	skin couple friction at the plates $\Pi_{1,2}$

1. Introduction

The recent industrial processes are characterized by the use of new materials which cannot be described by Newtonian fluids. Due to this reason, many non-Newtonian models have been proposed. Among these models, the micropolar fluids have been introduced by Eringen ([1]) in order to take into account the effects of local structure and micro-motions of the fluid particles which cannot be described by the classical models. The incompressible micropolar fluids represent liquids consisting of rigid, randomly oriented spherical particles suspended in a viscous medium, where the deformation of fluid particles is ignored. The related mathematical model is based on the introduction of a new vector field (the microrotation) which describes the total angular velocity field of the particles rotation. Hence, one new equation is added representing the balance law of local angular momentum.
There are many examples of flows of micropolar fluids that are relevant for practical applications as flows of biological fluids in thin vessels, polymeric suspensions, liquid crystals, slurries, colloidal fluids, exotic lubricants, etc. Extensive reviews of the theory and its applications can be found in [2] and [3].

The purpose of the present paper is to study the influence of an external uniform magnetic field on the mixed convection in the fully developed flow of a micropolar fluid filling a vertical channel under the Oberbeck-Boussinesq approximation. A systematic and rigorous derivation of this approximation is provided in [4].
Convection flow of an electrically conducting fluid in a channel under the effect of a transverse magnetic field has a relevant technical significance because of its many industrial applications such as geothermal reservoirs, cooling of nuclear reactors, electric transmission cables, thermal insulation and petroleum reservoirs, to name a few.

In our study we solve the problem of the mixed convection of a Bossinesquian electrically conducting micropolar fluid which steadily flows in a vertical channel under the action of a uniform magnetic field applied normal to the direction of the velocity. The walls are maintained at constant temperatures T_1 and T_2 ($T_1 \leq T_2$).

The first paper on the fully developed free convection of a micropolar fluid in a vertical channel is [5]; this work has been generalized in [6] in order to consider also the mass transfer. In [7], [8], [9] mixed convection flow with symmetric and asymmetric heating is examined. To the best of our knowledge, few results are known concerning the influence of an external magnetic field on the convective flow of a micropolar fluid in a vertical channel ([10], [11]), while in recent years the same situation in a double channel has been studied in [12]. However, in most of the previous papers, a restrictive condition on the material parameters has been imposed following the work of Ahmadi ([13]). We point out that in our research we have not required any condition so that two material parameters describe the micropolar nature of the fluid, instead of one as in the simplified Ahmadi's approach.

In our paper, as it is usual in the Oberbeck-Boussinesq approximation ([14]), we neglect the dissipation terms in the energy equation, so that we can obtain the explicit solution of the problem which takes into account the induced magnetic field. We point out that the induced magnetic field is neglected in most of the works concerning the convective flow in a vertical channel, also in the simpler case of a Newtonian fluid.

The paper is organized in this way:

In Section 2 we formulate the problem from the physical point of view. In order to determine the analytical solution, we have to distinguish three cases which are related to the strength of the external uniform magnetic field.

Section 3 is devoted to integrate the boundary value problem which describes the motion in the previous three cases.

In Section 4 we make some comments about the flow and we give the solution when the heating is symmetric, in the case of natural convection, in the absence of magnetic field and in the same problems for the Newtonian fluid.

The trend of the solution is plotted in Section 5 in order to show the influence of the relevant parameters on the flow. The behavior of the micropolar flow differs highly from the Newtonian one as the coupling number increases and the micropolar parameter M_p decreases. For suitable values of the buoyancy parameter λ, the reverse flow occurs near the coldest (hottest) wall if $\lambda > 0$ ($\lambda < 0$). The presence of the external magnetic field tends to prevent the occurrence of the reverse flow. If the buoyancy parameter vanishes (symmetric heating), then the phenomenon of the reverse flow does not appear.

Section 6 summarizes the results.

2. Formulation of the problem

Let us consider a Boussinesquian, electrically conducting micropolar fluid filling the region $\mathcal{S}$ between two infinite rigid, fixed, non-electrically conducting vertical plates $\Pi_1,\ \Pi_2$ separated by a distance $2\,d$ (Figure 1).

We assume the regions outside the plane to be a vacuum (free space). The

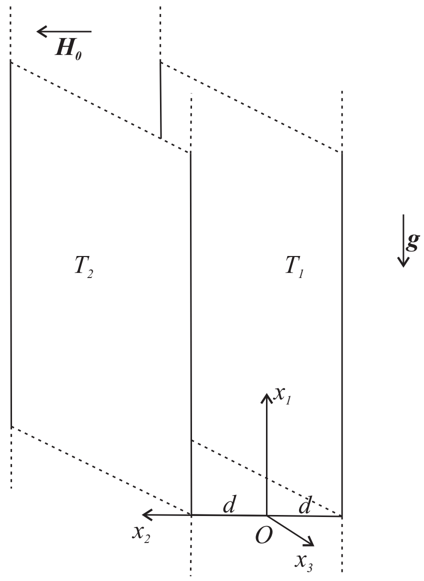

Figure 1: Physical configuration and coordinate system.

coordinate axes are fixed in order to have

$$
\begin{aligned}
&\mathcal{S} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_3) \in \mathbb{R}^2,\ x_2 \in (-d, d)\}, \\
&\Pi_i = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_3) \in \mathbb{R}^2,\ x_2 = (-1)^i d\}, \quad i = 1, 2
\end{aligned} \tag{1}
$$

and x_1-axis is vertical upward.

Our aim is to study the steady mixed convection in the fully developed flow of the fluid under the action of an external uniform magnetic field $H_0\mathbf{e}_2$ normal to planes $\Pi_{1,2}$ ($H_0 > 0$).

This flow in the absence of external mechanical body forces, body couples and free electric charges under the Oberbeck-Boussinesq approximation is governed by ([3], [1], [2])

$$
\begin{aligned}
&\rho_0 \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + (\mu + \mu_r) \triangle \mathbf{v} + 2\mu_r (\nabla \times \mathbf{w}) + \mu_e (\nabla \times \mathbf{H}) \times \mathbf{H} + \rho_0 [1 - \alpha_T (T - T_0)] \mathbf{g}, \\
&\rho_0 I \mathbf{v} \cdot \nabla \mathbf{w} = (c_a + c_d) \triangle \mathbf{w} + (c_0 + c_d - c_a) \nabla (\nabla \cdot \mathbf{w}) + 2\mu_r (\nabla \times \mathbf{v} - 2\mathbf{w}), \\
&\nabla \cdot \mathbf{v} = 0, \\
&\eta_e \triangle \mathbf{H} = \nabla \times (\mathbf{H} \times \mathbf{v}), \\
&\nabla \cdot \mathbf{H} = 0, \\
&\nabla T \cdot \mathbf{v} = b \triangle T, \qquad \text{in } \mathcal{S}.
\end{aligned} \tag{2}
$$

All the material parameters are positive constants and μ_e is equal to the magnetic permeability of free space. As it is usual in the Boussinesq approximation

([14]), in equation $(2)_6$ the dissipative terms have been neglected.
We note that in [1] and in [2] equations (2) are slightly different because they are deduced as a special case of a much more general model of microfluids. For the details we refer to [3], p.23.

We search $\mathbf{v}, \mathbf{w}, \mathbf{H}, T$ in the following form:

$$\mathbf{v} = v_1(x_2)\mathbf{e_1}, \quad \mathbf{w} = w_3(x_2)\mathbf{e_3}, \quad \mathbf{H} = H_0\mathbf{e_2} + H_1(x_2)\mathbf{e_1}, \quad T = T(x_2), \quad (3)$$

where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is the canonical base of $\mathbb{R}^3$. Thank to (3), $\mathbf{v}, \mathbf{w}, \mathbf{H}$ are divergence free.

The previous unknown functions satisfy the boundary conditions

$$v_1(\pm d) = 0, \quad w_3(\pm d) = 0, \quad H_1(\pm d) = 0, \quad T(-d) = T_1, \quad T(d) = T_2. \quad (4)$$

If $T_1 = T_2$, then the heating is called symmetric, otherwise it is asymmetric. We choice the reference temperature $T_0 = \dfrac{T_1 + T_2}{2}$ and, at the moment, we suppose $T_2 > T_1$.

By virtue of (3) and $(2)_1$, we deduce $P = P(x_1) = -Cx_1 + p_0$ (C, p_0 some constants) so that the governing equations give:

$$\begin{aligned}
&(\mu + \mu_r)v_1'' + 2\mu_r w_3' + \mu_e H_0 H_1' + \rho_0 \alpha_T (T - T_0) g \;=\; -C, \\
&(c_a + c_d)w_3'' - 2\mu_r v_1' - 4\mu_r w_3 \;=\; 0, \\
&\eta_e H_1'' + H_0 v_1' \;=\; 0, \\
&T'' \;=\; 0, \qquad\qquad \text{in } [-d, d]. \qquad (5)
\end{aligned}$$

By putting

$$\begin{aligned}
&N^2 = \frac{\mu_r}{\mu + \mu_r}, \quad l^2 = \frac{c_d + c_a}{4\mu}, \quad L = \frac{d}{l}, \quad M_p^2 = N^2 L^2, \quad M^2 = \frac{\sigma_e}{\mu}\mu_e^2 H_0^2 d^2, \\
&V_0 = \frac{Cd^2}{\mu}, \quad \nu_0 = \frac{\mu + \mu_r}{\rho_0}, \quad Re = \frac{V_0 d}{\nu_0}, \quad Gr = \frac{\alpha_T g (T_2 - T_1) d^3}{\nu_0^2}, \quad \lambda = \frac{Gr}{Re}, \quad y = \frac{x_2}{d}, \\
&v(y) = \frac{v_1(dy)}{V_0}, \quad w(y) = \frac{d w_3(dy)}{V_0}, \quad \vartheta(y) = \frac{T(dy) - T_0}{T_2 - T_1}, \quad h(y) = \frac{H_1(dy)}{V_0\sqrt{\sigma_e \mu}}, \quad (6)
\end{aligned}$$

equations (5) can be written in dimensionless form:

$$\begin{aligned}
&v'' + 2N^2 w' + M(1 - N^2)h' + \lambda\vartheta + 1 - N^2 \;=\; 0, \\
&w'' - \frac{M_p^2}{2(1 - N^2)}v' - \frac{M_p^2}{1 - N^2}w \;=\; 0, \\
&h'' + Mv' \;=\; 0, \\
&\vartheta'' \;=\; 0, \qquad\qquad \text{in } [-1, 1]. \qquad (7)
\end{aligned}$$

Boundary conditions (4) in dimensionless form become

$$v(\pm 1) = 0, \qquad w(\pm 1) = 0, \qquad h(\pm 1) = 0, \quad \vartheta(\pm 1) = \pm\frac{1}{2}. \tag{8}$$

Equation $(7)_4$ together with $(8)_4$ imply

$$\vartheta(y) = \frac{y}{2}, \quad \text{in } [-1,\, 1]. \tag{9}$$

By differentiating $(7)_{1,2}$ and using $(7)_3$ and (9), w has to satisfy the following linear ordinary differential equation:

$$w^{IV} - \alpha w'' + \beta w \;=\; \gamma, \tag{10}$$

where

$$\alpha = M_p^2 + (1 - N^2)M^2, \;\; \beta = M_p^2 M^2, \;\; \gamma = -\frac{\lambda M_p^2}{4(1 - N^2)}. \tag{11}$$

The general solution of equation (10) depends on the sign of the discriminant $\Delta = \alpha^2 - 4\beta$ of the algebraic equation

$$\xi^4 - \alpha \xi^2 + \beta = 0. \tag{12}$$

In the next section, we solve problem (7), (8) in the three cases: $\Delta > 0$, $\Delta = 0$, $\Delta < 0$. These cases are all possible from the physical point of view because they represent the following situations:

- if $\Delta > 0$, then $0 < H_0 < \sqrt{\dfrac{\mu}{\sigma_e}}\dfrac{M_p}{\mu_e d(1+N)}$ or $H_0 > \sqrt{\dfrac{\mu}{\sigma_e}}\dfrac{M_p}{\mu_e d(1-N)}$ weak or strong external uniform magnetic field;
- if $\Delta = 0$, then $H_0 = \sqrt{\dfrac{\mu}{\sigma_e}}\dfrac{M_p}{\mu_e d(1+N)}$ or $H_0 = \sqrt{\dfrac{\mu}{\sigma_e}}\dfrac{M_p}{\mu_e d(1-N)}$ critical external uniform magnetic field;
- if $\Delta < 0$, then $\sqrt{\dfrac{\mu}{\sigma_e}}\dfrac{M_p}{\mu_e d(1+N)} < H_0 < \sqrt{\dfrac{\mu}{\sigma_e}}\dfrac{M_p}{\mu_e d(1-N)}$ bounded external uniform magnetic field.

3. Solution of the flow

3.1. $\Delta > 0$*: weak or strong external uniform magnetic field*

In this situation, equation $(12)_1$ admits the following real routes

$$\begin{aligned}
\xi_1 &= \sqrt{\frac{\alpha - \sqrt{\Delta}}{2}} = \frac{1}{2}\left[\sqrt{(M_p + M)^2 - N^2M^2} - \sqrt{(M_p - M)^2 - N^2M^2}\right], \\
\xi_2 &= \sqrt{\frac{\alpha + \sqrt{\Delta}}{2}} = \frac{1}{2}\left[\sqrt{(M_p + M)^2 - N^2M^2} + \sqrt{(M_p - M)^2 - N^2M^2}\right], \\
\xi_3 &= -\xi_1, \;\; \xi_4 = -\xi_2,
\end{aligned} \tag{13}$$

so that the general solution of (10) is given by

$$w(y) = C_1 \cosh(\xi_1 y) + C_2 \sinh(\xi_1 y) + C_3 \cosh(\xi_2 y) + C_4 \sinh(\xi_2 y) + \frac{\gamma}{\beta}. \quad (14)$$

Thanks to this last equation, we obtain

$$\begin{aligned} v(y) = 2\Big\{ & A_1^+ [C_1 \sinh(\xi_1 y) + C_2 \cosh(\xi_1 y)] + \\ & A_2^+ [C_3 \sinh(\xi_2 y) + C_4 \cosh(\xi_2 y)] - \frac{\gamma}{\beta} y \Big\} + C_5, \\ h(y) = \frac{2}{M}\Big\{ & B_1^+ [C_1 \cosh(\xi_1 y) + C_2 \sinh(\xi_1 y)] + \\ & B_2^+ [C_3 \cosh(\xi_2 y) + C_4 \sinh(\xi_2 y)] + \frac{M^2}{2}\frac{\gamma}{\beta} y^2 - \frac{y}{2} + \frac{\gamma}{\beta} \Big\} + C_6, \quad (15) \end{aligned}$$

where

$$\begin{aligned} & A_1^+ = \frac{1-N^2}{M_p^2}\xi_1 - \frac{1}{\xi_1}, \quad A_2^+ = \frac{1-N^2}{M_p^2}\xi_2 - \frac{1}{\xi_2}, \\ & B_1^+ = 1 - \frac{\xi_1^2}{M_p^2}, \quad B_2^+ = 1 - \frac{\xi_2^2}{M_p^2}, \end{aligned} \quad (16)$$

and $C_i, \quad i = 1, \ldots, 6$ are arbitrary constants.
The solution of our problem is determined by asking that the functions given by (14), (15) satisfy the boundary conditions $(8)_{1,2,3}$. More precisely:

$$\begin{aligned} v(y) = & \frac{M_p^2}{\xi_2^2 - \xi_1^2} \frac{A_1^+ \sinh \xi_2 [\cosh(\xi_1 y) - \cosh \xi_1] - A_2^+ \sinh \xi_1 [\cosh(\xi_2 y) - \cosh \xi_2]}{\sinh \xi_1 \sinh \xi_2} \\ & + 2\frac{\gamma}{\beta} \frac{A_1^+ D_2^+ \sinh(\xi_1 y) - A_2^+ D_1^+ \sinh(\xi_2 y)}{A_1^+ \sinh \xi_1 \cosh \xi_2 - A_2^+ \sinh \xi_2 \cosh \xi_1} - 2\frac{\gamma}{\beta} y, \\ w(y) = & \frac{M_p^2}{2(\xi_2^2 - \xi_1^2)} \frac{\sinh \xi_2 \sinh(\xi_1 y) - \sinh \xi_1 \sinh(\xi_2 y)}{\sinh \xi_1 \sinh \xi_2} \\ & + \frac{\gamma}{\beta} \frac{D_2^+ \cosh(\xi_1 y) - D_1^+ \cosh(\xi_2 y)}{A_1^+ \sinh \xi_1 \cosh \xi_2 - A_2^+ \sinh \xi_2 \cosh \xi_1} + \frac{\gamma}{\beta}, \\ h(y) = & \frac{M_p^2}{M(\xi_2^2 - \xi_1^2)} \frac{B_1^+ \sinh \xi_2 \sinh(\xi_1 y) - B_2^+ \sinh \xi_1 \sinh(\xi_2 y)}{\sinh \xi_1 \sinh \xi_2} - \frac{y}{M} \\ & + \frac{2}{M}\frac{\gamma}{\beta} \frac{B_1^+ D_2^+ [\cosh(\xi_1 y) - \cosh \xi_1] - B_2^+ D_1^+ [\cosh(\xi_2 y) - \cosh \xi_2]}{A_1^+ \sinh \xi_1 \cosh \xi_2 - A_2^+ \sinh \xi_2 \cosh \xi_1} \\ & - M\frac{\gamma}{\beta}(1 - y^2). \end{aligned} \quad (17)$$

where

$$D_1^+ = \cosh \xi_1 + A_1^+ \sinh \xi_1, \quad D_2^+ = \cosh \xi_2 + A_2^+ \sinh \xi_2. \tag{18}$$

3.2. $\Delta = 0$*: critical external uniform magnetic field*

In this situation, equation $(12)_1$ admits the following real routes

$$\xi_1 = \xi_2 = \sqrt{\frac{\alpha}{2}} = \sqrt{M_p M} =: \xi, \quad \xi_3 = \xi_4 = -\xi_1. \tag{19}$$

After determining the general solution of $(7)_{1,2,3}$ and imposing the boundary conditions $(8)_{1,2,3}$, we arrive at:

$$\begin{aligned}
v(y) &= \frac{M_p^2}{2\xi} \frac{A_1^0[\cosh \xi \cosh(\xi y) - y \sinh \xi \sinh(\xi y) - 1] + A_2^0 \sinh \xi [\cosh \xi - \cosh(\xi y)]}{\sinh^2 \xi} \\
&\quad + 2\frac{\gamma}{\beta} \frac{(A_2^0 \cosh \xi - A_1^0 D_2^0) \sinh(\xi y) + A_1^0 D_1^0 y \cosh(\xi y)}{A_1^0 + A_2^0 \sinh \xi \cosh \xi} - 2\frac{\gamma}{\beta} y, \\
w(y) &= \frac{M_p^2}{4\xi} \frac{\cosh \xi \sinh(\xi y) - y \sinh \xi \cosh(\xi y)}{\sinh^2 \xi} \\
&\quad + \frac{\gamma}{\beta} \frac{D_1^0 y \sinh(\xi y) - (A_2^0 \sinh \xi + D_2^0) \cosh(\xi y)}{A_1^0 + A_2^0 \sinh \xi \cosh \xi} + \frac{\gamma}{\beta}, \\
h(y) &= \frac{M_p^2}{2\xi M} \frac{(B_1^0 \cosh \xi + 2B_2^0 \sinh \xi) \sinh(\xi y) - B_1^0 y \sinh \xi \cosh(\xi y)}{\sinh^2 \xi} - \frac{y}{M} \\
&\quad + \frac{2}{M} \frac{\gamma}{\beta} \left\{ B_1^0 \left[1 + \frac{D_1^0 y \sinh(\xi y) - (A_2^0 \sinh \xi + D_2^0) \cosh(\xi y)}{A_1^0 + A_2^0 \sinh \xi \cosh \xi} \right] + 2B_2^0 \frac{D_1^0 [\cosh \xi - \cosh(\xi y)]}{A_1^0 + A_2^0 \sinh \xi \cosh \xi} \right\} \\
&\quad - M \frac{\gamma}{\beta} (1 - y^2),
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
&A_1^0 = \frac{1 - N^2}{M_p^2} \xi - \frac{1}{\xi}, \quad A_2^0 = \frac{1 - N^2}{M_p^2} + \frac{1}{\xi^2}, \quad B_1^0 = 1 - \frac{\xi^2}{M_p^2}, \quad B_2^0 = \frac{\xi}{M_p^2}, \\
&D_1^0 = \cosh \xi + A_1^0 \sinh \xi, \quad D_2^0 = \sinh \xi + A_1^0 \cosh \xi.
\end{aligned} \tag{21}$$

3.3. $\Delta < 0$: bounded external uniform magnetic field

In this situation, equation $(12)_1$ admits the following complex routes

$$\xi_1 = \frac{1}{2}\left[\sqrt{(M_p+M)^2 - N^2M^2} - i\sqrt{N^2M^2 - (M_p-M)^2}\right],$$
$$\xi_2 = \frac{1}{2}\left[\sqrt{(M_p+M)^2 - N^2M^2} + i\sqrt{N^2M^2 - (M_p-M)^2}\right],$$
$$\xi_3 = -\xi_1, \quad \xi_4 = -\xi_2. \tag{22}$$

Proceeding as in the previous cases, the solution of the problem $(7)_{1,2,3}$, $(8)_{1,2,3}$ is given by:

$$\begin{aligned}
v(y) &= \frac{M_p^2}{2\delta\sigma}\frac{1}{\sin^2\sigma + \sinh^2\delta}\Big[(A_1^- \sin\sigma\cosh\delta - A_2^- \cos\sigma\sinh\delta)\cos(\sigma y)\cosh(\delta y) \\
&- (A_1^- \cos\sigma\sinh\delta + A_2^- \sin\sigma\cosh\delta)\sin(\sigma y)\sinh(\delta y) + A_2^- \sinh\delta\cosh\delta - A_1^- \cos\sigma\sin\sigma\Big] \\
&+ 2\frac{\gamma}{\beta}\frac{(A_2^- D_1^- - A_1^- D_2^-)\cos(\sigma y)\sinh(\delta y) + (A_1^- D_1^- + A_2^- D_2^-)\sin(\sigma y)\cosh(\delta y)}{A_1^- \sin\sigma\cos\sigma + A_2^- \sinh\delta\cosh\delta} - 2\frac{\gamma}{\beta}y, \\
w(y) &= \frac{M_p^2}{4\delta\sigma}\frac{\sin\sigma\cosh\delta\cos(\sigma y)\sinh(\delta y) - \cos\sigma\sinh\delta\sin(\sigma y)\cosh(\delta y)}{\sin^2\sigma + \sinh^2\delta} \\
&+ \frac{\gamma}{\beta}\frac{D_1^- \sin(\sigma y)\sinh(\delta y) - D_2^- \cos(\sigma y)\cosh(\delta y)}{A_1^- \sin\sigma\cos\sigma + A_2^- \sinh\delta\cosh\delta} + \frac{\gamma}{\beta}, \\
h(y) &= \frac{M_p^2}{2M\delta\sigma}\frac{1}{\sin^2\sigma + \sinh^2\delta}\Big[(B_1^- \sin\sigma\cosh\delta - B_2^- \cos\sigma\sinh\delta)\cos(\sigma y)\sinh(\delta y) \\
&- (B_1^- \cos\sigma\sinh\delta + B_2^- \sin\sigma\cosh\delta)\sin(\sigma y)\cosh(\delta y)\Big] - \frac{y}{M} \\
&+ \frac{2}{M}\frac{\gamma}{\beta}\frac{1}{A_1^- \sin\sigma\cos\sigma + A_2^- \sinh\delta\cosh\delta}\Big\{(B_2^- D_1^- - B_1^- D_2^-)[\cos(\sigma y)\cosh(\delta y) - \cos\sigma\cosh\delta] \\
&+ (B_1^- D_1^- + B_2^- D_2^-)[\sin(\sigma y)\sinh(\delta y) - \sin\sigma\sinh\delta]\Big\} - M\frac{\gamma}{\beta}(1-y^2),
\end{aligned} \tag{23}$$

where

$$\delta = \frac{1}{2}\sqrt{(M_p + M)^2 - N^2M^2}, \quad \sigma = \frac{1}{2}\sqrt{N^2M^2 - (M_p - M)^2},$$

$$A_1^- = \left(\frac{1-N^2}{M_p^2} - \frac{1}{M_pM}\right)\delta, \quad A_2^- = \left(\frac{1-N^2}{M_p^2} + \frac{1}{M_pM}\right)\sigma,$$

$$B_1^- = \frac{M_p^2 - M^2(1-N^2)}{2M_p^2}, \quad B_2^- = -\frac{2}{M_p^2}\sigma\delta,$$

$$D_1^- = A_1^- \cos\sigma \sinh\delta - A_2^- \sin\sigma \cosh\delta + \cos\sigma \cosh\delta,$$

$$D_2^- = A_1^- \sin\sigma \cosh\delta + A_2^- \cos\sigma \sinh\delta + \sin\sigma \sinh\delta. \tag{24}$$

4. Remarks on the flow

In this section, we first make some considerations on the flow which hold in all the three previous cases when the heating is asymmetric.

- It is easy to prove that all the denominators in (17), (20), (23) do not vanish.
- It is interesting to compute the electric field $\mathbf{E}$ associated to the magnetic field. From the Maxwell equation

$$\mathbf{E} = \frac{1}{\sigma_e}\nabla \times \mathbf{H} + \mu_e \mathbf{H} \times \mathbf{v}$$

and $(3)_{1,3}$ we get that $\mathbf{E}$ is parallel to $\mathbf{w}$:

$$\mathbf{E} = -\frac{\mu_e H_0 V_0}{M}[h'(y) + Mv(y)]\mathbf{e}_3.$$

Taking into account $(7)_3$ and the boundary conditions, after some straightforward and long calculations, we obtain that $\mathbf{E}$ is constant and when $\Delta > 0,\ \Delta = 0,\ \Delta < 0$ it assumes the following form

$$\mathbf{E} = E_0\mathbf{e}_3 = \frac{\mu_e H_0 V_0}{M^2}\left[1 - \frac{M^2M_p^2}{\xi_2^2 - \xi_1^2}\frac{A_2^+ \sinh\xi_1 \cosh\xi_2 - A_1^+ \sinh\xi_2 \cosh\xi_1}{\sinh\xi_1 \sinh\xi_2}\right]\mathbf{e}_3,$$

$$\mathbf{E} = E_0\mathbf{e}_3 = \frac{\mu_e H_0 V_0}{M^2}\left[1 - \frac{M^2M_p^2}{2\xi}\frac{A_2^0 \sinh\xi \cosh\xi - A_1^0}{\sinh^2\xi}\right]\mathbf{e}_3,$$

$$\mathbf{E} = E_0\mathbf{e}_3 = \frac{\mu_e H_0 V_0}{M^2}\left[1 - \frac{M^2M_p^2}{2\sigma\delta}\frac{A_2^- \sinh\delta \cosh\delta - A_1^- \sin\sigma \cos\sigma}{\sin^2\sigma + \sinh^2\delta}\right]\mathbf{e}_3. \tag{25}$$

The electric field does not depend on λ, i.e. on the difference of the temperature between the walls.

We notice that the computation of the electric field is omitted in most of the papers concerning MHD flows.

- Outside the planes where the vacuum is, by virtue of the usual transmission conditions for the electromagnetic field across $\Pi_{1,2}$, we have

$$\mathbf{E} = E_0 \mathbf{e}_3, \quad \mathbf{H} = H_0 \mathbf{e}_2.$$

- From the practical point of view, it is interesting to compute the Nusselt number at $\Pi_{1,2}$:

$$\mathrm{Nu}_{1,2} = \frac{k_{1,2}\, d}{k} = \frac{d}{T_2 - T_1} \frac{dT}{dx_2}|_{x_2=\pm d} = \vartheta'(\pm 1) = \frac{1}{2}.$$

  This allows us to compute the heat transfer coefficients $k_{1,2}$ evaluated at the walls.
  The Nusselt number is related to the heat flux vector in the channel, which is given by

$$\mathbf{q} = -\frac{(T_2 - T_1)k}{2d} \mathbf{e_2}.$$

  This expression is physically quite reasonable because the heat transfer occurs from the hot wall to the cold one.

- The skin friction ($\boldsymbol{\tau}_{1,2}$) and the skin couple friction ($\boldsymbol{\tau}_{p1,2}$) at both plates are given by

$$\boldsymbol{\tau}_{1,2} = (\mu + \mu_r) \frac{V_0}{d} v'(\pm 1) \mathbf{e}_1; \qquad \boldsymbol{\tau}_{p1,2} = (c_d + c_a) \frac{V_0}{d^2} w'(\pm 1) \mathbf{e}_3.$$

  The expression of $\tau_{\mathbf{1,2}}$ is related to the occurrence of the reverse flow, as we will see in the next section.

We now consider other interesting physical situations.

- It is very easy to obtain the corresponding results to Section 3.1, 3.2, 3.3 when $T_1 = T_2$ (symmetric heating). From $(5)_4$ we get $T = T_1 = T_2$ so that $\lambda = 0$ and the expression of v, w, h is deduced by putting $\gamma = 0$ in (17), (20), (23).

- In the previous Sections we have considered the mixed convection case assuming the constant $C \neq 0$. In the case of natural convection ($C = 0$), when the heating is asymmetric, the solution is obtained by (17), (20), (23) writing only the terms having coefficient $\frac{\gamma}{\beta}$. This fact implies that the electric field vanishes. Of course in the dimensionless variables, the reference velocity V_0 cannot be expressed in terms of C.
  If $C = 0$ and the heating is symmetric, then the fluid is at rest and the induced magnetic field vanishes ($v = w = h = 0$).

- In the absence of external magnetic field and when the heating is asymmetric, the motion is given by

$$
\begin{aligned}
v(y) =& \frac{1}{2}(1-y^2)\left[\frac{\lambda}{6(1-N^2)}y+1\right]+\frac{N^2}{M_p}\frac{\cosh(M_p y)-\cosh M_p}{\sinh M_p} \\
&+\frac{\lambda N^2[M_p^2+3(1-N^2)][\sinh(M_p y)-y\sinh M_p]}{6M_p^2(1-N^2)(M_p\cosh M_p - N^2\sinh M_p)}, \\
w(y) =& -\frac{\lambda(1-y^2)}{8(1-N^2)}+\frac{y\sinh M_p-\sinh(M_p y)}{2\sinh M_p} \\
&+\lambda\frac{[M_p^2+3(1-N^2)][\cosh M_p-\cosh(M_p y)]}{12M_p(1-N^2)(M_p\cosh M_p-N^2\sinh M_p)}.
\end{aligned} \tag{26}
$$

  If $T_1 = T_2$, then v, w can be deduced by putting $\lambda = 0$ in the previous expressions. This result is in agreement with the one obtained by Lukaszewicz for the Poiseuille flow of a homogeneous, incompressible micropolar fluid ([3]).

- If $\mathcal{S}$ is occupied by a Boussinesquian, electrically conducting Newtonian fluid, then its MHD mixed convective flow is governed by $(7)_1$ with $w = 0, N = 0$, $(7)_{3,4}$ and $(8)_{1,3,4}$ and it is given by

  $T_2 > T_1$:

$$
\begin{aligned}
\theta(y) &= \frac{y}{2}, \\
v(y) &= \frac{\cosh M-\cosh(My)}{M\sinh M}-\frac{\lambda[\sinh(My)-y\sinh M]}{2M^2\sinh M}, \\
h(y) &= \frac{\sinh(My)}{M\sinh M}-\frac{y}{M}+\frac{\lambda[\cosh(My)-\cosh M]}{2M^2\sinh M}+\frac{\lambda}{4M}(1-y^2), \\
\mathbf{E} &= E_0\mathbf{e}_3=\frac{\mu_e H_0 V_0}{M^2}\left[1-\frac{M\cosh M}{\sinh M}\right]\mathbf{e}_3;
\end{aligned}
$$

  $T_2 = T_1$:

$$
\begin{aligned}
T &= T_1 = T_2, \\
v(y) &= \frac{\cosh M-\cosh(My)}{M\sinh M}, \\
h(y) &= \frac{\sinh(My)}{M\sinh M}-\frac{y}{M}, \\
\mathbf{E} &= E_0\mathbf{e}_3=\frac{\mu_e H_0 V_0}{M^2}\left[1-\frac{M\cosh M}{\sinh M}\right]\mathbf{e}_3.
\end{aligned} \tag{27}
$$

  In the case of symmetric heating the previous solution coincides with Hartmann flow (see for example [15]).

In the case of natural convection, either when the heating is symmetrical or not, the solution can be easily obtained by the previous relations writing only the terms having coefficient λ.
Finally, as in [16] and [17] , in the absence of the external magnetic field we obtain

$$T_2 > T_1 : \ \theta(y) = \frac{y}{2}, \ \ v(y) = \frac{1}{2}(1-y^2)\left(\frac{\lambda}{6}y + 1\right),$$

$$T_2 = T_1 : \ T = T_1 = T_2, \quad v(y) = \frac{1}{2}(1-y^2). \tag{28}$$

Due to the geometry of the problem, all the previous relations hold for all $y \in [-1, 1]$.

5. Results and discussions

The problem of the mixed magnetoconvection in the fully developed flow of a micropolar fluid filling a vertical channel has been analytically solved. As it is proved in Section 3, in the general case of asymmetric heating, the solution is given by (17) or (20) or (23) according to the strength of the external magnetic field. In any case, the solution depends on the values of some relevant physical dimensionless parameters:

- the coupling parameter N^2 which is related to the Newtonian and microrotation viscosity coefficients. $0 < N < 1$ and when $N \to 0$ equation $(2)_1$ reduces to the corresponding equation for a Newtonian fluid;
- the micropolar parameter M_p which is related to N, to the geometry of the problem through L and to the particles size by means of l. Actually, the more the particles sizes are small, the more M_p is big;
- the Hartmann number M^2 which characterizes the strength of the external magnetic field and the electromagnetic properties of the fluid;
- the buoyancy coefficient λ which appears in the analytical solution through γ and which is related to the buoyancy forces due to the gravity. When the heating is symmetric, it vanishes. Its sign depends on the one of the characteristic velocity V_0.

The micropolar properties of the fluid are described by two parameters (N and M_p) unlike most of the papers in the literature because we have not employed any condition on the material constants.

The aim of this Section is to present a selected set of graphical results illustrating the effects on the flow of the various parameters involved in the problem. We first provide Figure 2 in order to show the influence of the parameter N on the velocity, on the microrotation and on the induced magnetic field.

The velocity v decreases as the coupling number increases. It can be noticed

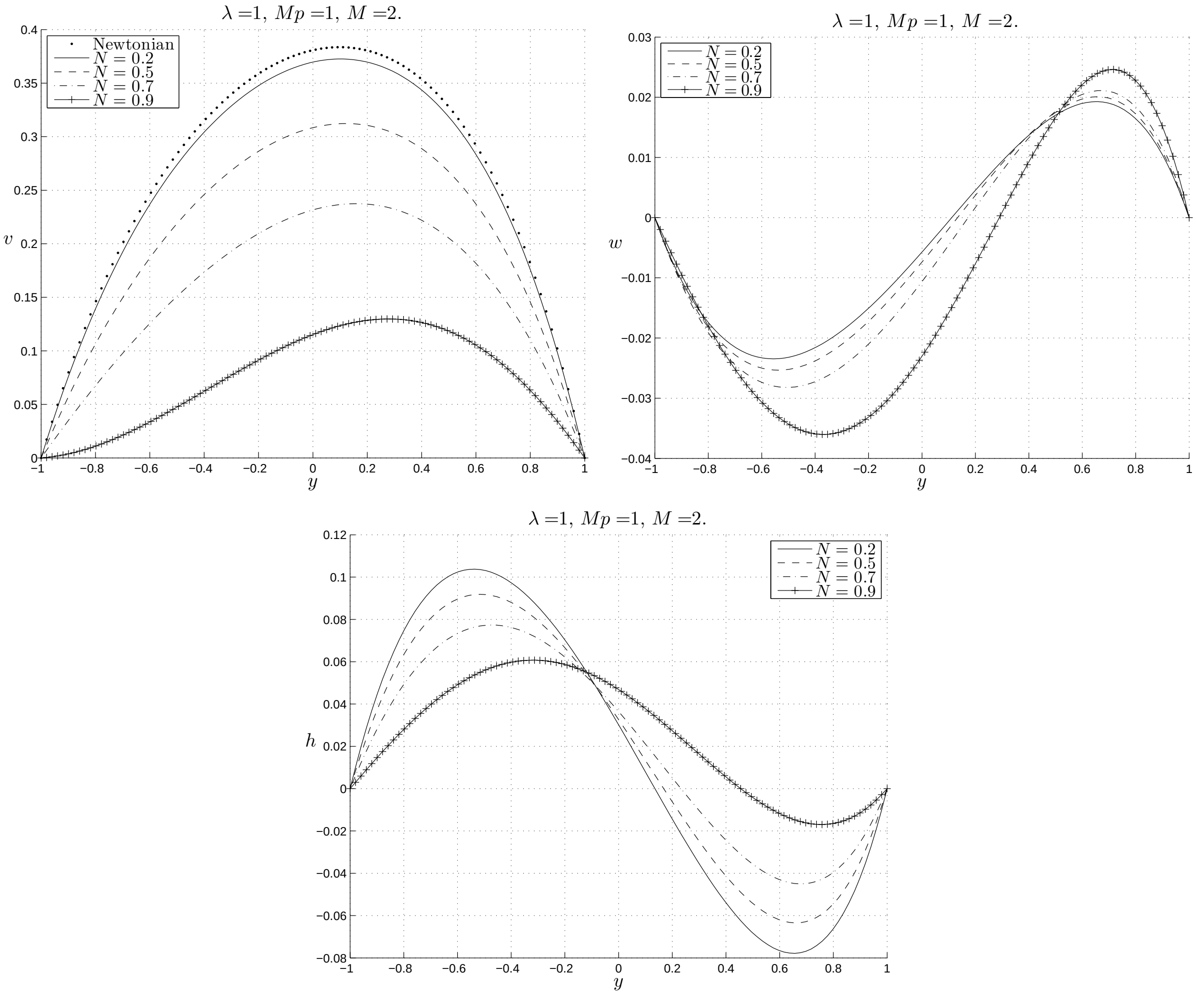

Figure 2: The effect of N on the velocity, on the microrotation and on the induced magnetic field. If $N = 0.2, 0.5, 0.7, 0.9$, then $\Delta > 0, \Delta = 0, \Delta < 0, \Delta < 0$, respectively.

that the velocity in the case of micropolar fluid is less than that of the Newtonian fluid. Since N is an increasing function in μ_r, the difference of the velocity between the micropolar and the Newtonian case grows with N, as it can be expected.

As far as the microrotation field is concerned, it takes a minimum and a maximum, which are more pronounced as N increases.
The absolute value of h (describing the induced magnetic field) decreases as N increases.

Figure 3 reveals the effect of the micropolar parameter M_p on the flow.
When N and the geometry of the problem are fixed, different values of M_p represent different values of c_a and c_d, i.e. different sizes of the particles. The smaller the particles sizes (M_p increases), the greater the non-Newtonian effects on the velocity. The minimum and the maximum of the microrotation and of the induced magnetic field are more pronounced as M_p increases.

We now provide Figure 4 in order to show the influence of the strength of the external magnetic field and the electromagnetic properties of the fluid on v, w, h.
Figure 4_1 illustrates that the velocity decreases as M increases. The main effect of the transverse external magnetic field is to generate electric currents which retard the fluid in the central regions and accelerate the fluid near the boundaries thus flattening the velocity profile in the absence of the magnetic field. This behavior is the same as in the Hartmann flow ([15]).
The absolute value of the microrotation decreases as M increases.
From picture 4_3, it appears that the absolute value of h is an increasing function in M until M reaches a critical values M^*. If $M > M^*$, then the absolute value of h becomes a decreasing function in M.
This behavior of h has never been observed previously in the study of the MHD flow of a fluid in a vertical channel.
The critical value M^* depends on the other parameters, as it is shown in Table 1.
It is interesting to compare these values with the corresponding value of M^* in the Newtonian case (Table 2).
In the micropolar fluid, M^* is always grater than in the Newtonian fluid and its value increases as N increases. This behavior can be expected because when $N \to 1$ the fluid differs highly from the Newtonian one.
Finally, the influence of the buoyancy parameter λ on the flow is provided in Figure 5.
As Figures $5_{1,2}$ reveal, the reverse flow occurs. This well known phenomenon has been first discovered for the Newtonian fluid in [16]. The reverse flow appears when the dimensional velocity and the gradient of P have the same direction. If $\lambda = 0$ (symmetric heating, i.e. $T_1 = T_2$), then the pictures show that v is always positive so that the dimensional velocity ($\mathbf{v} = v_1\mathbf{e_1}$) and the gradient of P ($\nabla P = -C\mathbf{e_1}$) have opposite direction, provided $C \neq 0$. Different choices of the values of the other parameters do not modify the profile of the velocity. Hence, in the case $\lambda = 0$ the reverse flow does not occur. Therefore, the occurrence of the reverse flow is a feature of the mixed convection in the case

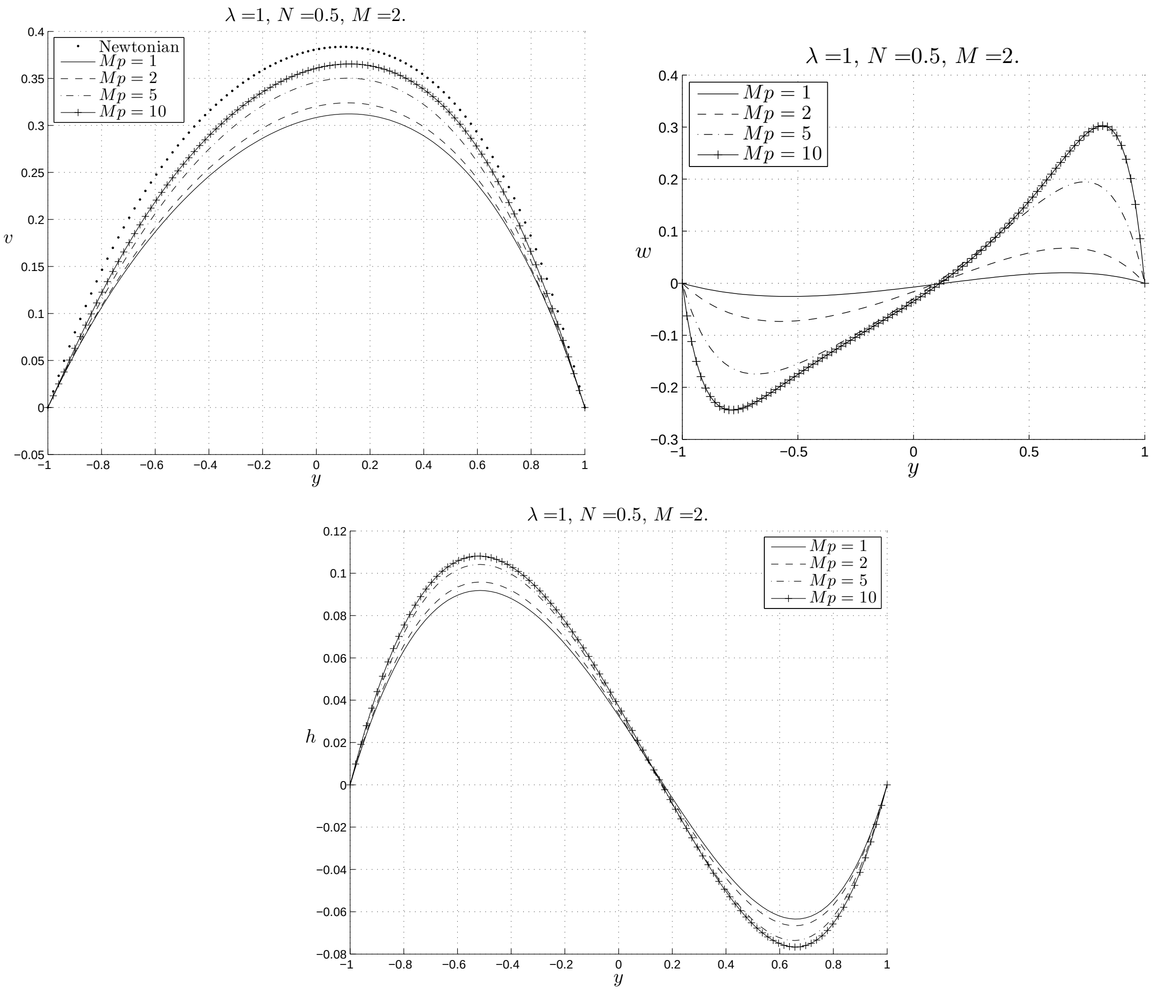

Figure 3: The effect of M_p on the velocity, on the microrotation and on the induced magnetic field. If $M_p = 1, 2, 5, 10$, then $\Delta = 0, \Delta < 0, \Delta > 0, \Delta > 0$, respectively.

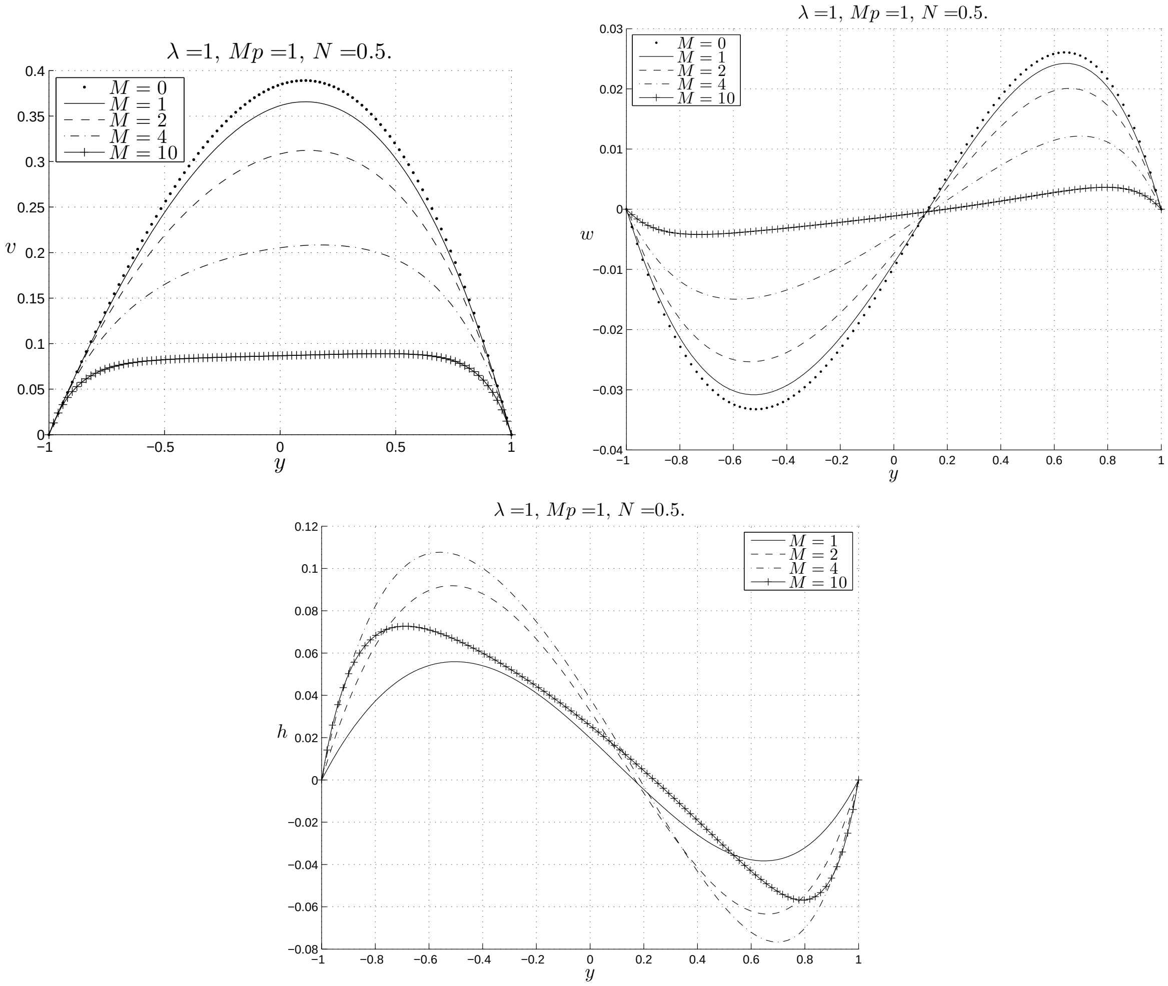

Figure 4: The effect of M on the velocity, on the microrotation and on the induced magnetic field. If $M = 1, 2, 4, 10$, then $\Delta < 0, \Delta = 0, \Delta > 0, \Delta > 0$, respectively.

Table 1: Micropolar: Critical value of M.

M_p	N	λ	M^*
1	0.20	1	3.3333
		2	3.2778
		5	3.2222
		10	3.2778
	0.90	1	6.6667
		2	6.6667
		5	6.9444
		10	6.9444
5	0.20	1	3.2778
		2	3.2222
		5	3.2222
		10	3.2778
	0.90	1	5.2778
		2	5.2778
		5	5.2778
		10	5.2778

Table 2: Newtonian: Critical value of M.

λ	M^*
1	3.2158
2	3.1603
5	3.1318
10	3.1731

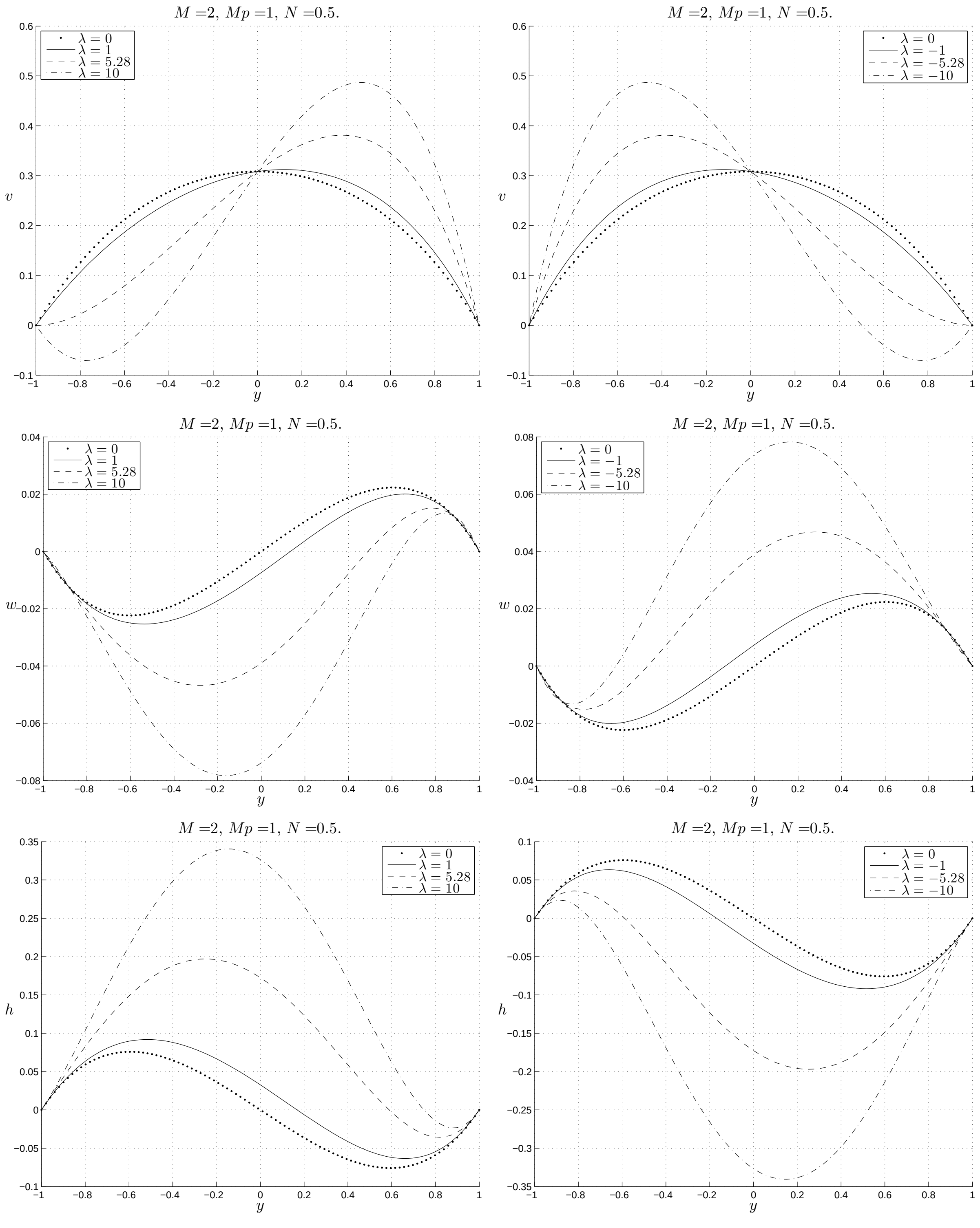

Figure 5: The effect of λ on the velocity, on the microrotation and on the induced magnetic field ($\Delta = 0$).

of asymmetric heating. Actually, this phenomenon appears for suitable values of λ. It is possible to compute a critical value λ^* of λ such that

- $C > 0$ ($\Longleftrightarrow \lambda > 0$): if $\lambda \leq \lambda^*$, then the reverse flow does not appear; if $\lambda > \lambda^*$, then the reverse flow occurs near the coldest wall;
- $C < 0$ ($\Longleftrightarrow \lambda < 0$): if $\lambda \geq \lambda^*$, then the reverse flow does not appear; if $\lambda < \lambda^*$, then the reverse flow occurs near the hottest wall.

The value of λ^* depends on the choice of the other parameters and it is computed by putting $\boldsymbol{\tau}_{1,2}$ equal to zero. In Table 3 we furnish the values of λ^* when $C > 0$. From this Table we can easily obtain the corresponding critical values of λ^* when $C < 0$ because the profiles of v for negative values of λ can be found by symmetry from the corresponding graphics of v when $\lambda > 0$.
From this Table it appears that

- λ^* increases as M increases;
- λ^* decreases as N increases;
- λ^* is not influenced in a relevant way by M_p.

Hence, the influence of M on λ^* shows that the presence of the external magnetic field tends to prevent the occurrence of the reverse flow. This behavior has been observed also in other physical situations ([18]).
For fixed values of M, when $N \to 1$ the value of λ^* differs highly from the corresponding value in the Newtonian case (see Tables 3 and 4).
We point out that the value of λ^* is never computed in the papers concerning micropolar fluid.

From pictures $5_{3,4,5,6}$ we see that the absolute values of w and h increase as $|\lambda|$ increases.

In order to complete the description of the flow, in Table 5 we furnish the values of $\frac{E_0}{\mu_e H_0 V_0}$ as M_p, N, and M change. We have that $\mathbf{E}$ has always opposite direction of $\mathbf{e}_3$.

For the sake of completeness, we provide Figure 6 which displays the behavior of the flow in the case of natural convection.

6. Conclusions

The analytical solution is obtained for the MHD mixed convection in the fully developed flow of an electrically conducting micropolar fluid filling a vertical channel with symmetric and asymmetric wall temperatures. In our analysis, we determine also the induced magnetic field, which is usually neglected in the literature.
The following facts have been reported:

1. The behavior of the micropolar flow differs highly from the Newtonian one as the coupling number N increases and the micropolar parameter M_p decreases.

Table 3: Micropolar: Critical value of λ.

M_p	N	M	λ^*
1	0.20	1	6.1086
		2	7.0873
		4	10.0861
		10	20.9480
	0.50	1	4.6599
		2	5.2791
		4	7.2492
		10	14.6721
	0.70	1	3.0595
		2	3.3612
		4	4.3715
		10	8.4447
	0.90	1	1.0547
		2	1.1027
		4	1.2785
		10	2.1232
5	0.20	1	6.0998
		2	7.0881
		4	10.1031
		10	20.9725
	0.50	1	4.6245
		2	5.2925
		4	7.3536
		10	14.8314
	0.70	1	3.0310
		2	3.4014
		4	4.5575
		10	8.7554
	0.90	1	1.0699
		2	1.1643
		4	1.4610
		10	2.4971

Table 4: Newtonian: Critical value of λ.

M	λ^*
1	6.3891
2	7.4444
4	10.6571
10	22.2222

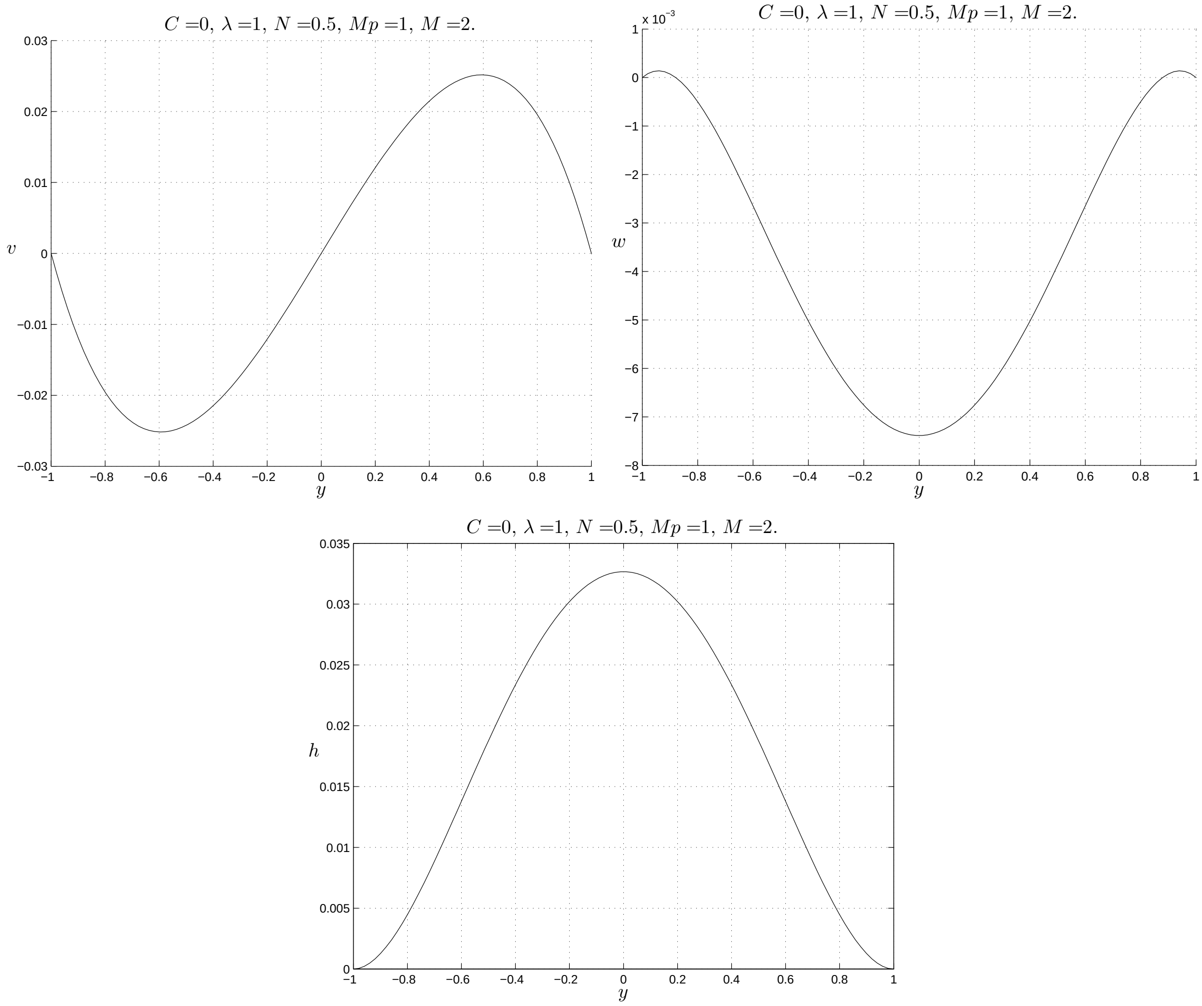

Figure 6: Behavior of the velocity, of the microrotation and of the induced magnetic field in the case of natural convection.

Table 5: Electric field.

M_p	N	M	$\frac{E_0}{\mu_e H_0 V_0}$
1	0.20	1	-0.3019
		2	-0.2602
		4	-0.1828
		10	-0.0880
	0.90	1	-0.0783
		2	-0.0743
		4	-0.0626
		10	-0.0357
5	0.20	1	-0.3071
		2	-0.2637
		4	-0.1842
		10	-0.0882
	0.90	1	-0.1920
		2	-0.1651
		4	-0.1119
		10	-0.0475

2. The absolute value of the function h describing the induced magnetic field is an increasing function in the Hartmann number M until M reaches a critical values M^*. If $M > M^*$, then the absolute value of h becomes a decreasing function in M.
3. For suitable values of the buoyancy parameter λ, the reverse flow occurs near the coldest (hottest) wall if $\lambda > 0$ ($\lambda < 0$). If the buoyancy parameter vanishes (symmetric heating), then the phenomenon of the reverse flow does not appear.
4. The presence of the external magnetic field tends to prevent the occurrence of the reverse flow.

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