Two-phase flow in porous media with hysteresis

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TWO-PHASE FLOW IN POROUS MEDIA WITH HYSTERESIS

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Abstract. Two-phase flow through a porous medium with hysteresis effects is considered. The model consists of a system of two coupled nonlinear equations: a transport equation for the water saturation and an evolution equation for the hysteresis variable. The latter is not in conservation form and contains discontinuous functions of the two unknown variables as coefficients. Some qualitative properties of piecewise smooth solutions of the system are proved. In particular, we show that the hysteresis variable satisfies a maximum principle, and its total variation is bounded by the total variation of its initial value. The traveling waves are investigated under the assumption that the convective term is convex. Riemann solvers for the inviscid system are constructed. Non-uniqueness due to hysteresis loops is finally discussed; several solutions are discarded by the maximum principle for the hysteresis variable.

Keywords: Hysteresis, porous media, traveling waves, Riemann solvers.
AMS Subject Classification: 76S05, 76T99, 35C07.

1. Introduction

Hysteresis is a complex physical phenomenon occurring in continuum mechanics, ferromagnetism and filtration through porous media, with phase transitions involved sometimes. The mathematical literature on this subject is very wide and applies to several phenomena by exploiting different models and techniques; we quote for instance [10, 14]. This paper focuses on a very simple model arising in oil recovery that was proposed in [13]. Such a model does not certainly aim at describing accurately the complex dynamics of the problem but rather to highlight some mathematical issues that characterize it.

More precisely, we consider a fluid flow through a porous medium, which is constituted by an aqueous phase, here formed by water, and a liquid phase, formed by oil. The flow is modeled by the diffusive equation

$$s_t + f_x = \epsilon s_{xx}, \tag{1.1}$$

where s is the water saturation, f the water fractional flow and ϵ the capillarity-induced diffusion coefficient. Both s and f are valued in $[0, 1]$. Hysteresis comes into play through the flow f, which depends not only on s but also on its history and current trend. Roughly speaking, f can be thought as a multi-valued function. For

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s fixed, these multiple values are parametrized by a new variable π, which encodes the behavior of s in the past and the actual increasing or decreasing of s. As a consequence, an equation for π is introduced. A more precise description of the model is provided in the following section.

In the case $\epsilon = 0$, the Riemann problem for equation (1.1) was briefly studied in [5]. A sketch of the construction is reported in [13]. The solution to such problem is far from being unique, not only because many combination of waves are possible for the same initial data, but in particular because *hysteresis loops* appear. For constant $\epsilon > 0$, a relaxation approximation for the equation of π is introduced in [13] and the authors determined which shock waves have a diffusive-relaxation profile. The drawback of that relaxation construction is that the relaxation's physical meaning is unknown, and that the flux function needs to be extended outside its natural domain, as well as the solutions. Such an approximation violates the famous subcharacteristic condition [12]; indeed, the failure of this condition is balanced by the presence of the diffusion term, and the whole effect is to allow the hysteresis loops.

More complicated flows have been considered in this framework. For instance, in [7] a three-component, two-phase flow is proposed, where polymer is added to the aqueous phase to increase the viscosity of this phase to enhance the extraction of oil from the porous medium. To take into account the presence of gas, which is common in porous rocks, authors of [5] introduced a three-component, three-phase flow.

The equation (1.1) is also used to model CO_2 plume migration in CO_2 sequestration [8], where the flux $f(u) = \sigma\left(\mathrm{sign}(u_t)\right) g(u)$ is discontinuous. In particular, $\sigma(r) = 1 - \epsilon$ if $r < 0$ and $\sigma(r) = 1$ if $r > 0$, for $\epsilon \in (0, 1]$. In [1], the authors introduced cross-hatch characteristics, and used them to investigate the structure of shock and rarefaction waves and the result of binary wave interactions. The difference between the equation studied in [1, 8] and the one studied in this paper is that the flux considered here is not of binary type, but has scanning curves joining the imbibition and drainage modes.

The plan of the paper is the following. In Section 2 we give precise details on the model and formulate the governing system of equations, which turns out to be nonlinear and nonconservative. This is a very difficult issue. On the one hand, the regularity of solutions cannot be much better than Lipschitz; on the other hand, the usual formulation for weak solutions does not apply. As a consequence, we mean solutions of the inviscid system as the vanishing viscosity limits of solutions to the viscous system. In preparation to prove the maximum principle for the hysteresis variable, we introduce a localized distributional way to look at the sign of a distribution and its evolution, in order to pinpoint the location and sign of concentrated masses. In Section 3, we show that the hysteresis variable satisfies the maximum principle and that its number of oscillations is nonincreasing as time t increases; hence, its total variation at time $t > 0$ is bounded by that at $t = 0$. In Section 4 we show the existence and uniqueness of viscous profiles; in this section and in the following one we require for simplicity that the convective term f is convex. Indeed, the physical case when f has an inflection point can be dealt analogously; difficulties are only technical. A complete set of Riemann solvers is constructed in Section 5. Often, these Riemann solvers are not unique, as discovered in earlier papers. However, using our result that the number of oscillations of the hysteresis variable in a viscous solution is nonincreasing, we can exclude many of them.

2. The model

In this section we explain the model presented in the Introduction and formulate our main assumptions; we refer to [13] for more details.

2.1. **The physical phenomenon.** As we mentioned above, the flux f in (1.1) depends not only on s but also on its history and current trend. For a given s, the range of f is a closed interval, whose maximum and minimum are denoted by $f^D(s)$ and $f^I(s)$, respectively; see Figure 2.1*(a)*. Below, we plot f^D and f^I as convex functions *only for making pictures clearer*: we stress that the model and all the results in Section 3 require *no* convexity assumptions. Convexity is only required (for simplicity) in Sections 4 and 5.

Let $x \in \mathbb{R}$ be fixed in the following discussion. Suppose that at the point x and time t_0 we have $f = f^I(s)$ and the saturation s increases as t increases. Then the flux f keeps the values $f = f^I(s)$ as long as s increases, and the fluid is said to be in *imbibition mode*. Analogously, if at (x, t_0) the flux f takes on the value $f^D(s)$ and s decreases as t increases, then f continues to take the value $f^D(s)$ as s decreases, and the fluid is said to be in *drainage mode*. As a consequence, we have

$$\begin{aligned} &\text{in imbibition mode:} \quad s_t(x,\cdot) > 0 \text{ and } f = f^I(s), \\ &\text{in drainage mode:} \quad s_t(x,\cdot) < 0 \text{ and } f = f^D(s), \end{aligned} \tag{2.1}$$

where the meaning of these inequalities will be made clear in Definition 2.2.

Now, suppose that the flow is in imbibition mode at (x, t), for $t_1 - \tau < t < t_1$ with $\tau > 0$, so that $\operatorname{sign} s_t(x, t) = 1$ for those t; we label the point $(s, f)(x, t_1)$ by A in Figure 2.1*(a)*. As t increases from $t_1 - \tau$ to t_1, the saturation $s(x, t)$ increases and (s, f) moves along $f = f^I(s)$ towards the point A. Moreover, we assume that $s_t(x, \cdot)$

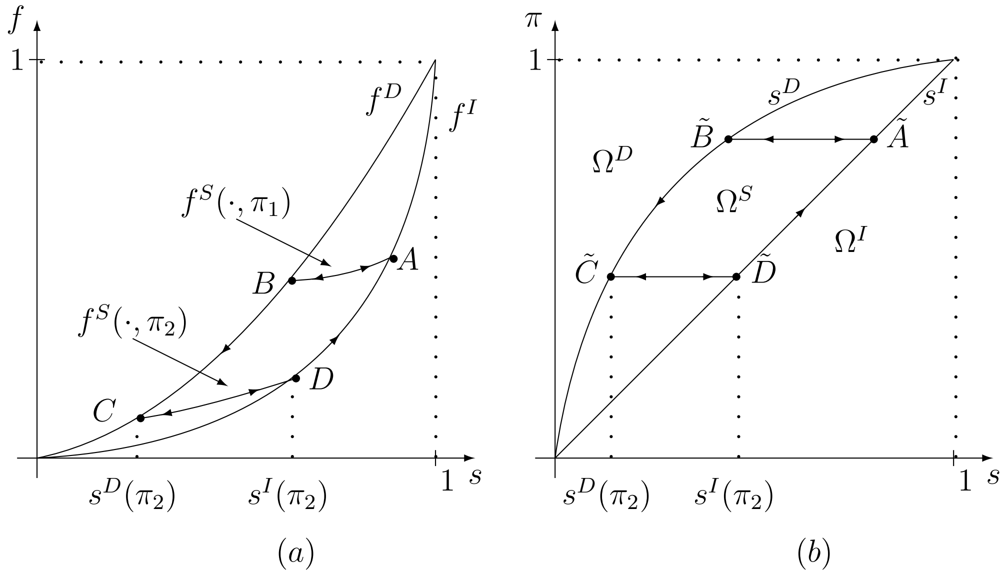


FIGURE 2.1. *(a):* The fluxes $f^D = f^D(s)$, $f^I = f^I(s)$ and $f = f^S(s,\pi)$. The parameter π is constant along the graph of $s \mapsto f(s,\pi)$ from A to B and from C to D. *(b):* the same loop in the plane (s,π).

reverses its sign at t_1 so that $\operatorname{sign} s_t(x,t) = -1$ for $t_1 < t < t_1 + \tau$. At these latter times the flux f does not follow $f = f^I(s)$ as s decreases, but rather takes another curve $f = f^S(s,\pi)$, located between $f^I(s)$ and $f^D(s)$, which intersects $f = f^I(s)$ at $s(x,t_1)$. Here, π is a parameter that characterizes such f^S curves by being constant along each of them. In this case the fluid is said to be in *scanning mode* and the curve $s \mapsto f^S(s,\pi)$ is called a *scanning curve*. An example of parametrization is $\pi = s$, where s is the s-coordinate of the point of intersection of $f = f^S(s,\pi)$ and $f = f^I(s)$; this choice dictates $0 \le \pi \le 1$. Without loss of generality, in this paper we use this parametrization and then $\pi \in [0,1]$.

Let π_1 be the π-value of the scanning curve through A. As $s(x,t)$ decreases from $s(x,t_1)$ as t increases, the flux changes according to the law $f = f^S(s,\pi_1)$ until this curve meets the curve $f = f^D(s)$ at the point labeled by B in Figure 2.1*(a)*. A further decrease of s forces the fluid into drainage mode where the flux changes according to the law $f = f^D(s)$. Assume now that the fluid at x is in drainage mode until t increases to $t_2 > t_1$ where the sign of $s_t(x,\cdot)$ reverses again, so that $s_t(x,t) > 0$ for $t_2 < t < t_2 + \tau$; the corresponding point in Figure 2.1*(a)* is labeled as C. Then the fluid enters into scanning mode again and the flux is governed by the law $f = f^S(s,\pi_2)$, where the curve $f = f^S(s,\pi_2)$ intersects the curve $f = f^D(s)$ at C.

If subsequently s increases, then the flux follows the curve $f = f^S(s,\pi_2)$ until it meets the imbibition curve at the point D. Further increases of s beyond this point bring the fluid to imbibition mode and the flux to be governed by the law $f = f^I(s)$ again.

Suppose that the flow at (x, t) is in scanning mode and hence the flux takes the form $f = f^S(s, \pi)$ for some π value. If $f^I(s) < f^S(s, \pi) < f^D(s)$, then $s(x, \cdot)$ can either increase or decrease while $f = f(s, \pi)$ for the same π. In other words, the flow can go both ways in scanning mode, unlike in imbibition or drainage modes where $s(x, \cdot)$ can only increase or decrease, respectively.

From the above discussion, we see that the parameter $\pi(x, t)$, which records the the parameter π to be used if the fluid at (x, t) is in scanning mode, influences $s(x, t)$. The state of the flow is described by $(s, \pi)(x, t)$ and hence a governing law for π must be specified to make (1.1) closed.

2.2. **The mathematical model.** Now, we state more precisely our assumptions. The imbibition and drainage flux-functions f^I and f^D are smooth in $[0, 1]$ and satisfy

$$f^D(0) = f^I(0) = 0, \quad f^D(1) = f^I(1) = 1, \tag{2.2}$$

$$f^D(s) > f^I(s), \quad f^D > f^I, \quad f^D_s > 0,\ f^I_s > 0, \quad \text{in } (0, 1), \tag{2.3}$$

Assumptions (2.2) and (2.3) are standard when studying conservation laws. We also assume that there exists a family of curves $\left\{(s, f^S(s, \pi))\right\}_{\pi\in[0,1]}$ providing a transversal foliation of the region located between $f^D(s)$ and $f^I(s)$. This means that at every point on $f = f^I(s)$, with $s \in (0, 1)$, there is a scanning curve (a leaf of the foliation) intersecting transversally $f = f^I(s)$ at that point and lying above it at the left of the intersection point; the same curve intersects transversally $f = f^D(s)$ at the other end point and lies below it at the right of the intersection point. We denote the abscissas of the points of intersection of $f = f^S(s, \pi)$ and $f = f^I(s)$ or $f = f^D(s)$ as $s^I(\pi)$ and $s^D(\pi)$, respectively; we have $s^D(\pi) < s^I(\pi)$ for $\pi \in (0, 1)$ because of (2.3). Thus, the physically feasible states (s, π) should satisfy

$$s^D(\pi) \le s \le s^I(\pi). \tag{2.4}$$

We define

$$\begin{aligned}
\Omega^I &= \left\{(s, \pi) \in [0, 1] \times [0, 1] \colon s = s^I(\pi)\right\}, \\
\Omega^D &= \left\{(s, \pi) \in [0, 1] \times [0, 1] \colon s = s^D(\pi)\right\}, \\
\Omega^S &= \left\{(s, \pi) \in (0, 1) \times (0, 1) \colon s^D(\pi) < s < s^I(\pi)\right\},
\end{aligned}$$

and finally $\Omega = \Omega^D \cup \Omega^S \cup \Omega^I$, see Figure 2.1*(b)*. We assume that f^S is defined and smooth in $\overline{\Omega^S}$ and satisfies

$$f^S_s(s,\pi) > 0,\ f^S_\pi(s,\pi) > 0, \quad \text{for } (s,\pi) \in \Omega^S, \tag{2.5}$$

$$f^S_s\left(s^I(\pi),\pi\right) < f^I_s\left(s^D(\pi)\right),\ f^S_s\left(s^D(\pi),\pi\right) < f^D_s\left(s^D(\pi)\right), \quad \text{for } \pi \in (0,1), \tag{2.6}$$

Assumption $f^S_\pi > 0$ in (2.5) is needed in order that the scanning curves do not intersect for different values of π, so that form a foliation. For future reference in the paper we gather the previous conditions under a single name.

Assumption (I). *We require conditions* (2.2)*,* (2.3)*,* (2.5) *and* (2.6)*.*

Lemma 2.1. *Assume (I). Then the functions $s^I(\pi)$ and $s^D(\pi)$ are strictly increasing in* $[0,1]$*.*

Proof. The function $s = s^I(\pi)$ solves the equation $f^S(s,\pi) - f^I(s) = 0$. By (2.6), we have $f^S_s(s,\pi) - f^I_s(s) < 0$ at $s = s^I(\pi)$. By the Implicit Function Theorem it follows that s^I is smooth and by (2.5) we have

$$s^I_\pi(\pi) = \frac{-f^S_\pi\left(s^I(\pi),\pi\right)}{f^S_s\left(s^I(\pi),\pi\right) - f^I_s\left(s^I(\pi)\right)} > 0. \qquad \square$$

Because of Lemma 2.1, both s^I and s^D are invertible in $[0,1]$; we denote by $\pi^I(s)$ and $\pi^D(s)$ their inverse functions, which are strictly increasing in $[0,1]$. The definition of $\pi^I(s)$ and $\pi^D(s)$ implies that

$$f^I(s) = f^S\left(s,\pi^I(s)\right) \quad \text{and} \quad f^D(s) = f^S\left(s,\pi^D(s)\right), \quad \text{for } s \in [0,1]. \tag{2.7}$$

Combining the behavior of the flux in different modes, the flux f in (1.1) can be written in $[0,1]\times[0,1]$ as

$$f = F(s,\pi) := \begin{cases} f^D(s) & \text{if } s \le s^D(\pi), \\ f^S(s,\pi) & \text{if } s^D(\pi) \le s \le s^I(\pi), \\ f^I(s) & \text{if } s \ge s^I(\pi), \end{cases} \tag{2.8}$$

see Figure 2.2*(a)*. Notice that, even if the physical flow is only defined in the domain $\Omega \subset [0,1]\times[0,1]$, the function f has been extended in a trivial way to the whole of $[0,1]\times[0,1]$. This will be needed, for instance, in the proof of the following Theorem 4.2. However, we show in Theorem 3.6 that the inequality $s^D(\pi) \le s \le s^I(\pi)$ is indeed enforced by the flow's governing system of equations; then, the extension in (2.8) does not alter solutions. We intentionally allow overlaps at $s = s^D(\pi)$ and $s = s^I(\pi)$ in definition (2.8) to emphasize that the scanning mode can occur when $s^D(\pi) \le s \le s^I(\pi)$, as we shall see just below. By the definition of $s^I(\pi)$ and $s^D(\pi)$, the function $F(s,\pi)$ is differentiable in $[0,1]\times[0,1]$ except along the curves $s = s^D(\pi)$

and $s = s^I(\pi)$, where it is continuous; it is then Lipschitz-continuous in $[0,1] \times [0,1]$. At any fixed π the graph of $F(\cdot,\pi)$ is provided in Figure 2.2*(b)*.

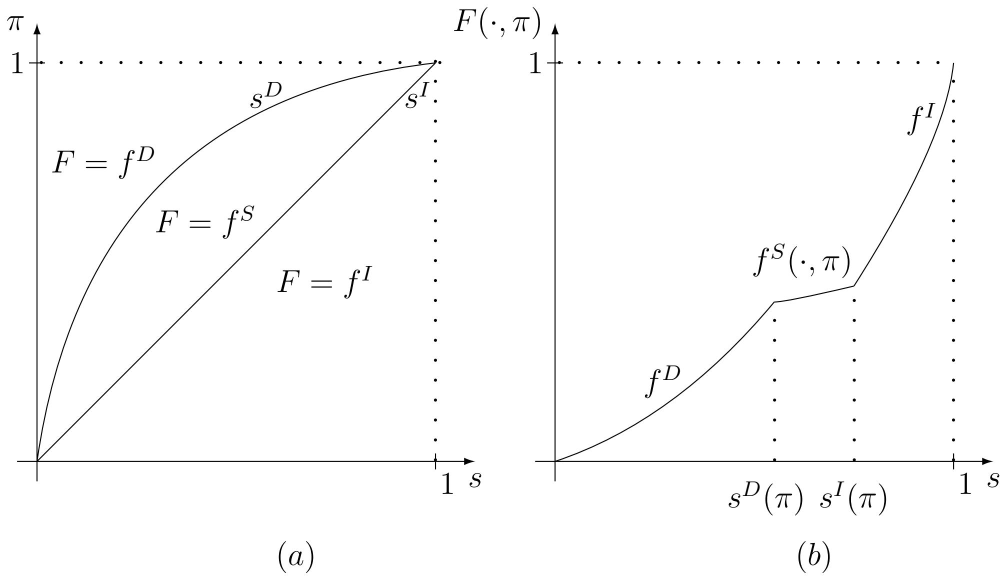


FIGURE 2.2. *(a)*: the function F. *(b)*: graph of the function $s \mapsto F(s,\pi)$ at π fixed.

In scanning mode, the pair (s,f) can move in either directions along the graph of $f = f^S(\cdot,\pi)$, for a fixed π. Since $\pi(x,t)$ records the value of π of the scanning flux to use if the fluid at (x,t) is in scanning mode, then $\pi_t = 0$ when the fluid is in scanning mode. The value $\pi(x,t)$ changes if and only if the fluid at x is in imbibition or drainage mode at time t; when this happens, then $\pi(x,t)$ is updated as $\pi^I\left(s(x,t)\right)$ and $\pi^D\left(s(x,t)\right)$, respectively. For given piecewise-continuous functions $s = s(x,t)$ and $\pi = \pi(x,t)$ defined in $\mathbb{R} \times [0,T]$ and valued in $[0,1]$, we introduce the boolean variables I, D and S, depending on (x,t), as

$$
\begin{aligned}
I &= \Big\{ s = s^I(\pi),\ s < 1 \text{ and } s_t > 0 \Big\},\\
D &= \Big\{ s = s^D(\pi)\ s > 0 \text{ and } s_t < 0 \Big\},\\
S &= \Big\{ s^D(\pi) < s < s^I(\pi) \Big\} \vee \big\{ s \in \{0,1\} \text{ and } s_t = 0 \big\}\\
&\quad \vee \Big\{ s = s^I(\pi),\ 0 < s < 1 \text{ and } s_t \le 0 \Big\} \vee \Big\{ s = s^D(\pi),\ 0 < s < 1 \text{ and } s_t \ge 0 \Big\}.
\end{aligned}
$$

Notice the particular care in handling the points $(0,0)$ and $(1,1)$ in the (s,π)-plane to avoid contradictions. Also notice that the cases when $s = 1, s_t > 0$ and $s = 0, s_t < 0$ are missing, as it is clear from a physical point of view. Indeed, we shall prove in

Corollary 3.7 that our equations prevent such cases. We also define the domains

$$\begin{aligned}\mathcal{D}^I &= \left\{(x,t)\in\mathbb{R}\times[0,T]\colon I(x,t)=\text{true}\right\},\\ \mathcal{D}^D &= \left\{(x,t)\in\mathbb{R}\times[0,T]\colon D(x,t)=\text{true}\right\},\\ \mathcal{D}^S &= \left\{(x,t)\in\mathbb{R}\times[0,T]\colon S(x,t)=\text{true}\right\}.\end{aligned}\tag{2.9}$$

From a physical point of view we must have $\mathbb{R}\times[0,T]=\mathcal{D}^I\cup\mathcal{D}^S\cup\mathcal{D}^D$. We shall prove, see Lemma 3.6, that the equations we propose below, see (2.16), indeed show that this is the case.

Since s typically has discontinuities, then the inequalities $s_t<0$ (≤ 0) and $s_t>0$ (≥ 0) are meant in the following distributional sense. The definition below differs from the usual definition of positivity/negativity of a distribution [9, page 38], which is given in open sets: our aim here is to pinpoint the sign of a distribution (indeed, of an atomic measure) at a point. To this aim we introduce the function space

$$\Psi := \left\{\psi\in C_0^\infty(\mathbb{R}) : \psi(x)\ge 0, \psi(0)>0, \operatorname{spt}\psi\subset[-1,1], \int_{-1}^1\psi(x)dx=1\right\}$$

and denote for $\psi\in\Psi$ its rescaled function

$$\psi_\mu(x)=\frac{1}{\mu}\psi\left(\frac{x}{\mu}\right).\tag{2.10}$$

In the following we use the notation

$$\psi^{(k)}(\xi):=\frac{d^k}{d\xi^k}\psi(\xi)\quad\text{ and hence }\quad\frac{d^k}{dx^k}\psi_\mu(x)=\frac{1}{\mu^k}\psi_\mu^{(k)}(x),\tag{2.11}$$

where $\psi_\mu^{(k)}(x)$ is the function $\psi^{(k)}(x)$ rescaled as in (2.10). The action of a distribution $u\in\mathcal{D}'(\mathbb{R})$ on a test function $\phi\in C_0^\infty(\mathbb{R})$ is denoted as

$$\langle u(x),\phi(x)\rangle =: \int_{\mathbb{R}} u(x)\phi(x)dx.\tag{2.12}$$

We also use this notation for distributions $v(x,t)$ and test functions $\phi(x,t)$ that are compactly supported in $\mathbb{R}\times[0,T]$, for some $T>0$:

$$\langle v(x,t),\phi(x,t)\rangle =: \iint_{\mathbb{R}\times[0,T]} v(x,t)\phi(x,t)\,dxdt.\tag{2.13}$$

Moreover, recall that by the Schwartz kernel Theorem [9, Th. 5.2.1] if $v\in\mathcal{D}'(\mathbb{R}\times[0,T])$ and $\phi(x)$, $\psi(t)$ are test functions, then the expression $\langle\tilde v(\phi),\psi\rangle:=\langle v,\phi(x)\psi(t)\rangle$ defines a a linear map $\tilde v : C_0^\infty(\mathbb{R})\to\mathcal{D}'([0,T])$ which is continuous in the sense that $\tilde v(\phi_j)\to 0$ in $\mathcal{D}'([0,T])$ if $\phi_j\to 0$ in $C_0^\infty(\mathbb{R})$ (and conversely). In the following we need to slightly extend this definition to the case of a function ϕ also depending on t and we define

$$\langle v(x,t),\phi(x,t)\rangle_x =: \int_{\mathbb{R}} v(x,t)\phi(x,t)\,dx,\tag{2.14}$$

Then $\langle v(x,t), \phi(x,t)\rangle_x$ is a distribution in $\mathcal{D}'([0,T])$. An analogous notation is used when v operates on test functions depending on the variable t.

Definition 2.1. *Consider $g \in \mathcal{D}'(E)$ for $E \subseteq \mathbb{R}$ and let $x_0 \in E$. We say that $g(x_0) > 0$ (or $g(x_0) \geq 0$) in the sense of distributions if*

$$\liminf_{\mu\to 0+} \langle g(x), \psi_\mu(x-x_0)\rangle > 0 \quad (\textit{or} \geq 0),$$

for any test function $\psi \in \Psi$. Similarly, we say that $g(x_0) < 0$ (or $g(x_0) \leq 0$) in the sense of distributions if for any test function $\psi \in \Psi$

$$\limsup_{\mu\to 0+} \langle g(x), \psi_\mu(x-x_0)\rangle < 0 \quad (\textit{or} \leq 0).$$

Notice that the actual *value* of $g(x_0)$ as a distribution (if it exists) does not affect the *sign* of $g(x_0)$ in the above sense. Clearly, if a distribution is non-negative in a neighborhood of x_0 according to the usual definition, then it is non-negative at x_0 according to Definition 2.1. We remark that the lim inf and lim sup above can be ∞ or $-\infty$, respectively.

Example 2.2. *The Heaviside function H satisfies $H(0) > 0$ in the sense of Definition 2.1, independently of how it is defined at $x = 0$. Definition 2.1 does not require $\langle H, \psi_\mu\rangle$ to be bounded away from 0 uniformly with respect to ψ. Analogously, the function* $\operatorname{sign}(x)$ *has no sign at* 0.

By integration by parts one easily proves that if $f(x) = 1 + \sin(1/x)$, then $f(0) > 0$ because $\langle \sin(1/x), \psi_\mu\rangle \to 0$ as $\mu \to 0+$. In the case $f(x) = \delta(x)$ we find that $\liminf_{\mu\to 0+}\langle \delta, \psi_\mu\rangle = \infty$ for any $\psi \in \Psi$ and then $\delta(0) > 0$ according to Definition 2.1.

We also define the boolean characteristic function

$$\chi(A) = \begin{cases} 1 & \text{if } A \text{ is true,} \\ 0 & \text{else.} \end{cases}$$

Then the equation for π_t is

$$\pi_t = \chi(I)\pi^I(s)_t + \chi(D)\pi^D(s)_t. \tag{2.15}$$

We notice that (2.15) is *fully nonlinear* because the functions $\chi(I)$ and $\chi(D)$ are *nonlinear discontinuous* functions of both s and s_t. At last, it is *not in conservation form*, which is expected because π is not a conserved quantity.

As the flow at a fixed x moves through states A, B, C, D in Figure 2.1 continuously, equation (2.15) forces $\pi(x,t) = \pi^I\big(s(x,t)\big)$ when the fluid at (x,t) is in imbibition mode, and $\pi(x,t) = \pi^D\big(s(x,t)\big)$ if the fluid is in drainage mode.

Therefore, by taking into account the hysteresis variable π, the complete system for (1.1) is the combination of (1.1), (2.8) and (2.15):

$$\begin{cases} s_t + F(s,\pi)_x = \epsilon\left(A(s,\pi)s_x\right)_x, \\ \pi_t = \chi(I)\pi^I(s)_t + \chi(D)\pi^D(s)_t, \end{cases} \tag{2.16}$$

where A is smooth, bounded away from 0 and $\epsilon > 0$ is a constant. Although we know of no papers stating the dependence of A on π, we just allowed it here in order to keep generality. It is unphysical to have a diffusion term in the second equation.

We also consider the inviscid case of (2.16), namely,

$$\begin{cases} s_t + F(s,\pi)_x = 0, \\ \pi_t = \chi(I)\pi^I(s)_t + \chi(D)\pi^D(s)_t. \end{cases} \tag{2.17}$$

Equation $(2.17)_2$ presents the classical mathematical challenge of giving a meaning to the product of distributions, in particular the meaning of the product of a step function and $\delta(x)$. Dal Maso, LeFloch and Murat [4] defined $L^\infty \cap BV_{\mathrm{loc}}$ solutions of nonconservative systems of quasilinear PDEs in the sense of Borel measures associated to a family of paths joining ends of approximate jumps of solutions. A direct application of the results of [4] requires $\chi(I)$ to be continuous, among other conditions, which system (2.17) fails to satisfy. Another possibility would be to study this problem in the framework of Colombeau's generalized functions [2] but we prefer a simpler and more physical approach.

Definition 2.2. *Consider system* (2.16) *and initial data*

$$\left(s(x,0),\pi(x,0)\right) = \left(s_0(x),\pi_0(x)\right) \tag{2.18}$$

satisfying

$$\left(s_0(x),\pi_0(x)\right) \in [0,1]\times[0,1] \quad \text{and} \quad s^D\left(\pi_0(x)\right) \le s_0(x) \le s^I\left(\pi_0(x)\right). \tag{2.19}$$

Assume that problem (2.16)-(2.18) *has a solution* $\left(s_{\epsilon_k},\pi_{\epsilon_k}\right)$*, for each* ϵ *in a sequence* $\{\epsilon_k > 0\}_{k\in\mathbb{N}}$ *where* $\epsilon_k \to 0+$ *as* $k\to\infty$*. If the sequence* $\left(s_{\epsilon_k},\pi_{\epsilon_k}\right)$ *is weakly convergent to some* (s,π)*, then we say that* (s,π) *is a solution of* (2.17)-(2.18).

Condition (2.19) is added to ensure consistency with the physical requirement (2.4). Note that different sequences $\{\epsilon_k\}$ could possibly provide different limits. It is known that for evolution equations not in conservation form, whose solutions have jump discontinuities, the two ends of a jump discontinuity are usually sensitive to the form of the vanishing viscosity terms, see [11]. We shall see that all results of existence of viscous profiles in Section 4, Theorem 4.2 to Theorem 4.5, show that the end states of viscous profiles of solutions of (2.17) are independent from the diffusivity A in (2.16).

Remark 2.1. Since the flux F takes on different forms when the fluid is in different modes, there could be possibilities for self-contradiction. For example, suppose that $\pi = \pi^I(s)$ holds near a point x at time t; then, the fluid is in scanning mode if $s_t \le 0$, or in imbibition mode if $s_t > 0$. If the equation $s_t + f^S(s,\pi)_x = 0$ yields $s_t > 0$ while $s_t + f^I(s)_x = 0$ shows $s_t \le 0$, then we have a self-contradiction. To avoid such self-contradictions, the signs of $s_t = -f^I_s(s)s_x$ and $s_t = -f^S_s(s)s_x$ must coincide when $\pi = \pi^I(s)$. Similar considerations are valid also when $\pi = \pi^D(s)$. Thus, the following *compatibility conditions*,

$$\begin{aligned} \operatorname{sign} f^I_s(s) &= \operatorname{sign} f^S_s(s,\pi) \quad \text{when } \pi = \pi^I(s),\\ \operatorname{sign} f^D_s(s) &= \operatorname{sign} f^S_s(s,\pi) \quad \text{when } \pi = \pi^D(s). \end{aligned}$$

should be required. Indeed, these conditions are already included in assumptions (2.3) and (2.5), where all the above signs are positive.

Now we formally discuss the implications of Definition 2.2 at the possible points of jump discontinuity of solutions (s,π) of system (2.17). Assume that a solution of (2.16) has a solution (s,π) with a discontinuity at (x_0,t_0) propagating with constant speed σ. In a neighborhood of (x_0,t_0) we introduce the change of variables

$$\xi = \frac{x - x_0 - \sigma(t-t_0)}{\epsilon}, \quad \tau = t,$$

and denote $\big(\bar{s}(\xi,\tau),\bar{\pi}(\xi,\tau)\big) = \big(s(x,t),\pi(x,t)\big)$, $\bar{F}(\bar{s},\bar{\pi}) = F(s,\pi)$ and so on. Since $s_x = \frac{1}{\epsilon}\bar{s}_\xi$, $s_t = \bar{s}_\tau - \frac{\sigma}{\epsilon}\bar{s}_\xi$, and analogously for π, system (2.16) becomes

$$\begin{cases} \epsilon\bar{s}_\tau - \sigma\bar{s}_\xi + \bar{F}(\bar{s},\bar{\pi})_\xi = \big(\bar{A}(\bar{s},\bar{\pi})\bar{s}_\xi\big)_\xi, \\ \epsilon\bar{\pi}_\tau - \sigma\bar{\pi}_\xi = \epsilon\chi(\bar{I})\bar{\pi}^I(\bar{s})_\tau - \sigma\chi(\bar{I})\bar{\pi}^I(\bar{s})_\xi + \epsilon\chi(\bar{I})\bar{\pi}^D(\bar{s})_\tau - \sigma\chi(\bar{D})\bar{\pi}^D(\bar{s})_\xi. \end{cases} \tag{2.20}$$

Notice that both (s,π) and $(\bar{s},\bar{\pi})$ depend on ϵ and $(\bar{s},\bar{\pi})(\xi,\tau_0) = (s,\pi)(x_0+\epsilon\xi,t_0)$. If $(s,\pi) \to (S,\Pi)$ and $(\bar{s},\bar{\pi}) \to (\bar{S},\bar{\Pi})$ as $\epsilon = \epsilon_k \to 0$ in the sense of distributions, then formally S has a jump discontinuity at x_0 while $(\bar{S},\bar{\Pi})$ become functions of the single variable ξ and satisfy

$$\begin{cases} -\sigma\bar{S}_\xi + \bar{F}(\bar{S},\bar{\Pi})_\xi = \big(\bar{A}(\bar{S},\bar{\Pi})\bar{S}_\xi\big)_\xi, \\ -\sigma\bar{\Pi}_\xi = -\sigma\left[\chi(\bar{I})\bar{\Pi}^I(\bar{S})_\xi + \chi(\bar{D})\bar{\Pi}^D(\bar{s})_\xi\right], \end{cases} \tag{2.21}$$

with boundary conditions

$$(\bar{S},\bar{\Pi})(\pm\infty) \doteq (\bar{S}_\pm,\bar{\Pi}_\pm) = (s,\pi)(x_0\pm,t_0) \quad \text{and} \quad (\bar{S}',\bar{\Pi}')(\pm\infty) = (0,0).$$

By integrating in ξ we find

$$\begin{cases} \sigma\left(\bar{S}_+ - \bar{S}_-\right) = \bar{F}_+ - \bar{F}_-, \\ \sigma(\Pi_+ - \Pi_-) = \sigma\left[\displaystyle\int_{-\infty}^{\infty}\chi(\bar{I})\,d\pi^I(\bar{s}) + \int_{-\infty}^{\infty}\chi(\bar{D})\,d\pi^D(\bar{s})\right]. \end{cases} \tag{2.22}$$

These equations can be understood as the Rankine-Hugoniot equations corresponding to discontinuities of system (2.17), see [4].

3. The Maximum Principle for the Viscous System (2.16)

In this section we deal with piecewise smooth solutions (s,π) of the viscous system (2.16) and prove that π satisfies the maximum principle. Furthermore, we prove that $TV\left(\pi(\cdot,t)\right) \le TV\left(\pi(\cdot,0)\right)$.

Clearly π_t suffers a jump discontinuity at every mode boundary because equation $(2.16)_2$ is discontinuous there; so, at best we expect π_t to be piecewise smooth. If π has a jump in x, then the left-hand side of $(2.16)_1$ contains a Dirac mass; this forces the term s_x in the right-hand side to be have a jump; as a consequence, s must be at least continuous. Indeed, if s has a jump in x then the equations are difficult to match. Then, we expect that s is continuous. We introduce the notation

$$\mathcal{S}\left(\mathbb{R}\times[0,T]\right) :=$$
$$=\left\{(s,\pi):\mathbb{R}\times[0,T]\mapsto\mathbb{R}^2\colon s\in C(\mathbb{R}\times[0,T];\mathbb{R})\ s,\pi\in\mathcal{PC}^2(\mathbb{R}\times[0,T];\mathbb{R})\right\},$$

where $\mathcal{PC}^k(\mathbb{R}\times[0,T];\mathbb{R})$, $k=0,1,\ldots$, denotes the collection of piecewise $\mathcal{C}^k$ functions $u:\mathbb{R}\times[0,T]\mapsto\mathbb{R}$. Here piecewise $\mathcal{C}^k$ means that $\mathbb{R}\times[0,T]$ can be partitioned locally into finitely many simply connected regions, with piecewise C^1 boundaries with finitely many pieces locally; the function $u(x,t)$ is $\mathcal{C}^k$ in the interior of each region and can be extended as a $\mathcal{C}^k$ function to the closure of each of them. The set for initial data is defined as

$$\mathcal{S}\left(\mathbb{R}\right) :=$$
$$=\left\{(s,\pi):\mathbb{R}\mapsto[0,1]^2\colon s\in C(\mathbb{R};[0,1])\ s,\pi\in\mathcal{PC}^2(\mathbb{R};[0,1]^2)\text{ and (2.19) holds}\right\}.$$

We shall consider solutions $(s,\pi)\in\mathcal{S}\left(\mathbb{R}\times[0,T]\right)$ and initial data $(s_0,\pi_0)\in\mathcal{S}\left(\mathbb{R}\right)$.

Lemma 3.1. *Assume (I). If $(s,\pi)\in\mathcal{S}\left(\mathbb{R}\times[0,T]\right)$ is a solution of system (2.16), then $\pi(x\pm,t)$ is continuous in t for every $x\in\mathbb{R}$.*

Proof. For a function $g\in\mathcal{S}$ the functions $g(x\pm,t)$, $g(x\pm,t)_t$ and $g(x\pm,t)_{tt}$, are well defined almost everywhere and are locally bounded. Thus, we can pass the limit through the integral as

$$\lim_{x\to x_0\pm}\int_{t_0-h_1}^{t_0+h_2} g_t(x,t)dt = \int_{t_0-h_1}^{t_0+h_2} g_t(x\pm,t)dt.$$

Fix $x_0\in\mathbb{R}$ and assume, for contradiction, that $\pi(x_0+,\cdot)$ (or $\pi(x_0-,\cdot)$) is discontinuous at $t=t_0$. Since $(s,\pi)\in\mathcal{PC}^2(\mathbb{R}\times[0,T])$, there is no other point of discontinuity for $\pi(x_0+,\cdot)$ in the interval $[t_0-h_1, t_0+h_2]$ if $h_1>0$ and $h_2>0$ are sufficiently small. Then $(s,\pi)(x_0+,\cdot)$ is C^2 over the interval $[t_0-h_1,t_0]$. By integrating $(2.16)_2$ we get

$$\begin{aligned} &\pi(x_0+,t_0+h_2)-\pi(x_0+,t_0-h_1) \\ &\quad = \int_{t_0-h_1}^{t_0+h_2} \Big[\chi(I)\pi^I\left(s(x_0+,t)\right)_t + \chi(D)\pi^D\left(s(x_0+,t)\right)_t\Big]\,dt =: \mathcal{I}+\mathcal{J}. \end{aligned} \tag{3.1}$$

The open set

$$E_1 := \big\{t\in[0,T]\colon s_t(x_0+,t)>0\big\}\cap(t_0-h_1,t_0)$$

is the union of at most countably many disjoint open subintervals, i.e.,

$$E_1=\bigcup_{j=0}^{N}(\tau_{2j},\tau_{2j+1}),$$

where $N\le\infty$ and $\tau_i<\tau_{i+1}$ for all $0\le i\le 2N+1$. Then we deduce

$$\begin{aligned} \int_{t_0-h_1}^{t_0}\chi(I)\pi^I\left(s(x_0+,t)\right)_t\,dt &= \int_{E_1}\pi^I\left(s(x_0+,t)\right)_t\,dt \\ &= \sum_{j=0}^{N}\Big[\pi^I\left(s(x_0+,\tau_{2j+1})\right)-\pi^I\left(s(x_0+,\tau_{2j})\right)\Big] = O(1)h_1, \end{aligned}$$

because $s(x,\cdot)$ is continuous and s_t is bounded a.e. in a neighborhood of $t=t_0$. Similarly, we have

$$\int_{t_0}^{t_0+h_2}\chi(I)\pi^I\left(s(x_0+,t)\right)_t\,dt = O(1)h_2. \tag{3.2}$$

Because $s(\cdot,t)$ is continuous, the item $\mathcal{I}$ in (3.1) becomes

$$\mathcal{I}=\left(\int_{t_0-h_1}^{t_0}+\int_{t_0}^{t_0+h_2}\right)\chi(I)\left(s(x_0+,t)\right)_t\,dt. \tag{3.3}$$

The estimates above then show that

$$\mathcal{I}=O(1)(h_1+h_2). \tag{3.4}$$

Analogously, we can prove that

$$\mathcal{J}=\int_{t_0-h_1}^{t_0+h_2}\chi(D)\pi^D\left(s(x_0+,t)\right)_t\,dt = O(1)(h_1+h_2). \tag{3.5}$$

The continuity of $\pi(x_0+,\cdot)$ at t_0 follows from (3.1), (3.4) and (3.5). □

If $(s,\pi)\in\mathcal{S}\left(\mathbb{R}\times[0,T]\right)$ then $\langle s_t,\psi_\mu(x-x_0)\rangle_x$ is in $\mathcal{D}'(\mathbb{R})$, see the discussion following (2.13). The following lemma, which will be used in the proof of Proposition 3.3, states that if (s,π) are solutions to (2.16) then $\langle s_t,\psi_\mu(x-x_0)\rangle_x$ is indeed a continuous function. However, we need the following assumption on the form of the viscosity term.

Assumption (II). *The viscosity term in* (2.16) *only depends on s, i.e., $A(s,\pi) = A(s)$.*

Lemma 3.2. *Assume (I) and (II); let $(s,\pi) \in \mathcal{S}\left(\mathbb{R} \times [0,T]\right)$ be a solution of system* (2.16)*. Then the expression $\langle s_t, \psi_\mu(x-x_0)\rangle_x$ is continuous in t.*

Proof. Since (s,π) is a solution of (2.16), then

$$\langle s_t, \psi_\mu(x-x_0)\rangle_x = \int \left[F(s,\pi)\psi_\mu(x-x_0)_x + \epsilon A(s)\psi_\mu(x-x_0)_{xx}\right] dx. \tag{3.6}$$

The right hand side of (3.6) is continuous with respect to t because s is so by assumption and π by Lemma 3.1. □

Remark 3.1. Lemma 3.2 implies that the integral

$$\int_{-\mu}^{\mu} s_t(y,t)\psi_\mu(y-x_0)\,dy$$

is well defined for $t \in [0,T]$ as the pointwise value of a continuous function. In this case we slightly extend Definition 2.1 to mean $s_t(x,t_0) \le 0$ at $x = x_0$ as

$$\limsup_{\mu\to 0+} \int_{-\mu}^{\mu} s_t(y,t_0)\psi_\mu(y-x_0)\,dy \le 0, \tag{3.7}$$

for every $\psi \in \Psi$.

Dealing with extrema of a piecewise discontinuous function may be difficult because of the value assumed by such a function at a discontinuity point. Consider $f \in \mathcal{PC}^0(\mathbb{R};\mathbb{R})$; a point $x_0 \in \mathbb{R}$ is said an *essential local-maximum point* if $\max\left\{f(x_0+), f(x_0-)\right\} \ge f(x)$ for every x in a neighborhood of x_0; the value $\max\left\{f(x_0+), f(x_0-)\right\}$ is called an *essential local maximum.* An analogous definition holds for the minimum. In the case f is continuous at x_0 we recover the usual definitions.

Proposition 3.3. *Assume (I) and (II). Let $(s,\pi) \in \mathcal{S}\left(\mathbb{R} \times [0,T]\right)$ be a solution of system* (2.16) *and $(x_0,t_0) \in \mathbb{R} \times (0,T)$. If x_0 is an essential local maximum point of both $s(\cdot,t_0)$ and $\pi(\cdot,t_0)$ satisfying* $\max\left\{\pi(x_0+,t_0), \pi(x_0-,t_0)\right\} = \pi^I\left(s(x_0,t_0)\right)$*, then $\pi(\cdot,t_0)$ is continuous at $x = x_0$.*

Furthermore, $s_t(x,t_0) \le 0$ holds at $x = x_0$ in the sense of Definition 2.1.

Proof. First, we prove that $\pi(\cdot,t_0)$ is continuous at x_0. Indeed, assume that x_0 is an essential local maximum point for $\pi(\cdot,t_0)$. Then, in a neighborhood of x_0 we have

$$\pi^I\left(s(x,t_0)\right) \le \pi(x,t_0) \le \max\left\{\pi(x_0+,t_0), \pi(x_0-,t_0)\right\} = \pi^I\left(s(x_0,t_0)\right). \tag{3.8}$$

Since s is continuous, the claim follows by passing to the limit for $x \to x_0$.

Second, we must show (3.7). We test equation $(2.16)_1$ with function $\psi_\mu(x-x_0)$, for $\psi \in \Psi$; by the change of variables $y := x - x_0$ we obtain

$$\begin{aligned}\int_{-\mu}^{\mu} s_t(y+x_0,t)\psi_\mu(y)dy &= \frac{1}{\mu}\int_{-\mu}^{\mu} F(s,\pi)(y+x_0,t)\psi'_\mu(y)dy \\ &\quad - \frac{\epsilon}{\mu}\int_{-\mu}^{\mu} A(s,\pi)(y+x_0,t_0)s_x(y+x_0,t_0)\psi'_\mu(y)\,dy.\end{aligned} \tag{3.9}$$

Consider the first summand in the right-hand side of (3.9) at $t=t_0$; we have

$$\frac{1}{\mu}\int_{-\mu}^{\mu} F(s,\pi)(y+x_0,t_0)\psi'_\mu(y)dy = \frac{1}{\mu}\int_{-\mu}^{\mu}[F(s,\pi)(y+x_0,t_0)-F(s,\pi)(x_0,t_0)]\psi'_\mu(y)dy,$$

beacuse $\int_{-\mu}^{\mu}\psi'(y)dy = 0$. Moreover,

$$\begin{aligned}&\frac{1}{\mu}\left|\int_{-\mu}^{\mu} F(s,\pi)(y+x_0,t_0)\psi'_\mu(y)dy\right| \\ &\le \frac{1}{\mu}\int_{-\mu}^{\mu}\big[F(s,\pi)(x_0,t_0)-F(s,\pi)(y+x_0,t_0)\big]\,|\psi'_\mu(y)|dy.\end{aligned} \tag{3.10}$$

We proved in (3.8) that $\pi(x_0,t_0) = \pi^I\big(s(x_0,t_0)\big)$; moreover, we always have $\pi(x,t) \ge \pi^I\big(s(x,t)\big)$. Then we obtain

$$\begin{aligned}&\frac{1}{\mu}\left|\int_{-\mu}^{\mu} F(s,\pi)(y+x_0,t_0)\psi'_\mu(y)dy\right| \\ &\le \frac{1}{\mu}\int_{-\mu}^{\mu}\left[F\left(s,\pi^I(s)\right)(x_0,t_0) - F\left(s,\pi^I(s)\right)(y+x_0,t_0)\right]|\psi'_\mu(y)|dy \\ &= \frac{1}{\mu}\int_{-\mu}^{\mu}\left[f^I\left(s(x_0,t_0)\right) - f^I\left(s(y+x_0,t_0)\right)\right]|\psi'_\mu(y)|dy \\ &= \frac{O(1)}{\mu}\int_{-\mu}^{\mu}\big[s(x_0,t_0)-s(y+x_0,t_0)\big]\,|\psi'_\mu(y)|dy,\end{aligned} \tag{3.11}$$

where $O(1)$ only depends on f. Since $s \in \mathcal{S}\big(\mathbb{R}\times[0,T]\big)$, and in particular s is continuous, then it has the Taylor expansion

$$s(y+x_0,t_0) = s(x_0,t_0)+s_x(x_0\pm,t_0)y+\frac{1}{2}s_{xx}(x_0\pm,t_0)y^2+o(1)y^2, \quad \text{for } \pm y > 0. \tag{3.12}$$

For brevity of notation we write in the following

$$\int_{-\mu}^{\mu} s_x(x_0\pm,t_0)g(y)\,dy := \int_{-\mu}^{0} s_x(x_0-,t_0)g(y)\,dy + \int_{0}^{\mu} s_x(x_0+,t_0)g(y)\,dy,$$

for some function $g = g(y)$ and so on. Then (3.9), (3.11) and (3.12) lead to

$$\begin{aligned}
&\int_{-\mu}^{\mu} s_t(y+x_0,t_0)\psi_\mu(y)dy \\
&= \frac{O(1)}{\mu}\int_{-\mu}^{\mu}\left[s_x(x_0\pm,t_0)y + \frac{1}{2}s_{xx}(x_0\pm,t_0)y^2 + o(1)y^2\right]\ |\psi'_\mu(y)|dy \\
&\quad -\frac{\epsilon}{\mu}\int_{-\mu}^{\mu} A(s,\pi)(y+x_0,t_0)\left[s_x(x_0\pm,t_0) + s_{xx}(x_0\pm,t_0)y + o(1)y\right]\psi'_\mu(y)\,dy.
\end{aligned} \tag{3.13}$$

There are two possibilities.

(a) Case $s_x(x_0-,t_0) = s_x(x_0+,t_0)$. Since x_0 is a local maximum point of $s(x,t_0)$, then $s_x(x_0,t_0) = 0$ and $s_{xx}(x_0\pm,t_0) \le 0$. If $s_{xx}(x_0+,t_0) = s_{xx}(x_0-,t_0) < 0$, then the conclusion is obvious. Otherwise we have two subcases.

(a1) If $s_{xx}(x_0+,t_0) = s_{xx}(x_0-,t_0) = 0$, then (3.13) equals $(\mu+\epsilon)\,o(1)$, where $o(1)$ tends to 0 as $\mu \to 0+$. This, together with the Bounded Convergence Theorem, proves (3.7) in this case.

(a2) If $s_{xx}(x_0+,t_0) \neq s_{xx}(x_0-,t_0)$, then at least one is < 0 and the other is ≤ 0. By (3.13) we deduce

$$\begin{aligned}
&\int_{-\mu}^{\mu} s_t(y+x_0,t_0)\psi_\mu(y)dy \\
&\le O(1)\,\frac{\mu}{2}\int_{-\mu}^{\mu}\left[s_{xx}(x_0\pm,t_0)\frac{y^2}{\mu^2} + o(1)\frac{y^2}{\mu^2}\right]\ |\psi'_\mu(y)|dy \\
&\quad -\epsilon\int_{-\mu}^{\mu} A(s,\pi)(y+x_0,t_0)\left[s_{xx}(x_0\pm,t_0)\frac{y}{\mu} + o(1)\frac{y}{\mu}\right]\psi'_\mu(y)\,dy.
\end{aligned} \tag{3.14}$$

By making the change of variables $\zeta = y/\mu$, it is easy to show that the first summand contributes for $O(1)\mu$ and then tends to 0 as $\mu \to 0+$. The second integral is dealt in the same way and contributes

$$-\epsilon\int_{-1}^{1} A(s,\pi)(x_0+\mu\zeta,t_0)\left[s_{xx}(x_0\pm,t_0) + o(1)\right]\zeta\psi'(\zeta)d\zeta,$$

where $o(1) \to 0$ and $A(s,\pi)(x_0+\mu\zeta,t_0) \to A(s,\pi)(x_0,t_0)$ as $\mu \to 0+$. Recall that we proved earlier in this proof that $(s,\pi)(x,t_0)$ is continuous at $x = x_0$. Since

$$-\int_{-1}^{0}\zeta\psi'(\zeta)\,d\zeta = \int_{-1}^{0}\psi(\zeta)\,d\zeta > 0 \quad \text{and} \quad -\int_{0}^{1}\zeta\psi'(\zeta)\,d\zeta = \int_{0}^{1}\psi(\zeta)\,d\zeta > 0,$$

we deduce that the second summand is strictly negative in the limit $\mu \to 0+$. This proves (3.7) in this case.

(b) Case $s_x(x_0-,t_0) \neq s_x(x_0+,t_0)$. Then, $s_x(x_0-,t_0) \ge 0$ and $s_x(x_0+,t_0) \le 0$ and at least one of them is not 0, implying $s_x(x_0+,t_0) - s_x(x_0-,t_0) < 0$. In this case,

estimate (3.13) can be further processed to yield

$$\int_{-\mu}^{\mu} s_t(y,t_0)\psi_\mu(y)dy \le O(1) + O(1)\mu$$
$$+\frac{\epsilon}{\mu}A(s,\pi)(x_0,t_0)\Big[\big(s_x(x_0+,t_0)-s_x(x_0-,t_0)\big)\psi(0)+O(1)\mu\Big]\to -\infty \tag{3.15}$$

as $\mu \to 0+$. This proves (3.7) in this case and then the proposition.

□

The following result is proved analogously to Proposition 3.3.

Proposition 3.4. *Assume (I) and (II). Let $(s,\pi) \in \mathcal{S}\big(\mathbb{R}\times(0,T)\big)$ be a solution of (2.16) and $(x_0,t_0) \in \mathbb{R}\times(0,T)$. If x_0 is an essential local-minimum point of both s and π satisfying* $\min\big\{\pi(x_0+,t_0),\pi(x_0-,t_0)\big\} = \pi^D\big(s(x_0,t_0)\big)$*, then $\pi(\cdot,t_0)$ is continuous at $x=x_0$. Furthermore, we have $s_t(x,t_0)\ge 0$ when $x=x_0$ in the sense of Definition 2.1.*

For $0<m<M<1$ we denote, see Figure 3.1,

$$\Omega_m^M := \Big\{(s,\pi)\in[0,1]\times[0,1]\colon s^D(\pi)\le s\le s^I(\pi),\ m\le\pi\le M\Big\}.$$

Now we show that Ω_m^M is an invariant domain for any $0<m<M<1$.

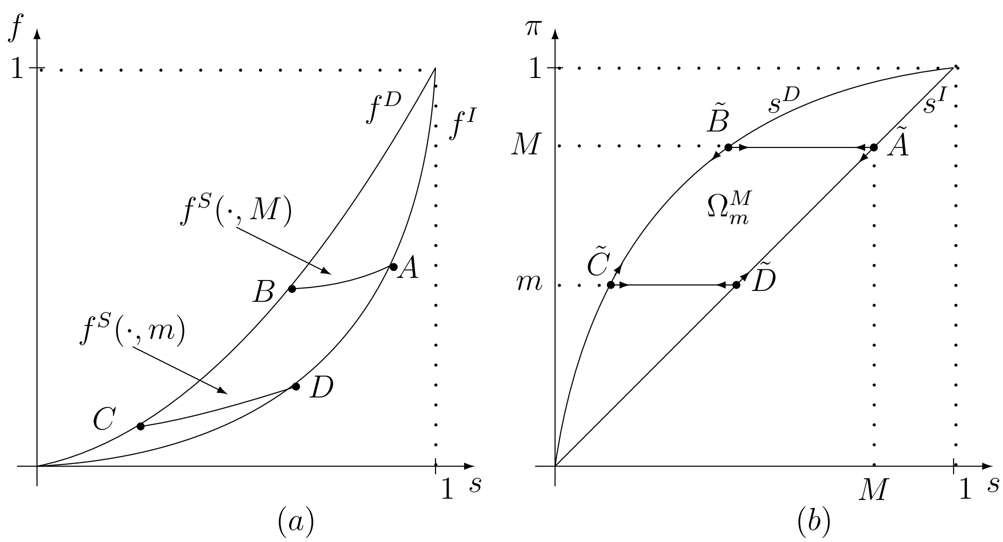


FIGURE 3.1. The set Ω_m^M. *(a)*: in the (s,f)-plane; *(b)*: in the (s,π)-plane, with the flow directions at points $\tilde{A},\ldots,\tilde{D}$.

Proposition 3.5. *Assume (I), (II) and $0<m<M<1$. If $(s_0,\pi_0)\in\mathcal{S}(\mathbb{R})$ are valued in Ω_m^M, then any solution $(s,\pi)\in\mathcal{S}\big(\mathbb{R}\times[0,T]\big)$ of (2.16)-(2.18) satisfies*

$$(s,\pi)(x,t)\in\Omega_m^M,\qquad (x,t)\in\mathbb{R}\times[0,T], \tag{3.16}$$

unless (3.16) is violated at $x\to\pm\infty$ for some $t\in[0,T)$ first.

Proof. It suffices to prove that if we have both $\limsup_{x\to\pm\infty}(s,\pi)(x,t\pm) \in \Omega_m^M$ and $\liminf_{x\to\pm\infty}(s,\pi)(x,t\pm) \in \Omega_m^M$, then $(s,\pi)(x,t\pm) \in \Omega_m^M$. In other words, it suffices to prove that $(s,\pi)(x,t\pm) \in \Omega_m^M$ cannot fail at $x \in (-\infty,\infty)$ at a time $t_1 > 0$ without it fails at $x = \infty$ or $-\infty$ at an earlier time $0 < t_0 < t_1$.

Consider the set Ω_m^M depicted in Figure 3.1. In the proof we exploit several times the continuity of s and $\pi(x,\cdot)$.

(a) First, we claim that a solution cannot exit Ω_m^M through the corner point $\tilde{A}$. Assume the contrary; then there is (x_0,t_0) such that $\big(s(x_0,t_0),\pi(x_0\pm,t_0)\big) = \tilde{A}$, $(s,\pi)(x,t_0) \in \Omega_m^M$ for every $x \in \mathbb{R}$ and, for some number $\gamma > 0$,

$$\pi(x\pm,t) > M = \pi(x_0,t_0) \quad \text{for some } (x,t) \in B_\gamma(x_0,t_0) \cap \{t > t_0\}. \tag{3.17}$$

Here and in the following the symbol "$\pm$" is understood as "either $+$ or $-$" and $B_\gamma(x_0,t_0)$ is the closed ball in $\mathbb{R}^2$ with radius γ and center (x_0,t_0). Since the point $x = x_0$ is an essential local-maximum point of both $s(\cdot,t_0)$ and $\pi(\cdot,t_0)$, by Proposition 3.3 we deduce $\pi(x_0-,t_0) = \pi(x_0+,t_0) = \pi^I\left(s(x_0,t_0)\right)$. Then (x_0,t_0) is indeed a local maximum point and so

$$(s,\pi)(x_0,t_0) = \tilde{A}, \quad (s,\pi)(x,t_0) \in \Omega_m^M \text{ for } x \in \mathbb{R}. \tag{3.18}$$

Proposition 3.3 also states that $s_t(x_0,t_0) \le 0$. Since $(s,\pi) \in \mathcal{S}\left(\mathbb{R}\times[0,T]\right)$, then (s,π) satisfies one of the following possibilities:

(i) $(s,\pi) \in \mathcal{C}^2\left(B_\gamma(x_0,t_0)\right)$, for a small enough $\gamma > 0$;
(ii) (s,π) is not $\mathcal{C}^2$ at $(x,t) = (x_0,t_0)$.

Recall that π increases in time only when s increases in imbibition mode.

In case *(i)*, assumption (3.17) implies that both functions $\pi(\cdot,t)$ and $s(\cdot,t)$ have a local (and global) maximum point in $B_\gamma^+(x_0,t_0) := B_\gamma(x_0,t_0) \cap \{t > t_0\}$, for $\gamma > 0$ small enough. These points satisfy $\pi(x,t) = \pi^I\left(s(x,t)\right)$; moreover, there is $(x_1,t_1) \in B_\gamma^+(x_0,t_0)$ such that $s(x_1,t_1) > M$. By (3.17) we cannot have $s_t \le 0$ for all $t_0 < t < t_1$ at global maximum points in $B_\gamma^+(x_0,t_0)$; hence, we have $s_t > 0$ at some local maximum point. This is in contradiction with Proposition 3.3 and then case *(i)* cannot occur.

Consider case *(ii)*. Pick a small enough constant $\delta > 0$. For each fixed $t \in [t_0,t_0+\delta]$, we denote a global maximum point of $s(\cdot,t)$ (for $-\infty < x < \infty$) in $B_\gamma(x_0,t_0) \cap \{t \ge t_0\}$ as $x = x_0(t)$. Then, we have $x_0(t_0) = x_0$. We are only interested in those $x_0(t)$'s that increase π to make (3.17) true. Hence it is necessary that

$$\pi\left(x_0(t)+,t\right) = \pi^I\left(s(x_0(t),t)\right) = \pi\left(x_0(t)-,t\right)$$

by the discussion following (3.17). Remark that there maybe several of such $x_0(t)$ points for each fixed t. We just select any one of them, maintaining continuity if possible, to make a path $x = x_0(t)$. We note that the path $x = x_0(t)$ may be discontinuous, but the maximum value $s\left(x_0(t), t\right)$ is continuous because $s(x,t)$ is continuous in t and hence its global maximum value cannot jump as t increases. Now, it suffices to show that $s\left(x_0(t), t\right)$ does not increase along none of these paths $x = x_0(t)$ to arrive at the desired contradiction to (3.17).

If the curve $x_0(t)$ is away from division boundaries of $\mathcal{PC}^2$ at some time $t_0 < t_1 < t_0+\delta$, then we are in case *(i)*, with t_0 replaced by t_1. There, we proved that $s\left(x_0(t), t\right)$ cannot increase at $t = t_1$. Thus, we can assume that $x = x_0(t)$, $t_0 < t < t_0+\delta$ lies on a piece of division boundary of $\mathcal{PC}^2$. Since division boundaries of $\mathcal{PC}^2$ are piecewise C^1 with locally finitely many pieces, we can assume that $x = x_0(t)$ is C^1 over $t_0 < t < t_0 + \delta$ for sufficiently small $\delta > 0$. If the local maximum value $s\left(x_0(t), t\right)$ exceeds M as t increases over the interval $[t_0, t_0 + \delta)$, then $\frac{d}{dt}s\left(x_0(t)\pm, t+\right) > 0$ at some point $t_1 \in [t_0, t_0 + \delta)$. Here we used the fact that s is C^1 up to the boundary. Without loss of generality, we assume, by re-assigning the notation for t_0 if necessary, that $t_1 = t_0$. Because s, being in $\mathcal{PC}^2$, is C^2 on either sides of $x = x_0(t)$ with locally bounded second derivatives on either side of the division curve $x = x_0(t)$, the chain rule applies to either sides of $x = x_0(t)$ to yield

$$\frac{d}{dt}s\left(x_0(t)\pm, t\right) = s_x\left(x_0(t)\pm, t\right)x_0'(t) + s_t\left(x_0(t)\pm, t\right). \tag{3.19}$$

Since $s(x,t)$ is continuous in x, and $x = x_0(t)$ is a local maximum point of $s(\cdot, t)$, then $s_x\left(x_0(t)+, t\right) \leq 0$ and $s_x\left(x_0(t)-, t\right) \geq 0$ must hold. If $x_0'(t) \geq 0$, then (3.19) implies that $s_t(x_0(t)+, t) > 0$ holds; if else $x_0'(t) < 0$, then $s_t(x_0(t)-, t) > 0$. Either way, at least one of $s_t\left(x_0(t)\pm, t\right)$ must be positive. However, we claim that this leads to a contradiction.

To prove this claim, assume for definiteness that $s_t(x_0+, t_0) > 0$. Take a test function $\psi \in \Psi$ with

$$|s_t(x_0-, t_0)| \int_{-1}^{0} \psi(y)dy < \frac{1}{2}s_t(x_0+, t_0)\int_0^1 \psi(y)dy.$$

Since s is Lipschitz continuous, then

$$\lim_{h\to 0+}\langle s_t(x_0+y, t_0), \psi_h(y)\rangle = s_t(x_0-, t_0)\int_{-1}^{0}\psi(y)dy + s_t(x_0+, t_0)\int_0^1 \psi(y)dy > 0,$$

violating Proposition 3.3 that states $s_t(x_0, t_0) \leq 0$ in the sense of Definition 2.1. This proves our claim.

Then, a solution cannot exit Ω_m^M through the corner point $\tilde{A}$. Similarly, the solution cannot exit Ω_m^M through the corner point $\tilde{C}$ either.

(b) Now, we prove that the solution cannot exit Ω_m^M through the side curves joining $\tilde{A}$ with $\tilde{D}$ and $\tilde{B}$ with $\tilde{C}$. Assume that the pair (s,π) lies at a point on $\pi = \pi^I(s)$; then we are either in imbibition or in scanning mode. In the former case we have $\pi_t = \pi^I(s)_t$ and $s_t > 0$; as a consequence, the pair (s,π) keeps on moving along the imbibition curve $\pi = \pi^I(s)$ and does not move outside Ω_m^M. In the latter case we have $\pi_t = 0$ and $s_t \le 0$ (otherwise we are in imbibition mode) and then (s,π) moves toward the interior of Ω_m^M. The proof in case (s,π) lies on $\pi = \pi^D(s)$ is analogous.

(c) Next, we claim that the solution cannot exit Ω_m^M through the top open line $s^D(M) < s < s^I(M)$, $\pi = M$, or the bottom line $s^D(m) < s < s^I(m)$, $\pi = m$. To this end, assume that for some (x_0, t_0) we have

$$\pi(x_0+, t_0) = \operatorname*{ess\,sup}_{x\in\mathbb{R}} \pi(x, t_0) = M, \qquad \text{with } s^D(M) < s(x_0, t_0) < s^I(M),$$

the case $\pi(x_0-, t_0) = M$ being completely analogous. Then (s,π) is in scanning mode on an interval $(x_0, x_0+\gamma) \times \{t = t_0\}$, for some small $\gamma > 0$, because s is continuous. About the $x = x_0-$ side, there are three possibilities.

-) If the side $x = x_0-$ is in scanning mode, then $(s,\pi)(x,t_0)$ is in scanning mode on both sides of x_0. Equation $(2.16)_2$ states $\pi_t = 0$ and hence π cannot exceed M in a neighborhood of (x_0, t_0).

-) If the side $x = x_0-$ is in imbibition mode, then $\pi(x_0-, t_0) = \pi^I\left(s(x_0,t_0)\right) < \pi^I\left(s^I(M)\right) = M$. The continuity of $\pi(x,\cdot)$, by Lemma 3.1, implies that $\pi(x_0-, t)$ cannot exceed M for t near t_0.

-) The remaining case where the $x = x_0-$ side is in drainage mode cannot occur because that would require $M = \pi(x_0-, t_0) = \pi^D\left(s(x_0,t_0)\right) > \pi^D\left(s^D(M)\right) = M$, which is a contradiction.

The above arguments show that the solution cannot exit Ω_m^M through the top open lines $s^D(M) < s < s^I(M)$, $\pi = M$. Similarly, it can be proved that the solution cannot exit Ω_m^M through the bottom open line $s^D(m) < s < s^I(m)$, $\pi = m$.

(d) At last, we show that the solution cannot exit Ω_m^M through the corner points $\tilde{B}$ or $\tilde{D}$. This case is much easier than case *(a)* because in that case the evolution of s (for $\tilde{A}$ we have $s_t > 0$ in imbibition mode, for $\tilde{C}$ we have $s_t < 0$ in drainage mode) could possibly move s *outside* Ω_m^M. For $\tilde{B}$ and $\tilde{D}$ the situation is opposite. Consider for instance the case of $\tilde{B}$. The solution is either in drainage or in scanning mode. In the former case we have $s_t < 0$ and the solution decreases along the drainage curve.

In the latter case the solution stays on the scanning curve. In any case, the solution does not exit Ω_m^M. $\square$

The well-known Tychonov's counter-example [6] shows that the uniqueness of solutions and the maximum principle for the Cauchy problem of the heat equation cannot hold without imposing a growth condition at $|x| \to \infty$. For system (2.16), being non-strict parabolic, we expect a similar phenomenon. Thus, we have to restrict to a set of functions relevant to the physical background of (2.16) when searching for solutions. For studying Riemann problem in later sections, we are interested in solutions, if any, of (2.16) in the class

$$X\left(\mathbb{R}\times[0,T]\right) := \left\{(s,\pi)\in\mathcal{S}\left(\mathbb{R}\times[0,T]\right) : \lim_{x\to\pm\infty} s(x) = s_\pm\right\}, \tag{3.20}$$

where $0 < s_\pm < 1$ are constants[1]. To be consistent, the initial data must belong to the space

$$X\left(\mathbb{R}\right) := \left\{(s,\pi)\in\mathcal{S}\left(\mathbb{R}\right) : \lim_{x\to\pm\infty} s(x,t) = s_\pm\right\}. \tag{3.21}$$

Theorem 3.6. *Assume (I), (II) and $0 < m < M < 1$. If the initial data $(s_0,\pi_0) \in X(\mathbb{R})$ are valued in Ω_m^M, then any solution in $X\left(\mathbb{R}\times[0,T]\right)$ of the equation (2.16) is also valued in Ω_m^M.*

Proof. Because π cannot change without s changes, the condition $\lim_{x\pm\infty} s(x,t) = s_\pm$ prevents π to exceed the range $[m,M]$ at $x\to\pm\infty$. Then Proposition 3.5 prevents $(s,\pi)(x,t)$ to exit Ω_m^M as t increases from 0. $\square$

Here follows a maximum-principle result about π. It is a straightforward consequence of Theorem 3.6.

Corollary 3.7. *Assume (I), (II) and $(s_0,\pi_0) \in X(\mathbb{R})$. Then any solution $(s,\pi) \in X\left(\mathbb{R}\times(0,T)\right)$ of the initial-value problem (2.16) with initial data (s_0,π_0) satisfies*

$$\operatorname*{ess\,inf}_{x\in\mathbb{R}} \pi(x,0) \le \pi(x,t) \le \operatorname*{ess\,sup}_{x\in\mathbb{R}} \pi(x,0) \quad a.e. \tag{3.22}$$

Remark 3.2. We cannot expect that s satisfies an analogous maximum principle. This is due to the fact that $t=0$ is an artificial choice of time, while the actual history of s, including the history of s for times $t \le 0$, is encoded in π. Thus, s can be expected to stay inside a region decided by $\pi(x,0)$, as stated in Theorem 3.6, but cannot be bounded by a region decided by $s(x,0)$ alone. Whether the invariant region described in Theorem 3.6 may be further reduced is left for future research.

[1]The strict inequalities $0 < s_\pm < 1$ are added to avoid the need of extending $f = F(s,\pi)$ outside of $[0,1]\times[0,1]$.

Similarly, we also have the following two theorems as immediate consequences of Proposition 3.5. The first one concerns the initial-boundary value problem for (2.16) over a bounded region of x; the second one deals with the initial-value problem for (2.16) under periodic initial and boundary conditions. Both problems avoid the possibility of the invariant region being reached as $x \to \pm\infty$. Below, the sets $\mathcal{S}\left([-L,L]\times[0,T]\right)$ and $\mathcal{S}\left([-L,L]\right)$ are defined analogously to $\mathcal{S}\left(\mathbb{R}\times[0,T]\right)$ and $\mathcal{S}\left(\mathbb{R}\right)$, with $\mathbb{R}$ replaced by the interval $[-L,L]$.

Theorem 3.8. *Assume (I), (II) and* $0<m<M<1$. *Let* $(s,\pi)\in\mathcal{S}\left([-L,L]\times[0,T]\right)$ *be a solution of system* (2.16) *in* $[-L,L]\times[0,T]$ *that satisfies the initial-boundary conditions*

$$\begin{cases} (s,\pi)(x,0)=(s_0,\pi_0)(x) & \text{if } -L<x<L,\\ (s,\pi)(\pm L,t)=(s_\pm,\pi_\pm)(t) & \text{if } 0<t<T. \end{cases} \tag{3.23}$$

Here $(s_0,\pi_0)\in\mathcal{S}([-L,L])$, $s_\pm\in C\left([0,T];[0,1]\right)\cap\mathcal{PC}^2\left([0,T];[0,1]\right)$ *and* $\pi(\pm L,t)$ *is determined by* $(2.16)_2$. *If both* (s_0,π_0) *and* $(s_\pm,\pi_\pm)$ *are valued in* Ω_m^M, *then* (s,π) *is also valued in* Ω_m^M.

Theorem 3.9. *Assume (I), (II) and let* $0<m<M<1$. *Let* $(s,\pi)\in\mathcal{S}\left(\mathbb{R}\times[0,T]\right)$ *be a periodic solution of system* (2.16) *over* $\mathbb{R}\times[0,T]$, *namely,* $(s,\pi)(x+L,t)=(s,\pi)(x,t)$ *for* $t\in[0,T]$, *with initial data*

$$(s,\pi)(x,0)=(s_0,\pi_0)(x). \tag{3.24}$$

If $(s_0,\pi_0)\in\mathcal{S}(\mathbb{R})$ *is valued in* Ω_m^M, *then* (s,π) *is also valued in* Ω_m^M.

In the remaining part of this section we assume for simplicity that piecewise continuous functions in $\mathcal{PC}^0(\mathbb{R})$ are continuous on the left.

Definition 3.1. *Let* $g\in\mathcal{PC}^0(\mathbb{R})$. *An* increasing stretch for g *is an interval* $[a,b]$ *such that*

- *(i) if* $a<b$, *then*
  - *(a)* $g(x_1-)\le g(x_2-)$ *if* $x_1<x_2$, *for* $x_1\in(a,b]$, $x_2\in(a,b]$;
  - *(b)* $g(x-)\le g(x+)$ *for every* $x\in(a,b)$;
- *(ii) if* $a=b$, *then* $g(a-)<g(a+)$;
- *(iii) the interval* $[a,b]$ *is maximal with respect to the properties above.*

A decreasing stretch for g *is an interval* $[a,b]$ *such that*

- *(i) if* $a<b$, *then*
  - *(a)* $g(x_1-)\ge g(x_2-)$ *if* $x_1<x_2$, *for* $x_1\in(a,b]$, $x_2\in(a,b]$;
  - *(b)* $g(x-)\ge g(x+)$ *for every* $x\in(a,b)$;

(ii) if $a = b$, then $g(a-) > g(a+)$;

(iii) the interval $[a, b]$ is maximal with respect to the properties above.

A stretch of monotonicity *for g is either an increasing or a decreasing stretch for g. An increasing and a decreasing stretch may overlap (for instance if g is constant in an interval); for $g \in \mathcal{PC}^0(\mathbb{R})$, the* number of monotone stretches N_g *is counted by meaning a possibly overlapping interval as being part of the monotone stretch on the left.*

In this way, if $g \in \mathcal{PC}^0(\mathbb{R})$, then the set $\mathbb{R}$ is covered by an at most countable number of monotone stretches for g. We refer to Figure 3.2 for some examples.

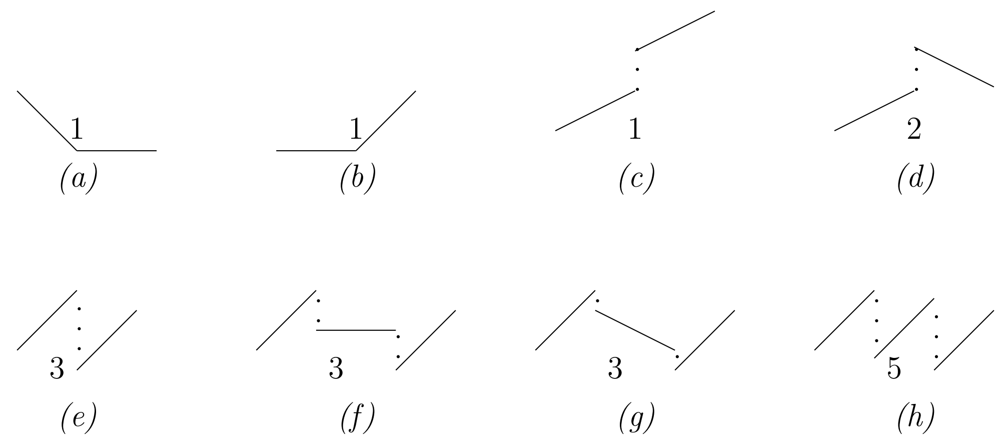


FIGURE 3.2. Monotone stretches of a function and their numbers.

Our next result aims at showing a monotonicity property of π. For brevity we denote by $N(t) := N_{\pi(\cdot,t)}$ the number of monotone stretches of $\pi(\cdot, t)$, which is assumed to be continuous on the left. As an example, we collect in Figure 3.3 some sketches of evolution patterns of the function $\pi(\cdot, t)$, with respect to t, that are either (possibly) allowed or disallowed by Propositions 3.3, 3.4 and Corollary 3.7. For instance, consider Figure 3.3*(a)* at time t_0 and denote by x_0 the corresponding minimum point. If we are in scanning mode, then that local pattern does not change for short times. If we are in imbibition mode, then π increases with time and this confirms the figure. If we are in drainage mode, then $\pi(x_0, t_0) = \pi^D\left(s(x_0, t_0)\right)$ and (x_0, t_0) also is be a local minimum point for $s(\cdot, t_0)$; Proposition 3.4 implies $s_t(x_0, t_0) \geq 0$ and then π increases again with t.

Theorem 3.10. *Assume (I) and (II). Let $(s, \pi) \in \mathcal{S}\left(\mathbb{R} \times [0, T]\right)$ be a solution of* (2.16), (2.18) *and suppose $N(0) < \infty$. Then $N(t)$ is nonincreasing as t increases.*

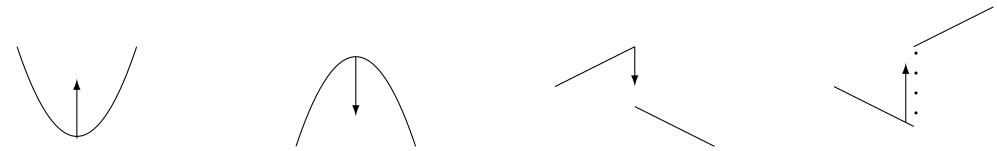


FIGURE 3.3. Allowed evolution patterns of $\pi(\cdot, t)$. Arrows denote the possible evolutions for a short interval of time. The converse evolution is disallowed.

Proof. Since the function $N(t)$ is integer valued, it can only change through jump discontinuities. For $N(t)$ to increase across a point $t = t_0$ there are two possibilities (recall that $\pi(\cdot, t)$ is assumed to be continuous on the left):

(i) a new local essential-maximum point $x = x_0(t)$ of $\pi(\cdot, t)$ is created as t increases across $t = t_0$ where either $\pi(x_0-, t)$ or $\pi(x_0+, t)$ (or both) is increasing at $t = t_0$;

(ii) a new local essential-minimum point $x = x_0(t)$ of $\pi(\cdot, t)$ is created as t increases across $t = t_0$ where $\pi(x_0-, t)$ or $\pi(x_0+, t)$ (or both) is decreasing at $t = t_0$.

However, both cases are disallowed by Propositions 3.3 and 3.4. In fact, consider for instance case *(i)* and assume that $\pi(x_0-, t)$ (or $\pi(x_0+, t)$) increases with t. By Proposition 3.3 we have $\pi(x_0\pm, t) = \pi(x_0, t) = \pi^I\left(s(x_0, t)\right)$ and (x_0, t_0) is also a local maximum point for $s(\cdot, t_0)$. Then $s_t(x_0, t_0) \leq 0$, which implies that $\pi(x_0, \cdot)$ cannot increase at $t = t_0$, a contradiction.

Thus, $N(t)$ cannot increase as t increases. The boundedness of the total variation of $\pi(\cdot, t)$ then follows from $N(t) \leq N(0)$ and Corollary 3.7. □

Corollary 3.11. *Assume (I) and (II). Let $(s, \pi) \in X\left(\mathbb{R} \times [0, T]\right)$ be a solution of* (2.16) *with initial data in $X(\mathbb{R})$. If $N(0) < \infty$, then the total variation of $\pi(\cdot, t)$ is bounded for every $t \in [0.T]$.*

4. Shock Profiles and Rarefaction Waves

In this section we study the viscous profiles of solutions to (2.16). Rewrite the system of equations (2.21) for shock profiles as

$$
\begin{cases} A(s, \pi)s' = -\sigma(s - s_-) + F(s, \pi) - F(s_-, \pi_-), \\ \sigma\pi' = \sigma\left[\chi(I)\pi^I(s)' + \chi(D)\pi^D(s)'\right], \\ (s, \pi)(\pm\infty) = (s_\pm, \pi_\pm), \qquad (s', \pi')(\pm\infty) = (0, 0) \end{cases} \tag{4.1}
$$

where we denoted $' = \frac{d}{d\xi}$ and by

$$
\sigma = \frac{F(s_+, \pi_+) - F(s_-, \pi_-)}{s_+ - s_-} \tag{4.2}
$$

the speed of the shock. Recall that the scaled viscosity $A > 0$ is assumed to be bounded from 0. With the traveling wave coordinate ξ, the definition of the boolean variables gives

$$I = \Big\{\sigma s' < 0\Big\} \wedge \Big\{s = s^I(\pi)\Big\}, \qquad D = \Big\{\sigma s' > 0\Big\} \wedge \Big\{s = s^D(\pi)\Big\}. \tag{4.3}$$

In the following, when a profile or a wave are valued in just one mode, then we call them single-mode profiles or single-mode waves, respectively. We recall the relation

$$\operatorname{sign}(s_t) = -\sigma \operatorname{sign}(s'), \tag{4.4}$$

which, together with (2.9), is fundamental to determine the mode of a profile.

From (4.1), it is clear that if (s, π) is bounded and measurable, then s is continuous. If $\sigma \neq 0$, then π' is also bounded and measurable and hence π is continuous. This further improves the regularity of s to $\mathcal{C}^1$. Because of the presence of characteristic functions $\chi(I)$ and $\chi(D)$ in the equation for π', the best possible regularity for π is piecewise $\mathcal{C}^1$ and uniform Lipschitz. However, if $\sigma = 0$, then there is not much restriction on $\pi(\xi)$, except the physical restrictions such as being bounded. In this case, the function s is Lipschitz. In the rest of this section, we show the existence and uniqueness of solutions of (4.1) in the class of continuous functions.

Lemma 4.1. *Assume (I) and consider $\sigma \neq 0$. If (4.1) has a continuous solution (s, π) and $s_- \neq s_+$, then s' is continuous and never vanishes. Therefore $s(\xi)$ is strictly monotone. Furthermore, the function π is Lipschitz continuous and monotone with the same type of monotonicity of s.*

Proof. It is clear from $(4.1)_1$ that s' is continuous. Then, π' is bounded and hence $\pi(\xi)$ is Lipschitz. Assume, by contradiction, that there is a point ξ_0 where $s'(\xi_0) = 0$, implying $\pi'(\xi_0) = 0$ by $(4.1)_2$. Therefore system (4.1) has a constant solution $\big(s(\xi_0), \pi(\xi_0)\big)$. We claim that this constant solution is the only solution. This requires a proof since the standard uniqueness results do not apply since the right-hand side of $(4.1)_2$ is discontinuous. Indeed, because of $s'(\xi_0) = 0$, equations $(4.1)_{1,2}$ can be rewritten as

$$\begin{cases} A(s,\pi)(s - s_0)' = -\sigma(s - s_0) + F(s,\pi) - F(s_0,\pi_0), \\ (\pi - \pi_0)' = \Big[\chi(I)(\pi^I(s) - \pi^I(s_0))' + \chi(D)(\pi^D(s) - \pi^D(s_0))'\Big], \end{cases} \tag{4.5}$$

where $(s_0, \pi_0) := \big(s(\xi_0), \pi(\xi_0)\big)$. Assume that $(s, \pi)(\xi)$ is another solution of (4.1) with $(s, \pi)(\xi_0) = (s_0, \pi_0)$ but $(s, \pi)(\xi) \neq (s_0, \pi_0)$ for some $\xi \neq \xi_0$, for contradiction. Because s' is continuous and not identically 0, then the set $\{\xi \in \mathbb{R} : s' \neq 0\}$ is open and non-empty. Thus, we can assume without loss of generality that $s'(\xi) \neq 0$ for $\xi \in (\xi_0, \xi_0 + \delta)$ for a sufficient small constant $\delta > 0$. Then either one of $I(\xi) = \mathit{true}$

and $D(\xi) = true$ holds for $\xi \in (\xi_0, \xi_0 + \delta)$, or both $I = D = false$. For the former case, $(4.5)_2$ yields

$$\pi(\xi) - \pi_0 = O(1)(s - s_0). \tag{4.6}$$

For the latter case, the identity

$$\pi(\xi) - \pi_0 = 0 \tag{4.7}$$

holds for $\xi \in (\xi_0, \xi_0 + \delta)$. Combining (4.6) and (4.7) with the integral of the $(4.5)_1$, we obtain that for every $\xi \in [\xi_0, \xi_0 + \delta]$ we have

$$\begin{aligned} \min\{|A|\} |s(\xi) - s_0| &\le O(1) \int_{\xi_0}^{\xi} \big(|s(\zeta) - s_0| + |\pi(\zeta) - \pi_0|\big) \, d\zeta \\ &\le O(1)\delta \max_{\zeta \in [\xi_0, \xi]} |s(\zeta) - s_0|. \end{aligned}$$

This implies $\big(s(\xi), \pi\big) \equiv (s_0, \pi_0)$ in $[\xi_0, \xi_0 + \delta]$ for sufficiently small $\delta > 0$. Then the constant solution is the only solution. However, this violates the requirement $(s, \pi)(\pm\infty) = (s_\pm, \pi_\pm)$. Thus s' cannot change sign and hence s is monotone.

Moreover, by $(4.1)_2$, it follows that if $s' > 0$ then $\pi' \ge 0$ while if $s' < 0$ then $\pi' \le 0$, for all $\xi \in \mathbb{R}$. Thus, π is monotone with the same type of monotonicity of s. $\square$

Though experimental data show that both functions f^D and f^I usually have a unique inflection point in $(0, 1)$, see [7, 13], we require here, for simplicity, the following convexity assumption to reduce the number of possible types of waves and simplify the structure of Riemann solvers.

Assumption (III). We assume

$$f^D_{ss}(s) > 0, \quad f^I_{ss}(s) > 0, \qquad \text{for } s \in (0, 1), \tag{4.8}$$

$$f^S_{ss}(s, \pi) > 0, \qquad \text{for } (s, \pi) \in \Omega^S. \tag{4.9}$$

Scanning-to-imbibition shocks. A scanning-to-imbibition shock (briefly, SIS) is a shock connecting $\big(s_- = s^I(\pi_-), \pi_-\big)$ to (s_+, π_+) in scanning mode. Then we have $s^D(\pi_+) < s_+ < s^I(\pi_+)$. The following theorem states when such shock waves have viscous profiles; indeed, for future reference, we treat as well the cases $s_+ = s^D(\pi_+)$ and $s_+ = s^I(\pi_+)$. The case $\pi_- = \pi_+$ concerns scanning waves and is not considered here.

Theorem 4.2. *Assume (I) and (III). Let $(s_\pm, \pi_\pm) \in \Omega$ satisfy $s_- = s^I(\pi_-)$ and $s^D(\pi_+) \le s_+ \le s^I(\pi_+)$.*

(i) If $\pi_+ < \pi_-$, then (4.1) has a solution $(s, \pi)(\xi)$ which is unique up to a shift. Furthermore, s is $\mathcal{C}^1$ and π is Lipschitz.

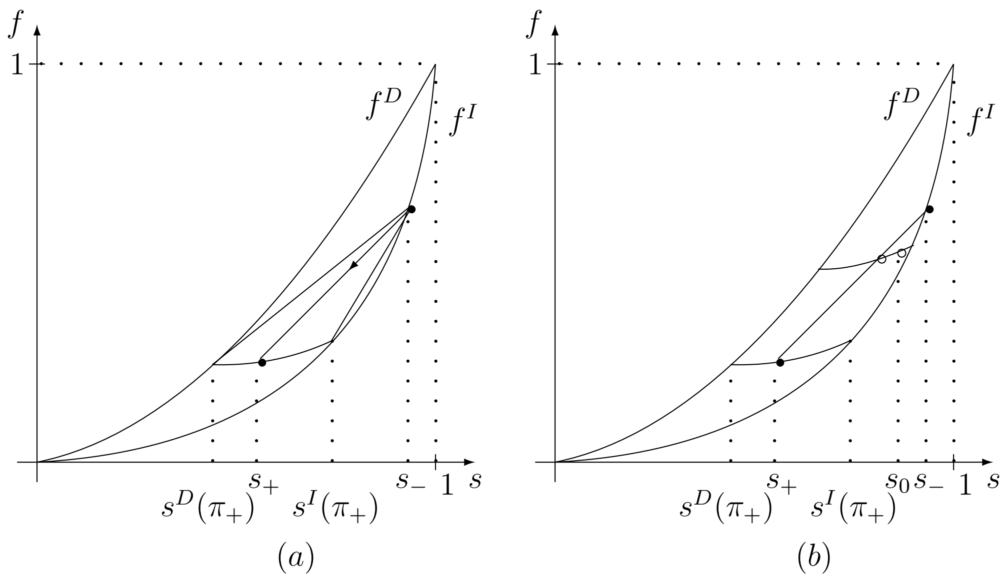


FIGURE 4.1. *(a):* a scanning-to-imbibition shock. *(b):* for the proof of Theorem 4.2.

(ii) If $\pi_+ > \pi_-$*, then* (4.1) *has no solution if* $\sigma \neq 0$*.*

Proof. We divide the proof into two steps and refer to Figure 4.1.

(i) Since $\pi_+ < \pi_-$, by Lemma 2.1 we have

$$s_+ \leq s^I(\pi_+) < s^I(\pi_-) = s_-. \tag{4.10}$$

As a consequence, by (2.8) we have

$$F(s_-, \pi_+) = F(s_-, \pi_-), \tag{4.11}$$

see Figure 4.1*(b)*; indeed, the extension of F was just done to this aim. Notice that $F(\cdot, \pi_+)$ is convex for $s \in (s_+, s_-)$. By (4.10) it follows that $\sigma > 0$. Consider the equation

$$A(s,\pi)s' = (s - s_-)\left(-\sigma + \frac{F(s,\pi_+) - F(s_-,\pi_+)}{s - s_-}\right), \tag{4.12}$$

where σ is given by (4.2) and $\pi(\xi)$ is defined as

$$\pi(\xi) := \begin{cases} \pi_+ & \text{if } s(\xi) < s^I(\pi_+), \\ \pi^I\left(s(\xi)\right) & \text{if } s(\xi) \geq s^I(\pi_+). \end{cases} \tag{4.13}$$

By (4.10) and the convexity assumptions (4.8), (4.9) (see also Figure 2.2*(b)*), we see that

$$-\sigma + \frac{F(s,\pi_+) - F(s_-,\pi_+)}{s - s_-} > 0$$

for $s \in (s_+, s_-)$. Pick any $s_0 \in (s_+, s_-)$ as initial point $s(0)$; the solution $s = s(\xi)$ of the corresponding initial-value problem for equation (4.12) (with (4.13)) is strictly decreasing and it is easily seen that it can be extended to the whole of $\mathbb{R}$. We claim that

$$\big(s(\xi),\pi(\xi)\big) := \begin{cases} \big(s(\xi),\pi_+\big) & \text{if } s(\xi) < s^I(\pi_+), \\ \big(s(\xi),\pi^I(s(\xi))\big) & \text{if } s(\xi) \geq s^I(\pi_+), \end{cases} \tag{4.14}$$

solves (4.1).

To prove this claim, consider first the case when $s_+ = s^I(\pi_+)$. In this case we have $\big(s(\xi),\pi(\xi)\big) = \big(s(\xi),\pi^I(s(\xi))\big)$ by (4.13) and (4.12) becomes

$$\begin{aligned} A\left(s,\pi^I(s)\right)s' &= -\sigma(s-s_-) + f^I(s) - f^I(s_-) \\ &= -\sigma(s-s_-) + F\left(s,\pi^I(s)\right) - F\left(s_-,\pi^I(s_-)\right), \end{aligned}$$

which is $(4.1)_1$. Equation $(4.1)_2$ is trivially satisfied. So (4.14) solves $(4.1)_1$ in this case.

When $s_+ < s^I(\pi_+)$, since $s(\xi)$ is continuous and decreasing, there is a point ξ_0 such that $s^D(\pi_+) < s(\xi) < s^I(\pi_+)$ for $\xi > \xi_0$ and $s(\xi) \geq s^I(\pi_+)$ for $\xi \leq \xi_0$. In the zone $\xi > \xi_0$ we have $S = true$, and hence (4.14) satisfies (4.1). In the $\xi \leq \xi_0$ side, $I = true$ holds for the function (4.14) and equation $(4.1)_2$ is satisfied; thus, the pair in (4.14), being continuous, solves $(4.1)_2$. It remains to show that (4.14) solves $(4.1)_1$. If $\xi < \xi_0$, this is true for the same reason given above for the case $s_+ = s^I(\pi_+)$; if $\xi > \xi_0$, this follows by (4.11), since equation $(4.1)_1$ coincide with (4.12). At $\xi = \xi_0$, the function (s,π) is continuous by the construction (4.14); thus, (4.14) solves $(4.1)_1$.

Since s is decreasing, we have $s(\xi) \to s_\pm$, the equilibrium points of (4.12), as $\xi \to \pm\infty$. Thus, $(4.1)_3$ is also satisfied. This proves our claim that (4.14) solves (4.1).

Now, we prove that the solution of (4.1) is unique up to a shift. To see this, let $(s_1,\pi_1)(\xi)$ be another continuous solution to (4.1). Since s_1 is continuous, it can be shifted so that $s_1(0) = s_0$, where s_0 can be chosen by (4.10) so that

$$s^I(\pi_+) < s_0 < s_-, \tag{4.15}$$

see Figure 4.1 *(b)*. We claim that

$$\pi_1(0) = \pi^I(s_0). \tag{4.16}$$

If we take (4.16) for granted, then this solution and the previous one constructed in (4.14), with a shift if necessary, have the same initial value $(s,\pi)(0) = (s_1,\pi_1)(0) = \big(s_0,\pi^I(s_0)\big)$.

We claim that these two solutions must coincide. Indeed, as ξ increases from 0, because $s_1' < 0$ as required by Lemma 4.1, then $s_1(\xi)$ will move from s_0 towards $s^I(\pi_+) \leq s_0$. If it was in S mode at some $\xi = \xi_1$ with $s^I(\pi_+) < s_1(\xi_1) \leq s_0$, then the pair $(s_1(\xi), \pi_1(\xi))$ would move into scanning zone and $\pi_1(\xi_1) \equiv \pi_1(\xi) > \pi_+$ would stay true until $(s_1(\xi), \pi_1(\xi))$ meets the the drainage curve $(s, \pi = \pi^D(s))$, say at $\xi = \xi_2 > \xi_1$. As ξ increases from ξ_2, the function $\pi_1(\xi)$ cannot decrease to π_+ because π_1 can decrease only in I-mode. Thus, $(s_1, \pi_1)(\xi)$ must stay in I-mode as ξ increases from 0 until $\xi = \xi_1 > 0$ where $s_1(\xi_1) = s^I(\pi_+)$. After ξ_1, $S = true$ must hold because, if else, then $\pi_1(\xi_1 + o(1)) < \pi_+$ would be true, making π non-monotone, in contradiction with Lemma 4.1. Therefore, the solution (s_1, π_1) coincides with the solution (4.14).

Thus, a proof of (4.16) will complete the proof of this theorem. To this end, assume by contradiction that $\pi_1(0) \neq \pi^I(s_0)$. Since $\pi^I(s) \leq \pi \leq \pi^D(s)$ for any $s \in [0,1]$, then we have $\pi^I(s_0) < \pi_1(0) \leq \pi^D(s_0)$. We recall that by Lemma 4.1 we have either $s_1'(0) > 0$ or $s_1'(0) < 0$.

If $s_1'(0) < 0$, then the inequality $\pi^I(s_0) < \pi_1(0)$ implies that the fluid at $\xi = 0$ is in scanning mode because of (4.3) and $\sigma > 0$. As a consequence, we have that both $\pi_1'(\xi) = 0$ and $s_1'(\xi) < 0$ hold as ξ increases until $(s_1, \pi_1)(\xi)$ enters into an equilibrium point of (4.1), which is the point of intersection between the secant line in Figure 4.1 *(b)* and $f = F\left(s, \pi_1(0)\right)$. This violates the requirement that $\pi_1(\infty) = \pi_+$.

If $s_1'(0) > 0$, then $s_1'(\xi) > 0$ for all $\xi \in \mathbb{R}$ by Lemma 4.1. On the other hand, by (4.15) it follows that $s_0 > s_+$. This excludes the possibility that $s_1(\infty) = s_+$. This proves (4.16) and thus the uniqueness.

(ii) Consider now the case $\pi_+ > \pi_-$. When $s_+ < s_-$, there is no solution for (4.1) because by Lemma 4.1, when $\sigma \neq 0$, s and π are either both increasing or both decreasing. If $s_- = s_+$, then problem (4.1) has not yet a solution, because the monotonicity of $s(\xi)$ dictates $s(\xi) \equiv s_-$ and hence, by $(4.1)_1$, that $\pi(\xi) \equiv \pi_-$, making $\pi_+ > \pi_-$ impossible to satisfy. It remains to show that (4.1) has no solution when $s_+ > s_-$.

When $s_+ > s_-$, we have $s'(\xi) > 0$ and $\sigma > 0$ from (4.2); by (4.3) we deduce $\chi(I) = 0$ for all ξ. Near $\xi = -\infty$, we have $s \sim s_- = s^I(\pi_-)$ and hence $\chi(D) = 0$. Thus, as ξ increases from $-\infty$, we have $\pi(\xi) = \pi_-$ while $s(\xi)$ increases and hence $s(\xi) > s_- = s^I\left(\pi(\xi)\right)$. This makes $\chi(D) = 1$ impossible to reach as ξ increases and hence $\pi(\xi) \equiv \pi_-$. Therefore, in this case, $\pi(\infty) = \pi_+$ is impossible. ☐

Remark 4.1. Under the assumptions of Theorem 4.2*(i)*, at a point $x >> 1$ and $t = 0$, i.e., for $\xi \sim \infty$, we have $\left(s(\xi), \pi(\xi)\right) \sim (s_+, \pi_+)$ and then the solution $\left(s(x,t), \pi(x,t)\right)$ is in scanning mode. As t increases, the wave (4.14) travels to the right, while the function $s(x, \cdot)$ increases and eventually goes into imbibition mode. For this reason, we call such shock a scanning-to-imbibition shock.

Remark 4.2. A special case of Theorem 4.2 is when (s_+, π_+) satisfies $\pi_+ = \pi^I(s_+)$. In this case, the profile is entirely in imbibition mode, i.e., $\pi(\xi) = \pi^I\left(s(\xi)\right)$ for all $\xi \in \mathbb{R}$. Although it is a single mode shock, it is also the limit of scanning-to-imbibition shocks where the limiting process is $s_+ \to s^I(\pi_+)-$. For simplicity, we still call this shock a scanning-to-imbibition shock to reduce the number of cases of Riemann solvers in next section.

Another special case of Theorem 4.2 is when (s_+, π_+) satisfies $\pi_+ = \pi^D(s_+)$. Since $-\sigma s' > 0$ near $\xi = \infty$, then (s_+, π_+) is in scanning mode. Thus, the shock is a scanning-to-imbibition shock.

Single-mode rarefaction waves. Consider potential solutions of the Riemann problem of the form $(s,\pi)(x,t) = (s,\pi)(x/t) =: (s,\pi)(\zeta)$, where $\zeta = x/t$. This leads to the system

$$\left\{\begin{array}{l} -\zeta s' + F(s,\pi)' = 0, \\ \zeta\pi' = \zeta\left(\chi(I)\pi^I(s)' + \chi(D)\pi^D(s)'\right), \end{array}\right. \tag{4.17}$$

with the boundary condition

$$(s,\pi)(-\infty) = (s_l, \pi_l), \ (s,\pi)(\infty) = (s_r, \pi_r), \tag{4.18}$$

where $I = \{s = s^I(\pi) \text{ and } \zeta s' < 0\}$ and $D = \{s = s^D(\pi) \text{ and } \zeta s' > 0\}$. Rarefaction waves are smooth solutions $(s,\pi)(\zeta)$ of (4.17) that are inside a single mode. For imbibition- and drainage-mode rarefaction waves, the equations for Riemann problem (4.17) are reduced to

$$\left\{\begin{array}{l} -\zeta s' + f^I(s)' = 0, \\ \zeta\pi' = \zeta\pi^I(s)', \end{array}\right. \tag{4.19}$$

and

$$\left\{\begin{array}{l} -\zeta s' + f^D(s)' = 0, \\ \zeta\pi' = \zeta\pi^D(s)', \end{array}\right. \tag{4.20}$$

respectively.

In the case of imbibition mode, a rarefaction wave is determined by $\zeta = {f^I}'(s)$. Since $f^I(s)$ is convex by (4.8), the rarefaction wave satisfies $s' > 0$. Furthermore, since f^I is increasing by (2.3), then $I =$*false*. This shows that there are no rarefaction waves in imbibition mode. On the contrary, there are shock waves in imbibition mode (IS) if $s_+ < s_-$, as stated in Remark 4.2.

In the case of drainage mode, analogous considerations lead to exclude shock waves; only rarefaction waves in drainage mode (DR) are possible.

In scanning mode, scanning shocks (SS) and scanning rarefaction waves (SR) are the ordinary shocks and rarefaction waves of (2.17) with $\pi =$ constant, and hence we need $\pi_- = \pi_+$. In this case, (2.17) reduces to the scalar conservation law $s_t + f^S(s, \pi_\pm)_x = 0$. As above, by (4.9), scanning shocks exist if and only if $s_- > s_+$, and scanning rarefaction waves exist if and only if $s_- < s_+$.

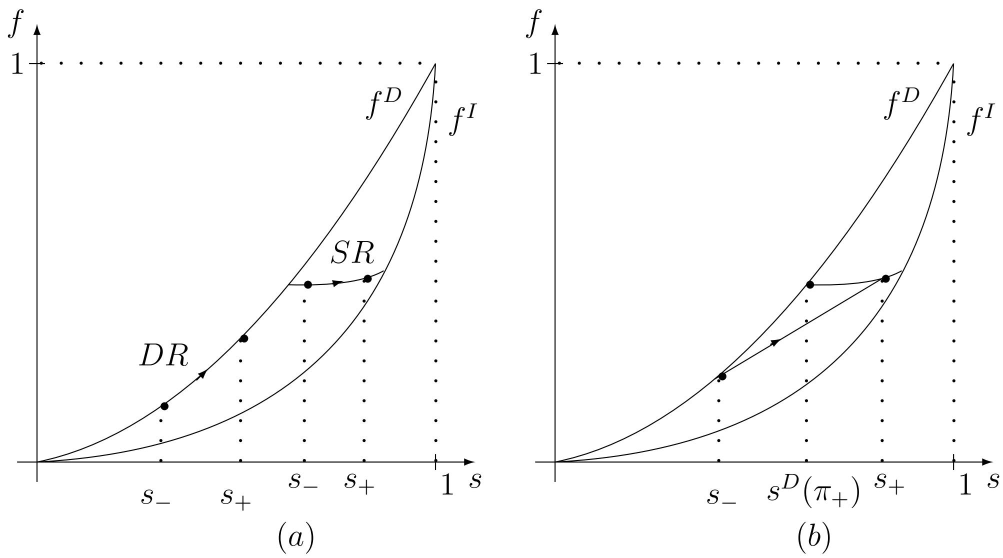


FIGURE 4.2. *(a)*: a drainage and a scanning rarefaction wave. *(b)*: a scanning-to-drainage shock wave.

A rarefaction wave containing a mode boundary, if such wave exists, can be considered as two single-mode rarefaction waves glued together. Thus there is no need to study such kind of waves.

Scanning-to-drainage shocks. A scanning-to-drainage shock (SDS) connects $\big(s_- = s^D(\pi^-), \pi_-\big)$ in drainage mode to (s_+, π_+) in scanning mode.

Theorem 4.3. *Assume (I) and (III). Let* $(s_\pm, \pi_\pm) \in \Omega$ *with* $s_- = s^D(\pi_-)$ *and* $s^D(\pi_+) \le s_+ \le s^I(\pi_+)$.

(i) If $\pi_- < \pi_+$, *then* (4.1) *has a solution if and only if condition*

$$\frac{F(s_+, \pi_+) - F(s_-, \pi_+)}{s_+ - s_-} < \frac{F(s, \pi_+) - F(s_-, \pi_+)}{s - s_-} \quad \text{for all } s_- < s < s_+ \tag{4.21}$$

is satisfied. In this case the solution is unique up to a shift.

(ii) If $\pi_- > \pi_+$ *and* $\sigma \neq 0$, *then* (4.1) *has no solution.*

Remark 4.3. In case *(i)*, since $\pi_- < \pi_+$, by Lemma 2.1 we have $s_- = s^D(\pi_-) < s^D(\pi_+) \le s_+$. Then the chord condition (4.21) is the usual Oleinik E-condition [3, 8.4.3] for the nonconvex flux $s \to F(s, \pi_+)$, see Figure 2.2*(b)*. Here it states that the line joining the points $\big(s_\pm, F(s_\pm, \pi_+)\big)$ lies below the graph of $F(s, \pi_+)$ in the interval (s_-, s_+).

Proof of Theorem 4.3. We split the proof into two cases and refer to Figure 4.2.

(i) We assume $\pi_- < \pi_+$; as a consequence, by (2.8) we have

$$F(s_-, \pi_+) = F(s_-, \pi_-), \tag{4.22}$$

see Figure 4.2*(b)*. Therefore σ coincides with the left-hand side of (4.21). The end states $s_\pm$ are equilibrium points of the equation

$$A(s,\pi)s' = (s - s_-)\left(-\sigma + \frac{F(s,\pi_+) - F(s_-,\pi_+)}{s - s_-}\right), \tag{4.23}$$

where π is defined as

$$\pi(\xi) := \begin{cases} \pi_+ & \text{if } s(\xi) > s^D(\pi_+), \\ \pi^D\big(s(\xi)\big) & \text{if } s(\xi) \le s^D(\pi_+), \end{cases} \tag{4.24}$$

and σ is given by (4.2). As a consequence of condition (4.21) we have

$$f_s^D(s_-) \ge \sigma \ge f_s^S(s_+, \pi_+). \tag{4.25}$$

Assume first that (4.25) holds with strict inequalities, namely,

$$f_s^D(s_-) > \sigma > f_s^S(s_+, \pi_+). \tag{4.26}$$

In this case the point s_- is an unstable equilibrium while s_+ is a stable equilibrium of (4.23). An unstable trajectory of s_- entering the $s > s_-$ side will increase as ξ increases and enter the first stable equilibrium point on the right side of s_-, which is s_+.

If (4.26) fails, then some equalities replace the strict inequalities: this means that the line joining the points $\big(s_\pm, F(s_\pm, \pi_\pm)\big)$, whose slope is σ by (4.22), is tangent to the graph of $s \mapsto F(s, \pi_+)$ either at the point of abscissa s_- or at that of abscissa s_+, possibly at both. In this case, we consider the straight line which is parallel to the line joining the points $\big(s_\pm, F(s_\pm, \pi_\pm)\big)$ but that is $\mu > 0$ higher than it in the (s, f)-plane. Then the chord condition (4.21) is satisfied with $(s_\pm, \pi_\pm)$ replaced by two points of intersections of the line and the graph of F, say $(s_\pm^\mu, \pi_\pm^\mu)$, which reestablish (4.25). Reconsider equation (4.23) with (s_-, π_-) and (s_+, π_+) replaced by $(s_-^\mu, \pi_-^\mu) := \big(s_-^\mu, \pi^D(s_-^\mu)\big)$ and $(s_+^\mu, \pi_+^\mu) := (s_+^\mu, \pi_+)$, respectively; we denote by $(4.23)^\mu$ this equation. Notice that σ is not affected by this replacement. Again by (4.8) and (4.9), condition (4.26) is true for $(s_\pm^\mu, \pi_\pm^\mu)$. Then, there is a solution $s^\mu(\xi)$ for $(4.23)^\mu$ and we can shift it in such a way that $s^\mu(0) = (s_+ + s_-)/2$. Since $\big\{s^\mu(\xi), \pi^\mu(\xi)\big\}$ is monotone in ξ and uniformly bounded between $(s_\pm, \pi_\pm)$, there is a sequence μ_k so that the pointwise limit $\big(s(\xi), \pi(\xi)\big) := \lim_{\mu_k \to 0+} \big(s^{\mu_k}(\xi), \pi^{\mu_k}(\xi)\big)$ exists. This limit must be a weak solution of (4.23), and hence a strong solution of (4.23). The datum $s^{\mu_k}(0) = (s_+ + s_-)/2$ translates into $s(0) = (s_+ + s_-)/2$ and implies that $s(\xi)$ is not a constant function, since $(s_+ + s_-)/2$ is not an equilibrium point of (4.23). Then $s(\pm\infty) = s_\pm$ follows. This proves that, under assumption (4.21), equation (4.23) has a solution $s(\xi)$ with $s(\pm\infty) = s_\pm$. As in the proof of Theorem 4.2, we can then prove that $\big(s(\xi), \pi(\xi)\big)$ solves (4.1).

Similarly, the uniqueness also follows as in that theorem: in this case we choose $s_- < s_0 < s^D(\pi_+)$. If $\pi(0) \neq \pi^D(s_0)$, then $\pi^I(s_0) \leq \pi_0 < \pi^D(s_0)$. If $s'(0) > 0$ we are in scanning mode; if $s'(0) < 0$ then we are either in scanning or in imbibition mode. In neither case the end state s_+ can be reached.

The above shows the existence and uniqueness of solutions of (4.1) when the chord condition (4.21) holds. It remains to show that when (4.21) fails, there is no solution for (4.1). Indeed, if (4.21) fails, then according to $(4.1)_1$, we have $s'(\xi_0) < 0$ for some $s(\xi_0) \in (s_-, s_+)$, hence $s(\xi)$ cannot be monotone. This contradicts Lemma 4.1.

(ii) The proof is similar to that of Theorem 4.2. It is omitted. $\square$

Stationary shocks (ST). Although the jump conditions (2.22) do not need shock profiles when $\zeta_0 = 0$, nevertheless the existence or non-existence of such profiles is still interesting. The Rankine-Hugoniot condition for stationary shocks (ST) is

$$F(s_+, \pi_+) - F(s_-, \pi_-) = 0.$$

These shocks are Lax contact discontinuities corresponding to the eigenvalue $\lambda_0 = 0$ of the scanning system. A possible pair $(s_\pm, \pi_\pm)$ of states giving rise to a stationary shock is shown in Figure 4.3.

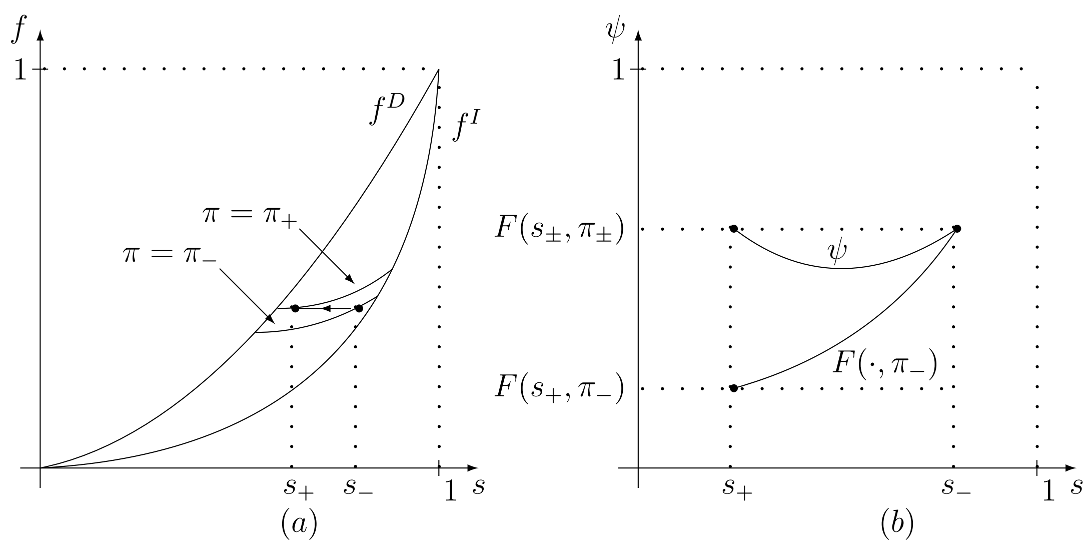


FIGURE 4.3. *(a):* a stationary shock wave. *(b):* the function ψ in the proof of Theorem 4.4.

Theorem 4.4. *Assume (I) and (III). Let* $(s_\pm, \pi_\pm) \in \Omega$ *with* $(s_-, \pi_-) \neq (s_+, \pi_+)$ *and assume* $F(s_-, \pi_-) = F(s_+, \pi_+)$*. Then, system* (4.1) *has infinitely many solution* $\big(s(\xi), \pi(\xi)\big)$ *with* $\sigma = 0$*.*

Proof. We assume $s_- > s_+$ for definiteness; the proof of the other case is similar. Since $F_s > 0$ and $F_\pi > 0$ by (2.5), conditions $F(s_-, \pi_-) = F(s_+, \pi_+)$ and $s_- > s_+$ imply $\pi_- < \pi_+$ and $F(s_+, \pi_-) < F(s_-, \pi_-)$, as shown in Figure 4.3. Consider the solution $s_0 = s_0(\xi)$ of (4.1) when $\sigma = 0$, i.e.,

$$\begin{cases} A(s,\pi)s' = F(s,\pi) - F(s_-, \pi_-), \\ (s,\pi)(\pm\infty) = (s_\pm, \pi_\pm). \end{cases} \tag{4.27}$$

For brevity we dropped the condition $(s', \pi')(\pm\infty) = (0, 0)$. Pick any smooth function $\psi(s)$ satisfying

$$F(s, \pi_-) < \psi(s) < F(s_-, \pi_-) \tag{4.28}$$

for $s \in (s_+, s_-)$ and $\psi(s_\pm) = F(s_\pm, \pi_\pm)$, see Figure 4.3 *(b)*.

Because $F_\pi > 0$, there is a unique $\pi = \pi(s)$ so that $F\left(s, \pi(s)\right) = \psi(s)$; moreover, the function $\pi(s)$ is also differentiable. Furthermore, we have $\pi(s_\pm) = \pi_\pm$ because of $\psi(s_\pm) = F(s_\pm, \pi_\pm)$. Then (4.27) with this choice of π becomes

$$\begin{cases} A\left(s, \pi(s)\right) s' = \psi(s) - F(s_-, \pi_-), \\ s(\pm\infty) = s_\pm. \end{cases} \tag{4.29}$$

Equation $(4.29)_1$ has two equilibrium points $s_\pm$ and $s' < 0$ for $s \in (s_+, s_-)$. Thus, it has a solution $s(\xi)$, resulting in a solution $\Big(s(\xi), \pi\left(s(\xi)\right)\Big)$ for the stationary-profile system (4.27). Since there are infinitely many choices of $\psi(s)$, and $s(\xi)$ obviously depends on ψ as indicated by (4.29), there are infinitely many solutions for (4.27).

The case $s_- < s_+$ is similar, with (4.28) changed to $F(s, \pi_-) > \psi(s) > F(s_-, \pi_-)$ for $s \in (s_-, s_+)$. $\square$

Theorem 4.5. *Assume (I) and (III). Let* $(s_\pm, \pi_\pm) \in \overline{\Omega^S}$ *and assume* $F(s_-, \pi_-) = F(s_+, \pi_+)$*. Then, system* (4.1) *has a solution* $\big(s(\xi), \pi(\xi)\big)$ *with* $\sigma = 0$*, where* $s_\xi(\xi)$ *has a jump discontinuity and* $\pi(\xi)$ *is a step function.*

Proof. We assume $s_- > s_+$ for definiteness; the proof of the other case is similar. Since $F_s > 0$ and $F_\pi > 0$ by (2.5) conditions $F(s_-, \pi_-) = F(s_+, \pi_+)$ and $s_- > s_+$ imply $\pi_- < \pi_+$ and $F(s_+, \pi_-) < F(s_-, \pi_-)$. Consider the solution $s_0 = s_0(\xi)$ of

$$\begin{cases} A(s, \pi_-)s' = F(s, \pi_-) - F(s_-, \pi_-) \quad \text{for } -\infty < x < 0, \\ s(0) = s_+. \end{cases} \tag{4.30}$$

Since $\pi_- < \pi_+$ and $s^D(\pi_+) \le s_+ \le s^I(\pi_+)$, the monotonicity of the function $s^D(\pi)$ implies the chain of inequalities $s^D(\pi_-) < s^D(\pi_+) \le s_+ < s_- \le s^I(\pi_-)$, see the scanning curves in Figure 4.3. As ξ decreases from 0 to $-\infty$, the function $s_0(\xi)$ increases from s_+ to s_-; moreover, $(s,\pi)(\xi) = \big(s_0(\xi),\pi_-\big)$ is in scanning mode, hence $\pi(\xi) = \pi_-$ is consistent with $(4.1)_2$ for $\xi < 0$. It is straightforward to check that the function $(s,\pi)(\xi)$ defined by

$$\big(s(\xi),\pi(\xi)\big) := \begin{cases} \big(s_0(\xi),\pi_-\big) & \text{if } -\infty < \xi < 0, \\ (s_+,\pi_+) & \text{else}, \end{cases} \tag{4.31}$$

is a solution of (4.1) with $\sigma = 0$ in the sense of distributions. □

Remark 4.4. When $\sigma = 0$, according to (4.1), there is not much constraint on $\pi(\xi)$. This provides the opportunity for infinitely solutions of (4.1) to exist for the same end states, as show in the last two theorems. Which one will appear in Riemann solvers? Our numerical experiments, which use upwinding schemes, indicate that it appears the one with a jump discontinuity in π given in Theorem 4.5. Systems (2.16) and (2.17) do not provide a smoothing mechanism for π in x direction if π has a stationary discontinuity. This leads to the possibility of stationary wave profiles with π discontinuous in x.

5. Solutions to Riemann Problems and Nonuniqueness

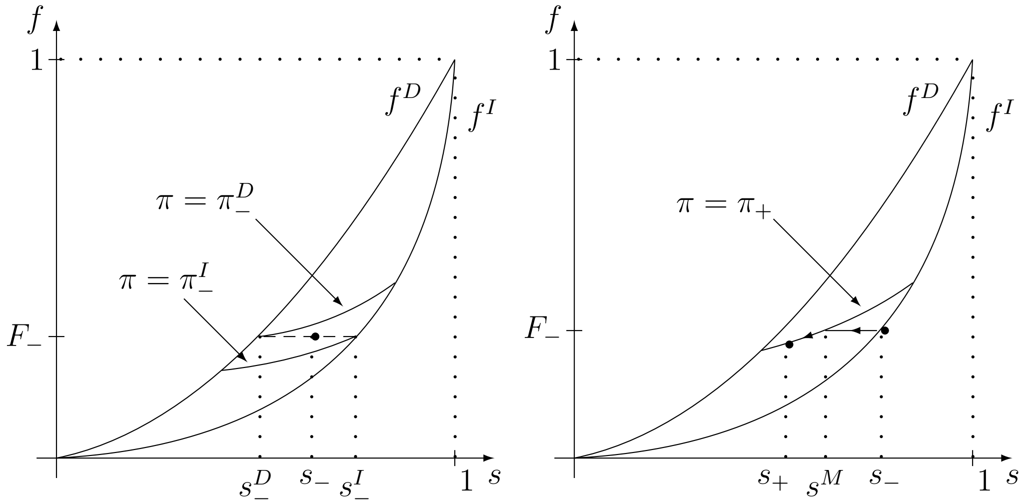
In this section, under Assumptions (I) and (III), we construct Riemann solvers by "gluing" the basic waves listed in Section 4. Although the Riemann problem (4.17), (4.18) has already been briefly studied in [13], here we list the Riemann solutions in full detail. In the following we use the notation (s_-,π_-), (s_+,π_+) for (s_l,π_l), (s_r,π_r), respectively.

We introduce the notation $F_\pm := F(s_\pm,\pi_\pm)$. Let s_-^D and s_-^I denote the solutions of the equations $F_- = f^D(s)$ and $F_- = f^I(s)$, respectively; the points s_-^D and s_-^I exist and are unique because the functions f^D and f^I are strictly increasing by (2.3). We also denote $\pi_-^D := \pi^D(s_-^D)$ and $\pi_-^I := \pi^I(s_-^I)$; see Figure 5.1. We point out that the points (s_-^D,π_-^D) and (s_-^I,π_-^I) are the intersection of the locus $F(s,\pi) = F_-$ with the graphs of the functions $s^D(\pi)$ and $s^I(\pi)$, respectively, see Figure 5.1*(a)*. The locus $F(s,\pi) = F_-$ in Ω^S implicitly defines a function $\pi = \pi(s)$; it satisfies $\frac{d\pi}{ds} < 0$, due to (2.5).

About the possible solutions of the Riemann problem (4.17)-(4.18), we consider several cases. They are classified first by the position of π_- and then by that of π_+.

Case 1. $\pi_- = \pi_-^I$ and hence $F(s_-,\pi_-) = f^I(s_-)$ by (2.8).

Case 1(a). $\pi_+ \le \pi_- = \pi_-^I$. A solution is a scanning-to-imbibition shock provided by Theorem 4.2. See Figure 4.1 *(a)*.



(a) (b)

FIGURE 5.1. *(a)*: the points s_-^D and s_-^I. *(b):* Case 1(b): When $\pi_-^I = \pi_- < \pi_+ \le \pi_-^D$, the solution depicted is a piecewise constant function $(s,\pi)(\xi)$ with constants being $(s_-,\pi_-) \xrightarrow{\text{ST}} (s^M,\pi_+) \xrightarrow{\text{SW}} (s_+,\pi_+)$ in the order of increasing $\xi \in \mathbb{R}$. Here ST stands for stationary shock given by Theorem 4.5, while SW for a scanning rarefaction wave or scanning shock.

The rest of Case 1 and Case 2, where (s_-,π_-) is in the scanning zone, are shown in Figures 5.2 and 5.3, respectively.

Case 3. When $\pi_- = \pi_-^D$, solutions are similar to those depicted in Figure 5.2 and Figure 5.3 *(a)*, except that s_- is moved to the drainage curve.

This concludes the analysis of the solutions of the Riemann problem; under Assumptions (I) and (III) we proved that for any two pairs of states (s_-,π_-) and (s_+,π_+) in Ω the Riemann problem (4.17), (4.18) has a solution. Moreover, the components π in Riemann solvers listed above are all monotone. On the other hand, due to the requirements in imbibition and drainage modes, the s component of solutions cannot be monotone in general.

In most cases listed above, however, there are *infinitely many solutions*, [13]. For example, in Case 1(c), the part of the solution corresponding to

$$(s_-,\pi_-) \xrightarrow{\text{ST}} (s_-^D,\pi_-^D) \tag{5.1}$$

can replaced by

$$(s_-,\pi_-) \xrightarrow{\text{SIS}} \Big(s^L,\pi^D(s^L)\Big) \tag{5.2}$$

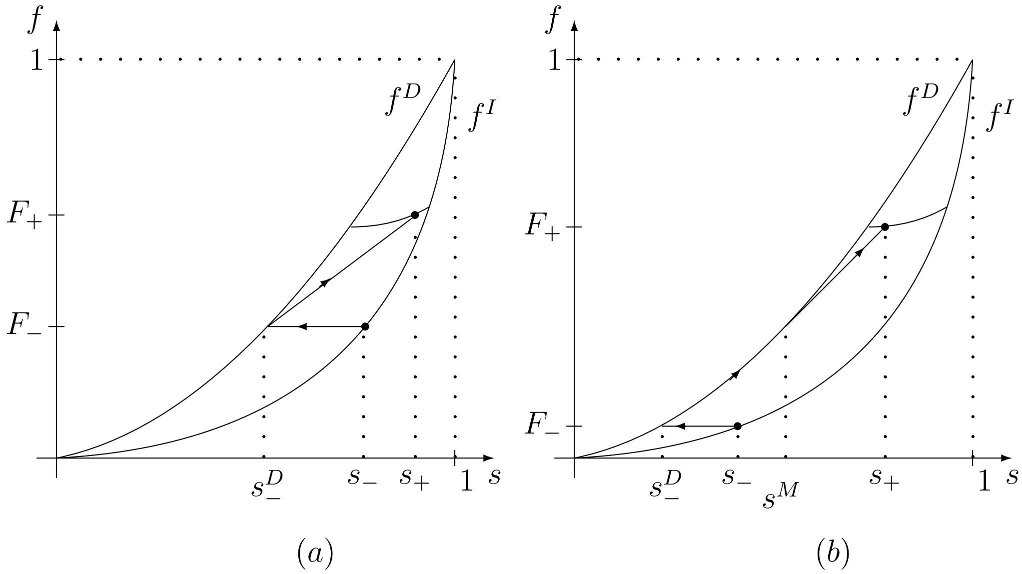


FIGURE 5.2. *(a):* Case 1(c1). A solution when $\pi_-^D < \pi_+$ and the chord condition

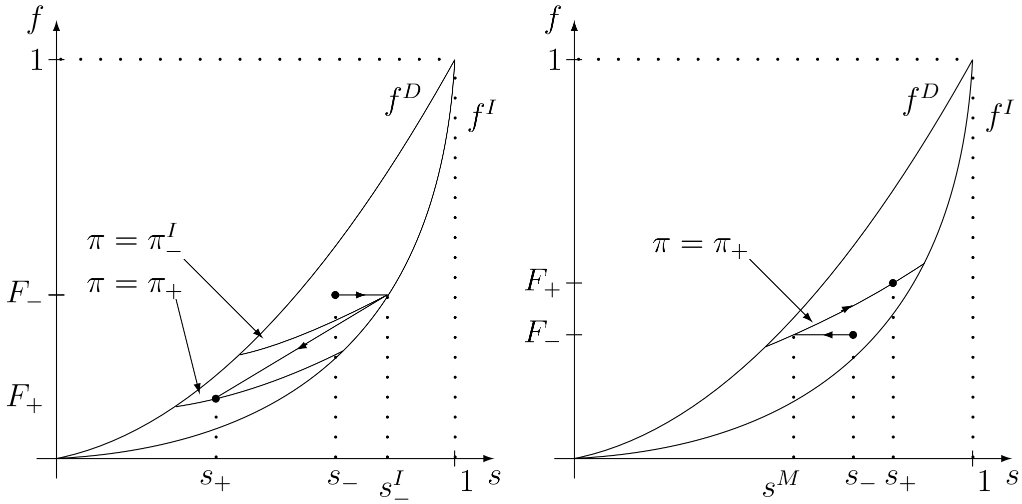
$$\frac{F(s_+,\pi_+)-F(s_-^D,\pi_+)}{s_+-s_-^D} < \frac{F(s,\pi_+)-F(s_-^D,\pi_+)}{s-s_-}$$

is satisfied for all $s_- < s < s_+$. *(b):* Case 1(c2). A solution for the case when $\pi_-^D < \pi_+$ and the chord condition in *(a)* fails. If the chord condition cannot be true from $\big(s^M,\pi^D(s^M)\big)$ to (s_+,π_+), then there exists s_M such that the tangent line of $f = f^D(s)$ at $s = s^M$ is also tangent to the scanning curve $\pi = \pi_+$ at some point $\big(s_N, f(s_N,\pi_+)\big)$. Then, the shock joining s^M to s_+ in the figure is replaced with a shock from s^M to s^N followed by a scanning wave to s_+.

with $s^L < s_-^D$ as long as $\big(F_- - f^D(s^L)\big)/(s_- - s^L) < f_s^D(s^L)$. The results are still solutions to the same Riemann problem. Similar constructions can be done for most of other cases listed above to result in nonuniqueness.

However, under the further Assumption (II), since Riemann initial data for π are monotone by Theorem 3.10, then $\pi(\xi)$ must also be monotone. Therefore, no Riemann solvers with non-monotone π can be limits of viscosity solutions and should be excluded.

Example 5.1. *The maximum principle stated in Corollary 3.7 can exclude some Riemann solvers. In* [13] *the authors discovered that there are infinitely many Riemann solutions for some Riemann initial data. An example is that there are many loop solutions, which are not constant solutions, for the same constant initial data. The maximum principle for the viscous system* (2.16) *states that only the constant*



(a) (b)

FIGURE 5.3. Case 2. $s^D(\pi_-) < s_- < s^I(\pi_-)$. *(a)*: A solution when $\pi_+ < \pi_-^I$. *(b)*: A solution when $\pi_-^I \leq \pi_+ \leq \pi_-^D$. This is just like Case 1(b) depicted in Figure 5.1*(b)* except that (s_-, π_-) is moved from the imbibition curve into the scanning zone. For the remaining case when $\pi_+ > \pi_-^D$, a solution is almost the same as in case 1(c) shown in Figure 5.2 except that now (s_-, π_-) is in the interior of the scanning zone.

solution for this initial data is the "good" solution from the vanishing viscosity point of view.

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