

# MONADIC DECOMPOSITIONS AND CLASSICAL LIE THEORY

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ABSTRACT. We show that the functor from bialgebras to vector spaces sending a bialgebra to its subspace of primitives has monadic length at most 2.

## INTRODUCTION

Given a functor  $R_0 : \mathcal{A} \rightarrow \mathcal{B}_0$  with left adjoint  $L_0 : \mathcal{B}_0 \rightarrow \mathcal{A}$  we get, following [AHW, MS], and under suitable hypotheses, a sequence of adjoint pairs of functors

$$\begin{array}{ccccccc}
 \mathcal{A} & \xleftarrow{\text{Id}_{\mathcal{A}}} & \mathcal{A} & \xleftarrow{\text{Id}_{\mathcal{A}}} & \mathcal{A} & \xleftarrow{\text{Id}_{\mathcal{A}}} & \dots \\
 \downarrow L_0 & & \downarrow L_1 & & \downarrow L_2 & & \\
 \mathcal{B}_0 & \xleftarrow{U_{0,1}} & \mathcal{B}_1 & \xleftarrow{U_{1,2}} & \mathcal{B}_2 & \xleftarrow{U_{2,3}} & \dots
 \end{array}$$

where for  $i \geq 0$ ,  $\mathcal{B}_{i+1}$  is the Eilenberg-Moore category of the monad  $(L_i, R_i)$ ,  $R_{i+1}$  is the comparison functor, and  $U_{i,i+1}$  is the corresponding forgetful functor. It is natural to inquire whether this process stops, as was done in [AHW, MS]. To be more specific, the monadic length of  $R_0$  is the first  $N$  such that  $U_{N,N+1}$  is an isomorphism of categories. In many basic examples, the functor  $R_0$  is monadic and, therefore, it has monadic length at most 1. In this note, we show that the functor  $P$  from bialgebras to vector spaces sending a bialgebra to its subspace of primitives has monadic length 2 (Theorem 2.4).

Section 1 contains some remarks on the monadic decompositions of functors studied in [AHW, MS] and their relationship with idempotent monads ([AT]). The basic case of the adjoint pair encoded by a bimodule over unital rings is described in Remark 1.14, with an eye on descent theory for modules. We also study the existence of comonadic decompositions under separability conditions (Proposition 1.16).

Section 2 contains the aforementioned monadic decomposition of monadic length at most 2 of the functor  $P$  from bialgebras to vector spaces.

## 1. MONADIC DECOMPOSITIONS

Consider categories  $\mathcal{A}$  and  $\mathcal{B}$ . Let  $(L : \mathcal{B} \rightarrow \mathcal{A}, R : \mathcal{A} \rightarrow \mathcal{B})$  be an adjunction with unit  $\eta$  and counit  $\epsilon$ , and consider the monad  $(RL, R\epsilon L, \eta)$  generated on  $\mathcal{B}$ . By  $\mathcal{B}_1$  we denote its Eilenberg-Moore category of algebras. Hence we can consider the so-called *comparison functor* of the adjunction  $(L, R)$  i.e. the functor

$$K : \mathcal{A} \rightarrow \mathcal{B}_1, \quad KX := (RX, R\epsilon X), \quad Kf := Rf.$$

Recall that the functor  $R : \mathcal{A} \rightarrow \mathcal{B}$  is called *monadic* (tripleable in Beck's terminology [Be, Definition 3', page 8]) whenever the comparison functor  $K : \mathcal{A} \rightarrow \mathcal{B}_1$  is an equivalence of categories.

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**1.1. Idempotent monads and monadic decompositions.** The notion of an idempotent monad is, as we will see below, tightly connected with the monadic length of a functor.

**DEFINITION 1.1.** [AT, page 231] A monad  $(Q, m, u)$  with multiplication  $m$  and unit  $u$  is called *idempotent* whenever  $m$  is an isomorphism. An adjunction  $(L, R)$  is called *idempotent* whenever the associated monad is idempotent.

There are several basic characterizations of idempotent adjunctions (see [AT, MS]). In particular, idempotency of an adjunction means equivalently that any of the four factors appearing in the two triangular identities of the adjoint pair  $(L, R)$  is an isomorphism ([MS, Proposition 2.8]).

In the following we will denote by  $U : \mathcal{B}_1 \rightarrow \mathcal{B}$  the forgetful functor and by  $F : \mathcal{B} \rightarrow \mathcal{B}_1$  the free functor associated to an adjunction  $(L, R)$ .

**REMARK 1.2.** Note that the adjunctions  $(F, U)$  and  $(L, R)$  have the same associated monad so that  $(L, R)$  is idempotent if and only if  $(F, U)$  is if and only if, by [MS, Proposition 2.8], one has  $\eta U$  is an isomorphism.

**PROPOSITION 1.3.** *For an adjunction  $(L : \mathcal{B} \rightarrow \mathcal{A}, R : \mathcal{A} \rightarrow \mathcal{B})$  with unit  $\eta$  and counit  $\epsilon$ , the following assertions are equivalent.*

- (a)  $(L, R)$  is idempotent,
- (b) the structure map of every object in  $\mathcal{B}_1$  is an isomorphism,
- (c)  $LU$  is a left adjoint of the comparison functor  $K : \mathcal{A} \rightarrow \mathcal{B}_1$  of  $(L, R)$ , and  $\eta U = U\eta_1$ , where  $\eta_1$  is the unit of the new adjunction (being the new counit still  $\epsilon$ ).

Moreover, if one of these conditions holds, then  $LU$  is full and faithful.

*Proof.* (a)  $\Rightarrow$  (b). By [MS, Proposition 2.8], we have that  $\eta RL = RL\eta$ . Now, for any algebra  $(X, \mu) \in \mathcal{B}_1$ , we have that  $\mu \circ \eta X = 1_X$ . Moreover, by naturality of  $\eta$ , we know that  $\eta X \circ \mu = RL\mu \circ \eta RLX = RL\mu \circ RL\eta X = 1_{RLX}$ . Therefore,  $\mu$  is an isomorphism.

(b)  $\Rightarrow$  (c). If  $(X, \mu)$  is an algebra over  $RL$ , then  $\mu : RLX \rightarrow X$  is a homomorphism of  $RL$ -algebras. Now, if  $\mu$  is an isomorphism, then, necessarily,  $\mu = (\eta X)^{-1}$ . We get easily that  $\eta X$  is a morphism in  $\mathcal{B}_1$ . Therefore,  $\eta$  will serve as the unit for an adjunction  $(LU, K)$  (being the counit  $\epsilon$ ).

(c)  $\Rightarrow$  (a). Given an object  $Y$  of  $\mathcal{A}$ , consider its free  $RL$ -algebra  $(RY, R\epsilon Y)$ . Since, by assumption,  $\eta RY$  is a homomorphism of  $RL$ -algebras, we get the identity  $R\epsilon LRY \circ RL\eta RY = \eta RY \circ R\epsilon Y$ , which implies, by the triangular identities for the adjunction  $(L, R)$ , that  $\eta RY \circ R\epsilon Y = 1_{RLRY}$ , whence  $R\epsilon Y$  is an isomorphism. With  $Y = LX$  for any object  $X$  of  $\mathcal{B}$ , one obtains that  $R\epsilon LX$  is an isomorphism. Hence,  $(L, R)$  is idempotent.

Let us prove the last part of the statement. By Remark 1.2, condition (a) is equivalent to  $\eta U$  isomorphism. From  $\eta U = U\eta_1$  and the fact that  $U$  reflects isomorphisms we deduce that  $\eta_1$  is an isomorphism so that  $LU$  is full and faithful.  $\square$

**REMARK 1.4.** By [MS, Proposition 2.8], if  $L$  is full and faithful, then the adjunction  $(L, R)$  is idempotent. On the other hand, since the units of the adjunctions  $(L, R)$  and  $(F, U)$  are equal, we get from Proposition 1.3 that  $L$  is full and faithful if and only if  $U$  is an isomorphism of categories if and only if  $U$  is an equivalence of categories.

Before defining monadic decompositions and monadic length, we derive the following consequence of Proposition 1.3, which is interpreted in the classical setting of epimorphisms of rings in Example 1.6.

**PROPOSITION 1.5.** *Let  $(L, R)$  be an adjunction. The following are equivalent.*

- (1)  $R$  is full and faithful.
- (2)  $(L, R)$  is idempotent and  $R$  is monadic.

*Proof.* Let  $\eta$  be the unit and  $\epsilon$  be the counit of the adjunction  $(L, R)$ .

(1)  $\Rightarrow$  (2). By assumption  $\epsilon$  is an isomorphism so that  $R\epsilon L$  is an isomorphism i.e.  $(L, R)$  is idempotent. By Proposition 1.3, we know that  $\Lambda := LU$  is a left adjoint of the comparison functor

$K : \mathcal{A} \rightarrow \mathcal{B}_1$  of  $(L, R)$ . Moreover  $\Lambda$  is full and faithful and  $\epsilon A = \epsilon_1 A$  where  $\epsilon_1$  is the counit of the adjunction  $(\Lambda, K)$ . Thus  $\epsilon_1$  is an isomorphism i.e.  $K$  is full and faithful too. Hence  $K$  is an equivalence.

(2)  $\Rightarrow$  (1). Since  $(L, R)$  is idempotent, by Proposition 1.3, we get  $\epsilon A = \epsilon_1 A$ . Since  $R$  is monadic, the comparison functor is an equivalence and hence  $\epsilon_1 A$  is an isomorphism. Hence  $\epsilon A$  is an isomorphism so that  $R$  is full and faithful.  $\square$

EXAMPLE 1.6. Let us consider a morphism of rings  $\pi : B \rightarrow A$ , and its canonical associated adjunction

$$L : \text{Mod-}B \rightarrow \text{Mod-}A, \quad R : \text{Mod-}A \rightarrow \text{Mod-}B.$$

By [St, Proposition 1.2, page 226],  $\pi$  is an epimorphism if and only if the counit of the adjunction is an isomorphism. This is equivalent to say that  $R$  is full and faithful. Thus, by Proposition 1.5,  $\pi$  is an epimorphism if and only if  $(L, R)$  is idempotent and  $R$  is monadic. Note that  $L$  needs not to be full and faithful. Thus, when  $\pi$  is an epimorphism, since  $R$  is monadic, it has a monadic decomposition of monadic length 1 in the sense of Definition 1.7 but a monadic decomposition of (essential) length 0 in the sense of [AHW, Definition 2.1].

DEFINITION 1.7. (See [AHW, Definition 2.1] and [MS, Definitions 2.10 and 2.14]) Fix a  $N \in \mathbb{N}$ . We say that a functor  $R$  has a *monadic decomposition of monadic length  $N$*  whenever there exists a sequence  $(R_n)_{n \leq N}$  of functors  $R_n$  such that

- 1)  $R_0 = R$ ;
- 2) for  $0 \leq n \leq N$ , the functor  $R_n$  has a left adjoint functor  $L_n$ ;
- 3) for  $0 \leq n \leq N - 1$ , the functor  $R_{n+1}$  is the comparison functor induced by the adjunction  $(L_n, R_n)$  with respect to its associated monad;
- 4)  $L_N$  is full and faithful while  $L_n$  is not full and faithful for  $0 \leq n \leq N - 1$ .

Compare with the construction performed in [Ma, 1.5.5, page 49].

Note that for functor  $R : \mathcal{A} \rightarrow \mathcal{B}$  having a monadic decomposition of monadic length  $N$ , we have a diagram

$$(1) \quad \begin{array}{ccccccc} \mathcal{A} & \xleftarrow{\text{Id}_{\mathcal{A}}} & \mathcal{A} & \xleftarrow{\text{Id}_{\mathcal{A}}} & \mathcal{A} & \xleftarrow{\text{Id}_{\mathcal{A}}} & \dots & \dots & \xleftarrow{\text{Id}_{\mathcal{A}}} & \mathcal{A} \\ \uparrow L_0 & & \uparrow L_1 & & \uparrow L_2 & & & & & \uparrow L_N \\ \mathcal{B}_0 & \xleftarrow{U_{0,1}} & \mathcal{B}_1 & \xleftarrow{U_{1,2}} & \mathcal{B}_2 & \xleftarrow{U_{2,3}} & \dots & \dots & \xleftarrow{U_{N-1,N}} & \mathcal{B}_N \end{array}$$

where  $\mathcal{B}_0 = \mathcal{B}$  and, for  $1 \leq n \leq N$ ,

- $\mathcal{B}_n$  is the category of  $(R_{n-1}L_{n-1})$ -modules  ${}_{R_{n-1}L_{n-1}}\mathcal{B}_{n-1}$ ;
- $U_{n-1,n} : \mathcal{B}_n \rightarrow \mathcal{B}_{n-1}$  is the forgetful functor  ${}_{R_{n-1}L_{n-1}}U$ .

We will denote by  $\eta_n : \text{Id}_{\mathcal{B}_n} \rightarrow R_n L_n$  and  $\epsilon_n : L_n R_n \rightarrow \text{Id}_{\mathcal{A}}$  the unit and counit of the adjunction  $(L_n, R_n)$  respectively for  $0 \leq n \leq N$ . Note that one can introduce the forgetful functor  $U_{m,n} : \mathcal{B}_n \rightarrow \mathcal{B}_m$  for all  $m \leq n$  with  $0 \leq m, n \leq N$ .

REMARKS 1.8. 1) Assume that  $R$  fits into a diagram such as (1). If  $R_{N-1}$  is monadic i.e.  $R_N$  is a category equivalence, then obviously  $L_N$  is full and faithful so that  $R_0$  has a monadic decomposition of monadic length at most  $N$ . Nevertheless if  $R_0$  has monadic length  $N$ , then  $R_N$  needs not to be an equivalence.

2) The notion of *comonadic decomposition of comonadic length  $N$*  can be easily introduced. In this case we will use the notations  $(L^n, R^n)_{n \in \mathbb{N}}$  with superscripts and require that  $R^N$  is full and faithful.

PROPOSITION 1.9. *Let  $(L : \mathcal{B} \rightarrow \mathcal{A}, R : \mathcal{A} \rightarrow \mathcal{B})$  be an idempotent adjunction. Then  $R : \mathcal{A} \rightarrow \mathcal{B}$  has a monadic decomposition of monadic length at most 1.*

*Proof.* By Proposition 1.3,  $L_1 = L_0 U_{0,1}$  is full and faithful.  $\square$

REMARK 1.10. It follows from Remark 1.4 that condition 4) in Definition 1.7 is equivalent to the requirement that the forgetful functor  $U_{N,N+1} : \mathcal{B}_{N+1} \rightarrow \mathcal{B}_N$  is an isomorphism of categories. Thus, if  $R : \mathcal{A} \rightarrow \mathcal{B}$  has a monadic decomposition of monadic length  $N \in \mathbb{N}$ , then we can consider the comparison functor  $R_{N+1} : \mathcal{A} \rightarrow \mathcal{B}_{N+1}$  of  $(L_N, R_N)$ . Moreover, still by Remark 1.4,  $L_N$  full and faithful implies that the adjunction  $(L_N, R_N)$  is idempotent. Hence, by Proposition 1.3,  $L_{N+1} := L_N U_{N,N+1}$  is a left adjoint of  $R_{N+1}$  (and  $L_{N+1}$  is full and faithful too). Note that the fact that  $R_{N+1}$  is a right adjoint is assumed from the very beginning in [AHW, Definition 2.1]. By Proposition 1.3 again, we deduce that  $\eta_N U_{N,N+1} = U_{N,N+1} \eta_{N+1}$  and  $\epsilon_N A = \epsilon_{N+1} A$  where  $\eta_n$  is the unit and  $\epsilon_n$  is the counit of the adjunction  $(L_n, R_n)$  for all  $n \leq N+1$ . Iterating this process we get that for all  $M \geq N$ , the tower in (1) can be extended with adjoints  $(L_M, R_M)$  where  $L_M$  is full and faithful so that  $U_{M,M+1} : \mathcal{B}_{M+1} \rightarrow \mathcal{B}_M$  is a category isomorphism. Moreover  $\eta_M U_{M,M+1} = U_{M,M+1} \eta_{M+1}$  and  $\epsilon_M A = \epsilon_{M+1} A$ . By the foregoing we have that

$$R = R_0 = U_{0,1} \circ U_{1,2} \circ \cdots \circ U_{N-1,N} \circ R_N$$

where  $U_{0,1}, U_{1,2}, \dots, U_{N-1,N}$  are monadic functors but not category isomorphisms. Moreover this is a maximal decomposition of this form. This is essentially [AHW, Remarks 2.2].

REMARK 1.11. If  $R : \mathcal{A} \rightarrow \mathcal{B}$  has a monadic decomposition of length  $N$ , then, since  $L_N : \mathcal{B}_N \rightarrow \mathcal{A}$  is full and faithful, the dual of Proposition 1.5 gives that  $L_N$  is a comonadic functor and  $(L_N, R_N)$  is coidempotent. Thus, the comparison functor  $C : \mathcal{B}_N \rightarrow \mathcal{A}^N$ , where  $\mathcal{A}^N$  denotes the category of  $L_N R_N$ -coalgebras, is an equivalence of categories.

**1.2. Essentially surjective.** The following result determines the objects which are images of right adjoint functors under suitable assumptions. This can be regarded as a sort of descent theory for these functors.

NOTATION 1.12. Let  $R : \mathcal{A} \rightarrow \mathcal{B}$ . We will denote by  $\text{Im}R$  the full subcategory of  $\mathcal{B}$  consisting of those objects  $B \in \mathcal{B}$  such that there is an object  $A \in \mathcal{A}$  and an isomorphism  $B \cong RA$  in  $\mathcal{B}$ .

Recall that a functor  $R : \mathcal{A} \rightarrow \mathcal{B}$  is essentially surjective if  $\text{Im}R = \mathcal{B}$ .

PROPOSITION 1.13. Suppose that  $R : \mathcal{A} \rightarrow \mathcal{B}$  has a monadic decomposition of monadic length  $N \in \mathbb{N}$ . Let  $n \in \{0, \dots, N\}$ . Then

- 1)  $\text{Im}R \subseteq \text{Im}U_{0,n}$ .
- 2)  $\text{Im}R = \text{Im}U_{0,n}$  whenever  $R_n$  is essentially surjective.
- 3)  $\text{Im}R = \text{Im}U_{0,N}$ .

*Proof.* It follows from the equalities  $U_{0,n} R_n = R_0 = R$ . □

REMARK 1.14. Proposition 1.13 can be considered as a “general dual descent theory” result. In fact the theorem states that the objects of  $\mathcal{B} = \mathcal{B}_0$  which are isomorphic to objects of the form  $RA$ , for some  $A \in \mathcal{A}$ , are exactly those of the form  $U_{0,N} B_N$  where  $B_N \in \mathcal{B}_N$ . In particular, when  $N = 1$ , i.e.  $L_1$  is full and faithful, we have that the objects of  $\mathcal{B}$  which are isomorphic to objects of the form  $RA$ , for some  $A \in \mathcal{A}$ , are exactly those of the form  $U_{0,1} B_1$  where  $B_1 \in \mathcal{B}_1$ . This is exactly the dual form of classical descent theory for (bi)modules. In fact, let  $S, T$  be rings and let  ${}_S M_T$  be a bimodule. Consider the following adjunction

$$\begin{aligned} L : \mathcal{M}_S &\rightarrow \mathcal{M}_T, & R : \mathcal{M}_T &\rightarrow \mathcal{M}_S \\ LX &= X \otimes_S M, & RY &= \text{Hom}_T(M, Y), \end{aligned}$$

between the category  $\mathcal{M}_S$  of right  $S$ -modules and the category  $\mathcal{M}_T$  of right  $T$ -modules. The category  $\mathcal{M}_T$  has (co)equalizers. By (dual) Beck’s Theorem [Be, Proof of Theorem 1], the comparison functors  $R_1$  and  $L^1$  have a left adjoint  $L_1$  and a right adjoint  $R^1$  respectively.

Assume that  $M$  is flat as a left  $S$ -module. Then  $L = L^0$  is exact so that, the dual of Beck’s Theorem ensures that  $R^1$  is full and faithful. Therefore,  $L$  admits a comonadic decomposition of comonadic length at most 1. Thus we have that the objects of  $\mathcal{M}_T$  which are isomorphic to objects of the form  $LX$ , for some  $X \in \mathcal{M}_S$ , are exactly those of the form  $U^{0,1} X^1$  where  $X^1$  is an object of the category of  $LR$ -coalgebras  $(\mathcal{M}_T)^1$ . Hence the category  $(\mathcal{M}_T)^1$  solves the descent problem for modules. When  $M_T$  is finitely generated and projective, we have an isomorphism of comonads

$LR \cong - \otimes_T M^* \otimes_S M$  where  $M^* \otimes_S M$  is the comatrix coring associated to  ${}_S M_T$  (see [EGT], and [GT, GTV] for more general bimodules). Coalgebras over  $LR$  are precisely the comodules over the  $T$ -coring  $M^* \otimes_S M$ .

Assume that  $M$  is projective as a right  $T$ -module. Then  $R = R_0$  is exact so that, Beck's Theorem ensures that  $L_1$  is full and faithful, and  $R$  has a comonadic decomposition of length at most 1.

**1.3. Separability.** Let  $(Q, m, u)$  be a monad on a category  $\mathcal{B}$ , with multiplication  $m$  and unit  $u$ . A *right module functor* on  $(Q, m, u)$  is a pair  $(W, \mu)$  where  $W : \mathcal{B} \rightarrow \mathcal{A}$  is a functor and  $\mu : WQ \rightarrow W$  is a natural transformation such that

$$\mu \circ \mu Q = \mu \circ Wm \quad \text{and} \quad \mu \circ Wu = \text{Id}_Q.$$

A morphism  $f : (W, \mu) \rightarrow (W', \mu')$  of right module functors is a natural transformation  $f : W \rightarrow W'$  such that  $\mu' \circ fQ = f \circ \mu$ .

It is clear that  $(WQ, Wm)$  is a right module functor on  $(Q, m, u)$  and that  $\mu : (WQ, Wm) \rightarrow (W, \mu)$  is morphism of right module functors. We will say that  $(W, \mu)$  is *relatively projective* whenever  $\mu : (WQ, Wm) \rightarrow (W, \mu)$  splits as a morphism of right module functors. Explicitly this means that there is a morphism  $\gamma : (W, \mu) \rightarrow (WQ, Wm)$  of right module functors such that  $\mu \circ \gamma = \text{Id}_{(W, \mu)}$  i.e. that there is a natural transformation  $\gamma : W \rightarrow WQ$  such that  $\mu \circ \gamma = \text{Id}_W$  and  $Wm \circ \gamma Q = \gamma \circ \mu$ .

Let  $(L : \mathcal{B} \rightarrow \mathcal{A}, R : \mathcal{A} \rightarrow \mathcal{B})$  be an adjunction with unit  $\eta$  and counit  $\epsilon$ . Then  $(L, \epsilon L)$  is a right module functor on  $(RL, R\epsilon L, \eta)$ . In fact  $\epsilon L \circ \epsilon LRL = \epsilon L \circ LR\epsilon L$  and  $\epsilon L \circ L\eta = \text{Id}_{RL}$ .

The notion of a separable functor was introduced in [NVV]. This concept is motivated by various examples, being perhaps the most fundamental the following. Given a homomorphism of rings  $R \rightarrow S$ , then the restriction of scalars functor  $\mathcal{M}_S \rightarrow \mathcal{M}_R$  is separable in the sense of [NVV] if and only if the extension  $R \rightarrow S$  is separable (i.e., the multiplication map  $S \otimes_R S \rightarrow S$  splits as an  $S$ -bimodule epimorphism). In general, if  $(L, R)$  is an adjunction, then  $R$  is a separable functor if and only if its counit is a split natural epimorphism ([Ra, Theorem 1.2]).

**LEMMA 1.15.** *Let  $(L : \mathcal{B} \rightarrow \mathcal{A}, R : \mathcal{A} \rightarrow \mathcal{B})$  be an adjunction. If  $R$  is separable, then  $(L, \epsilon L)$  is relatively projective as a right module functor on  $(RL, R\epsilon L, \eta)$*

*Proof.* By assumption, there is a natural transformation  $\sigma : \text{Id}_{\mathcal{A}} \rightarrow LR$  such that  $\epsilon \circ \sigma = \text{Id}_{\text{Id}_{\mathcal{A}}}$ . Set  $\gamma := \sigma L$ . Then  $\gamma$  is a natural transformation such that  $\epsilon L \circ \gamma = \text{Id}_L$  and  $LR\epsilon L \circ \gamma RL = \gamma \circ \epsilon L$ . Then  $\epsilon L : (LRL, LR\epsilon L) \rightarrow (L, \epsilon L)$  splits as a morphism of right module functors.  $\square$

In the following result, part 3) may be compared, in its dual version, with [Me, Proposition 3.16] and the results quoted therein.

**PROPOSITION 1.16.** *Let  $(L : \mathcal{B} \rightarrow \mathcal{A}, R : \mathcal{A} \rightarrow \mathcal{B})$  be an adjunction with unit  $\eta$  and counit  $\epsilon$ .*

- 1) *If  $R$  is a separable functor then the comparison functor  $R_1 : \mathcal{A} \rightarrow \mathcal{B}_1$  is full and faithful.*
- 2) *Suppose that the comparison functor  $R_1 : \mathcal{A} \rightarrow \mathcal{B}_1$  has a left adjoint  $L_1$ . If  $(L, \epsilon L)$  is relatively projective as a right module functor on  $(RL, R\epsilon L, \eta)$ , then  $L_1$  is full and faithful.*
- 3) *Suppose that the comparison functor  $R_1 : \mathcal{A} \rightarrow \mathcal{B}_1$  has a left adjoint  $L_1$ . If  $R$  is a separable functor, then  $R$  is monadic.*

*Proof.* 1) By assumption there is a natural transformation  $\sigma : \text{Id}_{\mathcal{A}} \rightarrow LR$  such that  $\epsilon \circ \sigma = \text{Id}_{\text{Id}_{\mathcal{A}}}$ . Let  $f, g : X \rightarrow Y$  morphisms in  $\mathcal{A}$  such that  $R_1 f = R_1 g$ . Since  $R = UR_1$ , we get

$$\sigma Y \circ f = LRf \circ \sigma X = LRg \circ \sigma X = \sigma Y \circ g$$

Now,  $\sigma Y$  is a monomorphism, whence  $f = g$ . Thus,  $R_1$  is faithful. To check that  $R_1$  is full, consider a morphism  $h : R_1 X \rightarrow R_1 Y$  in  $\mathcal{B}_1$ , and put  $h' = \epsilon Y \circ Lh \circ \sigma X$ . Since  $h$  is a morphism of algebras, we get

$$Rh' = R\epsilon Y \circ RLh \circ R\sigma X \stackrel{h \in \mathcal{B}_1}{=} h \circ R\epsilon X \circ R\sigma X = h.$$

2) By [Be, Proof of Theorem 1], since  $L_1$  exists, there exists a morphism  $\pi$  such that

$$(2) \quad LRLB \begin{array}{c} \xrightarrow{L\mu} \\ \xrightarrow{\epsilon LB} \end{array} LB \xrightarrow{\pi} L_1(B, \mu)$$

is a coequalizer for all  $(B, \mu) \in \mathcal{B}_1$ . Moreover  $L_1$  is full and faithful whenever

$$RLRLB \begin{array}{c} \xrightarrow{RL\mu} \\ \xrightarrow{R\epsilon LB} \end{array} RLB \xrightarrow{R\pi} RL_1(B, \mu)$$

is a coequalizer too. By assumption there is a natural transformation  $\gamma : L \rightarrow LRL$  such that  $\epsilon L \circ \gamma = \text{Id}_L$  and  $LR\epsilon L \circ \gamma RL = \gamma \circ \epsilon L$ . Clearly, we have  $\epsilon LB \circ \gamma B = \text{Id}_{LB}$ . Moreover

$$\begin{aligned} (L\mu \circ \gamma B) \circ L\mu &= L\mu \circ \gamma B \circ L\mu \stackrel{\text{nat } \gamma}{=} L\mu \circ LRL\mu \circ \gamma RL B \\ &= L\mu \circ (LR\epsilon LB \circ \gamma RL B) \\ &= L\mu \circ (\gamma B \circ \epsilon LB) \\ &= (L\mu \circ \gamma B) \circ \epsilon LB \end{aligned}$$

so that there is a unique morphism  $p : L_1(B, \mu) \rightarrow LB$  such that  $p \circ \pi = L\mu \circ \gamma B$ . We have

$$\pi \circ p \circ \pi = \pi \circ L\mu \circ \gamma B = \pi \circ \epsilon LB \circ \gamma B = \pi$$

so that, since  $\pi$  is an epimorphism, we get  $\pi \circ p = \text{Id}_{L_1(B, \mu)}$ . We have so proved that (2) is a contractible coequalizer. Thus it is preserved by any functor, in particular by  $R$ . Thus  $L_1$  is full and faithful too.

3) It follows from 1), 2) and Lemma 1.15 that both  $L_1$  and  $R_1$  are full and faithful.  $\square$

## 2. EXAMPLES

Let us fix a field  $\mathbb{k}$ . Vector spaces and algebras are meant to be over  $\mathbb{k}$ . From any vector space  $V$  we can construct its tensor algebra  $TV = \mathbb{k} \oplus V \oplus V^{\otimes 2} \oplus \dots$ . In fact, this is the object part of a functor  $T : Vect_{\mathbb{k}} \rightarrow Alg_{\mathbb{k}}$  from the category  $Vect_{\mathbb{k}}$  of vector spaces to the category  $Alg_{\mathbb{k}}$  of (associative and unital) algebras. By  $\Omega : Alg_{\mathbb{k}} \rightarrow Vect_{\mathbb{k}}$  we denote the forgetful functor.

### 2.1. Vector spaces and algebras.

EXAMPLE 2.1. If  $A$  is an algebra, and  $V$  a vector space, then the universal property of  $TV$  gives a bijection

$$(3) \quad Alg_{\mathbb{k}}(TV, A) \cong Vect_{\mathbb{k}}(V, \Omega A),$$

which is natural in both variables. In other words, the functor  $T : Vect_{\mathbb{k}} \rightarrow Alg_{\mathbb{k}}$  is left adjoint to the forgetful functor  $\Omega : Alg_{\mathbb{k}} \rightarrow Vect_{\mathbb{k}}$ . It is very well-known that  $\Omega$  is a monadic functor (cf. [Bo, Proposition 4.6.2]). Next, we check that  $T$  is a comonadic functor.

In fact, given  $V \in Vect_{\mathbb{k}}$ , consider the canonical projection  $\pi = \pi V : \Omega TV \rightarrow V$  on degree one. Let us check that it is natural in  $V$ . Let  $f : V \rightarrow V'$  be a morphism in  $Vect_{\mathbb{k}}$ . For all  $z \in V^{\otimes n}$  with  $n \neq 1$ ,

$$(\pi V' \circ \Omega T f)(z) = \pi V' (f^{\otimes n}(z)) = 0 = (f \circ \pi V)(z).$$

For  $v \in V$ , we have

$$(\pi V' \circ \Omega T f)(v) = \pi V' (f(v)) = f(v) = (f \circ \pi V)(v).$$

so that  $\pi V' \circ \Omega T f = f \circ \pi V$  and  $\pi V$  is natural in  $V$ . Moreover, we have  $\pi V \circ i_V = \text{Id}_V$ , where  $i_V : V \rightarrow \Omega TV$  is the canonical inclusion map for every  $V \in Vect_{\mathbb{k}}$ . Since  $i_V$  gives the unit of the adjunction at  $V$ , we can apply Rafael's Theorem [Ra, Theorem 1.2], to obtain that  $T$  is a separable functor. By the dual version of Proposition 1.16, in order to prove that  $T$  is comonadic it suffices to check that  $T^1$  has a right adjoint. This follows by Beck's Theorem [Be, Proof of Theorem 1] as  $Vect_{\mathbb{k}}$  has equalizers.

## 2.2. Vector spaces and bialgebras.

EXAMPLE 2.2. Let  $Bialg_{\mathbb{k}}$  be the category of bialgebras and  $\Omega : Bialg_{\mathbb{k}} \rightarrow Vect_{\mathbb{k}}$  be the forgetful functor. By  $P : Bialg_{\mathbb{k}} \rightarrow Vect_{\mathbb{k}}$  we denote the functor that sends a bialgebra  $A$  to its space  $PA$  of primitive elements. Obviously,  $P$  is a subfunctor of  $\Omega$ , let  $j : P \rightarrow \Omega$  denote the inclusion natural transformation. We know that the tensor algebra  $TV$  of a vector space  $V$  is already a bialgebra. Therefore, the bijection (3) gives, by restriction, a bijection

$$Bialg_{\mathbb{k}}(TV, A) \cong Vect_{\mathbb{k}}(V, PA)$$

which, of course, is natural.

In this way, we see that  $T$  is left adjoint to  $P$ . We will prove that  $P$  has a monadic decomposition of monadic length at most 2. First we need to prove a technical result.

LEMMA 2.3. *Let  $(L : \mathcal{B} \rightarrow \mathcal{A}, R : \mathcal{A} \rightarrow \mathcal{B})$  be an adjunction and let  $(B, \mu) \in \mathcal{B}_1$ . Let  $\zeta : B \rightarrow Z$  be a morphism in  $\mathcal{B}$ . Then*

$$\zeta \circ L\mu = \zeta \circ \epsilon LB \Leftrightarrow R\zeta \circ \eta B \circ \mu = R\zeta.$$

*Proof.* Consider the canonical isomorphism  $\Phi(X, Y) : \text{Hom}_{\mathcal{A}}(LX, Y) \rightarrow \text{Hom}_{\mathcal{B}}(X, RY)$  defined by  $\Phi(X, Y)f = Rf \circ \eta X$ . Then

$$\begin{aligned} \zeta \circ L\mu &= \zeta \circ \epsilon LB \Leftrightarrow \\ \Phi(RLB, Z)[\zeta \circ L\mu] &= \Phi(RLB, Z)[\zeta \circ \epsilon LB] \Leftrightarrow \\ R[\zeta \circ L\mu] \circ \eta RLB &= R[\zeta \circ \epsilon LB] \circ \eta RLB \Leftrightarrow \\ R\zeta \circ RL\mu \circ \eta RLB &= R\zeta \circ R\epsilon LB \circ \eta RLB \Leftrightarrow \\ R\zeta \circ \eta B \circ \mu &= R\zeta. \end{aligned}$$

□

THEOREM 2.4. *The functor  $P$  has a monadic decomposition of monadic length at most 2. Keep the notations of Definition 1.7 (so, in particular,  $\mathcal{B}_0 = Vect_{\mathbb{k}}$ ,  $R_0 = P$ , and  $L_0 = T$ ).*

- 1) *The functor  $L_1$  is given, for all  $(V_0, \mu_0) \in \mathcal{B}_1$ , by*

$$L_1(V_0, \mu_0) = \frac{L_0V_0}{(\text{Im}(\text{Id}_{R_0L_0V_0} - \eta_0V_0 \circ \mu_0))}.$$

- 2) *The adjunction  $(L_1, R_1)$  is idempotent.*

- 4) *For all  $V_2 := ((V_0, \mu_0), \mu_1) \in \mathcal{B}_2$ , we have the following cases.*

- $\text{char}\mathbb{k} = 0$ . *Then, for all  $x, y \in V_0$  we have that  $xy - yx \in R_0L_0V_0$ . Define a map  $[-, -] : V_0 \otimes V_0 \rightarrow V_0$  by setting  $[x, y] := \mu_0(xy - yx)$ . Then  $(V_0, [-, -])$  is an ordinary Lie algebra and  $L_2V_2$  is the universal enveloping algebra*

$$\mathfrak{u}V_0 := \frac{TV_0}{(xy - yx - [x, y] \mid x, y \in V_0)}.$$

- $\text{char}\mathbb{k} = p$ , a prime. *Then, for all  $x, y \in V_0$  we have that  $xy - yx, x^p \in R_0L_0V_0$ . Define two maps  $[-, -] : V_0 \otimes V_0 \rightarrow V_0$  and  $-^{[p]} : V_0 \rightarrow V_0$  by setting  $[x, y] := \mu_0(xy - yx)$  and  $x^{[p]} := \mu_0(x^p)$ . Then  $(V_0, [-, -], -^{[p]})$  is a restricted Lie algebra and  $L_2V_2$  is the restricted enveloping algebra*

$$\mathfrak{u}V_0 := \frac{TV_0}{(xy - yx - [x, y], x^p - x^{[p]} \mid x, y \in V_0)}.$$

*Proof.* Note that  $\mathcal{A} = Bialg_{\mathbb{k}}$  has coequalizers (see e.g. [Ag, page 1478]). Thus, using the notations of Definition 1.7, by Beck's Theorem [Be, Proof of Theorem 1], the functors  $L_1$  and  $L_2$  exist. By construction, for every  $V_1 := (V_0, \mu_0 : R_0L_0V_0 \rightarrow V_0) \in \mathcal{B}_1$  we have that  $L_1V_1$  is given by the coequalizer in  $\mathcal{A}$  of the diagram

$$L_0R_0L_0V_0 \begin{array}{c} \xrightarrow{L_0\mu_0} \\ \xrightarrow{\epsilon_0L_0V_0} \end{array} L_0V_0.$$

We want to compute explicitly this coequalizer. To this aim, we set

$$T_1V_1 := \frac{L_0V_0}{(S)},$$

where  $S := \text{Im}(\text{Id}_{R_0L_0V_0} - \eta_0V_0 \circ \mu_0)$ , and let us check it is a bialgebra. It is enough to check that

$$\begin{aligned} \Delta_{L_0V_0}S &\subseteq (S) \otimes L_0V_0 + L_0V_0 \otimes (S), \\ \varepsilon_{L_0V_0}S &= 0. \end{aligned}$$

Both equalities follow trivially since  $S \subseteq R_0L_0V_0 = PTV$ . Hence  $T_1V_1 \in \mathcal{A}$ . Let us check that

$$L_0R_0L_0V_0 \begin{array}{c} \xrightarrow{L_0\mu_0} \\ \xrightarrow{\varepsilon_0L_0V_0} \end{array} L_0V_0 \xrightarrow{\pi} T_1V_1$$

is a coequalizer in  $\mathcal{A}$ , where  $\pi$  is the canonical projection. Let  $\zeta : L_0V_0 \rightarrow Z$  be a morphism in  $\mathcal{A}$ . By Lemma 2.3

$$\zeta \circ L_0\mu_0 = \zeta \circ \varepsilon_0L_0V_0 \Leftrightarrow R_0\zeta \circ \eta_0V_0 \circ \mu_0 = R_0\zeta \Leftrightarrow \zeta \text{ vanishes on } S.$$

Hence we can take  $L_1V_1 := T_1V_1$ .

We need to describe  $L_1V_1$  in a different way for every  $V_1 := (V_0, \mu_0) \in \mathcal{B}_1$ . Note that  $R_0L_0V_0 = V_0 \oplus EV_0$  where  $EV_0$  denotes the subspace of  $\Omega L_0V_0$  spanned by primitive elements of homogeneous degree greater than one. Let  $x_1 = \eta_0V_0 : V_0 \rightarrow R_0L_0V_0$  and  $x_2 : EV_0 \rightarrow R_0L_0V_0$  be the canonical injections and set  $b := \mu_0 \circ x_2 : EV_0 \rightarrow V_0$ . Let  $c : V_0 \otimes V_0 \rightarrow V_0 \otimes V_0$  be the canonical flip. Then  $b$  is a bracket for the braided vector space  $(V_0, c)$  in the sense of [Ar1, Definition 3.2]. We compute

$$(\text{Id}_{R_0L_0V_0} - \eta_0V_0 \circ \mu_0) \circ x_1 = x_1 - \eta_0V_0 \circ \mu_0 \circ x_1 = \eta_0V_0 - \eta_0V_0 \circ \mu_0 \circ \eta_0V_0 = 0$$

so that

$$S = \text{Im}(\text{Id}_{R_0L_0V_0} - \eta_0V_0 \circ \mu_0) = \text{Im}[(\text{Id}_{R_0L_0V_0} - \eta_0V_0 \circ \mu_0) \circ x_2] = \text{Im}(x_2 - \eta_0V_0 \circ b)$$

and hence

$$L_1V_1 = \frac{L_0V_0}{(S)} = \frac{L_0V_0}{(\text{Im}(x_2 - \eta_0V_0 \circ b))} = \frac{L_0V_0}{(z - b(z) \mid z \in EV_0)}.$$

Therefore  $L_1V_1 = U(V_0, c, b)$  in the sense of [Ar1, Definition 3.5].

Let now  $V_2 := (V_1, \mu_1) \in \mathcal{B}_2$ . Then  $V_1$  is of the form  $(V_0, \mu_0)$ . By construction, the unit of the adjunction is the unique map  $\eta_1V_1 : V_1 \rightarrow R_1L_1V_1$  such that

$$U_{0,1}\eta_1V_1 = R_0\pi \circ \eta_0V_0.$$

Consider the canonical map  $i_U : V_0 \rightarrow U(V_0, c, b)$  i.e.

$$i_U = \Omega\pi \circ jL_0V_0 \circ \eta_0V_0 = jL_1V_1 \circ R_0\pi \circ \eta_0V_0 = jL_1V_1 \circ U_{0,1}\eta_1V_1$$

so that  $i_U$  corestricts to  $U_{0,1}\eta_1V_1$ . Now

$$U_{0,1}\mu_1 \circ U_{0,1}\eta_1V_1 = U_{0,1}(\mu_1 \circ \eta_1V_1) = \text{Id}_{V_0}$$

so that  $U_{0,1}\eta_1V_1$  is injective. Therefore  $i_U$  is injective. This means that  $(V_0, c, b)$  is a braided Lie algebra in the sense of [Ar1, Definition 4.1]. Let  $\mathcal{S}$  denote the class of braided vector spaces of combinatorial rank at most one. Then  $(V_0, c) \in \mathcal{S}$

(see [Ar2, Example 6.10], if  $\text{char}(\mathbb{k}) = 0$ , and [Ar3, Example 3.13], if  $\text{char}(\mathbb{k}) \neq 0$ ).

By [Ar1, Corollary 5.5], we have that  $U_{0,1}\eta_1V_1$  is an isomorphism. Since  $U_{0,1}$  reflects isomorphism, we get that  $\eta_1V_1$  is an isomorphism. We have so proved that  $\eta_1U_{1,2}$  is an isomorphism. By Remark 1.2, we have that the adjunction  $(L_1, R_1)$  is idempotent. By Proposition 1.9, the functor  $R_1$  has a monadic decomposition of monadic length at most 1 so that  $R$  has monadic decomposition of monadic length at most 2.

We have observed that  $(V_0, c, b)$  is a braided Lie algebra in the sense of [Ar1, Definition 4.1].

The last part of the statement follows by [Ar1, Remark 6.4] in case  $\text{char}\mathbb{k} = 0$  and by the same argument as in [Ar3, Example 3.13] in case  $\text{char}\mathbb{k} = p$ .  $\square$



REMARK 2.5. In the setting of Theorem 2.4,  $R = P : \mathcal{A} \rightarrow \mathcal{B}$  has a monadic decomposition of monadic length at most 2. Thus, by Theorem 1.13, we have that

$$\mathrm{Im}R = \mathrm{Im}U_{0,2}.$$

Note, since  $(L_1, R_1)$  is idempotent, we can apply Proposition 1.3 to get that an object in  $\mathrm{Im}U_{0,2}$  is isomorphic to an object of the form  $U_{0,2}(V_1, \mu_1) = U_{0,1}V_1$  for some  $V_1 \in \mathcal{B}_1$  such that  $\eta_1 V_1$  is an isomorphism.

REMARK 2.6. Let  $(L, R)$  be the adjunction considered in 2.2. For a moment let  $L'$  denote the left adjoint  $L$  of Example 2.1. Let  $W$  be the forgetful functor from the category of bialgebras to the category of algebras. Then  $W \circ L = L'$ . Hence, in view of [NVV, Lemma 1.1], from separability of  $L'$  we deduce separability of  $L$ . Since  $\mathcal{B}$  has all equalizers, by the dual version of Beck's Theorem [Be, Proof of Theorem 1], we have that the comparison functor  $L^1 : \mathcal{B} \rightarrow \mathcal{A}^1$  has a right adjoint  $R^1$ . Thus, by the dual version of Proposition 1.16, we have that  $L$  is comonadic.

Now, as observed in [Ag, Theorem 2.3], in view of [Sw, page 134], the functor  $W$  has a right adjoint, say  $\Gamma$ . Explicitly  $\Gamma A$  is the cofree bialgebra associated to  $A$ , for any algebra  $A$ . Now  $(WL, R\Gamma)$  is an adjunction as composition of adjunctions. Since  $W \circ L = L'$  and  $(L', R')$  is an adjunction, we deduce that  $R\Gamma$  is functorially isomorphic to  $R'$ .

### 2.3. Pretorsion theories.

EXAMPLE 2.7. Let  $A$  be a ring and let  $\mathcal{T}$  be a full subcategory of  $Mod\text{-}A$  closed under submodules, quotients and direct sums i.e.  $\mathcal{T}$  is an hereditary pretorsion class. Let  $t : Mod\text{-}A \rightarrow \mathcal{T}$  be the associated left exact preradical ([St, Corollary 1.8 page 138]). Then  $R = t$  is a right adjoint of the inclusion functor  $L = i : \mathcal{T} \rightarrow Mod\text{-}A$ . Note that  $RL = \mathrm{Id}_{\mathcal{T}}$  and  $\eta = \mathrm{Id}_{\mathrm{Id}_{\mathcal{T}}}$  so that  $L$  is full and faithful. Hence,  $R$  has a monadic decomposition of monadic length 0. By Remark 1.11, the comparison functor  $C : \mathcal{T} \rightarrow (Mod\text{-}A)^1$  is a category equivalence.

As a particular example we consider the case when  $A = C^*$  for some coalgebra  $C$  over a field  $\mathbb{k}$  and  $\mathcal{T}$  is the class of rational right  $C^*$ -modules i.e. the image of the canonical functor  $C\text{-CoMod} \rightarrow Mod\text{-}C^*$ .

### REFERENCES

- [Ag] A. L. Agore, *Categorical Constructions for Hopf Algebras*. Comm. Algebra, 1532-4125, Vol. **39**(4), (2011), 1476-1481.
- [Ar1] A. Ardizzoni, *On Primitively Generated Braided Bialgebras*, Algebr. Represent. Theory, Vol. **15**(4) (2012), 639-673.
- [Ar2] A. Ardizzoni, *A Milnor-Moore Type Theorem for Primitively Generated Braided Bialgebras*, J. Algebra, Vol. **327**(1) (2011), 337-365.
- [Ar3] A. Ardizzoni, *Universal Enveloping Algebras of PBW Type*, Glasg. Math. J., Vol. 54(1) (2012), 9-26.
- [AHW] J. Adámek, H. Herrlich, W. Tholen, *Monadic decompositions*. J. Pure Appl. Algebra **59** (1989), no. 2, 111-123.
- [AT] H. Appelgate, M. Tierney, *Categories with models*. 1969 Sem. on Triples and Categorical Homology Theory (ETH, zürich, 1966/67) pp. 156-244 Springer, Berlin.
- [Be] J.M. Beck, *Triples, algebras and cohomology*, Reprints in Theory and Applications of Categories **2** (2003), 1-59.
- [Bo] F. Borceux, *Handbook of categorical algebra. 2. Categories and structures*. Encyclopedia of Mathematics and its Applications, **51**. Cambridge University Press, Cambridge, 1994.
- [EGT] L. El Kaoutit and J. Gómez-Torrecillas, *Comatrix corings: Galois corings, descent theory, and a structure theorem for cosemisimple corings*, Math. Z. **244** (4), (2003) 887-906.
- [GT] J. Gómez-Torrecillas, *Comonads and Galois corings*, Appl. Categ. Structures **14** (5-6), (2006) 579-598.
- [GTV] J. Gómez-Torrecillas and J. Vercauteren, *Comatrix corings and Galois comodules over firm rings*, Algebr. Represent. Theory **10** (3), (2007) 271-306.
- [Ma] E. G. Manes, *A TRIPLE MISCELLANY: SOME ASPECTS OF THE THEORY OF ALGEBRAS OVER A TRIPLE*. Thesis (Ph.D.)-Wesleyan University. 1967.
- [Me] B. Mesablishvili, *Monads of effective type and comonadicity*. Theory Appl. Categ. **16** (2006), No. 1, 1-45
- [MS] J. L. MacDonald, A. Stone, *The tower and regular decomposition*. Cahiers Topologie Géom. Différentielle **23** (1982), no. 2, 197-213.
- [NVV] C. Năstăsescu, M. van den Bergh, and F. Van Oystaeyen, *Separable functors applied to graded rings*, J. Algebra **123** (1989), 397413.

- [Ra] M. D. Rafael, *Separable Functors Revisited*, Comm. Algebra **18** (1990), 1445–1459.
- [St] B. Stenström, *Rings of quotients* Die Grundlehren der Mathematischen Wissenschaften, Band **217**. An introduction to methods of ring theory. Springer-Verlag, New York-Heidelberg, 1975.
- [Sw] M. Sweedler, *Hopf Algebras*, Benjamin, New York, 1969.

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