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COHOMOLOGY AND COQUASI-BIALGEBRAS IN THE CATEGORY OF YETTER-DRINFELD MODULES

IVÁN ANGIONO, ALESSANDRO ARDIZZONI, AND CLAUDIA MENINI

Abstract. We prove that a finite-dimensional Hopf algebra with the dual Chevalley Property over a field of characteristic zero is quasi-isomorphic to a Radford-Majid bosonization whenever the third Hochschild cohomology group in the category of Yetter-Drinfeld modules of its diagram with coefficients in the base field vanishes. Moreover we show that this vanishing occurs in meaningful examples where the diagram is a Nichols algebra.

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Introduction

Let A be a finite-dimensional Hopf algebra over a field $\Bbbk$ of characteristic zero such that the coradical H of A is a sub-Hopf algebra (i.e. A has the dual Chevalley Property). Denote by $\mathcal{D}(A)$ the diagram of A. The main aim of this paper (see Theorem 5.6) is to prove that, if the third Hochschild cohomology group in ${}_{H}^{H}\mathcal{YD}$ of the algebra $\mathcal{D}(A)$ with coefficients in $\Bbbk$ vanishes, in symbols $\mathrm{H}^3_{\mathcal{YD}}(\mathcal{D}(A), \Bbbk) = 0$, then A is quasi-isomorphic to the Radford-Majid bosonization $E\#H$ of some connected bialgebra E in ${}_{H}^{H}\mathcal{YD}$ with $\mathrm{gr}(E) \cong \mathcal{D}(A)$ as bialgebras in ${}_{H}^{H}\mathcal{YD}$.

The paper is organized as follows. Let H be a Hopf algebra over a field $\Bbbk$. In Section 1 we investigate the properties of coalgebras with multiplication and unit in the category ${}_{H}^{H}\mathcal{YD}$ (in particular of coquasi-bialgebras) and their associated graded coalgebra. The main result of this section, Theorem 1.5, establishes that the associated graded coalgebra $\mathrm{gr}Q$ of a connected coquasi-bialgebra in ${}_{H}^{H}\mathcal{YD}$ is a connected bialgebra in ${}_{H}^{H}\mathcal{YD}$.

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In Section 2 we study the deformation of coquasi-bialgebras in ${}_H^H\mathcal{YD}$ by means of gauge transformations. In Proposition 2.5 we investigate its behaviour with respect to bosonization while in Proposition 2.6 with respect to the associated graded coalgebra.

In Section 3 we consider the associated graded coalgebra in case the Hopf algebra H is semisimple and cosemisimple (e.g. H is finite-dimensional cosemisimple over a field of characteristic zero). In particular, in Theorem 3.2, we prove that a f.d. connected coquasi-bialgebra Q in ${}_H^H\mathcal{YD}$ is gauge equivalent to a connected bialgebra in ${}_H^H\mathcal{YD}$ whenever $\mathrm{H}^3_{\mathcal{YD}}(\mathrm{gr}Q,\Bbbk)=0$. This result is inspired by [EG, Proposition 2.3].

In Section 4, we focus on the link between $\mathrm{H}^n_{\mathcal{YD}}(B,\Bbbk)$ and the invariants of $\mathrm{H}^n(B,\Bbbk)$, where B is a bialgebra in $\mathrm{H}^n_{\mathcal{YD}}(B,\Bbbk)$. In particular, in Proposition 4.7 we show that $\mathrm{H}^n_{\mathcal{YD}}(B,\Bbbk)$ is isomorphic to $\mathrm{H}^n(B,\Bbbk)^{D(H)}$, which is a subspace of $\mathrm{H}^n(B,\Bbbk)^H\cong\mathrm{H}^n(B\#H,\Bbbk)$, see Corollary 4.3.

Section 5 is devoted to the proof of the main result of the paper, the aforementioned Theorem 5.6.

In Section 6 we provide examples where $\mathrm{H}^n_{\mathcal{YD}}(B,\Bbbk)=0$ in case B is the Nichols algebra $\mathcal{B}(V)$ of a Yetter-Drinfeld module V. In particular we show that that $\mathrm{H}^3_{\mathcal{YD}}(\mathcal{B}(V),\Bbbk)$ can be zero although $\mathrm{H}^3(\mathcal{B}(V)\#H,\Bbbk)$ is non-trivial.

Preliminaries

Given a category $\mathcal{C}$ and objects $M,N\in\mathcal{C}$, the notation $\mathcal{C}(M,N)$ stands for the set of morphisms in $\mathcal{C}$. This notation will be mainly applied to the case $\mathcal{C}$ is the category of vector space $\mathbf{Vec}_\Bbbk$ over a field $\Bbbk$ or $\mathcal{C}$ is the category of Yetter-Drinfeld modules ${}_H^H\mathcal{YD}$ over a Hopf algebra H. The set of natural numbers including 0 is denoted by $\mathbb{N}_0$ while $\mathbb{N}$ denotes the same set without 0.

1. Yetter-Drinfeld

Definition 1.1. Let C be a coalgebra. Denote by C_n the n-th term of the coradical filtration of C and set $C_{-1}:=0$. For every $x\in C$, we set

$$|x|:=\min\{i\in\mathbb{N}_0:x\in C_i\}\qquad\text{and}\qquad\overline{x}:=x+C_{|x|-1}.$$

Note that, for $x=0$, we have $|x|=0$. One can define the associated graded coalgebra

$$\mathrm{gr}C:=\oplus_{i\in\mathbb{N}_0}\frac{C_i}{C_{i-1}}$$

with structure given, for every $x\in C$, by

$$\Delta_{\mathrm{gr}C}(\overline{x}) = \sum\nolimits_{0\leq i\leq|x|}(x_1+C_{i-1})\otimes\left(x_2+C_{|x|-i-1}\right), \tag{1}$$

$$\varepsilon_{\mathrm{gr}C}(\overline{x}) = \delta_{|x|,0}\varepsilon_C(x). \tag{2}$$

1.2. For every $i\in\mathbb{N}_0$, take a basis $\left\{\overline{x^{i,j}}\mid j\in B_i\right\}$ of the $\Bbbk$-module C_i/C_{i-1} with $\overline{x^{i,j}}\neq\overline{x^{i,l}}$ for $j\neq l$ and

$$\left|x^{i,j}\right|=i.$$

Then $\left\{x^{i,j}\mid 0\leq i\leq n, j\in B_i\right\}$ is a basis of C_n and $\left\{x^{i,j}\mid i\in\mathbb{N}_0, j\in B_i\right\}$ is a basis of C. Assume that C has a distinguished grouplike element $1=1_C\neq 0$ and take $i>0$. If $\varepsilon\left(x^{i,j}\right)\neq 0$ then we have that

$$\overline{x^{i,j}-\varepsilon\left(x^{i,j}\right)1}=\overline{x^{i,j}}$$

so that we can take $x^{i,j}-\varepsilon\left(x^{i,j}\right)1$ in place of $x^{i,j}$. In other words we can assume

$$\varepsilon\left(x^{i,j}\right)=0,\ \text{for every } i>0, j\in B_i. \tag{3}$$

It is well-known there is a $\Bbbk$-linear isomorphism $\varphi:C\to\mathrm{gr}C$ defined on the basis by $\varphi\left(x^{i,j}\right):=\overline{x^{i,j}}$.

We compute

$$\varepsilon_{\mathrm{gr}C}\varphi\left(x^{i,j}\right)=\varepsilon_{\mathrm{gr}C}\left(\overline{x^{i,j}}\right)\overset{(2)}{=}\delta_{i,0}\varepsilon\left(x^{0,j}\right)\overset{(3)}{=}\varepsilon\left(x^{i,j}\right).$$

Hence we obtain

$$\varepsilon_{\mathrm{gr}C}\circ\varphi=\varepsilon. \tag{4}$$

Let H be a Hopf algebra. A **coalgebra with multiplication and unit** in ${}_H^H\mathcal{YD}$ is a datum $(Q,m,u,\Delta,\varepsilon)$ where (Q,Δ,ε) is a coalgebra in ${}_H^H\mathcal{YD}$, $m:Q\otimes Q\to Q$ is a coalgebra morphism in ${}_H^H\mathcal{YD}$ called multiplication (which may fail to be associative) and $u:\Bbbk\to Q$ is a coalgebra morphism in ${}_H^H\mathcal{YD}$ called unit. In this case we set $1_Q:=u(1_\Bbbk)$.

Note that, for every $h\in H,k\in\Bbbk$, we have

$$h1_Q = hu(1_\Bbbk)=u(h1_\Bbbk)=u(\varepsilon_H(h)1_\Bbbk)=\varepsilon_H(h)u(1_\Bbbk)=\varepsilon_H(h)1_Q, \tag{5}$$
$$(1_Q)_{-1}\otimes(1_Q)_0 = (u(1_\Bbbk))_{-1}\otimes(u(1_\Bbbk))_0=(1_\Bbbk)_{-1}\otimes u((1_\Bbbk)_0)=1_H\otimes u(1_\Bbbk)=1_H\otimes 1_Q. \tag{6}$$

Proposition 1.3. *Let H be a Hopf algebra and let $(Q,m,u,\Delta,\varepsilon)$ be a coalgebra with multiplication and unit in ${}_H^H\mathcal{YD}$. If Q_0 is a subcoalgebra of Q in ${}_H^H\mathcal{YD}$ such that $Q_0\cdot Q_0\subseteq Q_0$, then Q_n is a subcoalgebra of Q in ${}_H^H\mathcal{YD}$ for every $n\in\mathbb{N}_0$. Moreover $Q_a\cdot Q_b\subseteq Q_{a+b}$ for every $a,b\in\mathbb{N}_0$ and the graded coalgebra* gr$Q$, *associated with the coradical filtration of* Q, *is a coalgebra with multiplication and unit in ${}_H^H\mathcal{YD}$ with respect to the usual coalgebra structure and with multiplication and unit defined by*

$$\begin{aligned} m_{\mathrm{gr}Q}((x+Q_{a-1})\otimes(y+Q_{b-1})) &:= xy+Q_{a+b-1},\\ u_{\mathrm{gr}Q}(k) &:= k1_Q+Q_{-1}\end{aligned} \tag{7}$$

Proof. The coalgebra structure of Q induces a coalgebra structure on gr$Q$. Since Q_0 is a subcoalgebra of Q in ${}_H^H\mathcal{YD}$ and, for $n\geq 1$, one has $Q_n=Q_{n-1}\wedge_Q Q_0$, then inductively one proves that Q_n is a subcoalgebra of Q in ${}_H^H\mathcal{YD}$. As a consequence one gets that gr$Q$ is a coalgebra in ${}_H^H\mathcal{YD}$ (this construction can be performed in the setting of monoidal categories under suitable assumptions, see e.g. [AM, Theorem 2.10]). Let us prove that gr$Q$ inherits also a multiplication and unit. Let us check that $Q_a\cdot Q_b\subseteq Q_{a+b}$ for every $a,b\in\mathbb{N}_0$. We proceed by induction on $n=a+b$. If $n=0$ there is nothing to prove. Let $n\geq 1$ and assume that $Q_i\cdot Q_j\subseteq Q_{i+j}$ for every $i,j\in\mathbb{N}_0$ such that $0\leq i+j\leq n-1$. Let $a,b\in\mathbb{N}_0$ be such that $n=a+b$. Since $\Delta(Q_a)\subseteq\sum_{i=0}^a Q_i\otimes Q_{a-i}$ and $c_{Q,Q}(Q_u\otimes Q_v)\subseteq Q_v\otimes Q_u$, where $c_{Q,Q}$ denotes the braiding in ${}_H^H\mathcal{YD}$, using the compatibility condition between Δ and m, one easily gets that $\Delta(Q_a\cdot Q_b)\subseteq Q_{a+b-1}\otimes Q+Q\otimes Q_0$.

Therefore $Q_a\cdot Q_b\subseteq Q_{a+b}$. This property implies we have a well-defined map in ${}_H^H\mathcal{YD}$

$$m_{\mathrm{gr}Q}^{a,b}:\frac{Q_a}{Q_{a-1}}\otimes\frac{Q_b}{Q_{b-1}}\to\frac{Q_{a+b}}{Q_{a+b-1}}$$

defined, for $x\in Q_a$ and $y\in Q_b$, by (7). This can be seen as the graded component of a morphism in ${}_H^H\mathcal{YD}$ that we denote by $m_{\mathrm{gr}Q}:\mathrm{gr}Q\otimes\mathrm{gr}Q\to\mathrm{gr}Q$. Let us check that $m_{\mathrm{gr}Q}$ is a coalgebra morphism in ${}_H^H\mathcal{YD}$. Consider a basis of Q with terms of the form $x^{i,j}$ as in 1.2. Hence we can write the comultiplication in the form

$$\Delta(x^{a,u})=\sum\nolimits_{s+t\leq a}\sum\nolimits_{l,m}\eta_{s,t,l,m}^{a,u}x^{s,l}\otimes x^{t,m}.$$

Now, using (1), one gets that

$$\Delta_{\mathrm{gr}Q}\left(\overline{x^{a,u}}\right)=\sum\nolimits_{0\leq i\leq a}\sum\nolimits_{l,m}\eta_{i,a-i,l,m}^{a,u}\overline{x^{i,l}}\otimes\overline{x^{a-i,m}}. \tag{8}$$

Using that $\Delta_{\mathrm{gr}Q\otimes\mathrm{gr}Q}=(\mathrm{gr}Q\otimes c_{\mathrm{gr}Q,\mathrm{gr}Q}\otimes\mathrm{gr}Q)(\Delta_{\mathrm{gr}Q}\otimes\Delta_{\mathrm{gr}Q})$ and (8), it is straightforward to check that $(m_{\mathrm{gr}Q}\otimes m_{\mathrm{gr}Q})\Delta_{\mathrm{gr}Q\otimes\mathrm{gr}Q}\left(\overline{x^{a,u}}\otimes\overline{x^{b,v}}\right)=\Delta_{\mathrm{gr}Q}m_{\mathrm{gr}Q}\left(\overline{x^{a,u}}\otimes\overline{x^{b,v}}\right)$.

Moreover, since $\varepsilon_{\mathrm{gr}Q\otimes\mathrm{gr}Q}=\varepsilon_{\mathrm{gr}Q}\otimes\varepsilon_{\mathrm{gr}Q}$, we get that $\varepsilon_{\mathrm{gr}Q}m_{\mathrm{gr}Q}\left(\overline{x^{a,u}}\otimes\overline{x^{b,v}}\right)=\varepsilon_{\mathrm{gr}Q\otimes\mathrm{gr}Q}\left(\overline{x^{a,u}}\otimes\overline{x^{b,v}}\right)$.

This proves that $m_{\mathrm{gr}Q}$ is a coalgebra morphism in ${}_H^H\mathcal{YD}$.

The fact that $u_{\mathrm{gr}Q}:\Bbbk\to\mathrm{gr}Q$, defined by $u_{\mathrm{gr}Q}(k):=k1_Q+Q_{-1}$ is a coalgebra morphism in ${}_H^H\mathcal{YD}$ easily follows by means of (5) and (6).

□

DEFINITION 1.4 ([ABM, Definition 5.2]). Let H be a Hopf algebra. Recall that a *coquasi-bialgebra* $(Q,m,u,\Delta,\varepsilon,\alpha)$ in the pre-braided monoidal category ${}_H^H\mathcal{YD}$ is a coalgebra (Q,Δ,ε) in ${}_H^H\mathcal{YD}$ together with coalgebra homomorphisms $m:Q\otimes Q\to Q$ and $u:\Bbbk\to Q$ in ${}_H^H\mathcal{YD}$ and a convolution invertible element $\alpha\in{}_H^H\mathcal{YD}\left(Q^{\otimes 3},\Bbbk\right)$ (*braided reassociator*) such that

$$\alpha\left(Q\otimes Q\otimes m\right)*\alpha\left(m\otimes Q\otimes Q\right)=\left(\varepsilon\otimes\alpha\right)*\alpha\left(Q\otimes m\otimes Q\right)*\left(\alpha\otimes\varepsilon\right), \tag{9}$$

$$\alpha\left(Q\otimes u\otimes Q\right)=\alpha\left(u\otimes Q\otimes Q\right)=\alpha\left(Q\otimes Q\otimes u\right)=\varepsilon_{Q\otimes Q}, \tag{10}$$

$$m\left(Q\otimes m\right)*\alpha=\alpha*m\left(m\otimes Q\right), \tag{11}$$

$$m\left(u\otimes Q\right)=\mathrm{Id}_Q=m\left(Q\otimes u\right). \tag{12}$$

Here $*$ denotes the convolution product, where $Q^{\otimes 3}$ is the tensor product of coalgebras in ${}_H^H\mathcal{YD}$ whence it depends on the braiding of this category. Note that in (10) any of the three equalities such as $\alpha\left(u\otimes Q\otimes Q\right)=\varepsilon_{Q\otimes Q}$ implies that α is unital.

THEOREM 1.5. *Let H be a Hopf algebra and let $(Q,m,u,\Delta,\varepsilon,\omega)$ be a connected coquasi-bialgebra in ${}_H^H\mathcal{YD}$. Then $\mathrm{gr}Q$ is a connected bialgebra in ${}_H^H\mathcal{YD}$.*

Proof. By Proposition 1.3, we know that $\mathrm{gr}Q$ is a coalgebra with multiplication and unit in ${}_H^H\mathcal{YD}$. We have to check that the multiplication is associative and unitary.

Given two coalgebras D,E in ${}_H^H\mathcal{YD}$ endowed with coalgebras filtration $\left(D_{(n)}\right)_{n\in\mathbb{N}_0}$ and $\left(E_{(n)}\right)_{n\in\mathbb{N}_0}$ in ${}_H^H\mathcal{YD}$ such that $D_{(0)}$ and $E_{(0)}$ are one-dimensional, let us check that $C_{(n)}:=\sum_{0\leq i\leq n}D_{(i)}\otimes E_{(n-i)}$ gives a coalgebra filtration on $C:=D\otimes E$ in ${}_H^H\mathcal{YD}$. First note that the coalgebra structure of C depends on the braiding. Thus, we have

$$\begin{aligned}
\Delta_C\left(C_{(n)}\right) &= \left(D\otimes c_{D,E}\otimes E\right)\left(\Delta_D\otimes\Delta_E\right)\left(\sum\nolimits_{i=0}^{n}D_{(i)}\otimes E_{(n-i)}\right)\\
&\subseteq \left(D\otimes c_{D,E}\otimes E\right)\left(\sum\nolimits_{i=0}^{n}\sum\nolimits_{a=0}^{i}\sum\nolimits_{b=0}^{n-i}D_{(a)}\otimes D_{(i-a)}\otimes E_{(b)}\otimes E_{(n-i-b)}\right)\\
&\subseteq \sum\nolimits_{i=0}^{n}\sum\nolimits_{a=0}^{i}\sum\nolimits_{b=0}^{n-i}D_{(a)}\otimes c_{D,E}\left(D_{(i-a)}\otimes E_{(b)}\right)\otimes E_{(n-i-b)}\\
&\subseteq \sum\nolimits_{i=0}^{n}\sum\nolimits_{a=0}^{i}\sum\nolimits_{b=0}^{n-i}D_{(a)}\otimes c_{D_{(i-a)},E_{(b)}}\left(D_{(i-a)}\otimes E_{(b)}\right)\otimes E_{(n-i-b)}\\
&\subseteq \sum\nolimits_{i=0}^{n}\sum\nolimits_{a=0}^{i}\sum\nolimits_{b=0}^{n-i}D_{(a)}\otimes E_{(b)}\otimes D_{(i-a)}\otimes E_{(n-i-b)}\\
&\subseteq \sum\nolimits_{i=0}^{n}\sum\nolimits_{w=0}^{n}\sum\nolimits_{\substack{0\leq a\leq i,\\ 0\leq b\leq n-i\\ a+b=w}}D_{(a)}\otimes E_{(b)}\otimes D_{(i-a)}\otimes E_{(n-i-b)}\\
&\subseteq \sum\nolimits_{w=0}^{n}C_{(w)}\otimes C_{(n-w)}.
\end{aligned}$$

Moreover, by [Sw, Proposition 11.1.1], we have that the coradical of C is contained in $D_{(0)}\otimes E_{(0)}$ and hence it is one-dimensional.

This argument can be used to produce a coalgebra filtration on $C:=Q\otimes Q\otimes Q$ using as a filtration on Q the coradical filtration. Let $n>0$ and let $w\in C_{(n)}=\sum_{i+j+k\leq n}Q_i\otimes Q_j\otimes Q_k$. By [AMS1, Lemma 3.69], we have that

$$\Delta_C\left(w\right)-w\otimes\left(1_Q\right)^{\otimes 3}-\left(1_Q\right)^{\otimes 3}\otimes w\in C_{(n-1)}\otimes C_{(n-1)}.$$

Thus we get

$$w_1\otimes w_2\otimes w_3-\Delta_C\left(w\right)\otimes\left(1_Q\right)^{\otimes 3}-\Delta_C\left(\left(1_Q\right)^{\otimes 3}\right)\otimes w\in\Delta_C\left(C_{(n-1)}\right)\otimes C_{(n-1)}$$

and hence, tensoring the first relation by $\left(1_Q\right)^{\otimes 3}$ on the right and adding it to the second one, we get

$$w_1\otimes w_2\otimes w_3-w\otimes\left(1_Q\right)^{\otimes 3}\otimes\left(1_Q\right)^{\otimes 3}-\left(1_Q\right)^{\otimes 3}\otimes w\otimes\left(1_Q\right)^{\otimes 3}-\left(1_Q\right)^{\otimes 6}\otimes w\in C_{(n-1)}\otimes C_{(n-1)}\otimes C_{(n-1)}.$$

For shortness, we set $\nu_n\left(z\right):=m\left(Q\otimes m\right)\left(z\right)+Q_{n-1}$ for every $z\in C$. Thus, by applying to the last displayed relation $C_{(n-1)}\otimes m\left(Q\otimes m\right)\otimes C_{(n-1)}$ and factoring out the middle term by Q_{n-1}, we get

$$\begin{bmatrix} w_1 \otimes \nu_n(w_2) \otimes w_3 - w \otimes \nu_n\left((1_Q)^{\otimes 3}\right) \otimes (1_Q)^{\otimes 3} + \\ -(1_Q)^{\otimes 3} \otimes \nu_n(w) \otimes (1_Q)^{\otimes 3} - (1_Q)^{\otimes 3} \otimes \nu_n\left((1_Q)^{\otimes 3}\right) \otimes w \end{bmatrix}$$

$$\in \quad C_{(n-1)} \otimes \left(\frac{\nu_n\left(C_{(n-1)}\right)}{Q_{n-1}}\right) \otimes C_{(n-1)} \subseteq C_{(n-1)} \otimes \frac{Q_{n-1}}{Q_{n-1}} \otimes C_{(n-1)} = 0.$$

Thus we can express the first term with respect to the remaining ones as follows

$$\begin{aligned} & w_1 \otimes \nu_n(w_2) \otimes w_3 \\ = \quad & w \otimes \nu_n\left((1_Q)^{\otimes 3}\right) \otimes (1_Q)^{\otimes 3} + (1_Q)^{\otimes 3} \otimes \nu_n(w) \otimes (1_Q)^{\otimes 3} + (1_Q)^{\otimes 3} \otimes \nu_n\left((1_Q)^{\otimes 3}\right) \otimes w \\ = \quad & w \otimes (1_Q + Q_{n-1}) \otimes (1_Q)^{\otimes 3} + (1_Q)^{\otimes 3} \otimes \nu_n(w) \otimes (1_Q)^{\otimes 3} + (1_Q)^{\otimes 3} \otimes (1_Q + Q_{n-1}) \otimes w \\ \overset{n \geq 0}{=} \quad & (1_Q)^{\otimes 3} \otimes \nu_n(w) \otimes (1_Q)^{\otimes 3}. \end{aligned}$$

We have so proved that for $n > 0$ and $w \in C_{(n)}$

$$w_1 \otimes \nu_n(w_2) \otimes w_3 = (1_Q)^{\otimes 3} \otimes \nu_n(w) \otimes (1_Q)^{\otimes 3}. \tag{13}$$

The same equation trivially holds also in the case $n = 0$ as $C_{(n)}$ is one-dimensional.

Let $x, y, z \in Q$. Then $x \otimes y \otimes z \in C_{(|x|+|y|+|z|)}$ so that

$$\begin{aligned} (\overline{x} \cdot \overline{y}) \cdot \overline{z} \quad &= \quad \left((x + Q_{|x|-1}) \cdot (y + Q_{|y|-1})\right) \cdot (z + Q_{|z|-1}) \\ &= \quad \left((xy) + Q_{|x|+|y|-1}\right) \cdot (z + Q_{|z|-1}) \\ &= \quad (xy)\, z + Q_{|x|+|y|+|z|-1} \\ &= \quad \omega^{-1}\left((x \otimes y \otimes z)_1\right) \nu_{|x|+|y|+|z|}\left((x \otimes y \otimes z)_2\right) \omega\left((x \otimes y \otimes z)_3\right) \\ &\overset{(13)}{=} \quad \omega^{-1}(1_Q \otimes 1_Q \otimes 1_Q)\, \nu_{|x|+|y|+|z|}(x \otimes y \otimes z)\, \omega(1_Q \otimes 1_Q \otimes 1_Q) \\ &= \quad \nu_{|x|+|y|+|z|}(x \otimes y \otimes z) \\ &= \quad x\,(yz) + Q_{|x|+|y|+|z|-1} = \overline{x} \cdot (\overline{y} \cdot \overline{z}). \end{aligned}$$

Therefore the multiplication is associative. It is also unitary as

$$\overline{x} \cdot \overline{1_Q} = \left(x + Q_{|x|-1}\right) \cdot (1_Q + Q_{-1}) = x \cdot 1_Q + Q_{|x|-1} = x + Q_{|x|-1} = \overline{x}$$

and similarly $\overline{1_Q} \cdot \overline{x} = \overline{x}$ for every $x \in Q$. □

2. Gauge deformation

Definition 2.1. Let H be a Hopf algebra and let $(Q, m, u, \Delta, \varepsilon, \omega)$ be a coquasi-bialgebra in ${}_H^H\mathcal{YD}$. A **gauge transformation** for Q is a morphism $\gamma : Q \otimes Q \to \Bbbk$ in ${}_H^H\mathcal{YD}$ which is convolution invertible in ${}_H^H\mathcal{YD}$ and which is also unitary on both entries.

Remark 2.2. For γ as above, let us check that γ^{-1} is unitary whence a gauge transformation too.

First note that for all $x \in Q$, by means of (6) and (5), one gets

$$(1_Q \otimes x)_1 \otimes (1_Q \otimes x)_2 \quad = \quad 1_Q \otimes x_1 \otimes 1_Q \otimes x_2 \tag{14}$$

$$(x \otimes 1_Q)_1 \otimes (x \otimes 1_Q)_2 \quad = \quad x_1 \otimes 1_Q \otimes x_2 \otimes 1_Q \tag{15}$$

Thus

$$\gamma^{-1}(1_Q \otimes x) = \gamma^{-1}(1_Q \otimes x_1)\, \varepsilon(x_2) = \gamma^{-1}(1_Q \otimes x_1)\, \gamma(1_Q \otimes x_2) = \left(\gamma^{-1} * \gamma\right)(1_Q \otimes x) = \varepsilon(x)$$

and similarly $\gamma^{-1}(x \otimes 1_Q) = \varepsilon(x)$.

Lemma 2.3. *Let H be a Hopf algebra and let C be a coalgebra in ${}_H^H\mathcal{YD}$. Given a map $\gamma \in {}_H^H\mathcal{YD}(C, \Bbbk)$, we have that γ is convolution invertible in ${}_H^H\mathcal{YD}(C, \Bbbk)$ if and only if it is convolution invertible in $\mathbf{Vec}_{\Bbbk}(C, \Bbbk)$. Moreover the inverse is the same.*

Proof. Assume there is a $\Bbbk$-linear map $\gamma^{-1} : C \to \Bbbk$ which is a convolution inverse of γ in $\mathbf{Vec}_{\Bbbk}(C, \Bbbk)$. By [ABM1, Remark 2.4(ii)], γ^{-1} is left H-linear. Let us check that γ^{-1} is left H-colinear:

$$\begin{aligned} c_{-1} \otimes \gamma^{-1}(c_0) &= (c_1)_{-1} 1_H \otimes \gamma^{-1}((c_1)_0)\gamma(c_2)\gamma^{-1}(c_3) \\ &= (c_1)_{-1}(c_2)_{-1} \otimes \gamma^{-1}((c_1)_0)\gamma((c_2)_0)\gamma^{-1}(c_3) \\ &\overset{(*)}{=} (c_1)_{-1} \otimes \gamma^{-1}\left(((c_1)_0)_1\right)\gamma\left(((c_1)_0)_2\right)\gamma^{-1}(c_2) \\ &= (c_1)_{-1} \otimes \left(\gamma^{-1} * \gamma\right)((c_1)_0)\gamma^{-1}(c_2) \\ &= (c_1)_{-1} \otimes \varepsilon_C((c_1)_0)\gamma^{-1}(c_2) \\ &\overset{(*)}{=} 1_H \otimes \varepsilon_C(c_1)\gamma^{-1}(c_2) = 1_H \otimes \gamma^{-1}(c) \end{aligned}$$

where in (*) we used that the comultiplication or the counit of C is left H-colinear. Thus γ is convolution invertible in ${}^H_H\mathcal{YD}(C, \Bbbk)$. The other implication is obvious. □

PROPOSITION 2.4. *Let H be a Hopf algebra and let $(Q, m, u, \Delta, \varepsilon, \omega)$ be a coquasi-bialgebra in ${}^H_H\mathcal{YD}$. Let $\gamma : Q \otimes Q \to \Bbbk$ be a gauge transformation in ${}^H_H\mathcal{YD}$. Then*

$$Q^\gamma := (Q, m^\gamma, u, \Delta, \varepsilon, \omega^\gamma)$$

is a coquasi-bialgebra in ${}^H_H\mathcal{YD}$, where

$$\begin{aligned} m^\gamma &:= \gamma * m * \gamma^{-1} \\ \omega^\gamma &:= (\varepsilon \otimes \gamma) * \gamma(Q \otimes m) * \omega * \gamma^{-1}(m \otimes Q) * \left(\gamma^{-1} \otimes \varepsilon\right). \end{aligned}$$

Proof. The proof is analogue to [K, Proposition XV.3.2] in its dual version. We include some details for the reader's sake. Note that Q^γ has the same underlying coalgebra of Q which is a coalgebra in ${}^H_H\mathcal{YD}$. The unit is also the same and hence it is a coalgebra map in ${}^H_H\mathcal{YD}$. Since m^γ is the convolution product of morphisms in ${}^H_H\mathcal{YD}$, it results that m^γ is in ${}^H_H\mathcal{YD}$ as well.

Since m is a coalgebra map in ${}^H_H\mathcal{YD}$ and γ is convolution invertible with convolution inverse γ^{-1}, it follows that m^γ is a coalgebra map in ${}^H_H\mathcal{YD}$.

By means of (14) and (15), one gets that $m^\gamma(1_Q \otimes x) = x = m^\gamma(x \otimes 1_Q)$.

Let us consider now ω^γ. Since it is the convolution product of morphisms in ${}^H_H\mathcal{YD}$, it results that ω^γ is in ${}^H_H\mathcal{YD}$ as well.

Let us check that ω^γ is unitary. Consider the map $\alpha_2 : Q \otimes Q \to Q \otimes Q \otimes Q$ defined by $\alpha_2(x \otimes y) = x \otimes 1_Q \otimes y$. The equalities (15) and (6) yield

$$\begin{aligned} (\alpha_2(x \otimes y))_1 \otimes (\alpha_2(x \otimes y))_2 &= \alpha_2\left(x_1 \otimes (x_2)_{-1} y_1\right) \otimes \alpha_2((x_2)_0 \otimes y_2) \\ &= \alpha_2((x \otimes y)_1) \otimes \alpha_2((x \otimes y)_2) \end{aligned}$$

so that α_2 is comultiplicative.

Thus

$$\omega^\gamma \alpha_2 := (\varepsilon \otimes \gamma)\alpha_2 * \gamma(Q \otimes m)\alpha_2 * \omega\alpha_2 * \gamma^{-1}(m \otimes Q)\alpha_2 * \left(\gamma^{-1} \otimes \varepsilon\right)\alpha_2$$

and computing the factors of this convolution products one gets

$$(\varepsilon \otimes \gamma)\alpha_2 = \varepsilon \otimes \varepsilon, \quad \gamma(Q \otimes m)\alpha_2 = \gamma, \quad \omega\alpha_2 = \varepsilon \otimes \varepsilon,$$
$$\gamma^{-1}(m \otimes Q)\alpha_2 = \gamma^{-1}, \quad \left(\gamma^{-1} \otimes \varepsilon\right)\alpha_2 = \varepsilon \otimes \varepsilon$$

and hence $\omega^\gamma \alpha_2 = \gamma * \gamma^{-1} = \varepsilon \otimes \varepsilon$, which means that $\omega^\gamma(x \otimes 1_Q \otimes y) = \varepsilon(x)\varepsilon(y)$ for every $x, y \in Q$.

Similarly, considering $\alpha_1 : Q \otimes Q \to Q \otimes Q \otimes Q$ defined by $\alpha_2(x \otimes y) = 1_Q \otimes x \otimes y$, one proves that $\omega^\gamma(1_Q \otimes x \otimes y) = \varepsilon(x)\varepsilon(y)$. A symmetric argument shows that $\omega^\gamma(x \otimes y \otimes 1_Q) = \varepsilon(x)\varepsilon(y)$.

Note that, by Lemma 2.3, ω^γ is convolution invertible in ${}^H_H\mathcal{YD}(D, \Bbbk)$ as it is convolution invertible in $\mathbf{Vec}_{\Bbbk}(D, \Bbbk)$.

Let us check that the multiplication is quasi-associative. By [ABM, Lemma 2.10 formula (2.7)], we have

$$m^\gamma\left(Q \otimes \gamma * m * \gamma^{-1}\right) = (\varepsilon \otimes \gamma) * m^\gamma(Q \otimes m) * \left(\varepsilon \otimes \gamma^{-1}\right),$$

$$\begin{aligned}
\left(\varepsilon\otimes\gamma^{-1}\right)*\left(\varepsilon\otimes\gamma\right) &= \varepsilon\otimes\left(\gamma^{-1}*\gamma\right)=\varepsilon\otimes\varepsilon\otimes\varepsilon,\\
m^{\gamma}\left(m^{\gamma}\otimes Q\right) &= m^{\gamma}\left(\gamma*m*\gamma^{-1}\otimes Q\right)=\left(\gamma\otimes\varepsilon\right)*m^{\gamma}\left(m*\gamma^{-1}\otimes Q\right)\\
&= \left(\gamma\otimes\varepsilon\right)*m^{\gamma}\left(m\otimes Q\right)*\left(\gamma^{-1}\otimes\varepsilon\right),\\
\left(\gamma^{-1}\otimes\varepsilon\right)*\left(\gamma\otimes\varepsilon\right) &= \left(\left(\gamma^{-1}*\gamma\right)\otimes\varepsilon\right)=\varepsilon\otimes\varepsilon\otimes\varepsilon.
\end{aligned}$$

By using these equalities one obtains

$$\begin{aligned}
m^{\gamma}\left(Q\otimes m^{\gamma}\right)*\omega^{\gamma} &= \left(\varepsilon\otimes\gamma\right)*\gamma\left(Q\otimes m\right)*m\left(Q\otimes m\right)*\omega*\gamma^{-1}\left(m\otimes Q\right)*\left(\gamma^{-1}\otimes\varepsilon\right),\\
\omega^{\gamma}*m^{\gamma}\left(m^{\gamma}\otimes Q\right) &= \left(\varepsilon\otimes\gamma\right)*\gamma\left(Q\otimes m\right)*\omega*m\left(m\otimes Q\right)*\gamma^{-1}\left(m\otimes Q\right)*\left(\gamma^{-1}\otimes\varepsilon\right)
\end{aligned}$$

so that $\omega^{\gamma}*m^{\gamma}\left(m^{\gamma}\otimes Q\right)=m^{\gamma}\left(Q\otimes m^{\gamma}\right)*\omega^{\gamma}$.

It remains to check that ω^{γ} is a reassociator. By [ABM, Lemma 2.10 formula (2.7)], we have

$$\begin{aligned}
\omega^{\gamma}\left(Q\otimes Q\otimes\gamma*m*\gamma^{-1}\right) &= \left(\varepsilon\otimes\varepsilon\otimes\gamma\right)*\omega^{\gamma}\left(Q\otimes Q\otimes m\right)*\left(\varepsilon\otimes\varepsilon\otimes\gamma^{-1}\right),\\
\omega^{\gamma}\left(\gamma*m*\gamma^{-1}\otimes Q\otimes Q\right) &= \left(\gamma\otimes\varepsilon\otimes\varepsilon\right)*\omega^{\gamma}\left(m\otimes Q\otimes Q\right)*\left(\gamma^{-1}\otimes\varepsilon\otimes\varepsilon\right),\\
\left(\gamma\otimes\varepsilon\otimes\varepsilon\right)*\left(\varepsilon\otimes\varepsilon\otimes\gamma\right) &= \gamma\otimes\gamma=\left(\varepsilon\otimes\varepsilon\otimes\gamma\right)*\left(\gamma\otimes\varepsilon\otimes\varepsilon\right).
\end{aligned}$$

By using these equalities one obtains

$$\begin{aligned}
&\omega^{\gamma}\left(Q\otimes Q\otimes m^{\gamma}\right)*\omega^{\gamma}\left(m^{\gamma}\otimes Q\otimes Q\right)\\
&=\left[\begin{array}{c}\left(\varepsilon\otimes\varepsilon\otimes\gamma\right)*\left(\varepsilon\otimes\gamma\left(Q\otimes m\right)\right)*\gamma\left(Q\otimes m\left(Q\otimes m\right)\right)\\ *\omega\left(Q\otimes Q\otimes m\right)*\omega\left(m\otimes Q\otimes Q\right)\\ *\gamma^{-1}\left(m\left(m\otimes Q\right)\otimes Q\right)*\left(\gamma^{-1}\left(m\otimes Q\right)\otimes\varepsilon\right)*\left(\gamma^{-1}\otimes\varepsilon\otimes\varepsilon\right)\end{array}\right]
\end{aligned}$$

and

$$\begin{aligned}
&\left(\varepsilon\otimes\omega^{\gamma}\right)*\omega^{\gamma}\left(Q\otimes m^{\gamma}\otimes Q\right)*\left(\omega^{\gamma}\otimes\varepsilon\right)\\
&=\left[\begin{array}{c}\left(\varepsilon\otimes\varepsilon\otimes\gamma\right)*\left(\varepsilon\otimes\gamma\left(Q\otimes m\right)\right)*\gamma\left(Q\otimes m\left(Q\otimes m\right)\right)\\ *\left(\varepsilon\otimes\omega\right)*\omega\left(Q\otimes m\otimes Q\right)*\left(\omega\otimes\varepsilon\right)\\ *\gamma^{-1}\left(m\left(m\otimes Q\right)\otimes Q\right)*\left(\gamma^{-1}\left(m\otimes Q\right)\otimes\varepsilon\right)*\left(\gamma^{-1}\otimes\varepsilon\otimes\varepsilon\right)\end{array}\right].
\end{aligned}$$

Therefore

$$\omega^{\gamma}\left(Q\otimes Q\otimes m^{\gamma}\right)*\omega^{\gamma}\left(m^{\gamma}\otimes Q\otimes Q\right)=\left(\varepsilon\otimes\omega^{\gamma}\right)*\omega^{\gamma}\left(Q\otimes m^{\gamma}\otimes Q\right)*\left(\omega^{\gamma}\otimes\varepsilon\right).$$

□

In analogy to the case of Hopf algebras, one can define the bosonization $E\#H$ of a coquasi-bialgebra in ${}_{H}^{H}\mathcal{YD}$ by a Hopf algebra H, see [ABM, Definition 5.4] for further details on the structure. The following result was originally stated for E a Hopf algebra. Yorck Sommerhäuser suggested the present more general form which investigates the behaviour of the bosonization under a suitable gauge transformation.

PROPOSITION 2.5. *Let H be a Hopf algebra and let $(E,m,u,\Delta,\varepsilon,\omega)$ be a coquasi-bialgebra in ${}_{H}^{H}\mathcal{YD}$. Let $\gamma:E\otimes E\rightarrow\Bbbk$ be a gauge transformation in ${}_{H}^{H}\mathcal{YD}$. Set*

$$\Gamma:\left(E\#H\right)\otimes\left(E\#H\right)\rightarrow\Bbbk:\left(x\#h\right)\otimes\left(x'\#h'\right)\mapsto\gamma\left(x\otimes hx'\right)\varepsilon_{H}\left(h'\right).$$

Then Γ is a gauge transformation and $\left(E\#H\right)^{\Gamma}=E^{\gamma}\#H$ as ordinary coquasi-bialgebras.

Proof. By [ABM, Lemma 2.15 and what follows], we have that Γ is convolution invertible H-bilinear and H-balanced. Moreover $\Gamma^{-1}\left(\left(x\#h\right)\otimes\left(x'\#h'\right)\right)=\gamma^{-1}\left(x\otimes hx'\right)\varepsilon_{H}\left(h'\right)$. If $\alpha:\left(E\#H\right)\otimes\left(E\#H\right)\rightarrow E\#H$ is H-bilinear and H-balanced, it is easy to check that $\Gamma*\alpha*\Gamma^{-1}$ is H-bilinear and H-balanced too.

In particular, since

$$m_{E\#H}\left(\left(x\#h\right)\otimes\left(x'\#h'\right)\right)=m\left(x\otimes h_{1}x'\right)\otimes h_{2}h'$$

we have that $m_{E\#H}$ is H-bilinear and H-balanced where $E\#H$ carries the left H-diagonal action and the right regular action over H.

Thus $m_{(E\#H)^{\Gamma}}=\Gamma*m_{E\#H}*\Gamma^{-1}$ is H-bilinear and H-balanced. Moreover, since E^{γ} is also a coquasi-bialgebra in ${}_{H}^{H}\mathcal{YD}$ we have that $m_{E^{\gamma}\#H}:(E\#H)\otimes(E\#H)\rightarrow E\#H$ is H-bilinear and H-balanced too.

Therefore, in order to check that $m_{(E\#H)^{\Gamma}}=m_{E^{\gamma}\#H}$, it suffices to prove that they coincide on elements of the form $(x\#1_{H})\otimes(x'\#1_{H})$.

Let us consider the multiplication

$$\begin{aligned}
& m_{(E\#H)^{\Gamma}}\left((x\#1_{H})\otimes(x'\#1_{H})\right)\\
= & \left(\Gamma*m_{E\#H}*\Gamma^{-1}\right)\left((x\#1_{H})\otimes(x'\#1_{H})\right)\\
= & \Gamma\left((x\#1_{H})_{1}\otimes(x'\#1_{H})_{1}\right)\cdot m_{E\#H}\left((x\#1_{H})_{2}\otimes(x'\#1_{H})_{2}\right)\cdot\Gamma^{-1}\left((x\#1_{H})_{3}\otimes(x'\#1_{H})_{3}\right).
\end{aligned}$$

Now, from

$$\Delta_{E\#H}(x\#h)=\sum\left(x^{(1)}\#x^{(2)}{}_{\langle-1\rangle}h_{1}\right)\otimes\left(x^{(2)}{}_{\langle0\rangle}\#h_{2}\right)$$

we get

$$\begin{aligned}
& (x\#1_{H})_{1}\otimes(x\#1_{H})_{2}\otimes(x\#1_{H})_{3}\\
= & \sum\left(x^{(1)}\#x^{(2)}{}_{\langle-1\rangle}x^{(3)}{}_{\langle-2\rangle}\right)\otimes\left(x^{(2)}{}_{\langle0\rangle}\#x^{(3)}{}_{\langle-1\rangle}\right)\otimes\left(x^{(3)}{}_{\langle0\rangle}\#1_{H}\right)
\end{aligned}$$

so that

$$\begin{aligned}
& m_{(E\#H)^{\Gamma}}\left((x\#1_{H})\otimes(x'\#1_{H})\right)\\
= & \Gamma\left((x\#1_{H})_{1}\otimes(x'\#1_{H})_{1}\right)\cdot m_{E\#H}\left((x\#1_{H})_{2}\otimes(x'\#1_{H})_{2}\right)\cdot\Gamma^{-1}\left((x\#1_{H})_{3}\otimes(x'\#1_{H})_{3}\right)\\
= & \left[\begin{array}{c}\sum\Gamma\left(x^{(1)}\#x^{(2)}{}_{\langle-1\rangle}x^{(3)}{}_{\langle-2\rangle}\otimes x'^{(1)}\#x'^{(2)}{}_{\langle-1\rangle}x'^{(3)}{}_{\langle-2\rangle}\right)\\ \cdot m_{E\#H}\left(x^{(2)}{}_{\langle0\rangle}\#x^{(3)}{}_{\langle-1\rangle}\otimes x'^{(2)}{}_{\langle0\rangle}\#x'^{(3)}{}_{\langle-1\rangle}\right)\\ \cdot\Gamma^{-1}\left(x^{(3)}{}_{\langle0\rangle}\#1_{H}\otimes x'^{(3)}{}_{\langle0\rangle}\#1_{H}\right)\end{array}\right]\\
= & \left[\begin{array}{c}\sum\gamma\left(x^{(1)}\otimes x^{(2)}{}_{\langle-1\rangle}x^{(3)}{}_{\langle-2\rangle}x'^{(1)}\right)\\ \cdot m_{E\#H}\left(x^{(2)}{}_{\langle0\rangle}\#x^{(3)}{}_{\langle-1\rangle}\otimes x'^{(2)}\#x'^{(3)}{}_{\langle-1\rangle}\right)\\ \cdot\gamma^{-1}\left(x^{(3)}{}_{\langle0\rangle}\otimes x'^{(3)}{}_{\langle0\rangle}\right)\end{array}\right]\\
= & \left[\begin{array}{c}\sum\gamma\left(x^{(1)}\otimes x^{(2)}{}_{\langle-1\rangle}x^{(3)}{}_{\langle-2\rangle}x'^{(1)}\right)\\ \cdot m\left(x^{(2)}{}_{\langle0\rangle}\otimes x^{(3)}{}_{\langle-2\rangle}x'^{(2)}\right)\otimes x^{(3)}{}_{\langle-1\rangle}x'^{(3)}{}_{\langle-1\rangle}\\ \cdot\gamma^{-1}\left(x^{(3)}{}_{\langle0\rangle}\otimes x'^{(3)}{}_{\langle0\rangle}\right)\end{array}\right]\\
= & \left[\begin{array}{c}\sum\gamma\left(x^{(1)}\otimes x^{(2)}{}_{\langle-1\rangle}x^{(3)}{}_{\langle-2\rangle}x'^{(1)}\right)\\ \cdot m\left(x^{(2)}{}_{\langle0\rangle}\otimes x^{(3)}{}_{\langle-1\rangle}x'^{(2)}\right)\otimes\left(x^{(3)}{}_{\langle0\rangle}\otimes x'^{(3)}\right)_{\langle-1\rangle}\\ \cdot\gamma^{-1}\left(x^{(3)}{}_{\langle0\rangle}\otimes x'^{(3)}{}_{\langle0\rangle}\right)\end{array}\right]\\
\overset{\gamma^{-1}\text{ colin.}}{=} & \left[\begin{array}{c}\sum\gamma\left(x^{(1)}\otimes x^{(2)}{}_{\langle-1\rangle}x^{(3)}{}_{\langle-2\rangle}x'^{(1)}\right)\cdot m\left(x^{(2)}{}_{\langle0\rangle}\otimes x^{(3)}{}_{\langle-1\rangle}x'^{(2)}\right)\otimes1_{H}\\ \cdot\gamma^{-1}\left(x^{(3)}{}_{\langle0\rangle}\otimes x'^{(3)}\right)\end{array}\right]\\
= & \left[\begin{array}{c}\sum\gamma\left(x^{(1)}\otimes x^{(2)}{}_{\langle-1\rangle}x^{(3)}{}_{\langle-2\rangle}x'^{(1)}\right)m\left(x^{(2)}{}_{\langle0\rangle}\otimes x^{(3)}{}_{\langle-1\rangle}x'^{(2)}\right)\\ \gamma^{-1}\left(x^{(3)}{}_{\langle0\rangle}\otimes x'^{(3)}\right)\end{array}\right]\otimes1_{H}.
\end{aligned}$$

Now we have

$$\sum(x\otimes y)^{(1)}\otimes(x\otimes y)^{(2)}=\sum x^{(1)}\otimes x^{(2)}{}_{\langle-1\rangle}y^{(1)}\otimes x^{(2)}{}_{\langle0\rangle}\otimes y^{(2)}$$

so that

$$\begin{aligned}
& \sum(x\otimes y)^{(1)}\otimes(x\otimes y)^{(2)}\otimes(x\otimes y)^{(3)}\\
= & \sum\left(x^{(1)}\otimes x^{(2)}{}_{\langle-1\rangle}x^{(3)}{}_{\langle-2\rangle}y^{(1)}\right)\otimes\left(x^{(2)}{}_{\langle0\rangle}\otimes x^{(3)}{}_{\langle-1\rangle}y^{(2)}\right)\otimes\left(x^{(3)}{}_{\langle0\rangle}\otimes y^{(3)}\right).
\end{aligned}$$

Using this equality we can proceed in our computation:

$$m_{(E\#H)^{\Gamma}}\left((x\#1_{H})\otimes(x'\#1_{H})\right)$$

$$
\begin{aligned}
&= \left[\begin{array}{c}\sum \gamma\left(x^{(1)} \otimes x^{(2)}{}_{\langle -1\rangle} x^{(3)}{}_{\langle -2\rangle} x'^{(1)}\right) \\ m\left(x^{(2)}{}_{\langle 0\rangle} \otimes x^{(3)}{}_{\langle -1\rangle} x'^{(2)}\right) \gamma^{-1}\left(x^{(3)}{}_{\langle 0\rangle} \otimes x'^{(3)}\right)\end{array}\right] \otimes 1_H \\
&= \left[\sum \gamma\left(\left(x \otimes x'\right)^{(1)}\right) \cdot m\left(\left(x \otimes x'\right)^{(2)}\right) \cdot \gamma^{-1}\left(\left(x \otimes x'\right)^{(3)}\right)\right] \# 1_H \\
&= \left(\gamma * m * \gamma^{-1}\right)\left(x \otimes x'\right) \# 1_H \\
&= m_{E^\gamma}\left(x \otimes x'\right) \# 1_H \\
&= m_{E^\gamma \# H}\left(\left(x \# 1_H\right) \otimes \left(x' \# 1_H\right)\right).
\end{aligned}
$$

Finally $u_{(E\#H)^\Gamma} = u_{E\#H} = 1_E \# 1_H = 1_{E^\gamma} \# 1_H = u_{E^\gamma \# H}$.

As a coalgebra $(E\#H)^\Gamma$ coincides with $E\#H$ and hence with $E^\gamma \# H$.

Finally let us check that $\omega_{E^\gamma \# H}$ and $\omega_{(E\#H)^\Gamma}$ coincide. To this aim, let us use the maps $\mho^*_{H,-}$ of [ABM, Lemma 2.15]. First note that $\omega_{E^\gamma \# H} = \mho^3_{H,E^\gamma}(\omega_{E^\gamma})$ by [ABM, Proposition 5.3]. Now

$$
\begin{aligned}
\omega_{(E\#H)^\Gamma} &= \left(\varepsilon_{E\#H} \otimes \Gamma\right) * \Gamma\left(E\#H \otimes m_{E\#H}\right) * \omega_{E\#H} * \Gamma^{-1}\left(m_{E\#H} \otimes E\#H\right) * \left(\Gamma^{-1} \otimes \varepsilon_{E\#H}\right) \\
&= \left(\mho^1_{H,E}(\varepsilon) \otimes \mho^2_{H,E}(\gamma)\right) * \mho^2_{H,E}(\gamma)\left(E\#H \otimes m_{E\#H}\right) * \mho^3_{H,E}(\omega) \\
&\quad * \mho^2_{H,E}\left(\gamma^{-1}\right)\left(m_{E\#H} \otimes E\#H\right) * \left(\mho^2_{H,E}\left(\gamma^{-1}\right) \otimes \mho^1_{H,E}(\varepsilon)\right)
\end{aligned}
$$

One easily checks that

$$
\begin{aligned}
\mho^1_{H,E}(\varepsilon) \otimes \mho^2_{H,E}(\gamma) &= \mho^3_{H,E^\gamma}(\varepsilon \otimes \gamma), \\
\mho^2_{H,E}(\gamma)\left(E\#H \otimes m_{E\#H}\right) &= \mho^3_{H,E^\gamma}(\gamma(E \otimes m)), \\
\mho^2_{H,E}\left(\gamma^{-1}\right)\left(m_{E\#H} \otimes E\#H\right) &= \mho^3_{H,E^\gamma}\left(\gamma^{-1}(m \otimes E)\right), \\
\mho^2_{H,E}\left(\gamma^{-1}\right) \otimes \mho^1_{H,E}(\varepsilon_E) &= \mho^3_{H,E^\gamma}\left(\gamma^{-1} \otimes \varepsilon\right).
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
\omega_{(E\#H)^\Gamma} &= \mho^3_{H,E^\gamma}(\varepsilon \otimes \gamma) * \mho^3_{H,E^\gamma}(\gamma(E \otimes m)) * \mho^3_{H,E}(\omega) * \mho^3_{H,E^\gamma}\left(\gamma^{-1}(m \otimes E)\right) * \mho^3_{H,E^\gamma}\left(\gamma^{-1} \otimes \varepsilon\right) \\
&= \mho^3_{H,E^\gamma}\left[(\varepsilon \otimes \gamma) * \gamma(E \otimes m) * \omega * \gamma^{-1}(m \otimes E) * \left(\gamma^{-1} \otimes \varepsilon\right)\right] \\
&= \mho^3_{H,E^\gamma}(\omega_{E^\gamma}) = \omega_{E^\gamma \# H}.
\end{aligned}
$$

□

PROPOSITION 2.6. *Let H be a Hopf algebra and let $(Q, m, u, \Delta, \varepsilon, \omega)$ be a connected coquasi-bialgebra in ${}^H_H\mathcal{YD}$. Let $\gamma : Q \otimes Q \to \Bbbk$ be a gauge transformation in ${}^H_H\mathcal{YD}$. Then $\mathrm{gr}(Q^\gamma)$ and $\mathrm{gr}(Q)$ coincide as bialgebras in ${}^H_H\mathcal{YD}$.*

Proof. By Proposition 2.4, Q^γ is a coquasi-bialgebra in ${}^H_H\mathcal{YD}$. It is obviously connected as it coincides with Q as a coalgebra. By Theorem 1.5, both $\mathrm{gr}Q$ and $\mathrm{gr}(Q^\gamma)$ are connected bialgebras in ${}^H_H\mathcal{YD}$. Let us check they coincide.

Note that, by Remark 2.2, we have that γ^{-1} is a gauge transformation, hence it is trivial on $\Bbbk 1_Q \otimes 1_Q$. Let $C := Q \otimes Q$. Let $n > 0$ and let $w \in C_{(n)} = \sum_{i+j \leq n} Q_i \otimes Q_j$. By [AMS1, Lemma 3.69], we have that $\Delta_C(w) - w \otimes (1_Q)^{\otimes 2} - (1_Q)^{\otimes 2} \otimes w \in C_{(n-1)} \otimes C_{(n-1)}$. Thus we get

$$
w_1 \otimes w_2 \otimes w_3 - \Delta_C(w) \otimes (1_Q)^{\otimes 2} - \Delta_C\left((1_Q)^{\otimes 2}\right) \otimes w \in \Delta_C\left(C_{(n-1)}\right) \otimes C_{(n-1)}
$$

and hence

$$
w_1 \otimes w_2 \otimes w_3 - w \otimes (1_Q)^{\otimes 2} \otimes (1_Q)^{\otimes 2} - (1_Q)^{\otimes 2} \otimes w \otimes (1_Q)^{\otimes 2} - (1_Q)^{\otimes 4} \otimes w \in C_{(n-1)} \otimes C_{(n-1)} \otimes C_{(n-1)}.
$$

Since $m\left(C_{(n-1)}\right) \subseteq Q_{n-1}$ we get

$$
w_1 \otimes m(w_2) \otimes w_3 - w \otimes 1_Q \otimes (1_Q)^{\otimes 2} - (1_Q)^{\otimes 2} \otimes m(w) \otimes (1_Q)^{\otimes 2} - (1_Q)^{\otimes 3} \otimes w \in C_{(n-1)} \otimes Q_{n-1} \otimes C_{(n-1)}
$$

and hence

$$
w_1 \otimes \left(m(w_2) + Q_{n-1}\right) \otimes w_3 = (1_Q)^{\otimes 2} \otimes \left(m(w) + Q_{n-1}\right) \otimes (1_Q)^{\otimes 2}. \tag{16}
$$

Let $x,y\in Q$. We compute

$$\begin{array}{rcl}\overline{x}\cdot_{\gamma}\overline{y} & = & \left(x+Q_{|x|-1}\right)\cdot_{\gamma}\left(y+Q_{|y|-1}\right)\\ & = & \left(x\cdot_{\gamma}y\right)+Q_{|x|+|y|-1}\\ & = & \gamma\left(\left(x\otimes y\right)_{1}\right)m\left(\left(x\otimes y\right)_{2}\right)\gamma^{-1}\left(\left(x\otimes y\right)_{3}\right)+Q_{|x|+|y|-1}\\ & = & \gamma\left(\left(x\otimes y\right)_{1}\right)\left(m\left(\left(x\otimes y\right)_{2}\right)+Q_{|x|+|y|-1}\right)\gamma^{-1}\left(\left(x\otimes y\right)_{3}\right)\\ & \overset{(16)}{=} & \gamma\left(\left(1_{Q}\right)^{\otimes 2}\right)\left(m\left(x\otimes y\right)+Q_{|x|+|y|-1}\right)\gamma^{-1}\left(\left(1_{Q}\right)^{\otimes 2}\right)\\ & = & m\left(x\otimes y\right)+Q_{|x|+|y|-1}=\left(x\cdot y\right)+Q_{|x|+|y|-1}=\overline{x}\cdot\overline{y}.\end{array}$$

Note that Q^{γ} and Q have the same unit so that $\mathrm{gr}Q$ and $\mathrm{gr}Q^{\gamma}$ have. □

3. (Co)semisimple case

Assume H is a semisimple and cosemisimple Hopf algebra (e.g. H is finite-dimensional cosemisimple over a field of characteristic zero). Note that H is then separable (see e.g. [Stf, Corollary 3.7] or [AMS1, Theorem 2.34]) whence finite-dimensional. Let $(Q,m,u,\Delta,\varepsilon)$ be a f.d. coalgebra with multiplication and unit in ${}^{H}_{H}\mathcal{YD}$. Assume that the coradical Q_0 is a subcoalgebra of Q in ${}^{H}_{H}\mathcal{YD}$ such that $Q_0\cdot Q_0\subseteq Q_0$. Let $y^{n,i}$ with $1\leq i\leq\dim\left(Q_n/Q_{n-1}\right)$ be a basis for Q_n/Q_{n-1}. Consider, for every $n>0$, the exact sequence in ${}^{H}_{H}\mathcal{YD}$ given by

$$0\longrightarrow Q_{n-1}\xrightarrow{s_n}Q_n\xrightarrow{\pi_n}\frac{Q_n}{Q_{n-1}}\longrightarrow 0$$

Now, since H is semisimple and cosemisimple, by [Ra2, Proposition 7] the Drinfeld double $D(H)$ is semisimple. By a result essentially due to Majid (see [Mo, Proposition 10.6.16]) and by [RT, Proposition 6], we get that the category ${}^{H}_{H}\mathcal{YD}\cong{}_{D(H)}\mathfrak{M}$ is a semisimple category. Therefore π_n cosplits i.e. there is a morphism $\sigma_n:(Q_n/Q_{n-1})\to Q_n$ in ${}^{H}_{H}\mathcal{YD}$ such that $\pi_n\sigma_n=\mathrm{Id}$. Let $u_n:\Bbbk\to Q_n$ be the corestriction of the unit $u:\Bbbk\to Q$ and let $\varepsilon_n=\varepsilon_{|Q_n}:Q_n\to\Bbbk$ be the counit of the subcoalgebra Q_n. Set

$$\sigma_n':=\sigma_n-u_n\circ\varepsilon_n\circ\sigma_n$$

This is a morphism in ${}^{H}_{H}\mathcal{YD}$. Moreover

$$\begin{array}{rcl}\pi_n\circ\sigma_n' & = & \pi_n\circ\sigma_n-\pi_n\circ u_n\circ\varepsilon_n\circ\sigma_n\overset{n\geq 0}{=}\mathrm{Id}_{Q_n/Q_{n-1}}-0=\mathrm{Id}_{Q_n/Q_{n-1}},\\ \varepsilon_n\circ\sigma_n' & = & \varepsilon_n\circ\sigma_n-\varepsilon_n\circ u_n\circ\varepsilon_n\circ\sigma_n=\varepsilon_n\circ\sigma_n-\varepsilon_n\circ\sigma_n=0.\end{array}$$

Therefore, without loss of generality we can assume that $\varepsilon_n\circ\sigma_n=0$. A standard argument on split short exact sequences shows that there exists a morphism $p_n:Q_n\to Q_{n-1}$ in ${}^{H}_{H}\mathcal{YD}$ such that $s_np_n+\sigma_n\pi_n=\mathrm{Id}_{Q_n}$, $p_ns_n=\mathrm{Id}_{Q_{n-1}}$ and $p_n\sigma_n=0$. We set

$$x^{n,i}:=\sigma_n\left(y^{n,i}\right).$$

Therefore

$$y^{n,i}=\pi_n\sigma_n\left(y^{n,i}\right)=\pi_n\left(x^{n,i}\right)=x^{n,i}+Q_{n-1}=\overline{x^{n,i}}.$$

These terms $x^{n,i}$ define a $\Bbbk$-basis for Q. As Q is finite-dimensional, there exists $d\in\mathbb{N}_0$ such that $Q=Q_d$; we fix d minimal. For all $0\leq a\leq b$, define the maps

$$\begin{array}{rclcl}p_{a,b} & : & Q_b\to Q_a, & & p_{a,b}:=p_{a+1}\circ p_{a+2}\circ\cdots\circ p_{b-1}\circ p_b,\\ s_{b,a} & : & Q_a\to Q_b, & & s_{b,a}:=s_b\circ s_{b-1}\circ\cdots\circ s_{a+2}\circ s_{a+1}.\end{array}$$

Clearly one has

$$p_{a,b}\circ s_{b,a}=\mathrm{Id}_{Q_a}.$$

Thus, for $0\leq i,a\leq b$ we have

$$p_{i,b}\circ s_{b,a}=\left\{\begin{array}{ll}p_{i,b}\circ s_{b,i}\circ s_{i,a} & i>a\\ p_{i,a}\circ p_{a,b}\circ s_{b,a} & i\leq a\end{array}\right.=\left\{\begin{array}{ll}s_{i,a} & i>a\\ p_{i,a} & i\leq a\end{array}\right.\tag{17}$$

Thus we get an isomorphism $\varphi:Q\to\mathrm{gr}Q$ of objects in ${}^{H}_{H}\mathcal{YD}$ given by

$$\varphi\left(x\right):=\quad p_{0,d}\left(x\right)+\pi_1p_{1,d}\left(x\right)+\pi_2p_{2,d}\left(x\right)+\cdots+\pi_{d-2}p_{d-2,d}\left(x\right)+\pi_{d-1}p_{d-1,d}\left(x\right)+\pi_d\left(x\right)$$

$$= \sum\nolimits_{0\leq t\leq d}\pi_t p_{t,d}\left(x\right), \text{ for every } x\in Q,$$

where we set

$$\pi_0 = \mathrm{Id}_{Q_0}, \qquad p_{d,d} = \mathrm{Id}_{Q_d}.$$

For $0 \leq n \leq d$, we have

$$\begin{aligned}
\varphi\left(x^{n,i}\right) &= \varphi\left(s_{d,n}\left(x^{n,i}\right)\right) = \varphi\left(s_{d,n}\sigma_n\left(y^{n,i}\right)\right) = \sum\nolimits_{0\leq t\leq d}\pi_t p_{t,d}s_{d,n}\left(\sigma_n\left(y^{n,i}\right)\right)\\
&= \sum\nolimits_{n<t\leq d}\pi_t p_{t,d}s_{d,n}\left(\sigma_n\left(y^{n,i}\right)\right) + \sum\nolimits_{0\leq t\leq n}\pi_t p_{t,d}s_{d,n}\left(\sigma_n\left(y^{n,i}\right)\right)\\
&\overset{(17)}{=} \sum\nolimits_{n<t\leq d}\pi_t s_{t,n}\left(\sigma_n\left(y^{n,i}\right)\right) + \sum\nolimits_{0\leq t<n}\pi_t p_{t,n}\left(\sigma_n\left(y^{n,i}\right)\right) + \pi_n p_{n,d}s_{d,n}\left(\sigma_n\left(y^{n,i}\right)\right)\\
&= \sum\nolimits_{n<t\leq d}\pi_t s_{t,t-1}s_{t-1,n}\left(\sigma_n\left(y^{n,i}\right)\right) + \sum\nolimits_{0\leq t<n}\pi_t p_{t,n-1}p_{n-1,n}\left(\sigma_n\left(y^{n,i}\right)\right) +\\
&\quad +\pi_n p_{n,d}s_{d,n}\left(\sigma_n\left(y^{n,i}\right)\right)\\
&= \sum\nolimits_{n<t\leq d}\pi_t s_t s_{t-1,n}\sigma_n\left(y^{n,i}\right) + \sum\nolimits_{0\leq t<n}\pi_t p_{t,n-1}p_n\sigma_n\left(y^{n,i}\right) + \pi_n\sigma_n\left(y^{n,i}\right)\\
&= 0+0+y^{n,i} = y^{n,i}.
\end{aligned}$$

Hence $\varphi\left(x^{n,i}\right) = y^{n,i}$. Since $y^{n,i}$ with $1 \leq i \leq \dim\left(Q_n/Q_{n-1}\right) =: d_n$ form a basis for Q_n/Q_{n-1} we have that

$$hy^{n,i} \in \frac{Q_n}{Q_{n-1}}, \qquad \left(y^{n,i}\right)_{-1}\otimes\left(y^{n,i}\right)_0 \in H\otimes\frac{Q_n}{Q_{n-1}}.$$

Therefore there are $\chi^n_{t,i} \in H^*$ and $h^n_{t,i} \in H$ such that

$$hy^{n,i} = \sum\nolimits_{1\leq t\leq d_n}\chi^n_{t,i}\left(h\right)y^{n,t}, \qquad \left(y^{n,i}\right)_{-1}\otimes\left(y^{n,i}\right)_0 = \sum\nolimits_{1\leq t\leq d_n}h^n_{i,t}\otimes y^{n,t}. \tag{18}$$

We have

$$\begin{aligned}
h\left(h'y^{n,i}\right) &= \sum\nolimits_{1\leq s\leq d_n}\chi^n_{s,i}\left(h'\right)hy^{n,s} = \sum\nolimits_{1\leq s\leq d_n}\chi^n_{s,i}\left(h'\right)\sum\nolimits_{1\leq t\leq d_n}\chi^n_{t,s}\left(h\right)y^{n,t}\\
&= \sum\nolimits_{1\leq s\leq d_n}\sum\nolimits_{1\leq t\leq d_n}\chi^n_{t,s}\left(h\right)\chi^n_{s,i}\left(h'\right)y^{n,t},\\
\left(hh'\right)y^{n,i} &= \sum\nolimits_{1\leq t\leq d_n}\chi^n_{t,i}\left(hh'\right)y^{n,t}
\end{aligned}$$

and hence

$$\chi^n_{t,i}\left(hh'\right) = \sum\nolimits_{1\leq s\leq d_n}\chi^n_{t,s}\left(h\right)\chi^n_{s,i}\left(h'\right).$$

Moreover

$$y^{n,i} = 1_H y^{n,i} = \sum\nolimits_{1\leq t\leq d_n}\chi^n_{t,i}\left(1_H\right)y^{n,t}$$

and hence

$$\chi^n_{t,i}\left(1_H\right) = \delta_{t,i}.$$

We also have

$$\begin{aligned}
\left(y^{n,i}\right)_{-1}\otimes\left(\left(y^{n,i}\right)_0\right)_{-1}\otimes\left(\left(y^{n,i}\right)_0\right)_0 &= \sum\nolimits_{1\leq s\leq d_n}h^n_{i,s}\otimes\left(y^{n,s}\right)_{-1}\otimes\left(y^{n,s}\right)_0\\
&= \sum\nolimits_{1\leq s\leq d_n}h^n_{i,s}\otimes\sum\nolimits_{1\leq t\leq d_n}h^n_{s,t}\otimes y^{n,t}\\
&= \sum\nolimits_{1\leq s\leq d_n}\sum\nolimits_{1\leq t\leq d_n}h^n_{i,s}\otimes h^n_{s,t}\otimes y^{n,t},\\
\left(\left(y^{n,i}\right)_{-1}\right)_1\otimes\left(\left(y^{n,i}\right)_{-1}\right)_2\otimes\left(y^{n,i}\right)_0 &= \sum\nolimits_{1\leq t\leq d_n}\Delta_H\left(h^n_{t,i}\right)\otimes y^{n,t}
\end{aligned}$$

so that

$$\Delta_H\left(h^n_{t,i}\right) = \sum\nolimits_{1\leq s\leq d_n}h^n_{i,s}\otimes h^n_{s,t}.$$

Moreover

$$y^{n,i} = \varepsilon_H\left(\left(y^{n,i}\right)_{-1}\right)\left(y^{n,i}\right)_0 = \sum\nolimits_{1\leq t\leq d_n}\varepsilon_H\left(h^n_{t,i}\right)y^{n,t}$$

and hence

$$\varepsilon_H\left(h^n_{t,i}\right) = \delta_{t,i}.$$

Finally

$$\begin{aligned}\left(h_1y^{n,i}\right)_{-1}h_2\otimes\left(h_1y^{n,i}\right)_0 &= \sum\nolimits_{1\leq s\leq d_n}\chi^n_{s,i}\left(h_1\right)\left(y^{n,s}\right)_{-1}h_2\otimes\left(y^{n,s}\right)_0\\ &= \sum\nolimits_{1\leq s\leq d_n}\chi^n_{s,i}\left(h_1\right)\sum\nolimits_{1\leq t\leq d_n}h^n_{s,t}h_2\otimes y^{n,t}\\ &= \sum\nolimits_{1\leq s\leq d_n}\sum\nolimits_{1\leq t\leq d_n}h^n_{s,t}\chi^n_{s,i}\left(h_1\right)h_2\otimes y^{n,t},\\ h_1\left(y^{n,i}\right)_{-1}\otimes h_2\left(y^{n,i}\right)_0 &= \sum\nolimits_{1\leq s\leq d_n}h_1h^n_{i,s}\otimes h_2y^{n,s}=\sum\nolimits_{1\leq s\leq d_n}h_1h^n_{i,s}\otimes\sum\nolimits_{1\leq t\leq d_n}\chi^n_{t,s}\left(h_2\right)y^{n,t}\\ &= \sum\nolimits_{1\leq s\leq d_n}\sum\nolimits_{1\leq t\leq d_n}h_1\chi^n_{t,s}\left(h_2\right)h^n_{i,s}\otimes y^{n,t}\end{aligned}$$

Therefore, we get

$$\sum\nolimits_{1\leq s\leq d_n}h^n_{s,t}\chi^n_{s,i}\left(h_1\right)h_2=\sum\nolimits_{1\leq s\leq d_n}h_1\chi^n_{t,s}\left(h_2\right)h^n_{i,s}.$$

We have

$$\begin{aligned}hx^{n,i} &= h\sigma_n\left(y^{n,i}\right)=\sigma_n\left(hy^{n,i}\right)=\sigma_n\left(\sum\nolimits_{1\leq t\leq d_n}\chi^n_{t,i}\left(h\right)y^{n,t}\right)=\sum\nolimits_{1\leq t\leq d_n}\chi^n_{t,i}\left(h\right)x^{n,t},\\ \left(x^{n,i}\right)_{-1}\otimes\left(x^{n,i}\right)_0 &= \left(\sigma_n\left(y^{n,i}\right)\right)_{-1}\otimes\left(\sigma_n\left(y^{n,i}\right)\right)_0=\left(y^{n,i}\right)_{-1}\otimes\sigma_n\left(\left(y^{n,i}\right)_0\right)=\sum\nolimits_{1\leq t\leq d_n}h^n_{i,t}\otimes x^{n,t},\\ \varepsilon_Q\left(x^{n,i}\right) &= \varepsilon_n\left(x^{n,i}\right)=\varepsilon_n\sigma_n\left(y^{n,i}\right)=0\text{ for }n>0.\end{aligned}$$

If Q is connected, then $d_0=1$ so we may assume $y^{0,0}:=1_Q+Q_{-1}$. Since $\pi_0=\mathrm{Id}_{Q_0}$ we get

$$\sigma_0=\mathrm{Id}_{Q_0}\circ\sigma_0=\pi_0\circ\sigma_0=\mathrm{Id}_{Q_0}$$

and hence

$$x^{0,0}=\sigma_0\left(y^{0,0}\right)=\sigma_0\left(1_Q+Q_{-1}\right)=1_Q.$$

Since, by Proposition 1.3, $Q_a\cdot Q_{a'}\subseteq Q_{a+a'}$ for every $a,a'\in\mathbb{N}_0$, we can write the product of two elements of the basis in the form

$$x^{a,l}x^{a',l'}=\sum\nolimits_{u\leq a+a'}\sum\nolimits_v\mu^{a,l,a',l'}_{u,v}x^{u,v}. \tag{19}$$

We compute

$$\begin{aligned}\overline{x^{a,l}}\cdot\overline{x^{a',l'}} &= \left(x^{a,l}+Q_{a-1}\right)\left(x^{a',l'}+Q_{a'-1}\right)\\ &= \left(x^{a,l}x^{a',l'}\right)+Q_{a+a'-1}\\ &\overset{(19)}{=} \left(\sum\nolimits_{u\leq a+a'}\sum\nolimits_v\mu^{a,l,a',l'}_{u,v}x^{u,v}\right)+Q_{a+a'-1}\\ &= \left(\sum\nolimits_v\mu^{a,l,a',l'}_{a+a',v}x^{a+a',v}\right)+Q_{a+a'-1}\\ &= \sum\nolimits_v\mu^{a,l,a',l'}_{a+a',v}\left(x^{a+a',v}+Q_{a+a'-1}\right)\\ &= \sum\nolimits_v\mu^{a,l,a',l'}_{a+a',v}\overline{x^{a+a',v}}.\end{aligned}$$

which gives

$$\overline{x^{a,l}}\cdot\overline{x^{a',l'}}=\sum\nolimits_v\mu^{a,l,a',l'}_{a+a',v}\overline{x^{a+a',v}}. \tag{20}$$

Remark 3.1. Let H be a Hopf algebra and let (A,m_A,u_A) be an algebra in ${}^H_H\mathcal{YD}$. Let $\varepsilon_A:A\to\Bbbk$ be an algebra map in ${}^H_H\mathcal{YD}$. The Hochschild cohomology in a monoidal category is known, see e.g. [AMS2]. Consider $\Bbbk$ as an A-bimodule in ${}^H_H\mathcal{YD}$ through ε_A. Then, following [AMS2, 1.24], we can consider an analogue of the standard complex

$${}^H_H\mathcal{YD}(\Bbbk,\Bbbk)\xrightarrow{\partial^0}{}^H_H\mathcal{YD}(A,\Bbbk)\xrightarrow{\partial^1}{}^H_H\mathcal{YD}(A^{\otimes2},\Bbbk)\xrightarrow{\partial^2}{}^H_H\mathcal{YD}(A^{\otimes3},\Bbbk)\xrightarrow{\partial^3}\cdots$$

Explicitly, given f in the corresponding domain of ∂^n, for $n=0,1,2,3$, we have

$$\partial^0\left(f\right) = f\left(1\right)\varepsilon_A-\varepsilon_Af\left(1\right)=0,$$

$$\begin{aligned}
\partial^1(f) &= f\otimes\varepsilon_A - f m_A + \varepsilon_A\otimes f,\\
\partial^2(f) &= f\otimes\varepsilon_A - f(A\otimes m_A) + f(m_A\otimes A) - \varepsilon_A\otimes f,\\
\partial^3(f) &= f\otimes\varepsilon_A - f(A\otimes A\otimes m_A) + f(A\otimes m_A\otimes A) - f(m_A\otimes A\otimes A) + \varepsilon_A\otimes f.
\end{aligned}$$

For every $n\geq 1$ denote by

$$\mathrm{Z}^n_{\mathcal{YD}}(A,\Bbbk) := \ker(\partial^n), \qquad \mathrm{B}^n_{\mathcal{YD}}(A,\Bbbk) := \mathrm{Im}\left(\partial^{n-1}\right) \qquad \text{and} \qquad \mathrm{H}^n_{\mathcal{YD}}(A,\Bbbk) := \frac{\mathrm{Z}^n_{\mathcal{YD}}(A,\Bbbk)}{\mathrm{B}^n_{\mathcal{YD}}(A,\Bbbk)}$$

the abelian groups of n-cocycles, of n-coboundaries and the n-th Hochschild cohomology group in ${}^H_H\mathcal{YD}$ of the algebra A with coefficients in $\Bbbk$. We point out that the construction above works for an arbitrary A-bimodule M in ${}^H_H\mathcal{YD}$ instead of $\Bbbk$.

Next result is inspired by [EG, Proposition 2.3]. Two coquasi-bialgebras Q and Q' in ${}^H_H\mathcal{YD}$ will be called **gauge equivalent** whenever there is some gauge transformation $\gamma: Q\otimes Q\to\Bbbk$ in ${}^H_H\mathcal{YD}$ such that $Q^\gamma\cong Q'$ as coquasi-bialgebras in ${}^H_H\mathcal{YD}$, see Proposition 2.4 for the structure of Q^γ.

THEOREM 3.2. *Let H be a semisimple and cosemisimple Hopf algebra and let $(Q,m,u,\Delta,\varepsilon,\omega)$ be a f.d. connected coquasi-bialgebra in ${}^H_H\mathcal{YD}$. If $\mathrm{H}^3_{\mathcal{YD}}(\mathrm{gr}Q,\Bbbk)=0$ then Q is gauge equivalent to a connected bialgebra in ${}^H_H\mathcal{YD}$.*

Proof. For $t\in\mathbb{N}_0$, and x,y,z in the basis of Q, we set

$$\omega_t(x\otimes y\otimes z) := \delta_{|x|+|y|+|z|,t}\,\omega(x\otimes y\otimes z).$$

Let us check it defines a morphism $\omega_t: Q\otimes Q\otimes Q\to\Bbbk$ in ${}^H_H\mathcal{YD}$. It is left H-linear as, by means of (18), the definition of ω_t and the H-linearity of ω, we can prove that $\omega_t\left(h\left(x^{n,i}\otimes x^{n',i'}\otimes x^{n'',i''}\right)\right) = \varepsilon_H(h)\,\omega_t\left(x^{n,i}\otimes x^{n',i'}\otimes x^{n'',i''}\right)$.

Moreover it is left H-colinear as, by means of (18), the definition of ω_t and the H-colinearity of ω, we can prove that

$$\left(x^{n,i}\otimes x^{n',i'}\otimes x^{n'',i''}\right)_{\langle -1\rangle}\otimes\omega_t\left(\left(x^{n,i}\otimes x^{n',i'}\otimes x^{n'',i''}\right)_{\langle 0\rangle}\right) = 1_H\otimes\omega_t\left(x^{n,i}\otimes x^{n',i'}\otimes x^{n'',i''}\right).$$

Clearly, for $x,y,z\in Q$ in the basis, one has

$$\sum_{t\in\mathbb{N}_0}\omega_t(x\otimes y\otimes z) = \sum_{t\in\mathbb{N}_0}\delta_{|x|+|y|+|z|,t}\,\omega(x\otimes y\otimes z) = \omega(x\otimes y\otimes z)$$

so that we can formally write

$$\omega = \sum_{t\in\mathbb{N}_0}\omega_t. \tag{21}$$

Since ε is trivial on elements in the basis of strictly positive degree, one gets

$$\omega_0 = \varepsilon\otimes\varepsilon\otimes\varepsilon. \tag{22}$$

If $\omega=\omega_0$ then Q is a (connected) bialgebra in ${}^H_H\mathcal{YD}$ and the proof is finished. Thus we can assume $\omega\neq\omega_0$ and set

$$\begin{aligned}
s &:= \min\{i\in\mathbb{N}:\omega_i\neq 0\},\\
\overline{\omega}_s &:= \omega_s\left(\varphi^{-1}\otimes\varphi^{-1}\otimes\varphi^{-1}\right),\\
\overline{Q} &:= \mathrm{gr}Q.
\end{aligned}$$

Note that $\overline{\omega}_s$ is a morphism in ${}^H_H\mathcal{YD}$ as a composition of morphisms in ${}^H_H\mathcal{YD}$.

Let $n\in\mathbb{N}_0$, let $C^4=Q\otimes Q\otimes Q\otimes Q$ and let $u\in C^4_{(n)}=\sum_{i+j+k+l\leq n}Q_i\otimes Q_j\otimes Q_k\otimes Q_l$.

A direct computation rewriting the cocycle condition using (21) proves that, for every $n\in\mathbb{N}_0$, and $u\in C^4_{(n)}$

$$\sum_{0\leq i+j\leq n}\left[\omega_i(Q\otimes Q\otimes m)*\omega_j(m\otimes Q\otimes Q)\right](u) \tag{23}$$

$$= \sum_{0\leq a+b+c\leq n} \left[(\varepsilon\otimes\omega_a) * \omega_b (Q\otimes m\otimes Q) * (\omega_c\otimes\varepsilon)\right](u).$$

Next aim is to check that $[\overline{\omega}_s] \in \mathrm{H}^3_{\mathcal{YD}}(\mathrm{gr}Q,\Bbbk)$ i.e. that

$$\overline{\omega}_s\left(m_{\overline{Q}}\otimes\overline{Q}\otimes\overline{Q}\right)+\overline{\omega}_s\left(\overline{Q}\otimes\overline{Q}\otimes m_{\overline{Q}}\right)=\left(\varepsilon_{\overline{Q}}\otimes\overline{\omega}_s\right)+\overline{\omega}_s\left(\overline{Q}\otimes m_{\overline{Q}}\otimes\overline{Q}\right)+\left(\overline{\omega}_s\otimes\varepsilon_{\overline{Q}}\right).$$

This is achieved by evaluating the two sides of the equality above on $\overline{u} := \overline{x}\otimes\overline{y}\otimes\overline{z}\otimes\overline{t}$ where x, y, z, t are elements in the basis and using (20). If $\overline{u}$ has homogeneous degree greater than s, then both terms are zero. Otherwise, i.e. if $\overline{u}$ has homogeneous degree at most s, one has $\overline{\omega}_s\left(m_{\overline{Q}}\otimes\overline{Q}\otimes\overline{Q}\right)(\overline{u}) = \omega_s(m_Q\otimes Q\otimes Q)(u)$ and similarly for the other pieces so that one has to check that

$$\omega_s(m\otimes Q\otimes Q)(u)+\omega_s(Q\otimes Q\otimes m)(u) = (\varepsilon\otimes\omega_s)(u)+\omega_s(Q\otimes m\otimes Q)(u)+(\omega_s\otimes\varepsilon)(u).$$

This equality follows by using (23) and the definition of s.

By assumption $\mathrm{H}^3_{\mathcal{YD}}(\mathrm{gr}Q,\Bbbk)=0$ so that there exists a morphism $\overline{v}:\overline{Q}\otimes\overline{Q}\to\Bbbk$ in ${}^H_H\mathcal{YD}$ such that

$$\overline{\omega}_s=\partial^2\overline{v}=\overline{v}\otimes\varepsilon_{\overline{Q}}-\overline{v}\left(\overline{Q}\otimes m_{\overline{Q}}\right)+\overline{v}\left(m_{\overline{Q}}\otimes\overline{Q}\right)-\varepsilon_{\overline{Q}}\otimes\overline{v}.$$

Explicitly, on elements in the basis we get

$$\overline{\omega}_s(\overline{x}\otimes\overline{y}\otimes\overline{z})=\overline{v}(\overline{x}\otimes\overline{y})\,\varepsilon_{\overline{Q}}(\overline{z})-\overline{v}(\overline{x}\otimes\overline{y}\cdot\overline{z})+\overline{v}(\overline{x}\cdot\overline{y}\otimes\overline{z})-\varepsilon_{\overline{Q}}(\overline{x})\,\overline{v}(\overline{y}\otimes\overline{z}).$$

Define $\overline{\zeta}:\overline{Q}\otimes\overline{Q}\to\Bbbk$ on the basis by setting

$$\overline{\zeta}(\overline{x}\otimes\overline{y}):=\delta_{|x|+|y|,s}\overline{v}(\overline{x}\otimes\overline{y}).$$

As we have done for ω_t, one can check that $\overline{\zeta}$ is a morphism in ${}^H_H\mathcal{YD}$.

Moreover on elements in the basis we get

$$\begin{aligned}
&\left(\partial^2\overline{\zeta}\right)(\overline{x}\otimes\overline{y}\otimes\overline{z})\\
=\ &\left(\overline{\zeta}\otimes\varepsilon_{\overline{Q}}\right)(\overline{x}\otimes\overline{y}\otimes\overline{z})-\overline{\zeta}\left(\overline{Q}\otimes m_{\overline{Q}}\right)(\overline{x}\otimes\overline{y}\otimes\overline{z})+\overline{\zeta}\left(m_{\overline{Q}}\otimes\overline{Q}\right)(\overline{x}\otimes\overline{y}\otimes\overline{z})-\left(\varepsilon_{\overline{Q}}\otimes\overline{\zeta}\right)(\overline{x}\otimes\overline{y}\otimes\overline{z})\\
=\ &\overline{\zeta}(\overline{x}\otimes\overline{y})\,\varepsilon_{\overline{Q}}(\overline{z})-\overline{\zeta}(\overline{x}\otimes\overline{y}\cdot\overline{z})+\overline{\zeta}(\overline{x}\cdot\overline{y}\otimes\overline{z})-\varepsilon_{\overline{Q}}(\overline{x})\,\overline{\zeta}(\overline{y}\otimes\overline{z}).
\end{aligned}$$

By using (20), one gets

$$\overline{\zeta}(\overline{x}\otimes\overline{y}\cdot\overline{z})=\delta_{|x|+|y|+|z|,s}\overline{v}(\overline{x}\otimes\overline{y}\cdot\overline{z})\qquad\text{and}\qquad\overline{\zeta}(\overline{x}\cdot\overline{y}\otimes\overline{z})=\delta_{|x|+|y|+|z|,s}\overline{v}(\overline{x}\cdot\overline{y}\otimes\overline{z}).$$

By means of these equalities one gets

$$\begin{aligned}
\left(\partial^2\overline{\zeta}\right)(\overline{x}\otimes\overline{y}\otimes\overline{z}) &= \delta_{|x|+|y|+|z|,s}\left(\partial^2\overline{v}\right)(\overline{x}\otimes\overline{y}\otimes\overline{z})=\delta_{|x|+|y|+|z|,s}\overline{\omega}_s(\overline{x}\otimes\overline{y}\otimes\overline{z})\\
&= \delta_{|x|+|y|+|z|,s}\omega_s(x\otimes y\otimes z)=\delta_{|x|+|y|+|z|,s}\delta_{|x|+|y|+|z|,s}\omega(x\otimes y\otimes z)\\
&= \delta_{|x|+|y|+|z|,s}\omega(x\otimes y\otimes z)=\omega_s(x\otimes y\otimes z)=\overline{\omega}_s(\overline{x}\otimes\overline{y}\otimes\overline{z}).
\end{aligned}$$

Therefore $\partial^2\overline{\zeta}=\overline{\omega}_s$. This means that we can assume that $\overline{v}(\overline{x}\otimes\overline{y})=0$ for $|x|+|y|\neq s$. Equivalently

$$\overline{v}(\overline{x}\otimes\overline{y})=\delta_{|x|+|y|,s}\overline{v}(\overline{x}\otimes\overline{y})\ \text{ for } x,y \text{ in the basis.}\tag{24}$$

Set

$$v:=\overline{v}\circ(\varphi\otimes\varphi)\qquad\text{and}\qquad\gamma:=(\varepsilon\otimes\varepsilon)+v.$$

In particular, one gets

$$v(x\otimes y)=\delta_{|x|+|y|,s}v(x\otimes y)\ \text{ for } x,y \text{ in the basis.}\tag{25}$$

Note also that both v and γ are morphisms in ${}^H_H\mathcal{YD}$ as they are obtained as composition or sum of morphisms in this category. Let us check that γ is a gauge transformation on Q in ${}^H_H\mathcal{YD}$.

Recall that $x^{0,0}=1_Q$ is in the basis. For x in the basis, we have $\gamma(x\otimes 1_Q)=\varepsilon(x)+v(x\otimes 1_Q)$. Note that

$$\begin{aligned}
0 &= \delta_{|x|,s}\varepsilon(x)=\delta_{|x|+|1_Q|+|1_Q|,s}\omega(x\otimes 1_Q\otimes 1_Q)\\
&= \omega_s(x\otimes 1_Q\otimes 1_Q)=\overline{\omega}_s\left(\overline{x}\otimes\overline{1_Q}\otimes\overline{1_Q}\right)
\end{aligned}$$

$$\begin{aligned}
&= \overline{v}\left(\overline{x}\otimes\overline{1_Q}\right)\varepsilon_{\overline{Q}}\left(\overline{1_Q}\right)-\overline{v}\left(\overline{x}\otimes\overline{1_Q}\cdot\overline{1_Q}\right)+\overline{v}\left(\overline{x}\cdot\overline{1_Q}\otimes\overline{1_Q}\right)-\varepsilon_{\overline{Q}}\left(\overline{x}\right)\overline{v}\left(\overline{1_Q}\otimes\overline{1_Q}\right)\\
&\overset{(24)}{=} \overline{v}\left(\overline{x}\otimes\overline{1_Q}\right)-\overline{v}\left(\overline{x}\otimes\overline{1_Q}\right)+\overline{v}\left(\overline{x}\otimes\overline{1_Q}\right)-\varepsilon_{\overline{Q}}\left(\overline{x}\right)\delta_{|1_Q|+|1_Q|,s}\overline{v}\left(\overline{1_Q}\otimes\overline{1_Q}\right)\\
&= v\left(x\otimes 1_Q\right)
\end{aligned}$$

so that $v\left(x\otimes 1_Q\right)=0$ and hence $\gamma\left(x\otimes 1_Q\right)=\varepsilon\left(x\right)+v\left(x\otimes 1_Q\right)=\varepsilon\left(x\right).$ Similarly one proves $\gamma\left(1_Q\otimes x\right)=\varepsilon\left(x\right).$ Hence γ is unital. Note that the coalgebra $C=Q\otimes Q$ is connected as Q is. Thus, in order to prove that $\gamma:Q\otimes Q\to\Bbbk$ is convolution invertible it suffices to check (see [Mo, Lemma 5.2.10]) that $\gamma_{|\Bbbk 1_Q\otimes\Bbbk 1_Q}$ is convolution invertible. But for $k,k'\in\Bbbk$ we have

$$\gamma\left(k1_Q\otimes k'1_Q\right)=kk'\gamma\left(1_Q\otimes 1_Q\right)=kk'\varepsilon\left(1_Q\right)=kk'=\left(\varepsilon\otimes\varepsilon\right)\left(k1_Q\otimes k'1_Q\right)$$

Hence $\gamma_{|\Bbbk 1_Q\otimes\Bbbk 1_Q}=\left(\varepsilon\otimes\varepsilon\right)_{|\Bbbk 1_Q\otimes\Bbbk 1_Q}$ which is convolution invertible. Thus there is a $\Bbbk$-linear map $\gamma^{-1}:Q\otimes Q\to\Bbbk$ and such that

$$\gamma*\gamma^{-1}=\varepsilon\otimes\varepsilon=\gamma^{-1}*\gamma.$$

Note that, by Lemma 2.3, $\gamma\in{}_H^H\mathcal{YD}$ implies $\gamma^{-1}\in{}_H^H\mathcal{YD}.$

Therefore γ is a gauge transformation in ${}_H^H\mathcal{YD}$. By Proposition 2.4, Q^γ is a coquasi-bialgebra in ${}_H^H\mathcal{YD}$. By Proposition 2.6, we have that $\mathrm{gr}Q^\gamma$ and $\mathrm{gr}Q$ coincide as bialgebras in ${}_H^H\mathcal{YD}$. Hence $\mathrm{H}^3_{\mathcal{YD}}\left(\mathrm{gr}Q^\gamma,\Bbbk\right)=\mathrm{H}^3_{\mathcal{YD}}\left(\mathrm{gr}Q,\Bbbk\right)=0$. Therefore Q^γ fulfills the same requirement of Q as in the statement. Let us check that $\left(\omega^\gamma\right)_t=0$ for $1\leq t\leq s$ (this will complete the proof by an induction process as Q is finite-dimensional).

Note that the definition of γ and (25) imply

$$\gamma\left(x\otimes y\right)=\delta_{|x|+|y|,0}\gamma\left(x\otimes y\right)+\delta_{|x|+|y|,s}\gamma\left(x\otimes y\right)\text{ for }x,y\text{ in the basis.}\tag{26}$$

Let $C^2=Q\otimes Q$ and let $C^2_{(n)}=\sum_{i+j\leq n}Q_i\otimes Q_j$. For $u\in C^2_{(2s-1)}$ we have

$$\left[\gamma*\left(\left(\varepsilon\otimes\varepsilon\right)-v\right)\right]\left(u\right)=\left(\varepsilon\otimes\varepsilon\right)\left(u\right)-v\left(u\right)+v\left(u\right)-v\left(u_1\right)v\left(u_2\right)\overset{(25)}{=}\left(\varepsilon\otimes\varepsilon\right)\left(u\right).$$

Therefore $\left[\gamma*\left(\left(\varepsilon\otimes\varepsilon\right)-v\right)\right]_{|C^2_{(2s-1)}}=\left(\varepsilon\otimes\varepsilon\right)_{|C^2_{(2s-1)}}$. By uniqueness of the convolution inverse, we deduce

$$\gamma^{-1}\left(u\right)=\left(\varepsilon\otimes\varepsilon\right)\left(u\right)-v\left(u\right),\text{ for }u\in C^2_{(2s-1)}.\tag{27}$$

Let x,y,z be in the basis. Set $\overline{u}:=\overline{x}\otimes\overline{y}\otimes\overline{z}$ and $u:=x\otimes y\otimes z$. We compute

$$\begin{aligned}
\left(\omega^\gamma\right)_s\left(u\right) &= \delta_{|x|+|y|+|z|,s}\omega^\gamma\left(u\right)\\
&= \delta_{|x|+|y|+|z|,s}\left[\left(\varepsilon\otimes\gamma\right)*\gamma\left(Q\otimes m\right)*\omega*\gamma^{-1}\left(m\otimes Q\right)*\left(\gamma^{-1}\otimes\varepsilon\right)\right]\left(u\right)\\
&= \delta_{|x|+|y|+|z|,s}\left[\left(\varepsilon\otimes\gamma\right)*\gamma\left(Q\otimes m\right)*\left(\omega_0+\omega_s\right)*\gamma^{-1}\left(m\otimes Q\right)*\left(\gamma^{-1}\otimes\varepsilon\right)\right]\left(u\right)\\
&\overset{(22)}{=} \delta_{|x|+|y|+|z|,s}\left[\begin{array}{c}\left(\varepsilon\otimes\gamma\right)*\gamma\left(Q\otimes m\right)*\gamma^{-1}\left(m\otimes Q\right)*\left(\gamma^{-1}\otimes\varepsilon\right)+\\ \left(\varepsilon\otimes\gamma\right)*\gamma\left(Q\otimes m\right)*\omega_s*\gamma^{-1}\left(m\otimes Q\right)*\left(\gamma^{-1}\otimes\varepsilon\right)\end{array}\right]\left(u\right)\\
&= \left[\begin{array}{c}\delta_{|x|+|y|+|z|,s}\left(\varepsilon\otimes\gamma\right)\left(u_1\right)\cdot\gamma\left(Q\otimes m\right)\left(u_2\right)\cdot\gamma^{-1}\left(m\otimes Q\right)\left(u_3\right)\cdot\left(\gamma^{-1}\otimes\varepsilon\right)\left(u_4\right)+\\ \delta_{|x|+|y|+|z|,s}\left(\varepsilon\otimes\gamma\right)\left(u_1\right)\cdot\gamma\left(Q\otimes m\right)\left(u_2\right)\cdot\omega_s\left(u_3\right)\cdot\gamma^{-1}\left(m\otimes Q\right)\left(u_4\right)\cdot\left(\gamma^{-1}\otimes\varepsilon\right)\left(u_5\right)\end{array}\right].
\end{aligned}$$

Now, all terms appearing in the last two lines, excepted ω_s, vanish out of degrees 0 and s and coincide with $\varepsilon\otimes\varepsilon\otimes\varepsilon$ on degree 0. On the other hand ω_s vanishes out of s. Since $\gamma:=\left(\varepsilon\otimes\varepsilon\right)+v$ and in view of (27), the term $\delta_{|x|+|y|+|z|,s}$ forces the following simplification

$$\left(\omega^\gamma\right)_s\left(u\right)=\left[\begin{array}{c}\delta_{|x|+|y|+|z|,s}\left[\left(\varepsilon\otimes v\right)\left(u\right)+v\left(Q\otimes m\right)\left(u\right)-v\left(m\otimes Q\right)\left(u\right)-\left(v\otimes\varepsilon\right)\left(u\right)\right]+\\ +\delta_{|x|+|y|+|z|,s}\omega_s\left(u\right)\end{array}\right].$$

Now $\omega_s\left(u\right)=\overline{\omega}_s\left(\overline{u}\right)$ while one proves that $\left(\varepsilon\otimes v\right)\left(u\right)=\left(\varepsilon_{\overline{Q}}\otimes\overline{v}\right)\left(\overline{u}\right),\ \delta_{|x|+|y|+|z|,s}v\left(m\otimes Q\right)\left(u\right)=\delta_{|x|+|y|+|z|,s}\overline{v}\left(m_{\overline{Q}}\otimes\overline{Q}\right)\left(\overline{u}\right)$ and similarly for the other pieces of the equality.

Thus one gets

$$\left(\omega^\gamma\right)_s\left(u\right) = \left[\begin{array}{c}\delta_{|x|+|y|+|z|,s}\left[\left(\varepsilon_{\overline{Q}}\otimes\overline{v}\right)\left(\overline{u}\right)+\overline{v}\left(\overline{Q}\otimes m_{\overline{Q}}\right)\left(\overline{u}\right)-\overline{v}\left(m_{\overline{Q}}\otimes\overline{Q}\right)\left(\overline{u}\right)-\left(\overline{v}\otimes\varepsilon_{\overline{Q}}\right)\left(\overline{u}\right)\right]+\\ +\delta_{|x|+|y|+|z|,s}\overline{\omega}_s\left(\overline{u}\right)\end{array}\right]$$

$$= \quad -\delta_{|x|+|y|+|z|,s}\partial^2\overline{v}+\delta_{|x|+|y|+|z|,s}\overline{\omega}_s\left(\overline{u}\right)=0.$$

For $0\leq t\leq s-1$, analogously to the above, we compute

$$\begin{aligned}
\left(\omega^{\gamma}\right)_t(u) &= \delta_{|x|+|y|+|z|,t}\omega^{\gamma}(u)\\
&= \delta_{|x|+|y|+|z|,t}\left[(\varepsilon\otimes\gamma)*\gamma(Q\otimes m)*\omega*\gamma^{-1}(m\otimes Q)*\left(\gamma^{-1}\otimes\varepsilon\right)\right](u)\\
&= \delta_{|x|+|y|+|z|,t}\left[(\varepsilon\otimes\gamma)*\gamma(Q\otimes m)*\omega_0*\gamma^{-1}(m\otimes Q)*\left(\gamma^{-1}\otimes\varepsilon\right)\right](u)\\
&\overset{(22)}{=} \delta_{|x|+|y|+|z|,t}\left[(\varepsilon\otimes\gamma)*\gamma(Q\otimes m)*\gamma^{-1}(m\otimes Q)*\left(\gamma^{-1}\otimes\varepsilon\right)\right](u)\\
&= \delta_{|x|+|y|+|z|,t}(\varepsilon\otimes\varepsilon\otimes\varepsilon)(u)=\delta_{0,t}(\varepsilon\otimes\varepsilon\otimes\varepsilon)(u).
\end{aligned}$$

Therefore we can now repeat the argument on ω^{γ} instead of ω. Deforming several times we will get a reassociator, say ω', whose first non trivial component ω'_t, with $t\neq 0$, exceeds the dimension of Q. In other words $\omega'=\omega'_0$ which is trivial. Hence Q is gauge equivalent to a connected bialgebra in ${}_H^H\mathcal{YD}$. $\square$

4. Invariants

Given a $\Bbbk$-algebra A, we denote by $\mathrm{H}^n(A,-)$ the n-th right derived functor of $\mathrm{Hom}_{A,A}(A,-)$ in the category of A-bimodules. In other words, for every A-bimodule M, $\mathrm{H}^n(A,M)$ is the Hochschild cohomology group of A with coefficients in M. Denote by $\mathrm{Z}^n(A,M)$ and $\mathrm{B}^n(A,M)$ the abelian groups of n-cocycles and of n-coboundaries respectively.

Let H be a Hopf algebra, let B be a left H-module algebra and let M be a $B\#H$-bimodule, where $B\#H$ denotes the smash product algebra, see e.g. [Mo, Definition 4.1.3]. Then $\mathrm{H}^n(B,M)$ becomes an H-bimodule as follows. Its structure of left H-module is given via ε_H and its structure of right H-module is defined, for every $f\in\mathrm{Z}^n(B,M)$ and $h\in H$, by setting

$$[f]\,h:=\left[\chi_n^h(M)(f)\right]$$

where, for every $k\in\Bbbk, b_1,\ldots,b_n\in B$, we set

$$\begin{aligned}
\chi_0^h(M)(f)(k) &:= (1_B\#S(h_1))\,f(k)\,(1_B\#h_2)\ \text{ for } n=0 \text{ while and for } n\geq 1\\
\chi_n^h(M)(f)(b_1\otimes b_2\otimes\cdots\otimes b_n) &:= (1_B\#S(h_1))\,f(h_2b_1\otimes h_3b_2\otimes\cdots\otimes h_{n+1}b_n)\,(1_B\#h_{n+2}).
\end{aligned}$$

Moreover

$$\partial^n\circ\chi_n^h(M)=\chi_{n+1}^h(M)\circ\partial^n,\ \text{ for every } n\geq -1, \tag{28}$$

where $\partial^n:\mathrm{Hom}_{\Bbbk}\left(B^{\otimes n},M\right)\to\mathrm{Hom}_{\Bbbk}\left(B^{\otimes(n+1)},M\right)$ denotes the differential of the usual Hochschild cohomology.

Denote by $\mathrm{H}^n(B,M)^H$ the space of H-invariant elements of $\mathrm{H}^n(B,M)$.

Proposition 4.1. *Let H be a semisimple Hopf algebra and let B be a left H-module algebra. Denote by $A:=B\#H$. Then, for each $n\in\mathbb{N}_0$ and for every A-bimodule M*

$$\mathrm{H}^n(B\#H,M)\cong\mathrm{H}^n(B,M)^H.$$

Proof. We will apply [Stf, Equation (3.6.1)]. To this aim we have to prove first that A/B is an H-Galois extension such that A is flat as left and right B-module. Now, $A=B\#_{\xi}H$ for $\xi:H\otimes H\to B$ defined by $\xi(x,y)=\varepsilon_H(x)\,\varepsilon_H(y)\,1_A$, cf. [Mo, Definition 7.1.1]. Moreover a direct computation shows that $\iota:B\to A:b\mapsto b\#1_H$ is a right H-extension where A is regarded as a right H-comodule via $\rho:A\to A\otimes H:b\#h\mapsto(b\#h_1)\otimes h_2$. Thus, by [Mo, Proposition 7.2.7], we know that $\iota:B\to A$ is H-cleft and hence, by [Mo, Theorem 8.2.4], it is H-Galois. The B-bimodule structure of A is induced by ι so that, explicitly, we have

$$\begin{aligned}
b'(b\#h) &= (b'\#1_H)(b\#h)=b'b\#h,\\
(b\#h)\,b' &= (b\#h)(b'\#1_H)=b\,(h_1b')\,\#h_2.
\end{aligned}$$

Note that $A=B\#H$ is flat as a left B-module as H is a free $\Bbbk$-module ($\Bbbk$ is a field). Now consider the map $\alpha : H\otimes B\to A$ defined by setting $\alpha(h\otimes b):=h_1 b\otimes h_2$ (note that it is defined as the braiding in ${}_H^H\mathcal{YD}$). We have

$$\alpha(h\otimes bb')=h_1(bb')\otimes h_2=(h_1 b)(h_2 b')\otimes h_3=(h_1 b\#h_2)b'=\alpha(h\otimes b)b'$$

so that α is right B-linear where $H\otimes B$ is regarded as a right module via $(h\#b)b':=h\#bb'$. Now H is semisimple and hence separable (see [Stf, Corollary 3.7]). Thus H is finite-dimensional and hence it has bijective antipode S_H. Thus α is invertible with inverse given by $\alpha^{-1}(b\#h):=h_2\otimes S_H^{-1}(h_1)b$. Therefore α is an isomorphism of right B-modules and hence A is flat as a right B-module as $H\otimes B$ is.

We have now the hypotheses necessary to apply [Stf, Equation (3.6.1)] and obtain

$$\mathrm{H}^n(A,M)\cong \mathrm{Hom}_{-,H}(\Bbbk,\mathrm{H}^n(B,M))=\mathrm{Hom}_{\Bbbk}(\Bbbk,\mathrm{H}^n(B,M))^H\cong \mathrm{H}^n(B,M)^H.$$

□

Remark 4.2. Proposition 4.1 in the particular case when $M=\Bbbk$ and B is finite-dimensional is [SV, Theorem 2.17]. Note that in the notations therein, one has $E(B)=\oplus_{n\in\mathbb{N}_0}E_n(B,\Bbbk)$ where $E_n(B,\Bbbk)=\mathrm{Ext}_B^n(\Bbbk,\Bbbk)\cong\mathrm{H}^n(B,\Bbbk)$. The latter isomorphism is [CE, Corollary 4.4, page 170].

Let H be a Hopf algebra and let B be a bialgebra in the braided category ${}_H^H\mathcal{YD}$. Denote by $A:=B\#H$ the Radford-Majid bosonization of B by H, see e.g. [Ra3, Theorem 1]. Note that A is endowed with an algebra map $\varepsilon_A : A\to\Bbbk$ defined by $\varepsilon_A(b\#h)=\varepsilon_B(b)\varepsilon_H(h)$ so that we can regard $\Bbbk$ as an A-bimodule via ε_A. Then we can consider $\mathrm{H}^n(B,\Bbbk)$ as an H-bimodule as follows. Its structure of left H-module is given via ε_H and its structure of right H-module is defined, for every $f\in\mathrm{Z}^n(B,\Bbbk)$ and $h\in H$, by setting

$$[f]h:=[fh],$$

where $(fh)(z)=f(hz)$, for every $z\in B^{\otimes n}$. The latter is the usual right H-module structure of $\mathrm{Hom}_{\Bbbk}(B^{\otimes n},\Bbbk)$. Indeed, for every $n\geq -1$, the vector space $\mathrm{Hom}_{\Bbbk}(B^{\otimes n},\Bbbk)$ is an H-bimodule with respect to this right H-module structure and the left one induced by ε_H.

Corollary 4.3. *Let H be a semisimple Hopf algebra and let B be a bialgebra in the braided category ${}_H^H\mathcal{YD}$. Set $A:=B\#H$. Then, for each $n\in\mathbb{N}_0$*

$$\mathrm{H}^n(B\#H,\Bbbk)\cong\mathrm{H}^n(B,\Bbbk)^H$$

and the differential $\partial^n:\mathrm{Hom}_{\Bbbk}(B^{\otimes n},\Bbbk)\to\mathrm{Hom}_{\Bbbk}\left(B^{\otimes(n+1)},\Bbbk\right)$ of the usual Hochschild cohomology is H-bilinear.

Proof. In the particular case $M=\Bbbk$, the right module H-structure used in Proposition 4.1 simplifies as follows. It is defined, for every $f\in\mathrm{Z}^n(B,\Bbbk)$ and $h\in H$, by setting

$$[f]h:=\left[\chi_n^h(\Bbbk)(f)\right]$$

where, for every $k\in\Bbbk, b_1,\ldots,b_n\in B$, we set

$$\begin{aligned}\chi_0^h(\Bbbk)(f)(k)\quad &: =\varepsilon_H(h)f(k)\ \text{ for } n=0 \text{ while and for } n\geq 1\\ \chi_n^h(\Bbbk)(f)(b_1\otimes b_2\otimes\cdots\otimes b_n)\quad &: =f(h_1b_1\otimes h_2b_2\otimes\cdots\otimes h_nb_n).\end{aligned}$$

More concisely $\chi_n^h(\Bbbk)(f)(z)=f(hz)$ for every $n\in\mathbb{N}_0$ and $z\in B^{\otimes n}$ i.e. $[f]h:=[fh]$ where $fh:=\chi_n^h(\Bbbk)(f)$.

Now consider the differential $\partial^n:\mathrm{Hom}_{\Bbbk}(B^{\otimes n},\Bbbk)\to\mathrm{Hom}_{\Bbbk}\left(B^{\otimes(n+1)},\Bbbk\right)$ of the usual Hochschild cohomology. Note that for each $n\in\mathbb{N}_0$, $\mathrm{Hom}_{\Bbbk}(B^{\otimes n},\Bbbk)$ is regarded as a bimodule over H using the left H-module structures of its arguments. By (28), we have

$$\partial^n\chi_n^h(\Bbbk)(f)=\chi_{n+1}^h(\Bbbk)\partial^n(f)$$

Since $\chi_n^h(\Bbbk)(f)=fh$, the last displayed equality becomes $\partial^n(fh)=\partial^n(f)h$ for every $n\in\mathbb{N}_0$. Thus ∂^n is right H-linear. Since $hf=\varepsilon_H(h)f$ for every $f\in\mathrm{Hom}_{\Bbbk}(B^{\otimes n},\Bbbk), h\in H$, we get that ∂^n is also left H-linear whence H-bilinear. □

Remark 4.4. Note that, in the context of the proof of [EG, Proposition 5.1], one has

$$\mathrm{H}^3\left(\mathcal{B}\left(V\right)\#\mathbb{C}\left[\mathbb{Z}_p\right],\mathbb{C}\right)\cong\mathrm{H}^3\left(\mathcal{B}\left(V\right),\mathbb{C}\right)^{\mathbb{Z}_p}.$$

This is a particular case of Corollary 4.3 where $H=\mathbb{C}\left[\mathbb{Z}_p\right], V\in{}_H^H\mathcal{YD}$ and $B=\mathcal{B}\left(V\right)$.

Proposition 4.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories. Let $r,\omega:\mathcal{C}\to\mathcal{D}$ be exact functors such that r is a subfunctor of ω i.e. there is a natural transformation $\eta:r\to\omega$ which is a monomorphism when evaluated on objects. If X is a subobject of Y then $r\left(X\right)=\omega\left(X\right)\cap r\left(Y\right)$. Moreover, for every morphism $f:X\to Y$ in $\mathcal{C}$ one has*

$$\begin{aligned}\ker\left(r\left(f\right)\right) &= r\left(\ker\left(f\right)\right)=\omega\left(\ker\left(f\right)\right)\cap r\left(X\right)=\ker\left(\omega\left(f\right)\right)\cap r\left(X\right),\\ \operatorname{Im}\left(r\left(f\right)\right) &= \operatorname{Im}\left(\omega\left(f\right)\right)\cap r\left(Y\right)=r\left(\operatorname{Im}\left(f\right)\right).\end{aligned}$$

Proof. The proof is similar to [Stn, Proposition 1.7, page 138]. □

Remark 4.6. From Corollary 4.3, we have

$$\begin{aligned}\mathrm{H}^n\left(B,\Bbbk\right)^H &= \left\{[f]\mid f\in\mathrm{Z}^n\left(B,\Bbbk\right),\varepsilon_H\left(h\right)[f]=[f]\,h,\text{ for every }h\in H\right\}\\ &= \left\{[f]\mid f\in\mathrm{Z}^n\left(B,\Bbbk\right),\left[\varepsilon_H\left(h\right)f\right]=[fh],\text{ for every }h\in H\right\}\end{aligned}$$

where, for every $z\in B^{\otimes n}$, we have

$$\left(fh\right)\left(z\right)=f\left(hz\right).$$

Note that, for any H-bimodule M one has

$$\mathrm{Hom}_{H,H}\left(H,M\right)\cong M^H=\left\{m\in M\mid hm=mh,\text{ for every }h\in H\right\}.$$

Note also that H is a separable $\Bbbk$-algebra whence it is projective in the category of H-bimodules. As a consequence $\mathrm{Hom}_{H,H}\left(H,-\right)\cong\left(-\right)^H:{}_H\mathfrak{M}_H\to\mathfrak{M}$ is an exact functor (here ${}_H\mathfrak{M}_H$ is the category of H-bimodules and $\mathfrak{M}$ the category of $\Bbbk$-vector spaces). By Proposition 4.5 applied to the case when $r:=\left(-\right)^H:{}_H\mathfrak{M}_H\to\mathfrak{M}$ and ω is the forgetful functor, for every morphism $f:X\to Y$ of H-bimodules one has

$$\ker\left(f^H\right)=\ker\left(f\right)\cap X^H=\left(\ker\left(f\right)\right)^H\qquad\text{and}\qquad\operatorname{Im}\left(f^H\right)=\operatorname{Im}\left(f\right)\cap Y^H=\left(\operatorname{Im}\left(f\right)\right)^H.$$

Still by Corollary 4.3, we know that the differential $\partial^n:\mathrm{Hom}_\Bbbk\left(B^{\otimes n},\Bbbk\right)\longrightarrow\mathrm{Hom}_\Bbbk\left(B^{\otimes(n+1)},\Bbbk\right)$ of the usual Hochschild cohomology is H-bilinear. Thus we can apply the argument above to get

$$\begin{aligned}\ker\left(\left(\partial^n\right)^H\right) &= \ker\left(\partial^n\right)\cap\mathrm{Hom}_\Bbbk\left(B^{\otimes n},\Bbbk\right)^H=\left(\ker\left(\partial^n\right)\right)^H\qquad\text{and}\\ \operatorname{Im}\left(\left(\partial^{n-1}\right)^H\right) &= \operatorname{Im}\left(\partial^{n-1}\right)\cap\mathrm{Hom}_\Bbbk\left(B^{\otimes n},\Bbbk\right)^H=\left(\operatorname{Im}\left(\partial^{n-1}\right)\right)^H.\end{aligned}$$

Now $\mathrm{Hom}_\Bbbk\left(B^{\otimes n},\Bbbk\right)^H=\mathrm{Hom}_{H,-}\left(B^{\otimes n},\Bbbk\right)$ so that we get

$$\begin{aligned}\mathrm{Z}^n_{H\text{-Mod}}\left(B,\Bbbk\right) &= \mathrm{Z}^n\left(B,\Bbbk\right)\cap\mathrm{Hom}_{H,-}\left(B^{\otimes n},\Bbbk\right)=\mathrm{Z}^n\left(B,\Bbbk\right)^H\qquad\text{and}\\ \mathrm{B}^n_{H\text{-Mod}}\left(B,\Bbbk\right) &= \mathrm{B}^n\left(B,\Bbbk\right)\cap\mathrm{Hom}_{H,-}\left(B^{\otimes n},\Bbbk\right)=\mathrm{B}^n\left(B,\Bbbk\right)^H.\end{aligned}$$

where $\mathrm{Z}^n_{H\text{-Mod}}\left(B,\Bbbk\right)$ and $\mathrm{B}^n_{H\text{-Mod}}\left(B,\Bbbk\right)$ denotes the the abelian groups of n-cocycles, of n-coboundaries for the cohomology of the algebra B with coefficients in $\Bbbk$ computed in the monoidal category H-Mod of left H-modules. The corresponding n-th Hochschild cohomology group is

$$\mathrm{H}^n_{H\text{-Mod}}\left(B,\Bbbk\right):=\frac{\mathrm{Z}^n_{H\text{-Mod}}\left(B,\Bbbk\right)}{\mathrm{B}^n_{H\text{-Mod}}\left(B,\Bbbk\right)}=\frac{\mathrm{Z}^n\left(B,\Bbbk\right)^H}{\mathrm{B}^n\left(B,\Bbbk\right)^H}\cong\left(\frac{\mathrm{Z}^n\left(B,\Bbbk\right)}{\mathrm{B}^n\left(B,\Bbbk\right)}\right)^H=\mathrm{H}^n\left(B,\Bbbk\right)^H.$$

Denote by $D\left(H\right)$ the Drinfeld double, see e.g. the first structure of [Maj, Theorem 7.1.1].

Proposition 4.7. *In the setting of Corollary 4.3 assume that H is also cosemisimple. Then, for $n\in\mathbb{N}_0$*

$$\mathrm{Z}^n_{\mathcal{YD}}\left(B,\Bbbk\right)=\mathrm{Z}^n\left(B,\Bbbk\right)^{D(H)},\quad\mathrm{B}^n_{\mathcal{YD}}\left(B,\Bbbk\right)=\mathrm{B}^n\left(B,\Bbbk\right)^{D(H)}\quad\text{and}\quad\mathrm{H}^n_{\mathcal{YD}}\left(B,\Bbbk\right)\cong\mathrm{H}^n\left(B,\Bbbk\right)^{D(H)}.$$

where $\mathrm{Z}^n\left(B,\Bbbk\right)$ and $\mathrm{B}^n\left(B,\Bbbk\right)$ are regarded as $D\left(H\right)$-subbimodules of $\mathrm{Hom}_\Bbbk\left(B^{\otimes n},\Bbbk\right)$ whose structure is induced by the left $D\left(H\right)$-module structures of its arguments.

Moreover $\mathrm{H}^n(B,\Bbbk)^{D(H)}$ *is a subspace of* $\mathrm{H}^n(B,\Bbbk)^H$.

Proof. For shortness, in this proof, we denote $D(H)$ by D. Consider the analogue of the standard complex as in Remark 3.1

$$ {}_H^H\mathcal{YD}(\Bbbk,\Bbbk) \xrightarrow{\partial^0} {}_H^H\mathcal{YD}(B,\Bbbk) \xrightarrow{\partial^1} {}_H^H\mathcal{YD}(B^{\otimes 2},\Bbbk) \xrightarrow{\partial^2} \cdots $$

where ∂^n is induced by the differential $\partial^n : \mathrm{Hom}_\Bbbk(B^{\otimes n},\Bbbk) \longrightarrow \mathrm{Hom}_\Bbbk\left(B^{\otimes (n+1)},\Bbbk\right)$ of the ordinary Hochschild cohomology. Now, since H is semisimple, it is finite-dimensional (whence it has bijective antipode) so that, by a result essentially due to Majid (see [Mo, Proposition 10.6.16]) and by [RT, Proposition 6], we get a category isomorphism ${}_H^H\mathcal{YD} \cong {}_D\mathfrak{M}$. Thus the complex above can be rewritten as follows

$$ \mathrm{Hom}_{D,-}(\Bbbk,\Bbbk) \xrightarrow{\partial^0} \mathrm{Hom}_{D,-}(B,\Bbbk) \xrightarrow{\partial^1} \mathrm{Hom}_{D,-}(B^{\otimes 2},\Bbbk) \xrightarrow{\partial^2} \cdots $$

Now, since, for each $n \in \mathbb{N}_0$, we have $\mathrm{Hom}_{D,-}(B^{\otimes n},\Bbbk) = \mathrm{Hom}_\Bbbk(B^{\otimes n},\Bbbk)^D$, we obtain the complex

$$ \mathrm{Hom}_\Bbbk(\Bbbk,\Bbbk)^D \xrightarrow{\partial^0} \mathrm{Hom}_\Bbbk(B,\Bbbk)^D \xrightarrow{\partial^1} \mathrm{Hom}_\Bbbk(B^{\otimes 2},\Bbbk)^D \xrightarrow{\partial^2} \cdots $$

We will write $(\partial^n)^D$ instead of ∂^n when we would like to stress that the map considered is the one induced on invariants. Thus we will write equivalently

$$ \mathrm{Hom}_\Bbbk(\Bbbk,\Bbbk)^D \xrightarrow{(\partial^0)^D} \mathrm{Hom}_\Bbbk(B,\Bbbk)^D \xrightarrow{(\partial^1)^D} \mathrm{Hom}_\Bbbk(B^{\otimes 2},\Bbbk)^D \xrightarrow{(\partial^2)^D} \cdots $$

Now, assume H is also cosemisimple. Since H is both semisimple and cosemisimple, by [Ra2, Proposition 7] the Hopf algebra D is semisimple as an algebra. Thus, as in Remark 4.6 in case of H, the functor $(-)^D : {}_D\mathfrak{M}_D \to \mathfrak{M}$ is exact (here ${}_D\mathfrak{M}_D$ is the category of D-bimodules and $\mathfrak{M}$ the category of $\Bbbk$-vector spaces). By Proposition 4.5 applied to the case when $r := (-)^D : {}_D\mathfrak{M}_D \to \mathfrak{M}$ and ω is the forgetful functor, for every morphism $f : X \to Y$ of D-bimodules one has

$$ \ker\left(f^D\right) = \ker(f) \cap X^D = (\ker(f))^D \qquad \text{and} \qquad \mathrm{Im}\left(f^D\right) = \mathrm{Im}(f) \cap Y^D = (\mathrm{Im}(f))^D. $$

In particular we get

$$ \begin{aligned} \ker\left((\partial^n)^D\right) &= \ker(\partial^n) \cap \mathrm{Hom}_\Bbbk\left(B^{\otimes n},\Bbbk\right)^D = \ker(\partial^n)^D \qquad \text{and} \\ \mathrm{Im}\left(\left(\partial^{n-1}\right)^D\right) &= \mathrm{Im}\left(\partial^{n-1}\right) \cap \mathrm{Hom}_\Bbbk\left(B^{\otimes n},\Bbbk\right)^D = \mathrm{Im}\left(\partial^{n-1}\right)^D \end{aligned} $$

and hence

$$ \begin{aligned} \mathrm{Z}^n_{\mathcal{YD}}(B,\Bbbk) &= \mathrm{Z}^n(B,\Bbbk) \cap \mathrm{Hom}_{D,-}\left(B^{\otimes n},\Bbbk\right) = \mathrm{Z}^n(B,\Bbbk)^D \qquad \text{and} \\ \mathrm{B}^n_{\mathcal{YD}}(B,\Bbbk) &= \mathrm{B}^n(B,\Bbbk) \cap \mathrm{Hom}_{D,-}\left(B^{\otimes n},\Bbbk\right) = \mathrm{B}^n(B,\Bbbk)^D \end{aligned} $$

Then we obtain

$$ \mathrm{H}^n_{\mathcal{YD}}(B,\Bbbk) = \frac{\mathrm{Z}^n_{\mathcal{YD}}(B,\Bbbk)}{\mathrm{B}^n_{\mathcal{YD}}(B,\Bbbk)} = \frac{\mathrm{Z}^n(B,\Bbbk)^D}{\mathrm{B}^n(B,\Bbbk)^D} \cong \mathrm{H}^n(B,\Bbbk)^D. $$

Let us prove the last part of the statement. The correspondence between the left D-module structure and the structure of Yetter-Drinfeld module over H is written explicitly in [Maj, Proposition 7.1.6]. In particular $D = H^* \otimes H$ and given $V \in {}_H^H\mathcal{YD}$, the two structures are related by the following equality $(f \otimes h) \rhd v = f\left((h \rhd v)_{-1}\right)(h \rhd v)_0$ for every $f \in H^*, h \in H, v \in V$. Thus $(\varepsilon_H \otimes h) \rhd v = h \rhd v$. Moreover H is a Hopf subalgebra of D via $h \mapsto \varepsilon_H \otimes h$, where D is considered with the first structure of [Maj, Theorem 7.1.1]. Since the D-bimodule structure of $\mathrm{H}^n(B,\Bbbk)$ is induced by the one of $\mathrm{Hom}_\Bbbk(B^{\otimes n},\Bbbk)$ which comes from the left D-module structures of its arguments and similarly for the H-bimodule structure of $\mathrm{H}^n(B,\Bbbk)$, we deduce that $\mathrm{H}^n(B,\Bbbk)^D$ is a subspace of $\mathrm{H}^n(B,\Bbbk)^H$. ∎

Example 4.8. In the setting of the proof of [An, Theorem 4.1.3], a Nichols algebra $\mathcal{B}(V)$ such that $\mathrm{H}^3(\mathcal{B}(V),\Bbbk)^{\mathbb{Z}_m}=0$ is considered where $\Bbbk$ is a field of characteristic zero. By Proposition 4.7 applied in the case $H=\Bbbk\mathbb{Z}_m$ and $B=\mathcal{B}(V)$, we have that $\mathrm{H}^3_{\mathcal{YD}}(\mathcal{B}(V),\Bbbk)\cong\mathrm{H}^3(\mathcal{B}(V),\Bbbk)^{D(H)}$ is a subspace of $\mathrm{H}^3(\mathcal{B}(V),\Bbbk)^H=\mathrm{H}^3(\mathcal{B}(V),\Bbbk)^{\mathbb{Z}_m}=0$. Thus we get $\mathrm{H}^3_{\mathcal{YD}}(\mathcal{B}(V),\Bbbk)=0$. Therefore, in view of Theorem 3.2, if $(Q,m,u,\Delta,\varepsilon,\omega)$ is a f.d. connected coquasi-bialgebra in ${}_H^H\mathcal{YD}$ such that $\mathrm{gr}Q\cong\mathcal{B}(V)$ (as above) as augmented algebras in ${}_H^H\mathcal{YD}$ (the counit must be the same in order to have the same Yetter-Drinfeld module structure on $\Bbbk$), then we can conclude that Q is gauge equivalent to a connected bialgebra in ${}_H^H\mathcal{YD}$.

Remark 4.9. Let A be a finite-dimensional coquasi-bialgebra with the dual Chevalley property i.e. the coradical H of A is a coquasi-subbialgebra of A (in particular H is cosemisimple). Assume the coquasi-bialgebra structure of H has trivial reassociator (i.e. it is an ordinary bialgebra) and also assume it has an antipode (i.e. it is a Hopf algebra). Then, by [AP, Corollary 6.4], $\mathrm{gr}A$ is isomorphic to $R\#H$ as a coquasi-bialgebra, where R is a suitable connected bialgebra in ${}_H^H\mathcal{YD}$. Note that $R\#H$ is the usual Radford-Majid bosonization as H has trivial reassociator, see [AP, Definition 5.4]. Hence we can compute

$$\mathrm{H}^3(\mathrm{gr}A,\Bbbk)=\mathrm{H}^3(R\#H,\Bbbk).$$

Assume further that H is semisimple. Then, by Corollary 4.3, we have

$$\mathrm{H}^n(R\#H,\Bbbk)\cong\mathrm{H}^n(R,\Bbbk)^H$$

so that $\mathrm{H}^3(\mathrm{gr}A,\Bbbk)\cong\mathrm{H}^3(R,\Bbbk)^H$. Thus, if $\mathrm{H}^3(R,\Bbbk)^H=0$, one gets $\mathrm{H}^3(\mathrm{gr}A,\Bbbk)=0$ which is the analogue of the condition [EG, Proposition 2.3] (note that our A is the dual of the one considered therein) which guarantees that A is gauge equivalent to an ordinary Hopf algebra, if A has an a quasi-antipode and $\Bbbk=\mathbb{C}$. Next we will give another approach to arrive at the same conclusion but just requiring $\mathrm{H}^3_{\mathcal{YD}}(R,\Bbbk)=0$. Note that a priori $\mathrm{H}^3_{\mathcal{YD}}(R,\Bbbk)\cong\mathrm{H}^3(R,\Bbbk)^{D(H)}$ is smaller than $\mathrm{H}^3(R,\Bbbk)^H$.

5. Dual Chevalley

The main aim of this section is to prove Theorem 5.6. Let A be a Hopf algebra over a field $\Bbbk$ of characteristic zero such that the coradical H of A is a sub-Hopf algebra (i.e. A has the dual Chevalley Property). Assume H is finite-dimensional so that H is semisimple. By [ABM, Theorem I], there is a gauge transformation $\zeta:A\otimes A\to\Bbbk$ such that A^ζ is isomorphic, as a coquasi-bialgebra, to the bosonization $Q\#H$ of a connected coquasi-bialgebra Q in ${}_H^H\mathcal{YD}$ by H. By construction ζ is H-bilinear and H-balanced: this follows from [ABM, Proposition 5.7] (note that gauge transformation $v_B:B\otimes B\to\Bbbk$, used therein for $B:=R\#_\xi H$, is H-bilinear and H-balanced, as observed in the proof) and the fact that there is an H-bilinear Hopf algebra isomorphism $\psi:B\to A$ (see [ABM, Proof of Theorem I, page 36 and Theorem 6.1] which is a consequence of [AMS1, Theorem 3.64]) where (R,ξ) is a suitable connected pre-bialgebra with cocycle in ${}_H^H\mathcal{YD}$ (note that $\zeta=v_B\circ(\psi^{-1}\otimes\psi^{-1})$): here by connected pre-bialgebra we mean that the coradical R_0 of R is $\Bbbk 1_R$ (by the properties of 1_R this implies that R_0 is a subcoalgebra in ${}_H^H\mathcal{YD}$ of R). Assume that A is finite-dimensional. Then $Q\#H$ and hence Q is finite dimensional.

Thus, by Theorem 3.2, if $\mathrm{H}^3_{\mathcal{YD}}(\mathrm{gr}Q,\Bbbk)=0$, then Q is gauge equivalent to a connected bialgebra in ${}_H^H\mathcal{YD}$.

First let us check which condition on A guarantee that $\mathrm{H}^3_{\mathcal{YD}}(\mathrm{gr}Q,\Bbbk)=0$. Note that by construction $Q=R^v$ (see [ABM, Proposition 5.7]) where $v:=(\lambda\xi)^{-1}$, the convolution inverse of $\lambda\xi$ and $\lambda:H\to\Bbbk$ denotes the total integral on H. Thus we can rewrite $\mathrm{gr}(Q)$ as $\mathrm{gr}(R^v)$.

Moreover v_B is given by $v_B((r\#h)\otimes(r'\#h'))=v(r\otimes hr')\varepsilon_H(h')$ for every $r,r'\in R,h,h'\in H$. By [AMStu, Proposition 2.5], $\mathrm{gr}(R)$ inherits the pre-bialgebra structure in ${}_H^H\mathcal{YD}$ of R. This is proved by checking that $R_i\cdot R_j\subseteq R_{i+j}$ for every $i,j\in\mathbb{N}_0$, where R_i denotes the i-th term of the coradical filtration of R. Moreover R_i is a subcoalgebra of R in ${}_H^H\mathcal{YD}$.

Lemma 5.1. *Keep the above hypotheses and notations. Then* $\mathrm{gr}(R^v)$ *and* $\mathrm{gr}(R)$ *coincide as bialgebras in* ${}_H^H\mathcal{YD}$ *where the structures of* $\mathrm{gr}(R)$ *are induced by the ones of* (R,ξ).

Proof. By Theorem 1.5, $\mathrm{gr}(R^v) = \mathrm{gr}(Q)$ is a connected bialgebras in ${}_H^H\mathcal{YD}$.

Note that R^v and R coincide as coalgebras in ${}_H^H\mathcal{YD}$ so that $\mathrm{gr}(R^v)$ and $\mathrm{gr}(R)$ coincide as coalgebras in ${}_H^H\mathcal{YD}$. They also have the same unit. It remains to check that their two multiplications coincide too.

Since ξ is unital, by [AMS1, Proposition 4.8], we have that v is unital and this is equivalent to v^{-1} unital (see the proof therein).

Let $C := R\otimes R$. Let $n>0$ and let $w\in C_{(n)} = \sum_{i+j\leq n} R_i\otimes R_j$. By [AMS1, Lemma 3.69], we have that

$$\Delta_C(w) - w\otimes(1_R)^{\otimes 2} - (1_R)^{\otimes 2}\otimes w \in C_{(n-1)}\otimes C_{(n-1)}.$$

Thus we get

$$w_1\otimes w_2\otimes w_3 - \Delta_C(w)\otimes(1_R)^{\otimes 2} - \Delta_C\left((1_R)^{\otimes 2}\right)\otimes w \in \Delta_C\left(C_{(n-1)}\right)\otimes C_{(n-1)}$$

and hence

$$w_1\otimes w_2\otimes w_3 - w\otimes(1_R)^{\otimes 2}\otimes(1_R)^{\otimes 2} - (1_R)^{\otimes 2}\otimes w\otimes(1_R)^{\otimes 2} - (1_R)^{\otimes 4}\otimes w \in C_{(n-1)}\otimes C_{(n-1)}\otimes C_{(n-1)}.$$

Since $m\left(C_{(n-1)}\right)\subseteq \sum_{i+j\leq n} m(R_i\otimes R_j)\subseteq R_{n-1}$ we get

$$w_1\otimes m(w_2)\otimes w_3 - w\otimes 1_R\otimes(1_R)^{\otimes 2} - (1_R)^{\otimes 2}\otimes m(w)\otimes(1_R)^{\otimes 2} - (1_R)^{\otimes 3}\otimes w \in C_{(n-1)}\otimes R_{n-1}\otimes C_{(n-1)}$$

and hence

$$w_1\otimes(m(w_2)+R_{n-1})\otimes w_3 = (1_R)^{\otimes 2}\otimes(m(w)+R_{n-1})\otimes(1_R)^{\otimes 2}. \tag{29}$$

Let $x,y\in R$. We compute

$$\begin{aligned}
\overline{x}\cdot_v\overline{y} &= \left(x+R_{|x|-1}\right)\cdot_v\left(y+R_{|y|-1}\right)\\
&= (x\cdot_v y)+R_{|x|+|y|-1} = m^v(x\otimes y)+R_{|x|+|y|-1}\\
&= v\left((x\otimes y)_1\right)m\left((x\otimes y)_2\right)v^{-1}\left((x\otimes y)_3\right)+R_{|x|+|y|-1}\\
&= v\left((x\otimes y)_1\right)\left(m\left((x\otimes y)_2\right)+R_{|x|+|y|-1}\right)v^{-1}\left((x\otimes y)_3\right)\\
&\overset{(29)}{=} v\left((1_R)^{\otimes 2}\right)\left(m(x\otimes y)+R_{|x|+|y|-1}\right)v^{-1}\left((1_R)^{\otimes 2}\right)\\
&= m(x\otimes y)+R_{|x|+|y|-1} = (x\cdot y)+R_{|x|+|y|-1} = \overline{x}\cdot\overline{y}.
\end{aligned}$$

□

The following result is inspired by [AMS1, Theorem 3.71].

Lemma 5.2. *Let* H *be a cosemisimple Hopf algebra. Let* C *be a left* H*-comodule coalgebra such that* C_0 *is a one-dimensional left* H*-comodule subcoalgebra of* C*. Let* $B=C\#H$ *be the smash coproduct of* C *by* H *i.e. the coalgebra defined by*

$$\begin{aligned}
\Delta_B(c\#h) &= \sum\left(c_1\#(c_2)_{\langle -1\rangle}h_1\right)\otimes\left((c_2)_{\langle 0\rangle}\#h_2\right),\\
\varepsilon_B(c\#h) &= \varepsilon_C(c)\,\varepsilon_H(h).
\end{aligned} \tag{30}$$

Then, for every $n\in\mathbb{N}_0$ *we have* $B_n = C_n\#H$.

Proof. Since C_0 is a subcoalgebra of C in ${}^H\mathfrak{M}$ and, for $n\geq 1$, one has $C_n = C_{n-1}\wedge_C C_0$, then inductively one proves that C_n is a subcoalgebra of C in ${}^H\mathfrak{M}$. Set $B_{(n)} := C_n\#H$ for every $n\in\mathbb{N}_0$. Let us check that $B_{(n)} = B_n$ by induction on $n\in\mathbb{N}_0$.

Let $n=0$. First note $B=\cup_{m\in\mathbb{N}_0}B_{(m)}$ and, since $\Delta_C(C_m)\subseteq\sum_{0\leq i\leq m}C_i\otimes C_{m-i}$, we also have

$$\begin{aligned}
\Delta_B\left(B_{(m)}\right) &= \Delta_B(C_m\#H)\subseteq\sum_{0\leq i\leq m}\sum\left(C_i\#(C_{m-i})_{\langle -1\rangle}(H)_1\right)\otimes\left((C_{m-i})_{\langle 0\rangle}\#(H)_2\right)\\
&\subseteq \sum_{0\leq i\leq m}(C_i\#H)\otimes(C_{m-i}\#(H)) = \sum_{0\leq i\leq m}B_{(i)}\otimes B_{(m-i)}.
\end{aligned}$$

Therefore $\left(B_{(m)}\right)_{m\in\mathbb{N}_0}$ is a coalgebra filtration for B and hence, by [Sw, Proposition 11.1.1], we get that $B_{(0)}\supseteq B_0$. Since C_0 is one-dimensional, there is a grouplike element $1_C\in C_0$ such that $C_0=\Bbbk 1_C$. Moreover one checks that C_0 is a subcoalgebra of C in ${}^{H}\mathfrak{M}$ implies $\sum(1_C)_{\langle -1\rangle}\otimes(1_C)_{\langle 0\rangle}=1_H\otimes 1_C$.

Let $\sigma:H\to C\otimes H:h\mapsto 1_C\otimes h$ be the canonical injection. We have

$$\begin{aligned}\Delta_B\sigma(h) &= \Delta_B(1_C\otimes h)=\sum\left(1_C\#(1_C)_{\langle -1\rangle}h_1\right)\otimes\left((1_C)_{\langle 0\rangle}\#h_2\right)\\ &= \sum(1_C\#1_Hh_1)\otimes(1_C\#h_2)=\sum\sigma(h_1)\otimes\sigma(h_2)=(\sigma\otimes\sigma)\Delta_H(h),\\ \varepsilon_B\sigma(h) &= \varepsilon_B(1_C\otimes h)=\varepsilon_C(1_C)\varepsilon_H(h)=\varepsilon_H(h)\end{aligned}$$

so that σ is a coalgebra map. Since H is cosemisimple and σ an injective coalgebra map we deduce that also $\sigma(H)=C_0\otimes H=B_{(0)}$ is a cosemisimple subcoalgebra of B whence $B_{(0)}\subseteq B_0$.

Let $n>0$ and assume that $B_i=B_{(i)}$ for $0\leq i\leq n-1$. Let $\sum\limits_{i\in I}c_i\#h_i\in B_n$. Then

$$\Delta_B\left(\sum_{i\in I}c_i\#h_i\right)\in B_{n-1}\otimes B+B\otimes B_0=C_{n-1}\otimes H\otimes C\otimes H+C\otimes H\otimes C_0\otimes H.$$

Let $p_n:C\to\frac{C}{C_n}$ be the canonical projection. If we apply $(p_{n-1}\otimes\varepsilon_H\otimes p_0\otimes H)$ we get

$$\begin{aligned}0 &= (p_{n-1}\otimes\varepsilon_H\otimes p_0\otimes H)\Delta_B\left(\sum_{i\in I}c_i\#h_i\right)\\ &= (p_{n-1}\otimes\varepsilon_H\otimes p_0\otimes H)\left(\sum_{i\in I}\left((c_i)_1\#((c_i)_2)_{\langle -1\rangle}(h_i)_1\right)\otimes\left(((c_i)_2)_{\langle 0\rangle}\#(h_i)_2\right)\right)\\ &= (p_{n-1}\otimes p_0\otimes H)\left(\sum_{i\in I}(c_i)_1\otimes(c_i)_2\otimes h_i\right)=((p_{n-1}\otimes p_0)\Delta_C\otimes H)\left(\sum_{i\in I}c_i\#h_i\right).\end{aligned}$$

Thus $\sum\limits_{i\in I}c_i\#h_i\in\ker((p_{n-1}\otimes p_0)\Delta_C\otimes H)=[\ker((p_{n-1}\otimes p_0)\Delta_C)]\otimes H=C_n\otimes H=B_{(n)}$. Thus $B_n\subseteq B_{(n)}$. On the other hand, form $\Delta_C(C_n)\subseteq C_{n-1}\otimes C+C\otimes C_0$ we deduce

$$\begin{aligned}\Delta_B\left(B_{(n)}\right) &= \Delta_B(C_n\otimes H)\\ &\subseteq \sum\left((C_n)_1\#((C_n)_2)_{\langle -1\rangle}(H)_1\right)\otimes\left(((C_n)_2)_{\langle 0\rangle}\#(H)_2\right)\\ &\subseteq \sum\left(C_{n-1}\#(C)_{\langle -1\rangle}H\right)\otimes\left((C)_{\langle 0\rangle}\#H\right)+\sum\left(C\#(C_0)_{\langle -1\rangle}H\right)\otimes\left((C_0)_{\langle 0\rangle}\#H\right)\\ &\subseteq (C_{n-1}\#H)\otimes(C\#H)+(C\#H)\otimes(C_0\#H)\\ &= B_{(n-1)}\otimes B+B\otimes B_{(0)}=B_{n-1}\otimes B+B\otimes B_0\end{aligned}$$

and hence $B_{(n)}\subseteq B_n$. □

Definition 5.3. Let A be a Hopf algebra over a field $\Bbbk$ such that the coradical H of A is a sub-Hopf algebra (i.e. A has the dual Chevalley Property). Set $G:=\mathrm{gr}(A)$. There are two canonical Hopf algebra maps

$$\begin{aligned}\sigma_G &: H\to\mathrm{gr}(A):h\mapsto h+A_{-1},\\ \pi_G &: \mathrm{gr}(A)\to H:a+A_{n-1}\mapsto a\delta_{n,0},\qquad n\in\mathbb{N}_0.\end{aligned}$$

The diagram of A (see [AS1, page 659]) is the vector space

$$\mathcal{D}(A):=\left\{d\in\mathrm{gr}(A)\mid\sum d_1\otimes\pi_G(d_2)=d\otimes 1_H\right\}.$$

It is a bialgebra in ${}_{H}^{H}\mathcal{YD}$ as follows. $\mathcal{D}(A)$ is a subalgebra of G. The left H-action, the left H-coaction of $\mathcal{D}(A)$, the comultiplication and counit are given respectively by

$$h\triangleright d:=\sum\sigma_G(h_1)d\sigma_GS(h_2),\qquad\rho(d)=\sum\pi_G(d_1)\otimes d_2,$$

$$\Delta_{D(A)}(d) := \sum d_1 \sigma_G S_H \pi_G(d_2) \otimes d_3, \qquad \varepsilon_{D(A)}(d) = \varepsilon_G(d).$$

Although the following result seems to be folklore, we include here its statement for future references.

Proposition 5.4. *Let A be a Hopf algebra over a field $\Bbbk$ such that the coradical H of A is a sub-Hopf algebra. Let A' be a Hopf algebra over a field $\Bbbk$. Let $f : A' \to A$ be an isomorphism of Hopf algebras. Then $H' := f^{-1}(H) \cong H$ is the coradical of A' and it is a sub-Hopf algebra of A'. Thus we can identify H' with H. Moreover f induces an isomorphism $\mathcal{D}(f) : \mathcal{D}(A') \to \mathcal{D}(A)$ of bialgebras in ${}^H_H\mathcal{YD}$.*

Proposition 5.5. *Keep the hypotheses and notations of the beginning of the section. Then $\mathcal{D}(A) \cong \mathcal{D}(R\#_\xi H) \cong \mathrm{gr}(R)$ as bialgebras in ${}^H_H\mathcal{YD}$.*

Proof. Apply Proposition 5.4 to the canonical isomorphism $\psi : B := R\#_\xi H \to A$ that we recalled at the beginning of the section to get that $\mathcal{D}(R\#_\xi H) \cong \mathcal{D}(A)$. Note that, by H-linearity we have

$$\psi(1_R\#h) = \psi((1_R\#1_H)(1_R\#h)) = \psi((1_R\#1_H)h) = \psi(1_R\#1_H)h = h$$

so that $\psi(\Bbbk 1_R \otimes H) = H$ and hence $H' = \psi^{-1}(H) = \Bbbk 1_R \otimes H$ with the notation of Proposition 5.4. Thus $B_0 = \Bbbk 1_R \otimes H = R_0 \otimes H$ so that we can identify B_0 with H via the canonical isomorphism $H \to R_0 \otimes H : h \mapsto 1_R \otimes h$. Its inverse is $R_0 \otimes H \to H : r \otimes h \mapsto \varepsilon_R(r)h$. With this identification and by setting $G := \mathrm{gr}(B)$, we can consider the canonical bialgebra maps

$$\begin{aligned} \sigma_G &: H \to \mathrm{gr}(B) : h \mapsto 1_R\#h + (R\#_\xi H)_{-1}, \\ \pi_G &: \mathrm{gr}(B) \to H : r\#h + (R\#_\xi H)_{n-1} \mapsto \varepsilon_R(r) h \delta_{n,0}, \text{ where } r\#h \in (R\#_\xi H)_n, n \in \mathbb{N}_0. \end{aligned}$$

Since the underlying coalgebra of B is exactly the smash coproduct of R by H and (R, ξ) is a connected pre-bialgebra with cocycle in ${}^H_H\mathcal{YD}$, by Lemma 5.2, we have that $B_n = R_n \otimes H$. Let us compute $\mathcal{D} := \mathcal{D}(B)$. As a vector space it is

$$\mathcal{D} := \left\{ d \in G \mid \sum d_1 \otimes \pi_G(d_2) = d \otimes 1_H \right\}.$$

By [AS1, Lemma 2.1], we have that $\mathcal{D} = \oplus_{n\in\mathbb{N}_0} \mathcal{D}^n$ where $\mathcal{D}^n = \mathcal{D} \cap G^n = \mathcal{D} \cap \frac{B_n}{B_{n-1}}$. Let $d := \overline{\sum_{i\in I} r_i\#h_i} \in \mathcal{D}^n$ where we can assume $\sum_{i\in I} r_i\#h_i \in B_n \backslash B_{n-1}$ and, for every $i \in I$, $r_i\#h_i \in B_n \backslash B_{n-1}$. Then $\overline{\sum_{i\in I} r_i\#h_i} = \sum_{i\in I} \overline{r_i\#h_i}$ and hence the fact that d is coinvariant rewrites as

$$\sum_{i\in I} \left(\overline{r_i\#h_i}\right)_1 \otimes \pi_G\left(\left(\overline{r_i\#h_i}\right)_2\right) = \sum_{i\in I} \overline{r_i\#h_i} \otimes 1_H. \tag{31}$$

By definition of π_G and (1), the left-hand side becomes

$$\sum_{i\in I} \left(\overline{r_i\#h_i}\right)_1 \otimes \pi_G\left(\left(\overline{r_i\#h_i}\right)_2\right) = \sum_{i\in I} \left(\left(r_i\#(h_i)_1\right) + B_{n-1}\right) \otimes (h_i)_2$$

so that (31) becomes

$$\sum_{i\in I} \left(\left(r_i\#(h_i)_1\right) + B_{n-1}\right) \otimes (h_i)_2 = \sum_{i\in I} \overline{r_i\#h_i} \otimes 1_H = \sum_{i\in I} \left(r_i\#h_i + B_{n-1}\right) \otimes 1_H$$

i.e.

$$\sum_{i\in I} \left(r_i\#(h_i)_1\right) \otimes (h_i)_2 - \sum_{i\in I} r_i\#h_i \otimes 1_H \in B_{n-1} \otimes H = R_{n-1} \otimes H \otimes H.$$

If we apply $R \otimes \varepsilon_H \otimes H$, we get

$$\sum_{i\in I} r_i \otimes h_i - \sum_{i\in I} r_i \varepsilon_H(h_i) \otimes 1_H \in R_{n-1} \otimes H = B_{n-1}.$$

Thus $\overline{\sum_{i\in I} r_i\#h_i} = \sum_{i\in I} \overline{r_i\#h_i} = \sum_{i\in I} \left(r_i\#h_i + B_{n-1}\right) = \sum_{i\in I} \left(r_i \varepsilon_H(h_i) \otimes 1_H\right) + B_{n-1}$.

Since $\sum_{i\in I} r_i\#h_i \in B_n\backslash B_{n-1}$ we get that $\left(\sum_{i\in I} r_i\varepsilon_H(h_i)\right)\otimes 1_H \notin B_{n-1}$ and hence $\sum_{i\in I} r_i\varepsilon_H(h_i) \notin R_{n-1}$ and we can write

$$\overline{\sum_{i\in I} r_i\#h_i} = \overline{\left(\sum_{i\in I} r_i\varepsilon_H(h_i)\right)\otimes 1_H}.$$

Therefore we have proved that the map

$$\varphi_n : \frac{R_n}{R_{n-1}} \to \mathcal{D}^n : \overline{r} \mapsto \overline{r\otimes 1_H},$$

which is well-defined as $\mathcal{D}^n = \mathcal{D}\cap G^n = \mathcal{D}\cap \frac{B_n}{B_{n-1}} = \mathcal{D}\cap \frac{R_n\otimes H}{R_{n-1}\otimes H}$, is also surjective.

It is also injective as $\varphi_n(\overline{r}) = \varphi_n(\overline{s})$ implies $r\otimes 1_H - s\otimes 1_H \in B_{n-1} = R_{n-1}\otimes H$ and hence, by applying $R\otimes\varepsilon_H$, we get $r-s\in R_{n-1}$ i.e. $\overline{r}=\overline{s}$. Therefore φ_n is an isomorphism such that $\overline{\sum_{i\in I} r_i\#h_i} = \varphi_n\left(\overline{\sum_{i\in I} r_i\varepsilon_H(h_i)}\right)$ and hence

$$\varphi_n^{-1}\left(\overline{\sum_{i\in I} r_i\#h_i}\right) = \overline{\sum_{i\in I} r_i\varepsilon_H(h_i)}.$$

Clearly this extends to a graded $\Bbbk$-linear isomorphism

$$\varphi : \mathrm{gr}(R) \to \mathcal{D}.$$

Let us check that φ is a morphism in ${}^H_H\mathcal{YD}$. First note that, for every $r\in R_n$, we have

$$\begin{aligned}\varphi(r+R_{n-1}) &= \delta_{|r|,n}\varphi(r+R_{n-1}) = \delta_{|r|,n}\varphi_n(r+R_{n-1}) = \delta_{|r|,n}\varphi_n(\overline{r})\\ &= \delta_{|r|,n}\overline{r\otimes 1_H} = \delta_{|r|,n}\left(r\otimes 1_H + (R\#_\xi H)_{n-1}\right) = r\otimes 1_H + (R\#_\xi H)_{n-1}.\end{aligned}$$

Thus

$$\varphi(r+R_{n-1}) = r\otimes 1_H + (R\#_\xi H)_{n-1}, \text{ for every } r\in R_n. \tag{32}$$

For every $r\in R_n\backslash R_{n-1}$, by using (32), it is straighforward to prove that $h\rhd\varphi(\overline{r}) = \varphi(h\overline{r})$.

Moreover, by applying (1), (30), the definition of π_G and (32), we get that $\rho\varphi(\overline{r}) = (H\otimes\varphi)\rho(\overline{r})$.

Let us check that φ is a morphism of bialgebras in ${}^H_H\mathcal{YD}$. Fix $r\in R_n\backslash R_{n-1}$.

Using the definition of $\Delta_{\mathcal{D}}$, (1), (30), the definition of π_G, the definition of σ_G, (32) and (1) again, we obtain $\Delta_{\mathcal{D}}\varphi(\overline{r}) = (\varphi\otimes\varphi)\Delta_{\mathrm{gr}(R)}(\overline{r})$.

Let us check φ is counitary:

$$\begin{aligned}\varepsilon_{\mathcal{D}}\varphi(\overline{r}) &= \varepsilon_G\varphi(\overline{r}) = \varepsilon_G\left(\overline{r\otimes 1_H}\right) \overset{(2)}{=} \delta_{n,0}\varepsilon_B(r\otimes 1_H)\\ &= \delta_{n,0}\varepsilon_R(r) \overset{(2)}{=} \varepsilon_{\mathrm{gr}(R)}(\overline{r}).\end{aligned}$$

Let us check φ is multiplicative. Let $s\in R_m\backslash R_{m-1}$. Then, by definition of φ, of $m_{\mathcal{D}}$ and of the multiplication of $R\#_\xi H$, we have that

$$m_{\mathcal{D}}(\varphi\otimes\varphi)(\overline{s}\otimes\overline{r}) = \sum\left(s^{(1)}\left(\left(s^{(2)}\right)_{\langle -1\rangle} r^{(1)}\right)\#\xi\left(\left(s^{(2)}\right)_{\langle 0\rangle}\otimes r^{(2)}\right)\right) + (R\#_\xi H)_{m+n-1}.$$

Now write $\sum s^{(1)}\otimes s^{(2)} = \sum_{0\leq i\leq m} s_i\otimes s'_{m-i}$ for some $s_i, s'_i\in R_i$ and similarly $\sum r^{(1)}\otimes r^{(2)} = \sum_{0\leq j\leq n} r_j\otimes r'_{n-j}$ for some $r_j, r'_j\in R_j$. Then

$$\begin{aligned}m_{\mathcal{D}}(\varphi\otimes\varphi)(\overline{s}\otimes\overline{r}) &= \sum\nolimits_{\substack{0\leq i\leq m\\0\leq j\leq n}}\left(s_i\left(\left(s'_{m-i}\right)_{\langle -1\rangle} r_j\right)\#\xi\left(\left(s'_{m-i}\right)_{\langle 0\rangle}\otimes r'_{n-j}\right)\right) + (R\#_\xi H)_{m+n-1}\\ &= \sum\nolimits_{\substack{0\leq i\leq m\\0\leq j\leq n}}\delta_{i,m}\delta_{j,n}\left(s_i\left(\left(s'_{m-i}\right)_{\langle -1\rangle} r_j\right)\#\xi\left(\left(s'_{m-i}\right)_{\langle 0\rangle}\otimes r'_{n-j}\right)\right) + (R\#_\xi H)_{m+n-1}\\ &= \sum\left(s_m\left((s'_0)_{\langle -1\rangle} r_n\right)\#\xi\left((s'_0)_{\langle 0\rangle}\otimes r'_0\right)\right) + (R\#_\xi H)_{m+n-1}\\ &\overset{R_0=\Bbbk 1_R}{=} \sum s_m\left((s'_0)_{\langle -1\rangle} r_n\right)\#\varepsilon_R\left((s'_0)_{\langle 0\rangle}\right)\varepsilon_R(r'_0)1_H + (R\#_\xi H)_{m+n-1}\end{aligned}$$

$$\begin{aligned}
&= \sum s_m\varepsilon_R(s'_0)\, r_n\varepsilon_R(r'_0)\,\#1_H + (R\#_\xi H)_{m+n-1}\\
&= \sum\nolimits_{\substack{0\le i\le m\\ 0\le j\le n}} \delta_{i,m}\delta_{j,n}\left(s_i\varepsilon_R\left(s'_{m-i}\right) r_j\varepsilon_R\left(r'_{m-j}\right)\#1_H\right) + (R\#_\xi H)_{m+n-1}\\
&= \sum\nolimits_{\substack{0\le i\le m\\ 0\le j\le n}} \left(s_i\varepsilon_R\left(s'_{m-i}\right) r_j\varepsilon_R\left(r'_{m-j}\right)\#1_H\right) + (R\#_\xi H)_{m+n-1}\\
&= \sum \left(s^{(1)}\varepsilon_R\left(s^{(2)}\right) r^{(1)}\varepsilon_R\left(r^{(2)}\right)\#1_H\right) + (R\#_\xi H)_{m+n-1}\\
&= (sr\#1_H) + (R\#_\xi H)_{m+n-1} \overset{(32)}{=} \varphi\left(sr + R_{m+n-1}\right)\\
&= \varphi\left((s+R_{m-1})(r+R_{n-1})\right) = \varphi m_{\mathrm{gr}(R)}\left(\overline{s}\otimes\overline{r}\right).
\end{aligned}$$

Let us check φ is unitary. We have

$$\varphi\left(1_{\mathrm{gr}(R)}\right) = \varphi\left(1_R + R_{-1}\right) = \varphi\left(\overline{1_R}\right) = \overline{1_R\otimes 1_H} = (1_R\otimes 1_H) + (R\#_\xi H)_{-1} = 1_B + B_{-1} = 1_G.$$

□

Summing up we have proved that

$$\mathrm{gr}(Q) \overset{Q=R^v}{=} \mathrm{gr}(R^v) \overset{\text{Lem. 5.1}}{\cong} \mathrm{gr}(R) \overset{\text{Pro. 5.5}}{\cong} \mathcal{D}(R\#_\xi H) \overset{\text{Pro. 5.4}}{\cong} \mathcal{D}(A)$$

as bialgebras in ${}^H_H\mathcal{YD}$. Therefore $\mathrm{H}^3_{\mathcal{YD}}(\mathcal{D}(A),\Bbbk) = 0$ (the Hochschild cohomology in ${}^H_H\mathcal{YD}$ of the algebra $\mathcal{D}(A)$ with values in $\Bbbk$) if, and only if, $\mathrm{H}^3_{\mathcal{YD}}(\mathrm{gr}Q,\Bbbk) = 0$. In this case, by the foregoing, we get that Q is gauge equivalent to a connected bialgebra in ${}^H_H\mathcal{YD}$.

Now let E be a connected bialgebra in ${}^H_H\mathcal{YD}$ and let $\gamma : E\otimes E\to\Bbbk$ be a gauge transformation in ${}^H_H\mathcal{YD}$ such that $Q = E^\gamma$. We proved that $A^\zeta \cong Q\#H \cong E^\gamma\#H$ as coquasi-bialgebras. By Proposition 2.5, we have that $(E\#H)^\Gamma = E^\gamma\#H$ as an ordinary coquasi-bialgebras. Recall that two coquasi-bialgebras A and A' are called **gauge equivalent** or **quasi-isomorphic** whenever there is some gauge transformation $\gamma : Q\otimes Q\to\Bbbk$ in $\mathbf{Vec}_\Bbbk$ such that $A^\gamma \cong A'$ as coquasi-bialgebras. We point out that, if A and A' are ordinary bialgebras and $A^\gamma\cong A'$, then γ comes out to be a unitary cocycle. This is encoded in the triviality of the reassociators of A and A'.

Theorem 5.6. *Let A be a finite-dimensional Hopf algebra over a field $\Bbbk$ of characteristic zero such that the coradical H of A is a sub-Hopf algebra (i.e. A has the dual Chevalley Property). If $\mathrm{H}^3_{\mathcal{YD}}(\mathcal{D}(A),\Bbbk) = 0$, then A is quasi-isomorphic to the Radford-Majid bosonization $E\#H$ of some connected bialgebra E in ${}^H_H\mathcal{YD}$ by H. Moreover $\mathrm{gr}(E)\cong\mathcal{D}(A)$ as bialgebras in ${}^H_H\mathcal{YD}$.*

Proof. By the foregoing $A^\zeta \cong Q\#H \cong E^\gamma\#H = (E\#H)^\Gamma$ as coquasi-bialgebras. Now A is quasi-isomorphic to A^ζ which is quasi-isomorphic to $E\#H$ so that A is quasi-isomorphic to $E\#H$. Moreover

$$\mathrm{gr}(E) = \mathrm{gr}(E^\gamma) = \mathrm{gr}(Q) \cong \mathcal{D}(A).$$

where the first equality holds by Proposition 2.6.

□

More generally, given A a (finite-dimensional) Hopf algebra whose coradical H is a sub-Hopf algebra, then if H is also semisimple, we expect that A is quasi-isomorphic to the Radford-Majid bosonization $E\#H$ of some connected bialgebra E in ${}^H_H\mathcal{YD}$ by H. See e.g. [GM, Corollary 3.4 and the proof therein] and [AAGMV, AAG] for a further clue in this direction.

6. Examples

We notice that the Hochschild cohomology of a finite-dimensional Nichols algebras has been computed in few examples. We consider here those Nichols algebras to compute $\mathrm{H}^3_{\mathcal{YD}}(\mathcal{B}(V),\Bbbk)$.

6.1. **Braidings of Cartan type.** Let $A = (a_{ij})_{1\le i,j\le\theta}$ be a finite Cartan matrix, Δ the corresponding root system, $(\alpha_i)_{1\le i\le\theta}$ a set of simple roots and W its Weyl group. Let $w_0 = s_{i_1}\cdots s_{i_M}$ be a reduced expression of the element $w_0 \in W$ of maximal length as a product of simple reflections, $\beta_j = s_{i_1}\cdots s_{i_{j-1}}(\alpha_{i_j})$, $1 \le j \le M$. Then $\beta_j \ne \beta_k$ if $j \ne k$ and $\Delta^+ = \{\beta_j | 1 \le j \le M\}$, see [H, page 25 and Proposition 3.6].

Let Γ be a finite abelian group, $\widehat{\Gamma}$ its group of characters. $\mathcal{D} = (\Gamma, (g_i)_{1\le i\le\theta}, (\chi_i)_{1\le i\le\theta}, A)$ is a *datum of finite Cartan type* [AS2] associated to Γ and A if $g_i \in \Gamma$, $\chi_j \in \widehat{\Gamma}$, $1 \le i,j \le \theta$, satisfy $\chi_i(g_i) \ne 1$, $\chi_i(g_j)\chi_j(g_i) = \chi_i(g_i)^{a_{ij}}$ for all i,j. Set $\mathfrak{q} = (q_{ij})_{1\le i,j\le\theta}$, where $q_{ij} = \chi_j(g_i)$.

In what follows V denotes the Yetter-Drinfeld module over $\Bbbk\Gamma$, $\dim V = \theta$, with a fixed basis $x_1,\dots,x_\theta$, where the action and the coaction over each x_i is given by χ_i and g_i, respectively. Then the associated braiding is $c(x_i \otimes x_j) = q_{ij}x_j \otimes x_i$ for all i,j. Let $\mathcal{B}_{\mathfrak{q}} = \mathcal{B}(V)$. The tensor algebra $T(V)$ is $\mathbb{N}_0^\theta$-graded with grading α_i for each x_i. For $\beta = \sum_{i=1}^{\theta} a_i\alpha_i \in \Delta^+$, set

$$g_\beta = g_1^{a_1}\cdots g_\theta^{a_\theta}, \qquad \chi_\beta = \chi_1^{a_1}\cdots\chi_\theta^{a_\theta}, \qquad q_\beta = \chi_\beta(g_\beta).$$

Given $\alpha,\beta \in \Delta^+$, we denote $q_{\alpha\beta} = \chi_\beta(g_\alpha)$.

We assume as in [AS2, MPSW] that *the order of q_{ii} is odd for all i, and not divisible by 3 for each connected component of the Dynkin diagram of A of type G_2*. Therefore the order of q_{ii} is the same for all the i in the same connected component J. Given $\beta \in J$, we denote by N_β the order of the corresponding q_{ii} in J, which is also the order of q_β.

By [L] there exist homogeneous elements x_β of degree β, $\beta \in \Delta^+$, such that the Nichols algebra $\mathcal{B}_{\mathfrak{q}}$ of V is presented by generators $x_1,\dots,x_\theta$ and relations

$$(\operatorname{ad}_c x_i)^{1-a_{ij}}x_j = 0, \qquad 1 \le i \ne j \le \theta;$$
$$x_\beta^{N_\beta} = 0, \qquad \beta \in \Delta_+.$$

Moreover $\{x_{\beta_1}^{n_1}\dots x_{\beta_M}^{n_M} | 0 \le n_i < N_{\beta_i}\}$ is a basis of $\mathcal{B}_{\mathfrak{q}}$.

We shall prove that $\mathrm{H}^3_{\mathcal{YD}}(\mathcal{B}_{\mathfrak{q}}, \Bbbk) = 0$. We need first some technical results.

Lemma 6.1. *Let $\alpha,\beta \in \Delta_+$. Then either $g_\alpha g_\beta^{N_\beta} \ne e$, or else $\chi_\alpha\chi_\beta^{N_\beta} \ne \epsilon$.*

Proof. Suppose on the contrary that $g_\alpha g_\beta^{N_\beta} = e$, $\chi_\alpha\chi_\beta^{N_\beta} = \epsilon$. Then

$$q_\alpha = \chi_\alpha^{-1}(g_\alpha^{-1}) = \chi_\beta^{N_\beta}(g_\beta^{N_\beta}) = q_\beta^{N_\beta^2} = 1,$$

since q_β is a root of unity of order N_β. But this is a contradiction, since $q_\alpha \ne 1$. □

Lemma 6.2. *Let $\alpha,\beta,\gamma \in \Delta^+$ be pairwise different. Then either $g_\alpha g_\beta g_\gamma \ne e$, or else $\chi_\alpha\chi_\beta\chi_\gamma \ne \epsilon$.*

Proof. Suppose on the contrary that $g_\alpha g_\beta g_\gamma = e$ and $\chi_\alpha\chi_\beta\chi_\gamma = \epsilon$. Then

$$q_\alpha = \chi_\alpha^{-1}(g_\alpha^{-1}) = \chi_\beta\chi_\gamma(g_\beta g_\gamma) = q_\beta q_\gamma q_{\beta\gamma}q_{\gamma\beta}, \quad q_\beta = q_\alpha q_\gamma q_{\alpha\gamma}q_{\gamma\alpha}, \quad q_\gamma = q_\alpha q_\beta q_{\alpha\beta}q_{\beta\alpha}. \tag{33}$$

Notice that α,β,γ belong to the same connected component. Indeed, if γ belongs to a different connected component, then $q_{\beta\gamma}q_{\gamma\beta} = q_{\alpha\gamma}q_{\gamma\alpha} = 1$. Thus $q_\beta = q_\alpha q_\gamma = q_\beta q_\gamma^2$, so $q_\gamma^2 = 1$, which is a contradiction. Therefore we may assume that the Dynkin diagram is connected.

One can prove that $q_{s_i(\alpha)} = q_\alpha$ for every $\alpha \in \Delta$. As we observed that $\Delta^+ = \{\beta_j | 1 \le j \le M\}$, we deduce that for every $\beta \in \Delta^+$ there is some j such that $q_\beta = q_j$. One can prove that there is some $q \in \Bbbk$ such that $q_\alpha = q^{(\alpha,\alpha)/2}$ and $q_{\alpha\gamma}q_{\gamma\alpha} = q^{(\alpha,\gamma)}$, where $(\cdot,\cdot)$ is the invariant bilinear form on the simple Lie algebra $\mathfrak{g}$ associated with the finite Cartan matrix [Bo, Ch. VI, §1, Proposition 3 and Definition 3] and the basis of the root systems given in [Bo, Ch. VI, §4] should be normalized in such a way that $q = q_\delta$, $(\delta,\delta) = 2$ for each short root $\delta \in \Delta$. Note that $q_\alpha = q^{(\alpha,\alpha)/2} \ne 1$ for all α as $(\alpha,\alpha) \ne 0$. Thus

- $q_\alpha = q_\beta = q_\gamma = q$ if the Dynkin diagram is simply laced,
- $q_\alpha, q_\beta, q_\gamma \in \{q, q^2\}$ if the Dynkin diagram has a double arrow,
- $q_\alpha, q_\beta, q_\gamma \in \{q, q^3\}$ if the Dynkin diagram is of type G_2.

If the Dynkin diagram is simply laced, then, by (33), we have $q_{\beta\gamma}q_{\gamma\beta} = q_{\alpha\gamma}q_{\gamma\alpha} = q_{\alpha\beta}q_{\beta\alpha} = q^{-1}$. Then $q^{(\alpha,\gamma)} = q^{-1}$. Now set $n(\alpha,\beta) := 2(\alpha,\beta)/(\beta,\beta) = (\alpha,\beta)$. Then $n(\alpha,\beta)$ is symmetric whence, by [Bo, Ch. VI, §1, page 148] we have $(\alpha,\gamma) = -1$ as the order of q is odd, so $\alpha+\gamma \in \Delta^+$, by [Bo, VI, §1, Corollary, page 149]. Now the same argument we used above shows that also $(\alpha,\beta) = -1 = (\gamma,\beta)$ and hence $(\alpha+\gamma,\beta) = -2$, so $\alpha+\beta+\gamma \in \Delta^+$, since $\alpha+\gamma \neq -\beta$ (as $\alpha+\gamma$ and β are both in Δ^+). But $q_{\alpha+\beta+\gamma} = q_\alpha q_\beta q_\gamma q_{\beta\gamma}q_{\gamma\beta}q_{\alpha\gamma}q_{\gamma\alpha}q_{\alpha\beta}q_{\beta\alpha} = 1$, which is a contradiction.

If the Dynkin diagram has a double arrow, then $q_\alpha,\ q_\beta,\ q_\gamma \in \{q, q^2\}$. If $q_\alpha = q_\beta = q_\gamma$, then the proof follows as for the simply-laced case because $n(u,v) = n(v,u)$ for $u,v \in \{\alpha,\beta,\gamma\}$. If $q_\alpha = q_\beta = q$ and $q_\gamma = q^2$, then $q_{\beta\gamma}q_{\gamma\beta} = q_{\alpha\gamma}q_{\gamma\alpha} = q^{-2}$, and $q_{\alpha\beta}q_{\beta\alpha} = 1$, by (33). Then a simple calculation yields $(\beta,\gamma) = -2$ so that $\beta+\gamma \in \Delta^+$. One also gets $(\alpha,\beta) = 0$ and $(\alpha,\gamma) = -2$ so that $(\alpha,\beta+\gamma) = (\alpha,\beta)+(\alpha,\gamma) = -2 < 0$ by the conditions on the order of q, so again $\alpha+\beta+\gamma \in \Delta^+$; but again we obtain $q_{\alpha+\beta+\gamma} = 1$, which is a contradiction. The proof for $q_\alpha = q_\beta = q^2$ and $q_\gamma = q$ follows analogously.

Finally, if the Dynkin diagram is of type G_2, then a similar analysis gives a contradiction. □

For each $1 \le k \le M$, set $\mathcal{B}_\mathfrak{q}(k)$ as the subspace of $\mathcal{B}_\mathfrak{q}$ spanned by $\{x_{\beta_1}^{n_1}\dots x_{\beta_k}^{n_k} | 0 \le n_i < N_{\beta_i}\}$. By [DP] this gives an algebra filtration, and the graded algebra $\operatorname{Gr}\mathcal{B}_\mathfrak{q}$ associated to this filtration is presented by generators $\mathbf{x}_\beta$, $\beta \in \Delta^+$, and relations

$$\mathbf{x}_\beta\mathbf{x}_\gamma = q_{\beta\gamma}\mathbf{x}_\gamma\mathbf{x}_\beta, \qquad \mathbf{x}_\beta^{N_\beta} = 0, \qquad \beta < \gamma \in \Delta_+.$$

In [MPSW] $\operatorname{Gr}\mathcal{B}_\mathfrak{q}$ is viewed as an algebra in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$, which (as an algebra) is the Nichols algebra of Cartan type $A_1 \times \cdots \times A_1$, M copies, with action and coaction on $\mathbf{x}_\beta$ given by χ_β, g_β, respectively. By [MPSW, Theorem 4.1], $\mathrm{H}^\bullet(\operatorname{Gr}\mathcal{B}_\mathfrak{q},\Bbbk)$ is the algebra generated by ξ_β, η_β, $\beta \in \Delta^+$, where $\deg\xi_\beta = 2$, $\deg\eta_\beta = 1$, and relations

$$\xi_\beta\xi_\gamma = q_{\beta\gamma}^{N_\beta N_\gamma}\xi_\gamma\xi_\beta, \qquad \eta_\beta\xi_\gamma = q_{\beta\gamma}^{N_\gamma}\xi_\gamma\eta_\beta, \qquad \eta_\beta\eta_\gamma = -q_{\beta\gamma}\eta_\gamma\eta_\beta, \qquad \beta,\gamma \in \Delta^+.$$

As we assume that all the q_{ii} have odd order, we deduce in particular from the last equality that $\eta_\beta^2 = 0$ for all $\beta \in \Delta^+$. As an algebra in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$, the action and coaction on ξ_β is given by $\chi_\beta^{-N_\beta}$, $g_\beta^{-N_\beta}$, while the action and coaction on η_β is given by χ_β^{-1}, g_β^{-1}.

Theorem 6.3. $\mathrm{H}^3_{\mathcal{YD}}(\mathcal{B}_\mathfrak{q},\Bbbk) = 0$.

Proof. First we will prove that $\mathrm{H}^3(\operatorname{Gr}\mathcal{B}_\mathfrak{q},\Bbbk)^D = 0$ for $D := D(\Bbbk\Gamma)$. Now, the invariants are with respect to the D-bimodule structure that $\mathrm{H}^3(\operatorname{Gr}\mathcal{B}_\mathfrak{q},\Bbbk)$ inherits from $\operatorname{Hom}\left((\operatorname{Gr}\mathcal{B}_\mathfrak{q})^{\otimes 3},\Bbbk\right)$ (this is a D-bimodule as its arguments are left D-modules). Since the left D-module structure is induced by the one of $\Bbbk$, it is trivial. Thus the invariants of $\mathrm{H}^3(\operatorname{Gr}\mathcal{B}_\mathfrak{q},\Bbbk)$ as a D-bimodule reduce to the its invariants as a right D-module. Since right D-modules are equivalent to left D-modules, via the antipode of D which is invertible as D is finite-dimensional, the right D-module structure of $\mathrm{H}^3(\operatorname{Gr}\mathcal{B}_\mathfrak{q},\Bbbk)$ becomes the structure of object in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ described above. Thus, in order to prove that $\mathrm{H}^3(\operatorname{Gr}\mathcal{B}_\mathfrak{q},\Bbbk)^D = 0$ we just have to check that the invariants of $\mathrm{H}^3(\operatorname{Gr}\mathcal{B}_\mathfrak{q},\Bbbk)$ as a left-left Yetter-Drinfeld modules are zero.

Now, by the defining relations of $\mathrm{H}^\bullet(\operatorname{Gr}\mathcal{B}_\mathfrak{q},\Bbbk)$, a basis B of $\mathrm{H}^3(\operatorname{Gr}\mathcal{B}_\mathfrak{q},\Bbbk)$ is given by $\{\xi_\alpha\eta_\beta\} \cup \{\eta_\alpha\eta_\beta\eta_\gamma | \alpha < \beta < \gamma\}$. If $v \in \mathrm{H}^3(\operatorname{Gr}\mathcal{B}_\mathfrak{q},\Bbbk)$ is invariant, then v is written as a linear combination of elements in the trivial component. Indeed, write $v = \sum_{b\in B} c_b\, b$ for some $c_b \in \Bbbk$, and let g_b, χ_b be the elements describing the component of $b \in B$. Then

$$v = g\cdot v = \sum\nolimits_{b\in B} c_b\, g\cdot b = \sum\nolimits_{b\in B} c_b\chi_b(g)\, b, \qquad \text{for all } g \in \Gamma,$$
$$1 \otimes v = \rho(v) = \sum\nolimits_{b\in B} c_b\,\rho\cdot b = \sum\nolimits_{b\in B} c_b g_b \otimes b.$$

If $c_b \neq 0$, then $\chi_b(g) = 1$ for all $g \in \Gamma$ so $\chi_b = \epsilon$, and $g_b = 1$. Thus b is invariant. We have so proved that the existence of $v \neq 0$ invariant implies the existence of $b \in B$ invariant. Hence, if B has no invariant element then there is no invariant element at all. Note that, for all $h \in H$, we have $h\cdot(\xi_\alpha\eta_\beta) = (\chi_\alpha^{-N_\alpha}\chi_\beta^{-1})(h)\xi_\alpha\eta_\beta$ and $\rho(\xi_\alpha\eta_\beta) = g_\alpha^{-N_\alpha}g_\beta^{-1}\otimes\xi_\alpha\eta_\beta$ so that, by Lemma 6.1, the element

$\xi_\alpha\eta_\beta$ is not D-invariant. A similar argument, using Lemma 6.2, shows that also $\eta_\alpha\eta_\beta\eta_\gamma$ is not D-invariant. Thus the elements in B are not D-invariant, so $\mathrm{H}^3\left(\mathrm{Gr}\,\mathcal{B}_{\mathfrak{q}}, \Bbbk\right)^D = 0$. Since the elements in $\{x_{\beta_1}^{n_1}\dots x_{\beta_k}^{n_k} | 0 \leq n_i < N_{\beta_i}\}$ are eigenvectors for D, we can mimic the argument in [MPSW, Section 5] by taking into account the spectral sequence associated to the filtration of algebras therein; see for example [MPSW, Corollary 5.5] for a similar argument. Thus $\mathrm{H}^3_{\mathcal{YD}}\left(\mathcal{B}_{\mathfrak{q}}, \Bbbk\right) \cong \mathrm{H}^3\left(\mathcal{B}_{\mathfrak{q}}, \Bbbk\right)^D = 0$. □

Remark 6.4. Notice that $\mathrm{H}^3_{\mathcal{YD}}\left(\mathcal{B}_{\mathfrak{q}}, \Bbbk\right) \cong \mathrm{H}^3\left(\mathcal{B}_{\mathfrak{q}}, \Bbbk\right)^{D(\Bbbk\Gamma)} = 0$ although $\mathrm{H}^3\left(\mathcal{B}_{\mathfrak{q}}\#\Bbbk\Gamma, \Bbbk\right) \cong \mathrm{H}^3\left(\mathcal{B}_{\mathfrak{q}}, \Bbbk\right)^\Gamma$ can be non-trivial, see for example [MPSW, Example 5.8].

6.2. **Braidings of non-diagonal type.** For $n \geq 3$, $\mathcal{FK}_n$ denotes the quadratic algebra [FK] with a presentation by generators $x_{(ij)}$, $1 \leq i < j \leq n$, and relations

$$\begin{aligned} x_{(ij)}^2 &= 0, && 1 \leq i < j \leq n,\\ x_{(ij)}x_{(jk)} &= x_{(jk)}x_{(ik)} + x_{(ik)}x_{(ij)}, && 1 \leq i < j < k \leq n,\\ x_{(jk)}x_{(ij)} &= x_{(ik)}x_{(jk)} + x_{(ij)}x_{(ik)}, && 1 \leq i < j < k \leq n,\\ x_{(ij)}x_{(kl)} &= x_{(kl)}x_{(ij)}, && \#\{i,j,k,l\} = 4. \end{aligned}$$

According to [MiS] each $\mathcal{FK}_n$ is a graded bialgebra in the category of Yetter-Drinfeld modules over the symmetric group S_n, generated as an algebra by the vector space V_n with basis $\{x_{(ij)} \mid 1 \leq i < j \leq n\}$. The action is described by identifying (ij) with the corresponding transposition in S_n and then consider the conjugation twisted by the sign, while the coaction is given by declaring x_σ a homogeneous element of degree σ. Then the braiding on V_n becomes

$$c(x_\sigma \otimes x_\tau) = \chi(\sigma,\tau)x_{\sigma\tau\sigma^{-1}} \otimes x_\sigma, \qquad \chi(\sigma,\tau) = \begin{cases} 1 & \sigma(i) < \sigma(j), \tau = (ij),\ i < j,\\ -1 & \text{otherwise,}\end{cases}$$

where σ and τ are transpositions. Moreover $\mathcal{FK}_n$ projects onto the Nichols algebra $\mathcal{B}(V_n)$. For $n = 3, 4, 5$, it is known that $\mathcal{FK}_n = \mathcal{B}(V_n)$ and has dimension, respectively, 12, 576 and 8294400.

The Hochschild cohomology of $\mathcal{FK}_3$ is a consequence of the results in [SV] as follows.

Theorem 6.5. $\mathrm{H}^\bullet_{\Bbbk S_3\text{-Mod}}\left(\mathcal{FK}_3, \Bbbk\right)$ *is isomorphic to the graded algebra*

$$\Bbbk[X,U,V]/(U^2V - VU^2), \qquad \textit{where } \deg U = \deg V = 2,\ \deg X = 4.$$

Proof. By [SV, Theorem 4.19], we have that $E(B\#\Bbbk S_3)$ is isomorphic to the algebra in the claim, where $B = \mathcal{FK}_3$. By [SV, Theorem 2.17], we know that $E(B\#\Bbbk S_3) \cong E(B)^{\Bbbk S_3}$ as graded algebras. As observed in Remark 4.2, we have that $E(B) \cong \mathrm{H}^\bullet\left(B, \Bbbk\right)$. By Remark 4.6, we have $\mathrm{H}^\bullet\left(B, \Bbbk\right)^{\Bbbk S_3} \cong \mathrm{H}^\bullet_{\Bbbk S_3\text{-Mod}}\left(\mathcal{FK}_3, \Bbbk\right)$. □

From this result we get $\mathrm{H}^3_{\Bbbk S_3\text{-Mod}}\left(\mathcal{FK}_3, \Bbbk\right) = 0$ so that, by Proposition 4.7 we conclude that

Corollary 6.6. $\mathrm{H}^3_{\mathcal{YD}}\left(\mathcal{FK}_3, \Bbbk\right) = 0$.

References

[ABM1] A. Ardizzoni, M. Beattie and C. Menini, *Cocycle Deformations for Hopf Algebras with a Coalgebra Projection*, J. Algebra, Vol. **324**(4) (2010) 673–705.

[ABM] A. Ardizzoni, M. Beattie and C. Menini, *Gauge Deformations for Hopf Algebras with the Dual Chevalley Property*, J. Algebra Appl., Vol. **11**(3) (2012) 1250051.

[AAG] N. Andruskiewitsch, I. E. Angiono, A. García Iglesias, *Liftings of Nichols algebras of diagonal type I. Cartan type A.* (arXiv:1509.01622)

[AAGMV] N. Andruskiewitsch, I. E. Angiono, A. García Iglesias, A. Masuoka and C. Vay, *Liftings via cocycle deformations*. J. Pure Appl. Alg. **218** (2014), 684-703.

[AM] A. Ardizzoni and C. Menini, *Associated Graded Algebras and Coalgebras*, Comm. Algebra, Vol. **30**(3) (2012), 862-896.

[AMS1] A. Ardizzoni, C. Menini and D. Ştefan, *A Monoidal Approach to Splitting Morphisms of Bialgebras*, Trans. Amer. Math. Soc., **359** (2007), 991-1044.

[AMS2] A. Ardizzoni, C. Menini and D. Ştefan, *Hochschild Cohomology And 'Smoothness' In Monoidal Categories*, J. Pure Appl. Algebra, Vol. **208** (2007), 297-330.

[AMStu] A. Ardizzoni, C. Menini and F. Stumbo, *Small Bialgebras with a Projection*, J. Algebra Vol. **314**(2) (2007), 613-663.

[An] I. E. Angiono, *Basic quasi-Hopf algebras over cyclic groups.* Adv. Math. **225** (2010), no. 6, 3545–3575.

[AP] A. Ardizzoni, A. Pavarin, *Bosonization for Dual Quasi-Bialgebras and Preantipode*, J. Algebra, Vol. **390** (2013), Pages 126–159.

[AS1] N. Andruskiewitsch, H.-J. Schneider, *Lifting of quantum linear spaces and pointed Hopf algebras of order* p^3. J. Algebra **209** (1998), no. 2, 658–691.

[AS2] N. Andruskiewitsch, H-J. Schneider, *On the classification of finite-dimensional pointed Hopf algebras*, Ann. of Math. **171** (2010), no. 1, 375–417.

[Bo] N. Bourbaki, *Groupes et algèbres de Lie*, Chapitres 4, 5 et 6, Hermann, Paris, 1968.

[Br] K. S. Brown, *Cohomology of groups.* Graduate Texts in Mathematics, **87**. Springer-Verlag, New York-Berlin, 1982.

[CE] H. Cartan, S. Eilenberg, *Homological algebra.* Princeton University Press, Princeton, N. **J.**, 1956.

[DNR] S. Dăscălescu, C. Năstăsescu, S. Raianu, *Hopf algebras. An introduction.* Monographs and Textbooks in Pure and Applied Mathematics, **235**. Marcel Dekker, Inc., New York, 2001.

[DP] C. De Concini, C. Procesi, *Quantum groups.* D-modules, representation theory, and quantum groups, 31–140, Lecture Notes in Math. **1565**, Springer, 1993.

[EG] P. Etingof, S. Gelaki, *Liftings of graded quasi-Hopf algebras with radical of prime codimension.* J. Pure Appl. Algebra **205** (2006), no. 2, 310–322.

[FK] S. Fomin, A. N. Kirillov, *Quadratic algebras, Dunkl elements and Schubert calculus*, Progr. Math. **172** (1999), 146–182.

[GiK] V. Ginzburg, S. Kumar, *Cohomology of quantum groups at roots of unity.* Duke Math. J. **69** (1993) 179–198.

[GM] L. Grunenfelder; M. Mastnak, *Pointed Hopf Algebras As Cocycle Deformations*, preprint. (`arXiv:1010.4976v1`)

[H] H. Hiller, *Geometry of Coxeter groups.* Research Notes in Mathematics, **54**. Pitman (Advanced Publishing Program), Boston, Mass.-London, 1982.

[L] G. Lusztig, *Quantum groups at roots of 1.* Geom. Dedicata **35** (1990), 89–113.

[K] C. Kassel, *Quantum groups.* Graduate Texts in Mathematics, **155**. Springer-Verlag, New York, 1995.

[Maj] S. Majid, *Foundations of quantum group theory.* Cambridge University Press, Cambridge, 1995.

[MPSW] M. Mastnak, J. Pevtsova, P. Schauenburg, S. Witherspoon, *Cohomology of finite-dimensional pointed Hopf algebras.* Proc. Lond. Math. Soc. **100** (2010), 377–404.

[MiS] A. Milinski, H.J. Schneider, *Pointed indecomposable Hopf algebras over Coxeter groups*, Contemp. Math. **267** (2000), 215–236.

[Mo] S. Montgomery, *Hopf Algebras and their actions on rings,* CMBS Regional Conference Series in Mathematics **82**, 1993.

[Ra1] D. E. Radford, *Hopf algebras.* Series on Knots and Everything, **49**. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012.

[Ra2] D. E. Radford, *Minimal quasitriangular Hopf algebras*, J. Algebra **157** (1993), 285–315.

[Ra3] D. E. Radford, *The structure of Hopf algebras with a projection.* J. Algebra **92** (1985), no. 2, 322–347.

[RT] D. E. Radford, J. Towber, *Yetter-Drinfel'd categories associated to an arbitrary bialgebra.* J. Pure Appl. Algebra **87** (1993), no. 3, 259–279.

[Sc] P. Schauenburg, *Faithful flatness over Hopf subalgebras: counterexamples.* Interactions between ring theory and representations of algebras (Murcia), 331–344, Lecture Notes in Pure and Appl. Math., **210**, Dekker, New York, 2000.

[Stf] D. Ştefan, *Hochschild cohomology on Hopf Galois extensions.* J. Pure Appl. Algebra **103** (1995), no. 2, 221–233.

[Stn] B. Stenström, *Rings of quotients Die Grundlehren der Mathematischen Wissenschaften, Band 217. An introduction to methods of ring theory.* Springer-Verlag, New York-Heidelberg, 1975.

[SV] D. Ştefan, C. Vay, *The cohomology ring of the 12-dimensional Fomin-Kirillov algebra.* arXiv:1404.5101.

[Sw] M. E. Sweedler, *Hopf algebras.* Mathematics Lecture Note Series W. A. Benjamin, Inc., New York 1969.

[W] X. Wang, *Semisimple connected Hopf Algebras.* (`arXiv:1212.0622v1`)

FaMAF-CIEM (CONICET), Universidad Nacional de Córdoba, Medina Allende s/n, Ciudad Universitaria, 5000 Córdoba, Argentina.
E-mail address: angiono@famaf.unc.edu.ar
URL: www.mate.uncor.edu/~angiono

University of Turin, Department of Mathematics "G. Peano", via Carlo Alberto 10, I-10123 Torino, Italy
E-mail address: alessandro.ardizzoni@unito.it
URL: sites.google.com/site/aleardizzonihome

University of Ferrara, Department of Mathematics and Computer Science, Via Machiavelli 35, Ferrara, I-44121, Italy
E-mail address: men@unife.it
URL: sites.google.com/a/unife.it/claudia-menini