T-norms with strong negations: Solution of two open problems

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Abstract

Two open problems of Jayaram are solved. Firstly, we prove that an Archimedean triangular norm with a strong associated negation is necessarily left-continuous. Secondly, as straightforward consequence of the first result, we show that an Archimedean triangular norm with a strong associated negation is necessarily conditionally cancellative.

Key words: T-norms, T-subnorms, Archimedean property, Conditional cancellativity, Left-continuity, Fuzzy negation.

1. Introduction

In a very recent paper ([3]), Jayaram presented two open problems connected to the relationships between an Archimedean triangular subnorm (t-subnorm, for short) whose associated negation is strong (see Definitions 2.2 and 2.1 below) and its analytic properties of left-continuity and conditional cancellativity (see Definition 2.3 below).

The investigation of t-norms and t-subnorms under the point of view of the associated negations has been treated in many papers. Particularly, the case of a t-subnorm with a strong associated negation has been deeply analyzed (see, for instance, [4] and [2]).

In this work, we will solve the two open problems proposed by Jayaram, showing that an Archimedean triangular subnorm with a strong associated negation is necessarily both left-continuous and conditionally cancellative.

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2. Preliminaries

In this section, we shall introduce some notions and properties which are crucial for the main results of this paper (for a thorough exposition of the theory, we recommend the monographs [5] and [1]). We warn the reader that throughout this paper the concept of monotonicity is intended in weak sense, otherwise we refer to strict monotonicity. In the sequel, we will denote the identity function by id, the unit real interval by $\mathbb{I}$ and the open unit interval by $\mathring{\mathbb{I}}$.

Definition 2.1. A fuzzy negation is a decreasing function $n : \mathbb{I} \to \mathbb{I}$ such that $n(0) = 1$ and $n(1) = 0$. Moreover, a fuzzy negation is called strong (or involutive) if $n \circ n = id_{\mathbb{I}}$.

Definition 2.2. A t-subnorm is an increasing function $T : \mathbb{I}^2 \to \mathbb{I}$ such that T is associative, commutative and satisfies $T(x, y) \leq \min\{x, y\}$ for all $x, y \in \mathbb{I}$.

It is well-known that a t-norm is a particular t-subnorm such that $T(x, 1) = x$ for all $x \in \mathbb{I}$.

Definition 2.3. Let T be a t-subnorm.

(1) T is said to be *Archimedean* if for all $x, y \in \mathring{\mathbb{I}}$ there exists an $n \in \mathbb{N}$ such that $x_T^{[n]} < y$, where, as usual, $x_T^{[1]} := x$ and $x_T^{[n]} = T(x, x_T^{[n-1]})$ for every $n > 1$;

(2) T is said to be *conditionally cancellative* if $T(x, y) = T(x, z) > 0$ implies $y = z$;

(3) An element $x \in \mathring{\mathbb{I}}$ is said to be a *nilpotent element* of T if there exists an $n \in \mathbb{N}$ such that $x_T^{[n]} = 0$.

Definition 2.4. Let T be a t-norm. We say that T is *nilpotent* if it is continuous and if each $x \in \mathring{\mathbb{I}}$ is a nilpotent element of T.

Definition 2.5. Let T be a t-subnorm. The associated negation of T is the function $n_T : \mathbb{I} \to \mathbb{I}$ defined as

$$n_T(x) = \sup\{t \in \mathbb{I} : T(x, t) = 0\}.$$

As emphasized in [3], n_T is a decreasing function, with $n_T(0) = 1$, but it need not be a fuzzy negation since $n_T(1)$ can be greater than zero. Remark that n_T is evidently a fuzzy negation when T is a t-norm.

The main result presented in [3], which is both a generalization of a theorem of Jenei (see [4]) and of Jayaram (see [2]), states that any t-subnorm with a strong associated negation n_T is actually a t-norm.

Theorem 2.1 (see Theorem 3.1 in [3])**.** *Let T be any t-subnorm whose associated n_T is strong. Then, T is a t-norm.*

Let us recall some important properties concerning the associated negation of a t-norm (cf. Proposition 2.3.4 in [1]).

Proposition 2.2. *Let T be any t-norm and n_T its associated negation. Then, we have the following:*

(i) $T(x, y) = 0$ implies $y \leq n_T(x)$;

(ii) $y < n_T(x)$ implies $T(x, y) = 0$;

(iii) if T is left-continuous, then $y = n_T(x)$ implies $T(x, y) = 0$.

Moreover, we present a classical result characterizing the properties of a strong associated negation of a t-norm (see Corollary 2.3.7 in [1]), due to its importance in what follows.

Lemma 2.3. *Let T be any t-norm and n_T its associated negation. Then, the following statements are equivalent:*

(a1) n_T is strictly decreasing and continuous;

(a2) n_T is strong.

3. Archimedean Property and Left-Continuity

In [3], the author poses the following, suitably rephrased, problem:

Problem 1. Is an Archimedean t-norm T whose n_T is strong necessarily left-continuous?

Well, the answer is yes and the purpose of this section is just to illustrate the proof of this statement. We need a preliminary result.

Proposition 3.1. *Let T be any t-norm whose associated n_T is strong. Then, T is left-continuous if, and only if, the following property is fulfilled:*

$$T(x, n_T(x)) = 0 \quad \text{for all } x \in \mathbb{I}. \tag{1}$$

Proof. Due to Proposition 2.2 (iii), it suffices to prove that Eq. (1) implies the left-continuity of T. Suppose *ab absurdo* that T is not left-continuous. Then, there exist $x_0 \in \mathbb{I} \setminus \{0\}$, $y_0 \in \mathring{\mathbb{I}}$ and a strictly increasing sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{I} \setminus \{1\}$ such that $x_n \to x_0$, but

$$\lim_{n\to\infty} T(x_n, y_0) < T(x_0, y_0).$$

Recalling that, by Lemma 2.3, n_T is continuous, from the last equation it follows that there exists a $z \in \mathring{\mathbb{I}}$ such that $T(x_n, y_0) < n_T(z) < T(x_0, y_0)$ for all $n \in \mathbb{N}$. Hence, Proposition 2.2 (i) evidently forces $T(z, T(x_0, y_0)) > 0$, while $T(z, T(x_n, y_0)) = 0$ for all $n \in \mathbb{N}$ directly follows from Proposition 2.2 (ii). Due to associativity and commutativity, the two previous conclusions may be rewritten as:

$$T(x_n, T(z, y_0)) = 0 \quad \text{for all } n \in \mathbb{N} \qquad \text{and} \qquad T(x_0, T(z, y_0)) > 0. \tag{2}$$

Therefore, employing the property of the sequence $\{x_n\}_{n\in\mathbb{N}}$, it is not difficult to see that Eq. (2) leads to $n_T(T(z, y_0)) = x_0$, so contradicting Eq. (1) for $x = T(z, y_0)$. □

Now, we are ready to present the main result of this section.

Theorem 3.2. *Let T be any t-norm whose associated n_T is strong. If T is Archimedean then it is left-continuous.*

Proof. By Proposition 3.1, it is enough to show that Eq. (1) is satisfied. Reasoning by contradiction, let $x_0 \in \mathring{\mathbb{I}}$ be such that $T(x_0, n_T(x_0)) > 0$. Due to the Archimedean property of T and the strongness of n_T, respectively, we have that $(x_0)_T^{[2]} < x_0 = n_T(n_T(x_0))$, hence

$$T((x_0)_T^{[2]}, n_T(x_0)) = 0$$

as consequence of Proposition 2.2 (ii). The last equation, combined with the associativity of T, leads to $T(x_0, T(x_0, n_T(x_0))) = 0$, which clearly excludes the possibility that $T(x_0, n_T(x_0))$ is equal to $n_T(x_0)$, due to the initial assumption $T(x_0, n_T(x_0)) > 0$. Therefore, we derive that $0 < T(x_0, n_T(x_0)) < n_T(x_0) < 1$ or, equivalently,

$$0 < x_0 < n_T(T(x_0, n_T(x_0))) < 1 \tag{3}$$

according to the strict monotonicity of n_T (see Lemma 2.3). For notational simplicity, set $s_T(x_0) := n_T(T(x_0, n_T(x_0)))$: now, we assert that $s_T(x_0) = \sup\{t \in \mathbb{I} : T(x_0, t) < x_0\}$. Indeed, by definition, we have that $s_T(x_0)$ is given by $\sup\{t \in \mathbb{I} : T(T(x_0, n_T(x_0)), t) = 0\}$, which in its turn coincides with $\sup\{t \in \mathbb{I} : T(T(x_0, t), n_T(x_0)) = 0\}$ by associativity and commutativity. By virtue of Proposition 2.2 (i), we know that $T(T(x_0, t), n_T(x_0)) = 0$ implies that $T(x_0, t) \leq x_0$, but the initial assumption evidently shows that the case $T(x_0, t) = x_0$ is impossible, so proving the assertion. As consequence of Eq. (3) and the assertion, there exists a $z \in [s_T(x_0), 1[$ such that $T(x_0, z) = x_0$: owing to associativity, we immediately get $T(z_T^{[2]}, x_0) = T(z, T(x_0, z)) = T(z, x_0) = x_0$ and, similarly,

$$T(z_T^{[n]}, x_0) = x_0 \quad \text{for all } n \in \mathbb{N}.$$

However, T being Archimedean, we have that $z_T^{[n]} < n_T(x_0)$ for a sufficiently large n and, consequently, $T(z_T^{[n]}, x_0) = 0$, so contradicting the previous equation and definitely concluding the proof. $\square$

4. Archimedean Property and Conditional Cancellativity

In [3], the author poses the following, suitably rephrased, problem:

Problem 2. Is an Archimedean t-norm T whose n_T is strong necessarily conditionally cancellative?

The answer is again affirmative, as straightforward consequence of Theorem 3.2.

Theorem 4.1. *Let T be any t-norm whose associated n_T is strong. If T is Archimedean then it is conditionally cancellative.*

Proof. Applying Theorem 3.2 yields the left-continuity of T. Now, employing Corollary 6.4 in [3], which is based upon a classical result by Kolesárová (see [6]), we derive that T is nilpotent, i.e. isomorphic to the Łukasiewicz t-norm $T_{\mathbf{Ł}}(x, y) = \max\{x + y - 1, 0\}$, which is clearly conditionally cancellative, so concluding the proof. □

Remark 4.1. We emphasize that, as a straightforward consequence of Theorem 2.1 and Theorem 4.1, every Archimedean t-subnorm with a strong associated negation is a nilpotent t-norm (isomorphic to the Łukasiewicz t-norm).

5. Concluding Remarks

We have solved two open problem of Jayaram ([3]). As straightforward consequence, we have found that an Archimedean triangular norm T with a strong associated negation n_T is necessarily both left-continuous and conditionally cancellative. Note that we cannot draw the same conclusion if we drop the assumption that T is Archimedean, as shown by the following example.

Example 5.1. Let $T : \mathbb{I}^2 \to \mathbb{I}$ be described as

$$T(x, y) = \begin{cases} 0, & \text{if } x + y < 1; \\ \min\{x, y\}, & \text{otherwise.} \end{cases}$$

It is not difficult to show that such T is a t-norm whose associated negation is given by $n_T(x) = 1 - x$, hence n_T is strong. Moreover, T is not Archimedean, because, for instance, $T(x, x) = x$ for all $x \geq \frac{1}{2}$. However, T is clearly neither left-continuous nor conditionally cancellative.

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