# Penalty functions based upon a general class of restricted dissimilarity functions

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**Abstract**

In this paper the notion of restricted dissimilarity function is discussed and some general results are shown. The relation between the concepts of restricted dissimilarity function and penalty function is presented. A specific model of construction of penalty functions by means of a wide class of restricted dissimilarity functions based upon automorphisms of the unit interval is studied. A characterization theorem of the automorphisms which give rise to 2-dimensional penalty functions is proposed. A generalization of the previous theorem to any dimension $n > 2$ is also provided. Finally, a not convex example of generator of penalty functions of arbitrary dimension is illustrated.

*Key words:* Restricted dissimilarity function; penalty function; quasi-convexity.

## 1 Introduction

The notion of *penalty based aggregation function* has become increasingly popular in the literature over the last few years (see, for instance, [4] and the references therein). In [3], the authors suggest the use of penalty functions for selecting alternatives in decision making problems. In particular, they build penalty functions by means of a particular class of *restricted dissimilarity functions* (see [1]), called *faithful restricted dissimilarity functions*, in order to generalize one of the most widely used methods in decision making, that is the weighted voting method (see, for example, [10,11]). From a mere theoretical point of view, a faithful restricted dissimilarity function is strictly related to a convex automorphism of the real unit interval up to a bijection. As the

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authors somehow admit, there is no real reason for imposing convexity restriction other than to assure that the corresponding penalty function is convex, so fulfilling, *a fortiori*, the crucial property of quasi-convexity demanded to all penalty functions.

This consideration has led us to the main goal of this paper: to characterize a class of restricted dissimilarity functions, wider than the faithful restricted dissimilarity functions, able to assure that the generated one-variable mappings, constructed in the same way as in [3], turn out to be penalty functions.

We have organized this paper as follows: In the next section we introduce the basic notions needed for subsequent developments. In Section 3 we analyze the limited cases in which a restricted dissimilarity function is also a distance and mostly we study a model of construction of penalty functions by means of a general subclass of restricted dissimilarity functions strictly connected to automorphisms of the real unit interval, also called generators. In Section 4, we characterize the class of generators of 2-dimensional penalty functions, while in Section 5 we extend the 2-dimensional result to any dimension $n > 2$. Finally, we present a conclusion and some references.

## 2 Basic notions and first results

In this article, we will make use of the following notations and assumptions.

Our domain of interest is the real unit interval, denoted by $\mathbb{I}$, being clear that it might be replaced without loss of generality by any closed, non-empty subinterval of the real line. We adopt the classical notation $\mathbf{x} = (x_1, \ldots, x_n)$ for any $n$-tuple $\mathbf{x}$ in $\mathbb{I}^n$, while $\mathbf{x}_{\nearrow} = (x_{(1)}, \ldots, x_{(n)})$ represents the result of the permutation of the components of $\mathbf{x}$ in increasing order, i.e. $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$. We denote by $W_n$ the set of *weighting vectors* of dimension $n$, i.e. $W_n = \{\mathbf{w} \in ]0,1[^n \colon \sum_{i=1}^n w_i = 1\}$. We will exclusively reserve the symbol $\mathbf{w}$ for any weighting vector: further, $\mathbf{w}_n^*$ stands for the weighting vector given by $(1/n, \ldots, 1/n)$.

**Definition 2.1** *Let $x_0 \in \mathbb{I}$, $\mu > 0$ and $E \subset \mathbb{I}$. Then, we set*

$$\mu E = \{\mu \cdot x : x \in E\} \tag{2.1}$$

*and*

$$E \pm \{x_0\} = \{t \pm x_0 : t \in E\}. \tag{2.2}$$

We warn the reader that throughout the paper the notion of monotonicity is intended in weak sense: otherwise, we speak of strict monotonicity. Moreover, when we say that a property holds almost everywhere (for short, a.e.), it is intended that it is true outside a set of Lebesgue measure, denoted by $\lambda$, equal to zero.

Consider now the following definition.

**Definition 2.2** [2,4] *Let $P : \mathbb{I}^{n+1} \to [0, \infty]$. We say that $P$ is a penalty function of dimension $n$ if, and only if, satisfies:*

(i) $P(\mathbf{x}, y) = 0$ *if $x_i = y$ for all $i \in \{1, \ldots, n\}$;*
(ii) *for every fixed $\mathbf{x} \in \mathbb{I}^n$, the set of minimizers of the mapping $y \mapsto P(\mathbf{x}, y)$ is a subinterval of $\mathbb{I}$, possibly reducible to a singleton.*

*The penalty based function $f : \mathbb{I}^n \to \mathbb{I}$ is given by*

$$f(\mathbf{x}) = \arg\min_{y} P(\mathbf{x}, y)$$

*if $y$ is the unique minimizer or $y = (a + b)/2$ if the set of minimizers is a subinterval of $\mathbb{I}$ with bounds $a$ and $b$.*

A penalty function is a tool for measuring the disagreement between the input $\mathbf{x}$ and the value $y$. The penalty based function $f$ associates with any input $\mathbf{x}$ the corresponding output value $y$ which just minimizes the chosen disagreement.

The first, prototypical model of penalty function appeared in the literature was of the form

$$P(\mathbf{x}, y) = \sum_{i=1}^{n} d_p(x_i, y), \tag{2.3}$$

where $d_p : \mathbb{I}^2 \to \mathbb{I}$ is given by $d_p(x, y) = |x - y|^p$, for $p \geq 1$. In particular, the cases $p = 1$ and $p = 2$ were already studied by Fermat, Laplace and Cauchy (see [9,16] and the references therein). When $p = 1$, the penalty function is obtained as sum of the (Euclidean) distances of the components of the input $\mathbf{x}$ to the value $y$. The model is then generalized replacing $d_1$ with $d_p$, with the crucial difference that only $d_1$ is a distance in mathematical sense. We will return to this point later.

The notion of penalty function is closely related to the dissimilarity function, as proposed and discussed in [13], even if this concept in its turn is inspired to a first formulation of penalty function given by Calvo et al. in [5] which is quite different from the above one. Our version fundamentally coincides with the most general one as it appears in [4], except for the redundant requirement $P(\mathbf{x}, y) \geq 0$ for all $\mathbf{x}, y$, here dropped.

In some papers, condition (ii) is replaced by the requirement that the mapping $y \mapsto P(\mathbf{x}, y)$ is quasi-convex for any fixed $\mathbf{x} \in \mathbb{I}^n$. Recall that a real function $g$ over a convex subset $X$ of $\mathbb{R}^n$ is quasi-convex if its level sets $L_c = \{\mathbf{x} \in X : g(\mathbf{x}) \leq c\}$ are convex (see [8] for a thorough exposition of the notion of quasi-convexity). It is well-known that if $g$ is a function of a single variable, then $g$ is quasi-convex if, and only if, either it is monotone or there exists a $x^* \in X$ such that $g$ is decreasing on $\{x \in X : x \leq x^*\}$ and increasing on $\{x \in X : x \geq x^*\}$. Therefore, it is clear that quasi-convexity is a stronger conditions than (ii).

The restricted dissimilarity functions were introduced by Bustince et al. in [1] and are inspired, among others, to the notions of proximity and dissimilarity measures, as they appear in [6] and [12], respectively. In the applications, they turn out to be very useful, for instance, in image processing, in order to measure the dissimilarity of two objects, while, from a theoretical point of view, they are more flexible than existing dissimilarity functions.

**Definition 2.3** *A mapping $d_R : \mathbb{I}^2 \to \mathbb{I}$ is called a restricted dissimilarity function if :*

(D1) $d_R(x, y) = d_R(y, x)$ *for every* $x, y \in \mathbb{I}$*;*
(D2) $d_R(x, y) = 1$ *if, and only if,* $\{x, y\} = \{0, 1\}$*;*
(D3) $d_R(x, y) = 0$ *if, and only if,* $x = y$*;*
(D4) $d_R(y, z) \leq d_R(x, t)$ *for all* $x, y, z, t \in \mathbb{I}$ *such that* $x \leq y \leq z \leq t$*.*

In the sequel, we will exclusively deal with continuous restricted dissimilarity functions. This is not a too strong assumption, since we know that, fixed any $x \in \mathbb{I}$, the mapping $t \mapsto d_R(x, t)$ is quasi-convex (see [2]), hence it is also continuous on $\mathbb{I}$ up to a subset $E_x$ of $\mathbb{I}$ such that $\lambda(E_x) = 0$ (see [8]). The next result just shows that if $E_x = \emptyset$ for all $x \in \mathbb{I}$, then $d_R$ is continuous as two-place function.

**Proposition 2.4** *Let $d_R$ be a restricted dissimilarity function. Then $d_R$ is continuous if, and only if, the mapping $t \mapsto d_R(x, t)$ is continuous on $\mathbb{I}$ for every fixed $x \in \mathbb{I}$.*

**PROOF.** First of all, notice that, by (D1), the assumption may be equivalently formulated as continuity of the mapping $t \mapsto d_R(t, x)$ for every fixed $x \in \mathbb{I}$. Given an arbitrary point $(x_0, y_0)$ of $\mathbb{I}^2$, we have to show that for any real $\varepsilon > 0$ there exists a neighborhood $U$ of $(x_0, y_0)$ such that

$$d_R(x_0, y_0) - \varepsilon \leq d_R(x, y) \leq d_R(x_0, y_0) + \varepsilon$$

for all $(x, y) \in U$. Let us divide the proof into four cases, according to the position of $(x_0, y_0)$.

Case (1): $0 < x_0 < y_0 < 1$. By continuity of $t \mapsto d_R(x_0, t)$ at $t = y_0$, we can always find a $\delta \in \, ]0, \min\{x_0, 1 - y_0, (y_0 - x_0)/2\}[$ such that

$$|d_R(x_0, t) - d_R(x_0, y_0)| \leq \varepsilon/2 \tag{2.4}$$

for all $t \in [y_0 - \delta, y_0 + \delta]$. Moreover, by continuity of $t \mapsto d_R(t, y_0 + \delta)$ at $t = x_0$, we can always find a $\delta_1 \in ]0, \delta[$ such that

$$d_R(x_0 - \delta_1, y_0 + \delta) \leq d_R(x_0, y_0 + \delta) + \varepsilon/2.$$

Employing Eq. (2.4), last inequality leads to

$$d_R(x_0 - \delta_1, y_0 + \delta) \leq d_R(x_0, y_0) + \varepsilon. \tag{2.5}$$

In the same way, starting with the continuity of the mapping $t \mapsto d_R(t, y_0 - \delta)$ at $t = x_0$, there exists a $\delta_2 \in ]0, \delta[$ such that

$$d_R(x_0 + \delta_2, y_0 - \delta) \geq d_R(x_0, y_0) - \varepsilon. \tag{2.6}$$

Set $\delta^* := \min\{\delta_1, \delta_2\}$: by the properties of $\delta$, it is quite easy to see that

$$x_0 - \delta_1 \leq x_0 - \delta^* < x_0 + \delta^* \leq x_0 + \delta_2 < y_0 - \delta,$$

hence, by (D4), one immediately finds

$$d_R(x_0 + \delta^*, y_0 - \delta) \leq d_R(x, y) \leq d_R(x_0 - \delta^*, y_0 + \delta)$$

for all $(x, y) \in [x_0 - \delta^*, x_0 + \delta^*] \times [y_0 - \delta, y_0 + \delta]$, and the claim directly follows from Eqs. (2.5) and (2.6).

Case (2): $0 < y_0 < x_0 < 1$. It directly follows from the previous case, taking into account (D1).

Case (3): $0 < x_0 = y_0 < 1$. The proof is the same (even simpler) as that of the first case.

Case (4): $(x_0, y_0)$ belongs to the boundary of $\mathbb{I}^2$. This is again a sub-case of the first one, so concluding the proof. □

In what follows, AC($\mathbb{I}$) denotes the family of absolutely continuous real functions over $\mathbb{I}$. We say that any $\varphi : \mathbb{I} \to \mathbb{I}$ belongs to $\mathcal{A}(\mathbb{I})$ if, and only if,

(A1) $\varphi \in$ AC($\mathbb{I}$);
(A2) $\varphi$ is an increasing bijection.

Obviously, any $\varphi \in \mathcal{A}(\mathbb{I})$ is an automorphism of the real unit interval (see, for instance, [7]). Recall that, as a consequence of (A1), the derivative of any $\varphi \in \mathcal{A}(\mathbb{I})$ exists and is defined on $\mathbb{I}$ up to a subset of measure zero (see, for instance, [17]).

Let us introduce now a special class of restricted dissimilarity functions by means of a construction illustrated in the next lemma, whose elementary proof is omitted. Later on, any continuous bijection $h : \mathbb{I} \to \mathbb{I}$ is simply called a *scaling function.*

**Lemma 2.5** *Let $\varphi \in \mathcal{A}(\mathbb{I})$. Let $h$ be a scaling function. Then, the mapping $d_R : \mathbb{I}^2 \to \mathbb{I}$ given by*

$$d_R(x, y) = \varphi(|h(x) - h(y)|) \tag{2.7}$$

*is a restricted dissimilarity function.*

Note that the class of restricted dissimilarity functions as in Eq. (2.7) properly contains the family of *faithful restricted dissimilarity functions*, as defined in [3]. Indeed, the construction is exactly the same: the only difference is that condition (A1) is replaced with the stronger requirement that $\varphi$ is convex. In fact, any convex and continuous $\varphi : \mathbb{I} \to \mathbb{I}$ is also absolutely continuous (see, for instance, [17, Theorem 14.13]).

Observe that the mappings $d_p$ previously introduced are exactly restricted dissimilarity functions of the form given in Eq. (2.7), with $\varphi(t) = t^p$ and $h \equiv \mathrm{id}$, where id is the identity function.

## 3 Construction of penalty functions based upon restricted dissimilarity functions

In this section, we address the problem of building penalty functions following the approach of Eq. (2.3), but with the crucial novelty that we replace the class of mappings $d_p$ with a larger family of (continuous) restricted dissimilarity functions.

Starting with an arbitrary continuous restricted dissimilarity function $d_R$ and fixing the dimension $n \in \mathbb{N}$, our candidate for being an $n$-dimensional penalty function is the mapping $P : \mathbb{I}^n \times W_n \times \mathbb{I} \to \mathbb{I}$ given by

$$P(\mathbf{x}, \mathbf{w}, y) = \sum_{i=1}^{n} w_i d_R(x_i, y). \tag{3.1}$$

When a mapping $P$ as in Eq. (3.1) is actually a penalty function, $d_R$ will be referred to as a generator of $P$.

**Remark 3.1** *Let $P$ be as in Eq.* (3.1)*: then, conditions* (i)-(ii) *have to be slightly modified as follows:*

(i') *for every fixed $\mathbf{w} \in W_n$, $P(\mathbf{x}, \mathbf{w}, y) = 0$ if $x_i = y$ for all $i \in \{1, \dots, n\}$;*

(ii') *for every fixed $\mathbf{x} \in \mathbb{I}^n$ and $\mathbf{w} \in W_n$, the set of minimizers of the mapping $y \mapsto P(\mathbf{x}, \mathbf{w}, y)$ is a subinterval of $\mathbb{I}$, possibly reducible to a singleton.*

Observe that condition (i') is trivially satisfied due to (D3), so the real difficulty is the verification of (ii').

Fixing any $\mathbf{x} \in \mathbb{I}^n$ and $\mathbf{w} \in W_n$, the mapping $y \mapsto P(\mathbf{x}, \mathbf{w}, y)$ is clearly continuous over a compact domain, so its set of minimizers, denoted by $S_{\mathbf{x},\mathbf{w}}(P)$, is always nonempty.

From now on, we will suppose first that $\mathbf{x} = \mathbf{x}_{\nearrow}$. Note that there is no loss of generality: otherwise it would suffice to relabel the indices of the components of the involved vectors. Secondly, assume that $x_i \neq x_{i+1}$ for all $i \in \{1, \ldots, n-1\}$ (otherwise, the dimension $n$ would be reduced). Under these assumptions, owing to (D4) it is easy to see that the mapping $y \mapsto P(\mathbf{x}, \mathbf{w}, y)$ is decreasing on $[0, x_1]$ and increasing on $[x_n, 1]$, whence condition (ii') is equivalent to showing that $S_{\mathbf{x},\mathbf{w}}(P)$ is a subinterval of $[x_1, x_n]$.

Now, we pose the following problem: may we make use of a distance as particular restricted dissimilarity function in order to generate $n$-dimensional penalty functions via Eq. (3.1)? The answer is decidedly negative even for $n = 2$ with only one exception, as we show in the following.

**Proposition 3.2** *Let $d_R$ be a continuous restricted dissimilarity function. Assume that $d_R$ is a generator of a 2-dimensional penalty function $P$ via Eq.* (3.1)*. Then, $d_R$ is a metric if, and only if, is of the form given in Eq.* (2.7) *with $\varphi \equiv \mathrm{id}$.*

**PROOF.** Let $d_R$ satisfy the assumptions. According to (D1) and (D3), it is easy to see that $d_R$ is a metric if, and only if, verifies the triangle inequality

$$d_R(x_1, x_2) \leq d_R(x_1, z) + d_R(x_2, z)$$

for all $x_1, x_2, z \in \mathbb{I}$. The only non-trivial case is when $x_1 < x_2$ and $z \in ]x_1, x_2[$. We have two mutually exclusive possibilities only: in the first one, assume that

$$d_R(x_1, x_2) = d_R(x_1, z) + d_R(x_2, z)$$

for all $x_1, x_2 \in \mathbb{I}$ such that $x_1 < x_2$ and for any $z \in [x_1, x_2]$. Then, define $h(t) := d_R(0, t)$: by (D4), we have that $h$ is a (continuous) increasing mapping, whilst (D2)-(D3) trivially force $h(t) = t$ when $t \in \{0, 1\}$. Now, consider any $x, y \in \mathbb{I}$ such that $0 \leq x \leq y$: if we apply the above equality with $x_1 = 0, x_2 = y$ and $z = x$, we immediately get $d_R(x, y) = h(y) - h(x)$. By (D3), this clearly implies the strict monotonicity of $h$, so closing the first case. The only other possibility is that $d_R$ fulfills the triangle inequality and, at the same time, there exists at least a pair of points $x_1, x_2 \in \mathbb{I}$, with $x_1 < x_2$, such that $d_R(x_1, x_2) < d_R(x_1, z_0) + d_R(x_2, z_0)$ for some $z_0 \in ]x_1, x_2[$. Let us show that this case is incompatible with the fact that $d_R$ acts as a generator of a 2-dimensional penalty function $P$ via Eq. (3.1). In fact, if this were true, we would immediately find that the generated penalty function $P(\mathbf{x}, \mathbf{w}_2^*, y)$ satisfies the inequality

$$P(\mathbf{x}, \mathbf{w}_2^*, y) \geq P(\mathbf{x}, \mathbf{w}_2^*, x_1) = P(\mathbf{x}, \mathbf{w}_2^*, x_2)$$

for all $y \in [x_1, x_2]$ and at the same time

$$P(\mathbf{x}, \mathbf{w}_2^*, z_0) > P(\mathbf{x}, \mathbf{w}_2^*, x_1) = P(\mathbf{x}, \mathbf{w}_2^*, x_2).$$

As a straightforward consequence, it would follow that $S_{\mathbf{x},\mathbf{w}_2^*}(P)$ certainly contains $x_1, x_2$, but not $z_0$, hence it could not be convex, so contradicting (ii'). $\square$

Since the whole class of continuous restricted dissimilarity functions is too wide, we focus on its relevant subclass introduced in Lemma 2.5.

Given any $\varphi \in \mathcal{A}(\mathbb{I})$, let $P_{h,\varphi} : \mathbb{I}^n \times W_n \times \mathbb{I} \to \mathbb{I}$ be described as

$$P_{h,\varphi}(\mathbf{x}, \mathbf{w}, y) = \sum_{i=1}^{n} w_i \varphi(|h(x_i) - h(y)|),$$

for any fixed scaling function $h$.

With a little abuse of terminology, when $P_{h,\varphi}$ is a penalty function, the mapping $\varphi$ will be referred to as generator of $P_{h,\varphi}$.

Our aim is to find the minimal requirements on $\varphi$ and $h$ in order for $P_{h,\varphi}$ to be an $n$-dimensional penalty function. A first, simple result shows the total irrelevance of the scaling functions. To avoid cumbersome symbols, in what follows we write $P_\varphi$ as the shorthand for $P_{h,\varphi}$ when $h \equiv \mathrm{id}$. More explicitly, $P_\varphi : \mathbb{I}^n \times W_n \times \mathbb{I} \to \mathbb{I}$ is given by

$$P_\varphi(\mathbf{x}, \mathbf{w}, y) = \sum_{i=1}^{n} w_i \varphi(|x_i - y|). \tag{3.2}$$

**Lemma 3.3** *Assume that $P_\varphi$ is an $n$-dimensional penalty function for some generator $\varphi \in \mathcal{A}(\mathbb{I})$. Then $P_{h,\varphi}$ is also an $n$-dimensional penalty function for any scaling function $h$.*

**PROOF.** Without loss of generality, suppose that $h$ is strictly increasing. Then, it suffices to show that, given any $\mathbf{x} \in \mathbb{I}^n$ and any $\mathbf{w} \in W_n$, the set $S_{\mathbf{x},\mathbf{w}}(P_{h,\varphi})$ is convex. By assumption, we know that $S_{\mathbf{u},\mathbf{w}}(P_\varphi)$ is a sub-interval of $\mathbb{I}$, where $\mathbf{u} = h(\mathbf{x}) := (h(x_1), \ldots, h(x_n))$. Then, it is quite easy to see that

$S_{\mathbf{x},\mathbf{w}}(P_{h,\varphi}) = h^{-1}(S_{\mathbf{u},\mathbf{w}}(P_\varphi))$, hence the claim is assured by the properties of $h$. $\square$

Due to the previous result, from now on, we will exclusively deal with the class of functions $P_\varphi : \mathbb{I}^n \times W_n \times \mathbb{I} \to \mathbb{I}$ of the form given by Eq. (3.2). However, because of the high complexity of our task, we try to achieve our goal through the pursuit of an intermediate target, that is the search for a subclass $\mathcal{S}$ of $\mathcal{A}(\mathbb{I})$ such that, whatever is the choice of $\varphi \in \mathcal{S}$, the mapping $y \mapsto P_\varphi(\mathbf{x}, \mathbf{w}, y)$ is quasi-convex for every fixed $\mathbf{x} \in \mathbb{I}^n$ and $\mathbf{w} \in W_n$ and for any dimension $n \geq 2$. Note that this is a well-posed problem, in the sense that $\mathcal{S}$ is not empty. In fact, if we denote by $\mathcal{CA}$ the set of all convex automorphisms of $\mathbb{I}$, it is clear that $\mathcal{CA} \subset \mathcal{S}$. To this end, note that the convexity of $\varphi$ easily implies convexity, and *a fortiori* quasi-convexity, of $P_\varphi$ (see, for instance, [3, Propositions 7 and 8]). Actually, our problem is twofold: first, the characterization of $\mathcal{S}$ through a differential condition on $\varphi$; second, relying on such characterization, to find out whether $\mathcal{CA} = \mathcal{S}$. We emphasize the fact that, as remarked in the previous section, any restricted dissimilarity function is quasi-convex in one variable, so our new task is strictly related to the more general problem of determining a non-trivial family of quasi-convex functions, containing nonconvex functions, which is closed under the operation of addition. As stated by the authors in [8], although many years have passed since then, to the best of our knowledge such problem is still open.

## 4 The 2-dimensional case

In this section, we solve the problem of characterizing the mappings $\varphi \in \mathcal{A}(\mathbb{I})$ such that every function $P_\varphi : \mathbb{I}^2 \times W_2 \times \mathbb{I} \to \mathbb{I}$ is a 2-dimensional penalty function.

First, we need to introduce some suitable notation and useful lemmata.

We begin to restrict the initial class of the candidates for being generators of 2-dimensional penalty functions: let $\mathcal{S}_0 = \{\varphi \in \mathcal{A}(\mathbb{I}) : \varphi'$ is strictly positive and continuous a.e. on $\mathbb{I}\}$. Observe that $\mathcal{CA} \subset \mathcal{S}_0$: in fact, given any convex $\varphi : \mathbb{I} \to \mathbb{I}$, $\varphi'$ is continuous on $\mathbb{I}$ up to a countable subset (see [15]). Moreover, the strict increasing monotonicity and convexity easily imply that $\varphi'(t) > 0$ for all $t \in ]0, 1]$ where the derivative exists.

**Remark 4.1** *In the sequel, we will exclusively reserve the symbol $D_{\varphi'}$ for the domain of the derivative of any $\varphi \in \mathcal{S}_0$. Note that $D_{\varphi'}$ is defined up to a zero measure subset of $\mathbb{I}$ : this precisely means that any subset $S$ of $\mathbb{I}$ is a proper domain of $\varphi'$ if $\lambda(\mathbb{I} \setminus S) = 0$ and $\varphi' : S \to ]0, \infty[$ is continuous.*

Recall a well-known property, useful for the sequel, which characterizes monotonicity of absolutely continuous functions.

**Lemma 4.2** *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function. Then, $f$ is decreasing if, and only if, $f'(x) \leq 0$ almost everywhere on $[a, b]$.*

Let us introduce a class of functions which turns out to be crucial for our purposes. Note that, throughout this section, we will reserve the symbol $X$ for denoting an arbitrary subset of $]0, 1]$ dense in $\mathbb{I}$. Given any $\varphi \in \mathcal{S}_0$, let $\mathcal{F}_2(X, \varphi) = \{\psi_{c,\mathbf{w}} : c \in X, \mathbf{w} \in W_2\}$ be the class of mappings $\psi_{c,\mathbf{w}} : [0, c] \to \mathbb{R}$ defined by

$$\psi_{c,\mathbf{w}}(t) = w_1\varphi(t) + w_2\varphi(c - t).$$

In the sequel, for sake of brevity, we will adopt the shorthand notation $\mathcal{F}_2(X)$, unless some confusion arises. Moreover, we will also adopt the shorthand notation $\psi_c$ for $\psi_{c,\mathbf{w}_2^*}$.

Recall that absolute continuity is closed under the operation of addition, therefore every function of $\mathcal{F}_2(X)$ is absolutely continuous, hence it is also a.e. differentiable.

Let us turn our attention to the notable subclass $\{\psi_c : c \in X\}$ of $\mathcal{F}_2(X)$. It is immediate to see that the graph of any member $\psi_c$ of this subclass is symmetrical with respect to the vertical line at $t = c/2$ and consequently $\psi_c$ is quasi-convex if, and only if, is decreasing on $[0, c/2]$. According to Lemma 4.2, this is equivalent to saying that $\psi_c$ is quasi-convex if, and only if,

$$\psi_c'(t) \leq 0 \quad \text{for almost all } t \in [0, c/2]. \tag{4.1}$$

**Remark 4.3** *Let $\varphi \in \mathcal{S}_0$ and $c \in X$. Based upon the properties of positively homogeneity and invariance under translation of the Lebesgue measure (see, for instance,* [17, Theorem 3.18] *and* [17, Theorem 3.16]) *and taking into account* Remark 4.1, *it is not difficult to check that the complement with respect to $\mathbb{I}$ of the set $D_{\varphi',c} := \{t \in \mathbb{I} : \varphi'(t), \varphi'(c - t) \text{ exist}\}$ has measure zero. This allows us to assume without loss of generality that the domain of the derivative of every member of the subclass of $\mathcal{F}_2(X)$ given by $\{\psi_{c,\mathbf{w}} \in \mathcal{F}_2(X) : \mathbf{w} \in W_2\}$ is exactly $D_{\varphi',c}$. Further, given any $\psi_c \in \mathcal{F}_2(X)$, an elementary computation shows that the inequality $\psi_c'(t) \leq 0$ amounts to $\varphi'(t) \leq \varphi'(c - t)$ for every $t \in D_{\varphi',c}$.*

The next result will be presented without including a proof: indeed, it is a straightforward consequence of both the uniform continuity of any $\varphi \in \mathcal{S}_0$ over $\mathbb{I}$ and the density of $X$ in $\mathbb{I}$.

**Lemma 4.4** *Let $\varphi \in \mathcal{S}_0$. Assume that every function of $\mathcal{F}_2(X)$ is quasi-convex for some dense subset $X \subset ]0, 1]$. Then, every member of $\mathcal{F}_2(]0, 1])$ is* *quasi-convex.*

Now, we shall proceed in two steps: in the first one, we consider the particular case $\mathbf{w} = \mathbf{w}_2^*$.

**Proposition 4.5** *Let $\mathbf{x} \in \mathbb{I}^2$ and $\varphi \in \mathcal{S}_0$. Then, the mapping $\varphi(y - x_1) + \varphi(x_2 - y)$ is quasi-convex in the variable $y$ if, and only if,*

$$\varphi'(t) \leq \inf\{\varphi'(z) : z \in ]t, 1 - t[\cap D_{\varphi'}\} \quad \text{for all } t \in D_{\varphi'} \cap [0, 1/2[. \tag{4.2}$$

**PROOF.** Set $t := y - x_1$: consequently $x_2 - y = c - t$, where $c := x_2 - x_1 \in ]0, 1]$. Using this change of variable and employing Lemma 4.4, we can equivalently consider the problem of characterizing quasi-convexity of every member of the subclass $\{\psi_c : c \in X\}$ for some subset $X$ of $]0, 1]$ dense in $\mathbb{I}$. Choose $X = \mathbb{Q} \cap ]0, 1]$: note that $X$ is countable and dense in $\mathbb{I}$. Owing to Remark 4.3, it is quite easy to see that $\mathbb{I} \setminus \bigcap_{c \in X} D_{\varphi',c}$ has measure zero, hence, according to Remark 4.1, we may assume that $D_{\varphi'} = \bigcap_{c \in X} D_{\varphi',c}$. This implies that, fixing any $t \in D_{\varphi'}$ and any $c \in X$, both $\varphi'(t)$ and $\varphi'(c - t)$ exist. As above stated, an arbitrary element of $\{\psi_c : c \in X\}$ is quasi-convex if, and only if, Eq. (4.1) is verified. Now, we assert that Eq. (4.1) holds for any $c \in X$ if, and only if, Eq. (4.2) is satisfied. Let us start with the necessity and suppose *ab absurdo* that Eq. (4.2) is violated for some $t_0 \in D_{\varphi'} \cap ]0, 1/2[$. This means that there exists a $z_0 \in ]t_0, 1 - t_0[\cap D_{\varphi'}$ such that $\varphi'(t_0) > \varphi'(z_0)$. Set $\epsilon := \varphi'(t_0) - \varphi'(z_0)$. Being $\varphi'$ continuous at $z_0$, there exists a sufficiently small $\delta > 0$ such that

$$\varphi'(z) < \varphi'(z_0) + \epsilon/2 \tag{4.3}$$

for all $z \in [z_0, z_0 + \delta] \cap D_{\varphi'}$. By the density of $X$ in $\mathbb{I}$ we can always find a $c_0 \in ]t_0 + z_0, t_0 + z_0 + \frac{\delta}{2}[\cap X$. Being $\varphi'$ continuous at $t_0$, there exists a sufficiently small $\delta_1 > 0$, with $\delta_1 < \frac{\delta}{2}$, such that

$$\varphi'(t) > \varphi'(t_0) - \epsilon/2 \tag{4.4}$$

for all $t \in [t_0 - \delta_1, t_0] \cap D_{\varphi'}$. Now, pick any $t \in [t_0 - \delta_1, t_0] \cap D_{\varphi'}$: being clearly $t < c_0/2$, Eq. (4.1) applies, so that, by Remark 4.3, we obtain

$$\varphi'(t) \leq \varphi'(c_0 - t). \tag{4.5}$$

Moreover, since it is not difficult to check that $c_0 - t \in ]z_0, z_0 + \delta[$, recalling that $\varphi'(c_0 - t)$ exists as consequence of $t \in D_{\varphi'}$, we can also apply Eq. (4.3) with $z = c_0 - t$. Finally, combining Eq. (4.5) and Eq. (4.3) with Eq. (4.4), we get the following chain of inequalities:

$$\varphi'(t_0) - \epsilon/2 < \varphi'(t) \leq \varphi'(c_0 - t) < \varphi'(z_0) + \epsilon/2.$$

The evident contradiction with $\epsilon = \varphi'(t_0) - \varphi'(z_0)$ closes the first part of the assertion. Conversely, let Eq. (4.2) hold. Given any $c \in X$, let $t \in ]0, c/2[ \cap D_{\varphi'}$:

after the assignment $z := c - t$, we immediately find that $z \in \,]t, 1-t[\cap D_{\varphi'}$, hence by applying Eq. (4.2) easily leads to $\psi_c'(t) \le 0$, so showing the assertion and concluding the proof. $\square$

Remark that Eq. (4.2) is equivalent to both convexity of $\varphi$ on $[0, 1/2]$ and $\varphi'(t) \le \mu_\varphi(t)$ a.e. on $[0, 1/2[$, where $\mu_\varphi : [0, 1/2[\to \mathbb{R}$ is defined as

$$\mu_\varphi(t) = \inf\{\varphi'(z) : z \in \,]1/2, 1-t[\cap D_{\varphi'}\}.$$

In fact, let Eq. (4.2) hold: consequently, given any $t_1, t_2 \in [0, 1/2[\cap D_{\varphi'}$ such that $t_1 < t_2$, if we apply Eq. (4.2) with $t = t_1$ we have

$$\varphi'(t_1) \le \inf\{\varphi'(z) : z \in \,]t_1, 1-t_1[\cap D_{\varphi'}\}.$$

Since it is clear that $t_2 \in \,]t_1, 1-t_1[\cap D_{\varphi'}$, we easily infer that $\varphi'(t_1) \le \varphi'(t_2)$. In conclusion, we have proved that $\varphi' : D_{\varphi'} \cap [0, 1/2[\to \mathbb{R}$ is an increasing function, thus (see, for instance, [17, Theorem 14.14]) $\varphi$ is a convex function on $[0, 1/2]$. As a straightforward consequence, it follows that for every $t \in [0, 1/2[\cap D_{\varphi'}$ we have $\varphi'(t) = \inf\{\varphi'(z) : z \in \,]t, 1/2]\}$. This obviously implies that Eq. (4.2) may be rewritten in the simplified form

$$\varphi'(t) \le \inf\{\varphi'(z) : z \in \,]1/2, 1-t[\cap D_{\varphi'}\} \quad \text{for all } t \in [0, 1/2[\cap D_{\varphi'}.$$

The converse is trivial. We have shown the following.

**Proposition 4.6** *Let $\mathbf{x} \in \mathbb{I}^2$ and $\varphi \in \mathcal{S}_0$. Then, the mapping $\varphi(y - x_1) + \varphi(x_2 - y)$ is quasi-convex in the variable $y$ if, and only if, $\varphi$ fulfills the following properties:*

(P1) *$\varphi$ is convex on $[0, 1/2]$;*
(P2) *$\varphi'(t) \le \mu_\varphi(t)$ almost everywhere on $[0, 1/2[$.*

The preceding result states that the subclass of $\mathcal{S}_0$ characterizing quasi-convexity of every member of $\{\psi_c : c \in \,]0, 1]\}$ is exactly given by $\mathcal{S}_1 = \{\varphi \in \mathcal{S}_0 : \varphi$ verifies (P1)-(P2)$\}$.

**Example 4.7** *Consider the function $\varphi : \mathbb{I} \to \mathbb{I}$ defined by*

$$\varphi(t) = \begin{cases} \frac{1}{2}t, & t \le \frac{3}{4}; \\ \frac{3}{8} + \frac{5}{4}\sqrt{t - \frac{3}{4}}, & \text{otherwise.} \end{cases}$$

*It is easy to check that $\varphi \in \mathcal{S}_0$ and that (P1) holds. Hence, it is really an elementary task to show that $\varphi \in \mathcal{S}_1$, being clearly $\varphi'(t) = \mu_\varphi(t) = 1/2$ for all $t \in \,]0, 1/2[$.*

**Remark 4.8** *Let $\varphi \in \mathcal{S}_0$. Suppose that $\varphi$ is convex on $[0,1/2]$ and is concave on $[1/2,1]$. Then, it is not difficult to show that $\varphi \in \mathcal{S}_1$ if, and only if, $\varphi'(t) \leq \varphi'(1-t)$ for almost all $t \in [0,1/2]$.*

In the second step, the weights come into play: in this case, as it was probably natural to expect, the subclass of $\mathcal{S}_0$ exclusively formed of generators of 2-dimensional penalty functions as in Eq. (3.2) is a proper subset of $\mathcal{S}_1$, as we show in the following example.

**Example 4.9** *Let $\varphi$ be as in* Example 4.7*, $\mathbf{x} = (0,1)$ and $\mathbf{w} = (5/6, 1/6)$. If we consider the corresponding $P_\varphi(\mathbf{x}, \mathbf{w}, y)$, it is only a matter of computation to see that $S_{\mathbf{x},\mathbf{w}}(P_\varphi) = \{0, 1/4\}$, hence $P_\varphi$ is not a penalty function although $\varphi \in \mathcal{S}_1$.*

Given any $\varphi \in \mathcal{S}_0$, let $\mathcal{Z}_2(X,\varphi) = \{\zeta_c : c \in X\}$ be the class of strictly positive functions defined a.e. on $[0,c]$ and described as

$$\zeta_c(t) = \frac{\varphi'(t)}{\varphi'(c-t)}.$$

We highlight that $\mathcal{Z}_2(X,\varphi)$ is made of continuous mappings, due to the continuity of $\varphi'$ on $D_{\varphi'}$. Similarly as above, we will adopt the shorthand notation $\mathcal{Z}_2(X)$ for $\mathcal{Z}_2(X,\varphi)$, unless some confusion arises.

**Remark 4.10** *Let $\varphi \in \mathcal{S}_0$ and $c \in X$. Remark that there is a natural, biunivocal correspondence between the subclass $\{\psi_{c,\mathbf{w}} \in \mathcal{F}_2(X,\varphi) : \mathbf{w} \in W_2\}$ of $\mathcal{F}_2(X,\varphi)$ and the mapping $\zeta_c \in \mathcal{Z}_2(X,\varphi)$. In particular, according to* Remark 4.3*, we may assume that the domain of $\zeta_c$ coincides with $D_{\varphi',c}$. Moreover, an elementary computation shows that the inequality $\psi'_{c,\mathbf{w}}(t) > 0$ amounts to $\zeta_c(t) > \frac{w_2}{w_1}$ for every $t \in D_{\varphi',c}$.*

We emphasize the fact that the increasing monotonicity of all mappings of $\mathcal{Z}_2(]0,1])$ will play a crucial role in characterizing the family of generators of 2-dimensional penalty functions of the form given by Eq. (3.2). In the next result, we show that such property is assured even if it occurs for a particular dense subset $X \subset ]0,1]$ only.

**Lemma 4.11** *Let $\varphi \in \mathcal{S}_0$. Assume that there exists a dense subset $X \subset ]0,1]$ such that every mapping of $\mathcal{Z}_2(X)$ is increasing. Then, every member of $\mathcal{Z}_2(]0,1])$ is increasing.*

**PROOF.** Reasoning by contradiction, suppose that $\zeta_{c_0}(t_1) > \zeta_{c_0}(t_2)$ for some $t_1, t_2 \in ]0, c_0[$, with $t_1 < t_2$, and for some $c_0 \in ]0,1] \setminus X$. It is only a matter of computation to see that $\zeta_{c_0}(t_1) > \zeta_{c_0}(t_2)$ amounts to

$$\frac{\varphi'(t_1)}{\varphi'(t_2)} > \frac{\varphi'(c_0 - t_1)}{\varphi'(c_0 - t_2)}.$$

The above inequality implies that

$$\frac{\varphi'(t_1) - \epsilon}{\varphi'(t_2) + \epsilon} > \frac{\varphi'(c_0 - t_1) + \epsilon}{\varphi'(c_0 - t_2) - \epsilon} \tag{4.6}$$

for a sufficiently small $\epsilon > 0$. Being $\varphi'$ continuous, there exists a sufficiently small $\delta > 0$ such that

$$|\varphi'(t) - \varphi'(t_i)| < \epsilon, \ \ i = 1, 2 \tag{4.7}$$

and

$$|\varphi'(t) - \varphi'(c_0 - t_i)| < \epsilon, \ \ i = 1, 2 \tag{4.8}$$

for almost all $t$ such that $|t - t_i| < \delta$ and $|t - (c_0 - t_i)| < \delta$ respectively and for $i \in \{1, 2\}$. Now, pick any $c \in X$ such that $|c - c_0| < \delta/2$: since $\mathbb{I} \setminus D_{\varphi',c}$ has measure zero, we can always find $u_1, u_2 \in D_{\varphi',c}$, with $u_1 < u_2$, such that $|u_i - t_i| < \delta/2$ for $i \in \{1, 2\}$. In this case, we have all the conditions to employ Eq. (4.7) and Eq. (4.8), so getting

$$\frac{\varphi'(u_1)}{\varphi'(u_2)} > \frac{\varphi'(t_1) - \epsilon}{\varphi'(t_2) + \epsilon}$$

and

$$\frac{\varphi'(c - u_1)}{\varphi'(c - u_2)} < \frac{\varphi'(c_0 - t_1) + \epsilon}{\varphi'(c_0 - t_2) - \epsilon}.$$

The last two inequalities, combined with Eq. (4.6), lead to

$$\frac{\varphi'(u_1)}{\varphi'(u_2)} > \frac{\varphi'(c - u_1)}{\varphi'(c - u_2)},$$

or, equivalently, $\zeta_c(u_1) > \zeta_c(u_2)$, so contradicting the assumption and concluding the proof. $\square$

Let us introduce the notable subclass of $\mathcal{S}_0$ given by $\mathcal{S}_2 = \{\varphi \in \mathcal{S}_0 :$ every member of $\mathcal{Z}_2(\,]0, 1])$ is increasing$\}$. The next two results just concern this subclass: in the first one, we show that the verification task of the increasing monotonicity of every member of $\mathcal{Z}_2(\,]0, 1])$ may be slightly simplified. In the second one, we will prove that $\mathcal{S}_1$ (strictly) includes $\mathcal{S}_2$. In order to reduce the complexity of the proof of both the results, an appropriate choice of the dense subset $X \subset ]0, 1]$ is required, as suggested in the next remark.

**Remark 4.12** *Observe that we can associate with any* $\varphi \in \mathcal{S}_0$ *a particular dense subset* $X \subset ]0,1]$*, denoted with* $X_2(\varphi')$*, characterized by the property that any* $x \in ]0,1]$ *belongs to* $X_2(\varphi')$ *if, and only if,* $\varphi'(x/2)$ *exists. Indeed, consider the set* $2((\mathbb{I} \setminus D_{\varphi'}) \cap [0,1/2])$ *(see* Eq. (2.1)*). Since the Lebesgue measure is positively homogeneous, we get*

$$\lambda(2((\mathbb{I} \setminus D_{\varphi'}) \cap [0,1/2])) = 2\lambda((\mathbb{I} \setminus D_{\varphi'}) \cap [0,1/2]) = 0.$$

*Now, the characterizing property of* $X_2(\varphi')$ *easily follows if we choose as* $X_2(\varphi')$ *the complement of* $2((\mathbb{I} \setminus D_{\varphi'}) \cap [0,1/2])$ *with respect to* $]0,1]$ *(for the density of* $X_2(\varphi')$ *see, for instance,* [14, Proposition 2.4]*).*

**Lemma 4.13** *Let* $\varphi \in \mathcal{S}_0$*. Assume that every mapping of* $\mathcal{Z}_2(X_2(\varphi'))$ *is increasing on* $[0,c/2]$*. Then, every member of* $\mathcal{Z}_2(]0,1])$ *is increasing.*

**PROOF.** Let $\zeta_c \in \mathcal{Z}_2(X_2(\varphi'))$: by assumption, we know that $\zeta_c$ is increasing on $[0,c/2]$. Accordingly, from the elementary property $\zeta_c(c/2-t)\cdot\zeta_c(c/2+t) = 1$ for almost all $t \in ]0,c/2[$, we infer that $\zeta_c$ is increasing also on $[c/2,c]$. Finally, according to Remark 4.12, we have that $c/2 \in D_{\varphi',c}$, hence we obtain

$$\lim_{t\uparrow c/2} \zeta_c(t) = \lim_{t\downarrow c/2} \zeta_c(t) = \zeta_c(c/2) = 1,$$

so obtaining that every mapping of $\mathcal{Z}_2(X_2(\varphi'))$ is increasing. Finally, the proof is concluded by applying Lemma 4.11. □

**Lemma 4.14** *Let* $\varphi \in \mathcal{S}_2$*. Then,* $\varphi \in \mathcal{S}_1$*.*

**PROOF.** By Propositions 4.5 and 4.6, the claim is equivalent to showing that Eq. (4.2) holds. Reasoning by contradiction, assume there exist $t_0, z_0 \in D_{\varphi'}$, with $t_0 \in ]0,1/2[$ and $z_0 \in ]t_0, 1-t_0[$, such that $\varphi'(t_0) > \varphi'(z_0)$. Set $\epsilon := \varphi'(t_0) - \varphi'(z_0)$. Being $\varphi'$ continuous at $t_0$, there exists a sufficiently small $\delta > 0$ such that $\varphi'(t) > \varphi'(t_0) - \frac{\epsilon}{2}$ for all $t \in [t_0-\delta, t_0] \cap D_{\varphi'}$. This immediately leads to

$$\varphi'(t) > \varphi'(z_0) + \epsilon/2 > \varphi'(z_0) \tag{4.9}$$

for all $t \in [t_0-\delta, t_0] \cap D_{\varphi'}$. Now, assign

$$F := [t_0 + z_0 - \delta, t_0 + z_0] \cap 2((\mathbb{I} \setminus D_{\varphi'}) \cap [0,1/2]).$$

Notice that $\lambda(F) = 0$, hence $F - \{z_0\}$ (see Eq. (2.2)) is a zero measure subset of $[t_0-\delta, t_0]$ due to the invariance under translation of the Lebesgue measure. Accordingly, fixing any $t \in ([t_0-\delta, t_0] \setminus (F - \{z_0\})) \cap D_{\varphi'}$, it is not difficult to see that $\varphi'(\frac{t+z_0}{2})$ exists or, equivalently, $t+z_0 \in X_2(\varphi')$. Now, assign $c := t+z_0$. Since it is clear that $c \in ]0,1]$, $t, c/2 \in D_{\varphi',c}$ and $t < c/2$, by applying Eq. (4.9) yields $\zeta_c(t) > 1 = \zeta_c(c/2)$, obtaining that not every member of $\mathcal{Z}_2(\,]0,1])$ is increasing. The contradiction closes the proof. $\square$

**Remark 4.15** *As noted above, the statement of the previous lemma cannot be reversed. Indeed, consider $\varphi$ as in* Example 4.7. *We know that such $\varphi \in \mathcal{S}_1$. Now, pick any $c \in \,]\frac{3}{4}, 1[$: then, by a simple computation, one finds that $\zeta_c(t) = \frac{4}{5}\sqrt{c - \frac{3}{4} - t}$ as $t \in [0, c - \frac{3}{4}[$, hence it is not increasing.*

Our purpose is to determine the class of generators of penalty functions as in Eq. (3.2) for $n = 2$. Summarizing the last results so far illustrated, we have shown that both the desired class and $\mathcal{S}_2$ are strictly included in $\mathcal{S}_1$. Further, we have somehow conjectured that these two classes might be coincident. It remains to be seen whether this conjecture is true.

**Theorem 4.16** *Let $\varphi \in \mathcal{S}_1$. Then, $\varphi$ is a generator of a 2-dimensional penalty function $P_\varphi : \mathbb{I}^2 \times W_2 \times \mathbb{I} \to \mathbb{I}$ if, and only if, $\varphi \in \mathcal{S}_2$.*

**PROOF.** Using the change of variable $t := y - x_1$ and setting $c := x_2 - x_1$, the claim is equivalent to showing that every member of $\mathcal{F}_2(\,]0,1], \varphi)$ is quasi-convex if, and only if, any $\zeta_c \in \mathcal{Z}_2(\,]0,1], \varphi)$ is increasing. First, we state that we can associate to any $\psi_{c,\mathbf{w}} \in \mathcal{F}_2^{(1)}(\,]0,1]) := \{\psi_{c,\mathbf{w}} \in \mathcal{F}_2(\,]0,1]) : w_1 \geq w_2\}$ a uniquely determined point $t^*_{c,\mathbf{w}} \in [0, c/2]$ such that $\psi'_{c,\mathbf{w}}(t) \geq 0$ for almost all $t \in [t^*_{c,\mathbf{w}}, c]$. Indeed, given any $\psi_{c,\mathbf{w}} \in \mathcal{F}_2^{(1)}(\,]0,1])$, set $J(\psi_{c,\mathbf{w}}) := \{u \in \,]0, c[\colon \psi'_{c,\mathbf{w}}(t) \geq 0$ almost everywhere on $[u, c]\}$. It is quite easy to see that $J(\psi_{c,\mathbf{w}})$ is a (possibly empty) subinterval of $\,]0, c[$. We assert that $J(\psi_{c,\mathbf{w}})$ is not empty, because $[c/2, c[\subseteq J(\psi_{c,\mathbf{w}})$. In fact any $\psi_{c,\mathbf{w}} \in \mathcal{F}_2^{(1)}(\,]0,1])$ is increasing on $[c/2, c]$, because, due to an elementary algebraic manipulation, it may be rewritten as

$$\psi_{c,\mathbf{w}}(t) = w_2(\varphi(t) + \varphi(c - t)) + (w_1 - w_2)\varphi(t),$$

that is a linear combination with non-negative coefficients of two increasing functions on $[c/2, c]$ (the increasing monotonicity of the mapping $\varphi(t)+\varphi(c-t)$ on that subdomain is assured by Proposition 4.6). If we bear in mind Lemma 4.2, this amounts to $\psi'_{c,\mathbf{w}}(t) \geq 0$ for almost all $t \in [c/2, c]$, so the assertion trivially holds as well as the statement, after the assignment $t^*_{c,\mathbf{w}} := \inf J(\psi_{c,\mathbf{w}})$. By the same token, given any $\psi_{c,\mathbf{w}} \in \mathcal{F}_2^{(2)}(\,]0,1]) := \{\psi_{c,\mathbf{w}} \in \mathcal{F}_2(\,]0,1]) : w_2 > w_1\}$, there exists a uniquely determined point $\widehat{t}_{c,\mathbf{w}} \in [c/2, c]$ such that $\psi'_{c,\mathbf{w}}(t) \leq 0$ for almost all $t \in [0, \widehat{t}_{c,\mathbf{w}}]$. Turning back to the claim, let us start with the sufficiency and suppose *ab absurdo* there exists a $\psi_{c,\mathbf{w}} \in \mathcal{F}_2(\,]0,1])$ such that $\psi_{c,\mathbf{w}}$ is not quasi-convex. We may limit ourselves to the case $\psi_{c,\mathbf{w}} \in \mathcal{F}_2^{(1)}(\,]0,1])$, since the remaining one may be treated analogously. Trivially, an arbitrary $\psi_{c,\mathbf{w}} \in \mathcal{F}_2^{(1)}(\,]0,1])$ is not quasi-convex if, and only if, $t^*_{c,\mathbf{w}} > 0$

and $\psi'_{c,\mathbf{w}}(t_0) > 0$ for some $t_0 \in ]0, t^*_{c,\mathbf{w}}[$. By Remark 4.10, this amounts to $\zeta_c(t_0) > \frac{w_2}{w_1}$. By the increasing monotonicity of $\zeta_c$, we immediately derive that for almost all $t \in [t_0, c]$ we have $\zeta_c(t) > \frac{w_2}{w_1}$. Again by Remark 4.10, this means that $\psi'_{c,\mathbf{w}}(t) > 0$ almost everywhere on $[t_0, c]$, so contradicting the specific nature of the point $t^*_{c,\mathbf{w}}$ and closing this step. Conversely, assume that every member of $\mathcal{F}_2(]0,1])$ is quasi-convex. Reasoning by contradiction, suppose that there exists a $c \in ]0,1]$ such that $\zeta_c(t_1) > \zeta_c(t_2)$ for some $t_1, t_2 \in ]0, c[$ such that $t_1 < t_2$. Suppose first that $\zeta_c(t_2) < 1$. Then, we can always choose a $\mathbf{w} \in W_2$ such that

$$w_2/w_1 \in ]\zeta_c(t_2), \zeta_c(t_1)[, \tag{4.10}$$

provided that $w_1 > w_2$. By Eq. (4.10), bearing in mind that $\zeta_c$ is continuous, we immediately deduce that there exists a sufficiently small $\delta > 0$ such that

$$\zeta_c(t) < w_2/w_1 \quad \text{almost everywhere on } [t_2 - \delta, t_2 + \delta], \tag{4.11}$$

and at the same time

$$\zeta_c(t) > w_2/w_1 \quad \text{almost everywhere on } [t_1 - \delta, t_1 + \delta]. \tag{4.12}$$

Owing to Remark 4.10 and Lemma 4.2, we know that Eq. (4.11) and Eq. (4.12) amount to the decreasing and increasing monotonicity of the corresponding $\psi_{c,\mathbf{w}}$ on the intervals $[t_2 - \delta, t_2 + \delta]$ and $[t_1 - \delta, t_1 + \delta]$ respectively, so showing that $\psi_{c,\mathbf{w}}$ is not quasi-convex and contradicting the assumption. The remaining case $\zeta_c(t_2) \geq 1$ may be treated analogously, choosing a $\mathbf{w} \in W_2$ such that

$$w_2/w_1 \in ]\zeta_c(t_2), \zeta_c(t_1)[,$$

provided that $w_2 > w_1$, so concluding the proof. □

**Remark 4.17** *We emphasize that $\mathcal{CA}$ is strictly included in $\mathcal{S}_2$. Indeed, let $\varphi \in \mathcal{S}_0$ : if $\varphi$ is convex, then $\varphi'$ is increasing and so is any $\zeta_c \in \mathcal{Z}_2(]0,1])$, being a ratio whose numerator and denominator are an increasing and a decreasing positive function, respectively. Now, consider the following $\varphi \in \mathcal{S}_0$ :*

$$\varphi(t) = \begin{cases} t^2, & t \leq \frac{3}{4}; \\ 1 - 7(t-1)^2, & \text{otherwise.} \end{cases}$$

*We claim that such mapping, although evidently not convex, belongs to $\mathcal{S}_2$. By* Lemma 4.13, *being $X_2(\varphi') = ]0,1]$, we may check the increasing monotonicity of an arbitrary $\zeta_c \in \mathcal{Z}_2(]0,1])$ on $[0, c/2]$. Fixing any $c \in ]0, \frac{3}{4}]$, $\zeta_c$ is clearly increasing because of the convexity of $\varphi$ on the subdomain $[0, \frac{3}{4}]$. When $c \in ]\frac{3}{4}, 1]$, by a direct computation one finds that*

$$\zeta_c(t) = \begin{cases} \frac{t}{7(t+1-c)}, & t \in [0, c - 3/4[; \\ \frac{t}{c-t}, & t \in ]c - 3/4, c/2], \end{cases}$$

*hence the claim easily follows.*

## 5 The $n$-dimensional case

In this section, we generalize the main results of the previous section in order to determine the generators of $n$-dimensional penalty functions of the form given by Eq. (3.2). For this reason, throughout this section, we will take for granted that $n > 2$, unless otherwise specified.

First, we present the $n$-dimensional versions of notations already introduced in the previous section for $n = 2$.

Let $E_n = \{\mathbf{x} \in ]0,1]^{n-1} : 0 < x_j < x_{j+1},\ j = 1,\dots,n-2\}$. For any vector $\mathbf{x} \in E_n$, we will follow the convention $x_0 := 0$. Moreover, throughout this section, we will reserve the symbol $X$ for denoting an arbitrary subset of $E_n$ dense in the topological closure of $E_n$. Given any $\varphi \in \mathcal{S}_0$, let $\mathcal{F}_n(X,\varphi) = \{\psi_{\mathbf{c},\mathbf{w}} : \mathbf{c} \in X, \mathbf{w} \in W_n\}$ be the class of mappings $\psi_{\mathbf{c},\mathbf{w}} : [0, c_{n-1}] \to \mathbb{R}$ defined by

$$\psi_{\mathbf{c},\mathbf{w}}(t) = \sum_{i=1}^{n} w_i \varphi(|c_{i-1} - t|).$$

We will adopt the shorthand notation $\mathcal{F}_n(X)$ for $\mathcal{F}_n(X,\varphi)$ unless otherwise stated. Obviously, $\mathcal{F}_n(X)$ coincides with the previously introduced $\mathcal{F}_2(X)$ for $n = 2$. Similar to the $n = 2$ case above, we can state that every mapping of $\mathcal{F}_n(X)$ is absolutely continuous and a.e. differentiable.

**Remark 5.1** *Let $\varphi \in \mathcal{S}_0$ and $\mathbf{c} \in X$. Following the same line of reasoning of* Remark 4.3*, it is easy to see that the complement of the set $D_{\varphi',\mathbf{c}} := \{t \in \mathbb{I} : \max\{c_j - t, t - c_j\} \in D_{\varphi'}, j = 0,\dots,n-1\}$ has zero measure, hence we may assume that the domain of the derivative of every member of the subclass $\{\psi_{\mathbf{c},\mathbf{w}} \in \mathcal{F}_n(X) : \mathbf{w} \in W_n\}$ of $\mathcal{F}_n(X)$ is just $D_{\varphi',\mathbf{c}}$.*

The next result is the $n$-dimensional version of Lemma 4.4 and consequently the proof will be omitted.

**Lemma 5.2** *Let $\varphi \in \mathcal{S}_0$. Assume that every function of $\mathcal{F}_n(X)$ is quasi-convex for some dense subset $X \subset E_n$. Then, every mapping of $\mathcal{F}_n(E_n)$ is quasi-convex.*

Let $A_n = \{\mathbf{p} \in ]0,\infty[^n\colon p_1 = p_n = 1\}$. Given any $\varphi \in \mathcal{S}_0$, let $\mathcal{Z}_n(X,\varphi) = \{\zeta_{\mathbf{c},\mathbf{p}} : \mathbf{c} \in X,\ \mathbf{p} \in A_n\}$ be the class of functions defined a.e. on $[0, c_{n-1}]$ and described as

$$\zeta_{\mathbf{c},\mathbf{p}}(t) = \frac{\sum\limits_{i=1}^{k+1} p_i \varphi'(t - c_{i-1})}{\sum\limits_{i=k+2}^{n} p_i \varphi'(c_{i-1} - t)}, \quad t \in ]c_k, c_{k+1}[,\ \ k \in \{0,\dots,n-2\}.$$

We will adopt the shorthand notation $\mathcal{Z}_n(X)$ for $\mathcal{Z}_n(X,\varphi)$ unless otherwise stated. Obviously, $\mathcal{Z}_n(X)$ coincides with the previously introduced $\mathcal{Z}_2(X)$ for $n = 2$.

**Remark 5.3** *Let $\varphi \in \mathcal{S}_0$ and $\mathbf{c} \in X$. Observe that there is a natural, bi-univocal correspondence between the subclass $\{\psi_{\mathbf{c},\mathbf{w}} \in \mathcal{F}_n(X,\varphi) : \mathbf{w} \in W_n\}$ of $\mathcal{F}_n(X,\varphi)$ and the subclass $\{\zeta_{\mathbf{c},\mathbf{p}} \in \mathcal{Z}_n(X,\varphi) : \mathbf{p} \in A_n\}$ of $\mathcal{Z}_n(X,\varphi)$. In particular, according to* Remark 5.1, *we may assume that the domain of any member of $\{\zeta_{\mathbf{c},\mathbf{p}} \in \mathcal{Z}_n(X,\varphi) : \mathbf{p} \in A_n\}$ coincides with $D_{\varphi',\mathbf{c}}$. Moreover, let $\psi_{\mathbf{c},\mathbf{w}} \in \mathcal{F}_n(X,\varphi)$ and $t \in ]c_k, c_{k+1}[\cap D_{\varphi',\mathbf{c}}$ for some $k \in \{0,\ldots,n-2\}$ : an elementary computation shows that the inequality $\psi'_{\mathbf{c},\mathbf{w}}(t) > 0$ amounts to $\zeta_{\mathbf{c},\mathbf{p_w}}(t) > \frac{w_n}{w_1}$, where the vector $\mathbf{p_w} \in A_n$ is given by*

$$(p_{\mathbf{w}})_j = \begin{cases} w_j/w_1, & j \in \{1,\ldots,k+1\}; \\ w_j/w_n, & j \in \{k+2,\ldots,n\}. \end{cases} \tag{5.1}$$

We emphasize the fact that the property of increasing monotonicity of all mappings of $\mathcal{Z}_n(E_n)$ will play a crucial role in characterizing the family of generators of penalty functions of the form given by Eq. (3.2). The next result, which states that it suffices to verify such property at least for a particular dense subset $X \subset E_n$, relies upon a similar argument as we have used in the proof of the corresponding Lemma 4.11, hence the proof will be omitted.

**Lemma 5.4** *Let $\varphi \in \mathcal{S}_0$. Assume that there exists a dense subset $X \subset E_n$ such that every mapping of $\mathcal{Z}_n(X)$ is increasing. Then, every member of $\mathcal{Z}_n(E_n)$ is increasing.*

Let us introduce the notable subclass of $\mathcal{S}_0$ given by $\mathcal{S}_n = \{\varphi \in \mathcal{S}_0 :$ every member of $\mathcal{Z}_n(E_n)$ is increasing$\}$. Note that $\mathcal{S}_n$ reduces to the previously introduced $\mathcal{S}_2$ when $n = 2$. Remark that the verification task of the characterizing property of $\mathcal{S}_n$ may be slightly simplified if we make use of a suitable dense subset $X$, illustrated in the next lemma.

**Lemma 5.5** *Let $\varphi \in \mathcal{S}_0$ and $n > 2$. Let $X_n(\varphi') = \{\mathbf{c} \in E_n : c_k - c_j \in D_{\varphi'}$ for all $k, j \in \{0,\ldots,n-1\}$ such that $k < j\}$. Then, $X_n(\varphi')$ is a dense subset of $E_n$.*

**PROOF.** Let $\mathbf{x} \in E_n$ and $\epsilon > 0$. Being $D_{\varphi'}$ a dense subset of $\mathbb{I}$ (see Remark 4.1), after selecting $\delta_1 = \min\{\epsilon, x_1\}$, we know that there exists a $c_1 \in ]x_1 - \delta_1, x_1]$ such that $0 < c_1 \leq x_1$, with $x_1 - c_1 < \epsilon$, and $c_1 \in D_{\varphi'}$. Now, consider the set $(\mathbb{I} \setminus D_{\varphi'}) + \{c_1\}$: evidently, this set has measure zero, hence its complement with respect to $D_{\varphi'}$ is dense in $\mathbb{I}$. Thus, after selecting $\delta_2 = \min\{\epsilon, x_2 - x_1\}$, there exists a $c_2 \in ]x_2 - \delta_2, x_2]$ such that $c_2 \in D_{\varphi'} \setminus ((\mathbb{I} \setminus D_{\varphi'}) + \{c_1\})$. While it is obvious that $x_1 < c_2 \leq x_2$, with $x_2 - c_2 < \epsilon$, and $c_2 \in D_{\varphi'}$, it is not difficult to check that $c_2 - c_1 \in D_{\varphi'}$. Repeating the same argument for a finite number of steps, it is quite easy to see that we can determine a vector $\mathbf{c} \in ]0,1]^{n-1}$ (following the convention $c_0 := 0$) satisfying the following properties:

(A1) $x_{k-1} < c_k \leq x_k,\ k = 1, \ldots, n-1$;
(A2) $c_k - c_j \in D_{\varphi'}$ for all $k, j \in \{0, \ldots, n-1\}$ such that $k < j$;
(A3) $x_k - c_k < \epsilon,\ k = 1, \ldots, n-1$.

From (A1)-(A2), it directly follows that $\mathbf{c} \in X_n(\varphi')$. Moreover, by (A3) we derive that $\|\mathbf{c} - \mathbf{x}\|_\infty < \epsilon$, where the symbol $\|\cdot\|_\infty$ stands for the sup norm, so closing the proof. $\square$

We emphasize that generally $X_n(\varphi')$ does not reduce to $X_2(\varphi')$ introduced in Remark 4.12 when $n = 2$.

**Lemma 5.6** *Let $\varphi \in \mathcal{S}_1$. Assume that every mapping of $\mathcal{Z}_n(X_n(\varphi'))$ is increasing on any subdomain $[c_k, c_{k+1}]$ for all $k \in \{0, \ldots, n-2\}$. Then, every member of $\mathcal{Z}_n(E_n)$ is increasing.*

**PROOF.** By Lemma 5.4, given any $\zeta_{\mathbf{c},\mathbf{p}} \in \mathcal{Z}_n(X_n(\varphi'))$, it suffices to show that $\zeta_{\mathbf{c},\mathbf{p}}(c_k{}^-) \leq \zeta_{\mathbf{c},\mathbf{p}}(c_k{}^+)$ for any $k \in \{1, \ldots, n-2\}$, where

$$\zeta_{\mathbf{c},\mathbf{p}}(c_k{}^-) := \lim_{t \uparrow c_k} \zeta_{\mathbf{c},\mathbf{p}}(t), \quad \zeta_{\mathbf{c},\mathbf{p}}(c_k{}^+) := \lim_{t \downarrow c_k} \zeta_{\mathbf{c},\mathbf{p}}(t).$$

Observe that $\zeta_{\mathbf{c},\mathbf{p}}(c_k{}^-)$ and $\zeta_{\mathbf{c},\mathbf{p}}(c_k{}^+)$ exist due to the monotonicity of $\zeta_{\mathbf{c},\mathbf{p}}$ on both the intervals $[c_{k-1}, c_k]$ and $[c_k, c_{k+1}]$. As a straightforward consequence of the characterizing property of $X_n(\varphi')$ and the continuity of $\varphi'$, it is quite easy to check that

$$\zeta_{\mathbf{c},\mathbf{p}}(c_k{}^-) = \frac{\sum\limits_{i=1}^{k} p_i \varphi'(c_k - c_{i-1})}{\sum\limits_{i=k+1}^{n} p_i \varphi'(c_{i-1} - c_k)}$$

and

$$\zeta_{\mathbf{c},\mathbf{p}}(c_k{}^+) = \frac{\sum\limits_{i=1}^{k+1} p_i \varphi'(c_k - c_{i-1})}{\sum\limits_{i=k+2}^{n} p_i \varphi'(c_{i-1} - c_k)}.$$

Note that both $\zeta_{\mathbf{c},\mathbf{p}}(c_k{}^-)$ and $\zeta_{\mathbf{c},\mathbf{p}}(c_k{}^+)$ contain the term $\varphi'(0)$, which certainly exists and is a finite non-negative number by (P1). Finally, the desired inequality easily follows from the fact that the numerator of $\zeta_{\mathbf{c},\mathbf{p}}(c_k{}^+)$ is the sum of the numerator of $\zeta_{\mathbf{c},\mathbf{p}}(c_k{}^-)$ and the non-negative term $p_{k+1}\varphi'(0)$ and the same holds for the denominators, with interchanged roles of $\zeta_{\mathbf{c},\mathbf{p}}(c_k{}^-)$ and $\zeta_{\mathbf{c},\mathbf{p}}(c_k{}^+)$. $\square$

Now we are ready to state the corresponding result of Theorem 4.16 for the $n$-dimensional case. Note that, being interested in the case $n > 2$, we can assume without loss of generality that any candidate $\varphi$ belongs to $\mathcal{S}_2$.

**Theorem 5.7** *Let $\varphi \in \mathcal{S}_2$. Then, $\varphi$ is a generator of a $n$-dimensional penalty function $P_\varphi : \mathbb{I}^n \times W_n \times \mathbb{I} \to \mathbb{I}$ if, and only if, $\varphi \in \mathcal{S}_n$.*

**PROOF.** Using the change of variable $t := y - x_1$ and setting $c_{i-1} := x_i - x_1$ for $i = 1, \dots, n$, the claim is equivalent to showing that every member of $\mathcal{F}_n(E_n, \varphi)$ is quasi-convex if, and only if, any $\zeta_{\mathbf{c},\mathbf{p}} \in \mathcal{Z}_n(E_n, \varphi)$ is increasing. First, we state that we can associate to any $\psi_{\mathbf{c},\mathbf{w}} \in \mathcal{F}_n^{(1)}(E_n) := \{\psi_{\mathbf{c},\mathbf{w}} \in \mathcal{F}_n(E_n) : w_1 \geq w_n\}$ a uniquely determined point $t^*_{\mathbf{c},\mathbf{w}} \in [0, c_{n-1}[$ such that $\psi'_{\mathbf{c},\mathbf{w}}(t) \geq 0$ for almost all $t \in [t^*_{\mathbf{c},\mathbf{w}}, c_{n-1}]$. Indeed, repeating same argument as we have used in the corresponding part of the proof of Theorem 4.16, let $\psi_{\mathbf{c},\mathbf{w}} \in \mathcal{F}_n^{(1)}(E_n)$ and set $J(\psi_{\mathbf{c},\mathbf{w}}) := \{u \in ]0, c_{n-1}[ : \psi'_{\mathbf{c},\mathbf{w}}(t) \geq 0 \text{ almost everywhere on } [u, c_{n-1}]\}$. Since it is clear that $J(\psi_{\mathbf{c},\mathbf{w}})$ is a subinterval of $]0, c_{n-1}[$, the statement easily follows from the assignment $t^*_{\mathbf{c},\mathbf{w}} := \inf J(\psi_{\mathbf{c},\mathbf{w}})$, provided $J(\psi_{\mathbf{c},\mathbf{w}})$ is not empty. To this end, by an elementary algebraic manipulation, any $\psi_{\mathbf{c},\mathbf{w}} \in \mathcal{F}_n^{(1)}(E_n)$ may be seen as the sum of two functions, i.e. $w_1\varphi(t) + w_n\varphi(c_{n-1} - t)$ and $\sum_{i=2}^{n-1} w_i\varphi(|c_{i-1} - t|)$, called in the sequel first and second summand, respectively. From the trivial identity

$$w_1\varphi(t) + w_n\varphi(c_{n-1} - t) = (w_1 + w_n) \cdot \left(\frac{w_1}{w_1 + w_n}\varphi(t) + \frac{w_n}{w_1 + w_n}\varphi(c_{n-1} - t)\right)$$

we easily deduce that, due to Theorem 4.16, the first summand is increasing at least on $[c_{n-1}/2, c_{n-1}]$. At the same time, since the second summand may be rewritten as $\sum_{i=2}^{n-1} w_i\varphi(t - c_{i-1})$ for $t \in [c_{n-2}, c_{n-1}]$, it is certainly increasing on the subdomain $[c_{n-2}, c_{n-1}]$ due to the monotonicity of $\varphi$. Accordingly, we derive that $\psi_{\mathbf{c},\mathbf{w}}$ is increasing at least on $[\max\{c_{n-1}/2, c_{n-2}\}, c_{n-1}]$, hence, due to Lemma 4.2, we can draw the desired conclusion. By the same token, given any $\psi_{\mathbf{c},\mathbf{w}} \in \mathcal{F}_n^{(2)}(E_n) := \{\psi_{c,\mathbf{w}} \in \mathcal{F}_n(E_n) : w_1 < w_n\}$, there exists a uniquely determined point $\widehat{t}_{\mathbf{c},\mathbf{w}} \in ]0, c_{n-1}]$ such that $\psi'_{\mathbf{c},\mathbf{w}}(t) \leq 0$ for almost all $t \in [0, \widehat{t}_{\mathbf{c},\mathbf{w}}]$. Turning back to the claim, let us start with the sufficiency and suppose *ab absurdo* there exists a $\psi_{\mathbf{c},\mathbf{w}} \in \mathcal{F}_n(E_n)$ such that $\psi_{\mathbf{c},\mathbf{w}}$ is not quasi-convex. We may limit ourselves to the case $\psi_{\mathbf{c},\mathbf{w}} \in \mathcal{F}_n^{(1)}(E_n)$, since the remaining one may be dealt with in analogous manner. Trivially, an arbitrary $\psi_{\mathbf{c},\mathbf{w}} \in \mathcal{F}_n^{(1)}(E_n)$ is not quasi-convex if, and only if, $t^*_{\mathbf{c},\mathbf{w}} > 0$ and $\psi'_{\mathbf{c},\mathbf{w}}(t_0) > 0$ for some $t_0 \in ]0, t^*_{\mathbf{c},\mathbf{w}}[$. By Remark 5.3, this amounts to $\zeta_{\mathbf{c},\mathbf{p_w}}(t) > \frac{w_n}{w_1}$, where the vector $\mathbf{p_w}$ is as in Eq. (5.1). By the increasing monotonicity of $\zeta_{\mathbf{c},\mathbf{p_w}}$, we immediately derive that for almost all $t \in [t_0, c_{n-1}]$ it holds $\zeta_{\mathbf{c},\mathbf{p_w}}(t) > \frac{w_n}{w_1}$. Again by Remark 5.3, this means that $\psi'_{\mathbf{c},\mathbf{w}}(t) > 0$ almost everywhere on $[t_0, c_{n-1}]$, so contradicting the specific nature of the point $t^*_{\mathbf{c},\mathbf{w}}$ and closing this step. Conversely, assume that every member of $\mathcal{F}_n(E_n)$ is quasi-convex. Reasoning by contradiction, suppose that there exists at least a member of $\mathcal{Z}_n(E_n)$ which is not increasing.

By Lemma 5.6, this entails that there exists a $\zeta_{\mathbf{c},\mathbf{p}} \in \mathscr{Z}_n(X_n(\varphi'))$ such that $\zeta_{\mathbf{c},\mathbf{p}}(t_1) > \zeta_{\mathbf{c},\mathbf{p}}(t_2)$ for some $c_k < t_1 < t_2 < c_{k+1}$ and for some $k \in \{0, \ldots, n-2\}$. We only study the case $\zeta_{\mathbf{c},\mathbf{p}}(t_2) < 1$, since the case $\zeta_{\mathbf{c},\mathbf{p}}(t_2) \geq 1$ may be treated analogously. Then, after the assignment $p_t := \sum_{i=1}^{n} p_i$, let us consider the family $U_n(p_t)$ of vectors $\mathbf{w}(\beta) \in \,]0,1]^n$ described as

$$w_j(\beta) = \begin{cases} p_j/p_t + \beta p_j \dfrac{\sum\limits_{i=k+2}^{n} p_i}{\sum\limits_{i=1}^{k+1} p_i}, & j = 1, \ldots, k+1; \\ p_j/p_t - \beta p_j, & j = k+2, \ldots, n, \end{cases} \tag{5.2}$$

for $\beta \in \,[0, p_t[$. By a simple computation, one easily finds that $U_n(p_t)$ is a subset of $W_n$. Now, if we consider the mapping $\beta \mapsto f(\beta) := w_n(\beta)/w_1(\beta)$, it is not a difficult task to check that $f$ is strictly decreasing on $[0, 1/p_t[$ with boundary values $f(0) = 1$ and $f(1/p_t) = 0$. Hence, we can always choose a $\beta^* \in \,]0, 1/p_t[$ such that

$$f(\beta^*) \in \,]\zeta_{\mathbf{c},\mathbf{p}}(t_2), \zeta_{\mathbf{c},\mathbf{p}}(t_1)[. \tag{5.3}$$

By Eq. (5.3), bearing in mind that $\zeta_{\mathbf{c},\mathbf{p}}$ is continuous, we immediately deduce that there exists a sufficiently small $\delta > 0$ such that

$$\zeta_{\mathbf{c},\mathbf{p}}(t) < f(\beta^*) \quad \text{almost everywhere on } [t_2 - \delta, t_2 + \delta], \tag{5.4}$$

and at the same time

$$\zeta_{\mathbf{c},\mathbf{p}}(t) > f(\beta^*) \quad \text{almost everywhere on } [t_1 - \delta, t_1 + \delta]. \tag{5.5}$$

According to Eq. (5.1) and Eq. (5.2), an elementary calculation shows that $\mathbf{p} = \mathbf{p}_{\mathbf{w}^*}$, where $\mathbf{w}^* := \mathbf{w}(\beta^*)$. Therefore, owing to Remark 5.3 and Lemma 4.2, we know that Eq. (5.4) and Eq. (5.5) amount to the decreasing and increasing monotonicity of the corresponding $\psi_{\mathbf{c},\mathbf{w}^*}$ on the intervals $[t_2 - \delta, t_2 + \delta]$ and $[t_1 - \delta, t_1 + \delta]$, respectively. This shows that $\psi_{\mathbf{c},\mathbf{w}^*}$ is not quasi-convex, so contradicting the assumption and definitely concluding the proof. □

Owing to the above result, it is not difficult to show that $\mathcal{S}_{n+1} \subset \mathcal{S}_n$ for all $n \geq 2$. In fact, by Theorem 5.7, it is enough to show that every member of $\mathcal{F}_n(E_n, \varphi)$ is quasi-convex for any fixed $\varphi \in \mathcal{S}_{n+1}$. The claim is trivially proved if, given any $\psi_{\mathbf{c},\mathbf{w}} \in \mathcal{F}_n(E_n, \varphi)$ and any $\epsilon > 0$, there exists a quasi-convex mapping $g_\epsilon : [0, c_{n-1}] \to \mathbb{R}$ such that $\|\psi_{\mathbf{c},\mathbf{w}} - g_\epsilon\|_\infty \leq \epsilon$. Let us choose as $g_\epsilon$ the function given by $g_\epsilon(t) = \psi_{\mathbf{c},\mathbf{w}}(t) + \epsilon \cdot \varphi(|c_\epsilon - t|)$, where $c_\epsilon$ is any real number belonging to $]0, c_1[$. It is immediate to check that $\|\psi_{\mathbf{c},\mathbf{w}} - g_\epsilon\|_\infty \leq \epsilon$, hence it remains to be seen whether $g_\epsilon$ or, equivalently, $g_\epsilon/(1+\epsilon)$ is quasi-convex. The last step, left to the reader, is to verify that $g_\epsilon/(1+\epsilon) = \psi_{\mathbf{c}_\epsilon,\mathbf{w}_\epsilon} \in \mathcal{F}_{n+1}(E_{n+1}, \varphi)$, where $\mathbf{c}_\epsilon \in E_{n+1}$ and $\mathbf{w}_\epsilon \in W_{n+1}$ are respectively given by

$$(c_\epsilon)_j = \begin{cases} c_\epsilon, & j = 1; \\ c_{j-1}, & j = 2, \ldots, n \end{cases}$$

and

$$(w_\epsilon)_j = \begin{cases} w_1/(1+\epsilon), & j = 1; \\ \epsilon/(1+\epsilon), & j = 2; \\ w_{j-1}/(1+\epsilon), & j = 3, \ldots, n+1. \end{cases}$$

We have proved the following.

**Lemma 5.8** *Let $\varphi \in \mathcal{S}_{n+1}$. Then, $\varphi \in \mathcal{S}_n$.*

Notice that $\mathcal{CA}$ is a subclass of $\mathcal{S}_n$ for any $n \geq 2$: this statement is exactly based upon the same argument as the one used in Remark 4.17.

Let $\mathcal{S} := \bigcap_{n=2}^{\infty} \mathcal{S}_n$: as above noted, $\mathcal{CA} \subseteq \mathcal{S}$. Now the problem is: does $\mathcal{S}$ exactly coincide with $\mathcal{CA}$ or not? Well, the answer is negative, as we show in the conclusive part of this section.

Consider the function $\varphi \in \mathcal{S}_2$ introduced in Remark 4.17: we will prove that such mapping, which is not convex, actually belongs to $\mathcal{S}_n$ for every fixed $n > 2$. Let $\zeta_{\mathbf{c},\mathbf{p}} \in \mathcal{Z}_n(X_n(\varphi'))$ and let $k \in \{0, \ldots, n-2\}$: by Lemma 5.6, the claim is equivalent to saying that $\zeta_{\mathbf{c},\mathbf{p}}$ is increasing on $[c_k, c_{k+1}]$. Consider the two finite sequences $\xi_i := c_{i-1} + 3/4$ for $i = 1, \ldots, k+1$ and $\eta_i := c_{i-1} - 3/4$ for $i = k+2, \ldots, n$. It is immediate to see that

$$\eta_{k+2} < \ldots < \eta_n < \xi_1 < \ldots < \xi_{k+1} \tag{5.6}$$

and that $\xi_i \neq c_k, c_{k+1}$ for all $i \in \{1, \ldots, k+1\}$ as well as $\eta_i \neq c_k, c_{k+1}$ for all $i \in \{k+2, \ldots, n\}$, because of the characterizing property of $X_n(\varphi')$. Note that $\varphi'$ exists at all $t \in \mathbb{I}$ except for $t = \xi_1$. Let us divide the analysis of $\zeta_{\mathbf{c},\mathbf{p}}$ on $[c_k, c_{k+1}]$ into two main cases:

(1) $\xi_1 < c_k$;
(2) $\xi_1 > c_k$.

In case 1, we immediately deduce that $k > 0$ and $[c_k, c_{k+1}] \subset D_{\varphi',\mathbf{c}}$. This case in its turn may be divided into the two following sub-cases:

(1A) there exists a unique $r \in \{1, \ldots, k\}$ such that $\xi_r < c_k < c_{k+1} < \xi_{r+1}$;
(1B) there exist two uniquely determined indices $r, s \in \{1, \ldots, k\}$, with $r < s$, such that $\xi_r < c_k < \xi_s < c_{k+1} < \xi_{s+1}$.

In the first sub-case, also taking into account Eq. (5.6), it is only a matter of computation to check that for any $t \in [c_k, c_{k+1}]$ we have

$$\zeta_{\mathbf{c},\mathbf{p}}(t) = 7\mu_r(t) + \nu_r(t), \tag{5.7}$$

where

$$\mu_j(t) := \frac{-(\sum_{i=1}^{j} p_i)t + \sum_{i=1}^{j} p_i(1 + c_{i-1})}{-(\sum_{i=k+2}^{n} p_i)t + \sum_{i=k+2}^{n} p_i c_{i-1}}$$

and

$$\nu_j(t) := \frac{\sum_{i=j+1}^{k+1} p_i(t - c_{i-1})}{\sum_{i=k+2}^{n} p_i(c_{i-1} - t)}$$

for any index $j \in \{1, \dots, k\}$. The increasing monotonicity of $\nu_r(t)$ is immediate, while the condition of (strict) increasing monotonicity for $\mu_r(t)$ is easily shown to be given by

$$\Big(\sum_{i=1}^{r} p_i\Big) \cdot \Big(\sum_{i=k+2}^{n} p_i c_{i-1}\Big) < \Big(\sum_{i=1}^{r} p_i(1 + c_{i-1})\Big) \cdot \Big(\sum_{i=k+2}^{n} p_i\Big). \tag{5.8}$$

Observe that Eq. (5.8) directly follows from the trivial inequalities

$$(1 + c_{i-1})p_i \geq p_i,\ i = 1, \dots, r; \qquad p_i \geq p_i c_{i-1},\ i = k + 2, \dots, n$$

and the fact that at least one of the above inequalities is necessarily strict. In sub-case 1B, it is not a difficult task to see that $\zeta_{\mathbf{c},\mathbf{p}}$, restricted to every subinterval of the type $[c_k, \xi_{r+1}]$,...,$[\xi_s, c_{k+1}]$, is again of the form described in Eq. (5.7), with the only difference that $\mu_r(t)$ and $\nu_r(t)$ must be replaced by $\mu_j(t)$ and $\nu_j(t)$ respectively for any $j \in \{r, \dots, s\}$, hence we may omit the proof. In case 2, we only discuss the sub-case in which there exist two uniquely determined indices $r \in \{k + 2, \dots, n - 1\}$ and $s \in \{1, \dots, k\}$ such that

$$\eta_r < c_k < \eta_{r+1} < \eta_n < \xi_1 < \xi_s < c_{k+1} < \xi_{s+1},$$

because any other sub-case is similar and even simpler. Note that $[c_k, c_{k+1}] \cap D_{\varphi',\mathbf{c}}$ is the finite union of three types of subintervals, i.e. $[a_j, \eta_{j+1}]$ for any $j \in \{r, \dots, n - 1\}$, where $a_j := \max\{c_k, \eta_j\}$, then $[\eta_n, \xi_1[$ and finally $[\xi_j, b_j] \setminus \{\xi_1\}$ for any $j \in \{1, \dots, s\}$, where $b_j := \min\{c_{k+1}, \xi_{j+1}\}$. For the first type, after some elementary algebra, one finds that

$$\frac{1}{\zeta_{\mathbf{c},\mathbf{p}}(t)} = 7\mu_j^*(t) + \nu_j^*(t),$$

where

$$\mu_j^*(t) := \frac{(\sum_{i=j+1}^{n} p_i)t + \sum_{i=j+1}^{n} p_i(1 - c_{i-1})}{(\sum_{i=1}^{k+1} p_i)t - \sum_{i=1}^{k+1} p_i c_{i-1}}$$

and

$$\nu_j^*(t) := \frac{\sum_{i=k+2}^{j} p_i(c_{i-1} - t)}{\sum_{i=1}^{k+1} p_i(t - c_{i-1})}.$$

The decreasing monotonicity of $\nu_j^*(t)$ is immediate as well as the (strict) decreasing monotonicity of $\mu_j^*(t)$ due to a trivial condition similar to Eq. (5.8), so closing this first step. The increasing monotonicity of $\zeta_{\mathbf{c},\mathbf{p}}$ on the subinterval $[\eta_n, \xi_1[$ is a straightforward consequence of the convexity of $\varphi$ on such subdomain. Finally, relatively to any subinterval of the third type, it is quite easy to see that Eq. (5.7) still holds, with $\mu_j(t)$ and $\nu_j(t)$ in place of $\mu_r(t)$ and $\nu_r(t)$, respectively. In order to conclude that the proposed $\varphi$ belongs to $\mathcal{S}$, it remains to be seen whether $\zeta_{\mathbf{c},\mathbf{p}}({\xi_1}^-) \leq \zeta_{\mathbf{c},\mathbf{p}}({\xi_1}^+)$: this directly follows from the fact that, after some trivial computation, we get

$$\frac{\zeta_{\mathbf{c},\mathbf{p}}({\xi_1}^-)}{\zeta_{\mathbf{c},\mathbf{p}}({\xi_1}^+)} = \frac{3/4 + \sum_{i=2}^{k+1} p_i(\xi_1 - c_{i-1})}{7/4 + \sum_{i=2}^{k+1} p_i(\xi_1 - c_{i-1})}.$$

## 6 Conclusion

We have discussed a model of construction of penalty functions by means of a general class of restricted dissimilarity functions, depending on automorphisms of the real unit interval also called generators, which properly includes the faithful restricted dissimilarity functions appeared in [3]. We have characterized the generators of penalty functions for any dimension $n \geq 2$. We have provided an explicit example which shows that convexity is not a necessary property for a generator of penalty functions of arbitrary dimension.

In the future, we plan to study the assumptions to be demanded to an arbitrary restricted dissimilarity function $d_R$ in order for a one-variable mapping of the type

$$y \mapsto P(\mathbf{x}, \mathbf{w}, y) = \sum_{i=1}^{n} w_i d_R(x_i, y)$$

to be a penalty function, so extending the results proposed in this paper.

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