`ArchNURBS`: NURBS-Based Tool for the Structural Safety Assessment of Masonry Arches in MATLAB®

Andrea Chiozzi[1]; Marcello Malagù [2]; Antonio Tralli [3]; and Antonio Cazzani [4]

ABSTRACT

A new approach toward a fully CAD-integrated structural analysis of arched masonry structures is proposed and a new MATLAB®-based computational tool, named `ArchNURBS`, is developed. It is addressed to professionals dealing with restoration or structural rehabilitation of historical constructions, who need to assess the safety of masonry arches under assigned load distributions. By using it, they can easily produce estimates of the carrying capacity of curved masonry members, and specifically arches of arbitrary shape. A Computer Aided Design (CAD) environment, which is very popular among professionals, can be employed to provide a Non-Uniform Rational B-Splines (NURBS) representation of the arch geometry. On the basis of such a representation it is then possible to perform both an elastic isogeometric analysis and a limit analysis of the structure up to the collapse load. Moreover, the developed tool is also devised for handling the presence of Fiber-Reinforced Polymers (FRP) reinforcement strips at the extrados and/or the intrados. This allows for the design of properly dimensioned reinforcement and its verification according to current building codes. The entire procedure relies upon a sound theoretical background. `ArchNURBS` is going to be freely distributed as an open-source project (`http://sourceforge.net/projects/archnurbs/`).

Keywords: Masonry, Arches, Computer Aided Design, Elastic analysis, Limit analysis, Fiber

---

[1]Corresponding Author, PhD, Postdoctoral Fellow, Department of Engineering, University of Ferrara, 1 Via Saragat, I-44122 Ferrara, Italy. E-mail: andrea.chiozzi@unife.it

[2]PhD Student, Department of Engineering, University of Ferrara, 1 Via Saragat, I-44122 Ferrara, Italy. E-mail: marcello.malagu@unife.it

[3]Full Professor, Department of Engineering, University of Ferrara, 1 Via Saragat, I-44122 Ferrara, Italy. E-mail: tra@unife.it

[4]Associate Professor, Department of Civil and Environmental Engineering and Architecture, University of Cagliari, 2 Via Marengo, I-09123 Cagliari, Italy. E-mail: antonio.cazzani@unica.it

reinforced polymers

INTRODUCTION

The paper is concerned with an ancient topic, the analysis of the structural behavior of curved masonry members like arches, which is here being revisited through modern tools, leading to the development of a new MATLAB®-based open-source computational tool named `ArchNURBS` for the safety assessment of masonry arches.

Currently, there is a large amount of literature regarding the analysis up to collapse of masonry arches and several methods are available for the assessment of the mechanical behaviour of historical masonry constructions. The interested reader is addressed to (Roca et al. 2007) and (Tralli et al. 2014) for an extensive state-of-the-art survey. A number of commercial software products which allow evaluating the bearing capacity of a masonry arch have been developed (LimitState Ltd. 2007; Gelfi 2008; AEDES 2014). However, these computer codes mainly cover those cases in which the arch shape can be assimilated to a polyline. Nonetheless, even though many arches may be correctly represented by a polyline, there is a wide class of arches which are not. For instance, this is the case of either masonry arches where the dimensions of the blocks are much smaller than the arch characteristic dimensions (see, for example, the three-centered masonry arch of Llanelltyd Bridge, Wales, portrayed in Fig. 1a) or arches composed by rounded stone voussoirs (see, for example, the arch of Porta Asinaria in Rome, Italy, depicted in Fig. 1b). Furthermore, a suitable approach capable of accurately and efficiently analyzing these cases is still lacking. Computer Aided Design (CAD) is a natural environment for developing a tool for the analysis of masonry arches which is both efficient and intuitive for professionals in the field of structural engineering and structural rehabilitation of historical masonry constructions, among which the use of CAD design representation techniques is widespread.

A CAD geometric representation of curved masonry members of arbitrary shape can be obtained through the use of Non-Uniform Rational B-Splines (NURBS) which consist of rational basis functions built upon common B-Splines basis functions in such a way that a given set of points lying in a known range is suitably approximated with a sufficiently high degree of regularity. Development of NURBS began in the 1950s and was carried out by engineers (mostly in the car manufacturing business) who needed a mathematically precise representation of free-form surfaces like those used for ship hulls, aerospace exterior surfaces, and car bodies, which could be exactly reproduced whenever it was technically needed. NURBS are commonly used in Computer Aided-Design (CAD), -Manufacturing (CAM), and -Engineering (CAE) systems and are part of numerous industry-wide standards. They can be efficiently handled by computer programs and yet allow for easy human interaction. In general, editing NURBS geometries is highly intuitive and predictable. Moreover, NURBS exactly represent particular geometries such as circles, parabolas and ellipses (Piegl and Tiller 1997).

In the last decade, NURBS have been extensively studied and developed for both describing the geometry of a structural model and for representing (with the role of basis functions) the displacement field within the Finite Element Method (FEM) (Hughes et al. 2005). Even if the use of polynomial functions belonging to the spline family for the approximate solution of boundary value problems dates back almost four decades (see e.g. (Prenter 1975; de Boor 1978; Benedetti and Tralli 1989; Gontier and Vollmer 1995)) this new method, which is known as Iso-Geometric Analysis (IGA), was precisely developed to cover the wide existing gap between the worlds of FEM and CAD (see e.g. (Hughes et al. 2005; Bazilevs et al. 2006; Cottrell et al. 2009; Benson et al. 2010; Auricchio et al. 2012)). As it is well-known, the term *isogeometric* is referred to a co-incidence of the geometric model, which is built in a CAD environment, and the structural model (i.e. the FEM model) used for performing stress analysis. In traditional FEM analysis structural model and geometric model never coincide since they are both representations of a true object but relying on different basis functions. This, in turn, produces accuracy-related issues in the computations, particularly for curved thin structures. Besides, if NURBS are used as basis functions, their smoothness is inherited by the FEM model, too: this is particularly important because it allows circumventing some serious difficulties in developing finite elements, e.g. flexible beams and plates where both bending and shear deformation must be accounted for. Moreover, the better a function is approximated, the smaller the error affecting its derivatives: since stress fields are not the primary solution variables, but need to be computed by differentiating displacements through post-processing techniques, smoother displacement fields ensure a more accurate approximation of the stresses.

On the other side, the growing interest in the preservation of masonry structures gave, in the past, an impulse towards the development of new efficient tools for evaluating the ultimate load-bearing capacity of these structures, in particular of masonry arches. From a mechanical point of view, the analysis of masonry arches begins with the contributions of the late 1600s English school (Hooke, Gregory) who stated the analogy between the inverted shape of a catenary and an arch subjected to compressive stresses. Nowadays a sound theoretical framework for the evaluation of masonry arches exists and it can be affirmed (following Huerta (Huerta 2001) and Como (Como 2013)) that the modern theory of limit analysis of masonry structures, which has been developed mainly by Heyman (Heyman 1966; Heyman 1982), is a powerful tool for properly understanding and analyzing curved masonry structures. Many other methods of analysis, other than limit analysis, can be used, of course, for determining the ultimate load carrying capacity of masonry arch bridges, e.g. non-linear FEM analysis, discrete element analysis, hybrid discrete/finite element methods etc. (see, for instance, (Crisfield 1997; Cundall and Strack 1979; Munjiza 2004)) and a number of commercial FEM codes have been developed (e.g. DIANA). However, with such methods the collapse load is identified as a by-product of an indirect (and potentially long) iterative non linear analysis procedure, which is often prone to numerical instabilities. Moreover, a non-linear incremental analysis of a masonry structure requires the definition of many material parameters which have to be precisely known in order to get reliable results. Finally, limit analysis may simply be extended to the case of masonry having a limited compressive strength (see e.g. (Livesley 1992; Orduna and Lourenço 2003)) and to the case of FRP (fiber-reinforced polymers) reinforced arches (see e.g. (Caporale et al. 2006; Basilio et al. 2014; Briccoli Bati et al. 2013)).

Nevertheless, a NURBS-based approach to limit analysis is still lacking. Moreover, a software product for the structural analysis of masonry arches which is capable of dealing with complex geometries which are not adequately approximated by a polyline has not been devised yet. Such a tool should allow the user to import the exact arch geometry, which can be easily generated within a CAD environment using NURBS curves. In addition, the tool should allow the user to carry out both an elastic and a limit analysis of the arch to be studied in order to assess its mechanical response respectively under service loads and at failure. In order to reach a new level of accuracy and to make the usage of the software simple and intuitive, such analyses should be based on the NURBS representation of the arch geometry.

In this paper, a new open-source CAD-based tool for the analysis of masonry arches which is specifically addressed to professionals in the field of structural engineering and structural rehabilitation of historical masonry constructions is proposed. The tool, named `ArchNURBS` and developed in MATLAB® environment, is based on a combination of IGA and limit analysis, both relying on the NURBS representation of the arch which can be easily obtained from CAD design products which are very popular among professional architects and civil engineers. As already discussed, NURBS representation of the arch guarantees a higher accuracy of analysis, especially when compared to a standard polyline representation of the same arch. An isogeometric finite element elastic analysis of the arch can be useful to assess the response of the arch under usual service loads which should not push the thrust line out of the arch depth. Even if standard curved finite elements could be used to accurately analyze an arbitrarily shaped arch, these tools are quite advanced and often out-of-reach for a professional engineer or architect, whereas IGA allows for greater accuracy without requiring the final user any particular effort. On the other hand, a NURBS-based limit analysis is used for assessing the ultimate bearing capacity of the arch. Therefore, the proposed tool allows for a fast evaluation of the safety level of a masonry arch under various loading conditions. Furthermore, algorithms which allows to take into account the effect of masonry crushing, sliding between blocks and additional FRP reinforcements placed either at the intrados or at extrados of the arch have been devised and implemented.

Such a tool could be particularly appreciated if one considers, for instance, the widespread damages that the 2012 Emilia (Italy) earthquake produced to ancient historical buildings, with a great loss for the Italian cultural heritage; after the earthquake, professionals engineers and architects have been called to assess the safety of a huge number of ancient masonry constructions, where arched and vaulted systems are recurrent, and to devise effective seismic retrofit interventions. Another reason lies in the fact that since the exact shape of the arch to be studied is usually not known in advance, a precise surveying of the structure is needed. This surveying is often carried out through the use of laser scanning techniques which may be imported in a CAD environment as a cloud of points. A CAD exact representation of the arch geometry is then possible and constitutes the basis upon which both an elastic and a limit analysis can be performed in an integrated way.

`ArchNURBS` is the first computational tool proposed in literature which allows for the evaluation of the load bearing capacity, and thus of the safety level, of arbitrarily loaded masonry arched structures, starting from a NURBS representation of the real arch generated within a CAD environment. To this aim, a new NURBS-based approach to limit analysis has been devised and extensions which allows to include finite masonry compressive strength, sliding between blocks and FRP reinforcement have been added to it. Finally, an isogeometric analysis has been made possible within `ArchNURBS` in order to allow the user to evaluate the elastic structural response of the arch under the action of service loads.

The paper is organized as follows: in Section 2 a synthetic survey on how the geometric shape of a masonry arch can be described by a NURBS representation is given. The adopted structural models and isogeometric elastic analysis are then recalled and commented upon in Section 3. In Section 4 we address the limit analysis based on the NURBS geometry representation of the arch and its application to the case of FRP reinforced arches and to masonry arches with a limited compressive strength. Section 5 is devoted to presenting a comparison with experimental results taken from literature and some meaningful numerical examples, all of them developed within `ArchNURBS`. Finally, conclusions will be drawn in Section 6.

GEOMETRY DESCRIPTION

`ArchNURBS` is a structural analysis tool for masonry arches having arbitrary shape based on a NURBS representation of the arch geometry which can be easily obtained within a CAD design environment. Description and computation of geometries in commercial CAD packages are based on B-splines and NURBS. More precisely, NURBS basis functions are built on B-splines basis functions which are piecewise polynomial functions defined by a sequence of coordinates $\Xi = \{\xi_1, \xi_2, ..., \xi_{n+p+1}\}$, also known as the *knot vector*, where the so-called *knots* $\xi_i \in [0, 1]$ are points in a parametric domain whereas p and n denote the polynomial order and the total number of basis functions, respectively. The distance between two consecutive knots is named knot span and it represents the equivalent of the element domain in traditional finite elements. Once the order of the basis functions and the knot vector are known, the i-th B-spline basis function $N_{i,p}$ can be computed by means of the Cox-de Boor recursion formula (Cox 1972; de Boor 1978), which is not reported here for the sake of brevity.

As previously mentioned, B-splines are the starting point for the computation of the NURBS basis functions. Indeed, given a set of weights $w_i \in \mathbb{R}$, the NURBS basis functions $R_{i,p}$ read

$$R_{i,p}(\xi) = \frac{N_{i,p}(\xi)\, w_i}{\sum_{i=1}^{n} N_{i,p}(\xi)\, w_i}. \tag{1}$$

NURBS share many properties with B-spline basis functions (Piegl and Tiller 1997). Among these, they are all non-negative, they have a compact support and build a partition of unity (PoU), that is

$$\sum_{i=1}^{n} N_{i,p}(\xi) = \sum_{i=1}^{n} R_{i,p}(\xi) = 1 \tag{2}$$

for each $\xi \in [0, 1]$ (see (Hughes et al. 2005)). Hence, it is noteworthy from Eqs. (1) and (2) that B-spline basis functions can be thought of as NURBS basis functions when all weights w_i are equal to one. However, NURBS basis functions have the great advantage of representing exactly the geometry of a wide set of curves such as circles, ellipses and parabolas (Piegl and Tiller 1997) and of the surfaces which can be generated by them.

Geometries which can be represented with B-spline and NURBS are obtained as linear combinations of basis functions (Piegl and Tiller 1997; Farin 2002). For instance, if we consider a set of B-spline basis functions $N_{i,p}$ (the same holds for the NURBS basis functions) with $i = 1, ..., n$, it is possible to define a curve $\mathbf{C}(\xi) \in \mathbb{R}^d$ as

$$\mathbf{C}(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi)\, \mathbf{B}_i, \tag{3}$$

where coefficients $\mathbf{B}_i \in \mathbb{R}^d$ are known as control points (in the following, $d = 2$ is assumed since this work focuses on planar curves). Differently from standard Lagrange and Hermite approximations, B-spline geometries do not usually interpolate these points. The continuity of the curve follows from that of the adopted basis functions (Hughes et al. 2005) which, in general, is $\mathscr{C}^{p-1}$ throughout the domain. However, if a knot has multiplicity m, the continuity decreases m times at that point (see (Piegl and Tiller 1997)).

Modeling CAD geometries inevitably involves several ingredients, such as knots, order of the approximation and control points. However, in many practical applications only few of these parameter are known *a priori*. In reverse engineering processes, for example, CAD models are created by interpolating or approximating a set of points $\mathbf{P}_i \in \mathbb{R}^2$ usually obtained from the real object by means of laser scanner records. Nonetheless, the parameterization of the input data for B-spline and NURBS geometries addresses a crucial issue concerning the quality of the final curve. Hence, there have been several attempts to improve the accuracy of B-spline and NURBS approximations and interpolations (de Boor 1978; Hartley and Judd 1980; Lee 1989; Sarkar and Menq 1991; Farin 2002). In particular, some of these parameterization techniques, such as the uniform method, the arch-length method and the centripetal method are available in several CAD programs (Autodesk, Inc. 2007).

The easiest way to assess the quality of the computed curve is to evaluate the distance between the CAD geometry $\mathbf{C}(\xi)$ and the analytical representation of the real curve $\mathbf{F}(\xi)$. Therefore, the distance between these two curves at the parametric point ξ is calculated as

$$d(\xi) = \min_{\xi} \{ |\mathbf{C}(t) - \mathbf{F}(\xi)| \}. \tag{4}$$

Once the value of $d(\xi)$ has been evaluated for n given data points, the errors in the L_∞ norm may be defined:

$$E_\infty = \max_{i=1}^{n} \{d_n\}, \tag{5}$$

and in the L_2 norm:

$$E_2 = \left[\frac{1}{n} \sum_{i=1}^{n} d_n^2 \right]^2. \tag{6}$$

Thus, for the sake of completeness we investigate the quality of the NURBS curve reported in Figure 2 in approximating a three-centered (or polycentric) arch composed of three circular arcs jointed together with $\mathscr{C}^1$ continuity. The radius R_i and the center of each portion are also shown. The NURBS curve has been drawn in AutoCAD® 2013 by interpolating the set of points $\mathbf{P}_i$ indicated by red circles in Figure 2 with cubic NURBS basis functions. In particular, the position of these points has been calculated by dividing each of the three circular arcs in equal parts. Table 1 summarizes both the maximum (i.e. E_∞) and the mean (E_2) errors obtained by approximating the exact geometry of the polycentric arch with its polyline and NURBS representations. As it is expected, the error decreases with the number of interpolating points and NURBS representation proves more accurate.

LINEAR ELASTIC ANALYSIS

As previously stated, a linear elastic analysis of a masonry arch may still be meaningful for several arches undergoing service loads, which usually are considerably lower than collapse loads and might not push the thrust line out of the shape of the arch. `ArchNURBS` allows for an elastic isogeometric analysis (IGA) of the masonry arch under study, based on its NURBS representation. In this Section an introduction to the IGA of a plane curved Timoshenko beam (Cazzani et al. 2014c) is given. Interesting studies on IGA of curved rods (even though Kirchhoff-Love rods) in the three dimensional space may be found in (Greco and Cuomo 2013; Greco and Cuomo 2014). In addition, some recent investigations of the application of IGA for the analysis of strongly curved beams are contained in (Cazzani et al. 2014b; Cazzani et al. 2014a).

As it is depicted in Figure 3, we consider a Cartesian reference system $O(x, y)$ and a local reference system $O'(t', n')$ where t' and n' are the unit-tangent and the unit-normal vectors to the beam axis. Further, we introduce the curvilinear abscissa $s \in [0, l]$ which spans the centroidal line of the plane curved beam, whose length is l, is defined by the parametric representation

$$\begin{cases} x(s) \;=\; \sum_{i=1}^{n} N_{i,p}(s(\xi))\, x_i \qquad \text{and} \\ y(s) \;=\; \sum_{i=1}^{n} N_{i,p}(s(\xi))\, y_i \end{cases} \tag{7}$$

where x_i and y_i are the control points coordinates. Thus, the unit tangent and normal vectors of a NURBS curve at a parametric point s are calculated as (Lipshultz 1969)

$$t' \;=\; \frac{(x_{,s}\,,\; y_{,s})}{\sqrt{x_{,s}^2 \,+\, y_{,s}^2}} \tag{8}$$

and

$$n' \;=\; (y_{,s}\,,\; -x_{,s}) \cdot \frac{x_{,ss} y_{,s} \;-\; x_{,s} y_{,ss}}{\left(x_{,s}^2 \,+\, y_{,s}^2\right)^2} \tag{9}$$

where comma denotes differentiation. Further, the curvature radius reads

$$R(s) \;=\; \frac{\left(x_{,s}^2 \,+\, y_{,s}^2\right)^{3/2}}{\left|x_{,s}\, y_{,ss} \;-\; x_{,ss}\, y_{,s}\right|}. \tag{10}$$

In order to describe the kinematics of a curved Timoshenko beam we consider the displacement and the load vectors

$$\mathbf{u} \;=\; [u,\, v,\, \phi]^T \qquad \text{and} \qquad \mathbf{p} \;=\; [q_t,\, q_r,\, m]^T\,, \tag{11}$$

referred to the local reference system (where $(\cdot)^T$ denotes the transpose). In particular, u and v are the tangential and normal displacement of the cross-section centroid, ϕ the cross-section rotation, q_t and q_r the tangential and radial distributed loads and m the distributed bending couples. Hence, by assuming small deformations, the equilibrium, compatibility and constitutive equations for the plane-curved Timoshenko beam are

$$N_{,s} - \frac{T}{R} + q_t = 0, \quad T_{,s} + \frac{N}{R} + q_r = 0 \quad \text{and} \quad M_{,s} - T + m = 0, \tag{12}$$

$$\varepsilon = u_{,s} - \frac{v}{R}, \quad \gamma = v_{,s} + \frac{u}{R} + \phi \quad \text{and} \quad \chi = \phi_{,s}, \tag{13}$$

$$N = EA\varepsilon, \quad T = GAk_s\gamma \quad \text{and} \quad M = EJ\chi, \tag{14}$$

where the generalized stresses N, T and M denote the axial force, the shear force and the bending moment, whereas the generalized strains ε, γ and χ are the axial, the shear and the curvature deformations. Finally, symbols E, G, A, J and k_s are respectively the Young's modulus, the shear modulus, the cross sectional area, the area moment of inertia and the shear-correction factor.

The first step towards a finite element solution of the problem is represented by the definition of the total potential energy of the system

$$\Pi = \frac{1}{2}\int_0^l \left(EA\varepsilon^2 + GAk_s\gamma^2 + EJ\,\chi^2\right)\,\mathrm{d}s - \int_0^l (q_t u + q_r v + m\phi)\,\mathrm{d}s. \tag{15}$$

Subsequently, by making use of the iso-parametric formulation, the discrete displacement field $\mathbf{u}^h(\xi) \in \mathbb{R}^2$ is defined as

$$\mathbf{u}^h(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi)\,\mathbf{u}_i, \tag{16}$$

where $\mathbf{u}_i = [u_i,\, v_i,\, \phi_i]$ are the displacements at the control points $\mathbf{B}_i$. It is worth noticing that, according to Eq. (3), the displacement field in Eq. (16) has been discretized with B-spline basis functions. Nonetheless, NURBS basis functions might have been used in cases where a NURBS-described curve is given. Hence, by making use of Eqs. (15) and (16) the discrete solution of the problem is defined as:

$$\underset{u,\,v,\,\phi}{\arg\min}\left\{\sum_{e=1}^{n_e}\left[\frac{1}{2}\int_{\xi_e}^{\xi_{e+1}}\left[EA\left(\varepsilon^h\right)^2 + GAk_s\left(\gamma^h\right)^2 + EJ\,\left(\chi^h\right)^2\right]\,\mathrm{d}s - \int_{\xi_e}^{\xi_{e+1}}\mathbf{p}^T\,\mathbf{u}^h\,\mathrm{d}s\right]\right\}, \tag{17}$$

where n_e is the number of spans whereas ξ_e and ξ_{e+1} are the knots which correspond to the e-th span. Once the numerical solution $\mathbf{u}^h$ is known, the generalized stresses and strains are calculated by means of Eqs. (13) and (14). Therefore, N, T and M and ε, γ and χ are defined with the same NURBS basis functions used for approximating the displacement field $\mathbf{u}^h$. Accordingly, the computation of the thrust line, which descends from the ratio M/N, is straightforward.

As in standard finite element discretizations, the numerical solution can be improved by refining the approximation. In particular, in IGA there are three different refinement techniques. The first two are knot insertion (h-refinement) and polynomial order elevation (p-refinement) which do not alter the geometry and the continuity of the curve. The third method,which is known as k-refinement, consists in order elevation of the basis functions and consequent knots insertion. This increases the continuity of the approximation without changing the geometry (Hughes et al. 2005; Cottrel et al. 2007). In `ArchNURBS` each of these methods may be used.

LIMIT ANALYSIS

As already discussed, limit analysis is a powerful tool to assess the structural safety level of a masonry construction. It is well established that when *mechanism* and *equilibrium* formulations of limit analysis are linearized, they produce dual Linear Programming (LP) problems (Charnes and Greenberg 1951). In particular Livesley (Livesley 1978) has shown that the *equilibrium* formulation can be applied to masonry arches. It involves the discretization of the arch into a number of rigid blocks. Many researchers have developed procedures to model masonry arches as discrete rigid blocks: among them we recall (Delbecq 1980; Boothby 1994; Gilbert and Melbourne 1994).

In `ArchNURBS` a joint equilibrium formulation, similar to that originally adopted by Livesley (Livesley 1978) and then proposed for masonry arches in (Gilbert 2007) is used. It should be incidentally observed that, while an equilibrium formulation has been formally used, assuming a finite number of blocks (and hence of interfaces) provides actually an upper bound estimate of the collapse multiplier.

The adopted model relies on the following traditional assumptions, originally proposed by Heyman (see (Heyman 1969)) for the limit analysis of masonry arches:

1. sliding failure of adjacent units in the arch cannot occur;
2. masonry has zero tensile strength;
3. masonry has infinite compressive strength.

Therefore, a procedure based on an *equilibrium formulation* and the above assumptions for the limit analysis of masonry arches is set out as follows.

The structure is divided into c elements (blocks) in much the same way as for elastic analysis. Subsequently to this subdivision, $d = c + 1$ interfaces are generated. For each block the equations of equilibrium are written, in such a way as to express contact forces $\mathbf{q} = [T_i, N_i, M_i]$ (which are respectively the shear force, the axial force and the bending moment) acting on the i-th inter-element boundary and any external load acting on the element $\mathbf{f}$, which can be either a dead load $\mathbf{f}_D$ or a live load $\lambda \mathbf{f}_L$. Such equations may be expressed as:

$$\mathbf{A}\mathbf{q} - \lambda \mathbf{f}_L = \mathbf{f}_D, \tag{18}$$

where $\mathbf{A}$ is a suitable $(3c \times 3d)$ equilibrium matrix containing the direction cosines of the unit-normal vector n' of the transversal section at each contact interface. These equations are the equilibrium constraints of the problem.

Yield constraints, in the no-tension material hypothesis, are then defined on $\mathbf{q}$:

$$\left.\begin{array}{l} M_i \leq 0.5 N_i t_i \\ \\ M_i \geq -0.5 N_i t_i \end{array}\right\} \forall \ \text{contact} \ \ i = 1, ..., c\,, \tag{19}$$

where t_i is the depth of the arch section at contact i. Finally the limit analysis problem for proportional loading is now written as *Maximize the load factor* λ*, subject to the equilibrium constraints* (18) *and to the yield constraints* (19):

$$\max\{\lambda\}. \tag{20}$$

Using this formulation the LP problem variables are the contact forces $(T_1, N_1, M_1, ..., T_c, N_c, M_c)$ and the unknown load factor λ. In `ArchNURBS` the linear programming problem is solved through the MATLAB® function `linprog.m` which is part of the MATLAB® Optimization Toolbox.

The yield constraints expressed in eq. (19) are valid only if the material exhibits an unlimited compressive strength. If, instead, it is assumed that masonry has a limited (i.e. finite) compressive strength σ_{crush}, and that thrust is transmitted, from one block to the next one, through a rectangular crush block, then, as it is suggested in (LimitState Ltd. 2011), Eq. (19) may be replaced by:

$$\left.\begin{aligned} M_i &\leq N_i\left(0.5t_i - \frac{N_i}{2\sigma_{\text{crush}}\,b}\right) \\ M_i &\geq -N_i\left(0.5t_i - \frac{N_i}{2\sigma_{\text{crush}}\,b}\right) \end{aligned}\right\} \forall \text{ contact } i = 1, ..., c, \tag{21}$$

where σ_{crush} is the masonry compressive strength and b is the width of the arch transversal section. However, the constraints in Eq. (21) are non-linear. Therefore, in order to continue using a Linear Programming (LP) solver, these constraints need to be approximated by a set of linear constraints (see e.g. (Gilbert 2007; LimitState Ltd. 2011)).

In order to minimize the number of constraints in the problem (and to maximize computational efficiency) an iterative solution algorithm which involves only refining the representation of the failure envelope where required is used. The algorithm can be summarized in the following steps:

1. Initially solve the global LP problem with the original linear constraints (19) plus the additional linear constraint $N_i < N_{i,max}$ on each contact i, where $N_{i,max}$ is the maximum axial force which the arch section can resist to before crushing occurs;
2. Substitute N_i from the last solution into the inequality constraints, eqs. (21), for each contact i. If a constraint is violated, calculate the violation factor e_i, i.e.

$$e_i = \frac{|M_i|}{N_i\left(0.5t_i - \frac{N_i}{2\sigma_{\text{crush}}\,b}\right)}, \tag{22}$$

and store, from the previous solution, the values of axial force corresponding to contacts where violation has occurred. These values are denoted by $N_{i,0}$;

3. For each contact with $e_i \geq 1.0$ (i.e. such that violation occurs) set an additional linear constraint which is tangential to the original failure envelope described by eqs. (21) at the point corresponding to $N = N_{i,0}$;
4. Solve the new global LP promblem;
5. Repeat from step 2 until the maximum value of $e_i < 1 + tol$ where the tolerance tol is taken as a suitably small value.

Moreover, if sliding between blocks has to be taken into account, additional sliding yield constraints to the linear programming problem (20) are needed. As suggested in (Melbourne and Gilbert 1995), it is possible to assume a simple associative friction model defining the following *linear constraints*:

$$\left.\begin{array}{l} T_i \leq \mu_i N_i \\ \\ T_i \geq -\mu_i N_i \end{array}\right\} \forall \ \text{contact} \ \ i = 1, ..., c \ , \tag{23}$$

where μ_i is a suitable friction coefficient for each contact interface i. This particular friction model has been chosen for simplicity reasons whereas in literature more advanced models exist, which involve non-associative friction laws and the use of both non-linear programming methods (see e.g. (Ferris and Tin-Loi 2001; Orduna and Lourenço 2003)) and iterative linear-programming methods (Gilbert et al. 2006). Nevertheless, applying the iterative procedure devised in (Gilbert et al. 2006) to brickwork masonry arch bridges analyzed previously with an associative friction model, (Gilbert and Ahmed 2004) found that the non-associative bridge strength predictions were at most 6 percent lower, largely justifying the initial associative friction assumption.

For the sake of simplicity, backfill (which is considered as a dead load and thus enters in Eq. (18)) is modeled as an external vertical force acting upon each block; it is given by the weight of the volume of the backfill portion lying above each block and is applied to the center of mass of the same volume. As discussed in (Callaway et al. 2012), it is necessary to point out that the influence of the backfill on the load capacity of masonry arches is a very complex topic. In literature, much more sophisticated models for backfill exist, which are capable of taking into account effects like load diffusion and the gradual build-up of passive pressures (see e.g. (Gilbert et al. 2007; Cavicchi and Gambarotta 2005)).

Finally, it is possible to modify the limit analysis in order to take into account the presence of Carbon Fiber Reinforced Polymer (CFRP) reinforcement strips at the intrados and/or at the extrados of the arch. Many researchers have proposed different solutions to this problem (see, e.g. (Briccoli Bati et al. 2013; Basilio et al. 2014; Caporale et al. 2006; Caporale and Luciano 2012; Caporale et al. 2014)). In the present paper we deal with the problem by modifying the original equilibrium formulation including two further variables $(F_{i,\text{intrados}}, F_{i,\text{extrados}})$ for each of the n CFRP reinforced interfaces. These variables represent the inner force acting within the FRP strip at the interface at the intrados and at the extrados respectively and enter into the equilibrium constraints (18). The new variables are subjected to the *additional yield constraints*:

$$\left.\begin{aligned} 0 < F_{i,\text{intrados}} < F_d \\ 0 < F_{i,\text{extrados}} < F_d \end{aligned}\right\} \forall \text{ reinforced contact } i = 1, ..., n \, , \tag{24}$$

where F_d is the design delamination resistance of the CFRP strip which may be evaluated, for example, following the prescriptions contained in Chapter 5 of (CNR2013).

NUMERICAL EXAMPLES

In this section a comparison with experimental results taken from literature and three numerical examples analyzed with the computational tool `ArchNURBS` are presented. The comparison with experimental results allows to validate the numerical results obtained with `ArchNURBS`. Furthermore, the influence of the geometric representation of the rigid blocks in which the arch is subdivided on the limit load multiplier λ is discussed. Then, the limit analysis for the three-centered arch described in Section 2 and a *real world* arch are taken into consideration in the second and third example, respectively.

Comparison with experimental results

In order to validate the results obtained with ArchNURBS a comparison with the experimental tests presented in (Vermeltfoort 2001) and later analyzed in (Milani et al. 2008) is carried out. (Vermeltfoort 2001) tested the ultimate strength of a segmental masonry arch with a clear span of 3 m, an inner radius of 2.5 m and a sagitta of 0.5 m. The arch is a one-head brick structure with depth equal to 0.10 m and width equal to 1.25 m. The test-arch had 51 layers and was built with Rijswaard soft mud bricks and 1:2:9 mortar. Brick compressive strength was 27 MPa and mortar compressive strength was 2.5 MPa. The test-arch was loaded with four concentrated loads, applied by four hydraulic jacks 600 mm centre to centre. In Fig. 4 the geometry of the test-arch and its loading conditions are reported. Only the second concentrated load from the left was increased until failure, whereas the remaining loads were maintained constant at the values of 5.9, 9.1 and 9.1 kN respectively. At failure, (Vermeltfoort 2001) observed a four hinges collapse mechanism which is depicted in Fig. 5a and measured a collapse load equal to 40.7 kN at the second jack.

The described test-arch has been modeled within ArchNURBS as a segmental arch formed by 51 blocks and the same point loads used by (Vermeltfoort 2001) in the experiments have been applied. Only the second point load from the left have been marked as a live load. After a limit analysis of the arch, ArchNURBS gives out a collapse value of 40.7 kN for the live load previously defined. Thus, the value of the collapse load calculated by ArchNURBS coincides with the collapse load measured during the experiments. Furthermore, a four hinges collapse mechanism has been numerically determined. Fig. 5b depicts the collapse mechanism numerically computed with ArchNURBS. The failure mechanism predicted by ArchNURBS is very close to the real failure mechanism experimentally observed by (Vermeltfoort 2001) when bringing to failure the test-arch. Thus, it can be concluded that ArchNURBS gives an accurate prediction of both collapse load and collapse mechanism of the masonry arch under study.

Influence of the voussoirs geometry

Many arches in the real-world occurrences are made of stone voussoirs which have a rounded shape rather than a quadrangular shape, as shown for example in Fig. 1b, representing the arch of

Porta Asinaria in Rome (Italy). When the size of these voussoirs is not small their exact geometric representation is of paramount importance in order to obtain accurate estimates of the collapse load multiplier λ.

`ArchNURBS` allows for an exact description of the arch rounded voussoirs by exploiting the features of NURBS geometries generated in CAD environments. On the contrary, most of existing commercial software codes approximate the shape of rounded voussoirs with simple quadrangular blocks.

In the case of a uniform vertical live load distribution, if the section depth of the arch is large enough (as is, for example, in the arch of Fig 1b) and unless taking into account finite stone compressive strength, the arch may result safe for every value of the applied load (Heyman 1969).

Nevertheless, let us consider a semi-circular arch with mean radius 2.125 m, section depth 0.250 m and width 0.500 m, loaded with a uniformly distributed vertical live load of 1 kN/m. In this case, section depth is not great enough to guarantee the safety of the arch for every value of the uniform vertical applied live load and a collapse load multiplier can be determined. The backfill height is assumed equal to 3.00 m. The arch is subdivided into ten voussoirs. Material properties of the stone-voussoirs and of the backfill are reported in Table 2.

`ArchNURBS`, which implements the limit analysis algorithms described in the previous Section, returns a collapse load multiplier $\lambda = 9.8$ for the arch model with rounded voussoirs. The obtained value is the exact collapse load multiplier for the rounded voussoirs arch here taken into consideration.

On the other hand, if the arch is modeled by means of ten quadrangular voussoirs a collapse load multiplier $\lambda = 8.9$ is computed (this result has been carried out with the commercial software LimitState:RING 3.0®). Therefore, the geometrical approximation of the rigid blocks with quadrangular elements leads to an error of 9.2 % on the estimate of λ.

Of course, the error could be greatly reduced if an higher number of quadrangular blocks was chosen to model the arch but then the number of interfaces between blocks (on which hinges positions are constrained to be) would be changed in respect to the original problem. In addition, computational efficiency would be clearly reduced.

In Figures 6b-c a comparison between the two arch models is shown along with a plot of the corresponding thrust line.

Three-centered arch

As shown in Fig. 1a, three-centered arches are recurrent in many masonry structures. The three-centered arch described in Section 2 is examined, as an example, in this Subsection. In particular, its depth and width are assumed equal to 0.560 m and 0.500 m, respectively. The exact arch geometry has been generated within a CAD environment using NURBS curves and then imported in `ArchNURBS`. Material properties used for masonry and for the backfill are reported in Table 2. The masonry mechanical properties chosen are typical for low quality masonry as suggested in the explicative circular (CIRC2009) related to the Italian Building Code (NTC2008).

The arch is supposed to be loaded by a downward linear uniform live load of 1 kN/m. This load is amplified by a load multiplier λ. The arch has been subdivided into 90 rigid blocks. First, limit analysis is performed without considering any backfill. In this case the collapse load multiplier is $\lambda = 0.78$. As it is illustrated in Figure 7a the resulting collapse mechanism is a symmetrical five hinges mechanism. The corresponding thrust line and position of hinges at collapse is indicated by a red line and red circles, respectively. A collapse load multiplier less than 1 indicates that the arch, in this configuration, is not safe under the action of the assigned linear uniform load of 1 kN/m.

The same analysis has been then carried considering a backfill having a height of 4.00 m and a specific weight as indicated in Table 2. In this case, despite the collapse mechanism and the position of the collapse thrust line being similar to those obtained in the previous case (see Figure 7b), the load multiplier increases to $\lambda = 21.62$. Therefore, the particular geometry of the arch studied in this example is very sensitive to the stabilizing effect of the backfill.

A masonry arch from *Torre Fornasini*

The third arch here analyzed is a real world masonry arch belonging to the groin vault which bears the first story of *Torre Fornasini*, a historical masonry tower construction in Poggio Renatico

(Italy), which was severely damaged by the earthquake which struck Emilia on May 2012. The tower, depicted in Figure 8, has been subjected to extensive seismic retrofit intervention which comprised reinforcement of the extrados of the vault with Carbon Fiber Reinforced Polymers (CFRP) strips (Milani et al. 2014). In particular, the analyzed arch is the segmental arch shown in Figure 9a. It is characterized by a span equal to 4.13 m, a midspan rise equal to 1.81 m, a depth equal to 0.14 m and a width equal to 0.25 m. After an accurate survey, the exact arch geometry has been generated within a CAD environment using NURBS curves and then imported in `ArchNURBS`. Again, material properties assumed for masonry and specific weight for the backfill are reported in Table 2. The arch is loaded by a downward acting linear uniform live load equal to 1 kN/m multiplied by a load multiplier λ. The arch has been subdivided into 90 rigid blocks. First, limit analysis is performed without taking into account any backfill. In this case a solution cannot be determined since the arch is not stable under its own weight. Then, the same analysis is carried out by considering a backfill with specific weight reported in Table 2 and a maximum height equal to 2.15 m. Under these assumptions, the optimization problem can be solved and the resulting collapse mechanism is a symmetrical five-hinges mechanism which is depicted in Figure 9b. The collapse load multiplier is $\lambda = 1.43$. In order to evaluate the effect of the limited compressive strength of masonry on the load capacity of the structure the same limit analysis has been carried out, allowing for a masonry compressive strength equal to 2.4 MPa, as indicated in Table 2 and as prescribed by the explicative circular (CIRC2009) related to the Italian Building Code (NTC2008), following the algorithm described in Section 4.In this case the collapse load multiplier drops to $\lambda = 0.86$. As it has been explained in the previous Section it is also possible to account for the effect of FRP reinforcement. Indeed, during the seismic retrofit intervention a 200 mm wide strip of carbon fiber tissue (MapeWrap C Uni-AX produced by MAPEI) was applied to the extrados of the vault. This tissue has thickness of 0.2 mm, Young's elastic modulus of 230 GPa (for tensile stress only) and ultimate strain of 2%. FRP delamination force has been calculated by following the Italian FRP Design Guidelines (CNR2013). By performing a limit analysis of the FRP reinforced arch, without taking into account the effect of limited compressive strength of masonry, the collapse load multiplier results $\lambda = 5.94$. In Figure 9c the symmetric five hinges collapse mechanism is shown: in this case the mechanism which develops only after FRP delamination has occurred at both sides of the arch. On the other hand, when the effect of finite compressive masonry strength is considered and coupled to the FRP reinforcement, the collapse load multiplier drops to $\lambda = 3.44$. In both cases FRP reinforcement is proven to be very effective in enhancing the safety level of the arch under study. Finally, an analysis of the original arch without reinforcement has been considered by accounting for possible sliding of masonry blocks as explained in Section 4. Therefore, after performing this limit analysis it can be observed that adopting a friction coefficient $\mu = 0.3$, as it is widely suggested in literature (see e.g. (Vasconcelos and Lourenço 2006)), the original solution does not change: collapse still occurs by formation of a five hinges mechanism and the collapse multiplier is still $\lambda = 1.43$. Besides, if the friction coefficient is reduced to $\mu = 0.275$ it is observed that collapse mechanism modifies since sliding occurs at the arch imposts. In this last case the collapse multiplier is $\lambda = 1.02$ and the corresponding collapse mechanism and collapsed thrust line at collapse are shown in Figure 9d. Results from the discussed analysis are summarized in Table 3.

CONCLUSIONS

This work introduced a new simple CAD-integrated computational tool for the safety assessment of masonry arches addressed to professionals in the field of structural rehabilitation of historical masonry constructions which cannot or do not want to get involved into more advanced and demanding computational methods. The proposed software, named `ArchNURBS`, provides a simple and very intuitive instrument for the accurate evaluation of the load bearing capacity of masonry arches. `ArchNURBS` has been implemented in MATLAB® and is freely available online at the address `http://sourceforge.net/projects/archnurbs/` as an open-source project.

`ArchNURBS` is the first computational tool for the analysis of masonry arches proposed in literature which is based on a NURBS description of the shape of the arch. Such representation can be easily generated in a CAD environment which is very popular among professional engineers and architects, possibly starting from topographical surveying data. NURBS representation allows for an exact shape description of the arch, which is especially useful for those arch shapes which are not well represented by a polyline. On the basis of such geometric representation, a preliminary isogeometric finite element elastic analysis (useful to determine the response under service loads which might not push the thrust line out of the shape of the arch) and a NURBS-based limit analysis of the masonry arch (which is needed to assess the safety level of the structure) are possible. Furthermore, limited compressive strength for masonry, sliding between blocks and presence of FRP reinforcement can be dealt with in `ArchNURBS`.

A comparison with experimental results has shown that `ArchNURBS` can predict with great accuracy the ultimate bearing capacity of masonry arches. In addition, some meaningful examples of NURBS-based limit analysis of masonry arches obtained with `ArchNURBS` have been presented.

The new approach on which `ArchNURBS` is based has proved to be effective in providing an accurate evaluation of the safety level of masonry arches of arbitrary shape, while requiring the professional user little effort compared to existing computational techniques.

AKNOWLEDGEMENTS

The financial support of RAS, the Autonomous Region of Sardinia, under grant number F71J09999350002-CRP1 475 (Legge Regionale 7/2007, bando 2008, Progetto MISC: Metodi Isogeometrici per Strutture Curve) is gratefully acknowledged by M. Malagù and by Prof. A. M. Cazzani. The researchers of the University of Ferrara gratefully acknowledge the financial support of ReLUIS3 programs WP2 (Analysis of the seismic response of masonry structures) and Task 2.2 (Horizontal structures, vaults, floors, roofs and their interaction with masonry structures).

REFERENCES

AEDES (2014). “SAV, Stabilità di archi e volte in muratura, <http://www.aedes.it>.

Auricchio, F., Beirão da Veiga, L., Hughes, T. J. R., Reali, A., and Sangalli, G. (2012). “Isogeometric collocation for elastostatics and explicit dynamics.” *Computer Methods in Applied Mechanics and Engineering*, 249–252, 2–14.

Autodesk, Inc. (2007). *Autodesk® MAYA® 8.5 - NURBS modeling*.

Basilio, I., Fedele, R., Lourenço, P., and Milani, G. (2014). “Assessment of curved FRP-reinforced masonry prisms: experiments and modeling.” *Construction and Building Materials*, 51, 492–505.

Bazilevs, Y., Calo, V. M., Zhang, Y., and Hughes, T. J. R. (2006). “Isogeometric fluid-structure interaction analysis with applications to arterial blood flow.” *Computational Mechanics*, 38, 310–322.

Benedetti, A. and Tralli, A. (1989). “A new hybrid F.E. model for Arbitrarily curved beam. I — Linear analysis.” *Computers and Structures*, 33(6), 1437–1449.

Benson, D. J., Bazilevs, Y., Hsu, M. C., and Hughes, T. (2010). “Isogeometric shell analysis: the Reissner-Mindlin shell.” *Computer Methods in Applied Mechanics and Engineering*, 199, 276–289.

Boothby, T. E. (1994). “Stability of masonry piers and arches including sliding.” *ASCE Journal of Engineering Mechanics*, 120, 304–319.

Briccoli Bati, S., Fagone, M., and Rotunno, T. (2013). “Lower bound limit analysis of masonry arches with CFRP Reinforcements: a numerical method.” *Journal of Composites for Construction*, 17, 543–553.

Callaway, P., Gilbert, M., and Smith, C. (2012). “Influence of backfill on the capacity of masonry arch bridges.” *Proceedings of the Institution of Civil Engineers: Bridge Engineering*, 165, 147–157.

Caporale, A., Feo, L., Hui, D., and Luciano, R. (2014). “Debonding of FRP in multi-span masonry arch structures via limit analysis.” *Composite Structures*, 108, 856–865.

Caporale, A. and Luciano, R. (2012). “Limit analysis of masonry arches with finite compressive strength and externally bonded reinforcement.” *Composites Part B: Engineering*, 43, 3131–3145.

Caporale, A., Luciano, R., and Rosati, L. (2006). “Limit analysis of masonry arches with externally bonded FRP reinforcements.” *Computer Methods in Applied Mechanics and Engineering*, 196, 247–260.

Cavicchi, A. and Gambarotta, L. (2005). “Collapse analysis of masonry bridges taking into account archfill interaction.” *Engineering Structures*, 27, 605–615.

Cazzani, A., Malagù, M., Stochino, F., and Turco, E. (2014a). “Constitutive models for strongly curves beams in the frame of isogeometric analysis.” *accepted for publication on Mathematics and Mechanics of Solids*.

Cazzani, A., Malagù, M., and Turco, E. (2014b). “Isogeometric analysis: a powerful numerical tool for the elastic analysis of historical masonry arches.” *accepted for publication on Continuum Mechanics and Thermodynamics*.

Cazzani, A., Malagù, M., and Turco, E. (2014c). “Isogeometric analysis of plane curved beams.” *Mathematics and Mechanics of Solids*, DOI 10.1177/1081286514531265.

Charnes, A. and Greenberg, H. J. (1951). “Plastic collapse and linear programming.” *Bulletin of the American Mathematical Society*, 57, 480–485.

CIRC2009. *Istruzioni per l’applicazione delle Nuove Norme Tecniche per le Costruzioni di cui al D.M. 14 gennaio 2008*. Circolare n. 617 02/02/2009, Gazzetta Ufficiale della Repubblica Italiana, (in Italian).

CNR2013. *Istruzioni per la Progettazione, l’Esecuzione ed il Controllo di Interventi di Consolidamento Statico mediante l’utilizzo di Compositi Fibrorinforzati*. DT200 R2013, Consiglio Nazionale delle Ricerche, 2013 (in Italian).

Como, M. (2013). *Statics of historic masonry constructions*. Springer Verlag, Berlin-Heidelberg.

Cottrel, J. A., Hughes, T. J. R., and Reali, A. (2007). “Studied of refinement and continuity in isogeometric structural analysis.” *Computer Methods in Applied Mechanics and Engineering*, 196, 4160–4183.

Cottrell, J. A., Hughes, T. J. R., and Bazilevs, Y. (2009). *Isogeometric Analysis: Toward Integration of CAD and FEA*. Wiley, New York.

Cox, M. G. (1972). “The numerical evaluation of B-splines.” *Journal of the Institute of Mathematics and its Applications*, 10, 134–149.

Crisfield, M. G. (1997). *Non-linear finite element analysis of solids and structures*, Vol. 1 and 2. Wiley, 2nd edition.

Cundall, P. A. and Strack, O. D. L. (1979). “A discrete numerical model for granular assemblies.” *Geotechnique*, 29, 47–65.

de Boor, C. (1978). *A practical guide to splines*. Springer Verlag, New York, 1st edition.

Delbecq, J. M. (1980). “Analyse de la stabilité des voûtes en maçonnerie et en béton non armé par le calcul á la rupture.” *Document interne SETRA*.

Farin, G. (2002). *Curves and Surfaces for CAGD: A Practical Guide*. Morgan Kaufmann Publishers, San Francisco, 5th edition.

Ferris, M. C. and Tin-Loi, F. (2001). “Limit analysis of frictional block assemblies as a mathematical program with complementarity constraints.” *International Journal of Mechanical Sciences*, 43, 209–227.

Gelfi, P. (2008). “Arco 2008, <http://dicata.ing.unibs.it/gelfi/software/>.

Gilbert, M. (2007). “Limit analysis applied to masonry arch bridges: state-of-the-art and recent developments.” *Proceedings of 5th International Conference on Arch Bridges*, Madeira, Portugal.

Gilbert, M. and Ahmed, H. M. (2004). “Developments to the RING masonry arch bridge analysis software.” *Proceedings of 4th International Conference on Arch Bridges*, Barcelona, Spain.

Gilbert, M., Casapulla, C., and Ahmed, H. (2006). “Limit analysis of masonry block structures with non-associative frictional joints using linear programming.” *Computers and structures*, 84, 873–887.

Gilbert, M. and Melbourne, C. (1994). “Rigid-block analysis of masonry structures.” *The structural engineer*, 72, 356–361.

Gilbert, M., Nguyen, D., and Smith, C. (2007). “Computational limit analysis of soil-arch interaction in masonry arch bridges.” *Proceedings of 5th International Conference on Arch Bridges*, Madeira, Portugal.

Gontier, C. and Vollmer, C. (1995). “A large displacement analysis of a beam using a CAD geometric definition.” *Computers and Structures*, 57.

Greco, L. and Cuomo, M. (2013). “B-Spline interpolation of Kirchhoff-Love space rods.” *Computer Methods in Applied Mechanics and Engineering*, 256, 251–269.

Greco, L. and Cuomo, M. (2014). “An implicit G1 multi patch B-spline interpolation for KirchhoffLove space rods.” *Computer Methods in Applied Mechanics and Engineering*, 269, 173–197.

Hartley, P. J. and Judd, C. J. (1980). “Parametrization and shape of B-spline curves for CAD.” *Computer-Aided Design*, 12, 235–238.

Heyman, J. (1966). “The stone skeleton.” *International Journal of Solids and Structures*, 2, 249–279.

Heyman, J. (1969). “The safety of masonry arches.” *International Journal of Mechanical Sciences*, 11, 363–385.

Heyman, J. (1982). *The masonry arch*. Ellis Horwood, Chichester.

Huerta, S. (2001). “Mechanics of masonry vaults: the equilibrium approach.” *Historical constructions*, R.B. Lourenço and P. Roca , ed., Guimarães, 47–69.

Hughes, T. J. R., Cottrell, J. A., and Bazilevs, Y. (2005). “Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement.” *Computer Methods in Applied Mechanics and Engineering*, 194, 4135–4195.

Lee, E. T. Y. (1989). “Choosing nodes in parametric curve interpolation.” *Computer-Aided Design*, 21, 363–370.

LimitState Ltd. (2007). “LimitState:RING 3.0, <http://www.limitstate.it/ring>.

LimitState Ltd. (2011). *LimitState:RING User Manual Version 3.0*. Sheffield UK.

Lipshultz, M. (1969). *Theory and problems of differential geometry*. McGraw-Hill.

Livesley, R. K. A. (1978). "Limit analysis of structures formed from rigid blocks." *International Journal for Numerical Methods in Engineering*, 12, 1853–1871.

Livesley, R. K. A. (1992). "A computational model for the limit analysis of three-dimensional masonry structures." *Meccanica*, 27, 161–172.

Melbourne, C. and Gilbert, M. (1995). "The behaviour of multi-ring brickwork arch bridges." *The Structural Engineer*, 73, 39–47.

Milani, E., Milani, G., and Tralli, A. (2008). "Limit analysis of masonry vaults by means of curved shell finite elements and homogenization." *International Journal of Solids and Structures*, 45, 5258–5288.

Milani, G., Marzocchi, S., Minghini, F., and Tralli, A. (2014). "Seismic assessment of a masonry tower in the region stricken by the 20-29 may 2012 emilia-romagna, italy, earthquake." *Proceedings of the 9th International Masonry Conference*, R.B. Lourenço and P. Roca , ed., Guimarães, Portugal.

Munjiza, A. (2004). *The combined finite-discrete element method*. Wiley, 1st edition.

NTC2008. *Norme tecniche per le costruzioni*. Decreto Ministeriale 14/01/2008, Gazzetta Ufficiale della Repubblica Italiana, (in Italian).

Orduna, A. and Lourenço, P. (2003). "Cap model for limit analysis and strengthening of masonry structures." *ASCE Journal of Structural Engineering*, 129, 1367–1375.

Piegl, L. and Tiller, W. (1997). *The NURBS book*. Springer Verlag, Berlin, 2nd edition.

Prenter, P. M. (1975). *Splines and variational methods*. Wiley, New York.

Roca, P., Cervera, M., G., G., and Pelà, L. (2007). "Structural analysis of masonry historical constructions." *Engineering Structures*, 17, 299–325.

Sarkar, B. and Menq, C. H. (1991). "Smooth-surface approximation and reverse engineering." *Computer-Aided Design*, 23, 623–628.

Tralli, A., Alessandri, C., and Milani, G. (2014). "Computational methods for masonry vaults: a review of recent results." *The Open Journal of Civil Engineering*, 8, 272–287.

Vasconcelos, G. and Lourenço, P. B. (2006). "Assessment of the in-plane shear strength of stone masonry walls by simplified models." *Proceedings of the 5th International Conference Structural Analysis of Historical Constructions*, R.B. Lourenç, P. Roca, C. Modena, S. Agrawal, ed., New Dehli, India.

Vermeltfoort, A. (2001). "Analysis and experiments of masonry arches." *Proceedings of Historical Constructions*, R.B. Lourenço and P. Roca , ed., Guimarães, Portugal, 489–498.

List of Tables

n° of $\mathbf{P}_i$	Polyline		NURBS	
	E_∞ [m]	E_2 [m]	E_∞ [m]	E_2 [m]
10	9.95e-2	6.61e-2	1.34e-2	1.64e-3
20	2.49e-2	1.66e-2	0.79e-2	3.73e-4
40	0.62e-2	4.15e-3	0.93e-3	2.74e-5

TABLE 1: Maximum and mean errors on the approximation of the polycentric arch with polyline and

NURBS representations.

Mechanical Properties	Example discussed in Section 5.2	Examples discussed in Sections 5.3 and 5.4
Masonry Young's modulus (E)	2800 MPa	1500 MPa
Masonry shear modulus (G)	860 MPa	500 MPa
Masonry mass density (ρ_m)	1800 kg/m^3	1800 kg/m^3
Masonry compressive strength (f_c)	6.0 MPa	2.4 Mpa
Backfill mass density (ρ_b)	1600 kg/m^3	1600 kg/m^3

TABLE 2: Masonry mechanical properties and backfill density for examples in Sections 5.

FRP reinforcement	Arch Configuration	Collapse Load Multiplier λ
no	no backfill and unlimited masonry compressive strength	-
	backfill and unlimited masonry compressive strength	1.43
	backfill and limited masonry compressive strength	0.86
	backfill, unlimited masonry compressive strength and limited friction between blocks ($\mu = 0.3$)	1.43
	backfill, unlimited masonry compressive strength and limited friction between blocks ($\mu = 0.275$)	1.02
yes	backfill and unlimited masonry compressive strength	5.94
	backfill and limited masonry compressive strength	3.44

TABLE 3: Collapse load multipliers resulting from limit analysis of the arch analyzed in Section 5.3, belonging to Torre Fornasini (Poggio Renatico, Italy) groin vault, under different assumptions

List of Figures

(a)

(b)

FIG. 1: Examples of historical arched structures: (a) three-centered masonry arch bridge in Llanelltyd, Wales (image courtesy of Bill Harvey) and (b) Porta Asinaria rounded stone voussoirs arch in Rome, Italy (image by authors). Both arches cannot be accurately modeled with polyline geometries.

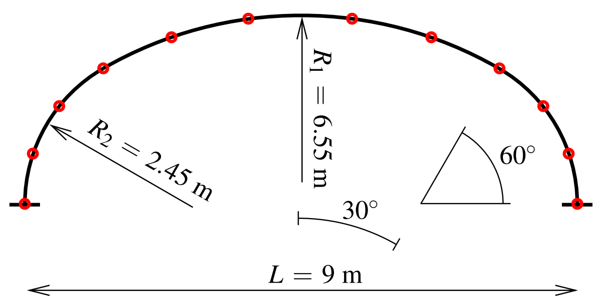


FIG. 2: Polycentric arch (solid circles denote the interpolating points $\mathbf{P}_i$)

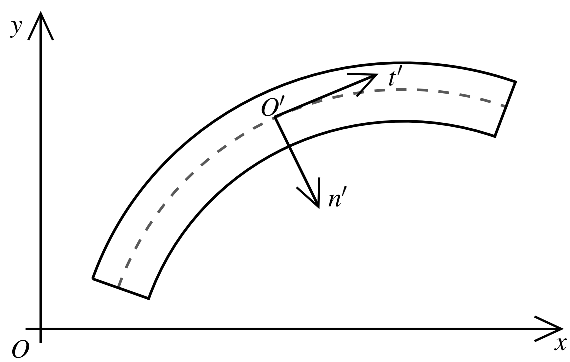


FIG. 3: Reference system $O(x, y)$ and local reference system $O'(t', n')$

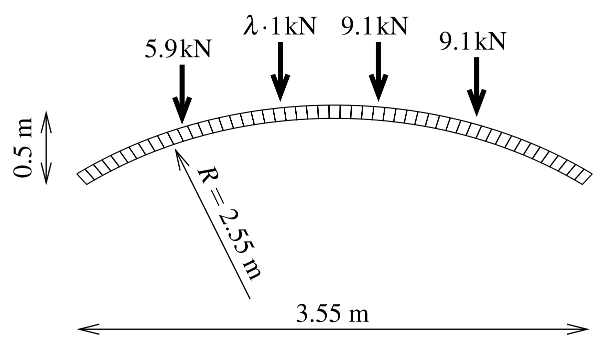
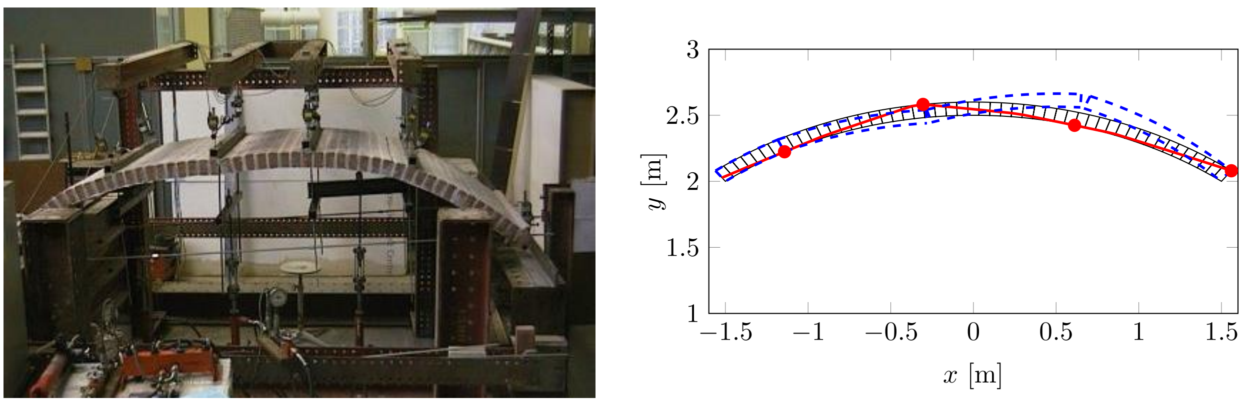


FIG. 4: Schematic representation of the segmental masonry test-arch used in (Vermeltfoort 2001): geometry and loading conditions.



(a) (b)

FIG. 5: (a) Failure mechanism experimentally obtained from (Vermeltfoort 2001). (b) Numerical failure mechanism (dashed lines) and thrust line at collapse (solid line) computed with `ArchNURBS`. Experimental failure photo is reported with the sole aim of showing the capabilities of the computational tool `ArchNURBS` (reprinted from (Vermeltfoort 2001), with permission.)

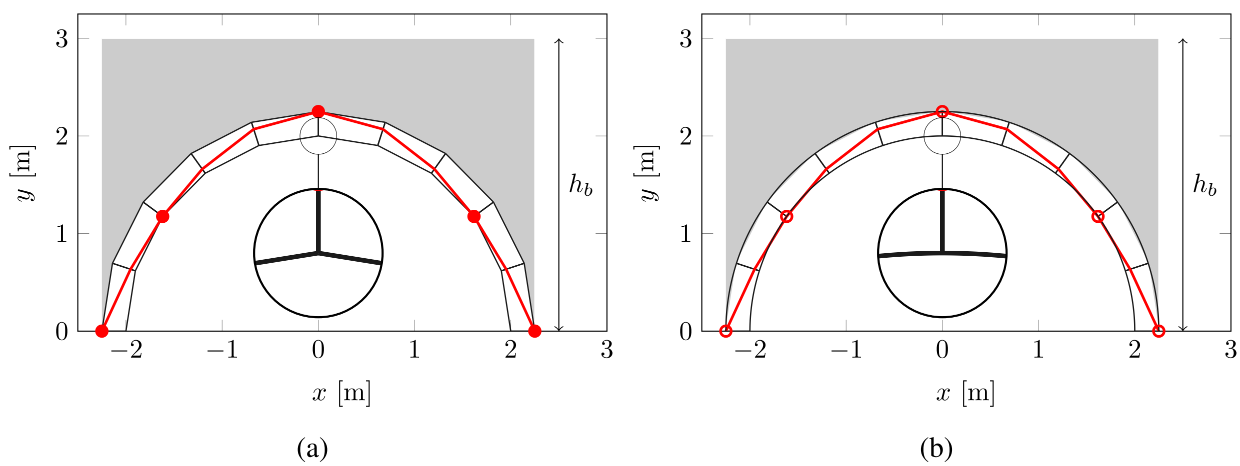


FIG. 6: Thrust lines for the example arch computed with (a) quadrangular voussoirs (LimitState:RING 3.0®) and (b) rounded voussoirs (`ArchNURBS`) models. Corresponding collapse load multipliers λ are equal to 8.9 and 9.8, respectively.

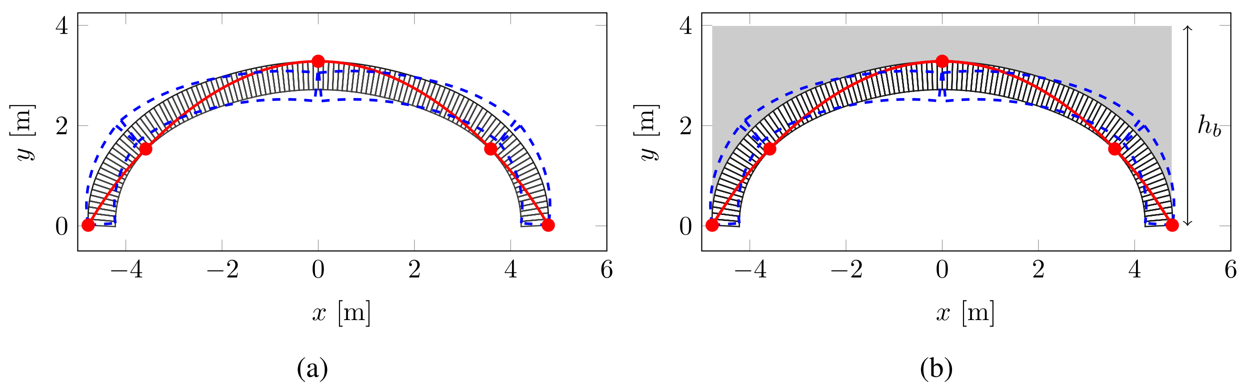


FIG. 7: Three-centered arch analyzed with `ArchNURBS`: thrust line (solid line) obtained without backfill (a) and with backfill (b). The collapse load multipliers corresponding to these configurations are 0.78 and 21.62, respectively.

(a) (b)

FIG. 8: (a) External view and (b) first story masonry groin vault of *Torre Fornasini* in Poggio Renatico, Italy (image by authors).

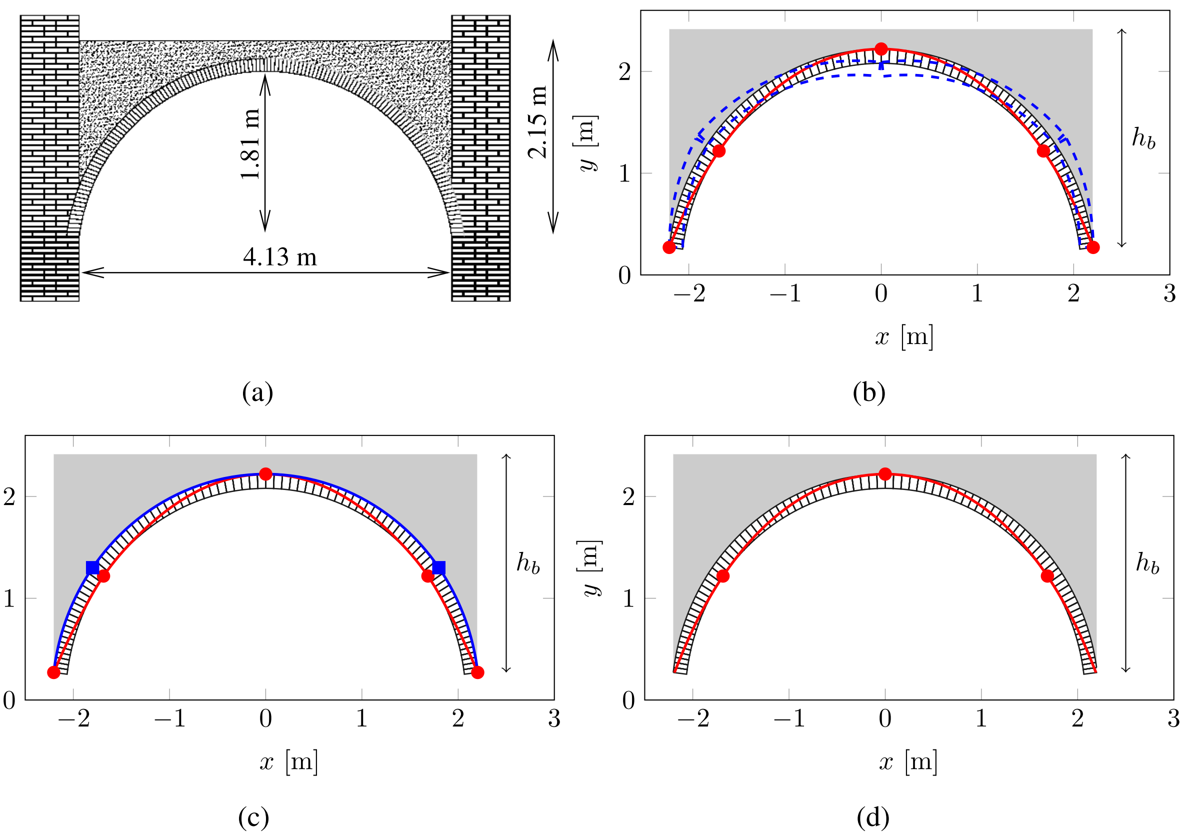


FIG. 9: A masonry arch from *Torre Fornasini* in Poggio Renatico, (Ferrara, Italy) and represented in (a) has been modeled within `ArchNURBS`. The thrust line (solid line) and the position of the hinges (solid circles) at collapse are illustrated in (b). Moreover, the solution of the limit analysis has been studied by considering (c) the FRP reinforcement indicated with a solid line at the extrados (solid squares denote the FRP delamination points) and (d) sliding between blocks (with $\mu = 0.275$).