New weight functions and second order approximation of the Oore-Burns integral for elliptical cracks subject to arbitrary normal stress field

Paolo Livieri[1], Fausto Segala[2]
[1]Dept of Engineering, University of Ferrara, via Saragat 1, 44122, Ferrara, Italy,
paolo.livieri@unife.it
[2]Dept of Physics, University of Ferrara, via Saragat 1, 44122, Ferrara, Italy,
fausto.segala@unife.it

ABSTRACT

In this paper, by means of a specific coordinate transformation, the singularity of the weight function is overcome. A strong advantage is obtained for a penny-shaped crack. In this case, a new exact formulation is given and a new alternative non-singular integral is proposed in terms of trigonometric functions. The new approach gives a remarkable streamlining of the Galin's function with the advantage of reducing the complexity of the double integral. Furthermore, we give a second order analytical approximation of Oore-Burns integral for elliptical cracks with respect to deviation from the disk. This approach drastically simplify the computational procedure without loss of accuracy.

KEYWORDS

three-dimensional cracks, weight function, stress intensity factor, analytic solution

NOMENCLATURE

δ	size of mesh over crack
Ω	crack shape
$\partial\Omega$	crack border
Q	point of Ω
Q'	point of crack border
Δ	distance between Q and $\partial\Omega$
K_I	mode I stress intensity factor
K_{I0}	mode I stress intensity factor in a circle
K_{Irw}	mode I stress intensity factor from Irwin's equation
K_{I1}	Taylor expansion up to first order of K_I for an ellipse
K_{I2}	Taylor expansion up to second order of K_I for an ellipse
ΔK_I	first order term of mode I stress intensity factor for an ellipse
ΔK_{II}	second order term of mode I stress intensity factor for an ellipse
$K_{m\,n}$	dimensionless stress intensity factor
$\overline{x}, \overline{y}$	actual Cartesian coordinate system
x, y	dimensionless Cartesian coordinate system
u, v	auxiliary dimensionless coordinate system
p, p_{mn}	reference constants (pressure)
$\overline{a}, \overline{b}$	actual semi-axis of an elliptical crack
a,b	dimensionless semi-axis of an elliptical crack
e	eccentricity of ellipse
K(e)	elliptical integral of first kind
E(e)	elliptical integral of second kind
σ_n	nominal tensile stress in $\overline{x}, \overline{y}$ actual Cartesian coordinate system
σ	nominal tensile stress in x, y dimensionless Cartesian coordinate system

1. INTRODUCTION

The weight functions technique introduced by Bueckner (1970) and Rice (1972) has been a crucial development, especially for cracks in a two dimensional body. The Stress Intensity Factors (SIFs) of planar cracks by means of weighted integrals can be calculated by means of exact or approximated equations proposed for many different geometries (see Fett and Munz (1997), Tada et al. (2000)). If the weight function is unknown, accurate results can be obtained by generalising the weight function derived from the displacement function of Petroski and Achenbach, as suggested by Glinka and Shen (1991). They consider the first three terms of the Taylor expansion of the weight function proposed by Sha and Yang (1986).

However, in many cases, for a more realistic simulation, cracks should be considered as planar two-dimensional defects in a three-dimensional body [Wen et al. (1998), J.L. Desjardins et al. (1991), Al-Falou and Ball (2000), Murakami and Endo (1983), Rice (1989), Livieri and Segala (2010) and (2012), Salvadori and Fantoni (2014)]. Usually the analytic expressions are involute and a numerical approach is hampered by the singular behaviour of the weight function. In many cases the weight function proposed in literature, is obtained from FE results of elliptic or semi-elliptic defects [Carpinteri at al. (2000), Zheng at al. (1995), Beghini et al. (1997)].

For a disk, the exact mode I weight function is known and was proposed by Galin (see Tada et al. (2000)). This equation is formulated in polar coordinates with the origin in the centre of the disk. Although this equation has an explicit analytic form, the integral of the weight function is not easy because of the singular nature of the weight function. Alternative ways were considered in literature in order to obtain the mode I stress intensity factor. In fact, Smith et al. (1967), by means of the stress function, gave the mode I stress intensity factor for a penny shape crack under quadratic or cubic expression of the nominal load [see Shah and Kobayashy (1971)].

A general weight function for three-dimensional cracks is known in the literature as the O-integral, given by Oore and Burns (1980). In the ASM Handbook (1996), the O-integral was recognised to be a general formally simple expression for SIF, which is suitable for any shape of the embedded crack.

In the case of circular or tunnel cracks, the O-integral perfectly agrees with the well known results of the literature. In order to strongly simplify the assessment of stress intensity factors, the designer often approximates defects with elliptical cracks in complex structures (see for example Hobbacher (1995) or BS 1991 in the case of welded joints).

For elliptical cracks, the agreement between O-integrl and known results is acceptably good (in the sense of a maximum errors of a few percent) in the range *major axis/minor axis* ≤ 2.5 (see Livieri and Segala (2012)).

The complexity of the evaluation of the O-integral suggested us to simplify the equation in order to obtain some powerful alternative formulation in closed analytical form.

In the case of general crack shape, the derivation of the first order approximation of the Oore-Burns integral is not immediate and it is heavily based on complex analysis, in particular on the residue theorem (see Livieri and Segala (2010)). Expression of the Oore-Burns integral accurate to the first order in the deviation from a circle, in terms of Fourier series could be useful for a quickly evaluation when the shape of crack moves away from the circular shape. From a mathematical point of view, the key to derive the Taylor expansion of the OB is complex analysis and especially the residue theorem.

In some recent works [see Livieri and Segala (2005, 2010)], for the O-integral, the authors gave the expression of the first order deviation from the circle for uniform pressures. In this paper we derive a careful closed-form representation of the Oore-Burns integral (hereinafter, OB integral) along elliptic cracks under general pressure. More precisely, we obtain the closed expression of the second order Taylor expansion of the stress intensity factors proposed by Oore-Burns (1980) with respect to deviation of the ellipse from the disk. The deviation of an ellipse from the disk is quantitatively described by the parameter $\varepsilon = 1 - b/a$, where a and b are the major and minor semi-axis, respectively. Furthermore, the paper presents a new approach for streamlining of the Galin's function with the advantage of reducing the complexity of the double integral giving a new way for the assessments of many analytical formulae.

2. Basic definitions

Let Ω be an open bounded simply connected subset of the plane as in Figure 1. We define:

$$f(Q)=\int_{\partial\Omega}\frac{ds}{\left| Q-P(s)\right|^2} \tag{1}$$

where $Q=Q(x,y)\in\Omega$, s is the arch-length parameter and the point P(s) runs over the boundary $\partial\Omega$. In their famous work in 1980, Oore-Burns proposed the following expression for the mode I stress intensity factor for a crack subjected to a nominal tensile loading $\sigma_n(Q)$ evaluated without the presence of the crack:

$$K_I(Q') = \frac{\sqrt{2}}{\pi}\int_{\Omega}\frac{\sigma_n(Q)}{\sqrt{f(Q)}\,\left|Q-Q'\right|^2}d\Omega \quad , \qquad Q'\in\partial\Omega \tag{2}$$

Under reasonable hypothesis on the function $\sigma_n(Q)$, the integral (2) is convergent and the proof is based on the asymptotic behaviour of f(Q). We recall that near ∂Ω (see Ascenzi et al. (2002) for details)

$$f(Q) \approx \frac{\pi}{\Delta} \tag{3}$$

where Δ is the distance between Q and $\partial\Omega$. By operating a rigid motion, we may assume Q'=0 and $\partial\Omega$ approximated by the parabola $y = k x^2$ in a small neighbourhood of Q. Then by applying the Lagrange multiplier method, for a fixed (x,y) with $y \geq k \cdot x^2$ we get

$$\Delta^2 = (u-x)^2 + (v-y)^2 \tag{4}$$

where *u* solves the implicit equation

$$2k^2 u^3 + (1-2ky)u - x = 0 \tag{5}$$

and $v = k u^2$.

By searching *u* in the form $\sum a_{jk} x^j y^k$, we obtain the expansion

$$\begin{aligned} u(x,y) &= x + 2kxy - 2k^2x^3 + 4k^2xy^2 - 16k^3x^3y + 8k^3xy^3 + 12k^4x^5 - 56k^4x^3y^2 + 16k^4xy^4 + \ldots = \\ &= x + 2kx(y - kx^2)[1 + 2ky - 6k^2x^2 + 4k^2y^2 + O(r^3)] \end{aligned} \tag{6}$$

where $r = \sqrt{x^2 + y^2}$

From (4), (5) and (6), it follows by means of some Taylor expansions

$$\Delta(x,y) = (y - kx^2)\,[1 - 2k^2x^2 + 4k^3x^2y + O(r^4)] \tag{7}$$

The generalisation of (7) is obvious. Let $\partial\Omega$ be locally a graph of a C^2 function v=v(u). Again, by applying multiplier theorem, equation (5) becomes u-x + (v-y) v'=0 with $v(u)=(v''(0)/2)u^2\,(1+o(1))$. Then

$$\Delta(x,y) = (y - v(x))\,[1 - \frac{v''(0)^2}{2}\,x^2 + O(r^3)] \tag{8}$$

Equation (8) can be easily tested on the circle.

In terms of polar coordinates, (r, ϑ), (8) can be rewritten in the form

$$\Delta = r\sin\vartheta - \frac{c\,r^2}{2}\cos^2\vartheta + O(r^3) \tag{9}$$

where $r = |Q - Q'|$, c is the curvature of $\partial\Omega$ at Q' and ϑ is the angle between Q-Q' and $\partial\Omega$. From (9) it follows $\sqrt{\Delta}/r = O(1/\sqrt{r}) \in L^1$ and the convergence of the OB integral is proved. The integral $K_I(Q')$ can be expressed in closed form only when Ω is a disk (see sections 4, 5).

Therefore, when Ω is an ellipse of semiaxis $(\bar{a}, \bar{b})$ $\frac{\bar{x}^2}{\bar{a}^2} + \frac{\bar{y}^2}{\bar{b}^2} \le 1$ ($\bar{b} \le \bar{a}$), a comparison between OB and the Irwin's solution K_{Irw}, given for uniform nominal stress σ_n, by

$$K_{Irw}(\alpha) = \frac{\sigma_n\sqrt{\pi\bar{b}}}{E(e)}\left(\sin^2\alpha + \frac{\bar{b}^2}{\bar{a}^2}\cos^2\alpha\right)^{1/4} \tag{10}$$

is not immediate.

In (10), $(x, y) = (\bar{a}\cdot\cos\alpha, \bar{b}\cdot\sin\alpha)$, $e = \sqrt{1 - \frac{\bar{b}^2}{\bar{a}^2}}$ is the eccentricity of the ellipse and E(e) is the elliptic integral of the second kind, i.e.

$$E(e) \;=\; \int_0^{\pi/2} \sqrt{1-e^2 \sin^2 t}\;\; dt \tag{11}$$

By following standard notation, we reserve symbol k to denote the elliptic integral of the first kind

$$K(e) \;=\; \int_0^{\pi/2} \frac{1}{\sqrt{1-e^2 \sin^2 t}}\;\; dt \tag{12}$$

3. A preliminary lemma

Lemma 3.1

Let (x,y) be a point of the plane with $\rho = x^2 + y^2 < 1$. Then for natural numbers *n* and *p*:

$$\int_{-\pi}^{\pi} \frac{e^{in\vartheta}}{\left[(x-\cos\vartheta)^2 + (y-\sin\vartheta)^2\;\right]^p}\;\; d\vartheta = \frac{\pi q(p,n,\rho^2)}{(1-\rho^2)^{2p-1}}\, w^n \tag{13}$$

where $w = x + i\,y$ and

$$q(1,n,t) \;=\; 2 \tag{14}$$

$$q(2,n,t) = 2\left[(n+1)-(n-1)t\right] \tag{15}$$

$$q(3,n,t) = n^3+3n+2+(8-2n^2)\, t+\left(n^2-3n+2\right)t^2 \tag{16}$$

Proof. We sketch the proof only for p=1. By calling $J_n(x,y)$ the integral on the l.h.s. when p=1, by the change of coordinates $x=\rho\cos\psi,\quad y=\rho\sin\psi$, we obtain

$$J_n(x,y)= e^{in\psi}\int_{-\pi}^{\pi}\frac{e^{in\vartheta}}{\rho^2-2\rho\cos\vartheta+1}\, d\vartheta \tag{17}$$

By setting $z=e^{i\theta}$, we can factor

$$\rho^2-2\rho\cos\vartheta+1=-\frac{\rho}{z}\left(z-\frac{1}{\rho}\right)(z-\rho) \tag{18}$$

Therefore, lemma follows from residue theorem, since by (17) and (18)

$$J_n(x,y)=\frac{i}{\rho}e^{in\psi}\oint\frac{z^n}{z-\frac{1}{\rho}}\ \frac{1}{z-\rho}dz=\frac{2\pi}{1-\rho^2}w^n \tag{19}$$

4. Oore-Burns on a disk

Let Ω be a fixed set. We may reconstruct OB integral along the front crack $\partial(\lambda\Omega)$ by its values along $\partial\Omega$ by the equation

$$K_I(\lambda, Q', \sigma_n) = \sqrt{\lambda}\, K_I(1, Q'/\lambda, \sigma) \tag{20}$$

where $\sigma = \sigma_n(\lambda Q)$ with $\lambda > 0$

If we "read" the boundary point Q' in terms of an angle α that is, for example Ω is star shaped with respect to the origin, (20) takes the simplest form

$$K_I(\lambda, \alpha, \sigma_n) = \sqrt{\lambda}\, K_I(1, \alpha, \sigma) \tag{21}$$

In the particular case when Ω is a disk of radius $\bar{a}$ centred at the origin of the plane $(\bar{x}, \bar{y})$, we denote by (x,y) the system in dimensionless coordinates $x = \bar{x}/\bar{a}$ and $y = \bar{y}/\bar{a}$ (see Figure 2).

By definition, for the unit disk, by (1) and (2) it follows:

$$K_{I,0}(\alpha) = \frac{\sqrt{2\,\bar{a}}}{\pi} \int_{x^2+y^2\le 1} \frac{\sigma(x,y)\; h(x,y)}{(x-\cos\alpha)^2 + (y-\sin\alpha)^2}\, dx\, dy \tag{22}$$

with

$$h(x,y) = 1/\sqrt{f(x,y)} \tag{23}$$

$$f(x,y) = \int_{-\pi}^{\pi} \frac{d\vartheta}{(x-\cos\vartheta)^2 + (y-\sin\vartheta)^2} \tag{24}$$

The function f(x,y) can be easily computed by using (13) and (14). Then, it follows

$$f(x,y) = \frac{2\pi}{1-\rho^2} \tag{25}$$

$$h(x,y) = \frac{\sqrt{1-\rho^2}}{\sqrt{2\pi}} \tag{26}$$

From now on, we will use the symbol Ξ

$$\Xi = (x-\cos\alpha)^2 + (y-\sin\alpha)^2 \tag{27}$$

Therefore

$$K_{I,0}(\alpha) = \frac{\sqrt{a}}{\pi\sqrt{\pi}} \int_{x^2+y^2\leq 1} \frac{\sigma(x,y)\sqrt{1-\rho^2}}{\Xi} dx\, dy \tag{28}$$

By introducing the change of variables $(\vartheta \in [0,\pi], \varphi \in [0,\pi/2])$

$$\begin{cases} x = (1 - r\sin\vartheta)\cos\alpha - r\cos\vartheta\sin\alpha \\ y = r\cos\vartheta\cos\alpha + (1 - r\sin\vartheta)\sin\alpha \end{cases} \tag{29}$$

where

$$r = 2\sin\vartheta\sin^2\varphi \tag{30}$$

by some simple calculations

$$\Xi = r^2 \tag{31}$$

$$dx\,dy = 4r\sin\vartheta\sin\varphi\cos\varphi\,d\vartheta\,d\varphi \tag{32}$$

$$\sqrt{1-\rho^2} = \sqrt{r}\,\sqrt{2\sin\vartheta - r} \tag{33}$$

Figure 3 shows the geometric meaning of the change of variables expressed in Eqs (29–30). For a given point P of coordinates (x,y), *r* is the distance $|PQ|$, ϑ is the angle between PQ and the tangent to the circle in Q. Figure 4 reports the geometrical meaning of angle φ introduced in (30).

By inserting (31), (32) and (33) in the integral (28), we have the following expression for the stress intensity factors on the unitary disk

$$K_{I,0}(\alpha) = \frac{4}{\pi\sqrt{\pi}} \int \sigma(x,y)\ \sin\vartheta\cos^2\varphi\,d\vartheta\,d\varphi \tag{34}$$

where the integral is computed on the “longitude” $\vartheta \in [0,\pi]$ and the “latitude” $\varphi \in [0,\pi/2]$.

Moreover, the pressure σ(x,y) is "read" in the new coordinates (ϑ, ϕ) for any fixed α, in the sense that *x* and *y* are given by (29), with *r* defined by (30). If the crack has a radius equal to $\bar{a}$ the stress intensity factor becomes

$$K_{I,0}(\alpha) = \frac{4\sqrt{\bar{a}}}{\pi\sqrt{\pi}} \int \sigma(x,y) \ \sin\vartheta \cos^2\varphi \ d\vartheta d\varphi \tag{35}$$

When $(\sigma \equiv 1)$, from (35) we obtain the well-known result:

$$K_{I,0}(\alpha) = \frac{2\sqrt{\bar{a}}}{\sqrt{\pi}} \approx 1.12837\sqrt{\bar{a}} \tag{36}$$

In order to show the efficiency of Eq. (35), now we look at some special cases of nominal stress distribution considered in the literature. Many others new examples can be obtained by changing the shape of the nominal stress *σ*.

Note that, the nominal stress *σ,* proposed in all the examples, is expressed in the x,y dimensionless coordinates system given by (29). If we know the stress in the actual $\bar{x}, \bar{y}$, by means of a simple transformation, the function $\sigma(x,y)$ can be calculated. Furthermore, in this paper only the mathematical aspects of Oore-Burns integral are taken into account if the two crack surfaces interfere with each other upon loading.

Example 4.1. Linear stress of coordinates x $\sigma(x,y) \equiv p\ x$.

In this case, the evaluation of the integral (35) is immediate:

$$K_{I,0}(\alpha) = \frac{4p\sqrt{a}}{3\sqrt{\pi}}\cos\alpha \tag{37}$$

Equation (37) perfectly agrees with that presented in Tata et al. (2000)

Example 4.2. Linear stress of coordinates y: $\sigma(x, y) \equiv p\,y$

Obviously, from (35)

$$K_{I,0}(\alpha) = \frac{4p\sqrt{a}}{3\sqrt{\pi}}\sin\alpha \tag{38}$$

Equation (38) perfectly agrees with that present in Tata et al. (2000)

Example 4.3. Hyperbolic stress: $\sigma(x, y) \equiv p\ x\,y$

Again, the evaluation of the integral (35) is very simple.

$$K_{I,0}(\alpha) = \frac{8p\sqrt{a}}{15\sqrt{\pi}}\sin 2\alpha \tag{39}$$

Equation (39) perfectly agrees with that present in R.C. Shah, A.S. Kobayashi (1971)

Example 4.4. Quadratic stress: $\sigma(x,y) \equiv p\ (x^2 - y^2)$

From Eq. (35):

$$K_{I,0}(\alpha) = \frac{16\ p\ \sqrt{a}}{15\ \sqrt{\pi}} \cos 2\alpha \tag{40}$$

Equation (40) perfectly agrees with that present in R.C. Shah, A.S. Kobayashi (1971)

Example 4.5. Exponential stress: $\sigma(x,y) \equiv p\ \left(x^2 + y^2\right)^{\frac{\gamma}{2}} = p\ \rho^\gamma$

By taking into account the identity $\rho^\gamma = \left(1 - \sin^2\vartheta\ \sin^2 2\varphi\right)^{\frac{\gamma}{2}}$ the equality between OB and the known result reported in Tada et al. (2000):

$$K_{I,0}(\alpha) = \sigma\sqrt{a}\ \frac{\Gamma\left(\frac{\gamma}{2}+1\right)}{\Gamma\left(\frac{\gamma}{2}+\frac{3}{2}\right)} \tag{41}$$

follows from the binomial series $(1+t)^\gamma = 1 + \gamma t + \frac{1}{2}\gamma(\gamma-1)t^2 + \ldots$

Example 4.6. Biquadratic stress $\sigma(x,y) \equiv p\ x^4$.

Again, the evaluation of the integral (35) gives the result:

$$K_{I,0}(\alpha) = \frac{4\,p\sqrt{\bar{a}}}{\pi^{3/2}}\left(\frac{\pi}{10}+\frac{4\pi}{35}\cos 2\alpha+\frac{8\pi}{315}\cos 4\alpha\right) \qquad (42)$$

Example 4.7. Biquadratic stress: $\sigma(x,y) \equiv p\; y^4$

Again the evaluation of the integral (35) gives:

$$K_{I,0}(\alpha) = \frac{4\,p\sqrt{\bar{a}}}{\pi^{3/2}}\left(\frac{\pi}{10}-\frac{4\pi}{35}\cos 2\alpha+\frac{8\pi}{315}\cos 4\alpha\right) \qquad (43)$$

5. Elliptical crack: first order deviation from the disk

For a given elliptical crack with half-axis $(\bar{a},\bar{b})$ and $\bar{b} \le \bar{a}$, in the system $(\bar{x},\bar{y})$, we call $x = \bar{x}/\bar{a}$, $y = \bar{y}/\bar{a}$ dimensionless coordinates. In the system (x,y), the half-axes becomes (1, b) with $b = \bar{b}/\bar{a}$ (see Figure 5). Then we consider the ellipse in non-dimensional form $x^2 + \frac{y^2}{b^2} \le 1,\ 0 < b \le 1$.

The value of the OB integral can be obtained from the integral over the normalised ellipse, by taking into account the change of scale in virtue of (21).

From the definition of OB, we emphasise the dependence of the integral from the half-axis *b*, by writing:

$$K_I(b,\alpha) = \frac{\sqrt{2}}{\pi} \int\limits_{x^2+\frac{y^2}{b^2}\le 1} \frac{\sigma(x,y)\; h(b,x,y)}{(x-\cos\alpha)^2+(y-b\sin\alpha)^2}\, dx\, dy \tag{44}$$

where

$$h(b,x,y) = 1/\sqrt{f(b,x,y)} \tag{45}$$

$$f(b,x,y) = \int_{-\pi}^{\pi} \frac{\sqrt{\sin^2\vartheta + b^2\cos^2\vartheta}}{(x-\cos\alpha)^2+(y-b\sin\alpha)^2}\, d\vartheta \tag{46}$$

By the change of variables $u = x$, $v = y/b$, we may rewrite the stress intensity factor as:

$$K_I(b,\alpha) = \frac{\sqrt{2}\, b}{\pi} \int\limits_{u^2+v^2\le 1} \frac{\sigma(u,bv)\; h(b,u,bv)}{(u-\cos\alpha)^2+b^2\,(v-\sin\alpha)^2}\, du\, dv \tag{47}$$

Only for reasons of pure simplicity, we prefer to recall by (x,y) the mute variables (u,v) in the integral (47), that is:

$$K_I(b,\alpha) = \frac{\sqrt{2}\, b}{\pi} \int\limits_{x^2+y^2\le 1} \frac{\sigma(x,by)\; h(b,x,by)}{(x-\cos\alpha)^2+b^2\,(y-\sin\alpha)^2}\, dx\, dy \tag{48}$$

In order to compute $\frac{\partial K_I(b,\alpha)}{\partial b}(1,\alpha)$,we begin, by assuming $\sigma\equiv 1$.

From now on, we will denote by ’ the derivation with respect to variable *b*.

For simplicity, we use the notation

$$\Lambda = (y - \sin\alpha)^2 \tag{49}$$

Then

$$K'_I(1,\alpha) = \frac{\sqrt{2}}{\pi}\left(\int_{x^2+y^2\le 1} \frac{h}{\Xi}\,dx\,dy + \int_{x^2+y^2\le 1} \frac{h'}{\Xi}\,dx\,dy - 2\int_{x^2+y^2\le 1} \frac{\Lambda h}{\Xi^2}\,dx\,dy \right) \tag{50}$$

where h and h’ are computed on b=1. We need the expression of $f' = \frac{\partial}{\partial b} f(b,x,by)$ on b=1. We have

$$f' = \int_{-\pi}^{\pi} \frac{\cos^2\vartheta}{\Xi}\,d\vartheta - 2\int_{-\pi}^{\pi} \frac{\Lambda}{\Xi^2}\,d\vartheta \tag{51}$$

and by taking into account that $\cos^2\vartheta = \frac{1}{2} + \mathrm{Re}\,\frac{e^{2i\vartheta}}{2}$, by lemma 3.1. It follows

$$\int_{-\pi}^{\pi} \frac{\cos^2\vartheta}{\Xi}\,d\vartheta = \frac{\pi}{1-\rho^2}\,(1 + x^2 - y^2) \tag{52}$$

$$\int_{-\pi}^{\pi} \frac{\Lambda}{\Xi^2}\,d\vartheta = \frac{\pi}{1-\rho^2} \tag{53}$$

and hence

$$f' = -\frac{\pi}{1-\rho^2}\,(1-x^2+y^2) \tag{54}$$

By the definition (46) of h:

$$h' = \frac{1}{4}\frac{\sqrt{1-\rho^2}}{\sqrt{2\pi}}\,(1-x^2+y^2) \tag{55}$$

By recalling the change of variables (29), (30) we have the following expressions:

$$x^2 = \frac{1}{2}\left(\rho^2 + A\cos 2\alpha - B\sin 2\alpha\right) \tag{56}$$

$$y^2 = \frac{1}{2}\left(\rho^2 - A\cos 2\alpha + B\sin 2\alpha\right) \tag{57}$$

$$x^2 - y^2 = A\cos 2\alpha - B\sin 2\alpha \tag{58}$$

$$\Lambda = \frac{r^2}{2}\left(1 + \cos 2\vartheta\cos 2\alpha - \sin 2\vartheta\sin 2\alpha\right) \tag{59}$$

where from now on

$$A = 1 - 2r\sin\vartheta - r^2\cos 2\vartheta \tag{60}$$

$$B = 2r\cos\vartheta - r^2\sin 2\vartheta \tag{61}$$

By taking into account (26), (33), (55)-(61), we deduce the equation

$$\left(\frac{h}{\Xi}+\frac{h'}{\Xi}-\frac{2\Lambda h}{\Xi^2}\right)r = \frac{1}{\sqrt{2\pi}}\frac{\sqrt{2\sin\vartheta - r}}{\sqrt{r}}\left[\frac{1}{4}-\left(\frac{A}{4}+\cos 2\vartheta\right)\cos 2\alpha+\left(\frac{B}{4}+\sin 2\vartheta\right)\sin 2\alpha\right] \tag{62}$$

Finally, by recalling the expression (30) of *r*:

$$K'_I(1,\alpha) = \frac{4}{\pi\sqrt{\pi}}\left(\int \sin\vartheta\cos^2\varphi\left[\frac{1}{4}-\left(\frac{A}{4}+\cos 2\vartheta\right)\cos 2\alpha+\left(\frac{B}{4}+\sin 2\vartheta\right)\sin 2\alpha\right]d\vartheta\, d\varphi\right) \tag{63}$$

where the integral is computed on $\vartheta\in[0,\pi]$ and $\varphi\in[0,\pi/2]$.

The integral (63) can be easily computed by inserting A and B:

$$K'_I(1,\alpha) = \frac{1}{\sqrt{\pi}}\left(\frac{1}{2}+\frac{2}{5}\cos 2\alpha\right) \tag{64}$$

By putting ε=1-b and by calling $K_{I,1}(\alpha)$ the first order Taylor expansion of $K_I(b,\alpha)$, from (64) and (36), we get

$$K_{I,1}(\alpha) = \frac{1}{\sqrt{\pi}}\left[2-\varepsilon\left(\frac{1}{2}+\frac{2}{5}\cos 2\alpha\right)\right] \tag{65}$$

In order to remove the assumption $\sigma\equiv 1$, we write

$$Q(\alpha,\vartheta,\varphi)=\frac{1}{4}-\left(\frac{1}{4}+\cos 2\vartheta-\frac{r}{2}\sin\vartheta-\frac{r^2}{4}\cos 2\vartheta\right)\cos 2\alpha+\left(\sin 2\vartheta+\frac{r}{2}\cos\vartheta-\frac{r^2}{4}\sin 2\vartheta\right)\sin 2\alpha \tag{66}$$

with *r* given by (30).

Then, for a general pressure σ(x,y), of class C^1, we are able to express by a closed form, the first order deviation from the disk:

$$\Delta K_{I,1}(\alpha) = \frac{-4\varepsilon}{\pi\sqrt{\pi}}\left(\int \sin\vartheta \cos^2\varphi \left(y\frac{\partial\sigma}{\partial y}+Q\sigma\right)d\vartheta d\varphi\right) \tag{67}$$

where the integral is computed on $\vartheta\in[0,\pi]$ and $\varphi\in[0,\pi/2]$ and $\sigma, \frac{\partial\sigma}{\partial y}$ are both evaluated on *x,y* dimensionless coordinate system are given by (29).

If the elliptical crack has a maximum axis equal to $\bar{a}$ the first order deviation becomes in virtue of (67)

$$\Delta K_{I,1}(\alpha) = \frac{-4\varepsilon\sqrt{\bar{a}}}{\pi\sqrt{\pi}}\left(\int \sin\vartheta \cos^2\varphi \left(y\frac{\partial\sigma}{\partial y}+Q\sigma\right)d\vartheta d\varphi\right) \tag{68}$$

Example 5.1. Power stress: $\sigma(x,y) \equiv p\, x^m y^n$

If the nominal stress σ over the crack is a polynomial in (x,y), that is a finite sum of terms of the type x^m y^n, since OB is a linear function of the pressure, it is useful to write (68) in the special case $\sigma(x,y) = p\, x^m\, y^n$, where *p* is a pressure that takes into account the physical dimension of σ.

Then, by taking into account (35), the first order deviation $\Delta K_{I,1}$, is given by

$$\Delta K_{I,1}(\alpha) = \frac{-4\varepsilon\sqrt{\bar{a}}}{\pi\sqrt{\pi}} \int \sin\vartheta \cos^2\varphi \left(n + Q\right) \sigma \, d\vartheta d\varphi \;\; = -\,\varepsilon\left(n\, K_{I,0}\,(\alpha) + \frac{4\sqrt{\bar{a}}}{\pi\sqrt{\pi}} \int \sin\vartheta \cos^2\varphi \; Q \; \sigma \; d\vartheta d\varphi \right) \tag{69}$$

where $K_{I,0}$ is the value of OB on the unitary disk. In conclusion, the first order expansion $K_{I,1}$ of the OB integral is the following

$$K_{I,1}(\alpha) = (1 - n\,\varepsilon)\,K_{I,0}\,(\alpha) - \frac{4\varepsilon\sqrt{\bar{a}}}{\pi\sqrt{\pi}} \int \sin\vartheta \cos^2\varphi \; Q \; \sigma \; d\vartheta d\varphi \tag{70}$$

Example 5.2. Linear stress of coordinates x: $\sigma(x, y) = p\, x$

We apply (69) by taking into account $\sigma(x, y) = p\, x$. Then, by putting

$$\begin{cases} C_1 = 1 - r\sin\vartheta \\ C_2 = r\cos\vartheta \\ C_3 = \dfrac{1}{4} + \cos 2\vartheta - \dfrac{r}{2}\sin\vartheta - \dfrac{r^2}{4}\cos 2\vartheta \\ C_4 = \sin 2\vartheta + \dfrac{r}{2}\cos\vartheta - \dfrac{r^2}{4}\sin 2\vartheta \end{cases} \tag{71}$$

with *r* given by (30), it follows

$$\begin{aligned} &\frac{4\varepsilon}{\pi\sqrt{\pi}} \int_{x^2+y^2\le 1} \sin\vartheta\cos^2\varphi \ x\, p\ Q\ d\vartheta d\varphi = \\ &= \frac{2\varepsilon p}{\pi\sqrt{\pi}} \int_{x^2+y^2\le 1} \sin\vartheta\cos^2\varphi \left[\left(\frac{C_1}{2} - C_1 C_3 - C_2 C_4\right)\cos\alpha - \left(C_1 C_3 - C_2 C_4\right)\cos 3\alpha\right] d\vartheta d\varphi \\ &= \frac{p\ \varepsilon}{\sqrt{\pi}}\left(\frac{4}{15}\cos\alpha + \frac{3}{35}\cos 3\alpha\right) \end{aligned} \tag{72}$$

By (37), (70), (72) we obtain

$$K_{I,1}(\alpha) = \frac{p\sqrt{a}}{105\sqrt{\pi}}\left[(140 - 21\varepsilon)\cos\alpha - 16\ \varepsilon\cos 3\alpha\right] \tag{73}$$

Example 5.3 Linear stress of coordinates y: $\sigma(x, y) = p\, y$

By using the notation of the previous example, we get

$$\frac{4\varepsilon}{\pi\sqrt{\pi}}\int \sin\vartheta\cos^2\varphi \; y\,p\; Q \; d\vartheta\, d\varphi \; =$$

$$=\frac{2\varepsilon p}{\pi\sqrt{\pi}}\int \sin\vartheta\cos^2\varphi\left[\left(\frac{C_1}{2}+C_1\,C_3+C_2\,C_4\right)\sin\alpha-\left(C_1\,C_3-C_2\,C_4\right)\sin 3\alpha\right]d\vartheta\, d\varphi \tag{74}$$

$$=\frac{p\;\varepsilon}{\sqrt{\pi}}\left(\frac{2}{5}\sin\alpha+\frac{3}{35}\sin 3\alpha\right)$$

By recalling (38) and by applying (70), by (74) we conclude

$$K_{I,1}(\alpha)=\frac{p\sqrt{\bar{a}}}{105\sqrt{\pi}}\left[\left(140-189\;\varepsilon\right)\sin\alpha-16\;\varepsilon\;\sin 3\alpha\right] \tag{75}$$

In the case σ linearly varying, the exact expression for stress intensity factors, was discussed by Shah and Kobayashi in (1971). It is interesting to look at a comparison between the Shah-Kobayashi formula in the special case $\sigma(x,y)=\;p\;y$ (*p* is given in the dimensionless coordinates x,y)

$$K_{ISB}(\alpha)\;=\;\frac{e^2\,p\;\sqrt{\pi\,\bar{a}}}{(1+e^2)\;E(e)-\frac{\bar{b}^2}{\bar{a}^2}K(e)}\;\left(\frac{\bar{b}}{\bar{a}}\right)^{\frac{3}{2}}\;\left(\sin^2\alpha+\frac{\bar{b}^2}{\bar{a}^2}\cos^2\alpha\right)^{1/4}\sin\alpha \tag{76}$$

and our approximation formula (75). The agreement is excellent (see section 7 for tables and details). For completeness, we report that in (76), K(e) and E(e) are the elliptic integral of the first and second kind computed for the eccentricity $e=\sqrt{1-\frac{\bar{b}^2}{\bar{a}^2}}$.

Example 5.4 Hyperbolic stress: $\sigma(x,y) \equiv p\ x\ y$

From (70) by setting n=m=1 it is possible to give the analytic evaluation of the integral relating to the first order approximation of K_I. $K_{I,0}$ is given by (40), so that

$$K_{I,0}(\alpha) = \frac{2\sqrt{\bar{a}}}{315\sqrt{\pi}}\left[(84-105\varepsilon)\sin 2\alpha - 10\ \varepsilon\ \sin 4\alpha\right] \tag{77}$$

6. Elliptical crack: second order deviation from the disk

From now on, we will denote by '' the second derivative with respect to the variable *b*. Therefore, from the definition (45) of the OB integral on the ellipse $x^2+\frac{y^2}{b^2}<1$, it follows by using the notation of section 5 and by assuming a unitary stress σ (in the following this assumption will be removed)

$$K''_I(1,\alpha) = \frac{\sqrt{2}}{\pi}\left(\int_{x^2+y^2\le 1}\left(\frac{h''}{\Xi}+\frac{2\ h'}{\Xi}-\frac{6\Lambda h}{\Xi^2}-\frac{4\Lambda h'}{\Xi^2}+\frac{8\Lambda^2 h}{\Xi^3}\right)dx\,dy\right) \tag{78}$$

Therefore, in order to compute the integral in the r.h.s. of (78), we need the expression of h''.

By (47) it follows that

$$f'' = \int_{-\pi}^{\pi} \left(\frac{\cos^2\vartheta \sin^2\vartheta}{\Xi} - \frac{2\,\Lambda}{\Xi^2} - \frac{4\Lambda\cos^2\vartheta}{\Xi^2} + \frac{8\Lambda^2}{\Xi^3} \right) d\vartheta \tag{79}$$

All integrals on the r.h.s. in (79) will be evaluated by means of lemma 3.1. Precisely

$$\int_{-\pi}^{\pi} \frac{\sin^2\vartheta\ \cos^2\vartheta}{\Xi}\, d\vartheta = \frac{\pi}{4(1-\rho^2)} \left[1 - \left(x^4 + y^4 - 6x^2y^2\right)\right] \tag{80}$$

$$\int_{-\pi}^{\pi} \frac{(y-\sin\vartheta)^2}{\Xi^2}\, d\vartheta = \frac{\pi}{1-\rho^2} \tag{81}$$

$$\int_{-\pi}^{\pi} \frac{(y-\sin\vartheta)^2}{\Xi^2} \cos^2\vartheta\, d\vartheta = \frac{\pi}{4(1-\rho^2)} (1+3x^2-y^2) \tag{82}$$

$$\int_{-\pi}^{\pi} \frac{(y-\sin\vartheta)^4}{\Xi^3}\, d\vartheta = \frac{3\pi}{4(1-\rho^2)} \tag{83}$$

Summing up, by means of easy arithmetic operations, we obtain

$$f'' = \frac{\pi}{4(1-\rho^2)}(13-12x^2+4y^2-x^4-y^4+6x^2y^2) \tag{84}$$

By recalling that $h = f^{-1/2}$, it follows that $h' = \frac{-1}{2} f^{-3/2} f'$ and consequently

$$h'' = \frac{3}{4} f^{-5/2} f'^2 - \frac{1}{2} f^{-3/2} f'' \tag{85}$$

By inserting (54) and (84) in (85), by taking into account that $\rho^2 = x^2 + y^2$, we deduce the expression of h''

$$h'' = \frac{1}{8}\frac{\sqrt{1-\rho^2}}{\sqrt{2\pi}}(-5+\rho^2+2\rho^4+2x^2-10\rho^2x^2+10x^4) \tag{86}$$

The next step is expressing h'' in terms of the coordinates (r,θ) given by (29). A simple calculation shows that

$$x^4 = \frac{1}{4}(\rho^4 + \frac{A^2+B^2}{2} + 2A\rho^2\cos 2\alpha - 2B\rho^2\sin 2\alpha + \frac{A^2-B^2}{2}\cos 4\alpha - AB\sin 4\alpha) \tag{87}$$

By inserting (87) in (86), h'' takes the form

$$h'' = \frac{\sqrt{1-\rho^2}}{8\sqrt{2\pi}}\,(-5+2\rho^2-\frac{\rho^4}{2}+\frac{5}{4}(A^2+B^2)+A\cos 2\alpha - B\sin 2\alpha+\frac{5}{4}(A^2-B^2)\cos 4\alpha-\frac{5}{2}AB\sin 4\alpha) \tag{88}$$

The expression of Λ^2 in terms of the coordinates (r,θ) is given by

$$\Lambda^2 = \frac{r^4}{4}\left(\frac{3}{2}+2\cos 2\vartheta\cos 2\alpha-2\sin\vartheta\sin 2\alpha+\frac{\cos 4\vartheta}{2}\cos 4\alpha-\frac{\sin 4\vartheta}{2}\sin 4\alpha\right) \tag{89}$$

By recalling the expressions of Λ, h, h' obtained in the section 5, by putting for the sake of simplicity

$$M = \cos 2\vartheta \tag{90}$$

$$N = \sin 2\vartheta \tag{91}$$

finally, we obtain

$$\frac{\sqrt{2}}{\pi} r\left(\frac{h''}{\Xi}+\frac{2\,h'}{\Xi}-\frac{6\Lambda h}{\Xi^2}-\frac{4\Lambda h'}{\Xi^2}+\frac{8\Lambda^2 h}{\Xi^3}\right)=\frac{1}{\pi\sqrt{\pi}}\frac{\sqrt{2\sin\vartheta-r}}{\sqrt{r}}\cdot$$

$$\cdot\left\{\begin{array}{l}-\frac{5}{8}+\frac{\rho^2}{4}-\frac{\rho^4}{16}+\frac{5}{32}(A^2+B^2)+\frac{AM+BN}{4}+\left(\frac{A}{8}+\frac{M}{2}\right)\cos 2\alpha-\left(\frac{B}{8}+\frac{N}{2}\right)\sin 2\alpha+\\ +\left[\frac{5}{32}\left(A^2-B^2\right)+\frac{AM-BN}{4}+M^2-N^2\right]\cos 4\alpha-\left[\frac{5}{16}AB+\frac{AN+BM}{4}+2MN\right]\sin 4\alpha\end{array}\right\} \tag{92}$$

The expressions of r^2 and r^4 in terms of (r, ϑ) easily follow the definition of $\rho^2 = x^2 + y^2$. Then

$$\rho^2 = 1-2r\sin\vartheta+r^2 \tag{93}$$

$$\rho^4 = 1-4r\sin\vartheta+2\,r^2\,(1+2\sin^2\vartheta)-4r^3\sin\vartheta+r^4 \tag{94}$$

By the definitions of A,B,M,N (see (60), (61), (90) and (91)) we are able to express all the coefficients in (88) in terms of (r,θ).

$$A^2+B^2 = 1-4r\sin\vartheta+2r^2\,(2-\cos 2\vartheta)-4r^3\,\sin\vartheta+r^4 \tag{95}$$

$$A^2-B^2 = 1-4r\sin\vartheta-6r^2\cos 2\vartheta+4r^3\,\sin 3\vartheta+r^4\cos 4\vartheta \tag{96}$$

$$AM+BN = \cos 2\vartheta+2r\sin\vartheta-r^2 \tag{97}$$

$$AM-BN = \cos 2\vartheta-2r\sin 3\vartheta-r^2\cos 4\vartheta \tag{98}$$

$$AB = 2r\cos\vartheta+2r^2\sin 2\vartheta-2r^3\cos 3\vartheta+\frac{r^4}{2}\sin 4\vartheta \tag{99}$$

$$AM+BN = \sin 2\vartheta+2r\cos 3\vartheta-r^2\sin 4\vartheta \tag{100}$$

By inserting (93–100) in (92), the integral $K''_I(1,\alpha)$ becomes

$K''_I(1,\alpha) =$

$$
=\frac{1}{\pi\sqrt{\pi}}\int\cdot\frac{\sqrt{2\sin\vartheta-r}}{\sqrt{r}}\left\{\begin{array}{l}
-\frac{9}{32}+\frac{\cos 2\vartheta}{4}-\frac{3}{8}r\sin\vartheta+\frac{3}{16}r^2(2-\cos 2\vartheta)-\frac{3}{8}r^3\sin\vartheta+\frac{3}{32}r^4+ \\
\left[\frac{1}{8}+\frac{\cos 2\vartheta}{2}-\frac{r}{4}\sin\vartheta-\frac{r^2}{8}\cos 2\vartheta\right]\cos 2\alpha+ \\
+\left[\begin{array}{l}\frac{5}{32}+\frac{\cos 2\vartheta}{4}+\cos 4\vartheta-\frac{r}{2}\left(\frac{5}{4}\sin\vartheta+\sin 3\vartheta\right)+ \\
-\frac{r^2}{4}\left(\frac{15}{4}\cos 2\vartheta+\cos 4\vartheta\right)+\frac{5}{8}r^3\sin 3\vartheta+\frac{5}{32}r^2\cos^4\vartheta\end{array}\right]\cos 4\alpha
\end{array}\right\}dr\,d\vartheta
\tag{101}
$$

The integrals $F_m(\vartheta) = \int_0^{2\sin\vartheta} r^{m-\frac{1}{2}}\sqrt{2\sin\vartheta-r}\,dr = 2^{m+2}\left[\int_0^{\pi/2}\sin^{2m}\varphi\ \cos^2\varphi\ d\varphi\right]\sin^{m+1}\vartheta$ can be expressed in the closed form

$$
F_m(\vartheta) = \frac{\pi\ (2m)!}{2^m\ m!\ (m+1)!}\sin^{m+1}\vartheta \tag{102}
$$

By inserting (102) in (101) the integrals $K''_I(1,\alpha)$ is reduced to a sum of integrals of the type $\int_0^{\pi} e^{im\vartheta}d\vartheta = \frac{1}{i\,m}\left[(-1)^m-1\right]$. In conclusion, when σ=1

$$K''_I(1,\alpha) \;=\; \frac{1}{\sqrt{\pi}}\left(-\frac{49}{60}-\frac{\cos 2\alpha}{5}-\frac{38}{315}\cos 4\alpha\right) \tag{103}$$

In others words the second order deviation from the unit disk is given by

$$\Delta K_{I,2} = \frac{-\varepsilon^2}{2\sqrt{\pi}}\left(\frac{49}{60}+\frac{\cos 2\alpha}{5}+\frac{38}{315}\cos 4\alpha\right) \tag{104}$$

Now we are able to remove the assumption $\sigma \equiv 1$ by taking σ of class C^2. By following previous computations, we infer that the second order expansion from the unit disk, for a general nominal stress σ, is given by

$$\Delta K_{I,2} = \frac{2\varepsilon^2}{\pi\sqrt{\pi}}\int \sin\vartheta \cos^2\varphi\left(y^2\frac{\partial^2\sigma}{\partial y^2}+2y\frac{\partial\sigma}{\partial y}Q+\sigma R\right)d\vartheta\, d\varphi \tag{105}$$

where Q is defined in (66) and

$$R = \begin{Bmatrix} -\frac{5}{8}+\frac{\rho^2}{4}-\frac{\rho^4}{16}+\frac{5}{32}(A^2+B^2)+\frac{AM+BN}{4}+\left(\frac{A}{8}+\frac{M}{2}\right)\cos 2\alpha-\left(\frac{B}{8}+\frac{N}{2}\right)\sin 2\alpha+ \\ \left[\frac{5}{32}(A^2-B^2)+\frac{AM-BN}{4}+M^2-N^2\right]\cos 4\alpha-\left[\frac{5}{16}AB+\frac{AN+BM}{4}+2MN\right]\sin 4\alpha \end{Bmatrix} \tag{106}$$

Moreover σ, $\frac{\partial\sigma}{\partial y}$, $\frac{\partial^2\sigma}{\partial y^2}$ are evaluated on dimensionless coordinates *x,y* given by (29).

If the elliptical crack has a maximum axis equal to $\bar{a}$ the second order deviation becomes

$$\Delta K_{I,2} = \frac{2\varepsilon^2 \sqrt{\bar{a}}}{\pi\sqrt{\pi}} \int \sin\vartheta \cos^2\varphi \left(y^2 \frac{\partial^2 \sigma}{\partial y^2} + 2y \frac{\partial \sigma}{\partial y} Q + \sigma R \right) d\vartheta\, d\varphi \tag{107}$$

Finally, by (35), (68) and (107) we obtain

$$K_{I,2} = \frac{4\sqrt{\bar{a}}}{\pi\sqrt{\pi}} \int \sin\vartheta \cos^2\varphi \left[\sigma - \varepsilon\left(\frac{\partial \sigma}{\partial y} + \sigma Q \right) + \frac{\varepsilon^2}{2}\left(y^2 \frac{\partial^2 \sigma}{\partial y^2} + 2y \frac{\partial \sigma}{\partial y} Q + \sigma R \right)\right] d\vartheta\, d\varphi \tag{108}$$

6.1 Linear stress of coordinate y: $\sigma(x, y) = p\, y$

By results of section 5, we only need a suitable expression of yR. By writing (106) in the form

$$R = D_0 + D_1 \cos 2\alpha - D_2 \sin 2\alpha + D_3 \cos 4\alpha - D_4 \sin 4\alpha \tag{109}$$

it follows

$$\text{odd terms of } \; y\,R = \left\{ \begin{array}{l} \left(C_1 D_0 - \dfrac{C_1 D_1}{2} - \dfrac{C_1 D_3}{2} - \dfrac{C_2 D_2}{2} \right) \sin\alpha + \dfrac{1}{2} \left(C_1 D_1 - C_2 D_2 - C_2 D_4 \right) \sin 3\alpha + \\ + \dfrac{1}{2} \left(C_1 D_3 - D_2 D_4 \right) \sin 5\alpha \end{array} \right\} \tag{110}$$

Where C_1 and C_2 are defined in (71). By inserting (110) in (108) and by taking into account (75), we get the second order Taylor expansion

$$K_{I,2}(\alpha) = \frac{p\sqrt{a}}{105\sqrt{\pi}} \left[(140 - 189\,\varepsilon) \sin\alpha - 16\,\varepsilon \sin 3\alpha \right] + \frac{2\,p\sqrt{a}\,\varepsilon^2}{\sqrt{\pi}} \left(\frac{27}{280} \sin\alpha + \frac{53}{630} \sin 3\alpha - \frac{29}{2310} \sin 5\alpha \right) \tag{111}$$

The performance of (111) will be tested in the tables of section 7.

6.2 Linear stress of coordinate x: $\sigma(x, y) = p\,x$

As in the previous case, by using (108), we get the second order Taylor expansion as:

$$\Delta K_{I,2}(\alpha) = \frac{-2\,p\sqrt{a}\,\varepsilon^2}{\sqrt{\pi}} \left(\frac{29}{280} \cos\alpha + \frac{29}{630} \cos 3\alpha + \frac{29}{2310} \cos 5\alpha \right) \tag{112}$$

Finally, by recalling (73), the Taylor expansion up to the second order becomes:

$$K_{I,2}(\alpha)=\frac{p\sqrt{a}}{105\sqrt{\pi}}\left[(140-21\varepsilon)\cos\alpha-16\,\varepsilon\cos 3\alpha\right]-\frac{2p\sqrt{a}\,\varepsilon^2}{\sqrt{\pi}}\left(\frac{29}{280}\cos\alpha+\frac{29}{630}\cos 3\alpha+\frac{29}{2310}\cos 5\alpha\right) \tag{113}$$

6.3 Power stress: $\sigma(x,y)=p\ \ x^m\, y^n$

When $\sigma(x,y)=p\ \ x^m\, y^n$, *e*quation (108) gives

$$K_{I,2}(\alpha)\ =\left[1-n\,\varepsilon+\frac{1}{2}n\,(n-1)\,\varepsilon^2\right]K_{I,0}\,(\alpha)\,-\,\frac{4\sqrt{a}\ \varepsilon}{\pi\sqrt{\pi}}\int\sin\vartheta\cos^2\varphi\left[(1-n\,\varepsilon)\ Q-\frac{\varepsilon}{2}R\right]\sigma\ d\vartheta\, d\varphi \tag{114}$$

The r.h.s. in (114) is a very good candidate to represent the stress intensity factor for elliptical cracks when the ratio major axis/minor axis is less than two.

6 4 Polynomial stress of degrees 2: $\sigma(x,y) = \sum_{m+n\le 2} p_{m\,n}\, x^m\, y^n$

We conclude this section by putting the explicit equation of $K_{I,2}(\alpha)$ when σ is a polynomial of degrees 2 and where $p_{m\,n}$ is a pressure that takes into account the physical dimension of σ. By applying (114) one has

$$K_{I,2} = \frac{\sqrt{a}}{\sqrt{\pi}} \sum_{m+n\le 2} p_{m\,n}\, K_{m\,n} \tag{115}$$

where

$$K_{0\,0}(\alpha) = 2 - \frac{\varepsilon}{5}\left(\frac{5}{2} + 2\cos 2\alpha\right) - \frac{\varepsilon^2}{10}\left(\frac{49}{12} + \cos\ 2\alpha + \frac{38}{63}\cos 4\alpha\right)$$

$$K_{10}(\alpha) = \frac{4}{3}\cos\alpha - \frac{\varepsilon}{5}\left(\cos\alpha + \frac{16}{21}\cos\ 3\alpha\right) - \frac{29}{35}\varepsilon^2\left(\frac{1}{4}\cos\ \alpha + \frac{1}{9}\cos\ 3\alpha + \frac{1}{33}\cos\ 5\alpha\right)$$

$$K_{01}(\alpha) = \frac{4}{3}\sin\alpha - \frac{\varepsilon}{5}\left(9\ \sin\alpha + \frac{16}{21}\sin\ 3\alpha\right) + \frac{1}{35}\varepsilon^2\left(\frac{27}{4}\sin\ \alpha + \frac{53}{9}\sin\ 3\alpha - \frac{29}{33}\sin\ 5\alpha\right)$$

$$K_{20}(\alpha) = \frac{2}{3} + \frac{8}{15}\cos 2\alpha - \varepsilon\left(\frac{1}{10} + \frac{16}{105}\cos 2\alpha + \frac{4}{63}\cos 4\alpha\right) - \frac{\varepsilon^2}{5}\left(\frac{29}{56} + \frac{4}{9}\cos\ 2\alpha + \frac{62}{231}\cos 4\alpha + \frac{164}{3003}\cos 6\alpha\right)$$

$$K_{11}(\alpha) = \frac{8}{15}\sin\ 2\alpha - \frac{2}{3}\varepsilon\left(\sin 2\alpha + \frac{2}{21}\sin\ 4\alpha\right) + \frac{\varepsilon^2}{21}\left(\frac{17}{30}\sin\ 2\alpha + \sin 4\alpha - \frac{164}{715}\sin\ 6\alpha\right)$$

$$K_{02}(\alpha) = \frac{2}{3} + \frac{8}{15}\cos 2\alpha + \frac{\varepsilon}{3}\left(\frac{-47}{10} + \frac{124}{35}\cos 2\alpha + \frac{4}{21}\cos 4\alpha\right) + \frac{\varepsilon^2}{21}\left(\frac{837}{40} - \frac{43}{3}\cos 2\alpha - \frac{172}{55}\cos 4\alpha + \frac{164}{715}\cos 6\alpha\right)$$

7. Application of second order approximation to some remarkable cases

7.1 Comparison between Eq. (108) and the Irwin equation for constant stress

By assuming $\sigma(x,y)$=*constant*, we denote by $K_{I,2}(\alpha)$ the second order Taylor expansion of $K_I(b=\bar{b}/\bar{a},\alpha)$ at the initial value b=1. The *nth* order Taylor expansion $K_{Irw,n}(\alpha)$ of Irwin (1962) solution $K_{Irw}(\alpha) = \frac{\sqrt{\pi b}}{E(e)}\left(\sin^2\alpha + b^2\cos^2\alpha\right)^{1/4}$ can be achieved by direct computation. In particular for n=2:

$$K_{Irw,2}(\alpha) = \frac{\sigma\sqrt{a}}{\sqrt{\pi}}\left[2-\varepsilon\left(\frac{1}{2}+\frac{1}{2}\cos 2\alpha\right)-\frac{\varepsilon^2}{2}\left(\frac{13}{16}+\frac{1}{4}\cos 2\alpha+\frac{3}{16}\cos 4\alpha\right)\right] \qquad (116)$$

The expression of $K_{I,2}$ was defined in the previous section (see (65) and (103)):

$$K_{I,2}(\alpha) = \frac{\sigma\sqrt{a}}{\sqrt{\pi}}\left[2-\varepsilon\left(\frac{1}{2}+\frac{2}{5}\cos 2\alpha\right)-\frac{\varepsilon^2}{2}\left(\frac{49}{60}+\frac{1}{5}\cos 2\alpha+\frac{38}{315}\cos 4\alpha\right)\right] \qquad (117)$$

Then, the difference between the Irwin and OB integral up to the second order approximation is expressed by:

$$\frac{K_{Irw}(\alpha)-K_{I,2}(\alpha)}{\sigma\sqrt{a}} = \frac{\varepsilon}{\sqrt{\pi}}\left[\frac{1}{10}\cos 2\alpha+\frac{\varepsilon}{2}\left(-\frac{1}{240}+\frac{1}{20}\cos 2\alpha+\frac{337}{5040}\cos 4\alpha\right)\right] \tag{118}$$

By observing that $\frac{337}{5040}=\frac{1}{15}+\frac{1}{5040}$, we may delete the insignificant contribution $\frac{\varepsilon^2}{10080\sqrt{\pi}}$ whose value for ε=0.4 is 0.0000089. Therefore

$$\frac{K_{Irw}(\alpha)-K_{I,2}(\alpha)}{\sigma\sqrt{a}} = \frac{\varepsilon}{10\sqrt{\pi}}\left[\cos 2\alpha+\varepsilon\left(-\frac{1}{48}+\frac{1}{4}\cos 2\alpha+\frac{1}{3}\cos 4\alpha\right)\right] \tag{119}$$

and finally, from (118)

$$\frac{K_{I,2}(\alpha)}{K_{Irw}(\alpha)} = 1+\frac{\varepsilon}{20}\left[\cos 2\alpha+\frac{\varepsilon}{2}\left(\frac{5}{24}+\cos 2\alpha+\frac{11}{12}\cos 4\alpha\right)\right]+O(\varepsilon^3) \tag{120}$$

Note that for α=0 the ratio in Eq. (119) is equal to $1+\frac{\varepsilon}{20}\left(1+\frac{17}{16}\varepsilon\right)$, whereas for $\alpha=\pi/2$, its value is $1+\frac{\varepsilon}{20}\left(-1+\frac{1}{16}\varepsilon\right)$. For example, when ε=0.5, the above values are respectively 1.038 and 0.975. The analytical evaluation of the second order approximation of the OB integral underlines the accuracy of the OB in the stress intensity factor predictions also for elliptical cracks. Due to the singularity nature of the weight function, from a numerical point of view, we need a greatly accurate mesh for estimating this very slight difference between Irwin and Oore-Burns when ε is relatively small (see Livieri and Segala 2014), so that the numerical errors can be greater than the theoretical accuracy of OB integral. Figure 6 shows the trend of $\frac{K_{I,2}(\alpha)}{K_{Irw}(\alpha)}$ given by (120). The maximum error is less than 4% up to ε=0.5.

7.2 Numerical verifications of the first and second order approximation

First of all, we take into account the error between the Oore-Burns K_I given by (2) and its approximations of first and second order. The OB integral (2) can be evaluated by a numerical procedure described in a previous paper (see Livieri and Segala 2014). Tables 1–3 show the comparison in the cases of nominal stress proportional to dimensionless coordinates: x, y and x·y. The analytical results are taken from Shah and Kobayashi (1967). Figure 7 reports a typical mesh used in numerical analysis for the evaluation of integral (2) when Ω is an ellipse. The mesh in dimensionless coordinates is of a type (M, N, 2M) with *M* and *N* integer numbers. The polar plane is divided into *M·N* subset of amplitude $\delta=\pi/M$. The point Q' is located at the origin of the mesh, and the contour of the ellipse is divided into *2·M* intervals.

The Riemann sums and the analytical values are essentially equal when the deviation from the disk is small.

However, when the *b/a* ratio reaches 0.5 the second order expansions of the OB integral, in the example proposed herein, gives maximum errors around 5% while the more significant parameter $\frac{\text{error}}{\max K_I}$ does not exceed 3%.

7.3 Comparison with an example by Atroshchenko *et al. (2010)*

In order to shown the accuracy of the second order approximation of equation (108), we now consider the example proposed by Atroshchenko et al. (2010) for an ellipse with a=1, b=0.4 and a nominal stress given by

$$\sigma(x, y) = \frac{p}{2 + 1.2\ x + 0.25\ y} \tag{121}$$

Now by using equation (108), we only need the expressions of $\frac{\partial \sigma}{\partial y}$ and $\frac{\partial^2 \sigma}{\partial y^2}$ because the coefficients Q and R are explicated in equations (66) and (106), respectively.

Figure 8 shows a comparison between Eq. (108) and Atroshchenko et al. (2010) solution. Despite of the example is on the border of the optimal range of application of Eq. (108), because the ratio *major axis/minor axis =2.5*, the agreement is very remarkable, with the advantage to evaluate the stress intensity factor directly with Eq (108) or alternatively (114), by taking into account the expansion:

$$\sigma(x, y) = \frac{p}{2} \sum_{0}^{\infty} \left(1 + \frac{3}{5}\ x + \frac{1}{8}\ y\right)^{K} \tag{122}$$

The difference in the maximum stress intensity factor prediction is around 7%.

8. Discussion

The first novelty of this paper is a significant improvement in the analytical approximation of the O-integral due to the computation of the second order Taylor expansion. A higher order expansion is not useful, because it is known in literature that the OB integral is a good approximation for cracks not too far from the disk, which means for relatively small ε. This is exactly the range where the second order Taylor expansion is very close to the true value of the integral.

The second novelty is the clearance of the uniformity by assuming a general pressure over the crack. The knowledge of an expression of the OB integral in closed form, avoids heavy numerical procedures. Indeed the computation of the integral by means of Riemann's sums in terms of the size of the mesh is very slow and requires a very large number of terms in order to have acceptable precision.

As a particular case, the third novelty of our paper is a new exact expression in closed form of the stress intensity factors on the disk, without singularity (see Eq. (34)). In the case of a crack in a two-dimensional body, the elimination of the singularity by means of a variable transformation was proposed, for example, by Mawatari and Nelson (2011) and Daniewicz (1994). We recall that on the disk, the definition of the Oore-Burns is correct, and agrees with the Galin's weight functions.

The core of our paper is Eq. (108) which expresses OB integral in terms of a finite linear combination of sinus and cosinus, accessible even to a non-specialist. Whit respect to the interesting work of Atroshchenko at al. (2010), our second order "final formula" is much more simple and handy. In order to compare our equation (108) with the results of the above work, at the end of section 7, we apply our formula to the same concrete example of pressure considered by those authors. Although the example is on the border of the optimal range of application of Eq. (108), because the ratio *major axis/minor axis =2.5*, the agreement is very remarkable.

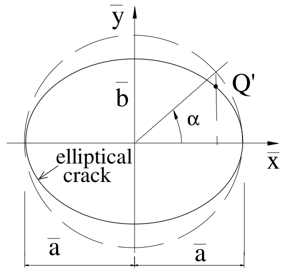
The results of this work, definitely confirm the accuracy of the OB integral as a tool for the investigation of the stress intensity factors for somewhat circular tensile cracks under remote mode I loadings. Despite of our approximations, the accuracy of Eqs. (108) is similar to those find by Krasowsky et al. 1999 developed specifically for elliptical defects.

6. CONCLUSIONS

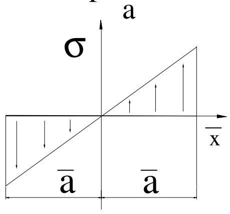
The main conclusions obtained in this paper are:

- By means of some mathematical tricks, we are able to derive a handy second order expansion of the stress intensity factor of an elliptical crack based on the Oore-Burns integral.
- For circular cracks, we obtain an analytic formulation of the Oore-Burns integral, which gives an alternative simplified full solution with the advantage to remove the singularity of the weight function.
- For elliptical cracks, the removal of the singularity allows a numerical approach, which is faster than previously available procedures. The only requirement is standard software without the use of any particular algorithms.

Table 1 Comparison among Oore-Burns' integral and Shah and Kobayashi (1971) (mesh for Riemann' sum $\delta/\bar{a} = 0.00503$; σ is the normal stress on the crack and p is a reference pressure)

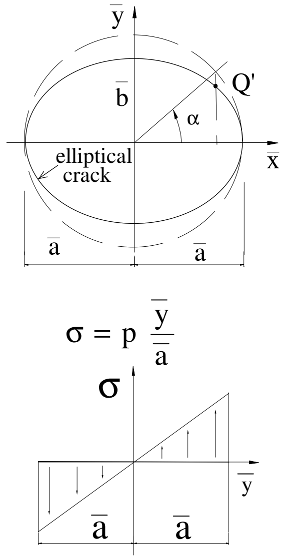


$$\sigma = p\,\frac{\bar{x}}{\bar{a}}$$



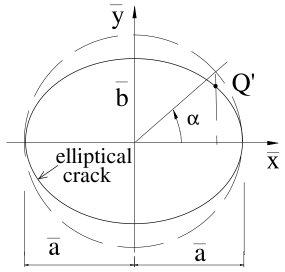
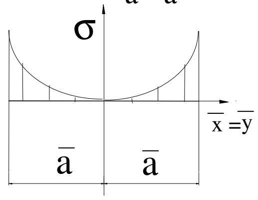
	α [deg]	$\frac{K_I}{p\sqrt{\bar{a}}}$ Riemann' sum of Eq. (2) (Livieri and Segala (2014))	$\frac{K_{I,1}}{p\sqrt{\bar{a}}}$ Eq. (73)	$\frac{K_{I,2}}{p\sqrt{\bar{a}}}$ Eq. (113)	$\frac{K_I}{p\sqrt{\bar{a}}}$ Shah and Kobayashi (1971)
0.9	0	0.723	0.732	0.731	0.731
0.9	30	0.644	0.642	0.641	0.642
0.9	60	0.381	0.379	0.379	0.381
0.8	0	0.699	0.712	0.705	0.706
0.8	30	0.630	0.632	0.628	0.632
0.8	60	0.383	0.382	0.381	0.385
0.7	0	0.668	0.693	0.676	0.673
0.7	30	0.614	0.622	0.614	0.618
0.7	60	0.384	0.385	0.384	0.389
0.6	0	0.629	0.673	0.643	0.633
0.6	30	0.594	0.612	0.598	0.601
0.6	60	0.383	0.388	0.386	0.391
0.5	0	0.578	0.653	0.607	0.581
0.5	30	0.569	0.603	0.580	0.578
0.5	60	0.378	0.391	0.388	0.390

Table 2 Comparison among Oore-Burns' integral and Shah and Kobayashi (1971) (mesh for Riemann' sum $\delta/\bar{a}$ = 0.00503; σ is the normal stress on the crack and p is a reference pressure)



	α [deg]	$\frac{K_I}{p\sqrt{\bar{a}}}$ Riemann' sum of Eq. (2) (Livieri and Segala (2014))	$\frac{K_{I,1}}{p\sqrt{\bar{a}}}$ Eq. (75)	$\frac{K_{I,2}}{p\sqrt{\bar{a}}}$ Eq. (111)	$\frac{K_I}{p\sqrt{\bar{a}}}$ Eq. (68) (Shah and Kobayashi (1971))
0.9	30	0.320	0.317	0.318	0.317
0.9	60	0.568	0.564	0.565	0.563
0.9	90	0.658	0.659	0.659	0.658
0.8	30	0.265	0.257	0.263	0.261
0.8	60	0.483	0.476	0.48	0.478
0.8	90	0.566	0.566	0.566	0.565
0.7	30	0.213	0.198	0.211	0.210
0.7	60	0.401	0.388	0.397	0.396
0.7	90	0.475	0.473	0.473	0.473
0.6	30	0.166	0.139	0.161	0.163
0.6	60	0.323	0.3	0.317	0.318
0.6	90	0.386	0.38	0.38	0.383
0.5	30	0.124	0.079	0.115	0.121
0.5	60	0.249	0.212	0.238	0.245
0.5	90	0.300	0.287	0.287	0.297

Table 3 Comparison among Oore-Burns' integral and Shah and Kobayashi (1971) (mesh for Riemann' sum $\delta/\bar{a}$ = 0.00503; σ is the normal stress on the crack and p is a reference pressure)

 $\sigma = p\,\frac{\bar{x}}{\bar{a}}\,\frac{\bar{y}}{\bar{a}}$ 	α [deg]	$\frac{K_I}{p\sqrt{\bar{a}}}$ Riemann sum of Eq. (2) (Livieri and Segala (2014))	$\frac{K_{I,1}}{p\sqrt{\bar{a}}}$ Eq. (77)	$\frac{K_{I,2}}{p\sqrt{\bar{a}}}$ Eq. (115)	$\frac{K_I}{p\sqrt{\bar{a}}}$ Shah and Kobayashi (1971)
0.9	0	0	0	0	0
0.9	30	0.227	0.225	0.225	0.225
0.9	60	0.233	0.231	0.231	0.231
0.8	0	0	0	0	0
0.8	30	0.192	0.189	0.191	0.191
0.8	60	0.203	0.202	0.201	0.201
0.7	0	0	0	0	0
0.7	30	0.159	0.154	0.157	0.157
0.7	60	0.173	0.172	0.171	0.172
0.6	0	0	0	0	0
0.6	30	0.127	0.118	0.124	0.126
0.6	60	0.143	0.143	0.141	0.142
0.5	0	0	0	0	0
0.5	30	0.097	0.082	0.091	0.096
0.5	60	0.113	0.113	0.111	0.112

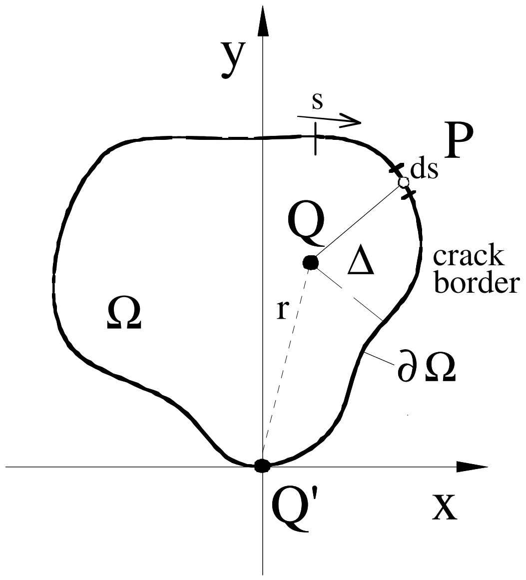
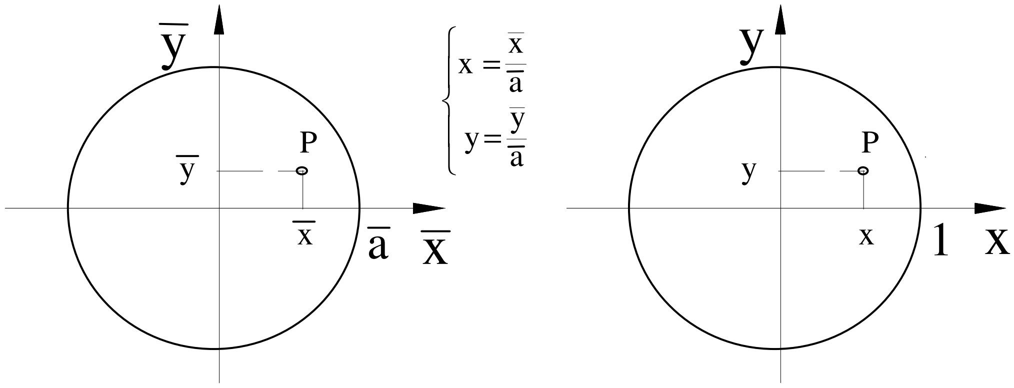


Figure 1. Crack shape Ω



a) b)

Figure 2. a) Actual circular crack; b) dimensionless circular crack integration domain

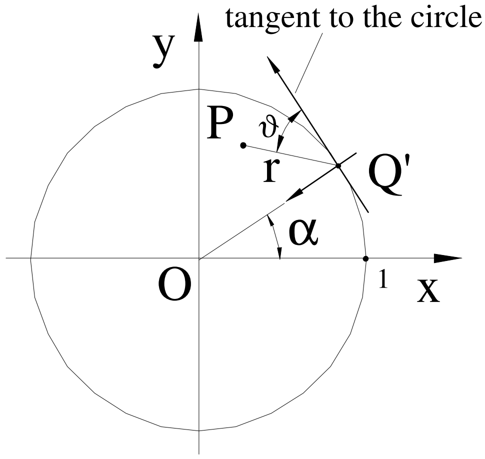


Figure 3. Co-ordinate system variable change

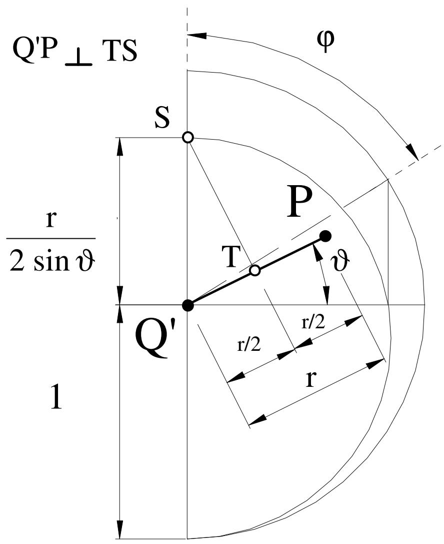


Figure 4: geometrical construction for the evaluation of latitude φ .

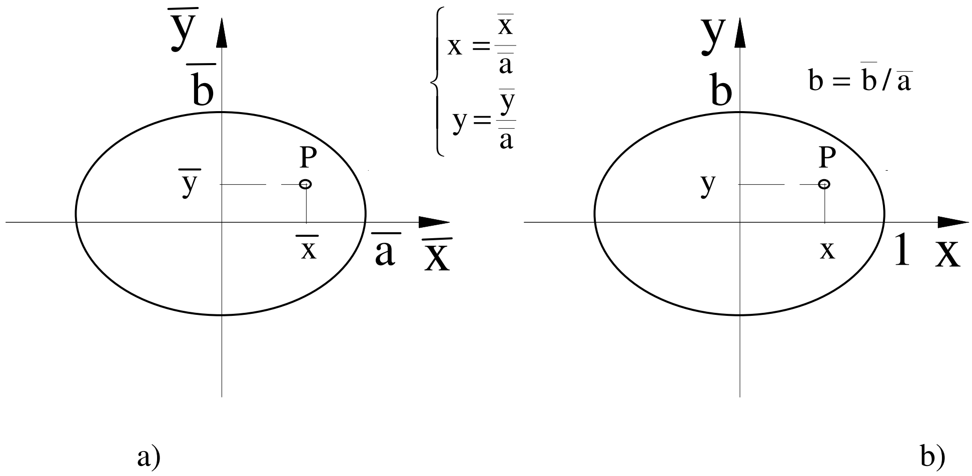


Figure 5. a) Actual elliptical crack; b) dimensionless elliptical crack

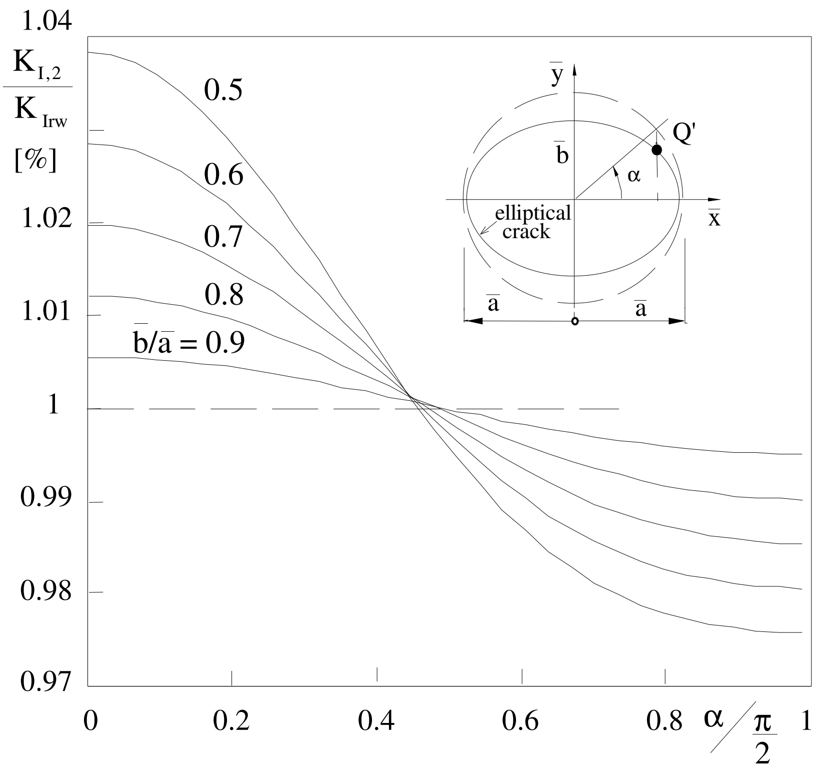


Figure 6. Difference between Irwin (1962) and Oore-Burns' integral up to the second order approximation

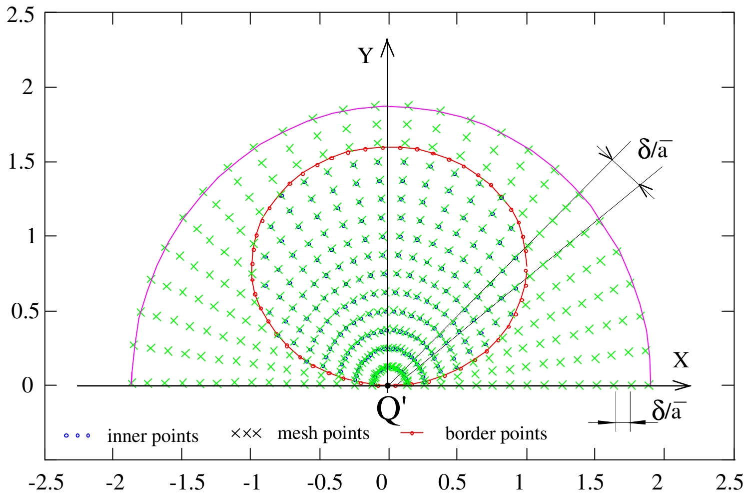


Figure 7. Typical mesh used for stress intensity factor evaluation of an elliptical cracks ($\delta/\bar{a}$ =0.126, $\bar{b}/\bar{a} = 0.8$)

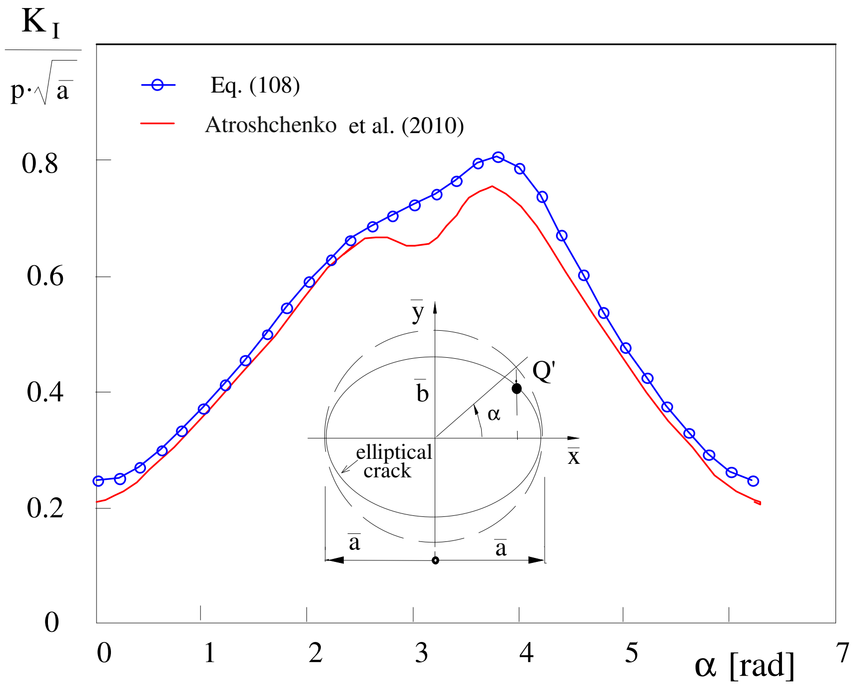


Figure 8. Comparison between Eq. (108) and Atroshchenko et al. (2010) solution. (elliptical crack a=1, b=0.4)

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