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Highlights:

- New frequency domain model to calculate damage degree with the use of the strain energy parameter;
- New probability distribution model for the strain energy density parameter;
- The strain energy parameter damage degree is calculated directly from the power spectral density;
- Theoretical proof of the equality of the energy and stress approach in frequency domain;
- The approach is verified with the use of experimental results for the narrowband and broadband loading case.

# A frequency-domain model assessing random loading damage by the strain energy density parameter

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***Abstract***

The paper presents a new fatigue damage assessment model defined for the strain energy density parameter. The model is defined for calculations in the frequency domain for the case of narrowband loading. Analytical expressions are derived for the probability density function of peaks, the level crossing spectrum, and for the expected damage that is directly linked to the power spectral density of the strain energy parameter. The paper presents a theoretical proof which shows that the damage obtained for the energy model is equal to the damage obtained for the stress model. A numerical example is used to verify the correctness of the proposed frequency-domain damage estimation formula by comparison with experimental and time-domain fatigue life estimation results for two simulated random time-histories for the case of narrowband and broadband loading. The comparison between experimental and calculation results confirm the very good agreement between frequency and time domain approaches.

**Keywords:** *fatigue, frequency domain, strain energy density parameter, fatigue assessment, power spectral density.*

***Nomenclature:***

| | |
|---|---|
| $a = 1/2E$ | scale parameter |
| $A$ | fatigue strength coefficient of the S-N curve of $\sigma(t)$ |
| $A_w$ | fatigue strength coefficient of the S-N curve of $W(t)E$ Young modulus of the material |
| $E[\Delta D_\sigma]$, $E[\Delta D_W]$ | expected damage per cycle for $\sigma(t)$ and $W(t)$ |
| $E[D_\sigma(T)]$, $E[D_W(T)]$ | expected damage in time for $\sigma(t)$ and $W(t)$ |
| $E[x^n]$ | central moment of order $n$ |
| $g[-]$, $g^{-1}[-]$ | direct and inverse nonlinear transformation |
| $G_\sigma(\omega)$ | power spectral density of $\sigma(t)$ |
| $k$ | inverse slope of the of the S-N curve of $\sigma(t)$ |
| $k'$ | inverse slope of the S-N curve of $W(t)$ |
| $m_n$ | $n$-th order spectral moment of $G_\sigma(\omega)$ |
| $m_0 = Var[\sigma(t)]$ | variance of $\sigma(t)$ |
| $N_f$ | number of cycles to failure |
| $N_{f0}$ | reference number of cycles to failure |
| $p_{a,\sigma}(\sigma_a)$ | PDF of stress amplitude |
| $p_\sigma(x)$, $p_W(w)$ | PDF of $\sigma(t)$ and $W(t)$ |
| $p_{p,\sigma}(u)$, $p_{p,W}(v)$ | PDF of the peaks in $\sigma(t)$ and $W(t)$ |
| $sgn\,(-)$ | signum function |
| $T$ | time duration |
| $W(t)$ | energy parameter time-history |
| $W_a$ | energy parameter amplitude |
| $W_{af}$ | energy parameter reference fatigue strength amplitude at the number of cycles $N_0$ |
| $\varepsilon(t)$ | strain time-history |
| $\varepsilon_a$ | strain amplitude |
| $\nu_0^+$ | rate of mean value up-crossings |
| $\nu_\sigma(x)$, $\nu_W(y)$ | level crossing spectrum of $\sigma(t)$ and $W(t)$ |
| $\sigma(t)$ | stress time-history |
| $\sigma_a$ | stress amplitude |
| $\sigma_{af}$ | reference fatigue strength amplitude in fully reversed tension-compression at the number of cycles $N_0$ |
| $ku$ | kurtosis |
| $sk$ | skewness |
| PDF | probability density function |
| PSD | power spectral density |

## 1. Introduction

Fatigue damage calculation algorithms can be divided into two major domains: time and frequency. The time domain uses cycle counting algorithms to assess the damage degree of the material. Beside this it has many advantages towards to the frequency domain which uses statistical information obtained from the power spectral density (PSD), but one huge disadvantage which is the computation time. During the last 70 years there has been an increasing growth in the use of advanced simulation and calculation approaches for random fatigue analysis that have been adapted to the use in the frequency domain. Since the early work of Bendat [1], which defined the basics for probability based random fatigue calculations, a great many spectral methods have been proposed. The first major paper by Dirlik [2] which incorporated the use of computers especially for calculations with the use of power spectral density (PSD) was a milestone for these techniques. Wirsching [3] was one of the first who defined probability based criteria for fatigue life estimation with the use of the frequency domain. Macha [4] was the first who defined the basics for multiaxial random calculations with the use of PSD. The frequency domain methods owe a lot of growth in renown to the papers by Pitoiset and Preumont [5,6], due to the fact that they adapted the popular von Mises method to the frequency domain regime. The problem of wide band loading in spectral methods was one of the main problems in papers by Kihl and Sarkani [7]. Rychlik et al [8] have worked on the damage degree model for non-Gaussian random loads. Then at the beginning of the 21th century we deal with the beginning of the renowned series of publications dealing with the topic of frequency defined methods presented by Benasciutti and Tovo [9] which vastly influenced the spectral methods theory and applicability due to the effects of narrowband, broadband and non-Gaussian [10–12] random loading and presenting their popular probability density function based on the damage degree similar to the solution working in the rainflow algorithm. The evolution of spectral methods in terms of multiaxial fatigue was also the one of the main tasks of the group led by Carpinteri [13,14]. Another problem which can be found in the literature on spectral methods is the non-stationarity effect which is widely discussed in the papers by Slavič et al [15,16]. Problems related to mean stress effects in frequency domain have been described by Böhm et al. [17,18]. A summation of the most important aspects in terms of variable amplitude loading and frequency domain calculations in terms of multiaxial loading can be found in the papers by Sonsino et al [19,20].

Nevertheless it is rare to stumble upon a paper on fatigue damage estimation in frequency domain with the use of the strain energy density parameter such as the Banvilett et al [21]. The strain energy density parameter model often referred as the energy parameter model describes the signed strain energy of the combined stress-strain state of the material. This parameter allows taking into account either the elastic or plastic state of the material or a combination of these states [22–24]. This parameter is popular among the engineers responsible for composite structures especially related to rubber. It is very often discussed and used for tire fatigue design procedures [25,26]. As it was noticed by many scientists like Garud or Kujawski the plastic strain amplitude alone has a major influence on the fatigue damage, but it is insufficient in many cases especially if we are analyzing the multiaxial state of material [27,28]. For this reason, the information of both strain and stress should be taken into account. Another important fact that tips the scale on the side of this method is that for low cycle and high cycle fatigue life regions the strain energy density is a constant damage parameter, where it is not necessary to distinguish or choose the appropriate calculation method for low or high cycle fatigue. That is why the energy of the hysteresis loop seems to be a good factor to describe fatigue damage.

There are still unsolved issues like, *inter alia*, the proper use of the strain energy density models directly with the proper domain description. If one would look at a simple case of strain energy density described with the use of the stress descriptors then we can notice that, even though the stress loading time history is Gaussian, the energy time history will be non-Gaussian. For those occupied with frequency domain methods it is well known that non-Gaussianity is a major issue in damage assessment, as described by Benasciutti and Tovo [29].

Therefore, it is required to compensate the information about the non-Gaussianity in another form by either using transformations or, like in the case of this paper, to propose a direct damage intensity model that takes into account the kurtosis and skewness values of the base signal.

By extending the work presented in [30], this paper obtains the main statistical properties of the strain energy parameter for the case of random loading; it also derives the expression for the damage intensity for the narrowband case. The model proposed in the paper is valid, as all frequency defined methods for the linear elastic material state. For verification of the proposed model a comparison between time and frequency domain results has been performed. The calculations are performed for the time domain damage assessment with the use of the rainflow and Palmgren-Miner hypothesis for stress as well as strain energy density for a narrowband and broadband case on the basis of the experimental results of S355JR steel. The frequency domain damage calculations are performed with the Benasciutti-Tovo model and with the use of a new model, that is using spectral moment information of the narrowband power spectral density of stress, which is used in the strain energy density description process. The formulation of the model is explained stepwise. The important fact of the model is that it takes into account the non-Gaussian characteristic of the strain energy signal. The obtained results show good compatibility between the damage models.

## 2. Strain energy density parameter

The strain energy density can be described with the use of a stress-strain relation. It describes the relationship between stress and strain and is a product of their multiplication:

$$W(t) = \frac{1}{2} \cdot \sigma(t) \cdot \varepsilon(t), \tag{1}$$

where: $\sigma(t)$-stress course, $\varepsilon(t)$- strain course.

$W$(t) has generally a positive mean and is non-Gaussian, even if $\sigma$(t) has a zero mean value and is Gaussian. Therefore, it is not possible to separate the cycle parts of the course which are responsible for tension or compression in the loading history. This is the main drawback of this approach. That is why a signed version of this course has been proposed which is referred as the strain energy parameter and allows us the separation of the tensile and compressive parts of the history. The strain energy parameter course can be described in the form of a simple formula with the use of a signum function [22]:

$$W(t) = \frac{1}{2} \cdot \sigma(t) \cdot \varepsilon(t) \cdot \frac{sgn\,[\varepsilon(t)] + sgn\,[\sigma(t)]}{2}, \tag{2}$$

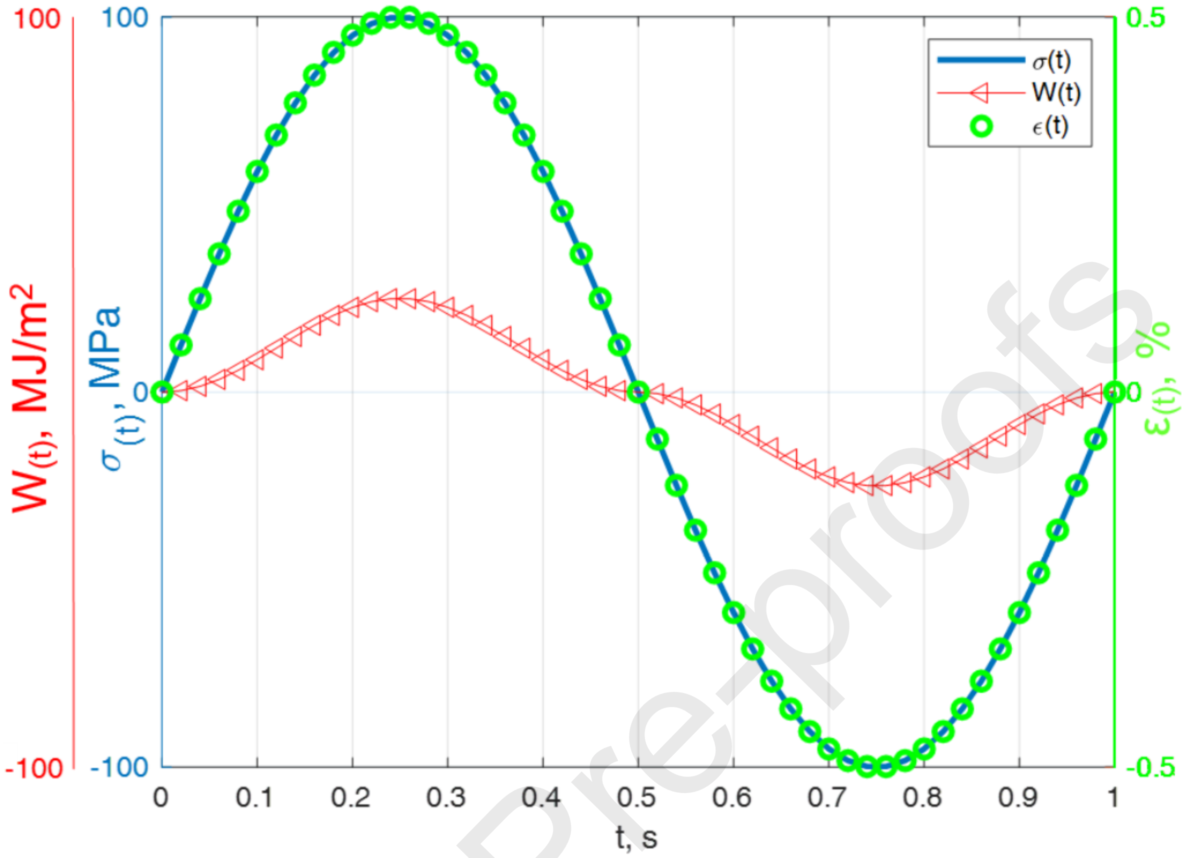



Figure 1. Sample of an energy parameter time history $W$(t) for a sinusoidal stress $\sigma$(t) and strain $\varepsilon$(t) history.

It can also be presented in a equivalent form, without the sgn[-] function but with the absolute value:

$$W(t) = \frac{|\sigma(t)| \cdot \varepsilon(t) + \sigma(t) \cdot |\varepsilon(t)|}{4}. \tag{3}$$

An example of time history of the strain energy density parameter of a sinusoidal load for an elastic material behavior is presented in Fig.1 for a time interval of 1 second.

The energy parameter amplitude can be described as presented in Eq. (4) if the amplitudes are the maximum values of the stress and strain:

$$W_a = 0.5 \cdot \sigma_a \cdot \varepsilon_a, \tag{4}$$

where:

$\sigma_a$- stress amplitude, $\varepsilon_a$- strain amplitude.

For materials in the linear elastic state:

$$W(t) = \frac{|\sigma(t)| \cdot \sigma(t)}{2E} = \frac{|\varepsilon(t)| \cdot \varepsilon(t)}{2} \cdot E, \quad (5)$$

so:

$$W_a = \frac{{\sigma_a}^2}{2E}. \quad (6)$$

Eq. (6) is the same as Eq. (4), because the material is linear elastic. Only for a linear elastic material the energy is the area of the triangle in the stress-strain curve. For the general case of nonlinear elastic material, the energy needs to be computed by an integral.

The reference fatigue strength amplitude at the number of cycles $N_0$ can be calculated with the use of the formula (for the linear elastic state):

$$W_{af} = \frac{\sigma_{af}^2}{2E}, \quad (7)$$

where:

$\sigma_{af}$ - reference fatigue strength amplitude at the number of cycles $N_0$ in fully reversed tension-compression, $E$-Young modulus of the material.

A fatigue curve described in the energy approach can be formed with the use of the formula:

$$A_w = {W_a}^{k'} \cdot N_f, \quad (8)$$

where:

$W_a$- amplitude of energy parameter amplitude, $N_f$ – number of cycles to failure, $k'$- slope of the energy fatigue curve [31]:

$$k' = \frac{k}{2}, \quad (9)$$

In addition, the relationship between the strength constant is:

$$A_w = W_{af}^{k'} \cdot N_{f0} = \frac{A}{\sqrt{2E}} \quad (10)$$

where $A = \sigma_{af}^k N_{f0}$ is the strength constant for the stress S-N curve. $k$- is the slope of the Basquin fatigue curve for stress $A = \sigma_a^k N_f$. The strength constant $A$ is usually defined as $A = \sigma_{af}^k N_{f0}$ in terms of the reference fatigue strength amplitude $\sigma_{af}$ at the reference number of cycles $N_{f0}$ (usually, $N_{f0} = 2 \times 10^6$). A similar definition holds for the energy parameter, for which the strength constant is $A_w = W_{af}^{k'} \cdot N_{f0}$.

The PSD of the energy parameter shown as time course in Fig. 1 is presented in Fig. 2. It can be noticed that the PSD of the energy parameter has peaks at two dominating frequencies $f$=1 and 3 Hz. The bimodal character of the PSD is related to the presence of the stress squared in the definition of $W$(t). The PSD of $W$(t) is not consistent with that of $\sigma$(t), as it shows a peak at double the central frequency of $\sigma$(t) which causes a “doubling” phenomenon.

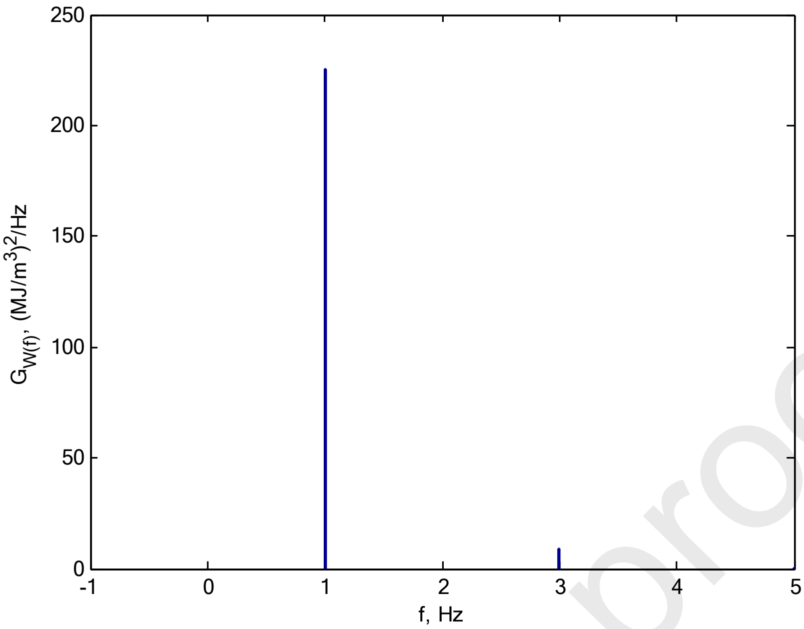



Figure 2. Power spectral density of the energy parameter shown in Fig.1

The PSD of $W$(t) provides additional information which can be used *inter alia* for signal decomposition techniques. While analyzing a simple sinusoidal stress signal and then its strain energy parameter, it is possible to see that there are two dominating frequencies as mentioned before. But a closer look would reveal that for the case of the obtained PSD of $W$(t) there are many more peaks that appear at frequencies $f_{0,n} = f_0 + n \cdot 2 \cdot f_0$ (for n=0, 1, 2, …).

## 3. Statistical properties of the energy parameter for random loading

This section extends the concept of strain energy parameter to the case of a stationary random loading; it obtains the closed-form expressions of some statistical properties of the energy density parameter (e.g. probability distribution of values, probability distribution of local peaks, level crossing spectrum).

Let $\sigma(t)$ represent a stationary Gaussian narrowband random stress with zero mean value. In the frequency domain, the random stress is characterized by the one-sided power spectral density $G_\sigma(\omega)$, which admits the family of spectral moments [32]:

$$m_n = \int_0^\infty \omega^n G_\sigma(\omega) d\omega \qquad (11)$$

The spectral moments, or their combinations, represent important statistical properties of the random stress $\sigma(t)$. For example, the stress variance is the zero-order moment $m_0 = Var[\sigma(t)]$, whereas $\nu_0^+ = \sqrt{m_2/m_0}/2\pi$ is the frequency of upward crossings of the mean value [32].

The instantaneous values of the Gaussian stress $\sigma(t)$ follows the normal probability density function:

$$p_\sigma(\sigma) = \frac{1}{\sqrt{2\pi m_0}} exp\left(-\frac{\sigma^2}{2m_0}\right) \quad (12)$$

It is now convenient to simplify the notation by introducing the scaling parameter $a = 1/2E$. From a mathematical point of view, the strain energy parameter given in Eq. (5) can be thought as a nonlinear time-invariant transformation of the Gaussian random stress:

$$W(t) = g[\sigma(t)] = a|\sigma(t)| \cdot \sigma(t), \quad (13)$$

The transformation $g[-]$ in Eq. (13) is displayed in Fig.3; it clearly shows an anti-symmetric behavior in the positive and negative values.

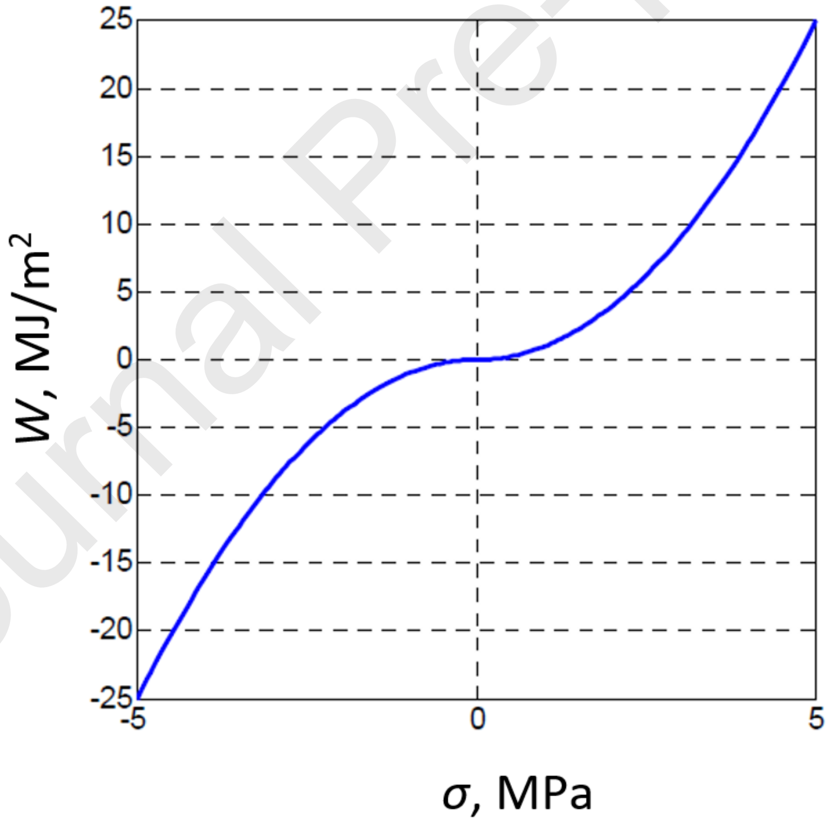



Figure 3. Transformation in Eq. (11) from $\sigma(t)$ (Gaussian) to energy parameter $W(t)$.

It is now straightforward to derive the closed-form expressions of some statistical properties of $W(t)$ by means of the transformation defined previously. The explicit dependence on the time variable *t* will be omitted.

3.1 Probability distribution of the values

From the definition in Eq. (13), the probability density function of $W(t)$ is obtained through the usual approach for transformed random variables [32]:

$$p_W(w) = \frac{p_\sigma\left(g^{-1}(w)\right)}{\left(\frac{dg(\sigma)}{d\sigma}\right)_{\sigma = g^{-1}(w)}} \tag{14}$$

in which $g^{-1}[-]$ denotes the inverse transformation; by inverting Eq. (13), it turns out to be:

$$\sigma = g^{-1}[W] = sgn\,(W)\sqrt{\frac{|W|}{a}} \tag{15}$$

The expression in Eq. (14) depends on the first derivative of the transformation $g[-]$. Owing to the presence of the absolute value in the definition of $g[-]$, the first derivative must be calculated separately by dividing the range of each random in two distinct regions (i.e. positive and negative values). For example, the derivative of the direct function is:

$$W' = \frac{dg(\sigma)}{dw} = \begin{cases} 2a\sigma & \sigma > 0,\ w > 0 \\ -2a\sigma & \sigma < 0,\ w < 0 \end{cases} \tag{16}$$

which can be written in a more compact form as:

$$W' = \frac{dg(\sigma)}{dW} = 2a|\sigma| \tag{17}$$

According to Eq. (14), this derivative must be computed at the value $\sigma = g^{-1}(w)$ that corresponds to a given value *w* of the strain energy parameter. By using the inverse transformation in Eq. (15), the result is:

$$\left(\frac{dg(x)}{dx}\right)_{x = g^{-1}(w)} = \begin{cases} 2a\sqrt{\frac{w}{a}} & \sigma > 0,\ w > 0 \\ 2a\sqrt{-\frac{w}{a}} & \sigma < 0,\ w < 0 \end{cases}$$

which simplifies into:

$$\left(\frac{dg(\sigma)}{d\sigma}\right)_{\sigma = g^{-1}(w)} = 2\sqrt{a|w|} \tag{18}$$

Similarly, Eq. (14) also requires that the probability distribution in Eq. (12) be computed at the values of $\sigma$ calculated through the inverse transformation, as $p_\sigma\left(g^{-1}(w)\right)$. On substituting the square of the inverse transformation $\sigma^2 = |w|/a$ into the Gaussian probability distribution in Eq. (12), along with the expression of the first derivative in Eq. (18), one gets the probability distribution of the strain energy parameter:

$$p_W(w) = \frac{1}{\sqrt{8\pi a m_0 |w|}} exp\left(-\frac{|w|}{2am_0}\right), \tag{19}$$

Here it is also possible to substitute directly the parameter a=1/2E in the formula, so that the equation shows the explicit dependence on the elastic modulus E.
This probability distribution is symmetric, therefore all its central moments $E[W^n]$ are zero for $n$ odd. For example, the expected value and the skewness are both zero, $\mu_W = E[W] = 0$ and $sk[W] = 0$. Instead, for the first two moments of even order, it is possible to demonstrate (see Appendix) that $E[W^2] = 3(am_0)^2$ and $E[W^4] = 105(am_0)^4$, from which it follows that for the strain energy parameter the variance is $Var[W] = 3(am_0)^2$ and the kurtosis $ku[W] = 105/9 \cong 11.7$. The random process $W(t)$ is then non-Gaussian with a 'leptokurtic' character, which characterizes a shift in probability towards the tails of the distribution. Figure 4 compares the Gaussian and non-Gaussian probability distribution in terms of their kurtosis and skewness.

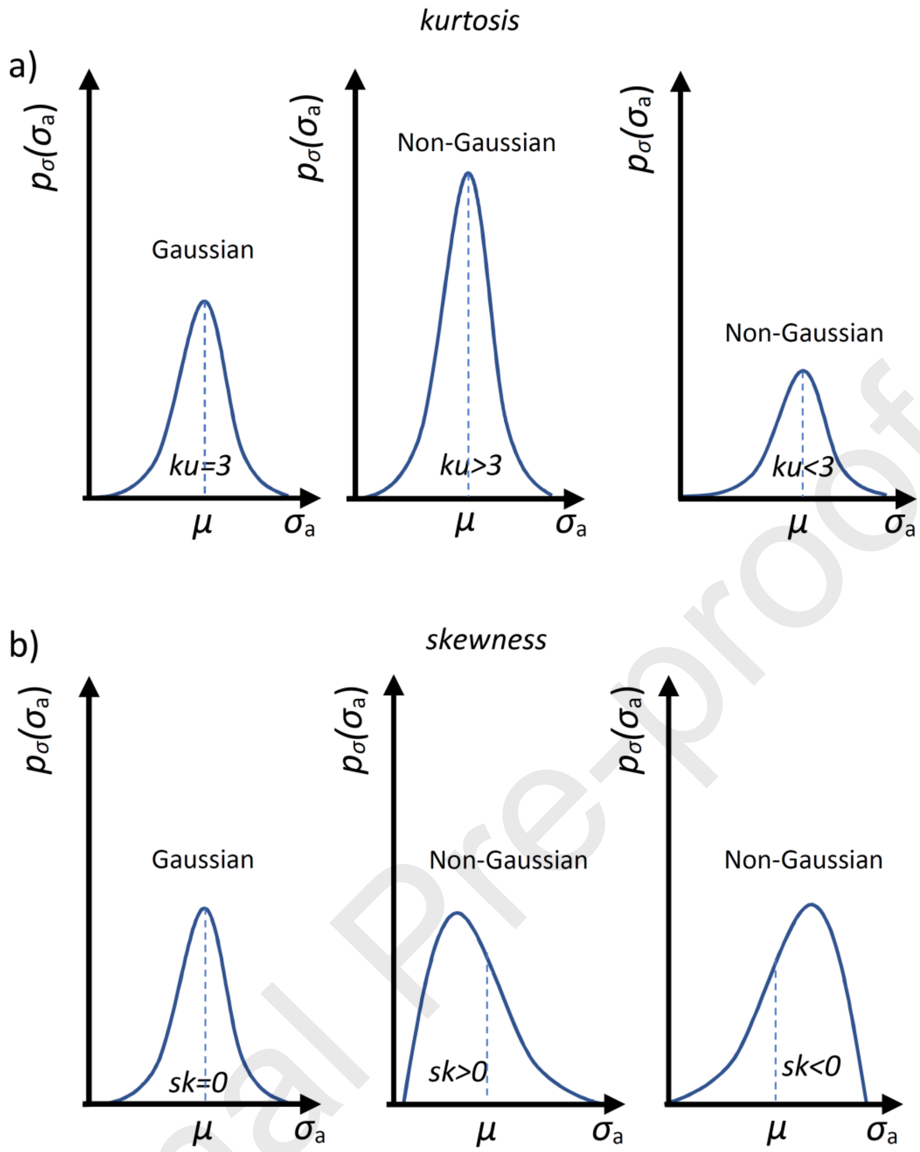



Figure 4. Comparison of Gaussian and non-Gaussian PDF's in terms of: a) kurtosis and b) skewness.

3.2 Probability distribution of peaks

In the hypothesis that the random stress $\sigma(t)$ is Gaussian and narrowband, each local peak $u$ is a random variable that takes on positive values and follows a Rayleigh probability distribution [32]:

$$p_{p,\sigma}(u) = \frac{u}{m_0} exp\left(-\frac{u^2}{2m_0}\right) \qquad u \geq 0 \tag{20}$$

In analogy with the approach presented in the previous section, the probability distribution of a local peak $v$ for the strain energy parameter is defined as:

$$p_{p,W}(v) = \frac{p_{p,\sigma}\left(g^{-1}(v)\right)}{\left(\frac{dg(\sigma)}{d\sigma}\right)_{u = g^{-1}(v)}} \tag{21}$$

The first derivative in the denominator has to be computed at the peak value $u = g^{-1}(v)$ that corresponds to a peak $v$ in the energy parameter. In these terms $u$ and $v$ represent the peaks of $\sigma$(t) and $W$(t). The expressions of the inverse transformation and its first derivative have already been determined in Eq. (15) and (18), respectively.
Upon substitution in Eq. (21), one gets the expression of the probability distribution of a local peak in the strain energy parameter:

$$p_{p,W}(v) = \frac{1}{2am_0} exp\left(-\frac{v}{2am_0}\right), \tag{22}$$

3.3 Level crossing spectrum

The level crossing spectrum is an important property often used in the structural durability analysis of a variable amplitude or random loading; it counts the number of times the loading crosses a fixed level and provides this measure for a range of levels.
For a Gaussian random loading, the level crossing spectrum is given by the celebrated Rice's formula [32]:

$$v_\sigma(x) = v_0^+ \, exp\left(-\frac{x^2}{2m_0}\right), \tag{23}$$

It provides the intensity $v_\sigma(x)$ (i.e. number per time unit) of the upward crossings of the level $x$. Symbol $v_0^+ = \sqrt{m_2/m_0}/2\pi$ is the frequency of upward crossings of the mean value of $\sigma(t)$; it depends on the zero and second order spectral moments of the stress PSD.
Since the strain energy parameter is linked to the stress by means of a monotonic transformation $W(t) = g[\sigma(t)]$, in both random processes the upward crossings of any level occur at the same instants of time. For example, if the random stress $\sigma(t)$ crosses the level $x_1$ at the time instant $t_1$, that is $\sigma(t_1) = x_1$, therefore $W(t)$ will cross the transformed level $y_1 = g[x_1]$ at the same time instant, that is $W(t_1) = y_1$. Accordingly, both processes have in common the same number of crossings of the mean, $v_{0,W}^+ = v_0^+$ .
The transformation then establishes a one-to-one relationship between the number of level crossings in $\sigma(t)$ and the number of crossings of the corresponding transformed levels in $W(t)$. This property allows the level crossing formula to be obtained from Eq. (22) simply as [33]:

$$v_W(y) = v_{0,W}^+ exp\left(-\frac{|y|}{2am_0}\right), \tag{24}$$

which results by simply substituting $x^2 = |y|/a$ in Eq. (21).

Figure 5 displays the level-crossing spectrum for the random stress $\sigma(t)$ and its corresponding energy parameter $W(t)$. Analytical results are compared with simulations.

The figure makes apparent that, except for the levels close to zero, elsewhere the random process $W(t)$ is characterized by a number of crossings that is far larger than that observed for $\sigma(t)$. Figure 6 presents the probability distribution of peaks for the process $\sigma(t)$ and its corresponding energy parameter $W(t)$.

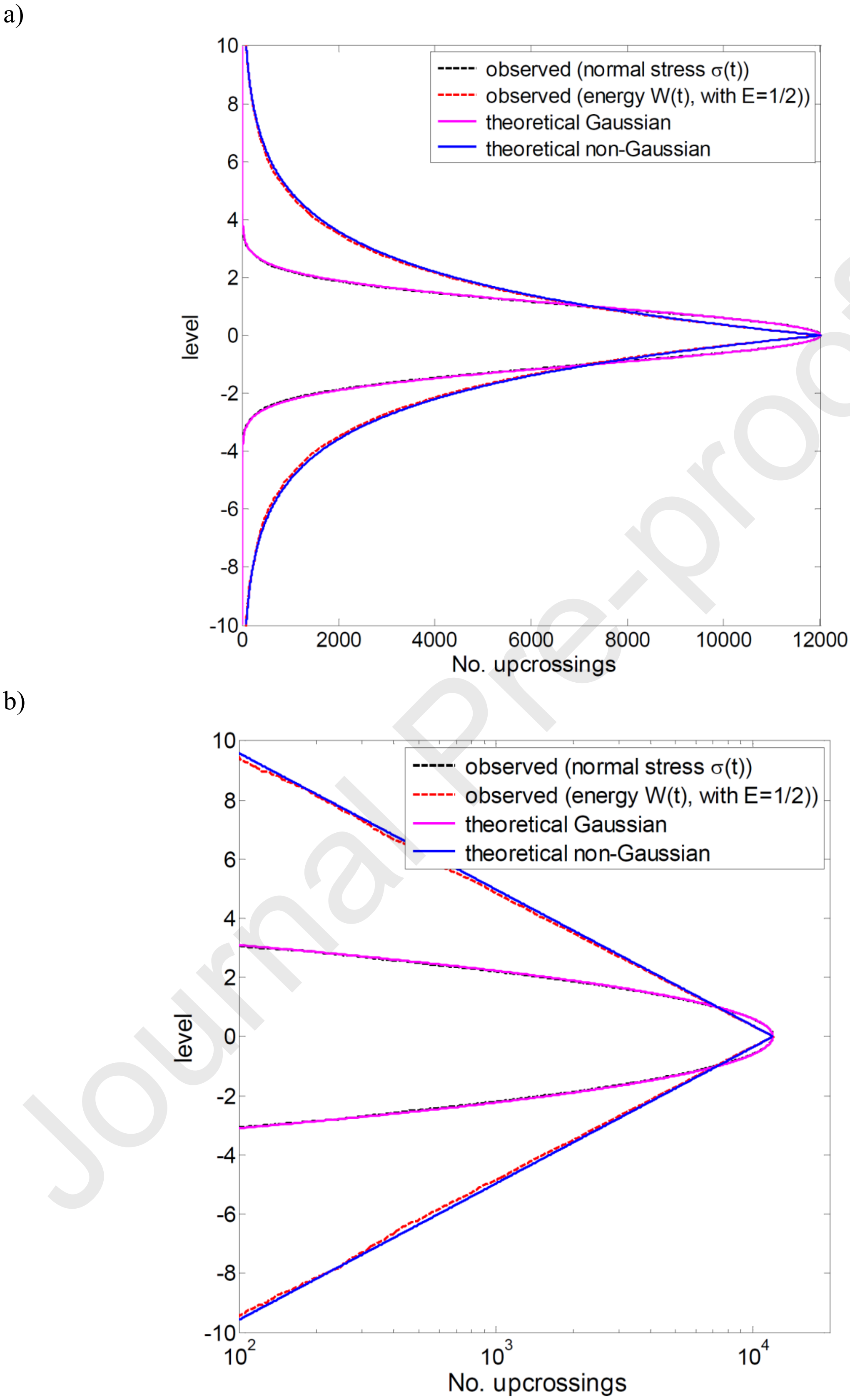



Figure 5. Level-crossing spectrum for stress and strain energy parameter a) linear scale; b) logarithmic scale.

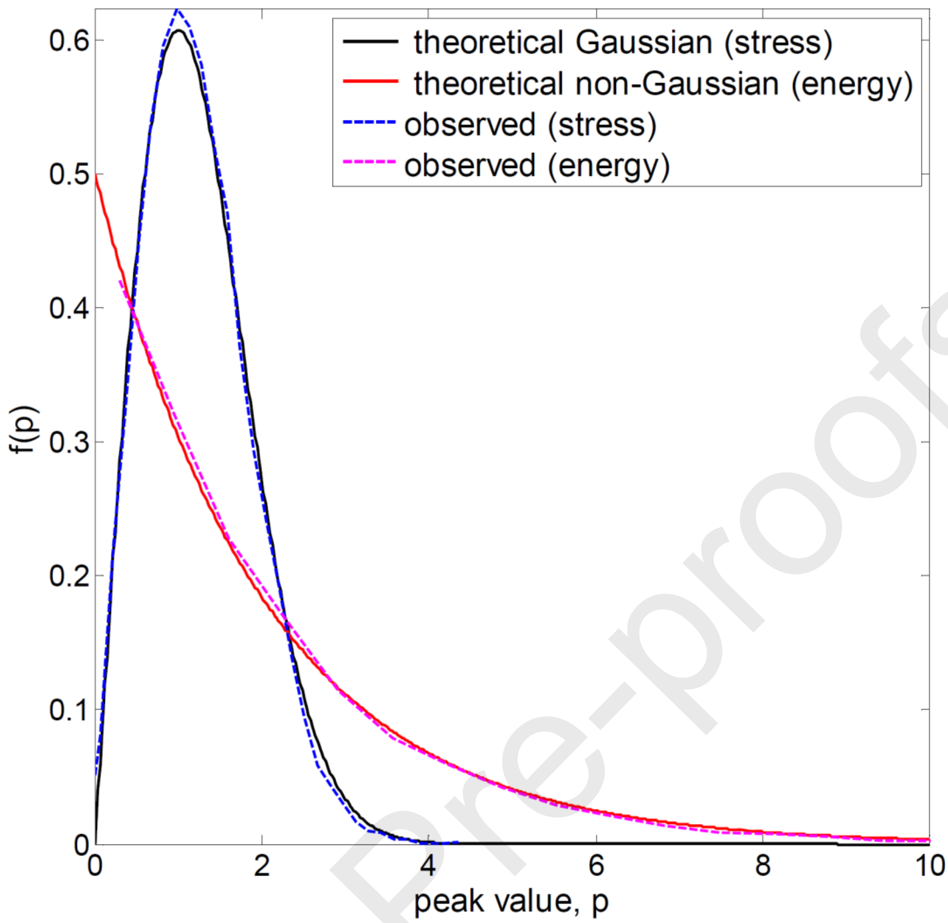



Figure 6. Probability distribution of peaks for the process $\sigma(t)$ and its corresponding energy parameter $W(t)$.

**4. Expected fatigue damage for the strain energy density parameter**

For a narrowband and Gaussian random stress $\sigma(t)$ of time duration $T$, the expected fatigue damage is [32]:

$$E[D_\sigma(T)] = N_c \cdot E[\Delta D_\sigma], \tag{25}$$

where $N_c = T\nu_0^+$ is the expected number of up-crossings of the mean value counted in $T$, and $E[\Delta D_\sigma]$ is the expected damage per cycle, which corresponds to the $k$-th moment $E[\sigma_a^k]$ of the amplitude probability distribution:

$$E[\Delta D_\sigma] = A^{-1}\int_0^\infty \sigma_a^k p_{a,\sigma}(\sigma_a)\, d\sigma_a. \tag{26}$$

Symbol $A = \sigma_a^k N_f$ denotes the strength coefficient of the S-N curve for stress.

Note that $T\nu_0^+$ well approximates the number of cycles because, in a narrow-band process, a cycle is counted after each up-crossing of the mean value. Furthermore, in a narrowband process the amplitude distribution coincides with the probability distribution of peaks in Eq. (20). From this result, and after combining together the previous two expressions, one gets the formula of the expected damage in time interval $T$ for a narrowband stress $\sigma(t)$:

$$E[D_\sigma(T)] = \frac{T\nu_{0,\sigma}^+}{A}\left(\sqrt{2m_0}\right)^k \Gamma\left(1+\frac{k}{2}\right). \tag{27}$$

Symbol $\Gamma(-)$ is the gamma function.

In analogy with Eq. (25), it is possible to write the expected fatigue damage of the strain energy parameter as $E[D_W(T)] = N_{c,W} \cdot E[\Delta D_W]$. In a previous section it has been demonstrated that, for the strain energy parameter, the number of mean value upward crossings is $\nu_{0,W}^+ = \nu_0^+$, which allows writing $N_{c,W} = N_c$. Therefore, only the expression of the expected damage per cycle $E[\Delta D_W]$ must be computed from the corresponding amplitude probability distribution.

From the S-N curve of the strain energy parameter, $A_w = {W_a}^{k'} \cdot N_f$, it follows that the damage of one cycle is $\left(1/N_f\right) = A_w^{-1} \cdot W_a^{k'}$; its expected value is:

$$E[\Delta D_W] = A_w^{-1}\int_0^\infty v^{k'} p_{p,W}(v)\,\mathrm{d}v = A_w^{-1}\int_0^\infty \frac{v^{k'}}{2am_0} exp\left(-\frac{v}{2am_0}\right)\mathrm{d}v \tag{28}$$

in which the probability distribution of peaks $p_{p,W}(v)$ in Eq. (22) is used. With the change of variable $q = \frac{v}{2am_0}$, the previous expression becomes:

$$E[\Delta D_W] = A_w^{-1}(2am_0)^{k'}\int_0^\infty q^{k'} exp(-q)\mathrm{d}q = A_w^{-1}(2am_0)^{k'}\Gamma(1+k') \tag{29}$$

In the expression, the integral corresponds to the definition of the 'gamma function'.
By multiplying the previous expression by the expected number of cycles, $\nu_{0,W}^{+}T$, one obtains the expected damage for the strain energy parameter $W(t)$ in time $T$:

$$E[D_W(T)] = \frac{T\,\nu_{0,W}^{+}}{A_w}(2am_0)^{k'}\Gamma(1+k') \tag{30}$$

in which $a = 1/2E$.

It is straightforward to demonstrate that this damage expression is equivalent to that of the stress process. Upon substituting the strength constant $A_w = W_{af}^{k'} \cdot N_{f0} = A/\sqrt{2E}$, the inverse slope $k' = k/2$, and the crossing intensity $\nu_{0,W}^{+} = \nu_0^{+}$, the result is:

$$E[D_W(T)] = \frac{T\,\nu_0^{+}}{A}\sqrt{2E}\left(\frac{1}{2E}\right)^{\frac{k}{2}}\left(\sqrt{2m_0}\right)^{k}\Gamma\left(1+\frac{k}{2}\right) = E[D_\sigma(T)] \tag{31}$$

which coincides with the formula in Eq. (27).

**5. Results and discussion**

Based on the previous paragraph it was possible to demonstrate that $E[D_\sigma(T)] = E[D_W(T)]$. This result was expected, because it states that the damage remains unchanged whichever quantity (stress or energy) is used to compute it. This statement is true for the High Cycle Fatigue regime as the fatigue slope values will remain constant for this area. This theoretical result is a formal proof of this concept, but it seems it would be interesting to check how the model behaves under broadband loading. This would allow emphasizing the range of applicability of the proposed frequency domain approach in comparison with time domain results.

The proposed solution should be appropriate for the case of a narrowband loading as it has been built on the assumptions of the narrowband model. Nevertheless the aim was to test it for two cases (i.e. narrowband and broadband loading characteristic) in order to estimate fatigue life. Therefore, the presented simulations are based on the frequency characteristic of such loading histories. In order to test the procedure, the experimental fatigue results of S355JR steel under uniaxial random loading conditions have been used [34]. The obtained experimental results serve as reference values for the calculations due to the fact that they have been performed under non-zero mean stress. The generated stress signals which have been used in the previous research have been rescaled to values of amplitudes that are adequate to the maximum values of the reference stress shown in that paper, but with a zero mean. The generated signals, along with the rainflow matrix are presented in Figure 7 and Figure 8 with their basic signal parameters underneath each signal in Table 1 and Table 2. Their PSDs are presented in Figure 9. The material data used in the fatigue estimation process is presented in Table 3. The results of calculations are presented in Table 4. The calculations have been performed for four cases, the first two with the use of the rainflow cycle counting method (stress and energy) and for the frequency domain calculations with the use of the presented method and the widely known Benasciutti-Tovo model [35] which is known to give good results for broad band loading cases. A comparison of experimental and calculation results has been presented in Figure 10 for the narrowband case and in Figure 11 for the broadband case.

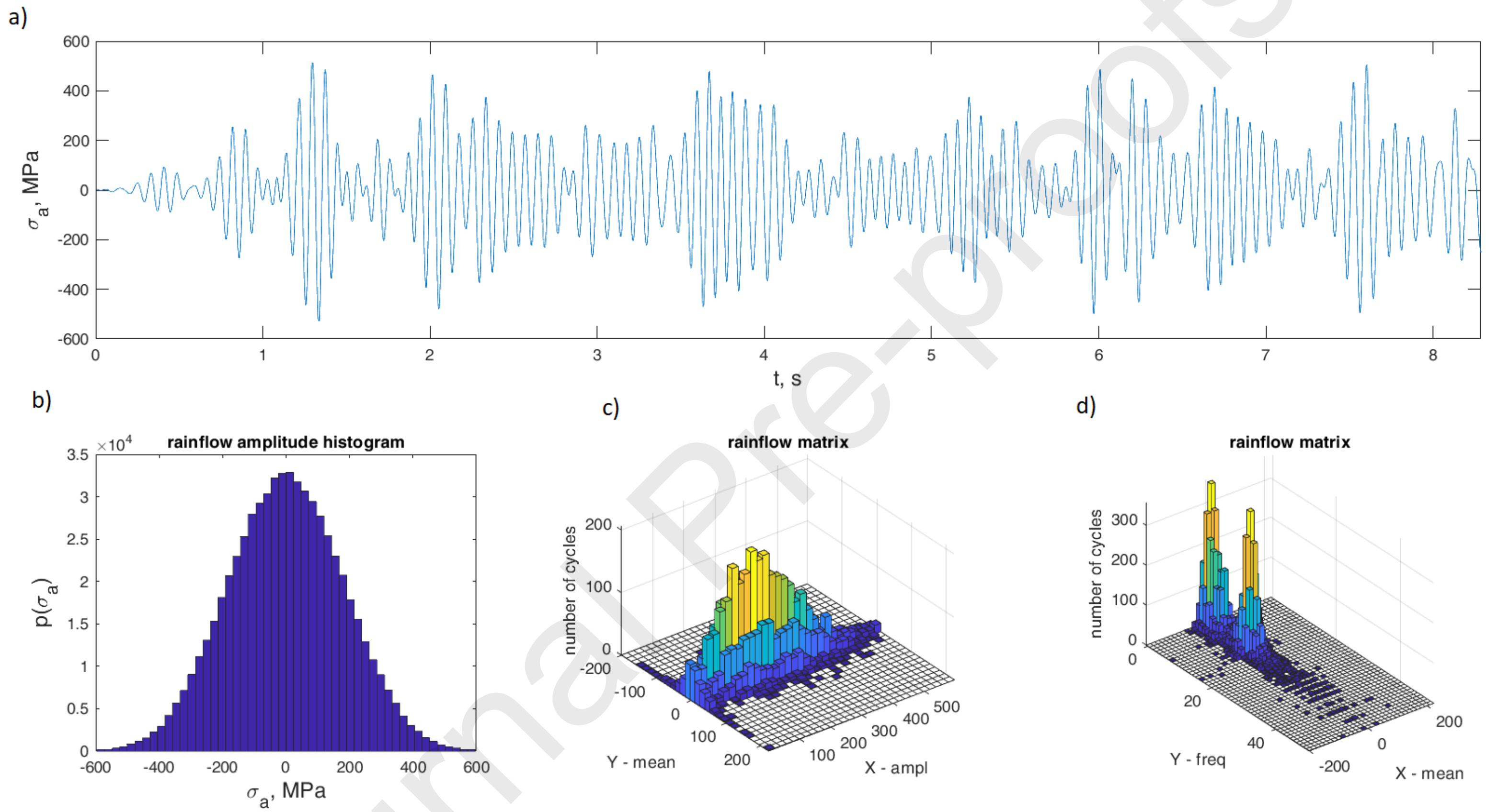



Figure 7. Narrowband generated stress signal: a) time history section; b) rainflow amplitude histogram; c) rainflow matrix amplitudes to mean; d) rainflow matrix frequency to mean.

Table 1. Basic signal parameters of the narrowband loading signal.

| | |
|---|---|
| **Length no. points** | 629999 |
| **No. of rainflow cycles** | 8361 |
| **Value max** | 596 |
| **Value min** | -595 |
| **Mean** | -0.006 |
| **Variance** | 32509 |
| **Standard deviation** | 180 |
| **Kurtosis** | 2.827 |
| **Skewness** | -0.001 |

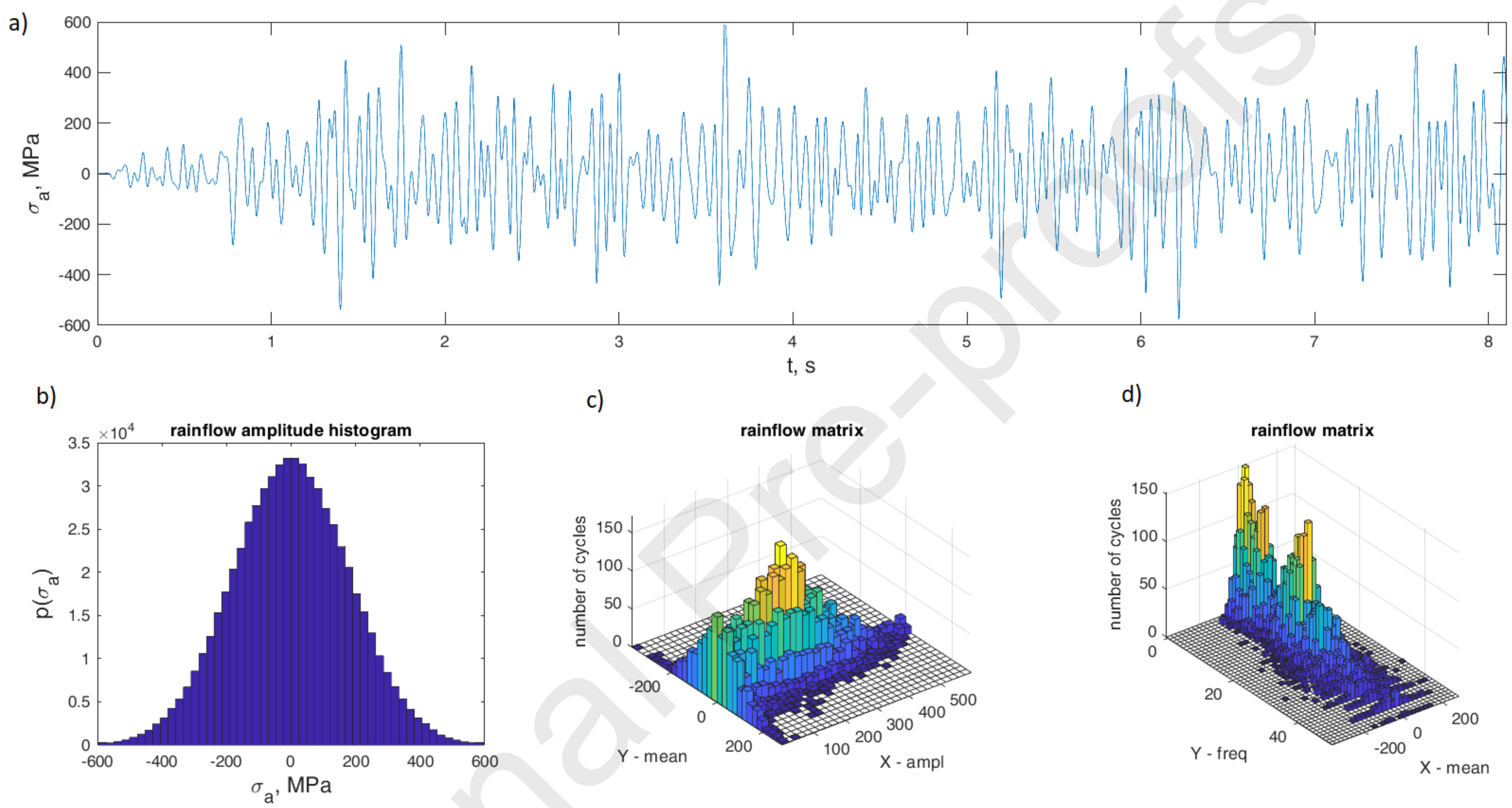
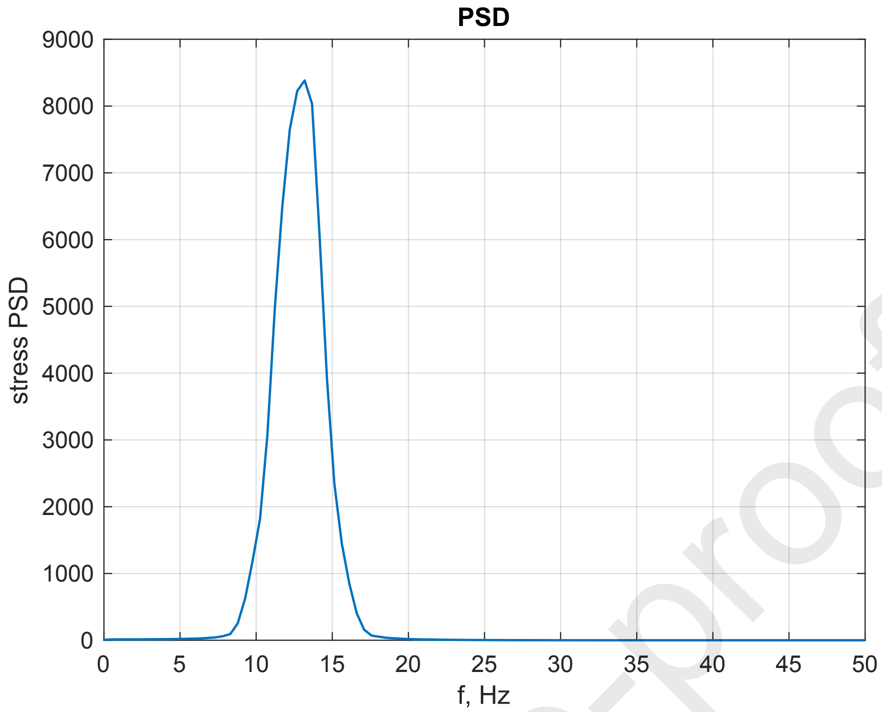



Figure 8. Broadband generated stress signal: a) time history section; b) rainflow amplitude histogram; c) rainflow matrix amplitudes to mean; d) rainflow matrix frequency to mean.

Table 2. Basic signal parameters of the broadband loading signal.

| | |
|---|---|
| **Length no. points** | 629999 |
| **No. of rainflow cycles** | 10982 |
| **Value max** | 596 |
| **Value min** | -595 |
| **Mean** | -0.002812 |
| **Variance** | 32468 |
| **Standard deviation** | 180 |
| **Kurtosis** | 2.9119 |
| **Skewness** | 0.0028 |

a)




b)

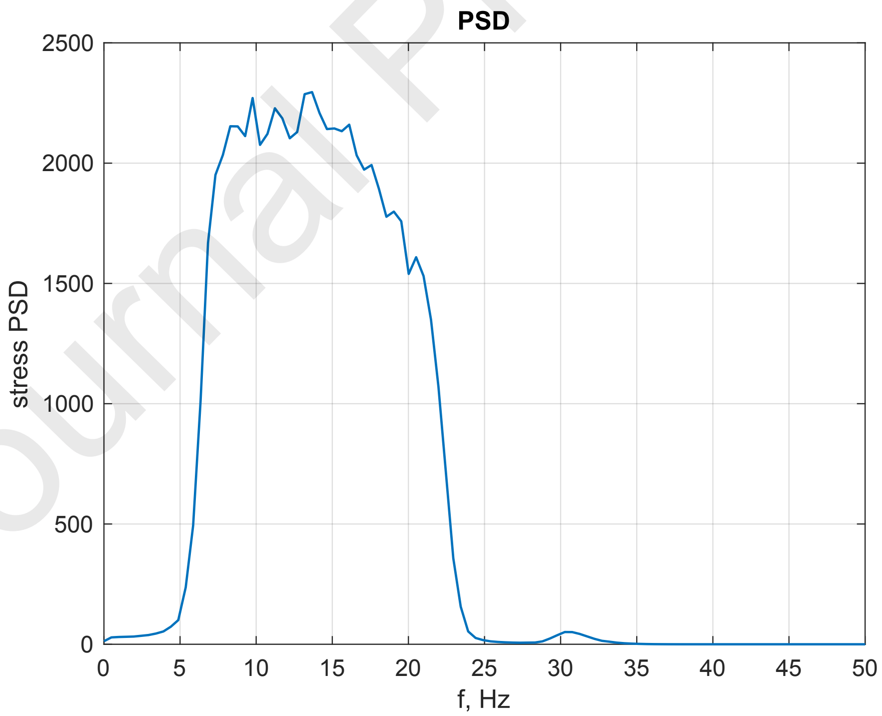



Figure 9. Generated signal Power Spectral Density: a) narrowband, b) broadband.

Table 3. Material data for S355JR steel.

| S355JR steel | | | | | |
|---|---|---|---|---|---|
| $\sigma_{af}$, $MPa$ | $N_0$, cycle | $k$ | $E$, $GPa$ | $R_m$, $MPa$ | $R_e$, $MPa$ |
| 204 | 1240000 | 8.2 | 213 | 630 | 394 |

Table 4. Results of calculations for the time domain calculations with the rainflow cycle counting method for stress $T_{RFstress}$, energy $T_{RFenergy}$ and frequency domain calculations for the Benasciutti-Tovo model $T_{B-T}$ and the calculation results obtained with the narrowband energy approach $T_{Energy}$.

| **Narrowband calculation results** | | | | | |
|---|---|---|---|---|---|
| **$\sigma_{amax}$ MPa** | **$T_{exp}$ s** | **$T_{RFstress}$ s** | **$T_{RFenergy}$ s** | **$T_{Energy}$ s** | **$T_{B-T}$ s** |
| **596** | 8640 | 11684 | 8974 | 10861 | 14085 |
| **596** | 5004 | 11684 | 8974 | 10861 | 14085 |
| **596** | 11340 | 11684 | 8974 | 10861 | 14085 |
| **556** | 29520 | 20654 | 15863 | 19200 | 24898 |
| **556** | 40788 | 20654 | 15863 | 19200 | 24898 |
| **556** | 23616 | 20654 | 15863 | 19200 | 24898 |
| **Broadband calculation results** | | | | | |
| **$\sigma_{amax}$ MPa** | **$T_{exp}$ s** | **$T_{RFstress}$ s** | **$T_{RFenergy}$ s** | **$T_{Energy}$ s** | **$T_{B-T}$ s** |
| **596** | 8964 | 13442 | 10158 | 9513 | 12144 |
| **596** | 11700 | 13442 | 10158 | 9513 | 12144 |
| **596** | 4176 | 13442 | 10158 | 9513 | 12144 |
| **556** | 26316 | 23762 | 17956 | 16816 | 21466 |
| **556** | 30024 | 23762 | 17956 | 16816 | 21466 |
| **556** | 36864 | 23762 | 17956 | 16816 | 21466 |

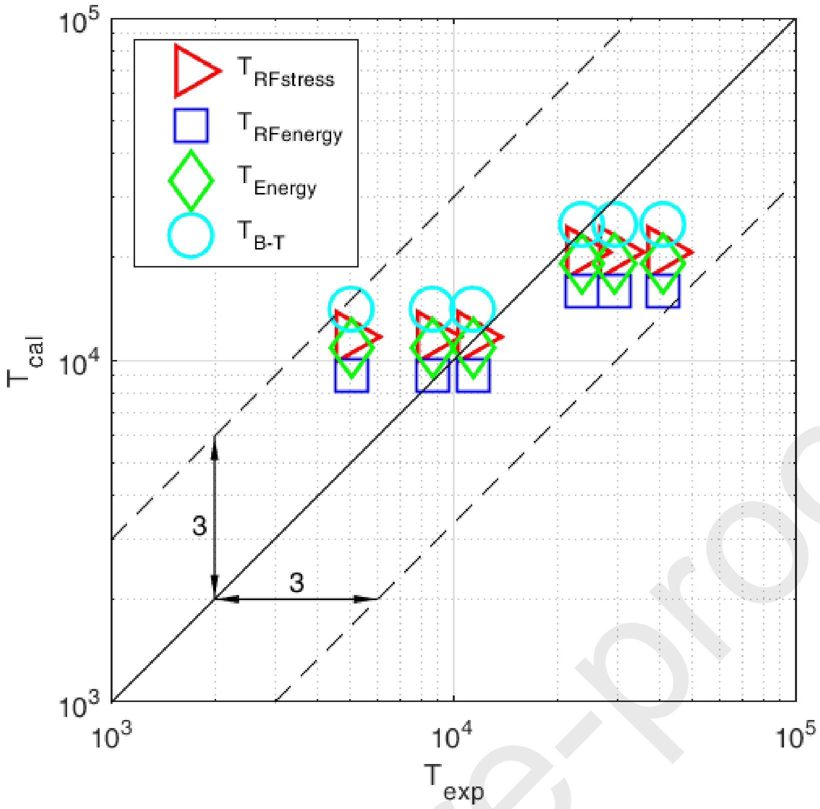



Figure 10. Comparison of experimental and calculated results for the narrowband case

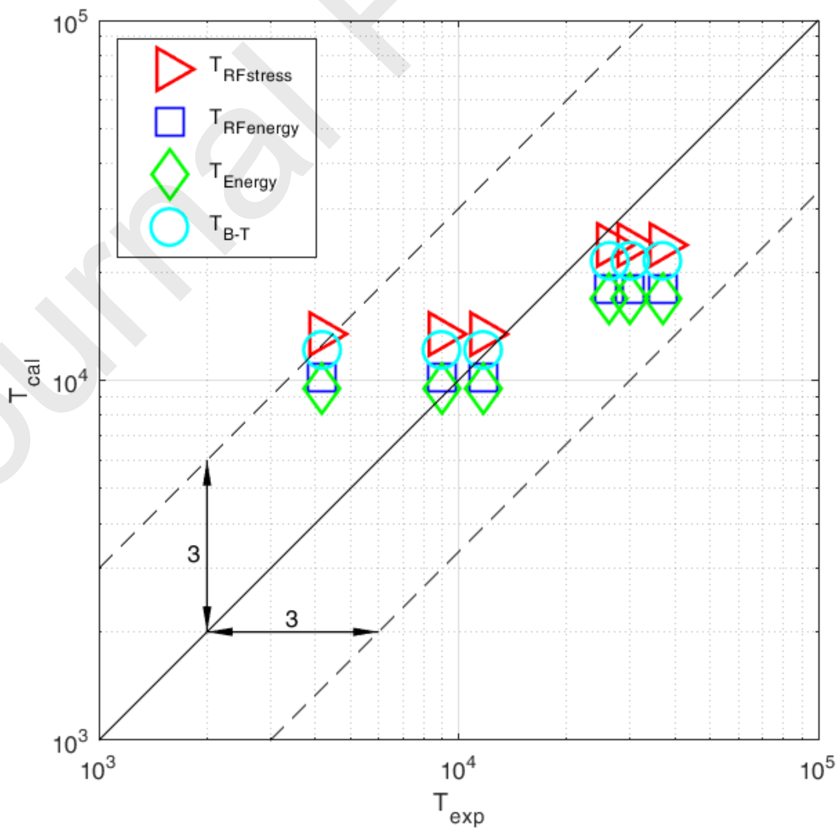



Figure 11. Comparison of experimental and calculated results for the broadband case.

Almost all results are within the scatter band area of 3, which is still acceptable for fatigue life estimation algorithms. It is favorable to obtain results beneath the center line, which means that the chosen calculation method does not overestimate the fatigue life in comparison to experimental results. In order to calculate individual scatter error for each used method the authors use the root mean square formula [17]:

$$E_{RMS} = \sqrt{\frac{\sum_{i=1}^{n} log^2 \frac{T_{exp\,i}}{T_{cal\,i}}}{n}}, \tag{32}$$

where $T_{\text{exp}}$ and $T_{\text{cal}}$ are the experimental and calculated fatigue life.

Finally, the mean deviation from the expected time to failure value is obtained by the use of the formula:

$$T_{RMS} = 10^{E_{RMS}}. \tag{33}$$

The results of these calculations are presented in Table 5. It can be noticed that as predicted the proposed energy approach, which is based on the narrowband model gave smaller scatter error values for this type of signal and had the highest value in comparison to other methods for the broadband signal.

Table 5. Scatter error for experimental and calculation comparison results

| | $T_{RFstress}$ | $T_{RFenergy}$ | $T_{Energy}$ | $T_{B\text{-}T}$ |
|---|---|---|---|---|
| Narrowband | 1.6265 | 1.7381 | 1.6350 | 1.6847 |
| Broadband | 1.7314 | 1.7171 | 1.7494 | 1.6974 |

## 6. Conclusions and observations

The paper presented a new fatigue damage calculation model for the strain energy density parameter $W(t)$ in which the fatigue damage can be estimated from the power spectral density in the frequency domain. The paper also derived the closed-form expressions of the main statistical properties of the random process $W(t)$ (i.e. the probability distribution of values, of peaks, and the level crossing spectrum). Some general conclusions and observations can be formulated as follows:

- the strain energy parameter $W(t)$ can be viewed as a nonlinear time-invariant transformation of the stress
- the probability density function of $W(t)$is symmetric and non-Gaussian, with a zero skewness and a kurtosis value of $\cong 11.7$ ;
- except for the levels close to zero, the level crossing spectrum of $W(t)$ is characterized by a far larger number of crossings, compared to that of the stress $\sigma(t)$;
- This method allows calculating the expected fatigue damage directly from the stress power spectral density (PSD), which speeds up the calculation process;

- As it was expected the proposed model gave more accurate results for the case of narrowband loading history with an individual scatter band error of 1.6350.
- The comparison of experimental data with calculation results show that the proposed method does not overestimate the fatigue lifetime and is within the scatter area of 3.
- It is expected to perform additional experimental results with the use of different signal characteristics which would allow expanding the model range of applicability.

**Appendix**

As the probability density function $p_W(w)$ is symmetric about the origin, all its central moments of odd order $n$ turn out to be zero, that is $E[W^n] = 0$ for $n$ odd. As a result, it is zero both the expected value $\mu_W = E[W] = 0$ and the third order moment, $E[W^3] = 0$, which implies that the skewness is zero, too.

Instead, the central moments of even order $n$ are defined as:

$$E[W^n] = \frac{2}{\sqrt{8\pi a m_0}} \int_0^{\infty} \frac{w^n}{w} exp\left(-\frac{w}{2am_0}\right) \mathrm{d}w \qquad \text{(for } n \text{ even)}$$

where the symmetry of $p_W(w)$ has been exploited to compute the integral as twice the integral from zero to infinity, and also to substitute $|w| = w$.

After the change of variable $q = \frac{w}{2am_0}$, the previous integral simplifies as:

$$E[W^n] = \frac{(2am_0)^n}{\sqrt{\pi}} \int_0^{\infty} q^n e^{-q} \mathrm{d}q \qquad \text{( for } n \text{ even)}$$

In the expression, the integral corresponds to the definition of the 'gamma function' of value $\Gamma\left(n + \frac{1}{2}\right)$; particular values of interest here are $\Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}$ (for $n = 2$) and $\Gamma\left(\frac{9}{2}\right) = \frac{105}{16}\sqrt{\pi}$ (for $n = 4$).

For these two $n$ values, the central moments turn out to be:

$$E[W^2] = 3(am_0)^2 \qquad E[W^4] = 105(am_0)^4$$

Since the expected value is zero, $\mu_W = 0$, the variance of $W(t)$ coincides with the second order moment, i.e. $Var[W] = E[W^2] - (E[W])^2 = E[W^2]$. Instead, the kurtosis is defined as [32]:

$$ku[W] = \frac{E[W^4]}{(Var[W])^2} = \frac{105}{9} \cong 11.7$$

In summary, the above results show that the strain energy parameter $W(t)$ is a non-Gaussian random process, with zero mean value and variance equal to $Var[W] = 3(am_0)^2$, and with zero skewness and kurtosis 11.7 (leptokurtic behavior).

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