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The Bénard Problem for Slightly Compressible Materials: Existence and Linear Instability

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Abstract. We study a generalization of the Oberbeck–Boussinesq system, which consists in a buoyancy term where the density depends also on the pressure. A new pressure equation is introduced, which is deduced from the divergence-free condition on the velocity; such an equation cannot be decoupled from the system and is studied under Robin's boundary conditions. Then, the existence of regular periodic solutions is proved for the full system. In Bénard's problem, the two-dimensional linear instability of the solution depends on a dimensionless parameter that is proportional to the compressibility factor: the related critical Rayleigh number decreases as it increases.

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1. Introduction

The mathematical modeling of some fluids as *incompressible* has been attracting the interest of researchers since a long time. The notion of incompressibility is just an idealization of several materials which can be deformed with almost no volume change during the isothermal processes. Nevertheless, in comparison with the modeling of the *compressible* fluids, the incompressible material theory has the advantage of being not only "simpler" but even more consistent with the macroscopic model of the continuous media. The incompressibility condition $\nabla \cdot \mathbf{v} = 0$, where $\mathbf{v}$ denotes the velocity, implies that the equations are less singular than those of the compressible flows [4].

In fact, a sort of in-between approach is put into practice, which consists in studying particular observable flows in which the motion of compressible fluids is *isochoric*: so that again $\nabla \cdot \mathbf{v} = 0$. When this condition is used to model the non-isothermal flows, it is suitable to choose the pressure p (instead of the density ρ) and the temperature T as thermodynamical independent variables [9,10]; the other quantities, such as the specific volume $V = 1/\rho$ and the internal energy ε, are identified by constitutive equations of the form

$$V \equiv V(p,T), \quad \varepsilon \equiv \varepsilon(p,T). \tag{1.1}$$

Two dimensionless parameters are important for the description of the fluid motion: the *thermal expansion coefficient* α and the *compressibility factor* β, which are defined by

$$\alpha = \frac{V_T}{V}, \quad \beta = -\frac{V_p}{V}. \tag{1.2}$$

The subscripts T, p denote partial derivatives with respect to the variables T, p, respectively.

The simplest model of the non-isothermal isochoric flow follows from the experimental evidence that for some fluids which are usually handled as incompressible—such as water—the volume changes a little with the temperature; while in practice, it remains unchanged by varying the pressure. Therefore, one can simplify (1.1) by assuming

$$V \equiv V(T). \tag{1.3}$$

For these fluids, the natural convection is modeled by the well-known *Oberbeck–Boussinesq approximation* [1,13,14], shortened as O–B in the following. The O–B approximation, in addition to the assumptions of isochoric motion $\nabla \cdot \mathbf{v} = 0$ and negligible self-dissipation $\mathbb{D} : \mathbb{D} \approx 0$ (where as usual $\mathbb{D}$ is the symmetric part of $\nabla \mathbf{v}$) postulates that in the balance equations, we have

$$\rho = \rho_0 \left(1 - \alpha_0 (T - T_0)\right) \text{ in the weight force,} \quad \rho = \rho_0 \text{ in all the other terms.} \tag{1.4}$$

Since condition (1.3) proves to be too strong in the light of the so-called Müller's paradox [9,10], it was recently proposed to assume at least $\alpha_0 = \alpha(T_0, p_0)$, see [11].

As we mentioned above, such an approximation is suited for fluids which can be assumed *thermally compressible but mechanically incompressible*; a theoretical justification can be found in [2,11]. More precisely, the papers [11] aim at identifying the most suitable range, as far as the dimensionless parameters are concerned, for the O–B approximation to hold. In particular, these papers show how the O–B approximation can be derived from the balance equations of a slightly compressible fluid with a buoyancy term in the momentum balance which does not depend on the pressure.

On the other hand, in many applications [3,7], the O–B approximation is also used for modeling air and other gases, see, for instance, [19], and though there is no theoretical explanation for this, it is reasonable to expect more realistic numerical outcomes by generalizing the model.

The O–B approximation has some drawbacks, as we discuss now. First, if all the thermodynamical functions are assumed to be independent of p, then the Gibbs equation

$$T\mathrm{d}S = \mathrm{d}\epsilon + p\mathrm{d}V \tag{1.5}$$

holds if and only if V is constant. As a consequence, one is led to assume again (1.3) but allowing the internal energy to depend also on the pressure, i.e., $\varepsilon \equiv \varepsilon(p, T)$ [9].

Second, assumption (1.3) implies instability in the wave propagation [12]; for thermodynamical stability, the compressibility factor must satisfy the inequality [10]

$$\beta > \beta_{\mathrm{cr}}, \quad \beta_{\mathrm{cr}} := \frac{\alpha^2 T V}{C_p} > 0, \tag{1.6}$$

where $C_p > 0$ denotes the specific heat at constant pressure.

To circumvent these difficulties, one can replace (1.3) with the *full* linear expansion of ρ at a reference state (T_0, p_0) [10], namely,

$$\rho = \rho_0 \left\{1 - \alpha_0 \left(T - T_0\right) + \beta_0 \left(p - p_0\right)\right\}. \tag{1.7}$$

Here, $\beta_0 = \beta(T_0, p_0)$. Expression (1.7) was used in [15] as constitutive equation in the weight force and tested on the simplest physical setting for the convection: Bénard's problem. In the 2D Bénard convection, a fluid is moving between two horizontal planes which are kept at two constant different temperatures. In the *classical* Bénard problem, i.e., under (1.4), there is a parabolic profile of the pressure which corresponds to the rest state: it does not fit with many experimental results and an alternative profile is given in [15] by exploiting (1.7). Moreover, the rest state with stratified linear temperature analytically shows a saddle structure—as a bifurcation—for all β_0, both under (1.4) and under (1.7). To be more precise, for any value of the characteristic non-dimensional parameters, there exists a set of initial conditions whose corresponding solutions decay exponentially to the basic solution. Further, still in [15], the linear instability of the thermal conductive solution is studied by analyzing the exponential time decay of the perturbations. Such a decay coincides with the classical one [5,6] in the limit $\beta_0 \to 0$. The investigation was performed by neglecting from the very beginning the terms of order β_0^3 (this simplifies the solution of the pressure equation).

If $\beta_0 > 0$, the convective motions still arise if the Rayleigh number is sufficiently large as a function of β_0. Again, as in the classic case, the first non-decreasing normal mode is a stationary state. Moreover, for small β_0, it is proved that the critical Rayleigh number—which is the one corresponding to zero rate of decay—decreases as β_0 increases. Therefore, the system of equations inferred by (1.7) admits a basic solution that is less stable than the one following (1.4).

In the present paper, we study the linear system of equations deduced by (1.7) to confirm the results in [15] by providing existence theorems. More precisely, in Sect. 2, we introduce the equations and the Boussinesq approximation we deal with in the following. We provide there a new equation for the pressure, which does not decouple from the full system, and we justify Robin's boundary conditions for it by following [17]. In Sect. 3, we find the optimal smallness conditions on β_0 for the existence and uniqueness of solutions to a reduced problem for the pressure. In Sect. 4, we first study the full time-dependent Stokes-like system and provide a result of existence, uniqueness and regularity of solutions using classical tools [16]. At last, we prove a stability result, which was firstly achieved only by numerical methods [15].

2. The Modified Boussinesq Approximation

If we assume that the body forces are only due to the gravity and no heat is supplied, then the balance equations of mass, momentum and energy are

$$\begin{cases} \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0, \\ \frac{\partial \rho\mathbf{v}}{\partial t} + \nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v} - \mathbb{T}) = \rho\mathbf{g}, \\ \frac{\partial\left(\rho\varepsilon + \frac{1}{2}\rho v^2\right)}{\partial t} + \nabla\cdot\left(\left(\rho\varepsilon + \frac{1}{2}\rho v^2\right)\mathbf{v} - \mathbb{T}(\mathbf{v}) + \mathbf{q}\right) = \rho\mathbf{g}\cdot\mathbf{v}, \end{cases} \tag{2.1}$$

where $\mathbf{v}$, $\mathbb{T}$ and $\mathbf{q}$ are the velocity, the stress tensor and the heat flux, respectively; t represents the time and $\mathbf{x}$ is the space variable. In the following, we only focus on smooth solutions of (2.6). In this case, system (2.1) can be rewritten as

$$\begin{cases} \frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\nabla\cdot\mathbf{v} = 0, \\ \rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \nabla\cdot\mathbb{T} + \rho\mathbf{g}, \\ \rho\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \mathbb{D}:\mathbb{T} - \nabla\cdot\mathbf{q}, \end{cases} \tag{2.2}$$

where $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla$ denotes the material derivative. Taking into account that $\mathbb{T} = -p\mathbb{I} + \mathbb{S}$, where $\mathbb{S}$ is the viscous stress tensor, we assume as constitutive equations the classical Fourier law $\mathbf{q} = -k\nabla T$, where k is the coefficient of thermal conductivity, and the definition of Navier–Stokes fluid

$$\mathbb{S} = \lambda\nabla\cdot\mathbf{v}\ \mathbb{I} + 2\mu\,\mathbb{D} = \nu\nabla\cdot\mathbf{v}\ \mathbb{I} + 2\mu\,\mathbb{D}^D.$$

Here $\nu = \lambda + 2\mu/3$, where λ and μ are the bulk and shear viscosity coefficients, respectively; the symbol $\mathbb{D}^D = \mathbb{D} - \frac{1}{3}\nabla\cdot v$ is the deviatoric part of $\mathbb{D}$. From the integrability conditions due to the Gibbs equation (1.5) we have

$$\left(\frac{\partial\varepsilon}{\partial\rho}\right)_T = \frac{1}{\rho^2}\left\{p - T\left(\frac{\partial p}{\partial T}\right)_\rho\right\},$$

where, with a slight abuse of notation, on the left-hand side, we mean the partial derivative of ε with respect to p at T fixed, and so on. Moreover, by taking into account the conservation of mass $(2.2)_1$, we find the identity

$$\rho\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \rho\left(\frac{\partial\varepsilon}{\partial T}\right)_\rho\frac{\mathrm{d}T}{\mathrm{d}t} - \left\{p - T\left(\frac{\partial p}{\partial T}\right)_\rho\right\}\nabla\cdot\mathbf{v}.$$

By denoting $C_V = (\partial\varepsilon/\partial T)_\rho$ the specific heat at constant volume, system (2.2) becomes

$$\begin{cases} \frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\nabla\cdot\mathbf{v} = 0, \\ \rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\nabla p + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{v}) + \mu\Delta\mathbf{v} + \rho\mathbf{g}, \\ \rho C_V\frac{\mathrm{d}T}{\mathrm{d}t} = -p\nabla\cdot\mathbf{v} + \lambda(\nabla\cdot\mathbf{v})^2 + 2\mu\mathbb{D}:\mathbb{D} + k\Delta T + \left\{p - T\left(\frac{\partial p}{\partial T}\right)_\rho\right\}\nabla\cdot\mathbf{v}. \end{cases} \tag{2.3}$$

We assume both

$$\nabla \cdot \mathbf{v} = 0 \quad \text{and} \quad \mathbb{D} : \mathbb{D} \approx 0, \tag{2.4}$$

as in the classical O–B approximation, but we replace (1.4) with

$$\begin{aligned} \rho &= \rho_0 \left(1 - \alpha(T - T_0) + \beta(p - p_0)\right) \text{ in the weight force,} \\ \rho &= \rho_0 \text{ in the other terms.} \end{aligned} \tag{2.5}$$

For simplicity of notations, with respect to (1.7), we wrote α and β for the constants α_0, β_0, respectively; the same notation is used in the following. As a consequence, system (2.3) can be written as

$$\begin{cases} \nabla \cdot \mathbf{v} = 0, \\ \rho_0 \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right) = -\nabla p + \mu \Delta \mathbf{v} + \rho_0 \left(1 - \alpha(T - T_0) + \beta(p - p_0)\right) \mathbf{g}, \\ \rho_0 C_V \left(\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T\right) = k \Delta T. \end{cases} \tag{2.6}$$

Now, we introduce the domain where we consider system (2.6). We denote the spatial variables by (x, y, z) and consider the region between the horizontal planes $z = 0$ and $z = h > 0$. We assume that the planes are thermostatically heated to

$$T_{|_{z=0}} = T_d > T_u = T_{|_{z=h}},$$

where T_d and T_u are constant, and denote $\delta T := T_d - T_u > 0$. A mathematical 2D setting for the study of the Bénard problem is physically meaningful, since the convection rolls of Bénard's convection are experimentally observed in the layer $0 \leq z \leq h$ and their invariant axes are normal to the random direction of the horizontal perturbation generating them [8]. We can assume that the direction of such horizontal perturbation coincides with the x axis and then look for solutions that are *periodic in* x with period $l > 0$; for simplicity, we assume

$$l = h. \tag{2.7}$$

Then, the dependence on y is dropped.

By taking as reference state (T_d, p_d), for some $p_d > 0$, the basic steady solution for (2.6) corresponding to $\mathbf{v} = 0$ is [15]

$$\bar{T}(z) = T_d - \frac{\delta T}{h} z, \tag{2.8}$$

$$\bar{p}(z) = p_d + \frac{1}{\beta^2} \frac{\alpha \delta T}{\rho_0 g h} \left(1 - e^{-\rho_0 g \beta z}\right) - \frac{1}{\beta} \left(1 - e^{-\rho_0 g \beta z} + \frac{\alpha \delta T}{h} z\right). \tag{2.9}$$

If we define the fields

$$\tau(x, z) = T(x, z) - \bar{T}(z), \tag{2.10}$$

$$P(x, z) = p(x, z) - \bar{p}(z), \tag{2.11}$$

which describe the perturbation with respect to the basic steady solution, then system (2.6) is fulfilled by $(\tau, P, \mathbf{v}) = 0$. At last, we introduce the dimensionless variables

$$t' = \frac{k}{h^2} t, \quad x' = \frac{x}{h}, \quad z' = \frac{z}{h}, \quad T' = \frac{T}{\delta T}, \quad P' = \frac{P\, h^2 \rho_0}{k \mu}. \tag{2.12}$$

Then, by exploiting (2.5) and (2.10), (2.11), (2.12), we end up with the system

$$\begin{cases} \nabla\cdot\mathbf{v}=0,\\ \frac{1}{\mathrm{Pr}}\left(\frac{\partial\mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}\right)=-\nabla P-\hat\beta P\mathbf{k}+\Delta\mathbf{v}+\mathrm{Ra}\,\tau\mathbf{k},\\ \frac{\partial\tau}{\partial t}+\mathbf{v}\cdot\nabla\tau=\Delta\tau+\mathbf{v}\cdot\mathbf{k}, \end{cases}\tag{2.13}$$

where $\mathbf{k}=(0,0,1)$. For simplicity, we dropped all $'$s; we keep this notation from here on. Prandtl and Rayleigh dimensionless parameters and the independent parameter $\hat\beta$ are defined by

$$\mathrm{Pr}:=\frac{\mu}{\rho_0 k},\quad \mathrm{Ra}:=\frac{\alpha\rho_0 h^3 g\delta T}{\mu k},\quad \hat\beta:=\rho_0 g\beta h.$$

Because of the domain considered above and the periodicity in x, system (2.13) can be studied either for (x,z) in $\Omega:=\mathbb{R}\times(0,1)$, under periodicity conditions in x, or in $\Omega_0:=(0,1)\times(0,1)$. Moreover, $t\in(0,\infty)$. We also denoted $\Delta=\partial_x^2+\partial_z^2$.

Before dealing with P, we consider the boundary conditions on $z=0$ and $z=1$ for both $\mathbf{v}$ and τ. To compare our results with the existing exact results on linear instability [8], we impose to $\mathbf{v}$ the same boundary conditions of [8], namely $\mathbf{v}\cdot\mathbf{k}=0$ and $(\mathbf{v}\cdot\mathbf{i})_z=0$, where $\mathbf{i}=(1,0,0)$; this means

$$v^z(x,0,t)=v^z(x,1,t)=0\quad\text{and}\quad v^x_z(x,0,t)=v^x_z(x,1,t)=0.\tag{2.14}$$

Here, v^x is the first component of $\mathbf{v}$ and so on. Concerning the temperature, we require

$$\tau(x,0,t)=\tau(x,1,t)=0.\tag{2.15}$$

Remark 2.1. *The null function $\mathbf{v}=0$, $\tau=P=0$, satisfies system (2.13) and the boundary conditions (2.14)–(2.15). Indeed, the space $\mathbb{S}_0$ of smooth solutions of (2.13) with $\mathbf{v}_x=0,\tau_x=0$ and $P_x=0$ is shown to be a stable subspace in [15]. In this case, we have $\mathbf{v}\equiv 0$ and solutions can be explicitly written. By Galilean invariance, the same holds for $\mathbf{v}=(C,0)$, $\tau=P=0$, for every costant C.*

For general initial conditions, system (2.13) cannot be solved separately for τ and $\mathbf{v}$ by simply writing the second equation in a weak form with divergence-free test functions because of the term $\hat\beta P\mathbf{k}$. Although the projection of $(2.13)_2$ still depends on P, an equation for P is obtained by taking the divergence of $(2.13)_2$; we have

$$\Delta P+\hat\beta P_z=-\frac{1}{\mathrm{Pr}}\nabla\cdot(\mathbf{v}\cdot\nabla\mathbf{v})+\mathrm{Ra}\,\tau_z.\tag{2.16}$$

By assumption (2.4), we deduce $\nabla\cdot(\mathbf{v}\cdot\nabla\mathbf{v})=(\nabla\mathbf{v}):(\nabla\mathbf{v})^T$. As a consequence, the first summand in the left-hand side of (2.16) cannot be neglected, unless linearizing the equations.

By following the arguments in [17] and, in particular, by inserting the pressure-depending buoyancy force at the right-hand side of equation $(2.13)_2$ in equation (40) of that paper, then the *natural* boundary condition for Eq. (2.16) is Robin's condition

$$P_z+\hat\beta P=0\quad\text{on } z=0 \text{ and } z=1.\tag{2.17}$$

Robin's condition is natural for the pressure in the present system as Neumann's condition is for the Navier–Stokes system. To be more precise, with the word *natural*, we mean that if the solutions are regular enough, then system (2.13) is equivalent to the one obtained by replacing $(2.13)_1$ with (2.16), and condition (2.17) is automatically verified by the solutions of the new system. Of course, if no *external force* is given, as in the present case, the regularity properties can be directly derived by the regularity of the boundary of Ω which is, in the present case, the largest possible. We are then led to the system

$$\begin{cases} \Delta P + \hat{\beta} P_z = -\frac{1}{\mathrm{Pr}} \nabla \cdot (\mathbf{v} \cdot \nabla \mathbf{v}) + \mathrm{Ra}\, \tau_z, \\ \frac{1}{\mathrm{Pr}} \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla P - \hat{\beta} P \mathbf{k} + \Delta \mathbf{v} + \mathrm{Ra}\, \tau \mathbf{k}, \\ \frac{\partial \tau}{\partial t} + \mathbf{v} \cdot \nabla \tau = \Delta \tau + \mathbf{v} \cdot \mathbf{k}. \end{cases} \tag{2.18}$$

If we define $\Pi = e^{\hat{\beta} z} P$, condition (2.17) turns into the simpler Neumann condition

$$\Pi_z = 0 \quad \text{ on } z = 0 \text{ and } z = 1, \tag{2.19}$$

and Eq. (2.16) into

$$\Delta \Pi - \hat{\beta} \Pi_z = -\frac{1}{\mathrm{Pr}} e^{\hat{\beta} z} \nabla \cdot (\mathbf{v} \cdot \nabla \mathbf{v}) + \mathrm{Ra}\; e^{\hat{\beta} z} \tau_z. \tag{2.20}$$

Notice that if $\hat{\beta} = 0$, then Eq. (2.20) with the Neumann condition on the boundary would reduce to the classical problem for the pressure. Using Π, system (2.18) becomes

$$\begin{cases} \Delta \Pi - \hat{\beta} \Pi_z = -\frac{1}{2\,\mathrm{Pr}} e^{\hat{\beta} z} (\nabla \mathbf{v}) : (\nabla \mathbf{v})^T + \mathrm{Ra}\; e^{\hat{\beta} z} \tau_z, \\ \frac{1}{\mathrm{Pr}} (\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v}) - \Delta \mathbf{v} = -e^{-\hat{\beta} z} \nabla \Pi + \mathrm{Ra}\, \tau \mathbf{k}, \\ \tau_t + \mathbf{v} \cdot \nabla \tau - \Delta \tau = \mathbf{v} \cdot \mathbf{k}. \end{cases} \tag{2.21}$$

As in the classic O–B framework, we first linearize (2.21) around zero to study instability. Then, we rearrange the pressure term to have a gradient and a buoyancy term. We obtain

$$\begin{cases} \Delta \Pi - \hat{\beta} \Pi_z = \mathrm{Ra}\; e^{\hat{\beta} z} \tau_z, \\ \frac{1}{\mathrm{Pr}} \mathbf{v}_t - \Delta \mathbf{v} = -\nabla (e^{-\hat{\beta} z} \Pi) - \hat{\beta} e^{-\hat{\beta} z} \Pi \mathbf{k} + \mathrm{Ra}\, \tau \mathbf{k}, \\ \tau_t - \Delta \tau = \mathbf{v} \cdot \mathbf{k}. \end{cases} \tag{2.22}$$

Second, since Eq. $(2.13)_1$ anyway holds true (again by the arguments in [17]), we introduce the *stream function* Φ, which is implicitly defined up to an additive constant by

$$\Phi_x = v^z, \quad \Phi_z = -v^x. \tag{2.23}$$

Now, we apply the operator $\nabla \times$ to $(2.22)_3$; by (2.23) and the identity $\nabla \times \nabla = 0$ we get

$$\begin{cases} \Delta \Pi - \hat{\beta} \Pi_z = \mathrm{Ra}\; e^{\hat{\beta} z} \tau_z, \\ \frac{1}{\mathrm{Pr}} \Psi_t - \Delta \Psi = \mathrm{Ra}\, \tau_x - \hat{\beta} e^{-\hat{\beta} z} \Pi_x, \\ \tau_t - \Delta \tau = \Phi_x, \end{cases} \tag{2.24}$$

for $\hat{\beta} > 0$, where $\Psi = \Delta\Phi$. The initial conditions are

$$\begin{cases} \Psi(x,z,0) = \Psi_0(x,z), \\ \tau(x,z,0) = \tau_0(x,z), \end{cases} \quad \text{for } (x,z) \in \Omega, \tag{2.25}$$

for given functions Ψ_0 and τ_0. About boundary conditions, the free-surface condition for the velocity $\mathbf{v}$ we assumed above leads to impose $\Psi = \Delta\Phi = 0$ at $z = 0$ and $z = 1$. Actually, by (2.23), condition (2.14) implies

$$\begin{aligned} \Phi_x(x,0,t) &= \Phi_x(x,1,t) = 0, \\ \Phi_{xx}(x,0,t) &= \Phi_{xx}(x,1,t) = 0, \\ \Phi_{zz}(x,0,t) &= \Phi_{zz}(x,1,t) = 0, \end{aligned} \tag{2.26}$$

for $(x,t) \in \mathbb{R} \times (0,\infty)$. As a consequence, the full set of boundary conditions associated to (2.24) is

$$\begin{cases} \Pi_z(x,0,t) = \Pi_z(x,1,t) = 0, \\ \Psi(x,0,t) = \Psi(x,1,t) = 0, \\ \tau(x,0,t) = \tau(x,1,t) = 0, \end{cases} \quad \text{for } (x,t) \in \mathbb{R} \times (0,\infty). \tag{2.27}$$

The solutions of the nonlinear system (2.13) mentioned in Remark 2.1 and belonging to the space $\mathbb{S}_0$, also solve the linear system (2.24) and the boundary condition (2.27). The aim of the present paper is to complete the results given in [15] by studying solutions to (2.24) in some complement of $\mathbb{S}_0$.

3. The Reduced Problem

We begin the study of system (2.24) by the pressure Eq. $(2.24)_1$ and more precisely with the reduced problem

$$\begin{cases} \Delta\Pi - \hat{\beta}\Pi_z = e^{\hat{\beta}z} f, & \text{in } \Omega, \\ \Pi_z(x,0) = \Pi_z(x,1) = 0, & \text{for } x \in \mathbb{R}, \end{cases} \tag{3.1}$$

for $f = f(x,z)$ periodic in x. The most important result of this section is an existence theorem for solutions to problem (3.1).

For given functions $F = F(x,z,t)$ and $G = G(x,z,t)$, we denote

$$\langle F \rangle = \int_{\Omega_0} F(x,z,t)\,\mathrm{d}x\mathrm{d}z, \quad (F,G) = \int_{\Omega_0} F(x,z,t)\,G(x,z,t)\,\mathrm{d}x\mathrm{d}z.$$

Clearly the same definitions apply, if F and G do not depend on t.

We shall assume that $\langle f \rangle = 0$; indeed, Eq. $(3.1)_1$ plays the role of Eq. $(2.24)_1$ and we have

$$\langle \tau_z \rangle = \int_0^1 (\tau(x,1,t) - \tau(x,0,t))\,\mathrm{d}x = 0,$$

by $(2.27)_3$. Notice that $\langle \tau_z \rangle = 0$ perfectly matches with $\langle \Delta P + \hat{\beta}P_z \rangle = 0$, which follows from (2.17) by taking into account the periodicity. In other words, if we multiply the left-hand side of $(3.1)_1$ by $e^{-\hat{\beta}z}$ and integrate, we see that the condition $\langle f \rangle = 0$ is necessary to have a solution. First, we notice that solutions of (3.1) are not unique: if Π is a solution, then also $\Pi + C$ is a solution for every constant C. Then we denote

$$\Pi' = \Pi - \langle \Pi \rangle. \tag{3.2}$$

Of course $\langle \Pi' \rangle = 0$; in the following, we simply write Π for Π'.

We consider the space of functions u in $C^\infty([0,1])$ satisfying $u_z(0) = u_z(1) = 0$ and consider its closure in the $W^{k,2}(0,1)$-norm, for $k = 0, 1, 2$. An orthogonal basis for such spaces is provided by $\{\cos(n\pi z)\}_{n \in \mathbb{N}_0}$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{N} = \{1, 2, \ldots\}$. Then, for $i = \pm 1$, we introduce the functions

$$\phi^i_{mn}(x,z) = \begin{cases} \cos(2\pi m x)\cos(\pi n z) & \text{if } i = 1, \\ \sin(2\pi m x)\cos(\pi n z) & \text{if } i = -1, \end{cases} \qquad m, n \in \mathbb{N}_0. \tag{3.3}$$

To deal with functions with zero mean value on the cell Ω_0, we need to exclude the functions ϕ^i_{0n}; then, we consider the reduced basis

$$\widetilde{\mathcal{B}} = \left\{ \phi^i_{mn} : \ (m,n) \in \mathbb{N} \times \mathbb{N}_0 \text{ for } i = \pm 1 \right\}.$$

The spaces $\widetilde{W}^{k,2}(\Omega_0)$, $k = 0, 1, 2$, are defined as the closures with respect to the $W^{k,2}(\Omega_0)$-norm of finite combinations of elements of the basis $\widetilde{\mathcal{B}}$; we denote $\widetilde{L}^2(\Omega_0) = \widetilde{W}^{0,2}(\Omega_0)$. By Sobolev's embedding theorem, it follows that $\widetilde{W}^{2,2}(\Omega_0) \subset C^0(\Omega_0)$; moreover, functions in $\widetilde{W}^{2,2}(\Omega_0)$ have traces in $W^{3/2,2}(\partial\Omega_0)$. Functions in $\widetilde{W}^{k,2}(\Omega_0)$ satisfy

$$\langle u \rangle = 0 \quad \text{ for } k = 0, 1, 2, \tag{3.4}$$
$$u_z(x,0) = u_z(x,1) = 0, \quad \text{ for } k = 2, \ x \in [0,1]. \tag{3.5}$$

We notice that the choice $m \neq 0$ in $\widetilde{\mathcal{B}}$ excludes the functions only depending on z from these spaces; then, $\widetilde{W}^{k,2}(\Omega_0) \cap \mathbb{S}_0 = \{0\}$. Moreover, since any u in $\widetilde{W}^{1,2}(\Omega_0)$ has zero mean value, it follows that it fulfills the Poincaré inequality [4, II, Theorem 4.3]

$$\|u\| \leq k_P \|\nabla u\|, \tag{3.6}$$

where k_P is the Poincaré constant. Here and in the following, we denote by $\|\cdot\|$ the norm in $L^2(\Omega_0)$.

Remark 3.1. *The functions ϕ^i_{mn} are eigenfunctions of the Laplace operator with the eigenvalues:*

$$\Delta \phi^i_{mn} = -\alpha_{mn} \phi^i_{mn}, \qquad \textit{for } \alpha_{mn} := 4\pi^2 m^2 + \pi^2 n^2. \tag{3.7}$$

After integration by parts, we obtain $(\nabla\phi^i_{mn}, \nabla\phi^i_{mn}) = \alpha_{mn}(\phi^i_{mn}, \phi^i_{mn})$. As a consequence, the optimal constant in the Poincaré inequality (3.6) is $k_P = (2\pi)^{-1}$.

Theorem 3.1. *Consider $f \in \widetilde{L}^2(\Omega_0)$ and assume $0 \leq \hat{\beta} < 2\pi$. Then, problem (3.1) has a unique solution $\Pi \in \widetilde{W}^{2,2}(\Omega_0)$; moreover, Π satisfies the estimate*

$$\|\nabla \Pi\| \leq \frac{1}{2\pi - \hat{\beta}} \|e^{\hat{\beta} z} f\|. \tag{3.8}$$

Proof. First, we prove (3.8). Assume that $\Pi \in \widetilde{W}^{1,2}(\Omega_0)$ solves (3.1) and notice that it is immediate to see that $e^{\hat{\beta}z} f \in \widetilde{L}^2(\Omega_0)$. We multiply $(3.1)_1$ by Π and integrate with respect to x and z to find

$$\int_{\Omega_0} \Delta\Pi \cdot \Pi \, \mathrm{d}x\mathrm{d}z - \hat{\beta} \int_{\Omega_0} \Pi_z \Pi \, \mathrm{d}x\mathrm{d}z = \int_{\Omega_0} e^{\hat{\beta}z} f \Pi \, \mathrm{d}x\mathrm{d}z.$$

We integrate by parts the first summand and exploit (3.4), (3.5) and the periodicity; we deduce

$$\int_{\Omega_0} |\nabla\Pi|^2 \, \mathrm{d}x\mathrm{d}z = -\hat{\beta} \int_{\Omega_0} \Pi_z \Pi \, \mathrm{d}x\mathrm{d}z - \int_{\Omega_0} e^{\hat{\beta}z} f \Pi \, \mathrm{d}x\mathrm{d}z.$$

Then, we use Cauchy–Schwarz inequality for both summands on the right-hand side and find

$$\|\nabla\Pi\|^2 \le k_P \left(\hat{\beta}\|\Pi_z\| + \|e^{\hat{\beta}z} f\|\right) \|\nabla\Pi\| \le \frac{1}{2\pi} \left(\hat{\beta}\|\nabla\Pi\| + \|e^{\hat{\beta}z} f\|\right) \|\nabla\Pi\|,$$

whence (3.8) follows.

Now, we prove the existence and uniqueness of the solution. Since

$$b_{mn} := (\phi^i_{mn}, \phi^i_{mn}) = \begin{cases} \frac{1}{4} & \text{if } n \neq 0, \\ \frac{1}{2} & \text{otherwise}, \end{cases} \tag{3.9}$$

we use the normalization $\psi^i_{mn} = \phi^i_{mn}/\gamma_{mn}$, for $\gamma_{mn} = \sqrt{\alpha_{mn} b_{mn}}$. The family $\{\psi^i_{mn}\}$ still is a basis for $\widetilde{W}^{1,2}(\Omega_0)$ and (3.7) holds with ψ^i_{mn} replacing ϕ^i_{mn}. By integration by parts, we obtain

$$\left(\nabla\psi^i_{mn}, \nabla\psi^i_{mn}\right) = 1. \tag{3.10}$$

We use the multi-index notation $\mu = (i, m, n)$ for $i = \pm 1$ and $(m, n) \in \mathbb{N} \times \mathbb{N}_0$; then, we order the elements ψ^i_{mn} of the basis according to the lexicographic order. As a consequence, the previous basis for $\widetilde{W}^{1,2}(\Omega_0)$ can be written as $\{\psi_\mu\}_{\mu \in \mathcal{M}}$, where we discarded the elements corresponding to multi-indices μ that do not appear in the definition of $\widetilde{\mathcal{B}}$. We use the same notation for the other functions. We write

$$\Pi^N = \sum_{|\mu| \le N} B_\mu \psi_\mu, \quad F^N = \sum_{|\mu| \le N} F_\mu \psi_\mu$$

with $F_\mu := (e^{\hat{\beta}z} f, \psi_\mu)$ and the B_μ's are to be found by imposing that equation $(3.1)_1$ with right-hand side F^N is satisfied. This means

$$\sum_{|\mu| \le N} B_\mu \Delta\psi_\mu = \hat{\beta} \sum_{|\mu| \le N} B_\mu \psi_{\mu,z} + \sum_{|\mu| \le N} F_\mu \psi_\mu. \tag{3.11}$$

By (3.7) we notice that

$$\Delta\Pi^N = - \sum_{|\mu| \le N} \alpha_\mu B_\mu \psi_\mu,$$

where we denoted $\alpha_\mu = \alpha_{i,m,n} := \alpha_{m,n}$ with a slight abuse of notation. If we multiply Eq. (3.11) by ψ_ν and integrate, we deduce by (3.10)

$$- B_\nu = \hat{\beta} \sum_{|\mu| \le N} B_\mu \left(\psi_{\mu,z}, \psi_\nu\right) + \frac{F_\nu}{\alpha_\nu}. \tag{3.12}$$

The coefficients B_ν depend in general also on N; for brevity, this dependence is omitted.

We now show that the algebraic system for the coefficients B_μ is solvable by a Leray's argument. Consider an algebraic system $\mathbf{P}(\mathbf{B}) = 0$, where $\mathbf{P} : \mathbb{R}^N \to \mathbb{R}^N$ is a continuous function; if there is R such that the scalar product $\mathbf{P}(\mathbf{B}) \cdot \mathbf{B}$ has a defined sign for $|\mathbf{B}| > R$, then there exists at least one solution of the system in $B(0, R)$, the ball of radius R and center 0. Therefore, we consider the quadratic form

$$\mathbf{P}(\mathbf{B}) \cdot \mathbf{B} = -\sum_{|\nu|\leq N} B_\nu^2 - \hat{\beta} \sum_{|\mu|,|\nu|\leq N} B_\mu B_\nu \left(\psi_{\mu,z}, \psi_\nu\right) - \sum_{|\nu|\leq N} \frac{F_\nu}{\alpha_\nu} B_\nu. \quad (3.13)$$

First, we compute $(\psi_{\mu,z}, \psi_\nu)$. We come back to the notation $\mu = (i, m, n)$ and $\nu = (j, r, s)$; by integration with respect to x and then to z we deduce

$$\left(\psi_{\mu,z}, \psi_\nu\right) = \left(\psi^i_{mn,z}, \psi^j_{rs}\right) = -\frac{\pi n}{2} \frac{\delta_{ij}\delta_{mr}}{\gamma_{mn}\gamma_{rs}} \left(\sin(\pi n z), \cos(\pi s z)\right),$$

where δ_{ij}, δ_{mr} are Kronecker's deltas. Moreover

$$(\sin(\pi n z), \cos(\pi s z)) = \begin{cases} \frac{1}{\pi}\left(\frac{1}{n+s} + \frac{1}{n-s}\right) & \text{if } n+s \geq 1 \text{ is odd}, \\ 0 & \text{otherwise}. \end{cases}$$

Consider the second term in the right-hand side of (3.13). We deduce

$$\sum_{|\mu|\leq N} B_\mu B_\nu \left(\psi_{\mu,z}, \psi_\nu\right) = -\frac{1}{2} \sum_{i,m,n,s} \frac{B^i_{mn} B^i_{ms}}{\gamma_{mn}\gamma_{ms}} \left(\frac{n}{n+s} + \frac{n}{n-s}\right) =: I + II, \quad (3.14)$$

where the sum on the right-hand side is done for $|\mu| \leq N$, $n+s \geq 1$ odd, and as usual we agree to discard the summands corresponding to indices that do not appear in the definition of $\widetilde{\mathcal{B}}$.

Fix i and m and consider the summation with respect to n and s in the right-hand side of (3.14). About I, if we denote $\tilde{B}^i_{mn} = B^i_{mn}/\gamma_{mm}$, we have

$$\begin{aligned} \sum_{\substack{n,s \\ n+s\geq 1 \text{ odd}}} \tilde{B}^i_{mn}\tilde{B}^i_{ms} \frac{n}{n+s} &= \Bigg(\sum_{\substack{n \text{ even} \\ s \text{ odd}}} + \sum_{\substack{n \text{ odd} \\ s \text{ even}}}\Bigg) \tilde{B}^i_{mn}\tilde{B}^i_{ms} \frac{n}{n+s} \\ &= \sum_{\substack{n \text{ even} \\ s \text{ odd}}} \tilde{B}^i_{mn}\tilde{B}^i_{ms} \frac{n}{n+s} \\ &\quad + \sum_{\substack{s \text{ odd} \\ n \text{ even}}} \tilde{B}^i_{mn}\tilde{B}^i_{ms} \frac{s}{n+s} = \sum_{\substack{n \text{ even} \\ s \text{ odd}}} \tilde{B}^i_{mn}\tilde{B}^i_{ms}. \end{aligned}$$

The analogous term in II gives exactly the same result. Then, by (3.14), we obtain

$$\sum_{|\mu|,|\nu|\leq N} B_\mu B_\nu \left(\psi_{\mu,z}, \psi_\nu\right) = -\sum_{\substack{|\mu|,|\nu|\leq N \\ n \text{ even}, s \text{ odd}}} \tilde{B}^i_{mn}\tilde{B}^i_{ms}. \quad (3.15)$$

Then, the expression in (3.13) can be written as

$$\mathbf{P}(\mathbf{B}) \cdot \mathbf{B} = -\sum_{|\nu|\leq N} B_\nu^2 + \hat{\beta} \sum_{\substack{|\mu|,|\nu|\leq N \\ n \text{ even}, s \text{ odd}}} \tilde{B}^i_{mn}\tilde{B}^i_{ms} - \sum_{|\nu|\leq N} \frac{F_\nu}{\alpha_\nu} B_\nu. \quad (3.16)$$

For fixed i, n and s, by Cauchy–Schwarz and Young inequalities we have

$$\sum_m \tilde{B}^i_{mn}\tilde{B}^i_{ms} \le |\tilde{B}^i_{\cdot n}|\,|\tilde{B}^i_{\cdot s}| \le \frac{1}{2}\left(|\tilde{B}^i_{\cdot n}|^2 + |\tilde{B}^i_{\cdot s}|^2\right),$$

for $|\tilde{B}^i_{\cdot n}|$ the euclidean norm of the vector $\{\tilde{B}^i_{mn}\}_m$. By summing over i, n and s, we obtain

$$\sum_{\substack{|\mu|,|\nu|\le N\\ n\,\text{even},\, s\,\text{odd}}} \tilde{B}^i_{mn}\tilde{B}^i_{ms} \le \frac{1}{2}\sum_{|\nu|\le N}\tilde{B}^2_\nu = \frac{1}{2}\sum_{|\nu|\le N}\frac{B^2_\nu}{\gamma^2_{mn}} \le \frac{1}{4\pi^2}\sum_{|\nu|\le N} B^2_\nu,$$

since $\gamma^2_{mn} = \alpha_{mn}b_{mn} \ge \min\{\alpha_{mn}b_{mn}\} = 2\pi^2$. Then, by (3.12) we deduce, because of $\hat{\beta} < 2\pi$,

$$\mathbf{P}(\mathbf{B})\cdot\mathbf{B} \le -\left(1 - \frac{\hat{\beta}^2}{4\pi^2}\right)\sum_{|\nu|\le N} B^2_\nu - \sum_{|\nu|\le N}\frac{F_\nu}{\alpha_\nu}B_\nu \le 0, \tag{3.17}$$

for $|\mathbf{B}| > R$ with R sufficiently large.

Since for all N the solutions Π^N corresponding to f^N are uniformly bounded in the norm of the gradient because of (3.8), then there exists a subsequence that weakly converges in that norm [4, Th. 1.2, §II]; this limit is a weak solution to problem (3.1). To prove that $\Pi \in \widetilde{W}^{2,2}(\Omega_0)$, we test $(3.1)_1$ with $\Delta\Pi$. By Cauchy–Schwarz inequality, one gets

$$\|\Delta\Pi\| \le \hat{\beta}\|\nabla\Pi\| + \|e^{\hat{\beta}z}f\|,$$

and then, by orthogonality,

$$\begin{aligned}\|\Pi_{xz}\|^2 &= \frac{1}{4}\sum_{i,m\ge1,n\ge1}\left((2\pi m)(\pi n)B^i_{mn}\right)^2\\ &\le \frac{1}{4}\sum_{i,m\ge1,n\ge1}\left(\frac{1}{2}\alpha_{mn}B^i_{mn}\right)^2 \le \frac{1}{16}\|\Delta\Pi\|.\end{aligned}$$

At last, uniqueness directly follows by (3.8). □

4. Existence Results for the Full Linear System

In this section, we study the existence and the stability of periodic solutions in $\Omega\times(0,\infty)$ to (2.21) under the initial conditions

$$(\mathbf{v}(x,0),\tau(x,0)) = (\mathbf{v}_0(x),\tau_0(x)) \tag{4.1}$$

and boundary conditions (2.14), (2.15) and (2.19). As we shown at the end of Sect. 2, this problem is equivalent to (2.24) under the initial-boundary conditions (2.25) and (2.27). Recalling Sect. 3, we look for Π in the space $\widetilde{W}^{2,2}(\Omega_0)$ and then expressed through the basis $\{\phi^i_{mn}\}_{(m,n)\in\mathbb{N}\times\mathbb{N}_0}$. On the other hand, a natural basis for τ, Ψ is

$$\zeta^i_{mn}(x,z) = \begin{cases}\cos(2\pi mx)\sin(\pi nz) & \text{if } i=1,\\ \sin(2\pi mx)\sin(\pi nz) & \text{if } i=-1,\end{cases} \tag{4.2}$$

for $(m, n) \in \mathbb{N}\times\mathbb{N}$. We denote by $\widehat{W}^{k,2}(\Omega_0)$ the space generated by (4.2) with the $W^{k,2}(\Omega_0)$-norm, for $k = 0, 1, 2$. Analogously, we denote by $\widehat{\mathcal{H}}(\Omega_0)$ the space of two-component vector functions generated by (4.2), and having null divergence, with respect to the $L^2(\Omega_0)$-norm; the space $\widehat{\mathcal{W}}^{k,2}(\Omega_0)$, $k = 1, 2$, is defined accordingly using the $W^{k,2}(\Omega_0)$-norm.

If $\mathbf{v} \in L^2\left((0, T), \widehat{\mathcal{W}}^{1,2}(\Omega_0)\right)$ and v^z vanishes on $z = 0$ and $z = 1$, as required by (2.14), then it is easy to see that

$$\|\mathbf{v}\| \le \|\nabla \mathbf{v}\|. \tag{4.3}$$

Further, the Poincaré inequality for $\tau \in L^2\left((0, T), \widehat{W}^{1,2}(\Omega_0)\right)$ is

$$\|\tau\| \le \frac{1}{\sqrt{5}\pi}\|\nabla\tau\|, \tag{4.4}$$

where the constant $1/(\sqrt{5}\pi)$ is optimal because the indices (m, n) in the basis $\{\zeta^i_{mn}\}$ run in $\mathbb{N} \times \mathbb{N}$.

At last, we introduce the energy

$$E(t) := \frac{1}{2}\left(\frac{1}{\mathrm{Pr}}\|\mathbf{v}(t)\|^2 + \mathrm{Ra}\,\|\tau(t)\|^2\right).$$

Theorem 4.1. *Assume $0 \le \hat{\beta} < 2\pi$. Then, for all $\mathrm{Ra} > 0$, $T > 0$ and $(\mathbf{v}_0, \tau_0) \in \widehat{\mathcal{H}}(\Omega_0) \times \widehat{L}^2(\Omega_0)$, system (2.22) with initial-boundary conditions (4.1), (2.14), (2.15) and (2.19) has a unique solution $(\Pi, \mathbf{v}, \tau)$ with*

$$\begin{aligned}
&\Pi \in L^2\left((0, T), \widetilde{W}^{1,2}(\Omega_0)\right),\\
&\mathbf{v} \in L^\infty\left((0, T), \widehat{\mathcal{H}}(\Omega_0)\right) \cap L^2\left((0, T), \widehat{\mathcal{W}}^{1,2}(\Omega_0)\right),\\
&\tau \in L^\infty\left((0, T), \widehat{L}^2(\Omega_0)\right) \cap L^2\left((0, T), \widehat{W}^{1,2}(\Omega_0)\right).
\end{aligned}$$

Moreover, for $t \in [0, T]$, the solution satisfies

$$E(t) \le e^{c_0 t}E(0), \tag{4.5}$$

where $c_0 > 0$ depends on Pr, Ra, $\hat{\beta}$; if Ra is sufficiently small, then $E(t)$ decays exponentially.

If the initial data satisfy the additional regularity $(\mathbf{v}_0, \tau_0) \in \widehat{\mathcal{W}}^{1,2}(\Omega_0) \times \widehat{W}^{1,2}(\Omega_0)$, then

$$\begin{aligned}
&\Pi \in L^2\left((0, T), \widetilde{W}^{2,2}(\Omega_0)\right),\\
&\mathbf{v} \in L^\infty\left((0, T), \widehat{\mathcal{W}}^{1,2}(\Omega_0)\right) \cap L^2\left((0, T), \widehat{\mathcal{W}}^{2,2}(\Omega_0)\right),\\
&\tau \in L^\infty\left((0, T), \widehat{W}^{1,2}(\Omega_0)\right) \cap L^2\left((0, T), \widehat{W}^{2,2}(\Omega_0)\right).
\end{aligned}$$

Proof. We split the proof into some steps.

An a priori inequality for system (2.22). We first prove an a priori energy estimate for solutions of (2.22). To this aim, we make the scalar product of

$(2.22)_2$ with $\mathbf{v}$ and multiply $(2.22)_3$ by τ. We deduce

$$\frac{1}{2\,\mathrm{Pr}}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}\|^2+\|\nabla\mathbf{v}\|^2=-\hat\beta(e^{-\hat\beta z}\Pi,v^z)+\mathrm{Ra}\,(\tau,v^z), \tag{4.6}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tau\|^2+\|\nabla\tau\|^2=(\tau,v^z), \tag{4.7}$$

where we denoted with $\frac{\mathrm{d}}{\mathrm{d}t}$ the usual time derivative. We multiply (4.7) by Ra and then sum the two equations. To estimate the term containing $\hat\beta$, we proceed as follows:

$$\hat\beta|(e^{-\hat\beta z}\Pi,v^z)|\le\hat\beta|(\Pi,v_z)|\le\hat\beta\|\Pi\|\|\mathbf{v}\|\le\frac{\hat\beta e^{\hat\beta}\,\mathrm{Ra}}{2\pi(2\pi-\hat\beta)}\|\nabla\tau\|\|\mathbf{v}\|$$
$$\le\frac{\mathrm{Ra}}{2M}\|\nabla\tau\|^2+c(\hat\beta,\mathrm{Ra},M)\|\mathbf{v}\|^2,$$

where M is precised later on and $c(\hat\beta,\mathrm{Ra},M)=\frac{\mathrm{Ra}\,M\hat\beta e^{\hat\beta}}{2\pi(2\pi-\hat\beta)}$. Here we used, in turn, Poincaré inequality for Π (see Remark 3.1), estimate (3.8) with $f=\mathrm{Ra}\,\tau_z$, Schwarz and Poincaré inequality (4.4) for τ. Moreover, by (4.4) we deduce

$$2\,\mathrm{Ra}\,|(\tau,v^z)|\le2\,\mathrm{Ra}\,\|\tau\|\|v^z\|\le\frac{2\,\mathrm{Ra}}{\sqrt5\pi}\|\nabla\tau\|\|\mathbf{v}\|\le\frac{\mathrm{Ra}}{5\pi^2}\|\nabla\tau\|^2+\mathrm{Ra}\,\|\mathbf{v}\|^2.$$

Now, we choose $M=\frac{5\pi^2}{5\pi^2-2}$ and we conclude that, for $c_0=2\,\mathrm{Pr}(c(\hat\beta,\mathrm{Ra},M)+\mathrm{Ra})$,

$$\frac{\mathrm{d}E}{\mathrm{d}t}(t)+\|\nabla\mathbf{v}(t)\|^2+\frac{\mathrm{Ra}}{2}\|\nabla\tau(t)\|^2\le c_0E(t). \tag{4.8}$$

Gronwall inequality shows that $E(t)\le E(0)e^{c_0t}$; this proves (4.5).

An a priori inequality for system (2.24). Assume that By (4.6) and (2.23) we deduce

$$\frac{1}{\mathrm{Pr}}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\Omega_0}|\nabla\Phi|^2\,\mathrm{d}x\mathrm{d}z+\iint_{\Omega_0}(\Delta\Phi)^2\,\mathrm{d}x\mathrm{d}z=-\hat\beta(e^{-\hat\beta z}\Pi,v^z)+\mathrm{Ra}\,(\tau,v^z). \tag{4.9}$$

For clarity, we briefly show how (4.9) can be deduced by system (2.24). By multiplying $(2.24)_2$ by Φ and integrating over Ω_0 we obtain

$$\frac{1}{\mathrm{Pr}}\iint_{\Omega_0}\Psi_t\Phi\,\mathrm{d}x\mathrm{d}z-\iint_{\Omega_0}\Delta\Psi\Phi\,\mathrm{d}x\mathrm{d}z=\mathrm{Ra}\iint_{\Omega_0}\tau_x\Phi\,\mathrm{d}x\mathrm{d}z$$
$$-\hat\beta\iint_{\Omega_0}e^{-\hat\beta z}\Pi_x\Phi\,\mathrm{d}x\mathrm{d}z. \tag{4.10}$$

About the first term on the left-hand side of (4.10), we have

$$\iint_{\Omega_0}\Psi_t\Phi\,\mathrm{d}x\mathrm{d}z=\iint_{\Omega_0}\mathrm{div}\,(\Phi\nabla\Phi_t)\,\mathrm{d}x\mathrm{d}z-\iint_{\Omega_0}\nabla\Phi\cdot\nabla\Phi_t\,\mathrm{d}x\mathrm{d}z$$
$$=\iint_{\partial\Omega_0}\Phi\,\nabla\Phi_t\cdot\mathbf{n}\,\mathrm{d}S-\frac{1}{2}\frac{d}{dt}\iint_{\Omega_0}|\nabla\Phi|^2\,\mathrm{d}x\mathrm{d}z$$
$$=-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\Omega_0}|\nabla\Phi|^2\,\mathrm{d}x\mathrm{d}z. \tag{4.11}$$

Above, we applied the divergence theorem, we denoted by $\mathbf{n}$ the exterior unit normal to Ω_0 and exploited the fact that, in the line integral, integrations along $z = 0$ and $z = 1$ give no contribution because of (4.15), while integrations along $x = 0$ and $x = 1$ have opposite signs because of periodicity and of $\mathbf{n}$. For the second summand on the left-hand side of (4.10), we have

$$\iint_{\Omega_0} \Delta\Psi\Phi \, \mathrm{d}x\mathrm{d}z = \iint_{\Omega_0} (\partial_x^4 + 2\partial_x^2\partial_z^2 + \partial_z^2)\Phi\,\Phi \, \mathrm{d}x\mathrm{d}z.$$

By integrating by parts, (2.26) and (4.15) we conclude

$$\iint_{\Omega_0} \Delta\Psi\Phi \, \mathrm{d}x\mathrm{d}z = \iint_{\Omega_0} (\Delta\Phi)^2 \, \mathrm{d}x\mathrm{d}z. \tag{4.12}$$

By (4.10)–(4.12) we deduce (4.9).

Equivalent norms. We claim that the norms $\|\nabla\mathbf{v}\|$ and $\|\Delta\Phi\| = \|\Psi\|$ are equivalent. Indeed,

$$v^x = \sum_{i,m,n} \pi n \frac{C^i_{mn}}{\alpha_{mn}} \phi^i_{mn}, \quad v^z = \sum_{i,m,n} 2\pi m \frac{-iC^{-i}_{mn}}{\alpha_{mn}} \zeta^i_{mn}.$$

We have, by orthogonality,

$$\Phi_{zx} = v^z_z = \sum_{i,m,n} (2\pi m)(\pi n) \frac{-iC^{-i}_{mn}}{\alpha_{mn}} \phi^i_{mn},$$

$$\|\Phi_{zx}\|^2 = \frac{1}{4} \sum_{i,m,n} \left(\frac{C^i_{mn}(2\pi m)(\pi n)}{\alpha_{mn}} \right)^2.$$

By Young's inequality, we deduce $\|\Phi_{zx}\|^2 \le \frac{1}{16}\left(\|\Phi_{xx}\|^2 + \|\Phi_{zz}\|^2\right)$; then, the norms $\|\nabla\mathbf{v}\|^2 = \|\Phi_{xx}\|^2 + \|\Phi_{zz}\|^2 + 2\|\Phi_{xz}\|^2$ and $\|\Delta\Phi\|^2 = \|\Phi_{xx}\|^2 + \|\Phi_{zz}\|^2 + 2\|\Phi_{xx}\|\|\Phi_{zz}\|$ are equivalent because they are both equivalent to $\|\Phi_{xx}\|^2 + \|\Phi_{zz}\|^2$. This proves the claim.

The ansatz. Now, we begin the proof of the existence of the solutions for system (2.24). We make the ansatz

$$\tau = \sum_{i,m,n} A^i_{mn}\zeta^i_{mn}, \quad \Pi = \sum_{i,m,n} B^i_{mn}\phi^i_{mn}, \quad \Psi = \sum_{i,m,n} C^i_{mn}\zeta^i_{mn}, \tag{4.13}$$

where the coefficients $A^i_{mn} = A^i_{mn}(t)$, $B^i_{mn} = B^i_{mn}(t)$ and $C^i_{mn} = C^i_{mn}(t)$ are to be found. Here, as in the following, we drop the dependence of the coefficients on both t and $\hat\beta$; moreover, we do not specify anymore the range of the indices (m, n).

To recover Φ from Ψ, we notice that by $(4.13)_3$ it follows

$$\Phi(x, z, t) = -\sum_{i,m,n} \frac{C^i_{mn}(t)}{\alpha_{mn}} \zeta^i_{mn}(x, z) + u(x, z, t), \tag{4.14}$$

where $\Delta u = 0$. Since τ and Φ must satisfy $(2.24)_3$, we look for u under the form

$$u(x, z, t) = \sum_{i,m,n} \eta^i_{mn}(t)\zeta^i_{mn}(x, z).$$

Then, $\eta^i_{mn} = 0$ for all $(m,n) \in \mathbb{N}\times\mathbb{N}_0$ because $\Delta u = 0$ and $\alpha_{mn} \neq 0$ for such (m,n). Then $u \equiv 0$.

As a consequence, in addition to the boundary values (2.26), the stream function Φ also satisfies

$$\Phi(x,0,t) = \Phi(x,1,t) = 0, \qquad \text{for } x \in [0,1],\, t \geq 0. \tag{4.15}$$

Approximate solutions. We proceed as in the proof of Theorem 3.1: we assume that we are given initial data

$$\tau_0^N(x,z) = \sum_{|\mu|\leq N} A^{iN}_{mn,0}\zeta^i_{mn}(x,z), \quad \Psi_0^N(x,z) = \sum_{|\mu|\leq N} C^{iN}_{mn,0}\zeta^i_{mn}(x,z), \tag{4.16}$$

where $\mu = (i,m,n)$, and look for τ, Π and Ψ as in (4.13), where, however, the sums are only made for $|\mu| \leq N$. For simplicity, we drop everywhere the index N.

By plugging $(4.13)_1$ and $(4.13)_2$ into $(2.24)_1$, one finds $B^i_{m0}(t) = 0$ for $n = 0$, so that from now on the indices (m,k) run in $\mathbb{N}\times\mathbb{N}$. Then,

$$\begin{aligned}\sum_{m,k}&\left(B^1_{mk}\cos 2\pi m x + B^2_{mk}\sin 2\pi m x\right)\left[-\alpha_{mk}\cos\pi k z + \hat\beta\pi k\sin\pi k z\right]\\ &= \mathrm{Ra}\sum_{m,k}\left(F^1_{mk}\cos 2\pi m x + F^2_{mk}\sin 2\pi m x\right)\cos\pi k z.\end{aligned} \tag{4.17}$$

The coefficients F^i_{mn} are defined as

$$\begin{aligned}F^i_{mk} &= \int_0^1\int_0^1 e^{\hat\beta z}\tau_z(x,z)\phi^i_{mk}(x,z)\mathrm{d}x\mathrm{d}z\\ &= \sum_{j,l,n}\pi n\int_0^1\int_0^1 e^{\hat\beta z}A^j_{ln}\phi^j_{ln}(x,z)\phi^i_{mk}(x,z)\mathrm{d}x\mathrm{d}z.\end{aligned}$$

By taking in Ω_0 the scalar product of (4.17) with $\cos(\pi k z)$ one deduces

$$\begin{aligned}B^i_{mk} &- \frac{2\hat\beta}{\alpha_{mk}}\sum_{\substack{n\\ n+k\ \mathrm{odd}}}\left(\frac{1}{n+k}+\frac{1}{n-k}\right)nB^i_{mn}\\ &= -\frac{\hat\beta\pi\,\mathrm{Ra}}{\alpha_{mk}}\sum_n A^i_{mn}n\left((-1)^{k+n}e^{\hat\beta}-1\right)\\ &\quad\times\left(\frac{1}{\pi^2(n+k)^2+\hat\beta^2}+\frac{1}{\pi^2(n-k)^2+\hat\beta^2}\right).\end{aligned} \tag{4.18}$$

Notice that the right-hand side of (4.18) becomes singular at $\hat\beta = 0$ when $n = k$; however, in this case, we also have $(-1)^{k+n}e^{\hat\beta} - 1 \sim \hat\beta$. This means that the right-hand side of (4.18) has a continuous extension to $\hat\beta = 0$ and, in this case, $B^i_{mk}(\hat\beta = 0) = \mathrm{Ra}\,\frac{\pi k}{\alpha_{mk}}A^i_{mk}$. This corresponds to the simpler and well-known classical case; hence, from now on, we focus on the case $\hat\beta > 0$.

If for $i = 1, 2$, $m \geq 1$, $n \geq 1$ and $k \geq 1$ we denote

$$\mathcal{D}^m_{nk} = \delta_{nk} + \hat{\beta} \begin{cases} -\frac{2}{\alpha_{mk}} \left(\frac{1}{n+k} + \frac{1}{n-k} \right) n, & \text{if } n+k \text{ odd}, \\ 0 & \text{if } n+k \text{ even}, \end{cases} \tag{4.19}$$

$$\mathcal{M}_{nk} = \pi n \left((-1)^{k+n} e^{\hat{\beta}} - 1 \right) \left(\frac{1}{\pi^2(n+k)^2 + \hat{\beta}^2} + \frac{1}{\pi^2(n-k)^2 + \hat{\beta}^2} \right), \tag{4.20}$$

then formula (4.18) can be written as

$$\sum_n B^i_{mn} \mathcal{D}^m_{nk} = -\frac{\mathrm{Ra}\, \hat{\beta}}{\alpha_{mk}} \sum\nolimits_n A^i_{mn} \mathcal{M}_{nk}. \tag{4.21}$$

Notice that the $N \times N$ matrix $\mathcal{D}^m$ has the form $\mathcal{D}^m(\hat{\beta}) = \mathbb{I} + \hat{\beta}\mathcal{X}$ for some matrix $\mathcal{X}$ and it is diagonal dominant for small $\hat{\beta}$. Then, $\mathcal{D}^m$ is invertible for small $\hat{\beta}$ [18, Cor. 5.4] and $(\mathcal{D}^m)^{-1}(\hat{\beta}) = \mathbb{I} + \hat{\beta}\mathcal{Y}(\hat{\beta})$, for some $\mathcal{Y}(\hat{\beta})$. Indeed, Eq. (4.21) is completely analogous to (3.12); by the part of the proof of Theorem 3.1 that follows (3.12) we deduce that the coefficients B^i_{mn} can be determined for any N if $\hat{\beta} < 2\pi$. Then we have, for $m \geq 1$, $l \geq 1$ and $\hat{\beta} < 2\pi$,

$$B^i_{mj} = -\frac{\mathrm{Ra}\, \hat{\beta}}{\alpha_{mj}} \sum_{n,k} A^i_{mn} \mathcal{M}_{nk} (\mathcal{D}^m)^{-1}_{kj}. \tag{4.22}$$

By (2.24_3) and (4.13) we deduce

$$C^i_{mk} = \frac{\alpha_{mk}}{2\pi m} \left(\dot{A}^{-i}_{mk} + \alpha_{mk} A^{-i}_{mk} \right). \tag{4.23}$$

We replace the coefficients (4.22) and (4.23) in $(2.24)_2$. Then, we exploit the linear independence of the functions of x by varying m, multiply by $\sin \pi j z$ and integrate for $z \in [0, 1]$. Since

$$-\hat{\beta} e^{-\hat{\beta} z} \Pi_x = \mathrm{Ra}\, \hat{\beta}^2 e^{-\hat{\beta} z} \sum_{i,m,j} \frac{2\pi m}{\alpha_{mj}} \sum_{l,k} \mathcal{M}_{lk} (\mathcal{D}^m)^{-1}_{kj} \left(-i A^i_{ml} \phi^{-i}_{mj} \right)$$

and moreover

$$\int_0^1 e^{-\hat{\beta} z} \cos \pi k z \sin \pi j z \, \mathrm{d}z = \frac{1}{2} \left(1 - e^{-\hat{\beta}} (-1)^{k+j} \right) \times \left(\frac{\pi(k+j)}{\pi^2(k+j)^2 + \hat{\beta}^2} - \frac{\pi(k-j)}{\pi^2(k-j)^2 + \hat{\beta}^2} \right),$$

then Eq. $(2.24)_2$ reduces to the second-order differential system

$$\ddot{A}^i_{mj} + (1 + \mathrm{Pr}) \alpha_{mj} \dot{A}^i_{mj} + \mathrm{Pr} \left(\alpha^2_{mj} - (2\pi m)^2 \frac{\mathrm{Ra}}{\alpha_{mj}} \right) A^i_{mj} = \mathrm{Pr}\, \mathrm{Ra}\, \hat{\beta}^2 \sum_n \mathcal{F}^n_{mj} A^i_{mn}, \tag{4.24}$$

where

$$\mathcal{N}_{kj} = \left(1 - e^{-\hat{\beta}}(-1)^{k+j}\right)\left(\frac{\pi(j+k)}{\pi^2(j+k)^2+\hat{\beta}^2} + \frac{\pi(j-k)}{\pi^2(j-k)^2+\hat{\beta}^2}\right),$$

$$\mathcal{F}^n_{mj} = \frac{(2\pi m)^2}{\alpha_{mj}} \sum_{k,l} \frac{1}{\alpha_{ml}} \mathcal{M}_{nk}(\mathcal{D}^m)^{-1}_{kl}\mathcal{N}_{lj}.$$

The initial data for system (4.24) are provided by $A^i_{mn,0}$ and $C^i_{mn,0}$ by exploiting (4.23) as an algebraic equation for $A^i_{mn,1} = \dot{A}^i_{mn}(0)$. Then, the initial-value problem for (4.24) has a unique solution; in turn, we determine the coefficients B^i_{mn} and C^i_{mn} by (4.22), (4.23). Thus, we have shown the existence of a unique solution (τ^N, Π^N, Ψ^N) to (2.24) corresponding to the initial data (4.16).

The weak solution. To deduce the convergence of the sequence $\{(\tau^N, \Pi^N, \Psi^N)\}$ to a weak solution, we need a uniform bound. We consider the estimate (4.8), corresponding to the sequence above, where $\mathbf{v}$ is deduced by (4.14) with $u = 0$; then we integrate with respect to t. The right-hand side is uniformly bounded with respect to N because of (4.5) and of the assumptions on the initial data. As a consequence, $\|\Psi^N\|$ is uniformly bounded; the same holds for $\|\nabla\tau^N\|$ and, by (4.4), also for $\|\tau^N\|$. A bound for $\|\Pi^N\|$ follows by (3.8) by means of inequality (3.6). By following a standard procedure [16, Theorem 3.2], the weak convergence of (a subsequence of) the approximate solutions to a weak solution follows and, in turn, the estimates stated in the theorem. At last, the time-continuity of the solution can be proved by parabolic embedding [16, Remark 3.3], so that the initial data are attained in a pointwise sense.

The exponential decay. At last, we prove the exponential decay of the L^2-norm of $E(t)$. We apply inequalities (4.3) and (4.4) to estimate (4.8) to deduce

$$\frac{\mathrm{d}E}{\mathrm{d}t} \le (c_0 - c_1)E,$$

where $c_1 = \min\left\{2\,\mathrm{Pr}, 5\pi^2\right\}$. If Ra is sufficiently small, then $c_0 - c_1 < 0$. This completes the proof of the theorem. □

We can now recover the instability result in [15], which is related to *normal mode solutions.* We consider initial data τ_0 and Ψ_0 that only involve one element of the basis, namely,

$$\tau_0(x,z) = A^i_{mj,0}\zeta^i_{mj}(x,z), \quad \Psi_0(x,z) = C^{-i}_{mj,0}\zeta^i_{mj}(x,z), \tag{4.25}$$

for some $i \in \{1,2\}$, $(m,j) \in \mathbb{N}\times\mathbb{N}$, $A^i_{mj0}, C^{-i}_{mj0} \in \mathbb{R}$. Clearly, for all $\mathrm{Ra} > 0$ and $0 \le \hat{\beta} < 2\pi$, Theorem 4.1 assures the existence and uniqueness of solutions to the initial-boundary value problem (2.24), (4.25), (2.27), in the class of solutions specified there. Because of (4.23), the linearity of the equations of system (2.24) and in particular of (4.24), the τ and Ψ components of the solution assume the form

$$\tau(x,z,t) = A^i_{mj}(t)\zeta^i_{mj}(x,z), \quad \Psi(x,z,t) = C^{-i}_{mj}(t)\zeta^i_{mj}(x,z), \tag{4.26}$$

for some functions A^i_{mj} and C^{-i}_{mj}. We shall see in the proof of Theorem 4.2 that Π has not such a simple expression; nevertheless it can be written as in $(4.13)_2$ with $B^i_{mn} = \bar{B}^i_{mn} A^i_{mj}(t)$, for some $\bar{B}^i_{mn}$ which does not depend on t. By (4.23), this shows that the time behavior of the solution is analogous for all components.

Before stating our last result, we make more precise what we mean by instability. Roughly speaking, this means that the time coefficients above (A^i_{mj}, for instance) do not decay. However, notice that in (2.7), we assumed $l = h$ for simplicity. A quick inspection to the normalization procedure (2.12) shows that, still keeping notation as in (2.12), in the general case, we have $\Omega_0 = (0, \omega) \times (0, 1)$, for $\omega = l/h$. Clearly, all previous results still hold by replacing the terms $\cos(2\pi m x)$, $\sin(2\pi m x)$ in (3.3) with $\cos(2\pi\omega m x)$, $\sin(2\pi\omega m x)$ and so on. While h is thought to be fixed, *instability* is referred here to the existence of some l, and then ω, for which the corresponding coefficient A^i_{mj} does not decay. In this sense, we recall [8] in the classical case $\hat{\beta} = 0$ instability occurs if $\mathrm{Ra} \geq \mathrm{Ra}_{\mathrm{cl}}$, for

$$\mathrm{Ra}_{\,\mathrm{cl}} \doteq \frac{27}{4}\pi^4. \tag{4.27}$$

Theorem 4.2. *For every sufficiently small $\hat{\beta}$, there exists $\mathrm{Ra}_*(\hat{\beta}) < \mathrm{Ra}_{\mathrm{cl}}$ such that problem (2.24), (4.25), (2.27) is unstable for $\mathrm{Ra} \geq \mathrm{Ra}_*(\hat{\beta})$.*

Proof. As in the proof of Theorem 4.1 we fix $N \in \mathbb{N}$; now, however, we study the solutions (τ^N, Π^N, Ψ^N) to system (2.24) corresponding to the initial data given by (4.25). For simplicity, we drop once more the dependence on N.

By uniqueness of solutions, system (4.24) decouples; we denote by $A^i_{mj}(t)$ the only nonvanishing coefficient, which satisfies the differential equation

$$\ddot{A}^i_{mj} + (\mathrm{Pr}+1)\alpha_{mj}\dot{A}^i_{mj} + \mathrm{Pr}\left(\alpha^2_{mj} - \mathrm{Ra}\,\frac{4\pi^2 m^2}{\alpha_{mj}}\right)A^i_{mj} = \mathrm{Pr}\,\mathrm{Ra}\,\hat{\beta}^2 \mathcal{F}^j_{mj} A^i_{mj}, \tag{4.28}$$

together with the initial conditions $A^i_{mn,0}$ and $A^i_{mn,1}$, where the latter is deduced by the algebraic Eqs. (4.23) and (4.25), namely,

$$A^i_{mj,1} = (-1)^i \frac{2\pi m}{\alpha_{mj}} C^i_{mj,0} - \alpha_{mj} A^i_{mj,0}.$$

In turn, Π is written as in $(4.13)_2$ with

$$B^i_{ml}(t) = -\frac{\mathrm{Ra}\,\hat{\beta}}{\alpha_{ml}} A^i_{mj}(t) \sum_{k\geq 1} \mathcal{M}_{jk} (\mathcal{D}^m)^{-1}_{kl}, \quad l \in \mathbb{N}, \tag{4.29}$$

and $B^i_{nl} = 0$ if $n \neq m$. Notice that in (4.29), the time behavior does not depend on l but only on j, so that Π and τ show the same dependence on t. This also accounts for the name given to these solutions. Notice that τ_0 does not depend on N but the coefficient $A^i_{mj}(t)$ does, because

$$\mathcal{F}^j_{mj} = \mathcal{F}^{jN}_{mj} = \frac{(2\pi m)^2}{\alpha_{mj}} \sum_{k,l=1}^{N} \frac{1}{\alpha_{ml}} \mathcal{M}_{jk} (\mathcal{D}^m)^{-1}_{kl} \mathcal{N}_{lj}.$$

Notice that the limit $\lim_{N\to\infty} \mathcal{F}^{jN}_{mj}(\hat\beta)$ is a real number for $\hat\beta \geq 0$ because of Theorem 4.1. Clearly, instability is equivalent to the existence of positive roots of the characteristic polynomial of Eq. (4.28), because they give rise to solutions $A^i_{mj}(t)$ that grow exponentially in time. To proceed further, we need some information on the term $\mathcal{F}^j_{mj}$. We claim that (emphasizing dependence on N)

(i) for every N, the function $\hat\beta \to \mathcal{F}^{jN}_{mj}(\hat\beta)$ has a continuous extension to $\hat\beta = 0$;
(ii) if $\hat\beta \geq 0$ is sufficiently small, then $\mathcal{F}^{1N}_{m1}(\hat\beta) > 0$ for every m and N;
(iii) if $\hat\beta$ is sufficiently small, then $\mathcal{F}^{1\infty}_{m1}(\hat\beta) := \lim_{N\to\infty} \mathcal{F}^{1N}_{m1}(\hat\beta) > 0$.

To prove these claims, we recall, as we pointed out in the proof of Theorem 4.1, that $(\mathcal{D}^m)^{-1}(\hat\beta) = (\mathbb{I}+\hat\beta\mathcal{X})^{-1} = \mathbb{I}+\hat\beta\mathcal{Y}(\hat\beta)$ for $\beta < 2\pi$, where $\mathcal{Y}$ is determined as a Neumann series and, hence, is nonsingular; as a consequence, we have

$$\mathcal{F}^j_{mj}(\hat\beta) = \widetilde{\mathcal{F}}^j_{mj}(\hat\beta) + O(\hat\beta), \tag{4.30}$$

where

$$\widetilde{\mathcal{F}}^j_{mj} = \frac{(2\pi m)^2}{\alpha_{mj}} \sum_{k=1}^{N} \frac{1}{\alpha_{mk}} \mathcal{M}_{jk}\mathcal{N}_{kj}.$$

Then, we first prove the analogous claims for $\widetilde{\mathcal{F}}^j_{mj}$; let us call them *(j)–(jjj)*. First, we observe that the sequences $\{\mathcal{M}_{jk}\}_k$, $\{\mathcal{N}_{kj}\}_k$, $\{\alpha_{mk}\}_k$ behave as k^{-2}, k^{-1}, k^2, respectively, as $k \to \infty$; as a consequence, the limit $\lim_{N\to\infty} \widetilde{\mathcal{F}}^{jN}_{mj}(\hat\beta)$ is a real number for $\hat\beta > 0$. Next, we notice that

$$\begin{aligned}\mathcal{M}_{jk}(\hat\beta)\mathcal{N}_{kj}(\hat\beta) &= -\pi j g_{j+k}(\hat\beta)\\ &\times \left(\frac{1}{\pi^2(j+k)^2+\hat\beta^2} + \frac{1}{\pi^2(j-k)^2+\hat\beta^2}\right)\\ &\times \left(\frac{\pi(j+k)}{\pi^2(j+k)^2+\hat\beta^2} + \frac{\pi(j-k)}{\pi^2(j-k)^2+\hat\beta^2}\right),\end{aligned} \tag{4.31}$$

where

$$\begin{aligned}g_{j+k}(\hat\beta) &= \left(1-(-1)^{j+k}e^{\hat\beta}\right)\left(1-(-1)^{j+k}e^{-\hat\beta}\right)\\ &= \begin{cases} 4+\hat\beta^2+O(\hat\beta^4) & \text{if } k+j \text{ is odd},\\ -\hat\beta^2+O(\hat\beta^4) & \text{if } k+j \text{ is even}.\end{cases}\end{aligned} \tag{4.32}$$

Above, the $O(\hat\beta)$ terms do not depend on j, k. If $k \neq j$, then for $\hat\beta \to 0$, we have

$$\mathcal{M}_{jk}(\hat\beta)\mathcal{N}_{kj}(\hat\beta) \sim -\pi j g_{j+k}(0)\left(\frac{1}{\pi^2(j+k)^2} + \frac{1}{\pi^2(j-k)^2}\right)$$
$$\times\left(\frac{\pi(j+k)}{\pi^2(j+k)^2} + \frac{\pi(j-k)}{\pi^2(j-k)^2}\right),$$

while, if $k = j$, then

$$\mathcal{M}_{jj}(\hat\beta)\mathcal{N}_{jj}(\hat\beta) \sim \frac{1}{2}\hat\beta^2\left(\frac{1}{4\pi^2 j^2} + \frac{1}{\hat\beta^2}\right) \sim \frac{1}{2} + O(\hat\beta^2).$$

An analogous argument was exploited in the proof of Theorem 4.1. Then, the function $\hat\beta \to \mathcal{M}_{jk}(\hat\beta)\mathcal{N}_{kj}(\hat\beta)$ has a continuous extension to $\hat\beta = 0$ for any j, k and this proves claim *(j)*.

To prove claim *(jj)*, we write

$$\begin{aligned}
&\sum_k \frac{1}{\alpha_{mk}}\mathcal{M}_{1k}(\hat\beta)\mathcal{N}_{k1}(\hat\beta)\\
&\quad = 2\pi^2 \sum_k \frac{1}{\alpha_{mk}} g_{1+k}(\hat\beta)\left(\frac{1}{\pi^2(1+k)^2+\hat\beta^2} + \frac{1}{\pi^2(1-k)^2+\hat\beta^2}\right)\times\\
&\quad\quad \times\left(\frac{\pi^2(k^2-1)-\hat\beta^2}{\left(\pi^2(1+k)^2+\hat\beta^2\right)\left(\pi^2(1-k)^2+\hat\beta^2\right)}\right).
\end{aligned} \tag{4.33}$$

We write

$$a_k(\hat\beta) = \frac{1}{\alpha_{mk}} g_{1+k}(\hat\beta) b_k(\hat\beta) c_k(\hat\beta)$$

for the summand in the series on the right-hand side of (4.33), with obvious notation. Then

$$c_k(\hat\beta) = \begin{cases} < 0 & \text{if } k = 1, \hat\beta > 0,\\ > 0 & \text{if } k \geq 2, \hat\beta \in (0, \sqrt{3}\pi).\end{cases} \tag{4.34}$$

By (4.32) and (4.34) we see that the sign of a_k is determined by the product $g_{1+k}c_k$: we have $a_1 = \frac{1}{\alpha_{m1}} g_2 b_1 c_1 > 0$ and $a_{2k} > 0$, $a_{2k+1} < 0$ if $k \geq 1$. We claim that

$$g_{2k+1}c_{2k} + g_{2k+2}c_{2k+1} > 0, \quad k = 1, 2, \ldots, \text{ for } \hat\beta \text{ sufficiently small.} \tag{4.35}$$

Since the sequences $\{1/\alpha_{mk}\}$ and $\{b_k\}$ are decreasing, by (4.35) we deduce $a_k > 0$ and then *(jj)*. To prove (4.35) we can argue by exploiting (4.32); otherwise, we show that

$$\{c_k(\hat\beta)\}_k \text{ decreases if } \hat\beta < 2.6970, \tag{4.36}$$

which gives an explicit and uniform (on both m and N) threshold for $\hat\beta$. To prove (4.36) we denote $\gamma = \hat\beta^2/\pi^2$ and consider the function $f(s) = \frac{s^2-1-\gamma}{((s+1)^2+\gamma)((s-1)^2+\gamma)}$. The function f decreases if $3\gamma^2 + 2(s^2+1)\gamma - (s^2-1)^2 < 0$, i.e., if $\gamma < h(s) = \frac{1}{3}\left(\sqrt{(s^2+1)^2 + 3(s^2-1)^2} - (s^2+1)\right)$. Since $\min_{s\in[2,\infty)} h(s) = h(2) \sim 0.7370$, claim (4.36) follows; of course, more precise bounds can be be given.

Claim *(jjj)* follows by (4.35) and $a_1 > 0$.

Now we deduce claims *(i)*–*(iii)*. Claim *(i)* follows by (4.30), claim *(ii)* by $\mathcal{F}^j_{mj}(0) = \widetilde{\mathcal{F}}^j_{mj}(0)$, claim *(iii)* by *(jjj)* since $a_1 > 0$. This completely proves the claims.

We now observe that

$$\left(\frac{\Pr+1}{2}\alpha_{mj}\right)^2+\Pr\left(\mathrm{Ra}\,\frac{4\pi^2m^2}{\alpha_{mj}}-\alpha_{mj}^2\right)$$
$$=\alpha_{mj}^2\left(\frac{\Pr-1}{2}\right)^2+\Pr\mathrm{Ra}\,\frac{4\pi^2m^2}{\alpha_{mj}}>0,$$

because $m\geq 1$. As a consequence, if $\hat\beta$ is sufficiently small (this is required by the fact that the terms $\mathcal{F}_{mj}^j$ have not a fixed sign for $j\geq 2$), then the roots of the characteristic polynomial of Eq. (4.28) are real and provided by the formula

$$\lambda_{mj}^{\pm}(\hat\beta)=-\frac{\Pr+1}{2}\alpha_{mj}\pm\sqrt{\left(\frac{\Pr+1}{2}\alpha_{mj}\right)^2+\Pr\left(\mathrm{Ra}\,\frac{4\pi^2m^2}{\alpha_{mj}}-\alpha_{mj}^2+\mathrm{Ra}\,\hat\beta^2\mathcal{F}_{mj}^j(\hat\beta)\right)}. \tag{4.37}$$

We remark that the root $\lambda_{mj}^+(0)$ is (strictly) positive if and only if $\mathrm{Ra}\,4\pi^2m^2-\alpha_{mj}^3>0$, i.e., $\mathrm{Ra}>\alpha_{mj}^3/(4\pi^2m^2)$. Since the inequality is strict, then $\lambda_{mj}^+(\hat\beta)>0$ for $\hat\beta>0$ sufficiently small. Recall that we are looking for the smallest Ra that leads to instability. Since $(m,j)\in\mathbb{N}\times\mathbb{N}$, then

$$\frac{\alpha_{mj}^3}{4\pi^2m^2}\geq\frac{\alpha_{m1}^3}{4\pi^2m^2}=\pi^4\frac{(4m^2+1)^3}{4m^2}.$$

Here m runs in $\mathbb{N}$; however, because of the remark on instability we made just before the statement of Theorem 4.2, we can understand it as $m\omega$ and, as a consequence, as varying in $(0,\infty)$. Then, we have the sharp inequality (writing again m for $m\omega$ with a slight abuse of notation)

$$\pi^4\frac{(4m^2+1)^3}{4m^2}\geq\min_{s>0}\pi^4\frac{(4s^2+1)^3}{4s^2}=\frac{27}{4}\pi^4,$$

and we have found the classical result that instability occurs if $\mathrm{Ra}>\mathrm{Ra}_{\,\mathrm{cl}}$.

A sharper instability inequality, whose threshold depends on $\hat\beta$, follows by exploiting claim *(iii)* above. Indeed, emphasizing again the dependence on N, we have $\lambda_{m1}^{N+}(\hat\beta)\geq 0$ if and only if

$$\mathrm{Ra}\,4\pi^2m^2-\alpha_{m1}^3+\mathrm{Ra}\,\hat\beta^2\alpha_{m1}\mathcal{F}_{m1}^{1N}(\hat\beta)\geq 0,$$

i.e., if and only if

$$\mathrm{Ra}\geq\pi^4\frac{(4m^2+1)^3}{4m^2+\hat\beta^2(4m^2+1)\mathcal{F}_{m1}^{1N}(\hat\beta)}.$$

We can now pass to the limit for $N\to\infty$ because of Theorem 4.1 and deduce that instability surely occurs if

$$\mathrm{Ra}>\mathrm{Ra}_{\,*m}^{\,1}(\hat\beta)\doteq\pi^4\frac{(4m^2+1)^3}{4m^2+\hat\beta^2(4m^2+1)\mathcal{F}_{m1}^{1\infty}(\hat\beta)}.$$

Since $\mathrm{Ra}_{\,*m}^{\,1}(\hat\beta)<\mathrm{Ra}_{\mathrm{cl}}$ for any m, any of these $\mathrm{Ra}_{\,*m}^{\,1}(\hat\beta)$ provides a threshold for instability that is sharper than the classical $\mathrm{Ra}_{\mathrm{cl}}$. More precisely, however, we can define analogously $\mathrm{Ra}_{*m}^{\,j}$ for $j = 1, 2, \ldots$; then the threshold of the statement is simply

$$\mathrm{Ra}_*(\hat{\beta}) = \inf_{\substack{m \in (0,\infty) \\ j=1,2,\ldots}} \mathrm{Ra}_{*m}^{\,j}(\hat{\beta}).$$

$\square$

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