Journal of Algebra 438 (2015) 85–89

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

A restriction theorem for stable rank two vector bundles on $\mathbb{P}^3$

Philippe Ellia [a,*], Laurent Gruson [b]

[a] Dipartimento di Matematica, 35 via Machiavelli, 44100 Ferrara, Italy
[b] L.M.V., Université de Versailles-St Quentin, 45 Av. des Etats-Unis, 78035 Versailles, France

ARTICLE INFO

Article history:
Received 4 September 2014
Available online 19 May 2015
Communicated by Steven Dale Cutkosky

MSC:
14J60

Keywords:
Rank two vector bundles
Projective space
Restriction
Plane

ABSTRACT

Let E be a normalized, rank two vector bundle on $\mathbb{P}^3$. Let H be a general plane. If E is stable with $c_2 \geq 4$, we show that $h^0(E_H(1)) \leq 2+c_1$. It follows that $h^0(E(1)) \leq 2+c_1$. We also show that if E is properly semi-stable and indecomposable, $h^0(E_H(1)) = 3$.

1. Introduction

We work over an algebraically closed field of characteristic zero. Let E denote a stable, normalized ($-1 \leq c_1(E) \leq 0$) rank two vector bundle on $\mathbb{P}^3$. By Barth's restriction theorem [1] if H is a general plane, then E_H is stable (i.e. $h^0(E_H) = 0$) except if E is a null-correlation bundle ($c_1 = 0$, $c_2 = 1$). In this note we prove:

* Corresponding author.
E-mail addresses: phe@unife.it (Ph. Ellia), Laurent.Gruson@uvsq.fr (L. Gruson).

http://dx.doi.org/10.1016/j.jalgebra.2015.05.009
0021-8693/

Theorem 1. *Let E be a stable, normalized, rank two vector bundle on $\mathbb{P}^3$. Assume $c_2(E) \geq 4$. Let H be a general plane, then:*

(a) $h^0(E_H(1)) \leq 1$ *if* $c_1 = -1$ *and*
(b) $h^0(E_H(1)) \leq 2$ *if* $c_1 = 0$.

In particular it follows that $h^0(E(1)) \leq 2 + c_1$.

The idea of the proof is as follows: if the theorem is not true then every general plane contains a unique line, L, such that E_L has splitting type $(r, -r + c_1)$, $r \geq c_2 - 1$. We call such a line a "super-jumping line". Then we show that these super jumping lines are all contained in a same plane, H. The plane H is very unstable for E. Performing a reduction step with H, we get a contradiction.

We observe (Remark 6) that the assumptions (and conclusions) of the theorem are sharp.

For sake of completeness we show (Proposition 7) that if E is properly semi-stable, indecomposable, then $h^0(E_H(1)) = 3$ for H a general plane.

2. Proof of the theorem

We need some definitions:

Definition 2. Let E be a stable, normalized rank two vector bundle on $\mathbb{P}^3$. A plane H is *stable* if E_H is stable; it is *semi-stable* if $h^0(E_H) \neq 0$ but $h^0(E_H(-1)) = 0$. A plane is *special* if $h^0(E_H(-m)) \neq 0$ with $m > 1$.

A line is general if the splitting type of E_L is $(0, c_1)$. A line L is a *super jumping line* (s.j.l.) if the splitting type of E_L is $(r, -r + c_1)$, with $r \geq c_2 - 1$.

Lemma 3. *Let E be a stable, normalized rank two vector bundle on $\mathbb{P}^3$. Assume $c_2(E) \geq 4$ and $h^0(E_H(1)) > 2 + c_1$ if H is a general plane. Then:*

(i) *Every stable plane contains a unique s.j.l., all the other lines are general or, if $c_1 = 0$, of type $(1, -1)$.*
(ii) *A semi-stable plane contains at most one s.j.l.*
(iii) *There is at most one special plane.*

Proof. (i) If H is a stable plane every section of $E_H(1)$ vanishes in codimension two:

$$0 \to \mathcal{O}_H \to E_H(1) \to \mathcal{I}_{Z,H}(2 + c_1) \to 0 \qquad (*)$$

We have $h^0(\mathcal{I}_{Z,H}(2 + c_1)) \geq 2 + c_1$ by the assumption. If $c_1 = -1$, Z has degree c_2 and is contained in a line L_H. If $c_1 = 0$, we have $h^0(\mathcal{I}_{Z,H}(2)) \geq 2$. Since $\deg(Z) = c_2 + 1 > 4$, the conics have a fixed line, L_H, and there is left a pencil of lines to contain the residual scheme of Z with respect to L_H. It follows that the residual scheme is one point and that $length(Z \cap L_H) = c_2$. So in both cases there is a line, L_H, containing a subscheme of Z of length c_2. Restricting (∗) to L_H we get $E_{L_H} \to \mathcal{O}_{L_H}(1 + c_1 - c_2)$. It follows that the splitting type of E_{L_H} is $(c_2 - 1, c_1 - c_2 + 1)$, hence L_H is a s.j.l. If $L \neq L_H$ is another line in H, let s be the length of $L \cap Z$. Restricting (∗) to L we get

$$0 \to \mathcal{O}_L(s-1) \to E_L \to \mathcal{O}_L(c_1 - s + 1) \to 0$$

This sequence splits except maybe if $c_1 = s = 0$ (in this case the splitting type is $(0, 0)$ or $(1, -1)$). If L is a s.j.l. then $s \geq c_2$, hence $L = L_H$. This shows that a stable plane contains a unique s.j.l. Since $s \leq 1$ (resp. $s \leq 2$) if $c_1 = -1$ (resp. $c_1 = 0$), a line different from L_H is general or has splitting type $(1, -1)$.

(ii) If H is semi-stable then we have

$$0 \to \mathcal{O}_H \to E_H \to \mathcal{I}_{T,H}(c_1) \to 0 \qquad (**)$$

Here $\deg(T) = c_2$. If L is a line in H let s denote the length of $L \cap T$. From (∗∗) we get $0 \to \mathcal{O}_L(s) \to E_L \to \mathcal{O}_L(c_1 - s) \to 0$. This sequence splits, so the splitting type of E_L is $(s, -s + c_1)$. If L is a s.j.l. then $s \geq c_2 - 1$ and L contains a subscheme of length at least $\deg(T) - 1$ of T. Since $c_2 \geq 4$, such a s.j.l. is uniquely defined. This shows that an unstable plane contains at most one s.j.l.

(iii) We may assume $h^0(E_H(-m-1)) = 0$. We have

$$0 \to \mathcal{O}_H \to E_H(-m) \to \mathcal{I}_{X,H}(c_1 - 2m) \to 0 \qquad (***)$$

If L is a general line of H ($L \cap X = \emptyset$) then E_L has splitting type $(m, -m + c_1)$, with $m > 1$.

Let's show that such a special plane, if it exists, is unique. Assume H_1, H_2 are two special planes. Let H be a general stable plane. If $L_i = H \cap H_i$, then L_1, L_2 are two lines of H with splitting type $(k_i, -k_i + c_1)$, $k_i > 1$. By (i) this is impossible. □

Remark 4. The referee points out that part (ii) of Lemma 3 follows also from Lemma 4 of [3].

We are ready for the proof of the theorem.

Proof of Theorem 1. Let $U \subset \mathbb{P}_3^*$ be the dense open subset of stable planes. We have a map $\varphi : U \to G(1, 3)$ defined by $\varphi(H) = L_H$ where L_H is the unique s.j.l. contained in H. So φ gives a rational map $\varphi : \mathbb{P}_3^* ---> G(1, 3)$. We claim that φ doesn't extend as a morphism to $\mathbb{P}_3^*$. Indeed in the contrary case we would have a section of the incidence variety $I = \{(H, L) \mid L \subset H\} \to \mathbb{P}_3^*$. Since $I \simeq Proj(\Omega_{\mathbb{P}_3^*}(1))$ (indeed the fiber at H of $\Omega_{\mathbb{P}_3^*}(1)$ is the hyperplane corresponding to H), such a section corresponds to an injective morphism of vector bundles $\mathcal{O}_{\mathbb{P}_3^*} \hookrightarrow T_{\mathbb{P}_3^*}(k)$, for some k. But there is no twist of $T_{\mathbb{P}_3^*}$ with a non-vanishing section. This can be seen by looking at $c_3(T_{\mathbb{P}_3^*}(k))$ or with the following argument: the quotient would be a rank two vector bundle with $H^1_* = 0$, hence, by Horrocks' theorem, a direct sum of line bundles which is absurd.

If H is a singular point of the "true" rational map φ, then, by Zariski's Main Theorem, H contains infinitely many s.j.l. This implies that H is the unique special plane (and that φ has a single singular point). We claim that every s.j.l. is contained in H. Indeed let R be a s.j.l. not contained in H. Let $z = R \cap H$. There exists a s.j.l. $L \subset H$ through z. The plane $\langle R, L\rangle$ contains two s.j.l. hence it is special: a contradiction.

Since there are ∞^2 s.j.l. we conclude that the general splitting type on the special plane H is $(c_2 - 1, -c_2 + c_1 + 1)$. So $m = c_2 - 1$ i.e. $h^0(E_H(-c_2 + 1)) \neq 0$ (and this is the least twist having a section). Now we perform a reduction step (see [8, Proposition 9.1]).

If $c_1 = 0$ we get

$$0 \to E' \to E \to \mathcal{I}_{W,H}(-c_2 + 1) \to 0$$

where E' is a rank two reflexive sheaf with Chern classes $c_1' = -1$, $c_2' = 1$, $c_3' = c_2^2 - c_2 + 1$. Since E is stable, E' is stable too. By [8, Theorem 8.2] we get a contradiction.

If $c_1 = -1$, since $E_H^* = E_H(1)$ we get

$$0 \to E'(-1) \to E \to \mathcal{I}_{R,H}(-c_2) \to 0$$

where the Chern classes of E' are $c_1' = 0$, $c_2' = 0$, $c_3' = c_2^2$. Since E is stable E' is semi-stable. By [8, Theorem 8.2] we get, again, a contradiction. □

Remark 5. The argument to show that φ doesn't extend to a morphism is taken from [6]. Another way to prove this is to consider the surfaces S_L defined in the following way: if L is a general line every plane through L is (semi-)stable, the general one being stable. So almost every plane of the pencil contains a unique s.j.l. taking the closure yields a ruled surface S_L. Then one shows that $S_L \neq S_D$ if L, D are general and then concludes by looking at $S_L \cap S_D$ (see [5]).

Remark 6. The assumption $c_2 \geq 4$ cannot be weakened in Theorem 1. If $c_1 = -1$ every stable rank vector bundle, E, with $c_1 = -1$, $c_2 = 2$ is such that $h^0(E_H(1)) = 2$ for a general plane H (see [9]). If $E(1)$ is associated to four skew lines, then $h^0(E_H(1)) = 3$ for H general and $c_i(E) = (0, 3)$.

On the other hand a special t'Hooft bundle ($E(1)$ associated to $c_2 + 1$ disjoint lines on a quadric) is stable with $c_1(E) = 0$ and, if $c_2 \geq 4$, satisfies $h^0(E_H(1)) = 2$ for H general.

By the way, Theorem 1 gives back $h^0(E(1)) \leq 2$ for an instanton, a result first proved by Boehmer and Trautmann (see [2] and also [10]).

Finally let $E(1)$ be associated to the disjoint union of $c_2/2$ double lines of arithmetic genus -2. Then E is stable with $c_1 = -1$ and, if $c_2 > 2$, $h^0(E_H(1)) = 1$ for H general.

Concerning properly semi-stable bundles ($c_1(E) = 0$, $h^0(E) \neq 0$, $h^0(E(-1)) = 0$) we have:

Proposition 7. *Let E be a properly semi-stable rank two vector bundle on $\mathbb{P}^3$. Assume E is indecomposable. If H is a general plane then $h^0(E_H(1)) = 3$.*

Proof. We have $0 \to \mathcal{O} \to E \to \mathcal{I}_C \to 0$, where C is a curve (E doesn't split) with $\omega_C(4) \simeq \mathcal{O}_C$. Twisting and restricting to a general plane: $0 \to \mathcal{O}_H(1) \to E_H(1) \to \mathcal{I}_{C\cap H,H}(1) \to 0$. If $h^0(\mathcal{I}_{C\cap H,H}(1)) \neq 0$ it follows from a theorem of Strano [11,4] that C is a plane curve, but this is impossible ($\omega_C(4) \not\simeq \mathcal{O}_C$ for a plane curve). □

Remark 8. To apply Strano's theorem we need $ch(k) = 0$ (see [7]). The previous argument gives a quick proof of Theorem 1 in case $c_1 = -1$, $h^0(E(1)) \neq 0$. In fact this remark has been the starting point of this note.

Remark 9. Let C be a plane curve of degree d. A non-zero section of $\omega_C(3) \simeq \mathcal{O}_C(d)$ yields $0 \to \mathcal{O} \to \mathcal{F}(1) \to \mathcal{I}_C(1) \to 0$, where $\mathcal{F}$ is a stable rank two reflexive sheaf with Chern classes $(-1, d, d^2)$. If H is a general plane, $h^0(\mathcal{F}_H(1)) = 2$ if $d > 1$ (resp. 3 if $d = 1$). Similarly, considering the disjoint union of a plane curve and of a line, we get stable reflexive sheaf with $c_1(\mathcal{F}) = 0$ and $h^0(\mathcal{F}_H(1)) = 3$. So Theorem 1 doesn't hold for stable reflexive sheaves. The interested reader can try to classify the exceptions.

References

[1] W. Barth, Some properties of stable rank-2 vector bundles on $\mathbb{P}^n$, Math. Ann. 226 (1977) 125–150.

[2] W. Boehmer, G. Trautmann, Special instanton bundles and Poncelet curves, in: Lecture Notes in Math., vol. 1273, Springer, Berlin, 1987, pp. 325–336.

[3] I. Coanda, On Barth's restriction theorem, J. Reine Angew. Math. 428 (1992) 97–110.

[4] Ph. Ellia, Sur les lacunes d'Halphen, in: Lecture Notes in Math., vol. 1389, Springer, 1989, pp. 43–65.

[5] Ph. Ellia, Some vanishing for the cohomology of stable rank two vector bundles on $\mathbb{P}^3$, J. Reine Angew. Math. 451 (1981) 1–14.

[6] D. Franco, Some vanishings for the cohomology of rank 2 stable reflexive sheaves on $\mathbb{P}^3$ and for their restriction to a general plane, Comm. Algebra 23 (6) (1995) 2281–2289.

[7] R. Hartshorne, The genus of space curves, Ann. Univ. Ferrara Sez. VII Sci. Mat. XL (1994) 207–223.

[8] R. Hartshorne, Stable reflexive sheaves, Math. Ann. 254 (1980) 121–176.

[9] R. Hartshorne, I. Sols, Stable rank two vector bundles on $\mathbb{P}^3$ with $c_1 = -1$, $c_2 = 2$, J. Reine Angew. Math. 325 (1981) 145–152.

[10] T. Nuessler, G. Trautmann, Multiple Koszul structures and instanton bundles, Internat. J. Math. 5 (3) (1994) 373–388.

[11] R. Strano, A characterization of complete intersection curves in $\mathbb{P}^3$, Proc. Amer. Math. Soc. 104 (1988) 711–715.