REGULARITY OF PARTIAL DIFFERENTIAL OPERATORS IN ULTRADIFFERENTIABLE SPACES AND WIGNER TYPE TRANSFORMS

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Abstract. We study the behaviour of linear partial differential operators with polynomial coefficients via a Wigner type transform. In particular, we obtain some results of regularity in the Schwartz space $\mathcal{S}$ and in the space $\mathcal{S}_\omega$ as introduced by Björck for weight functions ω. Several examples are discussed in this new setting.

1. Introduction

In this paper we are concerned with the regularity of linear partial differential operators with polynomial coefficients. This problem was introduced by Shubin [23], who says that a linear operator $A : \mathcal{S}' \to \mathcal{S}'$ is regular if the conditions $u \in \mathcal{S}'$, $Au \in \mathcal{S}$ imply that $u \in \mathcal{S}$. In [23, Chapter IV] global pseudodifferential operators on $\mathbb{R}^n$ are studied, giving a notion of (global) hypoellipticity (see formula (5.1)), that implies the above mentioned regularity in Schwartz spaces. Such global pseudodifferential operators are defined by treating in the same way variables and covariables, and have as basic examples linear partial differential operators with polynomial coefficients. The global hypoellipticity, on the other hand, is far from being a necessary condition for the regularity of an operator; some results have been obtained in this direction, we refer in particular to [25] who proved the regularity of the Twisted Laplacian (a non hypoelliptic operator in two variables), and to [21], who gave a characterization of the regularity of ordinary differential operators in the case when the roots of the corresponding Weyl symbol are suitably separated at infinity. Moreover, in [9] a class of non hypoelliptic regular partial differential operators with polynomial coefficients have been found, by using a technique related to transformations of Wigner type; such class includes as a particular case the Twisted Laplacian. The idea to use quadratic transformations for the study of general properties of partial differential equations (that underlies [9], as well as the present paper) goes back to some works related to engineering applications, cf. [13], [15], where the main aim is to understand the Wigner transform of the solution of a partial differential equation without finding the solution itself; the ideas of [13], [15] are developed and organically presented in [12]. In the present paper we study the regularity of linear partial differential operators, in the spirit of [9], developing the research in two directions; first, we consider a general representation in the Cohen class, defined as

$$Q[w] := \sigma * \mathrm{Wig}[w]$$

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for a kernel $\sigma \in \mathcal{S}'$, where $\mathrm{Wig}[w]$ is the Wigner transform, defined as

$$\mathrm{Wig}[w](x,y) := \int e^{-ity} w\left(x + \frac{1}{2}t, x - \frac{1}{2}t\right) dt.$$

The idea is that a linear partial differential operator B with polynomial coefficients is transformed into another one by a formula of the kind

$$Q[Bw] = \tilde{B}Q[w];$$

moreover, under suitable hypotheses on the kernel σ, the regularity is preserved by such transformation, and if we start from a global hypoelliptic operator B we find in general a non-global hypoelliptic operator $\tilde{B}$. Then, we can construct a large class of partial differential operators that are regular but not globally hypoelliptic. We also study regularity and the results just mentioned for the class $\mathcal{S}_\omega$ for a weight function ω, as introduced by Björck [2] (see also [14] for non subadditive weight functions), which gives a large scale of examples, working in particular for Gevrey weight functions. This requires a preliminary study of the Schwartz ultradifferentiable space $\mathcal{S}_\omega$ and of the Cohen class representation Q in $\mathcal{S}_\omega$ and $\mathcal{S}'_\omega$. In particular, we give a characterization of the spaces $\mathcal{S}_\omega$, improving a result of [11], introducing a new kind of seminorms in the spirit of the spaces of ultradifferentiable functions introduced by Braun, Meise and Taylor [8] (compare with Langenbruch [20]).

The examples that we can construct with our technique are quite general, we mention here some cases. We show for example that, if b is a polynomial in one variable that never vanishes, and $P(D_x, D_y)$ is an arbitrary partial differential operator with constant real coefficients, then the operator

$$b(x + P(D_x, D_y))$$

in $\mathbb{R}^2$ is regular in the sense of Shubin and in the sense of ultradifferentiable classes $\mathcal{S}_\omega$. The same is true for the operator in two variables

$$(x - D_y + Q(D_x))^2 + (y + R(D_y))^2,$$

for arbitrary ordinary differential operators $Q(D_x)$ and $R(D_y)$ with constant real coefficients. Observe in particular that the regularity here does not depend on the higher order terms, since the operators P, Q, R can have arbitrary order.

The paper is organized as follows. Section 2 is devoted to the study of some properties of the Wigner transform in $\mathcal{S}$, that we use in the following; in Sections 3 and 4 we study the global regularity through Cohen class representations in $\mathcal{S}$ and $\mathcal{S}_\omega$, respectively; finally, in the last section we analyze some examples. The results are proved in the case of dimension 2, for sake of simplicity, but they could easily be generalized to higher even dimension.

2. Some properties of the Wigner transform on $\mathcal{S}$

Let us define, following [9], the Wigner-like transform of a function $w \in \mathcal{S}(\mathbb{R}^2)$, by

$$\mathrm{Wig}[w](x,y) := \int e^{-ity} w\Big(x + \frac{1}{2}t, x - \frac{1}{2}t\Big) dt. \tag{2.1}$$

In this way

$$\begin{aligned} \mathrm{Wig}:\ & \mathcal{S} \to \mathcal{S} \\ \mathrm{Wig}:\ & \mathcal{S}' \to \mathcal{S}' \end{aligned}$$

is invertible, since it is the composition of a linear invertible change of variables and a partial Fourier transform, i.e.

$$\mathrm{Wig}[w](x,y) = \mathcal{F}_t\left(Tw(x,t)\right)(x,y) \tag{2.2}$$

with

$$Tw(x,t) = w\Big(x+\frac{1}{2}t, x-\frac{1}{2}t\Big), \qquad \mathcal{F}(f)(y) = \mathcal{F}_t(f(t))(y) = \int e^{-ity}f(t)dt. \tag{2.3}$$

Denote, as in [9],

$$\begin{aligned} M_1w(x,y) &= xw(x,y), & M_2w(x,y) &= yw(x,y), \\ D_1w(x,y) &= D_xw(x,y), & D_2w(x,y) &= D_yw(x,y), \end{aligned}$$

with $D_x = -i\partial_x$, $D_y = -i\partial_y$, and recall, from [9], the following properties:

$$D_1\,\mathrm{Wig}[w] = \mathrm{Wig}[(D_1+D_2)w] \tag{2.4}$$

$$D_2\,\mathrm{Wig}[w] = \mathrm{Wig}[(M_2-M_1)w] \tag{2.5}$$

$$M_1\,\mathrm{Wig}[w] = \mathrm{Wig}\left[\frac{1}{2}(M_2+M_1)w\right] \tag{2.6}$$

$$M_2\,\mathrm{Wig}[w] = \mathrm{Wig}\left[\frac{1}{2}(D_1-D_2)w\right]. \tag{2.7}$$

More generally, let $P(D_1,D_2) = \sum_{|(h,k)|\leq m} a_{hk}D_x^hD_y^k$ be a linear partial differential operator with constant coefficients and denote by

$$P(D_1+D_2, M_2-M_1) = \sum_{|(h,k)|\leq m} a_{hk}(D_1+D_2)^h(M_2-M_1)^k, \tag{2.8}$$

which is a linear partial differential operator with polynomial coefficients.

Note that

$$(D_1+D_2)^h(M_2-M_1)^k = (M_2-M_1)^k(D_1+D_2)^h \tag{2.9}$$

since

$$\begin{aligned} (D_1+D_2)(M_2-M_1)w =\ & D_1M_2w - D_1M_1w + D_2M_2w - D_2M_1w \\ =\ & M_2D_1w + iw - M_1D_1w - iw + M_2D_2w - M_1D_2w \\ =\ & (M_2-M_1)D_1w + (M_2-M_1)D_2w \\ =\ & (M_2-M_1)(D_1+D_2)w. \end{aligned}$$

We have the following

Lemma 2.1. *Let $P(D_x,D_y)$ be a linear partial differential operator with constant coefficients. Then, for every $w \in \mathcal{S}(\mathbb{R}^2)$,*

$$P(D_1,D_2)\,\mathrm{Wig}[w](x,y) = \mathrm{Wig}[P(D_1+D_2, M_2-M_1)w](x,y). \tag{2.10}$$

Proof. Let us prove by induction on $h \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ that

$$D_1^h D_2^k \operatorname{Wig}[w] = \operatorname{Wig}[(D_1 + D_2)^h (M_2 - M_1)^k w] \tag{2.11}$$

for all $k \in \mathbb{N}_0$.

Indeed, for $h = 0$ by (2.5) we have that

$$D_2^k \operatorname{Wig}[w] = \operatorname{Wig}[(M_2 - M_1)^k w].$$

Let us assume (2.11) to be true for h and prove it for $h+1$. By the inductive assumption (2.11) and (2.4):

$$\begin{aligned} D_1^{h+1} D_2^k \operatorname{Wig}[w] &= D_1 \operatorname{Wig}[(D_1 + D_2)^h (M_2 - M_1)^k w] \\ &= \operatorname{Wig}[(D_1 + D_2)^{h+1} (M_2 - M_1)^k w]. \end{aligned}$$

Moreover, since $\operatorname{Wig}[w] \in C^2(\mathbb{R}^2)$ we have that

$$D_1^h D_2^k \operatorname{Wig}[w] = D_2^k D_1^h \operatorname{Wig}[w] = \operatorname{Wig}[(D_1 + D_2)^h (M_2 - M_1)^k w]. \tag{2.12}$$

The thesis then follows from (2.12) and the definition of P. □

Analogous formulas hold for linear partial differential operators with polynomial coefficients:

Proposition 2.2. *Let $P(x, y, D_x, D_y)$ be a linear partial differential operator with polynomial coefficients. Then, for all $w \in \mathcal{S}(\mathbb{R}^2)$, the following formula holds:*

$$P(M_1, M_2, D_1, D_2) \operatorname{Wig}[w] = \operatorname{Wig}\left[P\left(\frac{1}{2}(M_2 + M_1), \frac{1}{2}(D_1 - D_2), D_1 + D_2, M_2 - M_1\right) w\right].$$

Proof. From Lemma 2.1 and (2.6),(2.7) we have that

$$M_1^m M_2^n D_1^h D_2^k \operatorname{Wig}[w] = \operatorname{Wig}\left[\frac{1}{2^{n+m}}(M_2 + M_1)^m (D_1 - D_2)^n (M_2 - M_1)^k (D_1 + D_2)^h w\right]$$

and hence the thesis, since $M_1^m M_2^n = M_2^n M_1^m$.

Note that, analogously to (2.9), we have that

$$(D_1 - D_2)^n (M_2 + M_1)^m = (M_2 + M_1)^m (D_1 - D_2)^n.$$

□

3. Properties and regularity of time-frequency representations in the Cohen's class with kernel in $\mathcal{S}'$

Let us now consider a time-frequency representation $Q[w]$ in the Cohen's class, i.e. of the form

$$Q[w] := \sigma * \operatorname{Wig}[w]$$

for $w \in \mathcal{S}(\mathbb{R}^2)$ and $\sigma \in \mathcal{S}'(\mathbb{R}^2)$.

By (2.4), (2.5) we have that

$$D_1 Q[w] = \sigma * D_1 \operatorname{Wig}[w] = Q[(D_1 + D_2)w] \tag{3.1}$$

$$D_2 Q[w] = \sigma * D_2 \operatorname{Wig}[w] = Q[(M_2 - M_1)w]. \tag{3.2}$$

Moreover, let us prove that

$$M_1Q[w] = Q\left[\frac{1}{2}(M_2 + M_1)w\right] + (M_1\sigma) * \mathrm{Wig}[w] \tag{3.3}$$

$$M_2Q[w] = Q\left[\frac{1}{2}(D_1 - D_2)w\right] + (M_2\sigma) * \mathrm{Wig}[w]. \tag{3.4}$$

Indeed, from (2.6) and (2.7):

$$\begin{aligned}
M_1Q[w](x,y) = & \int x\sigma(\alpha,\beta)\,\mathrm{Wig}[w](x-\alpha,y-\beta)d\alpha d\beta \\
= & \int \sigma(\alpha,\beta)(x-\alpha)\,\mathrm{Wig}[w](x-\alpha,y-\beta)d\alpha d\beta \\
& + \int \alpha\sigma(\alpha,\beta)\,\mathrm{Wig}[w](x-\alpha,y-\beta)d\alpha d\beta \\
= & \ \sigma * (M_1\,\mathrm{Wig}[w]) + (M_1\sigma) * \mathrm{Wig}[w] \\
= & \ \sigma * \mathrm{Wig}\left[\frac{1}{2}(M_2 + M_1)w\right] + (M_1\sigma) * \mathrm{Wig}[w]
\end{aligned}$$

and analogously

$$\begin{aligned}
M_2Q[w](x,y) = & \int y\sigma(\alpha,\beta)\,\mathrm{Wig}[w](x-\alpha,y-\beta)d\alpha d\beta \\
= & \ \sigma * (M_2\,\mathrm{Wig}[w]) + (M_2\sigma) * \mathrm{Wig}[w] \\
= & \ \sigma * \mathrm{Wig}\left[\frac{1}{2}(D_1 - D_2)w\right] + (M_2\sigma) * \mathrm{Wig}[w],
\end{aligned}$$

where the integrals are intended as the action of the distribution σ when σ is not a function.

In order to write also (3.3) and (3.4) in terms of Q applied to some $\tilde{P}(M_1, M_2, D_1, D_2)w$, for a linear partial differential operator $\tilde{P}$ with polynomial coefficients, we now choose $\sigma(\alpha,\beta)$ so that

$$\begin{cases} M_1\sigma(\alpha,\beta) = \alpha\sigma(\alpha,\beta) = P_1(D_\alpha, D_\beta)\sigma(\alpha,\beta) \\ M_2\sigma(\alpha,\beta) = \beta\sigma(\alpha,\beta) = P_2(D_\alpha, D_\beta)\sigma(\alpha,\beta) \end{cases} \tag{3.5}$$

for some linear partial differential operators P_1, P_2 with constant coefficients.

Let us solve (3.5) by Fourier transform:

$$\begin{cases} P_1(\xi,\eta)\widehat{\sigma}(\xi,\eta) = i\partial_\xi\widehat{\sigma}(\xi,\eta) \\ P_2(\xi,\eta)\widehat{\sigma}(\xi,\eta) = i\partial_\eta\widehat{\sigma}(\xi,\eta). \end{cases} \tag{3.6}$$

By simple computations, chosen any given real valued polynomial $P(\xi,\eta) \in \mathbb{R}[\xi,\eta]$, we can thus set

$$P_1(\xi,\eta) = \partial_\xi P(\xi,\eta), \qquad P_2(\xi,\eta) = \partial_\eta P(\xi,\eta)$$

and obtain that

$$\widehat{\sigma}(\xi,\eta) = e^{-iP(\xi,\eta)} \in \mathcal{S}'(\mathbb{R}^2) \tag{3.7}$$

solves (3.6) (note that $|\widehat{\sigma}| = 1$). Since the Fourier transform $\mathcal{F}: \mathcal{S}' \to \mathcal{S}'$ is invertible, we have that

$$\sigma(\alpha,\beta) = \mathcal{F}^{-1}\left(e^{-iP(\xi,\eta)}\right) \in \mathcal{S}'(\mathbb{R}^2) \tag{3.8}$$

solves (3.5).

For such a choice of σ, substituting in (3.3), by Lemma 2.1 we get:

$$\begin{aligned}
M_1 Q[w] &= Q\left[\frac{1}{2}(M_2+M_1)w\right] + P_1(D_1,D_2)\sigma * \mathrm{Wig}[w] && (3.9)\\
&= Q\left[\frac{1}{2}(M_2+M_1)w\right] + \sigma * P_1(D_1,D_2)\,\mathrm{Wig}[w]\\
&= Q\left[\frac{1}{2}(M_2+M_1)w\right] + \sigma * \mathrm{Wig}[P_1(D_1+D_2, M_2-M_1)w]\\
&= Q\left[\left(\frac{1}{2}(M_2+M_1) + P_1(D_1+D_2, M_2-M_1)\right)w\right]\\
&= Q\left[\left(\frac{1}{2}(M_2+M_1) + (iD_1P)(D_1+D_2, M_2-M_1)\right)w\right]. && (3.10)
\end{aligned}$$

Analogously, from (3.4):

$$\begin{aligned}
M_2 Q[w] &= Q\left[\frac{1}{2}(D_1-D_2)w\right] + P_2(D_1,D_2)\sigma * \mathrm{Wig}[w] && (3.11)\\
&= Q\left[\left(\frac{1}{2}(D_1-D_2) + P_2(D_1+D_2, M_2-M_1)\right)w\right]\\
&= Q\left[\left(\frac{1}{2}(D_1-D_2) + (iD_2P)(D_1+D_2, M_2-M_1)\right)w\right]. && (3.12)
\end{aligned}$$

Iterating this procedure we get the following:

Theorem 3.1. *Let $B(x,y,D_x,D_y)$ be a linear partial differential operator with polynomial coefficients and let $\sigma = \mathcal{F}^{-1}(e^{-iP(\xi,\eta)}) \in \mathcal{S}'(\mathbb{R}^2)$ for some $P \in \mathbb{R}[\xi,\eta]$. Then, for every $w \in \mathcal{S}(\mathbb{R}^2)$, the time-frequency representation $Q[w] = \sigma * \mathrm{Wig}[w]$ satisfies:*

$$B(M_1,M_2,D_1,D_2)Q[w] = Q[\bar{B}(M_1,M_2,D_1,D_2)w], \tag{3.13}$$

where $\bar{B}$ is the linear partial differential operator with polynomial coefficients defined by

$$\bar{B}(M_1,M_2,D_1,D_2) := B\left(\frac{M_2+M_1}{2} + P_1^*, \frac{D_1-D_2}{2} + P_2^*, D_1+D_2, M_2-M_1\right), \tag{3.14}$$

with

$$P_1^* = (iD_1P)(D_1+D_2, M_2-M_1), \qquad P_2^* = (iD_2P)(D_1+D_2, M_2-M_1).$$

Proof. By (3.1), (3.2) and (2.9) we immediately get

$$D_1^h D_2^k Q[w] = D_2^k D_1^h Q[w] = Q[(D_1+D_2)^h (M_2-M_1)^k w]. \tag{3.15}$$

Let us prove by induction on $m \in \mathbb{N}_0$ that

$$M_1^m Q[w] = Q\left[\left(\frac{M_2+M_1}{2}+P_1^*\right)^m w\right]. \tag{3.16}$$

Indeed, for $m=0$ there is nothing to prove. Let us assume (3.16) true for m and prove it for $m+1$. By the inductive assumption (3.16) and by (3.10) we have that

$$\begin{aligned} M_1^{m+1}Q[w] =&\ M_1 Q\left[\left(\frac{M_2+M_1}{2}+P_1^*\right)^m w\right]\\ =&\ Q\left[\left(\frac{M_2+M_1}{2}+P_1^*\right)\left(\frac{M_2+M_1}{2}+P_1^*\right)^m w\right]\\ =&\ Q\left[\left(\frac{M_2+M_1}{2}+P_1^*\right)^{m+1} w\right]. \end{aligned}$$

Analogously, by (3.12) we can prove by induction on $n \in \mathbb{N}_0$ that

$$M_2^n Q[w] = Q\left[\left(\frac{D_1-D_2}{2}+P_2^*\right)^n w\right]. \tag{3.17}$$

The thesis then follows from (3.15), (3.16) and (3.17). □

Reciprocally, we have the following:

Theorem 3.2. *Let $B(x,y,D_x,D_y)$ be a linear partial differential operator with polynomial coefficients and let $\sigma = \mathcal{F}^{-1}(e^{-iP(\xi,\eta)}) \in \mathcal{S}'(\mathbb{R}^2)$ for some $P \in \mathbb{R}[\xi,\eta]$. Then, for every $w \in \mathcal{S}(\mathbb{R}^2)$, the time-frequency representation $Q[w] = \sigma * \mathrm{Wig}[w]$ satisfies:*

$$Q[B(M_1,M_2,D_1,D_2)w] = \tilde{B}(M_1,M_2,D_1,D_2)Q[w], \tag{3.18}$$

where $\tilde{B}$ is the linear partial differential operator with polynomial coefficients defined by

$$\begin{aligned} &\tilde{B}(M_1,M_2,D_1,D_2)\\ &= B\left(M_1-\frac{1}{2}D_2-P_1, M_1+\frac{1}{2}D_2-P_1, \frac{1}{2}D_1+M_2-P_2, \frac{1}{2}D_1-M_2+P_2\right) \end{aligned} \tag{3.19}$$

with

$$P_1 = (iD_1P)(D_1,D_2), \qquad P_2 = (iD_2P)(D_1,D_2). \tag{3.20}$$

Proof. From (3.1), (3.2), (3.9) and (3.11) we have:

$$D_1Q[w] = Q[D_1w] + Q[D_2w] \tag{3.21}$$

$$D_2Q[w] = Q[M_2w] - Q[M_1w] \tag{3.22}$$

$$M_1Q[w] = \frac{1}{2}Q[M_2w] + \frac{1}{2}Q[M_1w] + P_1Q[w] \tag{3.23}$$

$$M_2Q[w] = \frac{1}{2}Q[D_1w] - \frac{1}{2}Q[D_2w] + P_2Q[w]. \tag{3.24}$$

Therefore, from (3.21) and (3.24):

$$Q[D_1 w] = \left(\frac{1}{2}D_1 + M_2 - P_2\right) Q[w]$$
$$Q[D_2 w] = \left(\frac{1}{2}D_1 - M_2 + P_2\right) Q[w];$$

from (3.22) and (3.23):

$$Q[M_1 w] = \left(M_1 - \frac{1}{2}D_2 - P_1\right) Q[w]$$
$$Q[M_2 w] = \left(M_1 + \frac{1}{2}D_2 - P_1\right) Q[w].$$

Iterating:

$$Q[M_1^m M_2^n D_1^h D_2^k w] = \left(M_1 - \frac{1}{2}D_2 - P_1\right)^m \left(M_1 + \frac{1}{2}D_2 - P_1\right)^n \cdot \left(\frac{1}{2}D_1 + M_2 - P_2\right)^h \left(\frac{1}{2}D_1 - M_2 + P_2\right)^k Q[w]. \tag{3.25}$$

Let us remark that $\left(M_1 - \frac{1}{2}D_2 - P_1(D_1, D_2)\right)$ and $\left(M_1 + \frac{1}{2}D_2 - P_1(D_1, D_2)\right)$ commute, as also $\left(\frac{1}{2}D_1 + M_2 - P_2(D_1, D_2)\right)$ and $\left(\frac{1}{2}D_1 - M_2 + P_2(D_1, D_2)\right)$. On the other hand, M_1 and M_2 commute and also $D_1 D_2 w = D_2 D_1 w$ since $w \in C^\infty(\mathbb{R}^2)$. The thesis follows therefore from (3.25). □

In order to prove further properties of Q, let us define the space C_p^∞ of C^∞ functions with polynomial growth:

$$C_p^\infty(\mathbb{R}^n) := \{\varphi \in C^\infty(\mathbb{R}^n) : \exists N \in \mathbb{N},\, c > 0 \text{ s.t. } |\partial^\gamma \varphi(x)| \le c(1 + |x|^2)^N \ \forall x \in \mathbb{R}^n,\ \gamma \in \mathbb{N}_0^n\}.$$

The last space is included in the space of multipliers $\mathcal{O}_M(\mathbb{R}^n)$ of the space $\mathcal{S}(\mathbb{R}^n)$, i.e., the space of smooth functions F such that $F\,\mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$. Indeed, it is known that $F \in \mathcal{O}_M(\mathbb{R}^n)$ if and only if for each $k \in \mathbb{N}$ there is $C > 0$ and $j \in \mathbb{N}$ such that $|F^{(\alpha)}(x)| \le C(1 + |x|)^j$ for all multi-index α with $|\alpha| \le k$. Then, the next lemma is obvious [17, 24].

Lemma 3.3. *Let $\varphi \in C_p^\infty(\mathbb{R}^n)$. If $u \in \mathcal{S}(\mathbb{R}^n)$, then $\varphi u \in \mathcal{S}(\mathbb{R}^n)$; if $w \in \mathcal{S}'(\mathbb{R}^n)$, then $\varphi w \in \mathcal{S}'(\mathbb{R}^n)$.*

We recall the notion of regularity from [23]:

Definition 3.4. *A linear operator A on $\mathcal{S}'(\mathbb{R}^n)$ is* regular *if*

$$Au \in \mathcal{S}(\mathbb{R}^n) \quad \Rightarrow \quad u \in \mathcal{S}(\mathbb{R}^n), \qquad \forall u \in \mathcal{S}'(\mathbb{R}^n).$$

We have the following:

Lemma 3.5. *For $\sigma = \mathcal{F}^{-1}(e^{-iP(\xi,\eta)}) \in \mathcal{S}'(\mathbb{R}^2)$ with $P \in \mathbb{R}[\xi, \eta]$ and $Q[w] = \sigma * \mathrm{Wig}[w]$, we have that:*

(i) *$Q : \mathcal{S}' \to \mathcal{S}'$ is invertible;*
(ii) *Q is* regular*;*
(iii) *$Q : \mathcal{S} \to \mathcal{S}$.*

Proof. Let us first prove that if $w \in \mathcal{S}'$ then $Q[w]$ is a well defined element of $\mathcal{S}'$. As a matter of fact, $\widehat{Q[w]} = \widehat{\sigma} \cdot \widehat{\mathrm{Wig}[w]} \in \mathcal{S}'$ because of Lemma 3.3, since $\widehat{\sigma} \in C_p^\infty$ and $\widehat{\mathrm{Wig}[w]} \in \mathcal{S}'$ for $w \in \mathcal{S}'$. Then $Q[w]$ is well defined as $\mathcal{F}^{-1}\left(\widehat{\sigma} \cdot \widehat{\mathrm{Wig}[w]}\right) \in \mathcal{S}'$.

The injectivity of $Q: \mathcal{S}' \to \mathcal{S}'$ is trivial. To prove the surjectivity, take $w \in \mathcal{S}'$. Then $\widehat{w} \in \mathcal{S}'$ and, by Lemma 3.3, also $\widehat{w}/\widehat{\sigma} \in \mathcal{S}'$ since $1/\widehat{\sigma} \in C_p^\infty$. By the surjectivity of the Fourier transform there exists $\psi \in \mathcal{S}'$ such that $\widehat{w}/\widehat{\sigma} = \widehat{\psi}$. By the surjectivity of the Wigner transform, $\psi = \mathrm{Wig}[u]$ for some $u \in \mathcal{S}'$ and therefore

$$\widehat{w} = \widehat{\sigma}\widehat{\psi} = \widehat{\sigma} \cdot \widehat{\mathrm{Wig}[u]} = \widehat{\sigma * \mathrm{Wig}[u]} = \widehat{Q[u]}$$

and by the injectivity of the Fourier transform $w = Q[u]$. This proves (i).

To prove condition (ii), assume that $Q[w] \in \mathcal{S}$ for some $w \in \mathcal{S}'$. From $\widehat{Q[w]} = \widehat{\sigma} \cdot \widehat{\mathrm{Wig}[w]} \in \mathcal{S}$ we thus have that $\widehat{\mathrm{Wig}[w]} \in \mathcal{S}$ since $|\widehat{\sigma}| = 1$. Therefore $\mathrm{Wig}[w] \in \mathcal{S}$ and hence $w \in \mathcal{S}$. This proves that Q is regular.

Finally, to prove (iii) let us remark that, for $w \in \mathcal{S}$,

$$Q[w] = \sigma * \mathrm{Wig}[w] = \mathcal{F}^{-1}\left(\widehat{\sigma} \cdot \widehat{\mathrm{Wig}[w]}\right) \in \mathcal{S}$$

because of Lemma 3.3, since $\widehat{\sigma} \in C_p^\infty$ and $\widehat{\mathrm{Wig}[w]} \in \mathcal{S}$ for $w \in \mathcal{S}$. □

Theorem 3.6. *Let $B(x, y, D_x, D_y)$ be a linear partial differential operator with polynomial coefficients and let $\sigma = \mathcal{F}^{-1}(e^{-iP(\xi,\eta)}) \in \mathcal{S}'(\mathbb{R}^2)$ for some $P \in \mathbb{R}[\xi, \eta]$.*

If B is regular and $\bar{B}$ is defined by (3.14)*, then also $\bar{B}$ is regular.*

Proof. Let us assume that $\bar{B}w \in \mathcal{S}$ for $w \in \mathcal{S}'$ and prove that $w \in \mathcal{S}$.

Indeed, $Q[w] = \sigma * \mathrm{Wig}[w] \in \mathcal{S}'$ by Lemma 3.5 (i) and, by Theorem 3.1 and Lemma 3.5 (iii), we get that $BQ[w] = Q[\bar{B}w] \in \mathcal{S}$. Since B is regular by assumption, we have that $Q[w] \in \mathcal{S}$ and hence $w \in \mathcal{S}$ by the regularity of Q from Lemma 3.5 (ii). □

Theorem 3.7. *Let $B(x, y, D_x, D_y)$ be a linear partial differential operator with polynomial coefficients and let $\sigma = \mathcal{F}^{-1}(e^{-iP(\xi,\eta)}) \in \mathcal{S}'(\mathbb{R}^2)$ for some $P \in \mathbb{R}[\xi, \eta]$.*

If B is regular and $\tilde{B}$ is defined by (3.19)*, then also $\tilde{B}$ is regular.*

Proof. Let us assume $\tilde{B}u \in \mathcal{S}$ for $u \in \mathcal{S}'$ and prove that $u \in \mathcal{S}$.

Indeed, by the surjectivity of $Q[w] = \sigma * \mathrm{Wig}[w]$ (cf. Lemma 3.5 (i)), there exists $w \in \mathcal{S}'$ such that $u = Q[w]$ and hence, from Theorem 3.2,

$$Q[Bw] = \tilde{B}Q[w] = \tilde{B}u \in \mathcal{S}.$$

By the regularity of Q (cf. Lemma 3.5 (ii)) we have that $Bw \in \mathcal{S}$ and hence $w \in \mathcal{S}$ by the regularity of B. Then also $u = Q[w] \in \mathcal{S}$ by Lemma 3.5 (iii). □

Let us now consider

$$\widehat{\sigma}_1(\xi, \eta) = q(\xi, \eta)\widehat{\sigma}(\xi, \eta) = q(\xi, \eta)e^{-iP(\xi,\eta)}, \tag{3.26}$$

where σ is defined by (3.8) for $P(\xi, \eta) \in \mathbb{R}[\xi, \eta]$ and $q(\xi, \eta) \in \mathbb{C}[\xi, \eta]$ is a polynomial that never vanishes on $\mathbb{R}^2$. Then

$$\sigma_1(x, y) = q(D_x, D_y)\sigma(x, y)$$

and, by Lemma 2.1:

$$\begin{aligned} Q^{(\sigma_1)}[w] := {} & \sigma_1 * \mathrm{Wig}[w] = (q(D_1,D_2)\sigma) * \mathrm{Wig}[w] \\ = {} & \sigma * (q(D_1,D_2)\,\mathrm{Wig}[w]) \\ = {} & \sigma * \mathrm{Wig}[q(D_1+D_2, M_2-M_1)w] = Q^{(\sigma)}[Aw], \end{aligned} \tag{3.27}$$

for $A(M_1,M_2,D_1,D_2) := q(D_1+D_2, M_2-M_1)$.

Proposition 3.8. *Let $\sigma = \mathcal{F}^{-1}(e^{-iP(\xi,\eta)}) \in \mathcal{S}'(\mathbb{R}^2)$ with $P \in \mathbb{R}[\xi,\eta]$, $\sigma_1 = q(D_1,D_2)\sigma$ for a polynomial $q(\xi,\eta)$ that never vanishes on $\mathbb{R}^2$, and set $Q^{(\sigma_1)}[w] = \sigma_1 * \mathrm{Wig}[w]$. Then:*

(i) *$Q^{(\sigma_1)}: \mathcal{S}' \to \mathcal{S}'$ is invertible;*
(ii) *$Q^{(\sigma_1)}$ is* regular*;*
(iii) *$Q^{(\sigma_1)}: \mathcal{S} \to \mathcal{S}$.*

Proof. The proof is analogous to that of Lemma 3.5, since $\widehat{\sigma}_1(\xi,\eta) = q(\xi,\eta)\widehat{\sigma}(\xi,\eta)$ and $q(\xi,\eta)$ never vanishes. □

Theorem 3.9. *Let $B(x,y,D_x,D_y)$ be a linear partial differential operator with polynomial coefficients. Let $\sigma = \mathcal{F}^{-1}(e^{-iP(\xi,\eta)}) \in \mathcal{S}'(\mathbb{R}^2)$ for some $P \in \mathbb{R}[\xi,\eta]$ and $\sigma_1 = q(D_1,D_2)\sigma$ for some $q \in \mathbb{C}[\xi,\eta]$ never vanishing on $\mathbb{R}^2$. Then $Q^{(\sigma)}[w] = \sigma * \mathrm{Wig}[w]$ and $Q^{(\sigma_1)}[w] = \sigma_1 * \mathrm{Wig}[w]$ satisfy, for $w \in \mathcal{S}'(\mathbb{R}^2)$,*

$$Q^{(\sigma_1)}[Bw] = \widetilde{AB}\, Q^{(\sigma)}[w], \tag{3.28}$$

where A is the operator defined by $A(M_1,M_2,D_1,D_2) = q(D_1+D_2, M_2-M_1)$, and $\widetilde{AB}$ is obtained from AB as in (3.19). Moreover, B is regular if and only if $\widetilde{AB}$ is regular.

Proof. The equality (3.28) follows from (3.27) and Theorem 3.2.

Assume, now, that B is regular. We prove that $\widetilde{AB}$ is regular. Let $\widetilde{AB}u \in \mathcal{S}$ for some $u \in \mathcal{S}'$. Since $Q^{(\sigma)}$ is surjective because of Lemma 3.5, there exists $w \in \mathcal{S}'$ such that $u = Q^{(\sigma)}[w]$. By (3.28)

$$Q^{(\sigma_1)}[Bw] = \widetilde{AB}\, Q^{(\sigma)}[w] = \widetilde{AB}u \in \mathcal{S}. \tag{3.29}$$

But $Q^{(\sigma_1)}$ is regular by Proposition 3.8 (*ii*) and hence $Bw \in \mathcal{S}$. Therefore $w \in \mathcal{S}$ since B is regular by assumption. Then also $u = Q^{(\sigma)}[w] \in \mathcal{S}$ by Lemma 3.5 (*iii*).

Reciprocally, let $\widetilde{AB}$ be regular. We prove that B is regular. Let $Bw \in \mathcal{S}$ with $w \in \mathcal{S}'$. Since $Q^{(\sigma_1)}: \mathcal{S} \to \mathcal{S}$ by Proposition 3.8 (*iii*), then $\widetilde{AB}Q^{(\sigma)}[w] = Q^{(\sigma_1)}[Bw] \in \mathcal{S}$ and hence $Q^{(\sigma)}[w] \in \mathcal{S}$ by the regularity of $\widetilde{AB}$. But $Q^{(\sigma)}$ is regular by Lemma 3.5 (*ii*) and therefore $w \in \mathcal{S}$. □

4. Time-frequency representations in the Cohen's class with kernel in $\mathcal{S}'_\omega$

We now want to obtain similar results in the class $\mathcal{S}_\omega$. We start by defining the class of weights that we consider.

Definition 4.1. *A* non-quasianalytic weight function *is a continuous increasing function* $\omega : [0, +\infty) \to [0, +\infty)$ *satisfying the following properties:*

(α) $\exists\ L > 0$ *s.t.* $\omega(2t) \leq L(\omega(t) + 1)\ \forall t \geq 0$;

(β) $\displaystyle\int_1^{+\infty} \frac{\omega(t)}{t^2} dt < +\infty$;

(γ) $\exists\ a \in \mathbb{R}, b > 0$ *s.t.* $\quad \omega(t) \geq a + b\log(1 + t) \qquad \forall t \geq 0$;

(δ) $\varphi_\omega :\ t \mapsto \omega(e^t)$ *is convex.*

We then define $\omega(\xi) = \omega(|\xi|)$ *for* $\xi \in \mathbb{C}^n$.

Remark 4.2. Condition (β) is the condition of non-quasianalyticity and guarantees that the spaces $\mathcal{D}_{(\omega)}(K)$ defined in (4.15) below are non-trivial for any compact set $K \subset \mathbb{R}^n$ with non-empty interior (see [8, Remark 3.2(1)]). When condition (β) is not satisfied we say that the weight ω is *quasianalytic*.

The function φ_ω of condition (δ) clearly depends on ω; for convenience we shall simply write φ instead of φ_ω.

Definition 4.3. *For a weight* ω *as in Definition 4.1 we define* $\mathcal{S}_\omega(\mathbb{R}^n)$ *as the set of all* $u \in L^1(\mathbb{R}^n)$ *such that* $u, \widehat{u} \in C^\infty(\mathbb{R}^n)$ *and*

(i) $\forall\lambda > 0,\ \alpha \in \mathbb{N}_0^n : \qquad \sup_{\mathbb{R}^n} e^{\lambda\omega(x)}|D^\alpha u(x)| < +\infty$

(ii) $\forall\lambda > 0,\ \alpha \in \mathbb{N}_0^n : \qquad \sup_{\mathbb{R}^n} e^{\lambda\omega(\xi)}|D^\alpha \widehat{u}(\xi)| < +\infty.$

As usual, the corresponding dual space is denoted by $\mathcal{S}'_\omega(\mathbb{R}^n)$ *and is the set of all the linear and continuous functionals* $u : \mathcal{S}_\omega(\mathbb{R}^n) \to \mathbb{C}$. *We say that an element of* $\mathcal{S}'_\omega(\mathbb{R}^n)$ *is an "*ω*-temperate distribution".*

Remark 4.4. In Definition 4.1 we consider weight functions in the sense of [8], then the weights are not necessarily subadditive in general as in [2]. On the other hand, we relax condition (γ) with respect to [8] since we work only in the Beurling setting, as in [2].

Following [8], we define the *Young conjugate* φ^* of φ as

$$\varphi^*(s) := \sup_{t \geq 0}\{st - \varphi(t)\},$$

for all $s \geq 0$. We notice that since we relax condition (γ) with respect to [8], the main properties of φ^* hold, but $\varphi^*(s)$ may take the value $+\infty$ for some s. In this case the expressions involving φ^* shall assume a formal meaning; for example, if $\varphi^*(s_0) = +\infty$, then $e^{\varphi^*(s_0)} = +\infty$, $e^{-\varphi^*(s_0)} = 0$, and so on. From Fenchel-Moreau Theorem (cf. for example [7]) we have that φ^* is convex and $\varphi^{**} = \varphi$. Moreover, since we can assume without loss of generality that ω vanishes on $[0, 1]$ we have that $\varphi^*(s)/s$ is increasing (cf. Lemma 1.5 of [8]).

We state the next result, that is well-known in the case of weights of Braun, Meise and Taylor [8], and it holds also for weights as in Definition 4.1 since it is independent of condition (γ) (for the proof we refer, for instance, to [5, Prop. 2.1(e) and Rem. 2.2]):

Lemma 4.5. *Let* ω *be a weight function and* D *be a constant such that* $\omega(et) \leq D\,(\omega(t) + 1)$ *for every* $t \geq 0$ *(such constant exists from condition* (α)*). Fix* $\lambda, \rho > 0$*; then for every* $0 < \lambda' \leq$

$\lambda/D^{[\log\rho+1]}$ *we have*

$$\rho^j e^{\lambda\varphi^*\left(\frac{j}{\lambda}\right)} \leq \Lambda_{\rho,\lambda} e^{\lambda'\varphi^*\left(\frac{j}{\lambda'}\right)}, \qquad \forall j \in \mathbb{N}_0,$$

with $\Lambda_{\rho,\lambda} = \exp\{\lambda[\log\rho+1]\}$*, where* $[\log\rho+1]$ *is the integer part of* $\log\rho+1$.

Remark 4.6. Observe that for $\omega_0(t) = \log(1+t)$ the corresponding space $\mathcal{S}_{\omega_0}(\mathbb{R}^n)$ coincides with the classical Schwartz space $\mathcal{S}(\mathbb{R}^n)$. Moreover, the condition (γ) in Definition 4.1 ensures us that for every weight ω the space $\mathcal{S}_\omega(\mathbb{R}^n)$ is contained in $\mathcal{S}(\mathbb{R}^n)$, and so we can rewrite the definition of $\mathcal{S}_\omega(\mathbb{R}^n)$ as

$$\mathcal{S}_\omega(\mathbb{R}^n) = \{u \in \mathcal{S}(\mathbb{R}^n) \text{ satisfying (i) and (ii) of Definition 4.3}\}.$$

The following characterization of the space $\mathcal{S}_\omega$ will be useful throughout this section. The theorem below extends the carachterizations of $\mathcal{S}_\omega$ given in [11, 16] and shows different equivalent systems of seminorms that can be used in such space. We begin with a lemma.

Lemma 4.7. *Let* ω *be a weight function as in Definition 4.1. Then, for every* $\lambda > 0$, $k \in \mathbb{N}$ *and* $t \geq 1$ *we have:*

(i) $t^k e^{-\lambda\omega(t)} \leq e^{\lambda\varphi^*\left(\frac{k}{\lambda}\right)}$,

(ii) $\inf_{j\in\mathbb{N}_0} t^{-j} e^{\lambda\varphi^*\left(\frac{j}{\lambda}\right)} \leq e^{-\left(\lambda-\frac{1}{b}\right)\omega(t)-a/b}$*, where* a, b *are the constants of condition* (γ) *of Definition 4.1.*

Proof. (i) For $t \geq 1$, we have:

$$k\log t - \lambda\omega(t) \leq \sup_{t\geq 1}\{k\log t - \lambda\omega(t)\} = \lambda \sup_{s\geq 0}\left\{\frac{k}{\lambda}s - \varphi(s)\right\} = \lambda\varphi^*\left(\frac{k}{\lambda}\right).$$

(ii) For all $s, \lambda > 0$ there is $j \in \mathbb{N}_0$ such that $j \leq s\lambda < j+1$ and hence (cf. [8]):

$$\begin{aligned}
\sup_{j\in\mathbb{N}_0}\left\{j\log t - \lambda\varphi^*\left(\frac{j}{\lambda}\right)\right\} &= \lambda \sup_{j\in\mathbb{N}_0}\left\{\frac{j+1}{\lambda}\log t - \varphi^*\left(\frac{j}{\lambda}\right)\right\} - \log t \\
&\geq \lambda \sup_{s\geq 0}\{s\log t - \varphi^*(s)\} - \log t \\
&= \lambda\varphi^{**}(\log t) - \log t = \lambda\omega(t) - \log t \\
&\geq \left(\lambda - \frac{1}{b}\right)\omega(t) + \frac{a}{b}.
\end{aligned}$$

□

Theorem 4.8. *Let* $u \in \mathcal{S}(\mathbb{R}^n)$ *and* ω *a non-quasianalytic weight function. Then* $u \in \mathcal{S}_\omega$ *if and only if one of the following equivalent conditions is satisfied:*

(1) *u satisfies the conditions:*
   - *(i)* $\forall\lambda > 0, \alpha \in \mathbb{N}_0^n: \sup_{x\in\mathbb{R}^n} e^{\lambda\omega(x)}|D^\alpha u(x)| < +\infty$;
   - *(ii)* $\forall\lambda > 0, \alpha \in \mathbb{N}_0^n: \sup_{\xi\in\mathbb{R}^n} e^{\lambda\omega(\xi)}|D^\alpha \widehat{u}(\xi)| < +\infty$.

(2) *u satisfies the conditions:*
   - *(i)'* $\forall\lambda > 0, \alpha \in \mathbb{N}_0^n: \sup_{x\in\mathbb{R}^n} e^{\lambda\omega(x)}|x^\alpha u(x)| < +\infty$;

(ii)′ $\forall\lambda>0,\alpha\in\mathbb{N}_0^n:\ \sup_{\xi\in\mathbb{R}^n} e^{\lambda\omega(\xi)}|\xi^\alpha\widehat{u}(\xi)|<+\infty.$

(3) *u satisfies the conditions:*

(i)″ $\forall\lambda>0:\ \sup_{x\in\mathbb{R}^n} e^{\lambda\omega(x)}|u(x)|<+\infty;$

(ii)″ $\forall\lambda>0:\ \sup_{\xi\in\mathbb{R}^n} e^{\lambda\omega(\xi)}|\widehat{u}(\xi)|<+\infty.$

(4) *u satisfies the conditions:*

(a) $\forall\beta\in\mathbb{N}_0^n,\lambda>0\ \exists C_{\beta,\lambda}>0:$

$$\sup_{x\in\mathbb{R}^n}|x^\beta D^\alpha u(x)|e^{-\lambda\varphi^*\left(\frac{|\alpha|}{\lambda}\right)}\leq C_{\beta,\lambda}\qquad\forall\alpha\in\mathbb{N}_0^n;$$

(b) $\forall\alpha\in\mathbb{N}_0^n,\mu>0\ \exists C_{\alpha,\mu}>0:$

$$\sup_{x\in\mathbb{R}^n}|x^\beta D^\alpha u(x)|e^{-\mu\varphi^*\left(\frac{|\beta|}{\mu}\right)}\leq C_{\alpha,\mu}\qquad\forall\beta\in\mathbb{N}_0^n.$$

(5) *u satisfies the condition:*

$$\begin{aligned}&\forall\mu,\lambda>0\ \exists C_{\mu,\lambda}>0\ s.t.\\&\sup_{x\in\mathbb{R}^n}|x^\beta D^\alpha u(x)|e^{-\lambda\varphi^*\left(\frac{|\alpha|}{\lambda}\right)}e^{-\mu\varphi^*\left(\frac{|\beta|}{\mu}\right)}\leq C_{\mu,\lambda}\qquad\forall\alpha,\beta\in\mathbb{N}_0^n.\end{aligned}$$

(6) *u satisfies the condition:*

$$\begin{aligned}&\forall\lambda>0\ \exists C_\lambda>0\ s.t.\\&\sup_{x\in\mathbb{R}^n}|x^\beta D^\alpha u(x)|e^{-\lambda\varphi^*\left(\frac{|\alpha+\beta|}{\lambda}\right)}\leq C_\lambda\qquad\forall\alpha,\beta\in\mathbb{N}_0^n.\end{aligned}$$

Proof. Note first that $u\in\mathcal{S}_\omega(\mathbb{R}^n)$ if and only if $u\in\mathcal{S}(\mathbb{R}^n)$ and satisfies (1) by Remark 4.6.

(1) ⇔ (3) is Corollary 2.9 of [16].

(2) ⇒ (3) follows taking $\alpha=0$ in (2).

(3) ⇒ (2) follows from condition (γ) of ω, since

$$|e^{\lambda\omega(x)}x^\alpha|\leq e^{-\frac{a\alpha}{b}}e^{\left(\frac{\alpha}{b}+\lambda\right)\omega(x)}.$$

(1) $\Rightarrow$ (4): let us first estimate

$$\begin{aligned}
|x^\beta D^\alpha u(x)| &= (2\pi)^{-n}\left|\int \xi^\alpha \widehat{u}(\xi) x^\beta e^{i\langle x,\xi\rangle} d\xi\right| \\
&= (2\pi)^{-n}\left|\int \xi^\alpha \widehat{u}(\xi) D^\beta_\xi e^{i\langle x,\xi\rangle} d\xi\right| \\
&= (2\pi)^{-n}\left|\int D^\beta_\xi \left(\xi^\alpha \widehat{u}(\xi)\right) e^{i\langle x,\xi\rangle} d\xi\right| \\
&\leq \sum_{\substack{\gamma\leq\beta\\ \gamma\leq\alpha}} \binom{\beta}{\gamma} \int |D^\gamma_\xi \xi^\alpha| \cdot |D^{\beta-\gamma}_\xi \widehat{u}(\xi)| d\xi \\
&\leq \sum_{\substack{\gamma\leq\beta\\ \gamma\leq\alpha}} \frac{\beta!}{\gamma!(\beta-\gamma)!}\frac{\alpha!}{(\alpha-\gamma)!} \int |D^{\beta-\gamma}_\xi \widehat{u}(\xi)| \cdot |\xi|^{|\alpha-\gamma|} d\xi \\
&\leq 2^{|\alpha|} \sum_{\substack{\gamma\leq\beta\\ \gamma\leq\alpha}} \frac{\beta!}{(\beta-\gamma)!} \int |D^{\beta-\gamma}_\xi \widehat{u}(\xi)| e^{2\lambda\omega(\xi)} e^{-\lambda\omega(\xi)} e^{-\lambda\omega(\xi)+|\alpha-\gamma|\log|\xi|} d\xi. \qquad (4.1)
\end{aligned}$$

Now, by condition (ii) of (1), for all $\gamma \leq \beta$,

$$|D^{\beta-\gamma}_\xi \widehat{u}(\xi)| e^{2\lambda\omega(\xi)} \leq C_{\beta,\lambda}$$

for some $C_{\beta,\lambda} > 0$. Since we can assume without loss of generality that $|\xi| \geq 1$, we have by Lemma 4.7(i),

$$e^{-\lambda\omega(\xi)+|\alpha-\gamma|\log|\xi|} \leq e^{-\lambda\omega(\xi)+|\alpha|\log|\xi|} \leq e^{\lambda\varphi^*\left(\frac{|\alpha|}{\lambda}\right)}.$$

Therefore, substituting in (4.1):

$$|x^\beta D^\alpha u(x)| \leq C'_{\beta,\lambda} 2^{|\alpha|} e^{\lambda\varphi^*\left(\frac{|\alpha|}{\lambda}\right)} \int e^{-\lambda\omega(\xi)} d\xi \qquad (4.2)$$

for some $C'_{\beta,\lambda} > 0$.

But from Lemma 4.5 we have that for all $0 < \lambda' \leq \lambda/D$, there exists $C_{\lambda'} > 0$ such that

$$2^{|\alpha|} e^{\lambda\varphi^*\left(\frac{|\alpha|}{\lambda}\right)} \leq C_{\lambda'} e^{\lambda'\varphi^*\left(\frac{|\alpha|}{\lambda'}\right)}, \qquad (4.3)$$

where $C_{\lambda'} = e^{\lambda' D}$. Moreover

$$\int e^{-\lambda\omega(\xi)} d\xi \in \mathbb{R} \qquad \text{for } \lambda \text{ sufficiently large}, \qquad (4.4)$$

by condition (γ).

Substituting (4.3) and (4.4) in (4.2) we finally have that for all $\beta \in \mathbb{N}_0^n$, $\lambda' > 0$ there exists $C_{\beta,\lambda'} > 0$ such that

$$|x^\beta D^\alpha u(x)| \leq C_{\beta,\lambda'} e^{\lambda'\varphi^*\left(\frac{|\alpha|}{\lambda'}\right)} \qquad \forall \alpha \in \mathbb{N}_0^n,$$

so that condition (a) of (4) is satisfied.

Condition (b) of (4) easily follows proceeding as before by condition (i) of (1):

$$\begin{aligned}|x^\beta D^\alpha u(x)|e^{-\mu\varphi^*\left(\frac{|\beta|}{\mu}\right)} \leq &\ |D^\alpha u(x)|e^{|\beta|\log|x|-\mu\varphi^*\left(\frac{|\beta|}{\mu}\right)}\\ \leq &\ |D^\alpha u(x)|e^{\mu\omega(x)} \leq C_{\alpha,\mu}.\end{aligned}$$

(4) $\Rightarrow$ (1): by (4)(b):

$$\begin{aligned}|D^\alpha u(x)| = &\ |x^\beta D^\alpha u(x)|e^{-\mu\varphi^*\left(\frac{|\beta|}{\mu}\right)}e^{-|\beta|\log|x|+\mu\varphi^*\left(\frac{|\beta|}{\mu}\right)}\\ \leq &\ C_{\alpha,\mu}e^{-|\beta|\log|x|+\mu\varphi^*\left(\frac{|\beta|}{\mu}\right)} \qquad \forall\alpha,\beta\in\mathbb{N}_0^n,\ \mu>0.\end{aligned}$$

Now, since the constant $C_{\alpha,\mu}$ of condition (b) of (4) does not depend on β, by Lemma 4.7(ii) we get condition (i) of (1):

$$|D^\alpha u(x)| \leq C'_{\alpha,\mu}e^{-(\mu-\frac{1}{b})\omega(x)} \qquad \forall\alpha\in\mathbb{N}_0^n,\ \mu>0,$$

where $C'_{\alpha,\mu} = C_{\alpha,\mu}e^{-a/b}$. Let us now prove also condition (ii) of (1):

$$\begin{aligned}|D^\beta_\xi\widehat{u}(\xi)| = &\ |\widehat{x^\beta u}(\xi)| = \left|\int x^\beta u(x)e^{-i\langle x,\xi\rangle}dx\right|\\ = &\ \left|\int D^\alpha_x(e^{-i\langle x,\xi\rangle})\frac{1}{\xi^\alpha}x^\beta u(x)dx\right|\\ = &\ \left|\int \frac{1}{\xi^\alpha}D^\alpha_x\left(x^\beta u(x)\right)e^{-i\langle x,\xi\rangle}dx\right|\\ \leq &\ \sum_{\substack{\gamma\leq\alpha\\ \gamma\leq\beta}}\binom{\alpha}{\gamma}\int |D^\gamma_x x^\beta|\cdot|D^{\alpha-\gamma}_x u(x)|e^{-|\alpha|\log|\xi|}dx\\ \leq &\ \sum_{\substack{\gamma\leq\alpha\\ \gamma\leq\beta}}\binom{\alpha}{\gamma}\frac{\beta!}{(\beta-\gamma)!}\int\langle x\rangle^{|\beta-\gamma|+n+1}|D^{\alpha-\gamma}_x u(x)|e^{-\lambda\varphi^*\left(\frac{|\alpha-\gamma|}{\lambda}\right)}e^{\lambda\varphi^*\left(\frac{|\alpha-\gamma|}{\lambda}\right)-|\alpha|\log|\xi|}\frac{1}{\langle x\rangle^{n+1}}dx\end{aligned}\tag{4.5}$$

where $\langle x\rangle := \sqrt{1+|x|^2}$.

By condition (a) of (4),

$$\langle x\rangle^{|\beta-\gamma|+n+1}|D^{\alpha-\gamma}_x u(x)|e^{-\lambda\varphi^*\left(\frac{|\alpha-\gamma|}{\lambda}\right)} \leq C_{\beta,\lambda}.\tag{4.6}$$

Moreover, by (4.3) for all $0<\lambda'\leq\lambda/D$ there exists $C_{\lambda'}>0$ such that:

$$2^{|\alpha|}e^{\lambda\varphi^*\left(\frac{|\alpha-\gamma|}{\lambda}\right)-|\alpha|\log|\xi|} \leq C_{\lambda'}e^{\lambda'\varphi^*\left(\frac{|\alpha|}{\lambda'}\right)-|\alpha|\log|\xi|}.\tag{4.7}$$

Since $\binom{\alpha}{\gamma}\leq 2^{|\alpha|}$, proceeding as before, taking the infimum in $|\alpha|$, by Lemma 4.7(ii), we have

$$|D^\beta\widehat{u}(\xi)| \leq C_{\beta,\lambda''}e^{-\lambda''\omega(\xi)} \qquad \forall\beta\in\mathbb{N}_0^n,\ \lambda''>0$$

since $\int\langle x\rangle^{-n-1}dx$ is a constant.

This proves condition (ii) of (1).

(5) $\Rightarrow$ (4) is trivial.

(4) ⇒ (5): let us first remark that there are relations between the L^∞ norms of $x^\beta D^\alpha u$ and the L^2 norms of $x^\beta D^\alpha u$. In fact, writing $N = \left[\frac{n+1}{4}\right] + 1$, we have

$$\begin{aligned}\|x^\beta D^\alpha u\|^2_{L^2(\mathbb{R}^n)} &= \int |x^\beta D^\alpha u(x)|^2 dx \\ &= \int \left|x^\beta (1+|x|^2)^N D^\alpha u(x)\right|^2 \frac{1}{(1+|x|^2)^{2N}} dx \\ &\le c \left\|x^\beta (1+|x|^2)^N D^\alpha u\right\|^2_{L^\infty(\mathbb{R}^n)}\end{aligned}$$

for some $c > 0$. We then have

$$\|x^\beta D^\alpha u\|^2_{L^2(\mathbb{R}^n)} \le c \sum_{|\gamma|\le N} \frac{N!}{\gamma!(N-|\gamma|)!} \left\|x^{\beta+2\gamma} D^\alpha u\right\|^2_{L^\infty(\mathbb{R}^n)}. \tag{4.8}$$

Reciprocally by Sobolev inequality (cf. [19, Ch. 3, Lemma 2.5]) there exists $C > 0$ such that

$$\|x^\beta D^\alpha u\|_{L^\infty(\mathbb{R}^n)} \le C \|x^\beta D^\alpha u\|_{H^s(\mathbb{R}^n)} \tag{4.9}$$

for $s > n/2$ (note that $x^\beta D^\alpha u \in L^\infty(\mathbb{R}^n)$ for every $\alpha, \beta \in \mathbb{N}_0^n$ implies that $x^\beta D^\alpha u \in H^s(\mathbb{R}^n)$ for every $\alpha, \beta \in \mathbb{N}_0^n$ and for every $s > 0$). From point (a) of (4) we then have from (4.8) that, for every $\lambda > 0$,

$$\|x^\beta D^\alpha u\|^2_{L^2(\mathbb{R}^n)} \le c \sum_{|\gamma|\le N} \frac{N!}{\gamma!(N-|\gamma|)!} C^2_{\beta+2\gamma,\lambda} e^{2\lambda\varphi^*\left(\frac{|\alpha|}{\lambda}\right)} = \tilde{C}^2_{\beta,\lambda} e^{2\lambda\varphi^*\left(\frac{|\alpha|}{\lambda}\right)}, \tag{4.10}$$

where

$$\tilde{C}^2_{\beta,\lambda} = c \sum_{|\gamma|\le N} \frac{N!}{\gamma!(N-|\gamma|)!} C^2_{\beta+2\gamma,\lambda}$$

depends only on β, λ and the dimension n. Now, from (4.8), the point (b) of (4) (rewritten for convenience with μ' instead of μ) implies that

$$\|x^\beta D^\alpha u\|^2_{L^2(\mathbb{R}^n)} \le c \sum_{|\gamma|\le N} \frac{N!}{\gamma!(N-|\gamma|)!} C^2_{\alpha,\mu'} e^{2\mu'\varphi^*\left(\frac{|\beta+2\gamma|}{\mu'}\right)};$$

from the convexity of φ^* we get:

$$e^{\mu'\varphi^*\left(\frac{|\beta+2\gamma|}{\mu'}\right)} \le e^{\frac{\mu'}{2}\varphi^*\left(\frac{2|\beta|}{\mu'}\right)} e^{\frac{\mu'}{2}\varphi^*\left(\frac{4|\gamma|}{\mu'}\right)}.$$

Then we obtain

$$\|x^\beta D^\alpha u\|^2_{L^2(\mathbb{R}^n)} \le c \sum_{|\gamma|\le N} \frac{N!}{\gamma!(N-|\gamma|)!} C^2_{\alpha,\mu'} e^{\mu'\varphi^*\left(\frac{2|\beta|}{\mu'}\right)} e^{\mu'\varphi^*\left(\frac{4|\gamma|}{\mu'}\right)} = C'_{\alpha,\mu'} e^{\mu'\varphi^*\left(\frac{2|\beta|}{\mu'}\right)},$$

where

$$C'_{\alpha,\mu'} = c \sum_{|\gamma|\le N} \frac{N!}{\gamma!(N-|\gamma|)!} C^2_{\alpha,\mu'} e^{\mu'\varphi^*\left(\frac{4|\gamma|}{\mu'}\right)}$$

depends only on α, μ' and the dimension n. Then, writing $\mu := \mu'/2$ we obtain that for every $\alpha \in \mathbb{N}_0^n$ and for every $\mu > 0$ there exists a constant $\tilde{C}_{\alpha,\mu} > 0$ satisfying

$$\|x^\beta D^\alpha u\|_{L^2(\mathbb{R}^n)} \leq \tilde{C}_{\alpha,\mu} e^{\mu\varphi^*\left(\frac{|\beta|}{\mu}\right)}. \tag{4.11}$$

Now, we will use that

$$\gamma! \leq C_\lambda e^{\lambda\varphi^*\left(\frac{|\gamma|}{\lambda}\right)}, \tag{4.12}$$

for all $\lambda > 0$, $\gamma \in \mathbb{N}_0^n$ and some constant C_λ. This is true because $\omega(t) = o(t)$ as $t \to \infty$ (from condition (β) of Definition 4.1). Therefore, from (4.10) and (4.11), and following the same idea as in [10], we thus estimate:

$$\begin{aligned}
\|x^\beta D^\alpha u\|^2_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \left|\left(x^{2\beta}\partial_x^\alpha u(x)\right)\cdot \partial_x^\alpha u(x)\right| dx \\
&\leq \sum_{\substack{\gamma\leq 2\beta\\ \gamma\leq\alpha}} \binom{\alpha}{\gamma}\binom{2\beta}{\gamma}\gamma! \|\partial^{2\alpha-\gamma}u(x)\|_{L^2(\mathbb{R}^n)}\|x^{2\beta-\gamma}u(x)\|_{L^2(\mathbb{R}^n)} \\
&\leq \sum_{\substack{\gamma\leq 2\beta\\ \gamma\leq\alpha}} \binom{\alpha}{\gamma}\binom{2\beta}{\gamma}\gamma! \tilde{C}_{0,\lambda} e^{\lambda\varphi^*\left(\frac{|2\alpha-\gamma|}{\lambda}\right)} \tilde{C}_{0,\mu} e^{\mu\varphi^*\left(\frac{|2\beta-\gamma|}{\mu}\right)} \\
&\leq \sum_{\substack{\gamma\leq 2\beta\\ \gamma\leq\alpha}} \binom{\alpha}{\gamma}\binom{2\beta}{\gamma} C_\lambda e^{\lambda\varphi^*\left(\frac{|\gamma|}{\lambda}\right)} e^{\lambda\varphi^*\left(\frac{|2\alpha-\gamma|}{\lambda}\right)} \tilde{C}_{0,\mu} e^{\mu\varphi^*\left(\frac{|2\beta|}{\mu}\right)} \\
&\leq 2^{|\alpha|}2^{2|\beta|} C_\lambda \tilde{C}_{0,\mu} e^{\lambda\varphi^*\left(\frac{|2\alpha|}{\lambda}\right)} e^{\mu\varphi^*\left(\frac{|2\beta|}{\mu}\right)} \\
&\leq C_{\lambda',\mu'} e^{\lambda'\varphi^*\left(\frac{|2\alpha|}{\lambda'}\right)} e^{\mu'\varphi^*\left(\frac{|2\beta|}{\mu'}\right)}
\end{aligned}$$

for some $\tilde{C}_{0,\lambda}, \tilde{C}_{0,\mu}, C_\lambda, C_{\lambda',\mu'} > 0$, because of the properties of φ^* and (4.3). Extracting the square root and writing $\lambda = \lambda'/2$ and $\mu = \mu'/2$ we have that for every $\lambda, \mu > 0$ there exists a constant $\tilde{C}_{\lambda,\mu} > 0$ such that

$$\|x^\beta D^\alpha u\|_{L^2(\mathbb{R}^n)} \leq \tilde{C}_{\lambda,\mu} e^{\lambda\varphi^*\left(\frac{|\alpha|}{\lambda}\right)} e^{\mu\varphi^*\left(\frac{|\beta|}{\mu}\right)}. \tag{4.13}$$

In order to prove that (5) holds, we have to estimate $\|x^\beta D^\alpha u\|_{L^\infty(\mathbb{R}^n)}$. Fix $\bar{s} = \left[\frac{n}{2}\right] + 1$; from (4.9) and (4.13) we have

$$\begin{aligned}
\|x^\beta D^\alpha u\|_{L^\infty(\mathbb{R}^n)} &\leq C \sum_{|\gamma|\leq\bar{s}} \|D^\gamma(x^\beta D^\alpha u)\|_{L^2(\mathbb{R}^n)} \\
&\leq C \sum_{|\gamma|\leq\bar{s}} \sum_{\substack{\sigma\leq\gamma\\ \sigma\leq\beta}} \binom{\gamma}{\sigma}\binom{\beta}{\sigma}\sigma! \|x^{\beta-\sigma}D^{\alpha+\gamma-\sigma}u\|_{L^2(\mathbb{R}^n)} \\
&\leq C \sum_{|\gamma|\leq\bar{s}} \sum_{\substack{\sigma\leq\gamma\\ \sigma\leq\beta}} \binom{\gamma}{\sigma}\binom{\beta}{\sigma}\sigma! \tilde{C}_{\lambda,\mu} e^{\lambda\varphi^*\left(\frac{|\alpha+\gamma-\sigma|}{\lambda}\right)} e^{\mu\varphi^*\left(\frac{|\beta-\sigma|}{\mu}\right)}.
\end{aligned} \tag{4.14}$$

Now, proceeding as in previous steps, using inequality (4.12), the convexity of φ^* and similar properties as before we easily get (5).

(5) $\Leftrightarrow$ (6) is trivial from the convexity of φ^*. $\square$

We recall quickly the definition of the space $\mathcal{E}_{(\omega)}(\Omega)$ of ω-ultradifferentiable functions of Beurling type in an open subset Ω of $\mathbb{R}^n$. It is the set

$$\mathcal{E}_{(\omega)}(\Omega) := \Big\{ f \in C^\infty(\Omega) : \ \forall K \subset\subset \Omega, \ \forall m \in \mathbb{N} \\ \sup_{\alpha\in\mathbb{N}^n} \sup_{x\in K} |D^\alpha f(x)| e^{-m\varphi^*\left(\frac{|\alpha|}{m}\right)} < +\infty \Big\}.$$

To define then the space of ω-ultradifferentiable functions of Beurling type with compact support, we first consider, for a compact set $K \subset \Omega$,

$$\mathcal{D}_{(\omega)}(K) := \{ f \in \mathcal{E}_{(\omega)}(\Omega) : \ \operatorname{supp} f \subseteq K \}. \tag{4.15}$$

This space is not trivial because of (β) of Definition 4.1 (considering the non-quasianalytic case; for quasianalytic weights the space (4.15) contains only the function identically 0). Finally, we set the space of test functions as follows

$$\mathcal{D}_{(\omega)}(\Omega) = \operatorname*{ind\,lim}_{K \nearrow \Omega} \mathcal{D}_{(\omega)}(K).$$

The spaces of Roumieu type are not used here and a definition can be found in [8] with a stronger condition instead of our (γ). The use of (γ) is clarified for the Beurling case in [3] (see also [14]).

We recall here some properties of the space $\mathcal{S}_\omega(\mathbb{R}^n)$, that we shall use in the following. For the proofs we refer to [14, Kap. I, §6] (see also [2]) .

Proposition 4.9. *Let ω be as in Definition 4.1.*

(a) *The Fourier transform is a continuous automorphism $\mathcal{F} : \mathcal{S}_\omega(\mathbb{R}^n) \to \mathcal{S}_\omega(\mathbb{R}^n)$. It can be extended to $\mathcal{S}'_\omega(\mathbb{R}^n)$ in the standard way, by the formula*

$$\langle \widehat{u}, \varphi \rangle = \langle u, \widehat{\varphi} \rangle \qquad \forall \varphi \in \mathcal{S}_\omega.$$

(b) *$\mathcal{S}_\omega(\mathbb{R}^n)$ is an algebra under multiplication and convolution.*

(c) *The differentiation D^α, the multiplication by x^α, for $\alpha \in \mathbb{N}_0^n$, the multiplication by $e^{i\langle \cdot, a\rangle}$ and the translation τ_a acting as $\tau_a u(x) := u(x-a)$, for $a \in \mathbb{R}^n$, are continuous on $\mathcal{S}_\omega(\mathbb{R}^n)$.*

(d) *The following inclusions hold: $\mathcal{D}_{(\omega)}(\mathbb{R}^n) \subset \mathcal{S}_\omega(\mathbb{R}^n) \subset \mathcal{E}_{(\omega)}(\mathbb{R}^n)$.*

(e) *$\mathcal{D}_{(\omega)}(\mathbb{R}^n)$ is dense in $\mathcal{S}_\omega(\mathbb{R}^n)$.*

(f) *For $\psi \in \mathcal{S}_\omega(\mathbb{R}^n)$ and $u \in \mathcal{S}'_\omega(\mathbb{R}^n)$ we have $\psi * u \in \mathcal{S}'_\omega(\mathbb{R}^n)$ and $\widehat{\psi * u} = \widehat{\psi} \cdot \widehat{u}$.*

We observe that Theorem 4.8 allows to define equivalent systems of seminorms for $\mathcal{S}_\omega$. For example, from condition (6) of this theorem it is clear that, given $u \in \mathcal{S}_\omega$ the family

$$p_\lambda(u) := \sup_{\alpha,\beta\in\mathbb{N}_0^n} \sup_{x\in\mathbb{R}^n} |x^\beta D^\alpha u(x)| e^{-\lambda\varphi^*\left(\frac{|\alpha+\beta|}{\lambda}\right)},$$

for all $\lambda > 0$, defines a fundamental system of seminorms for $\mathcal{S}_\omega$. In a similar way, we can construct different equivalent systems of seminorms from the other conditions of the theorem.

Remark 4.10. By Proposition 4.9 (a), $\mathcal{S}_\omega(\mathbb{R}^{2n})$ is invariant by Fourier transform $\mathcal{F} = \mathcal{F}_{(x,y)}$.

Moreover, it can be proved by direct calculation that $\mathcal{S}_\omega(\mathbb{R}^{2n})$ is also invariant by partial Fourier transform $\mathcal{F}_x$. This can also be deduced from the facts that it is clear for $\varphi \in \mathcal{S}_\omega(\mathbb{R}^{2n})$ of the form $\varphi(x,y) = f(x)\cdot g(y)$, with $f, g \in \mathcal{S}_\omega(\mathbb{R}^n)$, and $\mathcal{S}_\omega(\mathbb{R}^n)\otimes\mathcal{S}_\omega(\mathbb{R}^n)$ is dense in $\mathcal{S}_\omega(\mathbb{R}^{2n})$ by Proposition 4.9 (e) and [8, Thm. 8.1] (cf. also [3], since we assume condition (γ) of Definition 4.1 instead of $\log(t) = o(\omega(t))$ as $t \to \infty$).

Furthermore, the linear change of variable $T : \ \mathcal{S}_\omega \to \mathcal{S}_\omega$ defined in (2.3) is invertible and therefore from (2.2) we deduce that also the Wigner transform

$$\begin{aligned} \mathrm{Wig} : \ \ & \mathcal{S}_\omega \longrightarrow \mathcal{S}_\omega \\ & \mathcal{S}'_\omega \longrightarrow \mathcal{S}'_\omega \end{aligned}$$

is invertible.

The following lemma can be deduced as Lemma 3.3 above.

Lemma 4.11. *If $\varphi \in C_p^\infty(\mathbb{R}^n)$ and $u \in \mathcal{S}_\omega$ then $\varphi u \in \mathcal{S}_\omega$. If $w \in \mathcal{S}'_\omega$ then $\varphi w \in \mathcal{S}'_\omega$.*

Proposition 4.12. *For every non-quasianalytic weight function ω we have*

$$\mathcal{S}'(\mathbb{R}^n) \subset \mathcal{S}'_\omega(\mathbb{R}^n).$$

Proof. We already know that $\mathcal{S}_\omega(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$, cf. Remark 4.6. It is enough to prove that $\mathcal{S}_\omega(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$.

By [8, Prop. 3.9] we have that $\mathcal{D}_{(\omega)}(\mathbb{R}^n)$ is dense in $\mathcal{D}(\mathbb{R}^n)$. On the other hand, it is known that $\mathcal{D}(\mathbb{R}^n)$ is dense in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. Then $\mathcal{D}_{(\omega)}(\mathbb{R}^n)$ is also dense in $\mathcal{S}(\mathbb{R}^n)$. From the inclusions

$$\mathcal{D}_{(\omega)}(\mathbb{R}^n) \hookrightarrow \mathcal{S}_\omega(\mathbb{R}^n) \hookrightarrow \mathcal{S}(\mathbb{R}^n),$$

we can conclude that $\mathcal{S}_\omega$ is dense in $\mathcal{S}$. □

We give now the definition of regularity in the $\mathcal{S}_\omega$ frame and we extend to $\mathcal{S}_\omega$ the results of Sections 2 and 3.

Definition 4.13. *A linear operator A on $\mathcal{S}'_\omega(\mathbb{R}^n)$ is* ω-regular *if*

$$Au \in \mathcal{S}_\omega(\mathbb{R}^n) \quad \Rightarrow \quad u \in \mathcal{S}_\omega(\mathbb{R}^n), \qquad \forall u \in \mathcal{S}'_\omega(\mathbb{R}^n).$$

Proposition 4.14. *Let $\sigma = q(D_1, D_2)\mathcal{F}^{-1}(e^{-iP(\xi,\eta)})$ for some $P(\xi,\eta) \in \mathbb{R}[\xi,\eta]$ and $q(\xi,\eta) \in \mathbb{C}[\xi,\eta]$ with $q(\xi,\eta) \neq 0$ for all $\xi,\eta \in \mathbb{R}$. Let $u \in \mathcal{S}'_\omega$ for a non-quasianalytic weight function ω. Then $Q[u] = \sigma * \mathrm{Wig}[u]$ is well defined and satisfies:*

(i) *$Q : \ \mathcal{S}'_\omega \to \mathcal{S}'_\omega$ is invertible;*
(ii) *Q is ω-regular;*
(iii) *$Q : \ \mathcal{S}_\omega \to \mathcal{S}_\omega$.*

Proof. The proof is analogous to that of Lemma 3.5 (or Proposition 3.8), because of the invertibility of the Fourier transform and of the Wigner transform on $\mathcal{S}_\omega$ and $\mathcal{S}'_\omega$ (cf. Remark 4.10), and by means of Lemma 4.11, since $\widehat{\sigma}, 1/\widehat{\sigma} \in C_p^\infty$ and $|\widehat{\sigma}(\xi,\eta)| = |q(\xi,\eta)| \neq 0$ for all $\xi,\eta \in \mathbb{R}$. □

Theorem 4.15. *Let ω be a non-quasianalytic weight function, $P(\xi,\eta) \in \mathbb{R}[\xi,\eta]$ and $q(\xi,\eta) \in \mathbb{C}[\xi,\eta]$ with $q(\xi,\eta) \neq 0$ for all $\xi,\eta \in \mathbb{R}$. Let $\sigma = \mathcal{F}^{-1}(e^{-iP(\xi,\eta)}) \in \mathcal{S}' \subset \mathcal{S}'_\omega$, $\sigma_1 = q(D_1, D_2)\sigma$, $Q^{(\sigma)}[w] = \sigma * \mathrm{Wig}[w]$ and $Q^{(\sigma_1)}[w] = \sigma_1 * \mathrm{Wig}[w]$ for $w \in \mathcal{S}'_\omega$. Then, if $B(x,y,D_x,D_y)$ is a linear partial differential operator with polynomial coefficients, we have that*

$$Q^{(\sigma_1)}[Bw] = \widetilde{AB}Q^{(\sigma)}[w], \tag{4.16}$$

where A is the operator defined by $A(M_1, M_2, D_1, D_1) = q(D_1 + D_2, M_2 - M_1)$ and $\widetilde{AB}$ is obtained from AB as in (3.19)*. Moreover, B is ω-regular if and only if $\widetilde{AB}$ is ω-regular.*

Proof. Formula (4.16) has already been proved in Theorem 3.9.

Let B be ω-regular and prove that $\widetilde{AB}$ is ω-regular. So take $u \in \mathcal{S}'_\omega$ and assume that $\widetilde{AB}u \in \mathcal{S}_\omega$. By Proposition 4.14 (i) (with $q(\xi,\eta) \equiv 1$) there exists $w \in \mathcal{S}'_\omega$ such that $u = Q^{(\sigma)}[w]$. By (4.16) we have that $Q^{(\sigma_1)}[Bw] = \widetilde{AB}Q^{(\sigma)}[w] = \widetilde{AB}u \in \mathcal{S}_\omega$ and hence $Bw \in \mathcal{S}_\omega$ by Proposition 4.14 (ii). Since B is ω-regular by assumption, $w \in \mathcal{S}_\omega$. Finally, from Proposition 4.14 (iii), we have that $u = Q^{(\sigma)}[w] \in \mathcal{S}_\omega$ and we have proved that $\widetilde{AB}$ is ω-regular.

Reciprocally, assuming that $\widetilde{AB}$ is ω-regular, if $Bu \in \mathcal{S}_\omega$ for some $u \in \mathcal{S}'_\omega$, then $Q^{(\sigma_1)}[Bu] \in \mathcal{S}_\omega$ by Proposition 4.14 (iii) and therefore $\widetilde{AB}Q^{(\sigma)}[u] = Q^{(\sigma_1)}[Bu] \in \mathcal{S}_\omega$. By the ω-regularity of $\widetilde{AB}$ we have that $Q^{(\sigma)}[u] \in \mathcal{S}_\omega$ and hence $u \in \mathcal{S}_\omega$ by Proposition 4.14 (ii). This proves that B is ω-regular. □

Remark 4.16. Theorem 4.15 is an extension to $\mathcal{S}_\omega$ of Theorem 3.9. Observe in particular that for $q \equiv 1$, and hence $A \equiv I$, Theorem 4.15 implies that B is ω-regular if and only if $\tilde{B}$ is ω-regular, extending therefore to $\mathcal{S}_\omega$, for every weight function ω, the results obtained for $\mathcal{S}$ in the previous sections.

Remark 4.17. All the results of the present section may be proved also in the quasianalytic case, and more precisely when the weight function ω satisfies $\omega(t) = o(t)$ as $t \to +\infty$, instead of (β). In this case $\mathcal{S}_\omega$ does not contain functions with compact support, so that conditions (d) and (e) of Proposition 4.9 will drop. However, Proposition 4.12 is still valid, since the density of $\mathcal{S}_\omega(\mathbb{R}^n)$ in $\mathcal{S}(\mathbb{R}^n)$ can be proved by [20, Lemma 3.2], which shows that the Hermite functions, that are a Shauder basis in $\mathcal{S}(\mathbb{R}^n)$, are in $\mathcal{S}_\omega(\mathbb{R}^n)$ because of Theorem 4.8(6) and the following property:

$$\forall B > 0,\ \lambda > 0\ \exists C_{B,\lambda} > 0 \text{ s.t.} \qquad B^n n! \leq C_{B,\lambda} e^{\lambda\varphi^*\left(\frac{n}{\lambda}\right)}, \quad \forall n \in \mathbb{N},$$

which follows from (4.12) and Lemma 4.5.

5. Examples

In this section we give some examples of applications of our results in order to find classes of regular partial differential operators with polynomial coefficients. Recall from [23] that a polynomial $a(x,\xi)$ of order m, with $x,\xi \in \mathbb{R}^n$, is said to be hypoelliptic if there exists $m' \leq m$,

$\rho \in (0,1]$, and positive constants c, C such that for every $\alpha, \beta \in \mathbb{N}_0$,

$$
\begin{aligned}
&|a(x,\xi)| \geq c\langle(x,\xi)\rangle^{m'} \\
&|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C|a(x,\xi)|\,\langle(x,\xi)\rangle^{-\rho(|\alpha|+|\beta|)}
\end{aligned} \tag{5.1}
$$

for $|(x,\xi)| \geq B$, where $\langle(x,\xi)\rangle := (1+|x|^2+|\xi|^2)^{1/2}$.

Remark 5.1. From the results of [23] we have that an operator with polynomial coefficients $a(x,D)$ whose symbol $a(x,\xi)$ is hypoelliptic, is regular in $\mathcal{S}(\mathbb{R}^n)$, in the sense that it satisfies the condition of Definition 3.4. The question of proving regularity for non-hypoelliptic operators is not trivial. The results of the previous sections enable to find classes of regular (but not hypoelliptic) operators, and these classes are quite large due to the freedom we have in choosing the kernel σ of the representation in the Cohen's class. For example, using Theorem 3.7, we could consider a regular (possibly hypoelliptic) operator B and we immediately have regularity of the corresponding $\tilde{B}$, cf. (3.19). The operator $\tilde{B}$ in general is not hypoelliptic (cf. Remark 5.5 or [9] for more general examples of hypoelliptic operators B that are transformed, in the simple case when σ is the Dirac distribution, into regular operators $\tilde{B}$ which are never hypoelliptic).

It will be useful, for the discussion of examples, the following

Proposition 5.2. *Let ω be a non-quasianalytic weight function and let B be a continuous linear operator on $\mathcal{S}'_\omega(\mathbb{R})$ such that $B(\mathcal{S}_\omega(\mathbb{R})) \subseteq \mathcal{S}_\omega(\mathbb{R})$. Let I be the indentity operator on $\mathcal{S}'(\mathbb{R})$ and consider the operator $B\hat{\otimes}I$, interpreted as the "extension of B from one variable in $\mathbb{R}$ to two variables in $\mathbb{R}^2$". If $B\hat{\otimes}I$ is ω-regular in $\mathcal{S}'_\omega(\mathbb{R}^2)$, then B is ω-regular and injective in $\mathcal{S}'_\omega(\mathbb{R})$.*

Proof: Let $u \in \mathcal{S}'_\omega(\mathbb{R})$ with $Bu \in \mathcal{S}_\omega(\mathbb{R})$. We prove that $u \in \mathcal{S}_\omega(\mathbb{R})$. Indeed, for all $v \in \mathcal{S}_\omega(\mathbb{R})$ we have that $(B\hat{\otimes}I)(u\otimes v) = (Bu)\otimes v \in \mathcal{S}_\omega(\mathbb{R}^2)$, since $Bu \in \mathcal{S}_\omega(\mathbb{R})$. Then $u\otimes v \in \mathcal{S}_\omega(\mathbb{R}^2)$ for every $v \in \mathcal{S}_\omega(\mathbb{R})$, because $B\hat{\otimes}I$ is regular by assumption, and hence $u \in \mathcal{S}_\omega(\mathbb{R})$. This proves that B is ω-regular.

To prove that B is injective let us assume by contradiction that there exists $u \in \mathcal{S}'_\omega$ with $u \neq 0$ such that $Bu = 0$. Then, for the Dirac distribution δ we have that $(B\hat{\otimes}I)(u\otimes\delta) = 0 \in \mathcal{S}_\omega$ but $u\otimes\delta \notin \mathcal{S}_\omega$, and hence $B\hat{\otimes}I$ would not be regular. □

Proposition 5.2 has already been proved in [9] in the Schwartz case, i.e. $\omega(t) = \log(1+t)$. Under suitable assumptions also the converse is true in $\mathcal{S}'$, as it was proved in [9, Thm. 3].

Example 5.3. As first example consider the simple cases of a multiplication operator

$$B(x,y,D_x,D_y) = b(x,y),$$

where b is a polynomial. Then it is easy to prove that B is regular if and only if b never vanishes. We then have from Theorems 3.7 and 4.15 (cf. also Remark 4.16) that the operator

$$\tilde{B} = b\left(M_1 - \frac{1}{2}D_2 - P_1, M_1 + \frac{1}{2}D_2 - P_1\right)$$

is ω regular for each weight ω, for every P_1 as in (3.20); in particular it is regular in the sense of Schwartz spaces. Observe that $P_1 = P_1(D_1, D_2)$ is in fact an arbitrary partial differential operator with real constant coefficients in two variables. In the special case when the polynomial b depends only on one variable, we get that, if b never vanishes, the operator

$$b\,(x + P(D_x, D_y)) \tag{5.2}$$

is regular in Schwartz spaces and ω regular, for every partial differential operator $P = P(D_x, D_y)$ with constant real coefficients, without any other assumption on P.

The twisted Laplacian is an important example of a non hypoelliptic but regular operator. Its regularity (in Schwartz spaces) was proved in [25] and then re-obtained in [9] as a particular case of operators obtained as Wigner transformation of the harmonic oscillator. Applying the transformations in the Cohen's class considered in this paper we have the following example.

Example 5.4. We know from [9] that the operator

$$B(x, y, D_x, D_y) = x^2 + D_x^2 \tag{5.3}$$

is $\mathcal{S}$-regular, since it is the tensor product $B_1 \hat{\otimes} I$ of the harmonic oscillator $B_1(x, D_x) = x^2 + D_x^2$ in the x-variable (that is regular and one-to-one) and the identity in the y-variable. Then from Theorem 3.7 we have that the operator

$$\tilde{B} = \left(M_1 - \frac{1}{2}D_2 - P_1(D_1, D_2)\right)^2 + \left(M_2 + \frac{1}{2}D_1 - P_2(D_1, D_2)\right)^2 \tag{5.4}$$

is regular in Schwartz spaces, where

$$P_1 = (iD_1P)(D_1, D_2), \qquad P_2 = (iD_2P)(D_1, D_2)$$

and P is an arbitrary polynomial with real coefficients. In particular, if P is of the form $P(\xi, \eta) = P^{(1)}(\xi) + P^{(2)}(\eta)$, then P_1 and P_2 are arbitrary operators in D_1 and D_2, respectively, and so we have that the operator

$$\left(x - \frac{1}{2}D_y + Q(D_x)\right)^2 + \left(y + \frac{1}{2}D_x + R(D_y)\right)^2 \tag{5.5}$$

is regular in Schwartz spaces, for arbitrary partial differential operators $Q(D_x)$ and $R(D_y)$ with constant real coefficients. Another particular case of (5.4) is when $P(\xi, \eta) = \frac{1}{2}\xi\eta + P^{(1)}(\xi) + P^{(2)}(\eta)$ for polynomials $P^{(1)}$ and $P^{(2)}$ with real coefficients, and in this case we get the $\mathcal{S}$-regularity of

$$(x - D_y + Q(D_x))^2 + (y + R(D_y))^2\,, \tag{5.6}$$

for arbitrary differential operators $Q(D_x)$ and $R(D_y)$ with constant real coefficients.

The same results hold in the $\mathcal{S}_\omega$ frame, for a non-quasianalytic weight function ω. In order to prove this, we can show that, using the same technique as in [25], the twisted Laplacian

$$L = \left(D_x - \frac{1}{2}y\right)^2 + \left(D_y + \frac{1}{2}x\right)^2 \tag{5.7}$$

is ω-regular for every weight ω.

To this aim we first prove, following [25, Prop. 6.2], that there exists a constant $c > 0$ and, for every $s > 0$, there exists $C_s > 0$ such that

$$g(w) := \frac{1}{4\pi}\int_0^{+\infty} e^{-\frac{1}{4}|w|^2\cosh t}dt \leq C_s \frac{1}{|w|^s} e^{-c|w|^2} \qquad \forall w \in \mathbb{R}^2 \setminus \{0\}. \tag{5.8}$$

Indeed, for all $w \in \mathbb{R}^2 \setminus \{0\}$,

$$g(w) \leq \frac{1}{4\pi}\int_0^{+\infty} e^{-\frac{1}{4}|w|^2\frac{e^t}{2}}dt = \frac{1}{4\pi}\int_{|w|^2/8}^{+\infty}\frac{e^{-y}}{y}dy. \tag{5.9}$$

We then have that, for $0 < |w| \leq 1$,

$$\begin{aligned} g(w) &\leq \frac{1}{4\pi}\left(\int_{|w|^2/8}^{1}\frac{e^{-y}}{y}dy + \int_1^{+\infty}\frac{e^{-y}}{y}dy\right) \leq \frac{e^{-|w|^2/8}}{4\pi}\int_{|w|^2/8}^{1}\frac{1}{y}dy + D' \\ &= -\frac{1}{4\pi}\log\frac{|w|^2}{8}e^{-|w|^2/8} + D' \leq D''\left(1-\log\frac{|w|^2}{8}\right)e^{-|w|^2/8}; \end{aligned}$$

then for every $s > 0$ we can find a positive constant C'_s such that for every $0 < |w| \leq 1$

$$g(w) \leq C'_s\frac{1}{|w|^s}e^{-|w|^2/8}. \tag{5.10}$$

Consider now $w \in \mathbb{R}^2$ such that $|w| \geq 1$. From (5.9) we get

$$g(w) \leq \frac{2}{\pi|w|^2}\int_{|w|^2/8}^{+\infty}e^{-y}dy = \frac{2}{\pi|w|^2}e^{-|w|^2/8}.$$

Then, if we fix $c < 1/8$, for every $s > 0$ we can find a positive constant C''_s such that

$$g(w) \leq C''_s\frac{1}{|w|^s}e^{-c|w|^2} \tag{5.11}$$

for all $|w| \geq 1$. From (5.10) and (5.11) we finally have that (5.8) is satisfied for every $w \in \mathbb{R}^2 \setminus \{0\}$, with c as in (5.11) and $C_s = \max\{C'_s, C''_s\}$.
We prove now, following [25, Thm. 6.1], that if $f \in \mathcal{S}_\omega(\mathbb{R}^2)$ then the solution u of $Lu = f$ satisfies (i) and (ii) of Definition 4.3. Indeed, from [25] we have that

$$u(z) = \int_{\mathbb{R}^2} g(w)e^{\frac{1}{2}i(z_2w_1 - z_1w_2)}f(z-w)dw \qquad \forall z \in \mathbb{R}^2,$$

where g is defined in (5.8). By condition (α) in Definition 4.1, there is some constant $K > 1$ (see [8, 1.2 Lemma]) such that, for $\beta \in \mathbb{Z}^2_+$ we have

$$|e^{\lambda\omega(z)}(\partial_z^\beta u)(z)| \leq \int_{\mathbb{R}^2} e^{\lambda K(\omega(w)+1)}|g(w)|e^{\lambda K\omega(z-w)}\left|\partial_z^\beta\left(e^{\frac{i}{2}(z_2w_1 - z_1w_2)}f(z-w)\right)\right| dw. \tag{5.12}$$

The latter integral can be estimated by a sum of terms of the kind

$$\int_{\mathbb{R}^2} e^{\lambda K(\omega(w)+1)}|w|^{|\alpha|}|g(w)|e^{\lambda K\omega(z-w)}\left|\partial_z^\gamma f(z-w)\right| dw,$$

with $\alpha, \gamma \leq \beta$. Note that $e^{\lambda K\omega(z-w)}|\partial_z^\gamma f(z-w)|$ is bounded because $f = Lu \in \mathcal{S}_\omega(\mathbb{R}^2)$, moreover (5.8) implies that $e^{\lambda K(\omega(w)+1)}|w|^{|\alpha|}|g(w)|$ is summable either in $\{w \in \mathbb{R}^2 : |w| \leq 1\}$ for $s < 2$, or in $\{w \in \mathbb{R}^2 : |w| \geq 1\}$ since $\omega(t) = o(t)$ by condition (β). Therefore $\sup_z |e^{\lambda\omega(z)}(\partial_z^\beta u)(z)| < +\infty$, and so u satisfies (i) of Definition 4.3.

In order to prove that u satisfies also (ii) of Definition 4.3 we observe that u satisfies $Lu = f$ if and only if $\hat{u}$ satisfies $\hat{L}\hat{u} = \hat{f}$, where

$$\hat{L} = \left(\frac{1}{2}D_\eta + \xi\right)^2 + \left(\frac{1}{2}D_\xi - \eta\right)^2,$$

and this happens if and only if $v(\xi,\eta) := \hat{u}(\xi/2,\eta/2)$ satisfies the equation $Lv(2\xi,2\eta) = \hat{f}(\xi,\eta)$. Since the dilations do not affect the estimates (i) and (ii) of Definition 4.3 due to the fact that λ is arbitrary, we then have from the previous considerations that v satisfies (i) of Definition 4.3, and then u satisfies (ii) of Definition 4.3. So $u \in \mathcal{S}_\omega(\mathbb{R}^2)$, and L is ω-regular for every weight ω.

Looking at L, or equivalently at $\hat{L}$, as transformed operator $\tilde{B}$ (of the form (5.5) with $Q \equiv R \equiv 0$) we can apply Theorem 4.15 to obtain that B, defined by (5.3), is ω-regular for every ω. Applying again Theorem 4.15 we have that (5.4), and in particular (5.5) and (5.6), are ω-regular for every weight ω.

Moreover, the harmonic oscillator $B_1(x, D_x) = x^2 + D_x^2$ is ω-regular for every ω also for Proposition 5.2, from the ω-regularity of $B = B_1 \hat{\otimes} I$.

Remark 5.5. Note that the symbol $(\xi - \frac{1}{2}y)^2 + (\eta + \frac{1}{2}x)^2$ of the twisted Laplacian L defined in (5.7) is not hypoelliptic in the sense of (5.1), since it vanishes for $\xi = \frac{1}{2}y$, $\eta = -\frac{1}{2}x$.

Example 5.6. Another example comes from operators of the kind

$$A(x, y, D_x, D_y) = D_x + \alpha x^m,$$

for $\alpha \in \mathbb{C}$ and a positive integer m. The operator A is regular in Schwartz spaces for $(\operatorname{Im}\alpha)^m > 0$, cf. [9]. Then Theorem 3.7 gives us the regularity of

$$\tilde{B} = \frac{D_x}{2} + y - P_2 + \alpha\left(x - \frac{D_y}{2} - P_1\right)^m,$$

for P_1 and P_2 as in Example 5.4, cf. (3.20) also. In particular, if P is of the form $P(\xi,\eta) = \frac{1}{2}\xi\eta + P^{(1)}(\xi)$ or $P(\xi,\eta) = -\frac{1}{2}\xi\eta + P^{(2)}(\eta)$ for polynomials $P^{(1)}$ and $P^{(2)}$ with real coefficients, we obtain the regularity of

$$y + \alpha\left(x - D_y + Q(D_x)\right)^m$$

and

$$D_x + \alpha x^m + y + R(D_y)$$

respectively, for a positive integer m, $\alpha \in \mathbb{C}$ satisfying $(\operatorname{Im}\alpha)^m > 0$, and for arbitrary differential operators $Q(D_x)$ and $R(D_y)$ with constant real coefficients.

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