

# BIRATIONAL GEOMETRY OF RATIONAL QUARTIC SURFACES

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ABSTRACT. Two birational subvarieties of  $\mathbb{P}^n$  are called Cremona equivalent if there is a Cremona modification of  $\mathbb{P}^n$  mapping one to the other. If the codimension of the varieties is at least 2, they are always Cremona Equivalent. For divisors the question is much more subtle and a general answer is unknown. In this paper I study the case of rational quartic surfaces and prove that they are all Cremona equivalent to a plane.

ABSTRACT. Deux sous variétés birationnelles de  $\mathbb{P}^n$  sont dites Cremona équivalentes s'il existe une transformation de Cremona de  $\mathbb{P}^n$  qui envoie l'une sur l'autre. Si la codimension des variétés est au moins deux, elles sont toujours Cremona équivalentes. En ce qui concerne les diviseurs la question est plus subtile et l'on ne connaît pas de réponse générale. Dans cet article je considère le cas des surfaces quartiques rationnelles et je démontre qu'elles sont toutes Cremona équivalentes à un plan.

## INTRODUCTION

Let  $X \subset \mathbb{P}^n$  be an irreducible and reduced projective variety over an algebraically closed field. A classical question is to study the birational embedding of  $X$  in  $\mathbb{P}^n$  under the action of the Cremona group of  $\mathbb{P}^n$ . In other words considering  $X_1$  and  $X_2$ , two birationally equivalent projective varieties in  $\mathbb{P}^n$ , one wants to understand if there exists a Cremona transformation of  $\mathbb{P}^n$  that maps  $X_1$  to  $X_2$ . If this is the case  $X_1$  is said to be Cremona Equivalent (CE) to  $X_2$ , see Definition 1.1. This projective statement can also be interpreted in terms of log Sarkisov theory, [BM], and is related to the Abhyankar–Moh problem, [AM] and [Jel]. In the latter paper it is proved, using techniques derived from Abhyankar–Moh problem, that over the complex field the birational embedding is unique as long as  $\dim X < \frac{n}{2}$ . The general problem is then completely solved in [MP] where it is proved that this is the case as long as the codimension of  $X_i$  is at least 2, see also [CCMRZ] for a more algorithmic way to produce the required Cremona equivalence. Examples of inequivalent embeddings of divisors are well known, see also [MP], in all dimensions. The problem of Cremona equivalence is therefore reduced to study the action of the Cremona group on divisors. A class of divisors for which a reasonable answer is known is that of cones. In [Me2] it is proved that two cones are CE if their hyperplane sections are birational.

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*Date:* April 2019.

*1991 Mathematics Subject Classification.* Primary 14E25 ; Secondary 14E05, 14N05, 14E07.

*Key words and phrases.* Birational maps; Cremona equivalence; embeddings; hypersurfaces.

The special case of plane curves has been widely treated both in the old times, [Co], [SR], [Ju], and more recently, [Na], [Ii], [KM], [CC], and [MP2], see also [FLMN] for a nice survey. In [CC] and [MP2] a complete description of plane curves up to Cremona equivalence is given and in [CC] a detailed study of the Cremona equivalence for linear systems is furnished. In particular it is interesting to note that the Cremona equivalence of a plane curve is dictated by its singularities and cannot be guessed without a partial resolution of those, [MP2, Example 3.18]. Due to this it is quite hard even in the plane curve case to determine the Cremona equivalence class of a fixed curve simply by its equation.

It is then natural to investigate surfaces in  $\mathbb{P}^3$ . In this set up using the  $\sharp$ -Minimal Model Program, developed in [Me] or minimal model program with scaling [BCHM], a criterion for detecting surfaces Cremona equivalent to a plane has been given in [MP2]. The criterion, inspired by the previous work of Coolidge on curves Cremona equivalent to lines [Co], allows to determine all rational surfaces that are Cremona equivalent to a plane, [MP2, Theorem 4.15]. Unfortunately, worse than in the plane curve case, the criterion requires not only the resolution of singularities but also a control on different log varieties attached to the pair  $(\mathbb{P}^3, S)$ . So far it has been impossible to apply it to explicit examples of surfaces in  $\mathbb{P}^3$ . The main difficulty coming from the necessity to check the threshold, see Definition 1.2, on all good models, see Definition 1.4, of the pair  $(\mathbb{P}^3, S)$ . In this note I start removing this condition from the criterion in Corollary 1.7. This improvement gives back a result that can be applied in a wide range of cases. Next I concentrate on the case of quartic rational surfaces. The reason I study this special class of surfaces is twofold. Firstly it is quite easy to study the CE up to cubic surfaces, see Proposition 1.10. Then rational quartic surfaces are the first non immediate case having hundreds of non isomorphic families, [De] [Jes]. Beside this, there is an intrinsic complexity of quartic surfaces in  $\mathbb{P}^3$  that deserves to be studied under any perspective. Smooth quartic surfaces are the only smooth hypersurfaces with automorphisms not coming from linear automorphism of  $\mathbb{P}^n$ , [MM]. In a recent paper K. Oguiso produced examples of isomorphic smooth quartic surfaces that are not CE, [Og]. It is a long standing problem to determine which quartic surfaces are stabilized by subgroup of the Cremona group, that is for which quartic surface  $S \subset \mathbb{P}^3$  there is a Cremona modification  $\omega : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$  such that  $\omega$  is not an isomorphism and  $\omega(S) = S$ . The above problem has been studied by Enriques [En] and Fano [Fa] and also by Sharpe and coauthors in a series of papers, [MS] and [SS], at the beginning of the XX<sup>th</sup> century. More recently Araujo-Corti-Massarenti continued the study of mildly singular quartic surfaces admitting a non trivial stabilizers in the Cremona Group, [ACM], in the context of Calabi-Yau pairs preserving symplectic forms. In the light of these specialities of quartic surfaces the main theorem I prove is the following, quite surprising, result.

**Theorem 1.** *Let  $S \subset \mathbb{P}^3$  be an irreducible and reduced rational quartic surface. Then  $S$  is Cremona Equivalent to a plane.*

This shows that any rational quartic has a huge stabilizer in the Cremona group disregarding the type of singularity it may have. Indeed it is amazing that, even if there are hundreds of non isomorphic families of rational quartics, see [Jes] [De], the Cremona group of  $\mathbb{P}^3$  is playable enough to smooth any of them to a plane. A similar statement is not true for rational surfaces of degree at least 8, as a straightforward consequence of Noether-Fano inequalities. I have not a precise feeling on what

happens in the remaining degrees 5,6,7, but I think it is worthwhile to study them all.

The proof of the theorem is based on the simplified version of the criterion in [MP2] together with the analysis of some special Cremona modification associated to linear systems of quadrics. Indeed in many instances it is useful to produce a linear system of quadrics having multiplicity half the multiplicity of  $S$  along some valuation embedded in  $\text{Sing}(S)$ . Via this linear system the quartic is often simplified and can be linearized in an easier way.

I want to thank Ciro Ciliberto for reviving my interest in Cremona Equivalence for rational surfaces during a very pleasant stay in Cetraro and Igor Dolgachev for pointing out Jessop's book [Jes] and the Cyclides treated in the final Example. Thanks are due to the referee, whose careful reading prevented misprints and clarified the exposition.

## 1. PRELIMINARIES

I work over the complex field.

**Definition 1.1.** Let  $X, Y \subset \mathbb{P}^N$  be irreducible and reduced subvarieties of dimension  $r$ . I say that  $X$  is Cremona Equivalent (CE) to  $Y$  if there is a birational modification  $\phi : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$  such that  $\phi(X) = Y$  and  $\phi$  is well defined on the generic point of  $X$ .

It is clear that if  $X$  is CE to  $Y$  then  $X$  and  $Y$  are birational. This necessary condition is also sufficient as long as  $X$  is not a divisor by the main theorem in [MP], see also [CCMRZ]. In this note I am interested in studying the CE of rational surfaces of  $\mathbb{P}^3$ . For this reason I start with some definition and results about uniruled 3-folds.

**Definition 1.2.** Let  $(T, H)$  be a  $\mathbb{Q}$ -factorial uniruled 3-fold and  $H$  an irreducible and reduced effective Weil divisor on  $T$ . Let

$\rho(T, H) =: \sup \{m \in \mathbb{Q} \mid H + mK_T \text{ is } \mathbb{Q}\text{-linearly equivalent to an effective } \mathbb{Q}\text{-divisor} \}$   
be the (effective) threshold of the pair  $(T, H)$ .

*Remark 1.3.* The threshold is not a birational invariant of pairs and it is not preserved by blowing up. Consider a plane  $H \subset \mathbb{P}^3$  and let  $Y \rightarrow \mathbb{P}^3$  be the blow up of a point in  $H$  then  $\rho(Y, H_Y) = 0$ , while  $\rho(\mathbb{P}^3, H) = 1/4$ . For future reference note that both are less than one.

In [MP2], to overcome this problem it was introduced the notion of good models and of sup threshold  $\bar{\rho}(T, S)$ , [MP2]. These combined allowed to characterize the Cremona Equivalence to a plane, [MP2, Theorem 4.15]. The dark side of this characterization is the impossibility to check it on explicit examples.

Here, by a simple trick, I want to simplify the statement of [MP2, Theorem 4.15] to make it applicable in many instances. For this purpose I start recalling the following definition.

**Definition 1.4.** Let  $(Y, S_Y)$  be a 3-fold pair. The pair  $(Y, S_Y)$  is a birational model of the pair  $(T, S)$  if there is a birational map  $\varphi : T \dashrightarrow Y$  such that  $\varphi$  is well defined on the generic point of  $S$  and  $\varphi(S) = S_Y$ . A good model, [MP2], is a pair  $(Y, S_Y)$  with  $S_Y$  smooth and  $Y$  terminal and  $\mathbb{Q}$ -factorial.

*Remark 1.5.* Let  $(T, S)$  be a pair. To produce a good model it is enough to consider a log resolution of  $(T, S)$ . Clearly there are infinitely many good models for any pair and running a directed MMP one can find the one that is more suitable for the needs of the moment.

My first aim is to show that to check the Cremona Equivalence to a plane it is not necessary to go through all good models. In this direction the first technical result I am proving is that, even if the threshold is not a birational invariant of the pair as a number, there is the following useful property.

**Lemma 1.6.** *Let  $(T, S)$  and  $(T_1, S_1)$  be birational models of a pair. Assume that  $(T, S)$  has canonical singularities. If  $\rho(T, S) = a \geq 1$  then  $\rho(T_1, S_1) \geq a$ .*

*Proof.* Let  $\varphi : T \dashrightarrow T_1$  be a birational map with  $\varphi(S) = S_1$ . Let

$$\begin{array}{ccc} & Z & \\ q \swarrow & & \searrow p \\ T & \xrightarrow{\varphi} & T_1, \end{array}$$

be a resolution of the map  $\varphi$ .

I have

$$\begin{aligned} aK_Z + S_Z &= (a-1)K_Z + K_Z + S = p^*(a-1)K_T + \Delta + p^*(K_T + S) + \Delta_S = \\ &= p^*(aK_T + S_T) + \Delta + \Delta_S, \end{aligned}$$

for  $\mathbb{Q}$ -divisors  $\Delta$  and  $\Delta_S$ . The pairs  $(T, S)$  has canonical singularities, therefore  $\Delta$  and  $\Delta_S$  are effective divisors. By hypothesis  $aK_T + S$  is  $\mathbb{Q}$ -effective, thus  $aK_Z + S_Z$  is  $\mathbb{Q}$ -effective. Since

$$q_*(aK_Z + S_Z) \sim_{\mathbb{Q}} aK_{T_1} + S_1$$

this is enough to prove that  $\rho(T_1, S_1) \geq a$ .  $\square$

As a direct consequence of Lemma 1.6 I may reformulate the condition of being Cremona Equivalent to a plane, [MP2, Theorem 4.15], avoiding the check of the threshold of all good models.

**Corollary 1.7.** *A rational surface  $S \subset \mathbb{P}^3$  is Cremona equivalent to a plane if and only if there is a good model  $(T, S_T)$  of  $(\mathbb{P}^3, S)$  with  $0 < \rho(T, S_T) < 1$ .*

*Proof.* By Theorem 4.15 in [MP2]  $S$  is Cremona equivalent to a plane if and only if for all good models the threshold is bounded from the above by 1 and there is a good model with positive threshold. By Lemma 1.6 if there is a good model, say  $(T, S_T)$ , with  $0 < \rho(T, S_T) < 1$  all good models have threshold bounded by 1.  $\square$

*Remark 1.8.* Via a general projection  $S \subset \mathbb{P}^3$  of a quintic elliptic scroll in  $\mathbb{P}^4$ , it is possible to produce examples of good models  $(T, S_T)$  with  $T$  rational,  $S_T$  non rational and  $\rho(T, S_T) = 0$ . Allowing non rational singularities we may produce pairs with  $0 < \rho < 1$  and  $S$  non rational. Let  $S \subset \mathbb{P}^3$  be a cone over a smooth cubic curve then  $\rho(\mathbb{P}^3, S) = 3/4$  and the pair is not CE to a plane.

There is a class of surfaces, and more generally hypersurfaces, that are CE to a hyperplane.

*Remark 1.9.* Let  $X \subset \mathbb{P}^n$  be a monoid, that is an irreducible and reduced hypersurface of degree  $d$  with a point, say  $p$ , of multiplicity  $d - 1$ . Then I can write  $X = (x_0 F_{d-1} + F_d = 0)$ . Let us consider the linear system

$$\mathcal{L} := \{(F_{d-1}x_1 = 0), \dots, (F_{d-1}x_n = 0), X\}.$$

Then  $\varphi_{\mathcal{L}} : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$  is a birational modification and  $\varphi_{\mathcal{L}}(X)$  is a hyperplane. It follows that any monoid is CE to a hyperplane.

As a warm up I study rational surfaces of degree at most 3, see [MP2] and [Me2].

**Lemma 1.10.** *Let  $S \subset \mathbb{P}^3$  be an irreducible and reduced rational surface of degree at most 3. Then  $(\mathbb{P}^3, S)$  is CE to a plane.*

*Proof.* The statement is immediate in degree 2 by Remark 1.9, because any quadric is a monoid. Let  $S$  be a rational cubic. If  $S$  is smooth then  $(\mathbb{P}^3, S)$  is a good model with  $\rho(\mathbb{P}^3, S) = 3/4$ , hence I conclude by Corollary 1.7. If  $S$  has a double point, then it is a monoid and I conclude again by Remark 1.9. If  $S$  is a cone, then its plane section is a rational curve and I conclude by [Me2, Theorem 2.5].  $\square$

## 2. RATIONAL QUARTIC SURFACES

In this section I study the CE of rational quartic surfaces proving Theorem 1. The case of quartic surfaces singular along a twisted cubic is by far the most interesting from a geometric point of view.

*Remark 2.1.* There are two classes of these surfaces, a general projection of a rational scroll of degree 4 in  $\mathbb{P}^4$  and the tangential variety of the twisted cubic. The former has ordinary double points along  $\Gamma$  while the latter has cuspidal singularities. It is interesting to note that, from the point of view of CE they behave in the same way.

**Proposition 2.2.** *Let  $S \subset \mathbb{P}^3$  be a quartic surface singular along a twisted cubic  $\Gamma$ . Then  $S$  is CE to a plane.*

*Proof.* Let  $\nu : T \rightarrow \mathbb{P}^3$  be the blow up of  $\Gamma$  with exceptional divisor  $E$ . Then  $T$  has a scroll structure, say  $\pi : T \rightarrow \mathbb{P}^2$ , given by the secant lines of  $\Gamma$ , onto  $\mathbb{P}^2$ . In particular all fibers of  $\pi$  are irreducible and reduced and  $T = \mathbb{P}(\mathcal{E})$ , for a vector bundle  $\mathcal{E}$  on  $\mathbb{P}^2$  classically known to be defined by the following exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \rightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus 4} \rightarrow \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(1) \rightarrow 0.$$

Moreover I have  $\pi^* \mathcal{O}(1) = \nu^* \mathcal{O}(2) - E$  and  $S_T = \pi^* C$ , for  $C \subset \mathbb{P}^2$  an irreducible and reduced conic. Let  $S_T = \nu_*^{-1}(S)$  be the strict transform of the surface  $S$ . Then I have  $\rho(T, S_T) = 0$ . Note that at this point both the tangential variety and the general projection are pull backs of smooth conics. Therefore they are isomorphic as abstract varieties, but they differ in the scheme theoretic intersection with the exceptional divisor  $E$ . This is the only reason we end up with non isomorphic quartic surfaces in  $\mathbb{P}^3$ .

Next we want to extend a standard Cremona transformation of the base  $\mathbb{P}^2$  to a birational modification of  $T$ , following [Me, 5.7.4].

Let  $f_1, f_2, f_3 \subset S_T$  be three fibers of  $\pi$  in general position and  $l_i = \nu(f_i)$  the corresponding line in  $\mathbb{P}^3$ . Let  $Q_i \subset \mathbb{P}^3$  be the unique quadric containing  $\Gamma \cup l_j \cup l_k$ , with  $\{i, j, k\} = \{1, 2, 3\}$ , and  $D_i = \nu_*^{-1}(Q_i) \subset T$  its strict transform. Note that both  $Q_i$  and  $D_i$  are smooth quadrics and  $D_i \cap D_j = f_k$ , for  $\{i, j, k\} = \{1, 2, 3\}$ .

Let  $p : Z \rightarrow T$  be the blow up of the  $f_i$  with exceptional divisors  $E_i$ . Then by construction I have  $E_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ ,  $p_*^{-1}(D_i) \cong \mathbb{P}^1 \times \mathbb{P}^1$ , and in both cases the normal bundle is  $(0, -1)$ . This shows that there is a birational morphism  $q : Z \rightarrow T_1$  that blows down the  $D_i$ 's.

*Claim.*  $T_1 \cong T$  and  $S_{T_1} := (q \circ p^{-1})(S_T) \sim \pi_1^* \mathcal{O}(1)$ .

*Proof of the claim.* The varieties  $T$ ,  $Z$  and  $T_1$  are all scrolls.  $T$  and  $T_1$  over  $\mathbb{P}^2$ , and  $Z$  over the blow up of  $\mathbb{P}^2$  in 3 non collinear points, say  $W$ . Let  $T_1 = \mathbb{P}(\mathcal{E}_1)$  and  $\eta : W \rightarrow \mathbb{P}^2$  and  $\xi : W \rightarrow \mathbb{P}^2$  be the morphisms at the base level, induced by  $p$  and  $q$  respectively,

$$\begin{array}{ccccc}
 & & \mathbb{P}(\mathcal{E}_Z) = Z & & \\
 & \swarrow p & \downarrow \pi_Z & \searrow q & \\
 \mathbb{P}(\mathcal{E}) = T & & & & T_1 = \mathbb{P}(\mathcal{E}_1) \\
 \downarrow \pi & & \downarrow & & \downarrow \pi_1 \\
 \mathbb{P}^2 & \xleftarrow{\eta} & W & \xrightarrow{\xi} & \mathbb{P}^2,
 \end{array}$$

In particular I have  $\eta^* \mathcal{E} \cong \mathcal{E}_Z \cong \xi^* \mathcal{E}_1$ . The map  $\xi \circ \eta^{-1}$  is a standard Cremona modification, thus  $\eta$  and  $\xi$  are both the blow blow-ups of three non collinear points. Then I have  $\mathcal{E} = \mathcal{E}_1$  up to a twist. In particular  $T \cong T_1$ . Moreover, the choice of  $f_i \subset S_T$  yields  $(q \circ p^{-1})(S_T) \sim \pi_1^* \mathcal{O}(1)$ .  $\square$

By the Claim  $T_1 \cong T$  and  $S_{T_1} \sim \pi_1^*(\mathcal{O}(1))$ . By construction the 3-fold  $T$  had the contraction  $\nu : T \rightarrow \mathbb{P}^3$ . Therefore there is a contraction  $\nu_1 : T_1 \rightarrow \mathbb{P}^3$  that sends  $S_{T_1}$  to a quadric, This shows that  $S$  is CE to a quadric and therefore it is CE to a plane by Proposition 1.10.  $\square$

*Remark 2.3.* I want to give an alternative way to interpret the birational modification described in the proof of Proposition 2.2. Consider three secant lines  $f_1, f_2, f_3$  to  $\Gamma$  and the linear system  $\Lambda$  of cubics containing  $R := \Gamma \cup l_1 \cup l_2 \cup l_3$ . It is easy to see that  $\dim \Lambda = 3$ . Observe that a general element  $D \in \Lambda$  is a smooth cubic. In particular  $D$  is isomorphic to a plane blow up in 6 general points, say  $\{q_1, \dots, q_6\}$ . The reducible curve  $R$  is represented, in the plane, by the 4 conics passing through a subset of  $\{q_1, \dots, q_6\}$  in such a way that any point is triple for  $R$ . Therefore the plane model of  $\Lambda|_D$  is a linear system of dimension 2, degree 9, multiplicity 3 in  $\{q_1, \dots, q_6\}$ , and containing the degree 8 curve  $R$ . This shows that the plane model of  $\Lambda|_D$  is  $R + \mathcal{O}(1)$  and therefore it induces a birational map onto  $\mathbb{P}^2$ . That is  $\Lambda$  induces a birational map onto  $\mathbb{P}^3$ .

This modification is a degeneration of the classical cubo cubic Cremona modification centered on a curve of degree 6 and genus 3 of  $\mathbb{P}^3$ .

The following Theorem settles the CE problem for rational quartic surfaces.

**Proposition 2.4.** *Let  $S \subset \mathbb{P}^3$  be a rational surface of degree 4, then  $S$  is CE to a plane.*

*Proof.* If  $S$  is a cone then its general plane section is rational and I conclude by [Me2]. If  $S$  has a point of multiplicity 3, I conclude by Remark 1.9.

From now on I assume that  $S$  has only singular points of multiplicity 2. The first step in the proof is to show that rational quartic surfaces with isolated double points are CE to quartic surfaces with non isolated singularities.

Assume that  $S$  has isolated singularities. Then the rationality of  $S$  forces the presence of an elliptic point. By a computation on Leray spectral series, see [Um],  $S$  has a unique irrational singularity. Furthermore, by [Jes] and [De] classification, the irrational singularity is of the following type, in brackets the corresponding equation of  $S$ :

- (1) a double point with an infinitely near double line  
 $[x_0^2x_1^2 + x_0x_1Q_2(x_2, x_3) + F_4(x_1, x_2, x_3) = 0]$ ,
- (2) a tachnode with an infinitely near double line  
 $[x_0^2x_1^2 + x_0(x_2^3 + x_1Q_2(x_2, x_3)) + F_4(x_1, x_2, x_3) = 0]$ .

Let  $S$  be a rational quartic with a singular point of type  $(a)$ ,  $a \in \{1, 2\}$ , and let  $\Lambda_a \subset |\mathcal{O}(2)|$  be the linear system of quadrics having multiplicity  $a + 1$  on the valuation associated to the double line. Then it is easy to check that the map  $\varphi_{\Lambda_a} : \mathbb{P}^3 \dashrightarrow X_a \subset \mathbb{P}^{7-a}$  is birational.

As observed in [MP2, Example 4.3]  $X_1 \subset \mathbb{P}^6$  is the cone over the Veronese surface. With a similar argument,  $X_2 \subset \mathbb{P}^5$  is the projection of  $X_1$  from any smooth point  $z \in X_1$ . This shows that  $X_2$  is the cone over the cubic surface  $C \subset \mathbb{P}^4$ , where  $C$  is the projection of the Veronese surface, say  $V$ , from a point  $z \in V$ .

The main point here is that in both cases I have  $S_a := \varphi_{\Lambda_a}(S) \subset |\mathcal{O}_{\mathbb{P}^{7-a}}(2)|$ .

*Case 2.5 ( $S_2$ ).* Assume that  $S$  has a point of type (2). Then the pair  $(\mathbb{P}^3, S)$  is birational to  $(X_2, S_2)$ . The surface  $S_2 \subset X_2 \subset \mathbb{P}^5$  has degree 6 and  $X_2$  has degree 3. Let  $x \in S_2$  be a general point and  $\pi : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$  the projection from  $x$ . Then  $\pi(X_2) = Q$  is a quadric cone and  $S_x := \pi(S_2)$  is a surface of degree 5. Hence there is a cubic hypersurface  $D \subset \mathbb{P}^4$  such that

$$D|_Q = S_x + H,$$

for some plane  $H$ . Let  $y \in S_x$  be a general point and  $\pi_y : \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$  the projection from  $y$ .

*Claim.*  $\tilde{S} := \pi_y(S_x)$  is a quartic surface singular along a line.

*Proof.* The point  $y$  is general therefore  $\deg \tilde{S} = 4$ . The map  $\pi_{y|Q}$  is birational and it contracts the embedded tangent cone  $\mathbb{T}_y Q \cap Q = \Pi_1 \cup \Pi_2$  to a pair of lines  $l_1 \cup l_2$ . Up to reordering I may assume that  $H \cap \Pi_1$  is the vertex of the cone. Therefore  $\tilde{S} \cap \Pi_1$  is a cubic passing through  $y$ . Hence  $\tilde{S}$  has multiplicity 2 along  $l_1$ .  $\square$

The surface  $S$  is therefore CE to a quartic with non isolated singularities.

*Case 2.6 ( $S_1$ ).* Assume that  $S$  has a point of type (1). Then  $(\mathbb{P}^3, S)$  is birational to  $(X_1, S_1)$ .

*Claim.*  $S_1$  is in the smooth locus of  $X_1$  and  $S_1$  has at most ordinary double points.

*Proof.* Let me start describing the map  $\varphi := \varphi_{\Lambda_1} : \mathbb{P}^3 \dashrightarrow X_1$ , following [MP2, Example 4.3],

Let  $S \subset \mathbb{P}^3$  be the quartic, I may assume that the irrational singular point is  $p \equiv [1, 0, 0, 0] \in S$  and the equation of  $S$  is

$$(x_0^2x_1^2 + x_0x_1Q + F_4 = 0) \subset \mathbb{P}^3.$$

Let  $\epsilon : Y \rightarrow \mathbb{P}^3$  be the weighted blow up of  $p$ , with weights  $(2, 1, 1)$  on the coordinates  $(x_1, x_2, x_3)$ , and exceptional divisor  $E \cong \mathbb{P}(1, 1, 2)$ . Then I have:

- $\epsilon^*(x_1 = 0) = H + 2E$ ,  $\epsilon|_H : H \rightarrow (x_1 = 0)$  is an ordinary blow up and  $H|_E$  is a smooth rational curve;
- $\epsilon^*(S) = S_Y + 4E$ ,  $S_{Y|E}$  is a smooth elliptic curve, and  $S_{Y|H}$  is a union of four smooth disjoint rational curves.

In particular both  $H$  and  $S_Y$  are on the smooth locus of  $E$  and hence on the smooth locus of  $Y$ . The surface  $H$  is ruled by, the strict transforms of, the lines in the plane  $(x_1 = 0)$  passing through the point  $p$ . Let  $l_Y$  be a general curve in the ruling and  $\Lambda_Y = \epsilon_*^{-1}(\Lambda_1)$  the strict transform linear system. Then  $E \cdot l_Y = 1$  and by a direct computation I have

$$\begin{aligned} - \Lambda_Y \cdot l_Y &= (\epsilon^*(\mathcal{O}(2)) - 2E) \cdot l_Y = 0 \\ - S_Y \cdot l_Y &= (\epsilon^*(\mathcal{O}(4)) - 4E) \cdot l_Y = 0 \\ - K_Y \cdot l_Y &= (\epsilon^*(\mathcal{O}(-4)) + 3E) \cdot l_Y = -1. \\ - H \cdot l_Y &= (\epsilon^*(\mathcal{O}(1)) - 2E) \cdot l_Y = -1. \end{aligned}$$

Then  $H$  can be blown down to a smooth rational curve with a birational map  $\mu : Y \rightarrow X_1$  and by construction  $S_Y = \mu^*S_1$ . This shows that the unique singularity of  $X_1$  is the singular point in  $E$  and  $S_Y$  has at most isolated rational double points.  $\square$

If  $S_1$  is smooth then  $(X_1, S_1)$  is a good model of  $(\mathbb{P}^3, S)$  with threshold  $\rho(X_1, S) = 4/5$  and I conclude by Corollary 1.7. Assume that  $S_1$  is singular and let  $x \in \text{Sing}(S_1)$  be a singular point. Set  $\pi : \mathbb{P}^6 \dashrightarrow \mathbb{P}^5$  be the projection from  $x$ . Then  $\pi|_{X_1} : X_1 \dashrightarrow X_2$  is birational and  $\pi(S_1) \in |\mathcal{O}_{X_2}(2)|$ . I am therefore back to the previous case. This shows that  $(\mathbb{P}^3, S)$  is CE to a quartic with non isolated singularities.

To conclude the Proposition I am left to study the case of quartics with non isolated singularities.

From now on I fix a rational quartic  $S$  with a curve  $\Gamma$  of double points. Assume first that  $\Gamma$  contains a line  $l$ . Fix a general point  $x \in S$  and the linear system  $\Lambda$  of quadrics through  $l$  and  $x$ . Let  $\varphi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^5$  be the map associated to the linear system  $\Lambda$ . I have  $\varphi(\mathbb{P}^3) = Z \cong \mathbb{P}^1 \times \mathbb{P}^2$ , embedded via the Segre map, and  $\varphi(S) = \tilde{S}$  is a divisor of type  $(3, 2)$  in  $\mathbb{P}^1 \times \mathbb{P}^2$ . Note that divisors of type  $(1, 0)$  are planes and divisors of type  $(0, 1)$  are quadrics, then I have  $\deg \tilde{S} = 3 + 4 = 7$ . If  $\tilde{S}$  is smooth then  $(Z, \tilde{S})$  is a good model of  $(\mathbb{P}^3, S)$  with  $\rho(Z, \tilde{S}) = 2/3$  and I conclude by Corollary 1.7. If  $\tilde{S}$  is singular let  $y \in \text{Sing}(\tilde{S})$  be a point and  $\pi : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$  the projection from  $y$ . Then  $\pi|_Z$  is a birational map,  $Y := \pi(Z) \subset \mathbb{P}^4$  is a quadric of rank 4, and  $S_Q := \pi(\tilde{S})$  is a rational surface of degree  $7 - 2 = 5$ .

*Claim.* The vertex of the quadric is a smooth point of  $S_Q$ .

*Proof.* The surface  $\tilde{S}$  is a divisor of type  $(3, 2)$  in  $Z$  and it is singular in  $y$ . Let  $l$  and  $P$ , respectively, be the line and the plane passing through  $x$  in  $Z$ . The general choice of  $x \in S$  yields  $l \not\subset \tilde{S}$ . The line  $l$  is mapped to the vertex of the quadric and  $\tilde{S}|_l = 2x + p$  for some point  $p$ . This shows that  $S_Q$  contains the vertex of the quadric and it is smooth there.  $\square$

The 3-fold  $Q$  is a quadric cone. If  $S_Q$  is smooth let  $\nu : T \rightarrow Q$  be a  $\mathbb{Q}$ -factorialization of  $Q$  and  $S_T$  the strict transform of  $S_Q$ . Then  $(T, S_T)$  is a good model for  $(\mathbb{P}^3, S)$  and  $\rho(T, S_T) = 2/3$ . Therefore I conclude by Corollary 1.7. If  $S_Q$



is singular let  $z \in \text{Sing}(S_Q)$  be a point. By the Claim it is not the vertex of  $Q$ . Thus the projection from  $z$  produces a birational model of  $(Q, S_Q)$ , say  $(\mathbb{P}^3, Z)$ , with  $Z$  a rational cubic and I conclude by Lemma 1.10.

Assume that  $\Gamma$  does not contain a line. It is easy to see that  $\deg \Gamma \leq 3$ . Moreover if  $\deg \Gamma = 3$  the curve  $\Gamma$  is a twisted cubic. Therefore I am left to consider the following cases:  $\Gamma$  an irreducible conic,  $\Gamma$  a twisted cubic. If  $\Gamma$  is a conic let  $x \in S$  be a general point. Then the linear system of quadrics through  $\Gamma$  and  $x$  maps  $(\mathbb{P}^3, S)$  to the pair  $(\mathbb{P}^3, S')$  with  $S'$  a rational cubic surface, and I conclude again by Lemma 1.10. If  $\Gamma$  is a twisted cubic I conclude by Proposition 2.2.  $\square$

**Example 2.7.** *I finish giving an explicit example of linearization of quartics that has been classically studied for being envelopes of bitangent spheres, [Jes, Chapter V]: the Cyclides. I thank Igor Dolgachev for pointing out this special class of quartics and Alex Massarenti for working out the explicit equations with Macauley2*

Let  $S \subset \mathbb{P}^3$  be a quartic with the following equation

$$(x^2 + y^2 + z^2 - w^2)^2 + w^2q = 0,$$

where  $q$  is a polynomial of degree 2. The surface  $S$  is singular along the conic  $C = (w = x^2 + y^2 + z^2 = 0)$ . Assume first that there is a further singularity, say  $p \notin C$ , this is for instance the case of the Dupin's cyclid. Let  $Q$  be a general quadric through  $C \cup p$ . Then  $Q \cap S = 2C + R$  for some residual curve  $R$ . The residual curve  $R$  is a quartic rational curve. Fix a general point in  $q \in S$ . Then the linear system that linearizes  $S$  is

$$\Lambda = |\mathcal{I}_{C^2 \cup p^2 \cup R \cup q}(4)|.$$

If  $p$  is defined over the field of real numbers then the map is defined over the real numbers. Here is a sample with plausible equations. Let us start with

$$S = ((x^2 + y^2 + z^2 - w^2)^2 + w^2((w - x)^2 + y^2 - z^2) = 0)$$

the singular point is  $p = [1, 0, 0, 1]$  and the general point  $q = [0, 0, 1, 1]$ . Then a linear system that linearize  $S$  is

$$\{x^2zw + y^2zw + z^3w + 2x^2w^2 + y^2w^2 + xzw^2 - z^2w^2 - 4xw^3 - 2zw^3 + 2w^2, \\ x^2yw + y^3w + yz^2w + xyw^2 - 2yw^3, x^3w + xy^2w + xz^2w - 2x^2w^2 - 2y^2w^2 + xw^3, \\ x^4 + 2x^2y^2 + y^4 + 2x^2z^2 + 2y^2z^2 + z^4 - x^2w^2 - y^2w^2 - 3z^2w^2 - 2xw^3 + 2w^4\}.$$

If  $S$  has not further double points let  $\Gamma$  be a smooth rational quartic curve in  $S$ . Let  $p \in \Gamma \in S$  be a general point and consider the linear system

$$\Sigma = |\mathcal{I}_{C^3 \cup p^3 \cup \Gamma}(6)|.$$

Let  $D \in \Sigma$  be a general element then

$$D \cap S = 6C + \Gamma + R$$

This time the residual curve  $R$  has degree 8 and genus 3. The linear system that linearizes  $S$  is then

$$\Lambda = |\mathcal{I}_{C^3 \cup p^3 \cup R}(6)|.$$

Denote by  $E$  the cone over  $C$  with vertex  $p$ , then  $E + S \subset \Lambda$  and  $E$  is contracted by  $\varphi_\Lambda$ .

It is not clear to me if any such cyclid contains a rational quartic curve defined over the field of real numbers.

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