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On the Asymptotic Behavior of *D*-Solutions to the Displacement Problem of Linear Elastostatics in Exterior Domains

Vincenzo Coscia

Department of Mathematics and Computer Science, University of Ferrara, Via Machiavelli 30, 44121 Ferrara, Italy; vincenzo.coscia@unife.it

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Abstract: We study the asymptotic behavior of solutions with finite energy to the displacement problem of linear elastostatics in a three-dimensional exterior Lipschitz domain.

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1. Introduction

The displacement problem of elastostatics in an exterior Lipschitz domain Ω of $\mathbb{R}^3$ consists of finding a solution to the equations [1] (Notation—Unless otherwise specified, we will use the notation of the classical monograph [1] by M.E. Gurtin. In particular, $(\operatorname{div}\mathsf{C}[\nabla \boldsymbol{u}])_i = \partial_j(\mathsf{C}_{ijhk}\partial_k u_h)$, Lin is the space of second–order tensors (linear maps from $\mathbb{R}^3$ into itself) and Sym, Skw are the spaces of the symmetric and skew elements of Lin respectively; if $\boldsymbol{E} \in \mathrm{Lin}$ and $\boldsymbol{v} \in \mathbb{R}^3$, $\boldsymbol{Ev}$ is the vector with components $E_{ij}v_j$. $B_R = \{x \in \mathbb{R}^3 : r = |x| < R\}$, $T_R = B_{2R} \setminus B_R$, $\complement B_R = \mathbb{R}^3 \setminus \overline{B_R}$ and B_{R_0} is a fixed ball containing $\partial\Omega$. If $f(x)$ and $\phi(r)$ are functions defined in a neighborhood of infinity, then $f(x) = o(\phi(r))$ means that $\lim_{r\to+\infty}(f/\phi) = 0$. To lighten up the notation, we do not distinguish between scalar, vector, and second–order tensor space functions; c will denote a positive constant whose numerical value is not essential to our purposes.)

$$
\begin{aligned}
\operatorname{div}\mathsf{C}[\nabla \boldsymbol{u}] &= \boldsymbol{0} \quad \text{in } \Omega,\\
\boldsymbol{u} &= \hat{\boldsymbol{u}} \quad \text{on } \partial\Omega,\\
\lim_{R\to+\infty}\int_{\partial B} \boldsymbol{u}(R,\sigma)d\sigma &= \boldsymbol{0},
\end{aligned} \tag{1}
$$

where $\boldsymbol{u}$ is the (unknown) displacement field, $\hat{\boldsymbol{u}}$ is an (assigned) boundary displacement, B is the unit ball, $\mathsf{C} \equiv [\mathsf{C}_{ijhk}]$ is the (assigned) elasticity tensor, i.e., a map from $\Omega \times \mathrm{Lin} \to \mathrm{Sym}$, linear on Sym and vanishing in $\Omega \times \mathrm{Skw}$. We shall assume C to be symmetric, i.e.,

$$
\boldsymbol{E}\cdot\mathsf{C}[\boldsymbol{L}] = \boldsymbol{L}\cdot\mathsf{C}[\boldsymbol{E}] \quad \forall \boldsymbol{E},\boldsymbol{L} \in \mathrm{Lin}, \tag{2}
$$

and positive definite, i.e., there exists positive scalars μ_0 and μ_e (minimum and maximum elastic moduli [1]) such that

$$
\mu_0|\boldsymbol{E}|^2 \le \boldsymbol{E}\cdot\mathsf{C}[\boldsymbol{E}] \le \mu_e|\boldsymbol{E}|^2, \quad \forall\, \boldsymbol{E} \in \mathrm{Sym}, \;\; \textit{a.e. in } \Omega. \tag{3}
$$

Let $D^{1,q}(\Omega)$, $D_0^{1,q}(\Omega)$ ($q \in [1,+\infty)$) be the completion of $C_0^\infty(\overline{\Omega})$ and $C_0^\infty(\Omega)$, respectively, with respect to the norm $\|\nabla \boldsymbol{u}\|_{L^q(\Omega)}$.

We consider solutions $\boldsymbol{u}$ to equations (1) with finite Dirichlet integral (or with finite energy) that we call D-solutions analogous with the terminology used in viscous fluid dynamics (see [2]). More precisely, we say that $\boldsymbol{u} \in D^{1,2}(\Omega)$ is a D-solution to equation $(1)_1$

$$\int_\Omega \nabla \boldsymbol{\varphi} \cdot \mathbf{C}[\nabla \boldsymbol{u}] = 0, \quad \forall \boldsymbol{\varphi} \in D_0^{1,2}(\Omega). \tag{4}$$

A D-solution to system (1) is a D-solution to equation $(1)_1$, which satisfies the boundary condition in the sense of trace in Sobolev's spaces and tends to zero at infinity in a mean square sense [2]

$$\lim_{R\to+\infty} \int_{\partial B} |\boldsymbol{u}(R,\sigma)|^2 d\sigma = 0. \tag{5}$$

If $\boldsymbol{u}$ is a D-solution to $(1)_1$, then the traction field on the boundary is

$$\boldsymbol{s}(\boldsymbol{u}) = \mathbf{C}[\nabla \boldsymbol{u}]\boldsymbol{n}$$

where a well defined field of $W^{-1/2,2}(\partial\Omega)$ exists and the following generalized work and energy relation [1] holds

$$\int_{\Omega\cap B_R} \nabla \boldsymbol{u} \cdot \mathbf{C}[\nabla \boldsymbol{u}] = \int_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{s}(\boldsymbol{u}),$$

where abuse of notation $\int_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{s}(\boldsymbol{u})$ means the value of the functional $\boldsymbol{s}(\boldsymbol{u}) \in W^{-1/2,2}(\partial\Omega)$ at $\boldsymbol{u} \in W^{1/2,2}(\partial\Omega)$, and $\boldsymbol{n}$ is the unit outward (with respect to Ω) normal to $\partial\Omega$.

If $\hat{\boldsymbol{u}} \in W^{1/2,2}(\partial\Omega)$, denoting by $\boldsymbol{u}_0 \in D^{1,2}(\Omega)$ an extension of $\hat{\boldsymbol{u}}$ in Ω vanishing outside a ball, then $(1)_{1,2}$ is equivalent to finding a field $\boldsymbol{u} \in D_0^{1,2}(\Omega)$ such that

$$\int_\Omega \nabla \boldsymbol{\varphi} \cdot \mathbf{C}[\nabla \boldsymbol{v}] = -\int_\Omega \nabla \boldsymbol{\varphi} \cdot \mathbf{C}[\nabla \boldsymbol{u}_0], \quad \forall \boldsymbol{\varphi} \in D_0^{1,2}(\Omega). \tag{6}$$

Since the right-hand side of (6) defines a linear and continuous functional on $D_0^{1,2}(\Omega)$, and by the first Korn inequality (see [1] Section 13)

$$\int_\Omega |\nabla \boldsymbol{v}|^2 \le \frac{2}{\mu_0} \int_\Omega \nabla \boldsymbol{v} \cdot \mathbf{C}[\nabla \boldsymbol{v}],$$

by the Lax–Milgram lemma, (6) has a unique solution $\boldsymbol{v}$, and the field $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{u}_0$ is a D-solution to $(1)_{1,2}$. It satisfies $(1)_3$ in the following sense (see Lemma 1)

$$\int_{\partial B} |\boldsymbol{u}(R,\sigma)|^2 d\sigma = o(R^{-1}). \tag{7}$$

Moreover, $\boldsymbol{u}$ exhibits more regularity properties provided $\mathbf{C}$, $\partial\Omega$ and $\hat{\boldsymbol{u}}$ are more regular. In particular, if $\mathbf{C}$, $\hat{\boldsymbol{u}}$ and $\partial\Omega$ are of class C^∞, then $\boldsymbol{u} \in C^\infty(\overline{\Omega})$ [3].

If $\mathbf{C}$ is constant, then existence and regularity hold under the weak assumption of strong ellipticity [1], i.e.,

$$\mu_0|\boldsymbol{a}|^2|\boldsymbol{b}|^2 \le \boldsymbol{a} \cdot \mathbf{C}[\boldsymbol{a} \otimes \boldsymbol{b}]\boldsymbol{b}, \quad \forall\, \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3. \tag{8}$$

As far as we are aware, except for the property (7), little is known about the convergence at infinity of a D-solution and, in particular, whether or under what additional conditions (7) can be improved.

The main purpose of this paper is just to determine reasonable conditions on C assuring that (7) can be improved.

We say that C is *regular at infinity* if there is a constant elasticity tensor C_0 such that

$$\lim_{|x|\to+\infty}\mathsf{C}(x)=\mathsf{C}_0. \tag{9}$$

Let $\mathfrak{C}_0$ and $\mathfrak{C}$ denote the linear spaces of the D-solutions to the equations

$$\begin{aligned}\operatorname{div}\mathsf{C}[\nabla \boldsymbol{h}]&=\mathbf{0}\quad \text{in }\Omega,\\ \boldsymbol{h}&=\boldsymbol{\tau}\quad \text{on }\partial\Omega,\\ \lim_{R\to+\infty}\int_{\partial B}|\boldsymbol{h}(R,\sigma)|^2d\sigma&=\mathbf{0},\end{aligned} \tag{10}$$

for all $\boldsymbol{\tau}\in\mathbb{R}^3$ and

$$\begin{aligned}\operatorname{div}\mathsf{C}[\nabla \boldsymbol{h}]&=\mathbf{0}\quad \text{in }\Omega,\\ \boldsymbol{h}&=\boldsymbol{\tau}+\boldsymbol{A}\boldsymbol{x}\quad \text{on }\partial\Omega,\\ \lim_{R\to+\infty}\int_{\partial B}|\boldsymbol{h}(R,\sigma)|^2d\sigma&=\mathbf{0},\end{aligned} \tag{11}$$

for all $\boldsymbol{\tau}\in\mathbb{R}^3$, $\boldsymbol{A}\in\mathrm{Lin}$, respectively.

The following theorem holds.

Theorem 1. *Let $\boldsymbol{u}$ be the D–solution to (1). There is $q<2$ depending only on C such that*

$$\int_{\partial B}|\boldsymbol{u}(R,\sigma)|^q d\sigma=o(R^{q-3}). \tag{12}$$

If C is regular at infinity, then

$$\int_{\partial B}|\boldsymbol{u}(R,\sigma)|^q d\sigma=o(R^{q-3}),\quad \forall q\in(3/2,+\infty), \tag{13}$$

and

$$\int_{\partial B}|\boldsymbol{u}(R,\sigma)|^q d\sigma=o(R^{q-3}),\ \ \forall q\in(1,2]\iff\int_{\partial\Omega}\hat{\boldsymbol{u}}\cdot\boldsymbol{s}(\boldsymbol{h})=\mathbf{0},\ \ \forall\boldsymbol{h}\in\mathfrak{C}_0. \tag{14}$$

Moreover, if

$$\int_{\partial\Omega}\mathsf{C}[\hat{\boldsymbol{u}}\otimes\boldsymbol{n}]=\mathbf{0},\quad \int_{\partial\Omega}\hat{\boldsymbol{u}}\cdot\boldsymbol{s}(\boldsymbol{h})=\mathbf{0},\ \ \forall\boldsymbol{h}\in\mathfrak{C}, \tag{15}$$

then

$$\int_{\partial B}|\boldsymbol{u}(R,\sigma)|d\sigma=o(R^{-2}). \tag{16}$$

2. Preliminary Results

In this section, we collect the main tools we need to prove Theorem 1.

Lemma 1. *If $u\in D^{1,q}(\Omega)$, for $q\in[1,2]$, then*

$$\int_{\partial B}|\boldsymbol{u}(R,\sigma)|^q d\sigma\le c(q)R^{q-3}\int_{\complement B_R}|\nabla\boldsymbol{u}|^q. \tag{17}$$

Moreover, if $q = 2$*, then, for* $R \gg R_0$*,*

$$\int_{\complement B_R} \frac{|\boldsymbol{u}|^2}{r^2} \le 4 \int_{\complement B_R} |\nabla \boldsymbol{u}|^2. \tag{18}$$

Proof. Lemma 1 is well-known (see, e.g., [2,4] and [5] Chapter II). We propose a simple proof for the sake of completeness. Since $D^{1,2}(\Omega)$ is the completion of $C_0^\infty(\overline{\Omega})$ with respect to the norm $\|\nabla \boldsymbol{u}\|_{L^2(\Omega)}$, it is sufficient to prove (17) and (18) for a regular field $\boldsymbol{u}$ vanishing outside a ball. By basic calculus and Hölder inequality,

$$\begin{aligned}\int_{\partial B} |\boldsymbol{u}(R,\sigma)|^q d\sigma &= \int_{\partial B} \left| \int_R^{+\infty} \partial_r \boldsymbol{u}(r,\sigma) dr \right|^q d\sigma = \int_{\partial B} \left| \int_R^{+\infty} r^{2/q} r^{-2/q} \partial_r \boldsymbol{u}(r,\sigma) dr \right|^q d\sigma \\ &\le \left\{ \int_{\partial B} d\sigma \int_R^{+\infty} |\nabla \boldsymbol{u}(r,\sigma)|^q r^2 dr \right\} \left\{ \int_{\partial B} \left| \int_R^{+\infty} r^{-\frac{2}{q-1}} dr \right|^{q-1} d\sigma \right\}.\end{aligned}$$

Hence, (17) follows by a simple integration.

From

$$\int_{\complement B_R} \frac{|\boldsymbol{u}|^2}{r^2} = \int_R^{+\infty} \partial_r \left(r \int_{\partial B} |\boldsymbol{u}(r,\sigma)|^2 d\sigma \right) - 2 \int_{\complement B_R} \frac{\boldsymbol{u}}{r} \cdot \partial_r \boldsymbol{u}$$

by Schwarz's inequality, one gets

$$\int_{\complement B_R} \frac{|\boldsymbol{u}|^2}{r^2} \le 2 \left\{ \int_{\complement B_R} \frac{|\boldsymbol{u}|^2}{r^2} \int_{\complement B_R} |\nabla \boldsymbol{u}|^2 \right\}^{1/2}.$$

Hence, (18) follows at once. □

Let $\mathbf{C}_0$ be a constant and strongly elliptic elasticity tensor. The equation

$$\operatorname{div} \mathbf{C}_0[\nabla \boldsymbol{u}] = \mathbf{0} \tag{19}$$

admits a fundamental solution $\mathfrak{U}(x-y)$ [6] that enjoys the same qualitative properties as the well-known ones of homogeneous and isotropic elastostatics, defined by

$$\mathfrak{U}_{ij}(x-y) = \frac{1}{8\pi\mu(1-\nu)|x-y|} \left[(3-4\nu)\delta_{ij} + \frac{(x_i-y_i)(x_j-z_j)}{|x-y|^2} \right],$$

where μ is the shear modulus and ν the Poisson ratio (see [1] Section 51). In particular, $\mathfrak{U}(x) = O(r^{-1})$ and for $\boldsymbol{f}$ with compact support (say) the volume potential

$$\mathcal{V}[\boldsymbol{f}](x) = \int_{\mathbb{R}^3} \mathfrak{U}(x-y) \boldsymbol{f}(y) dv_y \tag{20}$$

is a solution (in a sense depending on the regularity of $\boldsymbol{f}$) to the system

$$\operatorname{div} \mathbf{C}_0[\nabla \boldsymbol{u}] + \boldsymbol{f} = \mathbf{0} \quad \text{in } \mathbb{R}^3. \tag{21}$$

Let $\mathcal{H}^1$ denote the Hardy space on $\mathbb{R}^3$ (see [7] Chapter III). The following result is classical (see, e.g., [7]).

Lemma 2. $\nabla_2 \mathcal{V}$ *maps boundedly* L^q *into itself for* $q \in (1, +\infty)$ *and* $\mathcal{H}^1$ *into itself.*

Lemma 3. *Let u be the D-solution to* (1)*, Then, for* $R \gg R_0$*,*

$$\int_{\partial\Omega} \boldsymbol{s}(\boldsymbol{u}) = \int_{\partial B_R} \boldsymbol{s}(\boldsymbol{u}), \tag{22}$$

and

$$\int_{\partial\Omega} \boldsymbol{h}\cdot\boldsymbol{s}(\boldsymbol{u}) = \int_{\partial\Omega} \hat{\boldsymbol{u}}\cdot\boldsymbol{s}(\boldsymbol{h}), \quad \forall \boldsymbol{h}\in\mathfrak{C}, \tag{23}$$

where $\mathfrak{C}$ *is the space of D-solutions to system* (11)*.*

Proof. Let

$$g(x) = \begin{cases} 0, & |x| > 2R, \\ 1, & |x| < R, \\ R^{-1}(R-|x|), & R \le |x| \le 2R, \end{cases} \tag{24}$$

with $R \gg R_0$. Scalar multiplication of both sides of $(1)_1$ by $g\boldsymbol{h}$, (2) and an integration by parts yield

$$\int_{\partial\Omega} \boldsymbol{h}\cdot\boldsymbol{s}(\boldsymbol{u}) - \int_{\partial\Omega} \hat{\boldsymbol{u}}\cdot\boldsymbol{s}(\boldsymbol{h}) = \frac{1}{R}\int_{T_R} \Big(\boldsymbol{u}\cdot\mathbf{C}[\nabla\boldsymbol{h}]\boldsymbol{e}_r - \boldsymbol{h}\cdot\mathbf{C}[\nabla\boldsymbol{u}]\boldsymbol{e}_r\Big),$$

where $\boldsymbol{e}_r = \boldsymbol{x}/r$. Since $R \le |x| \le 2R$, by Schwarz's inequality,

$$\begin{aligned} \left|\frac{1}{R}\int_{T_R} \boldsymbol{h}\cdot\mathbf{C}[\nabla\boldsymbol{u}]\boldsymbol{e}_r\right| &\le 2\int_{T_R} r^{-1}|\boldsymbol{h}\cdot\mathbf{C}[\nabla\boldsymbol{u}]\boldsymbol{e}_r| \le c\|r^{-1}\boldsymbol{h}\|_{L^2(T_R)}\|\nabla\boldsymbol{u}\|_{L^2(T_R)}, \\ \left|\frac{1}{R}\int_{T_R} \boldsymbol{u}\cdot\mathbf{C}[\nabla\boldsymbol{h}]\boldsymbol{e}_r\right| &\le 2\int_{T_R} r^{-1}|\boldsymbol{u}\cdot\mathbf{C}[\nabla\boldsymbol{h}]\boldsymbol{e}_r| \le c\|r^{-1}\boldsymbol{u}\|_{L^2(T_R)}\|\nabla\boldsymbol{h}\|_{L^2(T_R)}. \end{aligned}$$

Hence, letting $R \to +\infty$ and taking into account Lemma 1, (23) follows. □

Lemma 4. *Let* $\boldsymbol{u}$ *be the D-solution to* (1)*; then, for* $R \gg R_0$*,*

$$\int_{\partial\Omega} \big(\boldsymbol{x}\otimes\boldsymbol{s}(\boldsymbol{u}) - \boldsymbol{C}[\hat{\boldsymbol{u}}\otimes\boldsymbol{n}]\big) = \int_{\partial B_R} \big(\boldsymbol{x}\otimes\boldsymbol{s}(\boldsymbol{u}) - \boldsymbol{C}[\boldsymbol{u}\otimes\boldsymbol{e}_R]\big), \tag{25}$$

where $\mathfrak{C}$ *is the space of D-solutions to system* (11)*.*

Proof. (25) is easily obtained by integrating the identity

$$\boldsymbol{0} = \boldsymbol{x}\otimes\operatorname{div}\mathbf{C}[\nabla\boldsymbol{u}] = \operatorname{div}\big(\boldsymbol{x}\otimes\mathbf{C}[\nabla\boldsymbol{u}]\big) - \mathbf{C}[\nabla\boldsymbol{u}]$$

over B_R and using the divergence theorem. □

3. Proof of Theorem 1

Let $\vartheta(r)$ be a regular function, vanishing in B_R and equal to 1 outside B_{2R}, for $R \gg R_0$. The field $v = \vartheta\boldsymbol{u}$ is a D-solution to the equation

$$\operatorname{div}\mathbf{C}[\nabla v] + \boldsymbol{f} = \boldsymbol{0} \quad \text{in } \mathbb{R}^3, \tag{26}$$

with

$$\boldsymbol{f} = -\mathbf{C}[\nabla\boldsymbol{u}]\nabla\vartheta - \operatorname{div}\mathbf{C}[\boldsymbol{u}\otimes\nabla\vartheta]. \tag{27}$$

Of course, $\boldsymbol{f} \in L^2(\mathbb{R}^3)$ vanishes outside T_R. Let $\mathbf{C}_0$ be a strongly elliptic elasticity tensor. Clearly, $\boldsymbol{v}$ is a D-solution to the system

$$\operatorname{div} \mathbf{C}_0[\nabla \boldsymbol{v}] + \operatorname{div} (\mathbf{C} - \mathbf{C}_0)[\nabla \boldsymbol{v}] + \boldsymbol{f} = \mathbf{0} \quad \text{in } \mathbb{R}^3, \tag{28}$$

which coincides with $\boldsymbol{u}$ outside B_{2R}. Since

$$\nabla_k \mathcal{V}[\boldsymbol{f}](x) = O(r^{-1-k}), \quad k \in \mathbb{N}, \quad \nabla_k = \underbrace{\nabla \ldots \nabla}_{k-\text{times}}, \tag{29}$$

by Lemma 2, the map

$$\boldsymbol{w}'(x) = \nabla \mathcal{V}\big[(\mathbf{C} - \mathbf{C}_0)[\nabla \boldsymbol{w}]\big](x) + \mathcal{V}[\boldsymbol{f}](x) \tag{30}$$

is continuous from $D^{1,q}$ into itself, for $q \in (3/2, +\infty)$. Choose

$$\mathbf{C}_{0ijhk} = \mu_e \delta_{ih} \delta_{jk}.$$

Since

$$\|\nabla \mathcal{V}[(\mathbf{C} - \mathbf{C}_0)[\nabla \boldsymbol{w}]]\|_{D^{1,q}} \le c(q) \frac{\mu_e - \mu_0}{\mu_e} \|\boldsymbol{w}\|_{D^{1,q}}$$

and [7]

$$\lim_{q \to 2} c(q) = 1,$$

the map (30) is contractive in a neighborhood of 2 and its fixed point must coincide with $\boldsymbol{v}$. Hence, there is $q \in (1, 2)$ such that $\boldsymbol{u} \in D^{1,q}(\Omega)$ and (12) is proved.

If $\mathbf{C}$ is regular at infinity, then by Lemma 1 and the property of ϑ,

$$\|\boldsymbol{v}' - \boldsymbol{z}'\|_{D^{1,q}} \le c(q) \|\mathbf{C} - \mathbf{C}_0\|_{L^\infty(\complement S_{R_0})} \|\boldsymbol{v} - \boldsymbol{z}\|_{D^{1,q}}. \tag{31}$$

Since the constant $c(q)$ is uniformly bounded in every interval $[a, b]$ and $\|\mathbf{C} - \mathbf{C}_0\|_{L^\infty(\complement S_{R_0})}$ is sufficiently small, $u \in D^{1,q}$ for $\boldsymbol{q} \in (3/2, +\infty)$.

Assume that

$$\int_{\partial\Omega} \hat{\boldsymbol{u}} \cdot \boldsymbol{s}(\boldsymbol{h}) = \mathbf{0}, \quad \forall \boldsymbol{h} \in \mathfrak{C}_0. \tag{32}$$

By Lemma 3, for $R \gg R_0$,

$$\int_{\partial B_R} \boldsymbol{s}(\boldsymbol{u}) = \int_{\partial B_R} \mathbf{C}[\nabla \boldsymbol{u}] \boldsymbol{e}_R = \mathbf{0}.$$

Therefore, taking into account (27),

$$\int_{\mathbb{R}^3} \boldsymbol{f} = \int_{T_R} \boldsymbol{f} = \int_R^{2R} \vartheta'(r) dr \int_{\partial B_r} \mathbf{C}[\nabla \boldsymbol{u}] \boldsymbol{e}_r = \mathbf{0}, \tag{33}$$

Since

$$\mathcal{V}[f](x) = \int_{\mathbb{R}^3} [\mathcal{U}(x-y) - \mathcal{U}(y)] \boldsymbol{f}(y) dv_y + \mathcal{U}(x) \int_{\mathbb{R}^3} \boldsymbol{f},$$

by (33), Lagrange's theorem and (29)

$$\nabla \mathcal{V}[\boldsymbol{f}](x) = O(r^{-3}),$$

so that

$$\nabla \mathcal{V}[\boldsymbol{f}] \in L^q, \quad q \in (1, 2]. \tag{34}$$

Then, by (31), the map (30) is contractive for q in a right neighborhood of 1 so that

$$\boldsymbol{u} \in D^{1,q}(\Omega), \quad q \in (1,2]. \tag{35}$$

Conversely, if (35) holds, then a simple computation yields

$$\int_{\partial\Omega} \boldsymbol{s}(\boldsymbol{u}) = \int_{\mathbb{R}^3} \mathsf{C}[\nabla \boldsymbol{u}]\nabla g = -\frac{1}{R}\int_{T_R} \mathsf{C}[\nabla \boldsymbol{u}]\boldsymbol{e}_r, \tag{36}$$

where g is the function (24). By Hölder's inequality,

$$\frac{1}{R}\left|\int_{T_R} \mathsf{C}[\nabla \boldsymbol{u}]\boldsymbol{e}_r\right| \le \frac{c}{R}\left\{\int_{T_R} |\nabla \boldsymbol{u}|^{3/2}\right\}^{2/3}\left\{\int_{T_R} dv\right\}^{1/3} \le \left\{\int_{T_R} |\nabla \boldsymbol{u}|^{3/2}\right\}^{2/3}.$$

Therefore, letting $R \to +\infty$ in (36) yields

$$\int_{\partial\Omega} \boldsymbol{s}(\boldsymbol{u}) = \boldsymbol{0}$$

and this implies (32).

From

$$\begin{aligned}\mathcal{V}_i[f](x) = &\int_{\mathbb{R}^3} [\mathcal{U}_{ij}(x-y) - \mathcal{U}(y) - y_k\partial_k\mathcal{U}_{ij}(y)] f_j(y)dv_y \\ &+\mathcal{U}_{ij}(x)\int_{\mathbb{R}^3} f_j + \partial_k\mathcal{U}(x)\int_{\mathbb{R}^3} \mathcal{U}_{ij}(y)f_j(y)dv_y\end{aligned}$$

by (33), Lemma 4, Lagrange's theorem and (29)

$$\nabla \boldsymbol{\mathcal{V}}[\boldsymbol{f}](x) = O(r^{-4}),$$

so that $\nabla \boldsymbol{\mathcal{V}}[\boldsymbol{f}] \in L^1$. Since $\boldsymbol{f} \in L^2(\mathbb{R}^3)$ has compact support and satisfies (33), it belongs to $\mathcal{H}^1$ (see [7] p. 92) and by Lemma 2 $\boldsymbol{\mathcal{V}}[\boldsymbol{f}] \in \mathcal{H}^1$. Hence, it follows that (30) maps $\mathcal{H}^1$ into itself and

$$\|\boldsymbol{v}' - \boldsymbol{z}'\|_{\mathcal{H}^1} \le \|\mathsf{C} - \mathsf{C}_0\|_{L^\infty(\complement S_{R_0})}\|\boldsymbol{v} - \boldsymbol{z}\|_{\mathcal{H}^1}.$$

Since, by assumptions, $\|\mathsf{C} - \mathsf{C}_0\|_{L^\infty(\complement S_{R_0})}$ is small, (30) is a contraction and by the above argument its (unique) fixed point must coincide with $\boldsymbol{v}$ so that $\nabla \boldsymbol{u} \in L^1(\Omega)$. □

We aim at concluding the paper with the following remarks.

(i) It is evident that the hypothesis that C is regular at infinity can be replaced by the weaker one that $|\mathsf{C} - \mathsf{C}_0|$ is suitably small at a large spatial distance.

(ii) The operator $\boldsymbol{\mathcal{V}}$ maps boundedly the Hardy space $\mathcal{H}^q$ ($q \in (0,1]$) into itself [7]. Hence, the argument in the proof of (16) can be used to show that $\nabla \boldsymbol{v} \in \mathcal{H}^q$, $q > 3/4$. We can then use the Sobolev–Poincaré (see [8] p. 255) to see that $u \in L^q(\Omega)$ for $q > 1$.

(iii) Relation (16) is a kind of Stokes' paradox in nonhomogeneous elastostatics: *if* $\boldsymbol{\mathsf{C}}$ *is regular at infinity, then the system*

$$\begin{aligned}\operatorname{div} \boldsymbol{\mathsf{C}}[\nabla \boldsymbol{h}] = \boldsymbol{0} \quad &\text{in } \Omega,\\ \boldsymbol{h} = \boldsymbol{\tau} \quad &\text{on } \partial\Omega,\\ \int_{\partial B} \boldsymbol{h}(R,\sigma)d\sigma = o(R^{-1}),&\end{aligned}$$

with $\boldsymbol{\tau}$ nonzero constant vector, does not admit solutions.

(iv) If C is constant and strongly elliptic, then $\boldsymbol{u}$ is analytic in Ω and at large spatial distance admits the representation

$$\boldsymbol{u}(x) = \mathfrak{U}(x) \int\limits_{\partial\Omega} \boldsymbol{s}(\boldsymbol{u}) + \nabla\mathfrak{U}(x) \int\limits_{\partial\Omega} \Big(\boldsymbol{\xi} \otimes \boldsymbol{s}(\boldsymbol{u}) - \mathsf{C}[\hat{\boldsymbol{u}} \otimes \boldsymbol{n}]\Big)(\xi) + \boldsymbol{\psi}(x)$$

with $|\boldsymbol{x}|^3|\boldsymbol{\psi}(x)| \leq c$. Therefore, in the homogeneous case, the conclusions of Theorem 1 hold pointwise:

$$|\boldsymbol{x}|^2|\boldsymbol{u}(x)| \leq c \Longleftrightarrow \int\limits_{\partial\Omega} \hat{\boldsymbol{u}} \cdot \boldsymbol{s}(\boldsymbol{h}) = \boldsymbol{0}, \ \ \forall \boldsymbol{h} \in \mathfrak{C}_0,$$

$$\int\limits_{\partial\Omega} \mathsf{C}\,[\hat{\boldsymbol{u}} \otimes \boldsymbol{n}] = \boldsymbol{0}, \quad \int\limits_{\partial\Omega} \hat{\boldsymbol{u}} \cdot \boldsymbol{s}(\boldsymbol{h}) = \boldsymbol{0}, \ \ \forall \boldsymbol{h} \in \mathfrak{C} \Rightarrow |\boldsymbol{x}|^3|\boldsymbol{u}(x)| \leq c.$$

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References

1. Gurtin, M.E. The linear theory of elasticity. In *Handbuch der Physik*; Truesedell, C., Ed.; Springer: Berlin, Germany, 1972; Volume VIa/2.
2. Galdi, G.P. *An Introduction to the Mathematical Theory of the Navier—Stokes Equations. Steady—State Problems*; Springer: Berlin, Germany, 2011.
3. Fichera, G. Existence theorems in elasticity. In *Handbuch der Physik*; Truesedell, C., Ed.; Springer: Berlin, Germany, 1972; Volume VIa/2.
4. Russo, R. On Stokes' problem. In *Advances in Mathematical Fluid Mechanics*; Sequeira, R.R.A., Ed.; Springer: Berlin, Germany, 2010; pp. 473–512.
5. Kondratev, V.A.; Oleinik, O.A. Boundary value problems for a system in elasticity theory in unbounded domains. Korn inequalities. *Uspekhi Mat. Nauk* **1988**, *43*, 55–98. (In Russian)
6. John, F. *Plane Waves and Spherical Means Applied to Partial Differential Equations*; Interscience: New York, NY, USA, 1955.
7. Stein, E. *Harmonic Analysis: Real—Variable Methods, Orthogonality, and Oscillatory Integrals*; Princeton University Press: Princeton, NJ, USA, 1993.
8. Coifman, R.R.; Lions, P.L.; Meyer, Y.; Semmes, S. Compensated compactness and Hardy spaces. *J. Math. Pures Appl.* **1993**, *72*, 247–286.