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UNIQUENESS AND DECAY RESULTS FOR A BOUSSINESQUIAN NANOFLUID

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Abstract: In this paper a uniqueness theorem for classical solutions is proved in the case of the evolution of a nanofluid filling a bounded domain under the Boussinesq approximation. The mass density of the nanofluid depends on the temperature and on the nanoparticle volume fraction. A decay in time of a suitable energy is achieved assuming that the material parameters satisfy some conditions. These results are then generalized in the presence of a magnetic field.

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1. Introduction

Nanofluids are characterized by a high thermal conductivity obtained by suspending nanometer-sized (1-100 nm) solid metal (Cu, Fe, Au) or metal oxide (CuO, Al_2O_3) particles into a base liquid with low thermal conductivity, such

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as water, oils, ethylene glycol, etc. The presence of a few percents of nanoparticles produces a relevant enhancement of the effective thermal conductivity of nanofluids.

The concept of nanofluid was first introduced by Choi [5] in 1995 and it has been extensively studied recently (see for example [6, 7, 8, 9, 11, 12, 14, 13, 15, 16, 17, 22, 19, 23, 18, 25]) because of its application to a wide class of engineering problems, such as chemical processes and cooling of electronic equipments.

There are two main theories of modeling nanofluids: single-phase and two-phase. For an exhaustive review of the theory and the applications we refer to [20]. In this paper we adopt the two-phase model formulated by Buongiorno [4, 2] in which the nanofluid is considered as a two-component mixture and thermophoresis and Brownian motion are the principal mechanisms that produce a relative velocity between the nanoparticles and the base fluid.

While there is a vast literature concerning the study of specific problems involving nanofluids, to the best of our knowledge there are no results of uniqueness and decay in time of the energy.

In this paper we consider the evolution of a nanofluid filling a bounded domain under the Boussinesq approximation taking account that the mass density of the nanofluid depends on the temperature and on the nanoparticle volume fraction.

We first formulate the initial-boundary problem in dimensionless form in Section 2, we then state in Section 3 the uniqueness theorem for classical solutions without any restriction on the material parameters by introducing a suitable energy.

Section 4 is devoted to obtain a decay in time result of the energy supposing that the material parameters satisfy some inequalities.

Finally, in Section 5 we extend the previous results to the case in which a magnetic field is impressed to the nanofluid without modifying the hypotheses on the material parameters. To this regard, we notice that significant enhancement in the thermal conductivity of nanofluids has been experimentally shown when magnetic fields are applied (see for example [1, 21, 3, 10]).

2. Statement of the problem

In this section we introduce the initial-boundary value problem which we would like to study in the details.

Let us consider the evolution of a Boussinesquian homogeneous nanofluid filling a bounded domain Ω. We denote by $\mathbf{v}, p, T, \Phi$ the velocity, the modified pressure (given by the difference between the pressure and the hydrostatic pressure), the temperature and the nanoparticle volume fraction, respectively. We assume that the gravity $\mathbf{g}$ is aligned with the unit vector $\mathbf{e}_3$.

The governing field equations are

$$
\begin{aligned}
&\rho_R \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \triangle \mathbf{v} \\
&\qquad\qquad\qquad\qquad - \rho_R [\alpha_T (T - T_R) - \alpha_\Phi (\Phi - \Phi_R)] \mathbf{g}, \\
&\nabla \cdot \mathbf{v} = 0, \\
&\rho_R c \left(\frac{\partial T}{\partial t} + \nabla T \cdot \mathbf{v} \right) = k \triangle T + \rho_p c_p \left(D_B \nabla T \cdot \nabla \Phi + D_T \frac{|\nabla T|^2}{T} \right), \\
&\frac{\partial \Phi}{\partial t} + \nabla \Phi \cdot \mathbf{v} = \nabla \cdot \left(D_B \nabla \Phi + D_T \frac{\nabla T}{T} \right), \\
&\qquad\qquad\qquad \text{for all } (\mathbf{x},\, t) \in \Omega \times (0, +\infty).
\end{aligned} \tag{1}
$$

We have followed the Buongiorno two-phase model ([4]) for the description of the nanofluid, while, unlike Buongiorno, we suppose slightly compressibility in order to apply the Boussinesq approximation. More precisely, we take into account that the mass density ρ of the nanofluid depends on T and Φ and that it is the average of the nanoparticle density ρ_p and base fluid density ρ_{bf}:

$$\rho = \Phi \rho_p + (1 - \Phi) \rho_{bf}.$$

In (1) we have assumed that there exists a configuration in which the temperature T and the nanoparticle volume fraction Φ take the constant values T_R and Φ_R; $\rho_R = \rho(T_R, \Phi_R)$ and

$$\alpha_T = -\frac{1}{\rho_R} \frac{\partial \rho}{\partial T}(T_R, \Phi_R), \qquad \alpha_\Phi = \frac{1}{\rho_R} \frac{\partial \rho}{\partial \Phi}(T_R, \Phi_R).$$

We remark that from physical considerations the thermal coefficient of volume expansion α_T and the composition coefficient of volume expansion α_Φ are positive. Moreover, we have $\rho \simeq \rho_{bf}$ because $\Phi \ll 1$ ([24]).

In $(1)_{3,4}$ the parameter μ is the fluid dynamical viscosity coefficient, c, c_p are the nanofluid and nanoparticle specific heat and k is the nanofluid thermal conductivity. We take into account Brownian diffusion and thermophoresis by means of the coefficients D_B and D_T and, as it is usual, we neglect dissipative terms in $(1)_3$.

To (1) we associate the initial conditions

$$\begin{aligned} &\mathbf{v}(\mathbf{x},0)=\mathbf{v}_0(\mathbf{x}), \quad T(\mathbf{x},0)=T_0(\mathbf{x}), \quad \Phi(\mathbf{x},0)=\Phi_0(\mathbf{x}), \\ &\qquad \forall \mathbf{x}\in\overline{\Omega} \end{aligned} \tag{2}$$

and the boundary conditions

$$\begin{aligned} &\mathbf{v}|_{\partial\Omega\times[0,+\infty)}=\mathbf{v}^*, \quad T\,|_{\partial\Omega\times[0,+\infty)}=T^*, \\ &\Phi\,|_{\partial\Omega\times[0,+\infty)}=\Phi^*. \end{aligned} \tag{3}$$

The fields $\mathbf{v}_0, T_0, \Phi_0, \mathbf{v}^*, T^*, \Phi^*$ are prescribed on the appropriate domains and satisfy suitable regularity and compatibility conditions. Moreover we suppose that the domain Ω is bounded and that it is possible to apply the divergence theorem.

It is convenient to rewrite problem (1) in dimensionless form by using the following transformations:

$$\begin{aligned} &\mathbf{V}=\frac{\mathbf{v}}{\overline{V}}, \ \ \vartheta=\frac{T-T_R}{\overline{T}}, \ \ \varphi=\frac{\Phi-\Phi_R}{\overline{\Phi}}, \ \ P=\frac{p}{\rho_R\overline{V}^2}, \ \ \nu=\frac{\mu}{\rho_R}, \\ &\mathbf{x}'=\frac{\mathbf{x}}{L}, \quad t'=\frac{\overline{V}t}{L}, \quad \overline{V}=\frac{\nu}{L}, \quad \overline{T}=\sqrt{\frac{\nu^3\triangle_T}{\alpha_T\gamma L^3 g}}, \\ &\overline{\Phi}=\sqrt{\frac{\nu^3\Phi_R}{\alpha_\Phi D_B L^3 g}}, \end{aligned} \tag{4}$$

where $\overline{V}$, L, $\triangle_T(>0)$ are the reference velocity, length, characteristic temperature difference, respectively, while the thermal diffusivity γ is given by

$$\gamma=\frac{k}{\rho_R c}. \tag{5}$$

We assume that the change of temperature in the nanofluid is small comparing to the temperature T_R so that we can replace T by T_R in the denominator of $(1)_{3,4}$.

Problem (1) in dimensionless form becomes

$$\begin{aligned} &\frac{\partial\mathbf{V}}{\partial t}+\mathbf{V}\cdot\nabla\mathbf{V}=-\nabla P+\triangle\mathbf{V}+(R_T\vartheta-R_\Phi\varphi)\mathbf{e}_3, \\ &\nabla\cdot\mathbf{V}=0, \\ &P_r\left(\frac{\partial\vartheta}{\partial t}+\mathbf{V}\cdot\nabla\vartheta\right)=\triangle\vartheta+\frac{1}{Le}\left(\nabla\vartheta\cdot\nabla\varphi+\frac{1}{N_{BT}}|\nabla\vartheta|^2\right), \end{aligned}$$

$$Sc\left(\frac{\partial\varphi}{\partial t}+\mathbf{V}\cdot\nabla\varphi\right)=\triangle\varphi+\frac{1}{N_{BT}}\triangle\vartheta, \tag{6}$$

where we used the following dimensionless numbers:

$$\begin{aligned}
&P_r=\frac{\nu}{\gamma}\ \text{(Prandtl number)},\\
&R_T^2=\frac{\alpha_T L^3\triangle_T g}{\nu\gamma}\ \text{(temperature Rayleigh number)},\\
&R_\Phi^2=\frac{\alpha_\Phi L^3\Phi_R g}{\nu\gamma}\ \text{(nanoparticle volume fraction Rayleigh number)},\\
&Le=\frac{k}{\rho_P c_P D_B\overline{\Phi}}\ \text{(Lewis number)},\\
&N_{BT}=\frac{D_B T_R\overline{\Phi}}{D_T\overline{T}},\qquad Sc=\frac{\nu}{D_B}\ \text{(Schmidt number)}.
\end{aligned} \tag{7}$$

For the sake of brevity, we continue to denote by Ω the transformed of the domain occupied by the nanofluid, we omit to write the initial (2) and boundary conditions (3) in dimensionless form and in the sequel we will imply that they are dimensionless.

We confine ourselves to consider classical solutions to the initial-boundary value problem ($\mathbf{V}$, P, ϑ, φ), i.e.

$$\begin{aligned}
&\mathbf{V},\,\vartheta,\,\varphi\in C^{2,1}(\Omega\times(0,+\infty))\cap C(\overline{\Omega}\times[0,+\infty)),\\
&P\in C^{1,0}(\Omega\times(0,+\infty))\cap C(\overline{\Omega}\times[0,+\infty)),\\
&\nabla\mathbf{V},\,\nabla\vartheta,\,\nabla\varphi\in C(\overline{\Omega}\times[0,+\infty)).
\end{aligned}$$

3. Uniqueness Theorem

In this section we prove the uniqueness of the solution of the problem (6), (2), (3) without any restriction on the material parameters.

Theorem 1. *Let* $(\mathbf{V}_i,\,P_i,\vartheta_i,\varphi_i)$ $i=1,\,2$ *be two solutions of the problem* (6), (2), (3)*. Then*

$$\mathbf{V}_1=\mathbf{V}_2,\ \ P_1=P_2+\Pi,\ \ \vartheta_1=\vartheta_2,\ \ \varphi_1=\varphi_2\ \ in\ \ \Omega\times[0,+\infty),$$

with $\Pi=\Pi(t)$ *an arbitrary function.*

Proof. Let $(\mathbf{V}, \Pi, \vartheta, \varphi)$ denote the difference between the two solutions $(\mathbf{V}_i, P_i, \vartheta_i, \varphi_i)$ $i = 1, 2$ of the problem (6), (2), (3). Then the following system is satisfied

$$
\begin{aligned}
&\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}_1 \cdot \nabla \mathbf{V}_1 - \mathbf{V}_2 \cdot \nabla \mathbf{V}_2 = -\nabla \Pi + \triangle \mathbf{V} + (R_T \vartheta - R_\Phi \varphi)\mathbf{e}_3, \\
&\nabla \cdot \mathbf{V} = 0, \\
&P_r \left(\frac{\partial \vartheta}{\partial t} + \mathbf{V}_1 \cdot \nabla \vartheta_1 - \mathbf{V}_2 \cdot \nabla \vartheta_2 \right) = \triangle \vartheta \\
&+ \frac{1}{Le} \left[\nabla \vartheta_1 \cdot \nabla \varphi_1 - \nabla \vartheta_2 \cdot \nabla \varphi_2 + \frac{1}{N_{BT}} (|\nabla \vartheta_1|^2 - |\nabla \vartheta_2|^2) \right], \\
&Sc \left(\frac{\partial \varphi}{\partial t} + \mathbf{V}_1 \cdot \nabla \varphi_1 - \mathbf{V}_2 \cdot \nabla \varphi_2 \right) = \triangle \varphi + \frac{1}{N_{BT}} \triangle \vartheta
\end{aligned} \tag{8}
$$

together with homogeneous initial and boundary conditions for $\mathbf{V}$, ϑ, φ.

Let us multiply $(8)_1$ by $\mathbf{V}$, $(8)_3$ by ϑ and $(8)_4$ by $A\varphi$ with A some dimensionless positive constant to be specified and sum the resulting equations.

We now develop separately some terms:

$$
\begin{aligned}
&(\mathbf{V}_1 \cdot \nabla \mathbf{V}_1 - \mathbf{V}_2 \cdot \nabla \mathbf{V}_2) \cdot \mathbf{V} = \mathbf{V} \cdot \nabla \mathbf{V}_1 \cdot \mathbf{V} + \frac{1}{2} \mathbf{V}_2 \cdot \nabla \mathbf{V}^2, \\
&(\mathbf{V}_1 \cdot \nabla \vartheta_1 - \mathbf{V}_2 \cdot \nabla \vartheta_2)\vartheta = (\mathbf{V} \cdot \nabla \vartheta_1)\vartheta + \frac{1}{2} \mathbf{V}_2 \cdot \nabla \vartheta^2, \\
&(\nabla \vartheta_1 \cdot \nabla \varphi_1 - \nabla \vartheta_2 \cdot \nabla \varphi_2)\vartheta = (\nabla \vartheta_2 \cdot \nabla \varphi + \nabla \vartheta \cdot \nabla \varphi_1)\vartheta, \\
&(|\nabla \vartheta_1|^2 - |\nabla \vartheta_2|^2)\vartheta = \vartheta \nabla (\vartheta_1 + \vartheta_2) \cdot \nabla \vartheta, \\
&(\mathbf{V}_1 \cdot \nabla \varphi_1 - \mathbf{V}_2 \cdot \nabla \varphi_2)\varphi = \varphi \mathbf{V} \cdot \nabla \varphi_1 + \frac{1}{2} \mathbf{V}_2 \cdot \nabla \varphi^2.
\end{aligned} \tag{9}
$$

At this point we fix $\bar{t} \in (0, +\infty)$ and integrate by parts over Ω $\forall t \in [0, \bar{t}]$. Taking into account $(8)_2$, (9), the divergence theorem and the homogeneous boundary conditions (3), after some calculations we arrive at

$$
\begin{aligned}
&\frac{d}{dt} \frac{1}{2} \int_\Omega \left(\mathbf{V}^2 + P_r \vartheta^2 + A Sc \varphi^2 \right) d\Omega \\
&= \int_\Omega \left[-\mathbf{V} \cdot \nabla \mathbf{V}_1 \cdot \mathbf{V} - |\nabla \mathbf{V}|^2 + (R_T \vartheta - R_\Phi \varphi) V_3 \right] d\Omega \\
&+ \int_\Omega \Bigg[-P_r \vartheta \nabla \vartheta_1 \cdot \mathbf{V} - |\nabla \vartheta|^2 + \frac{1}{Le} \vartheta (\nabla \vartheta_2 \cdot \nabla \varphi + \nabla \vartheta \cdot \nabla \varphi_1) \\
&\qquad + \frac{1}{Le N_{BT}} \vartheta \nabla (\vartheta_1 + \vartheta_2) \cdot \nabla \vartheta \Bigg] d\Omega
\end{aligned}
$$

$$+\int_\Omega -A\left(Sc\varphi\nabla\varphi_1\cdot\mathbf{V}+|\nabla\varphi|^2+\frac{1}{N_{BT}}\nabla\vartheta\cdot\nabla\varphi\right)d\Omega. \tag{10}$$

In order to choose the constant A we consider the quadratic form

$$\omega(\xi)=\sum_{i=1}^{3}\xi_i^2+A\sum_{i=4}^{6}\xi_i^2+\frac{A}{N_{BT}}(\xi_1\xi_4+\xi_2\xi_5+\xi_3\xi_6),\quad \xi\in\mathbb{R}^6.$$

As it is easy to verify, ω is positive definite if and only if

$$\frac{A}{4N_{BT}^2}<1. \tag{11}$$

By virtue of this inequality, we deduce that if the material parameter $N_{BT}>\frac{1}{2}$, then $A=1$ assures that ω is positive definite; while if $N_{BT}\leq\frac{1}{2}$ then A must be chosen satisfying the inequality $A<4N_{BT}^2$.

In any case we have that

$$\omega(\xi)\geq\lambda_{min}|\xi|^2, \tag{12}$$

where λ_{min} is the smallest eigenvalue of the matrix associated to ω given by

$$\lambda_{min}=\frac{N_{BT}(A+1)-\sqrt{N_{BT}^2(A-1)^2+A^2}}{2N_{BT}}.$$

These considerations allow us to obtain the inequality

$$-|\nabla\vartheta|^2-A|\nabla\varphi|^2-A\frac{1}{N_{BT}}\nabla\vartheta\cdot\nabla\varphi\leq-\lambda_{min}(|\nabla\vartheta|^2+|\nabla\varphi|^2). \tag{13}$$

We can now estimate some terms at the right hand side of (10) on taking into account that the involved fields are bounded in $\overline{\Omega}\times[0,\bar{t}]$.

Putting

$$M_1=\max_{\overline{\Omega}\times[0,\bar{t}]}|\nabla\mathbf{V}_1|,\ M_2=\max_{\overline{\Omega}\times[0,\bar{t}]}|\nabla\vartheta_1|,$$

$$M_3=\max_{\overline{\Omega}\times[0,\bar{t}]}|\nabla\vartheta_2|,\ M_4=\max_{\overline{\Omega}\times[0,\bar{t}]}|\nabla\varphi_1|,$$

we arrive at

$$-\mathbf{V}\cdot\nabla\mathbf{V}_1\cdot\mathbf{V}-|\nabla\mathbf{V}|^2+(R_T\vartheta-R_\Phi\varphi)V_3-P_r\vartheta\nabla\vartheta_1\cdot\mathbf{V}-|\nabla\vartheta|^2$$
$$+\frac{1}{Le}\vartheta(\nabla\vartheta_2\cdot\nabla\varphi+\nabla\vartheta\cdot\nabla\varphi_1)+\frac{1}{LeN_{BT}}\vartheta\nabla(\vartheta_1+\vartheta_2)\cdot\nabla\vartheta$$

$$
\begin{aligned}
&- A\left(Sc\varphi\nabla\varphi_1\cdot\mathbf{V} + |\nabla\varphi|^2 + \frac{1}{N_{BT}}\nabla\vartheta\cdot\nabla\varphi\right)\\
&\le \frac{1}{2}(2M_1 + R_T - R_\Phi + P_r M_2 + AScM_4)\mathbf{V}^2\\
&+ \frac{1}{2}\left(R_T + M_2 + \frac{M_3}{Le\epsilon_1} + \frac{M_4}{Le\epsilon_2} + \frac{M_2+M_3}{LeN_{BT}\epsilon_2}\right)\vartheta^2\\
&+ \frac{1}{2}\left(AScM_4 - R_\Phi\right)\varphi^2 + \left[-\lambda_{min} + \frac{M_4\epsilon_2}{2Le} + \frac{(M_2+M_3)\epsilon_2}{2LeN_{BT}}\right]|\nabla\vartheta|^2\\
&+ \left(-\lambda_{min} + \frac{M_3\epsilon_1}{2Le}\right)|\nabla\varphi|^2, \qquad (14)
\end{aligned}
$$

where ϵ_1, ϵ_2 are arbitrary positive constants and we used inequality (13).

If we choose

$$
\epsilon_1 = \frac{2Le\lambda_{min}}{M_3}, \quad \epsilon_2 = \frac{2Le\lambda_{min}N_{BT}}{M_2 + M_3 + M_4 N_{BT}}, \qquad (15)
$$

then (10) and (14) lead to

$$
\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int_\Omega \left(\mathbf{V}^2 + P_r\vartheta^2 + ASc\,\varphi^2\right) d\Omega\\
&\le \frac{1}{2}\int_\Omega \left(B_1\mathbf{V}^2 + B_2 P_r\vartheta^2 + B_3 ASc\,\varphi^2\right) d\Omega, \qquad (16)
\end{aligned}
$$

where the coefficients $B_i,\ i = 1, 2, 3$, can be easily computed.

Putting

$$
B = \max\{B_1,\, B_2,\, B_3\}, \quad \mathcal{E}_A = \frac{1}{2}\int_\Omega \left(\mathbf{V}^2 + P_r\vartheta^2 + ASc\,\varphi^2\right) d\Omega,
$$

we arrive at

$$
\frac{d}{dt}\mathcal{E}_A(t) \le B\mathcal{E}_A(t), \qquad \forall t \in [0,\, \overline{t}]. \qquad (17)
$$

Therefore we deduce

$$
\mathcal{E}_A(\overline{t}) \le \mathcal{E}_A(0)e^{B\overline{t}}.
$$

By virtue of the initial conditions we have that $\mathcal{E}_A(0) = 0$, so that

$$
\mathbf{V}(\mathbf{x}, \overline{t}) = \mathbf{0}, \;\; \vartheta(\mathbf{x}, \overline{t}) = 0, \;\; \varphi(\mathbf{x}, \overline{t}) = 0, \quad \forall \mathbf{x} \in \overline{\Omega}.
$$

For the arbitrariness of $\overline{t}$, we get

$$\mathbf{V}_1 = \mathbf{V}_2, \ \ \vartheta_1 = \vartheta_2, \ \ \varphi_1 = \varphi_2 \ \text{ in } \overline{\Omega} \times [0, +\infty).$$

Finally, from $(10)_1$ we have

$$\nabla \Pi = \mathbf{0} \quad \Rightarrow \Pi \equiv \Pi(t),$$

which completes the proof. □

Remark 1. As it is easy to verify, the uniqueness theorem continues to hold even if we replace the boundary conditions (3) with

$$\frac{\partial \vartheta}{\partial \mathbf{n}}|_{\partial\Omega \times [0,+\infty)} = q^*_\vartheta, \quad \frac{\partial \varphi}{\partial \mathbf{n}}|_{\partial\Omega \times [0,+\infty)} = q^*_\varphi,$$

or we maintain $(3)_2$ and replace $(3)_3$ with

$$\frac{1}{N_{BT}} \frac{\partial \vartheta}{\partial \mathbf{n}}|_{\partial\Omega \times [0,+\infty)} + \frac{\partial \varphi}{\partial \mathbf{n}}|_{\partial\Omega \times [0,+\infty)} = j^*,$$

where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$ and q^*_ϑ, q^*_φ, j^* are sufficiently smooth prescribed fields (see for example [12], [21]).

Actually, we get again an inequality analogous to (16): we have to modify in a suitable way the expressions of B_2 and B_3 in the first case and of B_3 in the latter one by inserting $\max_{\overline{\Omega} \times [0,\overline{t})} |\mathbf{V}_2|$.

Remark 2. The function $\mathcal{E}_A$ represents a suitable (dimensionless) energy of the nanofluid.

4. Decay in time of the energy $\mathcal{E}_A$

In this section we prove the exponential decay in time of the energy $\mathcal{E}_A$ in $[0, \overline{t}]$ with $\overline{t} \in (0, \infty)$ for solutions of problem (6), (2), (3) with homogeneous boundary conditions provided that some suitable inequalities hold.

Theorem 2. *Let $(\mathbf{V}, P, \vartheta, \varphi)$ be the solution of the problem (6), (2), (3) with homogeneous boundary conditions and $\overline{t} \in (0, \infty)$. If the following inequalities*

$$R_T + R_\Phi < \frac{2}{\beta}, \quad \lambda_{min} > \frac{\beta}{2} \max\{R_T, R_\Phi\},$$

$$M_{\bar{t}} < 2\frac{Le}{\beta}\min\left\{\frac{\sqrt{\beta}N_{BT}}{\sqrt{\beta}N_{BT}+2}\left(\lambda_{min}-\frac{\beta R_T}{2}\right),\, \lambda_{min}-\frac{\beta R_\Phi}{2}\right\}, \tag{18}$$

are verified, then there exists a positive constant C, depending on the material parameters, Poincaré constant of Ω and $M_{\bar{t}} = \max_{\Omega\times[0,\bar{t}]}|\nabla\vartheta|$, such that

$$\mathcal{E}_A(t) \le \mathcal{E}_A(0)\, e^{-Ct}, \qquad \forall t \in [0,\bar{t}]. \tag{19}$$

Proof. Fix $\bar{t}$ and let $(\mathbf{V},\, P, \vartheta, \varphi)$ be the solution of the problem (6), (2), (3) with homogeneous boundary conditions. The equations $(6)_1$, $(6)_3$ and $(6)_4$ multiplied by $\mathbf{V}, \vartheta$ and $A\varphi$ respectively are summed and the resulting equation is integrated over Ω to see that, thanks to integration by parts, we have

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_A &= \int_\Omega \left[-|\nabla\mathbf{V}|^2 + (R_T\vartheta - R_\Phi\varphi)V_3 - |\nabla\vartheta|^2\right] d\Omega\\ &+ \int_\Omega \left[\frac{1}{Le}\vartheta\left(\nabla\vartheta\cdot\nabla\varphi + \frac{1}{N_{BT}}|\nabla\vartheta|^2\right) - A\left(|\nabla\varphi|^2 + \frac{1}{N_{BT}}\nabla\vartheta\cdot\nabla\varphi\right)\right] d\Omega. \end{aligned}\tag{20}$$

Now, by using Cauchy and Poincaré inequalities, we get

$$\begin{aligned}&\int_\Omega (R_T\vartheta - R_\Phi\varphi)V_3 d\Omega\\ &\le \frac{\beta}{2}\int_\Omega [R_T(|\nabla\mathbf{V}|^2 + |\nabla\vartheta|^2) + R_\Phi(|\nabla\mathbf{V}|^2 + |\nabla\varphi|^2)]d\Omega,\end{aligned}\tag{21}$$

where β is the Poincaré constant for Ω.

Applying Cauchy, Schwarz and Poincaré inequalitie,s we obtain

$$\begin{aligned}&\int_\Omega \vartheta\left(\nabla\vartheta\cdot\nabla\varphi + \frac{1}{N_{BT}}|\nabla\vartheta|^2\right) d\Omega\\ &\le M_{\bar{t}}\int_\Omega \left[\left(\frac{\beta}{2} + \frac{\sqrt{\beta}}{N_{BT}}\right)|\nabla\vartheta|^2 + \frac{\beta}{2}|\nabla\varphi|^2\right] d\Omega.\end{aligned}\tag{22}$$

Taking into account inequalities (13), (21) and (22) we deduce in $[0,\bar{t}]$

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_A &\le \int_\Omega \left[\frac{\beta}{2}(R_T + R_\Phi) - 1\right]|\nabla\mathbf{V}|^2 d\Omega\\ &+ \int_\Omega \left(\frac{\beta}{2}R_T + \frac{\beta M_{\bar{t}}}{2Le} + \frac{\sqrt{\beta}M_{\bar{t}}}{LeN_{BT}} - \lambda_{min}\right)|\nabla\vartheta|^2 d\Omega\\ &+ \int_\Omega \left(\frac{\beta M_{\bar{t}}}{2Le} - \frac{\beta}{2}R_\Phi - \lambda_{min}\right)|\nabla\varphi|^2 d\Omega.\end{aligned}\tag{23}$$

In order to get the thesis, the coefficients of $|\nabla \mathbf{V}|^2, |\nabla \vartheta|^2,\ |\nabla \varphi|^2$ must be negative, i.e. inequalities (18) must be verified where it is convenient to choose in λ_{min} the value of A that makes it maximum. This value of λ_{min} is given by $\frac{N_{BT}^2}{1+N_{BT}^2}$.

If Poincaré inequality is again applied, one gets

$$\frac{d}{dt}\mathcal{E}_A(t) \leq -C\mathcal{E}_A(t), \qquad \forall t \in [0,\, \bar{t}], \tag{24}$$

where $C = \frac{1}{\beta}\min\{C_1,\ \frac{C_2}{P_r},\ \frac{C_3}{ASc}\}$, being $C_i,\ i = 1, 2, 3$ the opposite of the coefficients of the three gradients in (23). From (24) follows that

$$\mathcal{E}_A(t) \leq \mathcal{E}_A(0)e^{-Ct}, \qquad \forall t \in [0,\, \bar{t}],$$

which is the desired conclusion. □

Remark 3. If inequality $(18)_3$ holds for any $\bar{t} > 0$, then $\mathcal{E}_A$ decays exponentially in $[0, +\infty)$. This result may be regarded as a universal stability result for the basic flow $\mathbf{v} = \mathbf{0}$, $T = T_R, \Phi = \Phi_R$ and the basic flow is asymptotically stable in the mean. Finally, we notice that if we do not consider the Boussinesq approximation $(\alpha_T = 0, \alpha_\Phi = 0)$ then the previous theorems continue to hold with the suitable simplifications.

5. Extension of the results to the magnetohydrodynamic case

In this section we suppose that the nanofluid is embedded in a magnetic field and denote by $\mathbf{H}$ the total magnetic field. If the Hall and the displacement currents are disregarded, in order to formulate the new initial-boundary value problem we have to modify system (1) by adding in $(1)_1$ the term $\mu_e(\nabla\times\mathbf{H})\times\mathbf{H}$ and the equations

$$\begin{aligned} &\nabla \cdot \mathbf{H} = 0, \\ &\frac{\partial \mathbf{H}}{\partial t} = \eta \triangle \mathbf{H} + \nabla \times (\mathbf{v} \times \mathbf{H}), \end{aligned} \tag{25}$$

where μ_e is the magnetic permeability and η the magnetic diffusivity given by

$$\eta = \frac{1}{\sigma\mu_e}, \qquad \sigma = \text{electrical conductivity}.$$

Of course we must consider the initial and boundary conditions for the magnetic field:

$$\begin{aligned} &\mathbf{H}(\mathbf{x},0)=\mathbf{H}_0(\mathbf{x}), \quad \forall \mathbf{x}\in\overline{\Omega},\\ &\mathbf{H}|_{\partial\Omega\times[0,+\infty)}=\mathbf{H}^*. \end{aligned} \tag{26}$$

As it is usual, we modify p by adding the term $\mu_e\frac{\mathbf{H}^2}{2}$ and continue to denote by p the new expression of the modified pressure.

Then, in order to write the problem in dimensionless form we use the following transformation for the magnetic field:

$$\mathbf{h}=\frac{\mathbf{H}}{\overline{V}\sqrt{\mu\sigma}}.$$

After some calculations, the dimensionless equations that govern the magnetohydrodynamic flow of the nanofluid become

$$\begin{aligned} &\frac{\partial \mathbf{V}}{\partial t}+\mathbf{V}\cdot\nabla\mathbf{V}=-\nabla P+\triangle\mathbf{V}+(R_T\vartheta-R_\Phi\varphi)\mathbf{e}_3+R_m\mathbf{h}\cdot\nabla\mathbf{h},\\ &\nabla\cdot\mathbf{V}=0,\\ &P_r\left(\frac{\partial\vartheta}{\partial t}+\mathbf{V}\cdot\nabla\vartheta\right)=\triangle\vartheta+\frac{1}{Le}\left(\nabla\vartheta\cdot\nabla\varphi+\frac{1}{N_{BT}}|\nabla\vartheta|^2\right),\\ &Sc\left(\frac{\partial\varphi}{\partial t}+\mathbf{V}\cdot\nabla\varphi\right)=\triangle\varphi+\frac{,1}{N_{BT}}\triangle\vartheta,\\ &\nabla\cdot\mathbf{h}=0,\\ &R_m\frac{\partial\mathbf{h}}{\partial t}=\triangle\mathbf{h}+R_m(\mathbf{h}\cdot\nabla\mathbf{V}-\mathbf{V}\cdot\nabla\mathbf{h}), \end{aligned} \tag{27}$$

where $R_m=\frac{\nu}{\eta}$ is the Reynolds magnetic number and $\mathbf{h}$ has the same regularity properties as $\mathbf{V}$.

By proceeding as in the previous sections we can state a uniqueness theorem and a decay result for the energy

$$\mathcal{E}_{Am}=\frac{1}{2}\int_\Omega\left(\mathbf{V}^2+P_r\vartheta^2+ASc\,\varphi^2+R_m\mathbf{h}^2\right)d\Omega. \tag{28}$$

Theorem 3. *Let* $(\mathbf{V}_i,\, P_i,\vartheta_i,\varphi_i,\mathbf{h}_i),\ i=1,2,$ *be two solutions of the problem* (27), (2), (3), (26). *Then*

$$\mathbf{V}_1=\mathbf{V}_2,\ \ P_1=P_2+\Pi,\ \ \vartheta_1=\vartheta_2,\ \ \varphi_1=\varphi_2,$$

$$\mathbf{h}_1=\mathbf{h}_2\ \ in\ \ \Omega\times[0,+\infty)$$

with $\Pi=\Pi(t)$ *an arbitrary function.*

Proof. The theorem is proved by similar arguments as for Theorem 1; it is sufficient to consider only the new terms involving the magnetic field.

If we proceed as in the proof of Theorem 1 and set $\mathbf{h} = \mathbf{h}_1 - \mathbf{h}_2$, we obtain an equation deducible from (10) by replacing at the left hand side $\mathcal{E}_A$ with $\mathcal{E}_{Am}$ and adding at the right hand side the following integral

$$\int_\Omega \left[-|\nabla \mathbf{h}|^2 + R_m(\mathbf{h}\cdot\nabla\mathbf{h}_1\cdot\mathbf{V} - \mathbf{V}\cdot\nabla\mathbf{h}_1\cdot\mathbf{h} + \mathbf{h}\cdot\nabla\mathbf{V}_1\cdot\mathbf{h})\right] d\Omega.$$

After putting $M_5 = \max_{\Omega\times[0,\bar{t}]} |\nabla\mathbf{h}_1|$, the function under the previous integral can be estimated as

$$\begin{aligned} & -|\nabla \mathbf{h}|^2 + R_m(\mathbf{h}\cdot\nabla\mathbf{h}_1\cdot\mathbf{V} - \mathbf{V}\cdot\nabla\mathbf{h}_1\cdot\mathbf{h} + \mathbf{h}\cdot\nabla\mathbf{V}_1\cdot\mathbf{h}) \\ & \leq R_m\left[M_5\mathbf{V}^2 + (M_1+M_5)\mathbf{h}^2\right]. \end{aligned} \tag{29}$$

From the equation (10) modified, (13), (14), (15) and (29) we get

$$\begin{aligned} & \frac{d}{dt}\frac{1}{2}\int_\Omega \left(\mathbf{V}^2 + P_r\vartheta^2 + ASc\,\varphi^2 + R_m\mathbf{h}^2\right) d\Omega \\ & \leq \frac{1}{2}\int_\Omega \left(B_1'\mathbf{V}^2 + B_2P_r\vartheta^2 + B_3ASc\,\varphi^2 + B_4R_m\mathbf{h}^2\right) d\Omega, \end{aligned} \tag{30}$$

where the coefficients B_2, B_3 are the same as in (16) while B_1' is obtained from B_1 by means of a suitable modification and B_4 is new because of the presence of the magnetic field.

Inequality (30) leads to

$$\frac{d}{dt}\mathcal{E}_{Am}(t) \leq B\mathcal{E}_{Am}(t), \qquad \forall t\in[0,\,\bar{t}], \tag{31}$$

where $B = \max\{B_1',\, B_2,\, B_3,\, B_4\}$. By using (31) it is easy to complete the proof of the theorem. □

Theorem 4. *Let* $(\mathbf{V},\, P, \vartheta, \varphi, \mathbf{h})$ *be the solution of the problem* (27), (2), (3), (26) *with homogeneous boundary conditions and* $\bar{t} \in (0,\infty)$. *If the inequalities* (18) *are verified, then there exists a positive constant* C_m, *depending on the material parameters, Poincaré constant of* Ω *and* $M_{\bar{t}} = \max_{\Omega\times[0,\bar{t}]} |\nabla\vartheta|$, *such that*

$$\mathcal{E}_{Am}(t) \leq \mathcal{E}_{Am}(0)\, e^{-C_m\, t}, \qquad \forall t\in[0,\bar{t}]. \tag{32}$$

Proof. By the same arguments used in the proof of Theorem 2, it is easily to show that

$$
\frac{d}{dt}\mathcal{E}_{Am} = \int_\Omega \left[-|\nabla \mathbf{V}|^2 + (R_T\vartheta - R_\Phi\varphi)V_3 - |\nabla\vartheta|^2 - |\nabla \mathbf{h}|^2\right] d\Omega + \\
\int_\Omega \left[\frac{1}{Le}\vartheta\left(\nabla\vartheta\cdot\nabla\varphi + \frac{1}{N_{BT}}|\nabla\vartheta|^2\right)\right. \\
\left. -A\left(|\nabla\varphi|^2 + \frac{1}{N_{BT}}\nabla\vartheta\cdot\nabla\varphi\right)\right] d\Omega, \tag{33}
$$

because the other terms involving the magnetic field do not make any contribution.

An application of Poincaré inequality to $\mathbf{h}$ and the arguments of the proof of Theorem 2 lead us to conclude that under conditions (18)

$$
\mathcal{E}_{Am}(t) \leq \mathcal{E}_{Am}(0)\, e^{-C_m\, t}, \qquad \forall t \in [0, \bar{t}],
$$

where $C_m = \frac{1}{\beta}\min\{C_1, \frac{C_2}{P_r}, \frac{C_3}{ASc}, \frac{1}{R_m}\}$, which completes the proof. □

Remark 4. We underline that the presence of the magnetic field may only modify the decay constant of the energy by means of R_m.

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References

[1] R. Azizian, E. Dorooduchi, T. McKrell, J. Buongiorno, L.W. Hu, B. Moghtaderi, Effect of magnetic field on laminar convective heat transfer of magnetite nanofluids, *Int. J. Heat Mass Transfer*, **68** (2014), 94-109.

[2] J. Buongiorno et al., A benchmark study on the thermal conductivity of nanofluids, *J. Appl. Phys.*, **106** (2009), 1-14.

[3] A. Borrelli, G. Giantesio, M.C. Patria, A.V. Rosca, N.C. Rosca, I. Pop, Influence of temperature and magnetic field on the oblique stagnation-point flow for a nanofluid past a vertical stretching/shrinking sheet, *Intern. J. of Numerical Methods for Heat & Fluid Flow*, **28** (2018), 2874-2894.

[4] J. Buongiorno, Convective transport in nanofluids, *ASME J. Heat Transfer*, **128** (2006), 240-250.

[5] S.U.S. Choi, Enhancing thermal conductivity of fluids with nanoparticles, In: *Proc. of the 1995 ASME Intern. Mechanical Engineering Congress and Exposition 66, ASME*, San Francisco, CA, 99-105.

[6] S.K. Das, S.U.S. Choi, W. Yu, Y. Pradeep, *Nanofluids: Science and Technology*, Wiley, New Jersey (2008).

[7] J. Fan, L. Wang, Review of heat conduction in nanofluids, *ASME J. Heat Transfer*, **133** (2011), 1-14.

[8] D.H. Fontes, D. Dall'Onder dos Santos, E.L. Martìnez Padilla, E.P. Bandarra Filh, Two numerical modelings of free convection heat transfer using nanofluids inside a square enclosure, *Mech. Res. Comm.*, **66** (2015), 34-43.

[9] S. Kakaç, A. Pramuanjaroenkij, Review of convective heat transfer enhancement with nanofluids, *Int. J. Heat Mass Transfer*, **52** (2009), 3187-3196.

[10] A.U. Khan, S. Nadeem, S.T. Hussain, Phase flow study of MHD nanofluid with slip effects on oscillatory oblique stagnation point flow in view of inclined magnetic field, *J. of Molecular Liquids*, **224** (2016), 1210-1219.

[11] D. Khurana, R. Choudhary, S. Subudhi, A critical review of forced convection heat transfer and pressure drop of and CuO nanofluids, *Heat Mass Transfer*, **53** (2017), 343-361.

[12] A.V. Kuznetsov, D.A. Nield, Natural convective boundary-layer flow of a nanofluid past a vertical plate: A revised model, *Int. J. Thermal Sci.*, **77** (2014), 126-129.

[13] O. Mahian, A. Kianifar, S.A. Kalogirou, I. Pop, S. Wongwises, A review of the applications of nanofluids in solar energy, *Int. J. Heat Mass Transfer*, **57** (2013), 582-594.

[14] O. Manca, Y. Jaluria, D. Poulikakos, Heat transfer in nanofluids, *Adv. Mechanical Engng.*, (2010), # 380826.

[15] A.A. Minea, A study on Brinkman number variation on water based nanofluid heat transfer in partially heated tubes, *Mech. Res. Comm.*, **73** (2016), 7-11.

[16] W.J. Minkowycz, E.M. Sparrow, J.P. Abraham (Eds.), *Nanoparticle Heat Transfer and Fluid Flow*, CRC Press, Taylor & Fracis Group, New York (2013).

[17] T.G. Myers, H. Ribera, V. Cregan, Does mathematics contribute to the nanofluid debate?, *Int. J. Heat Mass Transfer*, **111** (2017), 279-288.

[18] M.M. Rahman, T. Grosan, I. Pop. Oblique stagnation-point flow of a nanofluid past a shrinking sheet, *Int. J. Numer Methods Heat Fluid Flow*, **26** (2016), 189-213.

[19] N.C. Roşca, I. Pop, Axisymmetric rotational stagnation point flow impinging radially a permeable stretching/shrinking surface in a nanofluid using Tiwari and Das model, *Scientific Reports*, **7** (2017), # 40299; DOI: 10.1038/srep40299.

[20] M. Sheikholesslami, D.D. Ganji, *Applications of Nanofluid for Heat Transfer and Enhancemen*, Elsevier, 2017.

[21] M. Sheikholeslami, H. B. Rokni, Nanofluid two phase model analysis in existence of induced magnetic field, *Int. J. of Heat and Mass Transfer*, **107** (2017), 288-299.

[22] A. Shenoy, M. Sheremet, I. Pop, *Convective Flow and Heat Transfer from Wavy Surfaces: Viscous Fluids, Porous Media and Nanofluids*, CRC Press, Taylor & Francis Group, New York, 2016.

[23] R.K. Tiwari, M.K. Das, Heat transfer augmentation in a two-sided lid-driven differentially heated square cavity utilizing nanofluids, *Int. J. Heat Mass Transfer*, **50** (2007), 2002-2018.

[24] D. Y. Tzou, Thermal instability of nanofluids in natural convection, *Int. J. Heat Mass Transfer*, **51** (2008), 2967-2979.

[25] N.A.Yacob, A. Ishak, I. Pop, K. Vajravelu, Boundary layer flow past a stretching/shrinking surface beneath an external uniform shear flow with a convective surface boundary condition in a nanofluid. *Nanoscale Res. Lett.*, **6** (2011), 1-7.