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FRACTIONAL LAPLACIANS, PERIMETERS AND HEAT SEMIGROUPS IN CARNOT GROUPS

Fausto Ferrari

Dip. di Matematica, Università di Bologna
Piazza di Porta San Donato, 5, 40126
Bologna, Italy

Michele Miranda Jr.

Dip. di Matematica e Informatica, Università di Ferrara
via Machiavelli 30
44121 Ferrara, Italy

Diego Pallara

Dip. di Matematica e Fisica "Ennio De Giorgi", Università del Salento
P.O.B. 193, and INFN
73100 Lecce, Italy

Andrea Pinamonti

Dip. di Matematica, Università di Trento
via Sommarive, 14
38123 Povo (TN), Italy

Yannick Sire*

Department of Mathematics
Johns Hopkins University, 404 Krieger Hall, 3400 N. Charles Street
Baltimore, MD 21218, USA

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Abstract. We define and study the fractional Laplacian and the fractional perimeter of a set in Carnot groups and we compare the perimeter with the asymptotic behaviour of the fractional heat semigroup.

1. **Introduction.** The aim of this paper is to define the fractional perimeter of a set in Carnot groups and to investigate some relations between fractional perimeter and the asymptotic behaviour of the fractional heat semigroup (i.e., the semigroup generated by the fractional Laplacian in L^2) as $t \to 0$. Our results generalise those in [21, 2, 7, 1], where semigroups generated by *local* elliptic operators are considered. We discuss in an informal way the case of $\mathbb{R}^n$ in this introduction. The proofs are given in the more general case of Carnot groups.

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* Corresponding author.

In the whole paper α is a parameter in $(0,1)$. We start by defining the *Gagliardo seminorm* of $f:\mathbb{R}^n\to\mathbb{R}$

$$[f]^p_{W^{\alpha,p}}=\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|f(x)-f(y)|^p}{|x-y|^{n+p\alpha}}\,dxdy,\quad 1\le p<\infty, \tag{1}$$

the *fractional perimeter* of a Borel set $E\subset\mathbb{R}^n$ is

$$P_\alpha(E)=\frac{1}{2}[\chi_E]^2_{W^{\alpha/2,2}}=\int_{E\times E^c}\frac{1}{|x-y|^{n+\alpha}}\,dxdy, \tag{2}$$

where χ_E is the characteristic function of E and $E^c=\mathbb{R}^n\setminus E$, and the *fractional Laplacian* is

$$(-\Delta)^\alpha=\frac{1}{\Gamma(-\alpha)}\int_0^\infty\left(e^{t\Delta}-\mathrm{Id}\right)\frac{dt}{t^{1+\alpha}},\qquad \Gamma(-\alpha)=-\frac{\Gamma(1-\alpha)}{\alpha}, \tag{3}$$

where $e^{t\Delta}$ is the classical heat semigroup generated by Δ. As usual in functional calculus (see [29]), definition (3) is motivated by the elementary equality

$$\lambda^\alpha=\frac{1}{\Gamma(-\alpha)}\int_0^\infty\left(e^{-\lambda t}-1\right)\frac{dt}{t^{1+\alpha}},\quad \lambda>0.$$

Another way of defining the fractional Laplacian is via the Fourier transform, $\widehat{(-\Delta)^\alpha f}=|\xi|^{2\alpha}\hat{f}$, but the inversion formula (up to a constant depending on n and α) leads formally to a convolution against the singular kernel $|x|^{-(n+2\alpha)}$, the inverse Fourier transform of the symbol $|\xi|^{2\alpha}$. Nevertheless, it is possible to put the fractional Laplacian into a classical PDE framework. Indeed, according to [9, 29], the *nonlocal* operator $(-\Delta)^\alpha$ can be recovered from a boundary value problem concerning a (degenerate) *local* operator as follows. Let $f\in\mathrm{dom}((-\Delta)^\alpha)$. Consider the extension problem

$$\begin{cases}\mathrm{div}(y^{1-2\alpha}\nabla u)=0 & \text{in } \mathbb{R}^n\times(0,+\infty)\\ u(x,0)=f & \text{in } \mathbb{R}^n,\end{cases} \tag{4}$$

which is the Euler equation of the functional

$$J(u)=\int_{\mathbb{R}^n\times\mathbb{R}^+}\left(|Du|^2+|\partial_y u|^2\right)y^{1-2\alpha}\,dxdy, \tag{5}$$

u in the space $W^{1,2}(\mathbb{R}^n\times\mathbb{R}^+,dx\otimes y^{1-2\alpha}dy)$ defined by

$$W^{1,2}(\mathbb{R}^n\times\mathbb{R}^+,dx\otimes y^{1-2\alpha}dy)=\left\{u\in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n\times\mathbb{R}^+):\int_{\mathbb{R}^n\times\mathbb{R}^+}\left(|u|^2+|Du|^2+|\partial_y u|^2\right)y^{1-2\alpha}\,dxdy<\infty\right\},$$

under the constraint $u(\cdot,0)=f(\cdot)$. A solution of (4) is given by the Poisson type formula

$$u(x,y)=\frac{y^{2\alpha}}{4^\alpha\Gamma(\alpha)}\int_0^\infty e^{t\Delta}f(x)e^{-y^2/4t}\frac{dt}{t^{1+\alpha}} \tag{6}$$

and the following limit relation

$$\lim_{y\to0^+}y^{1-2\alpha}\partial_y u(x,y)=-\frac{2\alpha\Gamma(-\alpha)}{4^\alpha\Gamma(\alpha)}(-\Delta)^\alpha f(x) \tag{7}$$

holds in $L^2(\mathbb{R}^n)$. From (6), computing the derivative in (7) gives

$$(-\Delta)^\alpha f(x)=C(n,\alpha)\ \mathrm{P.V.}\int_{\mathbb{R}^n}\frac{f(x)-f(y)}{|x-y|^{n+2\alpha}}\,dy, \tag{8}$$

where the integral has to be understood in the usual *principal value* sense, i.e.,

$$\text{P.V.}\int_{\mathbb{R}^n} \frac{f(x)-f(y)}{|x-y|^{n+2\alpha}}\,dy = \lim_{\varepsilon\to 0}\int_{(B(x,\varepsilon))^c} \frac{f(x)-f(y)}{|x-y|^{n+2\alpha}}\,dy$$

and the constant $C(n,\alpha)$ is

$$C(n,\alpha) = \frac{4^\alpha\Gamma(n/2+\alpha)}{\pi^{n/2}\Gamma(-\alpha)}.$$

Integrating (8) the following equalities easily follow

$$\begin{aligned}\|(-\Delta)^{\alpha/2}f\|^2_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} f(-\Delta)^\alpha f\,dx = C(n,\alpha)\int_{\mathbb{R}^n} f(x)\int_{\mathbb{R}^n}\frac{f(x)-f(y)}{|x-y|^{n+2\alpha}}\,dydx \\ &= \frac{C(n,\alpha)}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^2}{|x-y|^{n+2\alpha}}\,dydx = \frac{C(n,\alpha)}{2}[f]^2_{W^{\alpha,2}}.\end{aligned} \tag{9}$$

Multiplying the differential equation in (4) by its solution u and integrating by parts leads to

$$\begin{aligned}[f]^2_{W^{\alpha,2}} &= 2C(n,\alpha)^{-1}\int_{\mathbb{R}^n} f(-\Delta)^\alpha f\,dx \\ &= \frac{\Gamma(\alpha)\pi^{n/2}}{\alpha\Gamma(n/2+\alpha)}\inf\Big\{J(u),\ u\in W^{1,2}(\mathbb{R}^n\times\mathbb{R}^+, dx\otimes y^{1-2\alpha}dy),\ u(\cdot,0)=f(\cdot)\Big\},\end{aligned} \tag{10}$$

so that the Gagliardo seminorm of f comes from a minimum problem. We refer to [18] for much more on the subject. Now, it is time for the *fractional heat semigroup* to come into play. Let $h_\alpha(t,z)$ be the fundamental solution of the fractional heat equation in $\mathbb{R}_+\times\mathbb{R}^n$

$$u_t + (-\Delta)^\alpha u = 0. \tag{11}$$

Setting $\tilde h_\alpha(z) = h_\alpha(1,z)$, h_α is known to satisfy

$$\int_{\mathbb{R}^n} h_\alpha(t,z)\,dz = 1 \qquad \forall\, t>0, \qquad h_\alpha(t,z) = \frac{1}{t^{n/2\alpha}}\tilde h_\alpha(t^{-1/2\alpha}z) \tag{12}$$

and

$$\lim_{t\to 0}\frac{h_\alpha(t,x)}{t} = \frac{C_{n,\alpha}}{|x|^{n+2\alpha}}, \tag{13}$$

see [3, Theorem 2.1], where the exact value of the constant is given. The fractional heat semigroup that gives the solution of (11) with initial datum f is given by

$$e^{-t(-\Delta)^\alpha}f(x) = \int_{\mathbb{R}^n} h_\alpha(t,y)f(x-y)\,dy, \qquad f\in L^1(\mathbb{R}^n),$$

and, since the kernel h_α has integral one, we have

$$e^{-t(-\Delta)^\alpha}f(x) - f(x) = \int_{\mathbb{R}^n} h_\alpha(t,y)\Big(f(x-y)-f(x)\Big)\,dy.$$

Recalling (9), we may consider the following quantity

$$Q^\alpha_t(f) = \int_{\mathbb{R}^n\times\mathbb{R}^n} h_\alpha(t,y)\Big(f(x-y)-f(x)\Big)^2\,dx\,dy.$$

Using the previous asymptotics on the fractional heat kernel allows us to get

$$\lim_{t\to 0}\frac{Q^\alpha_t(f)}{t} = C_{n,\alpha}[f]^2_{W^{\alpha,2}(\mathbb{R}^n)}.$$

Hence we have that $f \in W^{\alpha,2}(\mathbb{R}^n)$ if and only if

$$\lim_{t\to 0} \frac{Q_t^\alpha(f)}{t} < \infty.$$

Let us mention that similar results have been obtained in a stochastic analysis framework in [12].

The starting point of our investigation is the remark that all the previous relations between the fractional seminorms and perimeters on one hand and the fractional heat semigroups on the other rely on the asymptotic behaviour of the fundamental solution h_α given in (13) and on the asymptotic behaviour of the kernel $|z|^{-(n+2\alpha)}$, which appears in (1) and elsewhere, as $z \to 0$. Therefore, we were wondering whether similar computations can be done when suitable kernels exhibiting the same asymptotic behaviour are available. This is possible in Carnot groups, even though a limit relation as (13) seems not to be available and it has to be replaced by a two-sides estimate, see Section 3.

Notation. Henceforth, we denote by c all the constants that depend only on α and the dimension of the space. The value of c can vary from line to line.

2. **Preliminary results.** A connected and simply connected stratified nilpotent Lie group $(\mathbb{G},\cdot)$ is said to be a *Carnot group of step k* if its Lie algebra $\mathfrak{g}$ admits a *step k stratification*, i.e., there exist linear subspaces $V_1,...,V_k$ such that

$$\mathfrak{g} = V_1 \oplus ... \oplus V_k, \quad [V_1, V_i] = V_{i+1}, \quad V_k \neq \{0\}, \quad V_i = \{0\} \text{ if } i > k, \tag{2.1}$$

where $[V_1, V_i]$ is the subspace of $\mathfrak{g}$ generated by the commutators $[X,Y]$ with $X \in V_1$ and $Y \in V_i$.

In the last few years, Carnot groups have been largely studied in several respects, such as differential geometry [10], subelliptic differential equations [4, 17, 16, 27], complex variables [28].

For a general introduction to Carnot groups from the point of view of the present paper and for further examples, we refer, e.g., to [4, 17, 28].

Set $m_i = \dim(V_i)$, for $i = 1,\dots,k$ and $h_i = m_1 + \dots + m_i$, so that $h_k = n$. For sake of simplicity, we write also $h_0 = 0$, $m := m_1$. We denote by Q the *homogeneous dimension* of $\mathbb{G}$, i.e., we set

$$Q := \sum_{i=1}^k i \dim(V_i).$$

We choose now a basis $e_1,\dots,e_n$ of $\mathbb{R}^n$ adapted to the stratification of $\mathfrak{g}$, i.e., such that $e_{h_{j-1}+1},\dots,e_{h_j}$ is a basis of V_j for each $j = 1,\dots,k$. Moreover, let $X = \{X_1,\dots,X_n\}$ be the family of left invariant vector fields such that $X_i(0) = e_i$, $i = 1,\dots,n$.

The sub-bundle of the tangent bundle $T\mathbb{G}$ that is spanned by the vector fields $X_1,\dots,X_m$ plays a particularly important role in the theory, it is called the *horizontal bundle* $H\mathbb{G}$; the fibers of $H\mathbb{G}$ are

$$H_x\mathbb{G} = \operatorname{span}\{X_1(x),\dots,X_m(x)\}, \qquad x \in \mathbb{G}.$$

We can endow each fiber of $H\mathbb{G}$ with an inner product $\langle\cdot,\cdot\rangle$ and with a norm $|\cdot|$ that make the basis $X_1(x),\dots,X_m(x)$ an orthonormal basis. The sections of $H\mathbb{G}$ are called *horizontal sections* and a vector of $H_x\mathbb{G}$ a *horizontal vector*. Each horizontal section is identified by its canonical coordinates with respect to this moving frame $X_1(x),\dots,X_m(x)$. This way, a horizontal section ϕ is identified with a function

$\phi = (\phi_1, \dots, \phi_m) : \mathbb{R}^n \to \mathbb{R}^m$. For any $x \in \mathbb{G}$, the *(left) translation* $\tau_x : \mathbb{G} \to \mathbb{G}$ is defined as

$$\tau_x z = x \cdot z.$$

For any $\lambda > 0$, the *dilation* $\delta_\lambda : \mathbb{G} \to \mathbb{G}$, is defined as

$$\delta_\lambda(x_1, ..., x_n) = (\lambda \xi_1, \dots, \lambda^k \xi_k), \tag{2.2}$$

where $x = (\xi_1, \dots, \xi_k) \in \mathbb{R}^{m_1} \times \dots \times \mathbb{R}^{m_k} \equiv \mathbb{G}$.

The Haar measure of $\mathbb{G} = (\mathbb{R}^n, \cdot)$ is the Lebesgue measure in $\mathbb{R}^n$. If $A \subset \mathbb{G}$ is Lebesgue measurable, we write $|A|$ to denote its Lebesgue measure.

Once an orthonormal basis $X_1, \dots, X_m$ of the horizontal layer is fixed, we define, for any function $f : \mathbb{G} \to \mathbb{R}$ for which the partial derivatives $X_j f$ exist, the horizontal gradient of f, denoted by $\nabla_{\mathbb{G}} f$, as the horizontal section

$$\nabla_{\mathbb{G}} f := \sum_{i=1}^m (X_i f) X_i,$$

whose coordinates are $(X_1 f, ..., X_m f)$.Given $X_1, \dots, X_m$, we denote by $\mathcal{L}$ the associated positive sub-Laplacian, namely

$$\mathcal{L} := -\sum_{j=1}^m X_j^2.$$

Let $\| \cdot \| : \mathbb{G} \to [0, \infty)$ denote a symmetric homogeneous norm on $\mathbb{G}$ [4]. Since any two continuous homogeneous norm are equivalent [4], from now on we denote by $\| \cdot \|$ any one of them; all the estimates that we give are then the same up to changes in the constants. We denote by

$$B(x, r) = \{y \in \mathbb{G} : \|y^{-1} \cdot x\| < r\}$$

the ball centred at $x \in \mathbb{G}$ with radius $r > 0$ and by $B(r) = B(0, r)$.

2.1. **Heat kernels and norms.** Consider $\hat{\mathbb{G}} := \mathbb{R} \times \mathbb{G}$ as a Carnot group, where the group operation in the first coordinate is the usual addition, the dilations are $\hat{\delta}_\lambda(t, x) = (\lambda^2 t, \delta_\lambda(x))$, and its Lie algebra $\hat{\mathfrak{g}}$ admits the stratification

$$\hat{\mathfrak{g}} = \hat{V}_1 \oplus V_2 \oplus \cdots \oplus V_k, \tag{2.3}$$

where $\hat{V}_1 = \text{span}\,\{\hat{T}, V_1\}$ and $\hat{T} = \partial_t$. Since the basis $\{X_1, \dots, X_m\}$ of V_1 has been already fixed once and for all, the associated basis for $\hat{V}_1$ is $\{\hat{T}, X_1, \dots, X_m\}$. The heat operator in $\hat{\mathbb{G}}$:

$$\mathcal{H} := \partial_t + \mathcal{L}$$

is translation invariant, homogeneous of degree 2, i.e., $\mathcal{H}(u \circ \hat{\delta}_\lambda) = \lambda^2 \mathcal{H} u$, and hypoelliptic. It is well known (see [17]) that $\mathcal{H}$ admits a fundamental solution h, usually called the heat kernel in $\mathbb{G}$. In the following theorem we collect the main properties of h, the proofs can be found in [17, 4].

Theorem 2.1. *There exists a function h defined in $\hat{\mathbb{G}}$ such that:*

(i) $h \in C^\infty(\hat{\mathbb{G}} \setminus \{(0, 0)\})$;
(ii) $h(\lambda^2 t, \delta_\lambda(x)) = \lambda^{-Q} h(t, x)$ *for every* $t > 0$, $x \in \mathbb{G}$ *and* $\lambda > 0$;
(iii) $h(t, x) = 0$ *for every* $t < 0$ *and* $\int_{\mathbb{G}} h(t, x) dx = 1$ *for every* $t > 0$;
(iv) $h(t, x) = h(t, x^{-1})$ *for every* $t > 0$ *and* $x \in \mathbb{G}$;

(v) there exists $c>0$ such that for every $x\in\mathbb{G}$ and $t>0$

$$c^{-1}t^{-Q/2}\exp\Big(-\frac{\|x\|^2)}{c^{-1}t}\Big)\le h(x,t)\le ct^{-Q/2}\exp\Big(-\frac{\|x\|^2}{ct}\Big). \tag{2.4}$$

As in the Euclidean case, we can introduce the heat semigroup by

$$e^{-t\mathcal{L}}f(x):=\int_{\mathbb{G}}h(t,y^{-1}\cdot x)f(y)dy,\quad f\in L^1(\mathbb{G}). \tag{2.5}$$

Then for every $f\in L^1(\mathbb{G})$ and $t>0$, we have $e^{-t\mathcal{L}}f\in C^\infty(\mathbb{G})$, the function $v(t,x):=e^{-t\mathcal{L}}f(x)$ solves $\mathcal{H}v=0$ in $(0,\infty)\times\mathbb{G}$, $v(t,x)\to f(x)$ strongly in $L^1(\mathbb{G})$ as $t\to 0$ and $-\mathcal{L}$ is the infinitesimal generator of $e^{-t\mathcal{L}}$. As proved in [5, Proposition 3.2.1], the heat semigroup preserves one-sided bounds, namely

$$f\le C \text{ a.e. in } \mathbb{G}\Longrightarrow e^{-t\mathcal{L}}f\le C \text{ a.e. in } \mathbb{G},\ \forall t\ge 0. \tag{2.6}$$

As in [15], for every $\alpha>0$ we define

$$\tilde R_\alpha(x):=-\frac{\alpha}{2\Gamma(-\alpha/2)}\int_0^\infty t^{-\frac{\alpha}{2}-1}h(t,x)dt,\quad R_\alpha(x)=\frac{1}{\Gamma(\alpha/2)}\int_0^\infty t^{\frac{\alpha}{2}-1}h(t,x)dt. \tag{2.7}$$

As pointed out in [15], $\tilde R_\alpha$ and R_α are smooth functions in $\mathbb{G}\setminus\{0\}$ and $\mathcal{L}R_{2-\alpha}=\tilde R_{-\alpha}$. In addition, $\tilde R_\alpha$ is positive and homogeneous of degree $-\alpha-Q$. Moreover, using (iv) and (v) in Theorem 2.1 we get

$$\tilde R_\alpha(x)=\tilde R_\alpha(x^{-1}), \tag{2.8}$$

and

$$c^{-1}\|x\|^{-\alpha-Q}\le\tilde R_\alpha(x)\le c\|x\|^{-\alpha-Q}\qquad\forall x\in\mathbb{G}. \tag{2.9}$$

We define

$$\|x\|_\alpha:=\Big(\tilde R_\alpha(x)\Big)^{-\frac{1}{\alpha+Q}}, \tag{2.10}$$

which turns out to be a homogeneous symmetric norm, smooth out of the origin. Using (2.9), we find a constant $c>0$ depending only on α such that for every $x\in\mathbb{G}$

$$c^{-1}\|x\|\le\|x\|_\alpha\le c\|x\|.$$

Remark 1. If $(\mathbb{G},\cdot)=(\mathbb{R}^n,+)$ then for every $\alpha>0$

$$\tilde R_\alpha(x)=-\frac{\alpha}{2\Gamma(-\alpha/2)}\frac{1}{(4\pi)^{\frac{n}{2}}}\int_0^\infty t^{-\alpha/2-n/2-1}e^{-\frac{|x|^2}{4t}}dt.$$

Changing variables we obtain:

$$\tilde R_\alpha(x)=-\frac{\alpha}{2\Gamma(-\alpha/2)}\frac{4^{\frac{\alpha}{2}+\frac{n}{2}}}{(4\pi)^{\frac{n}{2}}}|x|^{-\alpha-n}\int_0^\infty y^{\frac{1+\alpha}{2}}e^{-y}dy=-\frac{\alpha}{2\Gamma(-\alpha/2)}\frac{4^{\frac{\alpha}{2}+\frac{n}{2}}}{(4\pi)^{\frac{n}{2}}}\Gamma\Big(\frac{\alpha+n}{2}\Big)|x|^{-\alpha-n}.$$

Setting $c:=-\frac{\alpha}{2\Gamma(-\alpha/2)}\frac{4^{\frac{\alpha}{2}}}{\pi^{\frac{n}{2}}}\Gamma\Big(\frac{\alpha+n}{2}\Big)$, we get

$$\tilde R_\alpha(x)=c\frac{1}{|x|^{n+\alpha}}\quad\text{and}\quad\|x\|_\alpha=c^{-\frac{1}{n+\alpha}}|x|.$$

2.2. **Fractional Sobolev Spaces.** The theory of fractional Besov and Sobolev spaces in Carnot groups is well developed and is presented in [16], [26]. Let $\alpha \in (0,1)$ and $p \geq 1$. In analogy with the Euclidean case, we consider the *fractional Gagliardo seminorm* in $\mathbb{G}$ defined by

$$[f]^p_{\alpha,p;\mathbb{G}} := \int_{\mathbb{G}\times\mathbb{G}} \frac{|f(x)-f(y)|^p}{\|y^{-1}\cdot x\|_\alpha^{Q+\alpha p}}\, dxdy$$

and the corresponding norm defined by

$$\|f\|_{\alpha,p;\mathbb{G}} := \left(\|f\|^p_{L^p(\mathbb{G})} + [f]^p_{\alpha,p;\mathbb{G}}\right)^{1/p}.$$

Let $W^{\alpha,p}(\mathbb{G})$ be the space of measurable functions $f : \mathbb{G} \to \mathbb{R}$ such that $\|f\|_{\alpha,p;\mathbb{G}} < \infty$. Following [17] and [26] it is readily seen that $(W^{\alpha,p}(\mathbb{G}), \|\cdot\|_{\alpha,p;\mathbb{G}})$ is a Banach space and the Schwartz space $\mathcal{S}(\mathbb{G})$ is a subspace of $W^{\alpha,p}(\mathbb{G})$.

We can also define, for $\alpha \in (0,1)$ and $p \geq 1$ the space

$$H^{\alpha,p}(\mathbb{G}) := \overline{\mathcal{S}(\mathbb{G})}^{\|\cdot\|_{\alpha,p;\mathbb{G}}}.$$

By the preceding discussion, $H^{\alpha,p}(\mathbb{G})$ is a Banach space. For $\mathbb{G} = \mathbb{R}^n$ this space is consistent with the usual fractional space $W^{\alpha,p}(\mathbb{R}^n)$, see [8]. As in the Euclidean case, we have the equality $W^{\alpha,p}(\mathbb{G}) = H^{\alpha,p}(\mathbb{G})$, see [16, Theorem (4.5)].

2.3. **Fractional Laplacian.** From now on, we deal only with the case $p = 2$, where the theory of fractional perimeters is settled.

As $\mathcal{L}$ is a self-adjoint operator on $L^2(\mathbb{G})$, we can define the operator $\mathcal{L}^\alpha : W^{2\alpha,2}(\mathbb{G}) = D(\mathcal{L}^\alpha) \subset L^2(\mathbb{G}) \to L^2(\mathbb{G})$ by functional calculus (see e.g. [30, §IX.11] and [16, §3] for the present case) as in (3)

$$\mathcal{L}^\alpha = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \left(e^{-t\mathcal{L}} - \mathrm{Id}\right) \frac{dt}{t^{1+\alpha}}. \tag{2.11}$$

Notice that the operator $\mathcal{L}^\alpha$ can be equivalently defined via the *spectral resolution* $\{E(\lambda)\}$ of $\mathcal{L}$ in $L^2(\mathbb{G})$,

$$\mathcal{L}^\alpha = c(\alpha) \int_0^\infty \lambda^\alpha \, dE(\lambda) \tag{2.12}$$

for a suitable constant $c(\alpha)$.

Let us recall that on smooth functions $\mathcal{L}^\alpha$ has a pointwise representation. In view of (2.12), the proof of the following result is contained in [15, Theorem 3.11], where the kernel is written with a different notation.

Proposition 1. *For every $u \in \mathcal{S}(\mathbb{G})$ and $\alpha \in (0,1)$ it holds*

$$\mathcal{L}^\alpha u(x) = \mathrm{P.V.} \int_{\mathbb{G}} \frac{u(x)-u(y)}{\|y^{-1}\cdot x\|_\alpha^{Q+2\alpha}} dy.$$

Since $\mathcal{L}^\alpha$ is self-adjoint and $\mathcal{L}^\alpha \circ \mathcal{L}^\beta = \mathcal{L}^{\alpha+\beta}$ we immediately have

$$\|\mathcal{L}^{\frac{\alpha}{2}} f\|^2_{L^2(\mathbb{G})} = \int_{\mathbb{G}} f(x)\mathcal{L}^\alpha f(x) dx, \quad \forall f \in \mathcal{S}(\mathbb{G}).$$

Remark 2. As for the Euclidean case (4), the equation $\mathcal{L}^\alpha u(x) = f(x)$ in $\mathbb{G}$ is related to the following elliptic degenerate problem

$$\begin{cases} \mathrm{div}_{\hat{G}}(y^\beta \nabla_{\hat{\mathbb{G}}} v) = 0 & \text{in } \mathbb{G}\times(0,+\infty) \\ \\ v(x,0) = f & \text{in } \mathbb{G}\times\{0\}, \end{cases} \tag{2.13}$$

$\beta = 1 - 2\alpha$. Precisely, the following result has bee proved in [15].

Theorem 2.2. *Let $u \in W^{2\alpha,2}(\mathbb{G})$ be given, $u \geq 0$ and $\mathcal{L}^\alpha u = 0$ in an open set $\Omega \subseteq \mathbb{G}$. Let us define*

$$v(\cdot, y) := u * P_{\mathbb{G}}(\cdot, y) \tag{2.14}$$

where $P_{\mathbb{G}}(\cdot, y)$ is as in [15, Theorem 4.4]. *We denote by $\hat{v}$ the function on $\hat{\mathbb{G}}$ obtained continuing v by parity across $\{y = 0\}$. Then,*

(i) $\hat{v} \geq 0$;
(ii) $\hat{v} \in W^{1,2}_{\mathbb{G},loc}(\hat{\Omega}, y^{1-2\alpha} dx \otimes dy)$ *where* $\hat{\Omega} = \Omega \times (-1, 1)$;
(iii) $\hat{v}$ *is a weak solution of* $\mathrm{div}_{\hat{\mathbb{G}}}(y^{1-2\alpha}\nabla_{\hat{\mathbb{G}}} v) = 0$ *in* $\hat{\Omega}$.

In the following proposition we show another expression of the fractional Laplacian.

Proposition 2. *If $u \in \mathcal{S}(\mathbb{G})$ then*

$$\mathcal{L}^\alpha u(x) = -\frac{1}{2}\int_{\mathbb{G}} \frac{u(x \cdot y) + u(x \cdot y^{-1}) - 2u(x)}{\|y\|_\alpha^{Q+2\alpha}} dy, \tag{2.15}$$

.

Proof. We proceed as in [14]. By definition,

$$\mathcal{L}^\alpha u(x) = -\mathrm{P.V.}\int_{\mathbb{G}} \frac{u(y) - u(x)}{\|y^{-1} \cdot x\|_\alpha^{Q+2\alpha}} dy,$$

by choosing $z := y^{-1}x$ we get

$$\mathcal{L}^\alpha u(x) = -\mathrm{P.V.}\int_{\mathbb{G}} \frac{u(x \cdot z^{-1}) - u(x)}{\|z\|_\alpha^{Q+2\alpha}} dz. \tag{2.16}$$

Moreover, by substituting $w = z^{-1}$ and using (iv) in Theorem 2.1, we have

$$\begin{aligned}\mathrm{P.V.}\int_{\mathbb{G}} \frac{u(x \cdot z^{-1}) - u(x)}{\|z\|_\alpha^{Q+2\alpha}} dz &= \mathrm{P.V.}\int_{\mathbb{G}} \frac{u(x \cdot w) - u(x)}{\|w^{-1}\|_\alpha^{Q+2\alpha}} dw \\ &= \mathrm{P.V.}\int_{\mathbb{G}} \frac{u(x \cdot w) - u(x)}{\|w\|_\alpha^{Q+2\alpha}} dw. \end{aligned} \tag{2.17}$$

Relabeling $w = z$ and adding (2.16) and (2.17) we have

$$2\mathcal{L}^\alpha u(x) = -\mathrm{P.V.}\int_{\mathbb{G}} \frac{u(x \cdot z^{-1}) + u(x \cdot z) - 2u(x)}{\|z\|_\alpha^{Q+2\alpha}} dz.$$

To conclude it suffices to prove that for every $x \in \mathbb{G}$ the map $G : \mathbb{G} \to \mathbb{R}$ defined by

$$G(z) = \frac{u(x \cdot z^{-1}) + u(x \cdot z) - 2u(x)}{\|z\|_\alpha^{Q+2\alpha}}$$

belongs to $L^1(\mathbb{G})$. In [15, Theorem 3.1] it is proved that for every $x \in \mathbb{G}$ the maps

$$\begin{aligned} G_1 &: \mathbb{G} \to \mathbb{R} \\ z &\mapsto \frac{u(x \cdot z) - u(x) - \omega(z)\langle \nabla_{\mathbb{G}} u(x), \pi_x(z)\rangle}{\|z\|_\alpha^{Q+2\alpha}} \end{aligned}$$

and

$$\begin{aligned} G_2 &: \mathbb{G} \to \mathbb{R} \\ z &\mapsto \frac{u(x \cdot z^{-1}) - u(x) - \omega(z^{-1})\langle \nabla_{\mathbb{G}} u(x), \pi_x(z^{-1})\rangle}{\|z\|_\alpha^{Q+2\alpha}}, \end{aligned}$$

where π_x denotes the orthogonal projection on $H_x\mathbb{G}$, belong to $L^1(\mathbb{G})$, where $\omega(x)$ denotes the characteristic function of the ball $\{y \in \mathbb{G} \mid \|y^{-1}\cdot x\|_\alpha \le 1\}$. By using (iv) in Theorem 2.1 and the fact that $\|\cdot\|_\alpha$ is a homogeneous norm we get

$$\frac{\omega(z^{-1})\left\langle \nabla_{\mathbb{G}} u(x), \pi_x(z^{-1})\right\rangle}{\|z^{-1}\|_\alpha^{Q+2\alpha}} = -\frac{\omega(z)\left\langle \nabla_{\mathbb{G}} u(x), \pi_x(z)\right\rangle}{\|z\|_\alpha^{Q+2\alpha}}$$

therefore $G = G_1 + G_2 \in L^1(\mathbb{G})$. □

Proposition 3. *For any $u \in \mathcal{S}(\mathbb{G})$ the following statement holds:*

$$\lim_{\alpha\to 1^-}(1-\alpha)\mathcal{L}^\alpha u(x) = \mathcal{L}u(x), \qquad \forall x \in \mathbb{G}.$$

Proof. Using the change of variables $\xi = t^{1-\alpha}$ in (2.11) we get

$$(1-\alpha)\mathcal{L}^\alpha u(x) = \frac{1}{\Gamma(-\alpha)}\int_0^\infty (e^{-\xi^{\frac{1}{1-\alpha}}\mathcal{L}}u(x) - u(x))\xi^{-\frac{1}{1-\alpha}}d\xi. \tag{2.18}$$

Clearly, if $0 \le \xi < 1$ then $\lim_{\alpha\to1^-}\xi^{\frac{1}{1-\alpha}} = 0$. Since $u \in \mathcal{S}(\mathbb{G})$ then by [16, (ii) in Theorem 3.1] (notice that $\mathcal{D}(\mathbb{G})$ is L^∞-dense in $\mathcal{S}(\mathbb{G})$) we have

$$\left\|\frac{e^{-h\mathcal{L}}u - u}{h} + \mathcal{L}u\right\|_{L^\infty(\mathbb{G})} \to 0 \quad \text{as } h \to 0. \tag{2.19}$$

Set $c = 1/\Gamma(-\alpha) = -\alpha/\Gamma(1-\alpha)$. We claim that for every $x \in \mathbb{G}$

$$\lim_{\alpha\to1^-} c\int_0^1 (e^{-\xi^{\frac{1}{1-\alpha}}\mathcal{L}}u(x) - u(x))\xi^{-\frac{1}{1-\alpha}}d\xi = \mathcal{L}u(x). \tag{2.20}$$

Indeed, denoting by $f_{\alpha,x}(\xi) := (e^{-\xi^{\frac{1}{1-\alpha}}\mathcal{L}}u(x) - u(x))\xi^{-\frac{1}{1-\alpha}}$ and using (2.19) we get

$$f_{\alpha,x}(\xi) \to -\mathcal{L}u(x) \quad \forall x \in \mathbb{G}. \tag{2.21}$$

Moreover,

$$|f_{\alpha,x}(\xi)| = \left|(e^{-\xi^{\frac{1}{1-\alpha}}\mathcal{L}}u(x) - u(x))\xi^{-\frac{1}{1-\alpha}}\right| = \left|\frac{1}{\xi^{\frac{1}{1-\alpha}}}\int_0^{\xi^{\frac{1}{1-\alpha}}}\frac{d}{d\tau}e^{-\tau\mathcal{L}}u(x)d\tau\right| \tag{2.22}$$
$$= \left|\frac{1}{\xi^{\frac{1}{1-\alpha}}}\int_0^{\xi^{\frac{1}{1-\alpha}}} e^{-\tau\mathcal{L}}\mathcal{L}u(x)d\tau\right| \le \|\mathcal{L}u\|_{L^\infty(\mathbb{G})},$$

where in the last inequality we used $u \in \mathcal{S}(\mathbb{G})$ and (2.6). Thus, (2.20) follows using the definition of c, (2.21) and (2.22). Furthermore, using (2.6), we obtain

$$c\int_1^\infty |(e^{-\xi^{\frac{1}{1-\alpha}}\mathcal{L}}u(x) - u(x))|\xi^{-\frac{1}{1-\alpha}}d\xi \le 2c\|u\|_{L^\infty(\mathbb{G})}\int_1^\infty \xi^{-\frac{1}{1-\alpha}}d\xi = -\frac{(1-\alpha)}{\Gamma(1-\alpha)}\|u\|_{L^\infty(\mathbb{G})}$$

and the thesis follows letting $\alpha \to 1^-$. □

3. **Fractional heat semigroup and perimeter.** This section is devoted to the presentation of our main results. After describing the relevant properties of the fractional heat semigroup in $\mathbb{G}$ and of the related kernel, we define the fractional perimeter and show their connections, as sketched in the Introduction.

We start by recalling a few properties of the semigroups generated by fractional powers of generators of strongly continuous semigroup and refer to [30, §IX.11] for more information and the missing proofs. To this aim, let $\{e^{tA};\quad t \ge 0\}$ be an equicontinuous semigroup of class C_0 in a Banach space X, whose generator is the operator $A : D(A) \subset X \to X$. For $\sigma > 0, t > 0, \alpha \in (0,1)$ we define the function

$$f_{t,\alpha}(\lambda) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{\sigma - i\infty}^{\sigma + i\infty} e^{z\lambda - tz^\alpha} dz, & \text{if } \quad \lambda > 0 \\ 0 & \text{if } \lambda < 0. \end{cases}$$

We define $\{e^{tA_\alpha}; \quad t \ge 0\}$, a new equicontinuous semigroup of class C_0, as follows

$$e^{tA_\alpha} u = \int_0^\infty f_{t,\alpha}(s) e^{sA} u \, ds, \qquad u \in X. \tag{3.1}$$

The infinitesimal generator A_α of e^{tA_α} is the corresponding fractional power of the generator of e^{tA}, i.e., $A_\alpha = -(-A)^\alpha$, where $(-A)^\alpha$ is defined as in (2.11). Moreover, e^{tA_α} is a holomorphic semigroup and a change of variable in the integral shows that

$$e^{tA_\alpha} u = \int_0^\infty f_{1,\alpha}(\tau) e^{\tau t^{1/\alpha} A} u \, d\tau. \tag{3.2}$$

Let us come to the heat semigroup. In this case, $\mathcal{L}^\alpha$ is defined in (2.11), its domain is $W^{2\alpha,2}(\mathbb{G})$ and we may write

$$e^{t\mathcal{L}^\alpha} u(x) = \int_0^\infty f_{t,\alpha}(s) \left(\int_{\mathbb{G}} h(s, y^{-1} \cdot x) u(y) dy \right) ds, \qquad u \in L^2(\mathbb{G}),$$

so that using (3.2) we have

$$\begin{aligned} e^{t\mathcal{L}^\alpha} u(x) &= \int_0^\infty e^{\tau t^{1/\alpha} \mathcal{L}} u(x) f_{1,\alpha}(\tau) d\tau = \int_0^\infty \left(\int_{\mathbb{G}} h(\tau t^{1/\alpha}, y^{-1} \cdot x) u(y) dy \right) f_{1,\alpha}(\tau) d\tau \\ &= \int_{\mathbb{G}} \left(\int_0^\infty h(\tau t^{1/\alpha}, y^{-1} \cdot x) f_{1,\alpha}(\tau) d\tau \right) u(y) dy. \end{aligned}$$

Thus, the function

$$h_\alpha(t, y) = \int_0^\infty h(\tau t^{\frac{1}{\alpha}}, y) f_{1,\alpha}(\tau) d\tau \tag{3.3}$$

is the integral kernel of the semigroup $e^{t\mathcal{L}^\alpha}$, i.e.,

$$e^{t\mathcal{L}^\alpha} u(x) = \int_{\mathbb{G}} h_\alpha(t, y^{-1}x) u(y) \, dy, \qquad u \in L^2(\mathbb{G}). \tag{3.4}$$

Remark 3. If h is the classical heat kernel in $\mathbb{R}^n$, i.e.,

$$h(t, y) = \frac{1}{(2\pi t)^{n/2}} e^{-|y|^2/4t} = \frac{1}{(2\pi t)^{n/2}} \tilde{h}(|y|/\sqrt{t}), \tag{3.5}$$

where $\tilde{h}(r) = e^{-\frac{r^2}{4}}$, we get

$$h_\alpha(t, y) = \int_0^\infty h(\tau t^{1/\alpha}, y) f_{1,\alpha}(\tau) d\tau = \int_0^\infty \frac{1}{(2\pi \tau t^{1/\alpha})^{n/2}} e^{-|y|^2/(4\tau t^{1/\alpha})} f_{1,\alpha}(\tau) d\tau = \frac{1}{t^{n/2\alpha}} \tilde{h}_\alpha(t^{-1/2\alpha} y),$$

where, according to (12),

$$\tilde{h}_\alpha(z) = \int_0^\infty \frac{1}{(2\pi\tau)^{n/2}} e^{-|z|^2/(4\tau)} f_{1,\alpha}(\tau) d\tau.$$

Some more estimates are available, and we are going to exploit some of them.

Remark 4. Recalling (2.4) we can give Gaussian type estimate of h_α. Indeed, let us argue on the upper estimates; we get:

$$\begin{aligned} h_\alpha(t,y) &= \int_0^\infty h(\tau t^{1/\alpha}, y) f_{1,\alpha}(\tau)d\tau \\ &\le c\int_0^\infty \tau^{-\frac{Q}{2}} t^{-\frac{Q}{2\alpha}} \exp\left(-\frac{\|y\|^2}{c\tau t^{\frac{1}{\alpha}}}\right) f_{1,\alpha}(\tau)d\tau \\ &= ct^{-\frac{Q}{2\alpha}} \int_0^\infty \tau^{-\frac{Q}{2}} \exp\left(-\frac{\|y\|^2}{c\tau t^{\frac{1}{\alpha}}}\right) f_{1,\alpha}(\tau)d\tau \end{aligned}$$

where the integral above converges thanks to [30, Proposition IX.11.3]. Arguing in the same way for the lower estimates, we eventually get

$$c^{-1}t^{-\frac{Q}{2\alpha}} \int_0^\infty \tau^{-\frac{Q}{2}} \exp\Big(-\frac{\|y\|^2}{c\tau t^{\frac{1}{\alpha}}}\Big) f_{1,\alpha}(\tau)d\tau \le h_\alpha(t,y) \le ct^{-\frac{Q}{2\alpha}} \int_0^\infty \tau^{-\frac{Q}{2}} \exp\Big(-\frac{\|y\|^2}{c\tau t^{\frac{1}{\alpha}}}\Big) f_{1,\alpha}(\tau)d\tau. \tag{3.6}$$

Beside the above Gaussian estimates, we recall that the following also hold, (see [11]):

$$c^{-1}\Big(t^{-Q/2\alpha} \wedge \frac{t}{\|z\|^{Q+2\alpha}}\Big) \le h_\alpha(t,z) \le c\Big(t^{-Q/2\alpha} \wedge \frac{t}{\|z\|^{Q+2\alpha}}\Big). \tag{3.7}$$

Let us now introduce a notion of α-horizontal perimeter.

Definition 3.1. For a Borel set $E \subset \mathbb{G}$ and $\alpha \in (0,1)$ the fractional $\alpha-$horizontal perimeter of E is

$$\mathrm{Per}_{\alpha,\mathbb{G}}(E) := \int_E \int_{E^c} \frac{1}{\|y^{-1}\cdot x\|_\alpha^{Q+\alpha}} dxdy.$$

We say that $E \subset \mathbb{G}$ has finite fractional $\alpha-$horizontal perimeter if $\mathrm{Per}_{\alpha,\mathbb{G}}(E) < \infty$.

In the following Proposition we collect some elementary properties of the fractional $\alpha-$horizontal perimeter.

Lemma 3.2. *Let $\alpha \in (0,1)$.*

(i) (Subadditivity) Let $E, F \subset \mathbb{G}$ be Borel sets, then

$$\mathrm{Per}_{\alpha,\mathbb{G}}(E \cup F) \le \mathrm{Per}_{\alpha,\mathbb{G}}(E) + \mathrm{Per}_{\alpha,\mathbb{G}}(F).$$

(ii) (Translation invariance) Let $E \subset \mathbb{G}$ be a Borel set; for any $z \in \mathbb{G}$ we have

$$\mathrm{Per}_{\alpha,\mathbb{G}}(\tau_z(E)) = \mathrm{Per}_{\alpha,\mathbb{G}}(E).$$

(iii) (Scaling) Let $E \subset \mathbb{G}$ be a Borel set; for any $\lambda > 0$ we have

$$\mathrm{Per}_{\alpha,\mathbb{G}}(\delta_\lambda(E)) = \lambda^{Q-\alpha}\mathrm{Per}_{\alpha,\mathbb{G}}(E).$$

(iv) (Equivalence with the fractional norm) Let $E \subset \mathbb{G}$ be a Borel set and $p \ge 1$; if $\mathrm{Per}_{\alpha,\mathbb{G}}(E) < \infty$ then

$$\mathrm{Per}_{\alpha,\mathbb{G}}(E) = \frac{1}{2}[\chi_E]^p_{\frac{\alpha}{p},p;\mathbb{G}}.$$

Proof. (i). If E and F are Borel sets, then

$$\begin{aligned}\mathrm{Per}_{\alpha,\mathbb{G}}(E\cup F) &= \int_{(E\cup F)}\int_{(E\cup F)^c}\frac{1}{\|y^{-1}\cdot x\|_\alpha^{Q+\alpha}}dxdy\\ &\leq \int_E\int_{(E\cup F)^c}\frac{1}{\|y^{-1}\cdot x\|_\alpha^{Q+\alpha}}dxdy+\int_F\int_{(E\cup F)^c}\frac{1}{\|y^{-1}\cdot x\|_\alpha^{Q+\alpha}}dxdy\\ &\leq \int_E\int_{E^c}\frac{1}{\|y^{-1}\cdot x\|_\alpha^{Q+\alpha}}dxdy+\int_F\int_{F^c}\frac{1}{\|y^{-1}\cdot x\|_\alpha^{Q+\alpha}}dxdy\\ &= \mathrm{Per}_{\alpha,\mathbb{G}}(E)+\mathrm{Per}_{\alpha,\mathbb{G}}(F).\end{aligned}$$

(ii) and (iii) follow from a change of variables, noticing that for every $\lambda>0$ and $z\in\mathbb{G}$, $\delta_\lambda(E^c)=(\delta_\lambda(E))^c$ and $\tau_z(E^c)=(\tau_z(E))^c$.
(iv). By definition

$$\begin{aligned}[\chi_E]^p_{\frac{\alpha}{p},p;\mathbb{G}} &= \int_{\mathbb{G}}\int_{\mathbb{G}}\frac{|\chi_E(x)-\chi_E(y)|^p}{\|y^{-1}\cdot x\|_\alpha^{Q+\alpha}}dxdy\\ &= \int_E\int_{\mathbb{G}}\frac{|\chi_E(x)-1|^p}{\|y^{-1}\cdot x\|_\alpha^{Q+\alpha}}dxdy+\int_{E^c}\int_{\mathbb{G}}\frac{|\chi_E(x)|^p}{\|y^{-1}\cdot x\|_\alpha^{Q+\alpha}}dxdy\\ &= 2\int_E\int_{E^c}\frac{1}{\|y^{-1}\cdot x\|_\alpha^{Q+\alpha}}dxdy=2\mathrm{Per}_{\alpha,\mathbb{G}}(E).\end{aligned}$$

□

The function

$$Q_t^\alpha(\chi_E)=\int_{\mathbb{G}\times\mathbb{G}}h_\alpha(t,y)|\chi_E(y^{-1}\cdot x)-\chi_E(x)|dxdy$$

allows us to present a relationship between the fractional heat semigroup and the fractional perimeter.

The following Lemma is equivalent to saying that the embedding of Besov spaces $B^\alpha_{2,2}(\mathbb{G})\subset B^\alpha_{2,\infty}(\mathbb{G})$ is continuous, see [26, Th. 9 and 14]. We use it in the proof of Theorem 3.4 below.

Lemma 3.3. *Let $u\in W^{\alpha/2,2}(\mathbb{G})$; then there exists $c_\alpha>0$ such that for all $z\in\mathbb{G}$, denoting by $\tau_z u(x):=u(z^{-1}x)$, there holds*

$$\|\tau_z u-u\|^2_{L^2(\mathbb{G})}\leq c_\alpha\|z\|^\alpha\int_{\mathbb{G}\times\mathbb{G}}\frac{|u(x)-u(w^{-1}\cdot x)|^2}{\|w\|^{Q+\alpha}}dxdw.$$

The following results relates the fractional perimeter with the short-time behaviour of the fractional heat semigroup, as sketched in the Introduction.

Theorem 3.4. *There are constants $c_1(\alpha),c_2(\alpha)>0$ such that for every Borel set E there holds*

$$c_1(\alpha)\mathrm{Per}_{\alpha,\mathbb{G}}(E)\leq\liminf_{t\to0}\frac{Q_t^{\alpha/2}(\chi_E)}{t}\leq\limsup_{t\to0}\frac{Q_t^{\alpha/2}(\chi_E)}{t}\leq c_2(\alpha)\mathrm{Per}_{\alpha,\mathbb{G}}(E).\quad(3.8)$$

Proof. Let us start from the upper estimate. By (3.7) and Lemma 3.3 we immediately get

$$\begin{aligned}
Q_t^{\alpha/2}(\chi_E) =& \int_{\mathbb{G}\times\mathbb{G}} h_{\alpha/2}(t,z)|\chi_E(z^{-1}\cdot x)-\chi_E(x)|dxdz \\
&= \int_{B(t^{1/\alpha})} h_{\alpha/2}(t,z)\|\tau_z\chi_E-\chi_E\|^2_{L^2(\mathbb{G})}dz + \int_{B^c(t^{1/\alpha})} h_{\alpha/2}(t,z)\|\tau_z\chi_E-\chi_E\|^2_{L^2(\mathbb{G})}dz \\
&\le c_2 t^{-Q/\alpha}\int_{B(t^{1/\alpha})}\|z\|^\alpha[\chi_E]^2_{W^{\alpha/2,2}}dz + c_2 t\int_{B^c(t^{1/\alpha})\times\mathbb{G}}\frac{|\chi_E(z^{-1}\cdot x)-\chi_E(x)|}{\|z\|^{Q+\alpha}}dxdz \\
&\le 2tc_2|B(1)|\mathrm{Per}_{\alpha,\mathbb{G}}(E) + c_2 t\int_{B^c(t^{1/\alpha})\times\mathbb{G}}\frac{|\chi_E(z^{-1}\cdot x)-\chi_E(x)|}{\|z\|^{Q+\alpha}}dxdz \\
&\le c_2(\alpha)t\mathrm{Per}_{\alpha,\mathbb{G}}(E).
\end{aligned}$$

Let us come to the lower bound. By (3.7), since on the complement of the ball $B(t^{1/\alpha})$, we have the estimate

$$h_{\alpha/2}(t,y) \ge c_1\frac{t}{\|y\|^{Q+\alpha}},$$

we deduce that

$$\begin{aligned}
Q_t^{\alpha/2}(\chi_E) =& \int_{\mathbb{G}\times\mathbb{G}} h_{\alpha/2}(t,y)|\chi_E(y^{-1}\cdot x)-\chi_E(x)|dxdy \\
&\ge \int_{\mathbb{G}\setminus B(t^{1/\alpha})} h_\alpha(t,y)\int_{\mathbb{G}}|\chi_E(y^{-1}\cdot x)-\chi_E(x)|dxdy \\
&\ge ct\int_{\mathbb{G}\setminus B(t^{1/\alpha})}\int_{\mathbb{G}}\frac{|\chi(y^{-1}\cdot x)-\chi_E(x)|}{\|y\|^{Q+\alpha}}dxdy \\
&\ge c_1(\alpha)t\int_{\mathbb{G}\setminus B(t^{1/\alpha})}\int_{\mathbb{G}}\frac{|\chi(y^{-1}\cdot x)-\chi_E(x)|}{\|y\|_\alpha^{Q+\alpha}}dxdy.
\end{aligned}$$

It follows

$$c_1(\alpha)\int_{\mathbb{G}\times\mathbb{G}}\frac{|\chi(y^{-1}\cdot x)-\chi_E(x)|}{\|y\|_\alpha^{Q+\alpha}} = \lim_{t\to 0} c_1\int_{\mathbb{G}\setminus B(t^{1/\alpha})}\int_{\mathbb{G}}\frac{|\chi(y^{-1}\cdot x)-\chi_E(x)|}{\|y\|_\alpha^{Q+\alpha}} \le \liminf_{t\to 0}\frac{Q_t^{\alpha/2}(\chi_E)}{t}$$

and the proof is complete. □

The same kind of result can be extended for functions in $W^{\alpha,2}(\mathbb{G})$ considering the quantity

$$Q_t^\alpha(u) = \int_{\mathbb{G}\times\mathbb{G}} h_\alpha(t,y)|u(y^{-1}\cdot x)-u(x)|^2dxdy.$$

Notice that since $\chi_E(y^{-1}\cdot x)-\chi_E(x)\in\{-1,0,1\}$, then $|\chi_E(y^{-1}\cdot x)-\chi_E(x)| = |\chi_E(y^{-1}\cdot x)-\chi_E(x)|^2$ and then $Q_t^\alpha(u)$ coincides with $Q_t^\alpha(\chi_E)$

Theorem 3.5. *There are constants $c_1(\alpha)$ and $c_2(\alpha)$ such that for any $u\in L^2(\mathbb{G})$*

$$c_1(\alpha)[u]^2_{\alpha,2;\mathbb{G}} \le \liminf_{t\to 0}\frac{Q_t^\alpha(u)}{t} \le \limsup_{t\to 0}\frac{Q_t^\alpha(u)}{t} \le c_2(\alpha)[u]^2_{\alpha,2;\mathbb{G}};$$

then $u\in W^{\alpha,2}(\mathbb{G})$ if and only if

$$\limsup_{t\to 0}\frac{Q_t^\alpha(u)}{t} < +\infty.$$

Proof. The proof is exacly the same as in the proof of Theorem 3.4 by using estimates (3.7). □

4. **Open Problems.** As mentioned in the Introduction one of the main motivation of the present paper was to study the asymptotic behaviour of the heat semigroup in a non-Euclidean context. By using similar techniques, one could probably introduce and study the fractional perimeter on Riemannian manifolds. Another line of research are asympotics. Indeed, it is well known (see [6, 13, 25]) that for every bounded Borel set $E \subset \mathbb{R}^n$ of finite perimeter it holds:

$$\lim_{\alpha \to 1^-} (1-\alpha) P_\alpha(E) = C_n P(E) \tag{4.1}$$

where $P(E)$ is the perimeter of E and C_n is a positive constant depending only on n. Similar results have been proved also for different notions of fractional perimeters, see [22, 23, 24, 19, 20]. We conjecture that a similar property is satisfied also by $\mathrm{Per}_{\alpha,\mathbb{G}}(\cdot)$. A similar limit when $\alpha \to 0^+$ could be investigated. Another interesting question is to understand whether or not in a general Carnot group the function h_α satisfies a relation similar to (13). We plan to investigate this question in the Heisenberg group.

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E-mail address: fausto.ferrari@unibo.it
E-mail address: michele.miranda@unife.it
E-mail address: diego.pallara@unisalento.it
E-mail address: andrea.pinamonti@unitn.it
E-mail address: sire@math.jhu.edu