ON AUTOMORPHISMS OF MODULI SPACES OF PARABOLIC VECTOR BUNDLES

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ABSTRACT. Fix $n \geq 5$ general points $p_1, \ldots, p_n \in \mathbb{P}^1$, and a weight vector $\mathcal{A} = (a_1, \ldots, a_n)$ of real numbers $0 \leq a_i \leq 1$. Consider the moduli space $\mathcal{M}_{\mathcal{A}}$ parametrizing rank two parabolic vector bundles with trivial determinant on $(\mathbb{P}^1, p_1, \ldots, p_n)$ which are semistable with respect to $\mathcal{A}$. Under some conditions on the weights, we determine and give a modular interpretation for the automorphism group of the moduli space $\mathcal{M}_{\mathcal{A}}$. It is isomorphic to $\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^k$ for some $k \in \{0, \ldots, n-1\}$, and is generated by admissible elementary transformations of parabolic vector bundles. The largest of these automorphism groups, with $k = n-1$, occurs for the central weight $\mathcal{A}_F = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$. The corresponding moduli space $\mathcal{M}_{\mathcal{A}_F}$ is a Fano variety of dimension $n-3$, which is smooth if n is odd, and has isolated singularities if n is even.

CONTENTS

1. INTRODUCTION

Let C be a smooth projective curve and fix distinct points $p_1, \ldots, p_n \in C$, which we refer to as parabolic points. Let $S = p_1 + \cdots + p_n$ be the effective reduced divisor determined by these points. A *quasi parabolic vector bundle* on (C, S) is a vector bundle E on C with the additional data of a flag on the fiber over each parabolic point. If in addition we attach some weights to theses flags we call it a *parabolic vector bundle*. Parabolic vector bundles were introduced by Mehta and Seshadri ([Ses77], [MS80]) in order to generalize to curves with cusps the Narasimhan-Seshadri correspondence between stable vector bundles on smooth projective curves and unitary representations of their fundamental groups ([NS65]). As in the classical case, once one fixes a line bundle L and a notion of slope-stability, there is a moduli space of semistable parabolic vector bundles having determinant L. The notion of slope-stability depends on sets of weights assigned to the parabolic flags. Different choices of weights usually yield different moduli spaces, coming from variation of GIT.

There is one case in which the theory has been extensively investigated, and the different moduli spaces are well described. This is the case when $C \cong \mathbb{P}^1$ is the complex projective line, the vector bundles have rank 2 and the flags are given by parabolic directions $V_i \subset E_{p_i}$ over each parabolic point. In this case, we may assume that the vector bundles have trivial determinant, and the slope-stability condition depends on the choice of a weight vector $\mathcal{A} = (a_1, \ldots, a_n)$ of real numbers $0 \leq a_i \leq 1$ (see Section 2). We denote by $\mathcal{M}_{\mathcal{A}}$ the corresponding moduli space of semistable parabolic vector bundles.

The goal of this paper is to determine and give a modular interpretation of the automorphism groups of the moduli spaces $\mathcal{M}_{\mathcal{A}}$. Descriptions of automorphisms of moduli spaces in terms of the objects that they parametrize were obtained in many cases. See for instance [BM13, Mas14, MM14, MM17, FM17, Mas17, BM17, FM18, Lin11, Roy71], for moduli spaces of pointed curves and other configuration

spaces, [BGM13] for moduli spaces of vector bundles over curves, and [BM16] for generalized quot schemes.

In [Bau91], Bauer described the weight polytope $\Delta \subset [0,1]^n$ consisting of weight vectors $\mathcal{A}$ for which $\mathcal{M}_{\mathcal{A}} \neq \emptyset$. He also exhibited a wall-and-chamber decomposition on Δ corresponding to the variation of GIT for the moduli spaces $\mathcal{M}_{\mathcal{A}}$, and described the birational maps between models corresponding to different chambers. The weight polytope Δ is the polytope generated by the even vertices of the hypercube $[0,1]^n$, where the parity of a vertex is the parity of the set of its coordinates that equal 1. This polytope is called *demi-hypercube*. It is the weight polytope for the root system of D_n, and its symmetry group

$$\mathrm{Aut}(\Delta) \ \cong \ W(D_n) \ \cong \ \left(\mathbb{Z}/2\mathbb{Z}\right)^{n-1} \rtimes S_n$$

is generated by reflections along pairs of coordinate axes centered at the middle point $\left(\frac{1}{2}, \dots, \frac{1}{2}\right)$, and permutations of the n coordinate axes.

1.1. Elementary transformations. The normal subgroup $\left(\mathbb{Z}/2\mathbb{Z}\right)^{n-1} \lhd \mathrm{Aut}(\Delta)$ of reflections admits a modular realization as a group of *elementary transformations*, which we now describe. Let $(E, \mathbf{v})$ be a rank 2 quasi parabolic vector bundle on $(\mathbb{P}^1, S)$ of degree 0, and let $R \subset \{1, \dots, n\}$ be a subset of cardinality $r \geq 0$. Identifying vector bundles with their associated locally free sheaves, we consider the natural exact sequence of sheaves

$$0 \ \to \ E' \ \overset{\alpha}{\to} \ E \ \to \ \bigoplus_{i \in R}(E_{p_i}/V_i) \otimes \mathcal{O}_{p_i} \ \to \ 0 \ .$$

Note that we have the following equality

$$\det E' = \det E \otimes \mathcal{O}_{\mathbb{P}^1}\big(-\sum_{i \in R} p_i \big).$$

In particular, E' is a vector bundle of rank 2 and degree $-r$. We view E' as a quasi parabolic vector bundle on $(\mathbb{P}^1, S)$ as follows. If $i \notin R$, then $\alpha_{p_i} : E'_{p_i} \longrightarrow E_{p_i}$ is an isomorphism and

$$V'_i = (\alpha_{p_i})^{-1}(V_i) \subset E'_{p_i}$$

is the parabolic direction at p_i. If $i \in R$, then $V'_i = \ker(\alpha_{p_i})$ is the parabolic direction at p_i. This operation corresponds to the birational transformation of ruled surfaces $\mathbb{P}(E) \dashrightarrow \mathbb{P}(E')$ obtained by blowing-up the points $\mathbb{P}(V_i) \in \mathbb{P}(E_{p_i})$ and then blowing-down the strict transforms of the fibers $\mathbb{P}(E_{p_i})$ to the points $\mathbb{P}(V'_i) \in \mathbb{P}(E'_{p_i})$, $i \in R$. When r is even, we obtain a correspondence

$$el_R \ : \ (E, \mathbf{v}) \ \mapsto \ \Big(E' \otimes \mathcal{O}_{\mathbb{P}^1}(r/2), \{V'_i\}\Big)$$

between rank 2 quasi parabolic vector bundles on $(\mathbb{P}^1, S)$ of degree 0. We call it the *elementary transformation centered at the parabolic points* $\{p_i\}_{i \in R}$. Note that el_R is not the identity unless $R = \emptyset$. Elementary transformations are involutions and $el_R \circ el_S = el_{R \cup S \setminus R \cap S}$. So they form a group

$$\mathbf{El} \ = \ \big\{ el_R \mid R \subset \{1, \dots, n\} \text{ of even cardinality} \big\} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{n-1} .$$

When we perform an elementary transformation, the stability condition is preserved after an appropriate modification of weights. For a weight vector $\mathcal{A} = (a_1, \dots, a_n) \in \Delta$ and a subset $R \subset \{1, \dots, n\}$ of even cardinality, we set

$$\mathcal{A}^R := (a'_1, \dots, a'_n) \in \Delta$$

where $a'_i = a_i$ if $i \notin R$, and $a'_i = 1 - a_i$ if $i \in R$. If $(E, \mathbf{v})$ is semistable with respect to $\mathcal{A}$, then $el_R(E, \mathbf{v})$ is semistable with respect to $\mathcal{A}^R$. This follows from the following observation. If $L \subset E$ is a line bundle, then its image $L' \subset E'$ is

$$L' = L \otimes \mathcal{O}_C(-D)$$

where D is the reduced divisor supported on the points p_i such that $V_i \not\subseteq L$. We conclude that the correspondence el_R defines an isomorphism between moduli spaces

$$el_R : \mathcal{M}_{\mathcal{A}} \longrightarrow \mathcal{M}_{\mathcal{A}^R}.$$

The moduli space $\mathcal{M}_{\mathcal{A}_F}$ associated to the central weight $\mathcal{A}_F = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$ is specially interesting. It is a Fano variety of dimension $n - 3$ that is smooth if n is odd, and has isolated singularities if n is even. If follows from the above discussion that el_R induces an automorphism of $\mathcal{M}_{\mathcal{A}_F}$ for every subset $R \subset \{1, \ldots, n\}$ of even cardinality. In other words, we have:

$$\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{n-1} \cong \mathbf{El} \subset \mathrm{Aut}\left(\mathcal{M}_{\mathcal{A}_F}\right).$$

Our first result is the following.

Theorem 1.2. *Fix $n \geq 5$ general points $p_1, \ldots, p_n \in \mathbb{P}^1$ and let $\mathcal{M}_{\mathcal{A}_F}$ be the moduli space of rank two parabolic vector bundles with trivial determinant on $\left(\mathbb{P}^1, S\right)$ wich are semistable with respect to the weight vector $\mathcal{A}_F = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$. Then*

$$\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{n-1} \cong \mathbf{El} = \mathrm{Aut}\left(\mathcal{M}_{\mathcal{A}_F}\right).$$

We remark that for n odd, the isomorphism

$$\mathrm{Aut}\left(\mathcal{M}_{\mathcal{A}_F}\right) \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{n-1}$$

was proved in [AC17, Proposition 1.9], without the modular description as elementary transformations. For $n = 5$, the moduli space $\mathcal{M}_{\mathcal{A}_F}$ is isomorphic to a del Pezzo surface of degree four and its automorphism group is classically known ([Dol12, Section 8.6.4]).

For an arbitrary weight $\mathcal{A} \in \Delta$, let $\mathcal{C}_{\mathcal{A}}$ denote the subset of Δ consisting of weight vectors defining the same stability condition as $\mathcal{A}$. It can be explicitly read off from Bauer's wall-and-chamber decomposition on Δ. Consider the subgroup of $\mathcal{A}$-*admissible elementary transformations*:

$$\mathbf{El}_{\mathcal{A}} = \left\{ el_R \in \mathbf{El} \mid \mathcal{A}^R \in \mathcal{C}_{\mathcal{A}} \right\} \subset \mathbf{El}.$$

Then

(1.3) $$\mathbf{El}_{\mathcal{A}} \subset \mathrm{Aut}\left(\mathcal{M}_{\mathcal{A}}\right).$$

In general one does not have equality in (1.3). For instance, there are weight vectors $\mathcal{A} \in \Delta$ for which $\mathcal{M}_{\mathcal{A}} \cong \mathbb{P}^n$ (see [Bau91] and Section 4). However, there is an open sub-polytope of Δ for which equality in (1.3) holds. It can be described as follows. For every vertex v of Δ, let $H_v \subset \mathbb{R}^n$ be the hyperplane spanned by those vertices of Δ that are adjacent to v. Let Π be the sub-polytope obtained from Δ by chopping off each vertex v of Δ with the hyperplane H_v (see Section 4). It contains in its interior the subset $\mathcal{C}_{\mathcal{A}_F}$ consisting of weight vectors defining the same stability condition as $\mathcal{A}_F$.

Corollary 1.4. *Fix $n \geq 5$ general points $p_1, \ldots, p_n \in \mathbb{P}^1$ and let $\mathcal{A}$ be a weight vector in the interior of the polytope Π defined above. Let $\mathcal{M}_{\mathcal{A}}$ be the moduli space of rank two parabolic vector bundles with trivial determinant on $\left(\mathbb{P}^1, S\right)$ which are semistable with respect to the weight vector $\mathcal{A}$. Then*

$$\mathbf{El}_{\mathcal{A}} = \mathrm{Aut}\left(\mathcal{M}_{\mathcal{A}}\right).$$

The polytope Π has a natural description from the point of view of birational geometry. Namely, for weights $\mathcal{A}$ in the interior of the polytope Π, the moduli spaces $\mathcal{M}_{\mathcal{A}}$ are small modifications of the Fano variety $\mathcal{M}_{\mathcal{A}_F}$.

This paper is organized as follows. In Section 2, we revise basic properties of moduli spaces of parabolic vector bundles, Hitchin systems and spectral curves. This theory is used in Section 3 to prove Theorem 1.2. In Section 4, we use birational geometry and the small equivalence of models to prove Corollary 1.4. In Section 5, we describe the automorphism group of moduli spaces of involutional vector bundles on hyperelliptic curves by relating them to $\mathcal{M}_{\mathcal{A}}$.

2. MODULI SPACES OF PARABOLIC VECTOR BUNDLES ON $\mathbb{P}^1$

Fix $p_1, \ldots, p_n \in \mathbb{P}^1$ general points and denote by $S = p_1 + \cdots + p_n$ the effective reduced divisor determined by them.

2.1. Quasi parabolic vector bundles. A *quasi parabolic vector bundle* $(E, \mathbf{v})$, $\mathbf{v} = \{V_i\}$, of rank two on $(\mathbb{P}^1, S)$ consists of

- a vector bundle E of rank two on $\mathbb{P}^1$; and
- for each $i = 1, \ldots, n$, a 1-dimensional linear subspace $V_i \subset E_{p_i}$.

By abuse of notation we often write E for $(E, \mathbf{v})$. We refer to the points p_i's as parabolic points, and to the suspace $V_i \subset E_{p_i}$ as the parabolic direction of E at p_i.

Let $(E, \mathbf{v})$ and $(E', \mathbf{v}')$ be quasi parabolic vector bundles. A homomorphism of vector bundles $f : E \longrightarrow E'$ is called *parabolic* if $f(V_i) \subseteq V_i'$ for every $i = 1, \ldots, n$. It is called *strongly parabolic* if $f(E_{p_i}) \subseteq V_i'$ and $f(V_i) = 0$ for every $i = 1, \ldots, n$. We denote by $\mathcal{PHom}(E, E')$ and $\mathcal{SPHom}(E, E')$ the sheaves of parabolic and strongly parabolic homomorphisms, by $\mathcal{PEnd}(E) := \mathcal{PHom}(E, E)$ and $\mathcal{SPEnd}(E) := \mathcal{SPHom}(E, E)$ the sheaves of parabolic and strongly parabolic endomorphisms of $(E, \mathbf{v})$, and by $\mathcal{PEnd}_0(E)$ and $\mathcal{SPEnd}_0(E)$ their subsheaves of traceless endomorphisms.

By taking the trace of the product of two endomorphisms, one defines symmetric $\mathcal{O}_{\mathbb{P}^1}$-bilinear sheaf homomorphisms

$$\mathcal{E}nd(E) \times \mathcal{E}nd(E) \to \mathcal{O}_{\mathbb{P}^1} \quad \text{and} \quad \mathcal{E}nd_0(E) \times \mathcal{E}nd_0(E) \to \mathcal{O}_{\mathbb{P}^1}.$$

A simple linear algebra computation then yields the following parabolic dualities

$$(2.1) \qquad \mathcal{PEnd}(E)^\vee \cong \mathcal{SPEnd}(E) \otimes \mathcal{O}_{\mathbb{P}^1}(S) \quad \text{and} \quad \mathcal{PEnd}_0(E)^\vee \cong \mathcal{SPEnd}_0(E) \otimes \mathcal{O}_{\mathbb{P}^1}(S).$$

2.2. Weights and stability conditions. Fix a weight vector $\mathcal{A} = (a_1, \ldots, a_n)$ of real numbers $0 \leq a_i \leq 1$. The *parabolic slope* of $(E, \mathbf{v})$ with respect to $\mathcal{A}$ is

$$\mu_{\mathcal{A}}(E) = \frac{\deg E + \sum_{i=1}^n a_i}{2}.$$

Let $L \subset E$ be a line subbundle. For each $i = 1, \ldots, n$, set

$$a_i(L, E) \quad = \quad \begin{cases} a_i & \text{if } L_{p_i} = V_i, \\ 0 & \text{if } L_{p_i} \neq V_i. \end{cases}$$

The *parabolic slope* of $L \subset E$ with respect to $\mathcal{A}$ is

$$\mu_{\mathcal{A}}(L, E) = \deg(L) + \sum_{i=1}^n a_i(L, E).$$

A quasi parabolic vector bundle $(E, \mathbf{v})$ is $\mu_{\mathcal{A}}$-*semistable* (respectively $\mu_{\mathcal{A}}$-*stable*) if for every line subbundle $L \subset E$ we have $\mu_{\mathcal{A}}(L, E) \leq \mu_{\mathcal{A}}(E)$ (respectively $\mu_{\mathcal{A}}(L, E) < \mu_{\mathcal{A}}(E)$). A *parabolic vector bundle* is a quasi parabolic vector bundle together with a weight vector $\mathcal{A}$. We say that a parabolic vector bundle is semistable if the corresponding quasi parabolic vector bundle is $\mu_{\mathcal{A}}$-semistable.

By [MS80], for each fixed degree $d \in \mathbb{Z}$, there is a moduli space $\mathcal{M}_{\mathcal{A}}(d)$ parametrizing rank two degree d quasi parabolic vector bundles on $(\mathbb{P}^1, S)$ which are $\mu_{\mathcal{A}}$-semistable. It is a normal projective variety. By twisting vector bundles with a fixed line bundle, we see that $\mathcal{M}_{\mathcal{A}}(d) \cong \mathcal{M}_{\mathcal{A}}(d')$ whenever d and d' have the same parity. By performing an elementary transformation centered at one parabolic point p_i, as described in the introduction, we see that $\mathcal{M}_{\mathcal{A}}(d) \cong \mathcal{M}_{\mathcal{A}^i}(d-1)$, where

$$\mathcal{A}^i = (a_1, \ldots, 1 - a_i, \ldots, a_n).$$

So from now on we assume that $d = 0$ and write simply $\mathcal{M}_{\mathcal{A}}$ for the corresponding moduli space.

Let $\mathcal{M}_{\mathcal{A}}^s \subset \mathcal{M}_{\mathcal{A}}$ be the Zariski open subset parametrizing stable parabolic vector bundles. If it is not empty, then it is an irreducible smooth quasi-projective variety of dimension $n - 3$. We describe the tangent space of $\mathcal{M}_{\mathcal{A}}^s$ at a point $(E, \mathbf{v})$). We denote by $\omega_{\mathbb{P}^1}$ the canonical sheaf of $\mathbb{P}^1$. For any

invertible sheaf $\mathcal{L}$ and $S \in \text{Div}(\mathbb{P}^1)$ we write $\mathcal{L}(S)$ instead of $\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}^1}(S)$. We also write $T_E \mathcal{M}_{\mathcal{A}}^s$ for the tangent space of $\mathcal{M}_{\mathcal{A}}^s$ at $(E, \mathbf{v})$. By [Yok95, Theorem 2.4],

$$(2.2) \qquad T_E \mathcal{M}_{\mathcal{A}}^s \cong \text{H}^1(\mathbb{P}^1, \mathcal{PE}nd(E)) \cong \text{H}^0(\mathbb{P}^1, \mathcal{SPE}nd(E) \otimes \omega_{\mathbb{P}^1}(S))^{\vee},$$

where the second isomorphism holds by (2.1) and Serre duality. We refer to the combination of these two dualities as *parabolic Serre duality*.

Let $\theta \in \text{H}^0(\mathbb{P}^1, \mathcal{SPE}nd(E) \otimes \omega_{\mathbb{P}^1}(S))$ be a global section. For each parabolic point p_i, the residual endomorphism $\text{Res}(\theta, p_i) \in \text{End}(E_{p_i})$ is well defined. The strongly parabolic condition implies that these endomorphisms are nilpotent for each parabolic point p_i. In particular the trace $\text{tr}(\theta) \in \text{H}^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}(S))$ of θ vanishes at $p_1, \ldots, p_n$, and thus $\text{tr}(\theta) = 0$. So we have an isomorphism

$$T_E \mathcal{M}_{\mathcal{A}}^s \cong \text{H}^0(\mathbb{P}^1, \mathcal{SPE}nd_0(E) \otimes \omega_{\mathbb{P}^1}(S))^{\vee}.$$

2.3. Parabolic Higgs bundles. Given a parabolic vector bundle $(E, \mathbf{v})$ on $(\mathbb{P}^1, S)$, a *Higgs field* on $(E, \mathbf{v})$ is a section

$$\theta \in \text{H}^0(\mathbb{P}^1, \mathcal{SPE}nd(E) \otimes \omega_{\mathbb{P}^1}(S)).$$

In order to simplify notation we shall denote the vector space above by

$$Higgs(E, \mathbf{v}) := \text{H}^0(\mathbb{P}^1, \mathcal{SPE}nd(E) \otimes \omega_{\mathbb{P}^1}(S)).$$

In view of (2.2), there is an isomorphism

$$Higgs(E, \mathbf{v}) \cong T_E^* \mathcal{M}_{\mathcal{A}}^s$$

for each $(E, \mathbf{v}) \in \mathcal{M}_{\mathcal{A}}^s$. As we noted above, the trace of a Higgs field vanishes. This implies that the minimal polynomial of θ is $t^2 + \det(\theta)$.

A *parabolic Higgs bundle* (E, θ) on $(\mathbb{P}^1, S)$ consists of a parabolic vector bundle $(E, \mathbf{v})$ together with a Higgs field θ on E. It is $\mu_{\mathcal{A}}$-*semistable* (respectively $\mu_{\mathcal{A}}$-*stable*) if for every line subbundle $L \subset E$ invariant under θ, we have $\mu_{\mathcal{A}}(L, E) \leq \mu_{\mathcal{A}}(E)$ (respectively $\mu_{\mathcal{A}}(L, E) < \mu_{\mathcal{A}}(E)$).

We denote by $\mathcal{M}_{\mathcal{A}}^{Higgs}$ the moduli space of $\mu_{\mathcal{A}}$-semistable parabolic Higgs bundles of rank two and trivial determinant. It is a normal, quasiprojective variety of dimension $2n - 6$. By (2.2), $\mathcal{M}_{\mathcal{A}}^{Higgs}$ contains as an open subset the total space of the cotangent bundle of $\mathcal{M}_{\mathcal{A}}^s$.

2.4. The Hitchin map. Let (E, θ) be a parabolic Higgs bundle on $(\mathbb{P}^1, S)$, and consider $\det(\theta) \in \text{H}^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes 2}(2S))$. Since $\text{Res}(\theta, p_i)$ is nilpotent for every parabolic point $p_i \in \mathbb{P}^1$, $\det(\theta)$ lies in the linear subspace $V \subset \text{H}^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes 2}(2S))$ consisting of sections vanishing at $p_1, \ldots, p_n$. Identifying V with $\text{H}^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes 2}(S))$, the *Hitchin map* is defined as

$$
\begin{array}{rccc}
H: & \mathcal{M}_{\mathcal{A}}^{Higgs} & \longrightarrow & \text{H}^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes 2}(S)) \\
& (E, \mathbf{v}, \theta) & \longmapsto & \det(\theta).
\end{array}
$$

Our next goal is to describe the fibers of the Hitchin map, and of its restriction to the total space of the cotangent bundle of $\mathcal{M}_{\mathcal{A}}^s$, which we denote by

$$h: T^* \mathcal{M}_{\mathcal{A}}^s \longrightarrow \text{H}^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes 2}(S)).$$

For this purpose we recall the properties of spectral curves associated to the Hitchin map.

2.5. Spectral curves. Denote by $\mathbb{V}$ the total space of the sheaf $\omega_{\mathbb{P}^1}(S)$, with natural map $\pi : \mathbb{V} \longrightarrow \mathbb{P}^1$. There is a tautological section $s \in \text{H}^0(\mathbb{V}, \pi^*(\omega_{\mathbb{P}^1}(S)))$. Given $a \in V \subset \text{H}^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes 2}(2S))$, we define the spectral curve C_a associated to a as the zero locus of the section

$$s^2 + \pi^* a \in \text{H}^0(\mathbb{V}, \pi^*(\omega_{\mathbb{P}^1}^{\otimes 2}(2S))).$$

We denote by $\pi_a : C_a \longrightarrow \mathbb{P}^1$ the restriction of π to C_a. It is a $2 : 1$ map branched over the zero locus of the global section a.

2.6. The Fano model. The central weight vector $\mathcal{A}_F = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$ yields a distinguished moduli space $\mathcal{M}_{\mathcal{A}_F}$. For $n = 4$, we have $\mathcal{M}_{\mathcal{A}_F} \cong \mathbb{P}^1$. So from now on we assume that $n \geq 5$.

The moduli space $\mathcal{M}_{\mathcal{A}_F}$ is a Fano variety of dimension $n - 3$ (see [Muk05, Cas15, AM16]). If n is odd, then there are no strictly $\mu_{\mathcal{A}}$-semistable bundles and so $\mathcal{M}_{\mathcal{A}_F}$ is smooth. If n is even, then

$$\text{Sing}(\mathcal{M}_{\mathcal{A}_F}) = \mathcal{M}_{\mathcal{A}_F} \setminus \mathcal{M}_{\mathcal{A}_F}^s$$

consists of a finite set of points (see [BHK10, Section 2]).

We summarize in the following proposition the description of the fibers of the Hitchin map

$$H : \mathcal{M}_{\mathcal{A}_F}^{Higgs} \longrightarrow \mathrm{H}^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes 2}(S))$$

in terms of spectral curves.

Proposition 2.7 ([BHK10, Section 2, Proposition 2.2, Lemma 3.1]). *Let the notation be as above and fix a general section $a \in \mathrm{H}^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes 2}(S))$. Then*

  (i) *The spectral curve C_a is a smooth and connected curve of genus $n - 3$.*
  (ii) *The fiber $H^{-1}(a)$ is an abelian variety isomorphic to $\mathrm{Pic}^{n-2}(C_a)$.*
  (iii) *The codimension of $H^{-1}(a) \setminus h^{-1}(a)$ in $H^{-1}(a)$ is at least two.*
  (iv) *Denote by $p : h^{-1}(a) \longrightarrow \mathcal{M}_{\mathcal{A}_F}^s$ the restriction of the natural projection $T^* \mathcal{M}_{\mathcal{A}_F}^s \longrightarrow \mathcal{M}_{\mathcal{A}_F}^s$, and by Θ the theta divisor on $\mathrm{Pic}^{n-2}(C_a) \supset h^{-1}(a)$. Then*

$$p^*\big(- K_{\mathcal{M}_{\mathcal{A}_F}^s} \big) \ = \ 4^{n-3} \Theta_{|h^{-1}(a)} \ .$$

Remark 2.8. It follows from Proposition 2.7 that the Hitchin map $h : T^* \mathcal{M}_{\mathcal{A}_F}^s \longrightarrow \mathrm{H}^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes 2}(S))$ is the affinization of $T^* \mathcal{M}_{\mathcal{A}_F}^s$. In other words, viewed as an affine variety, $\mathrm{H}^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes 2}(S))$ is the spectrum of the ring of regular functions on $T^* \mathcal{M}_{\mathcal{A}_F}^s$.

2.9. The natural involution on the fibers of the Hitchin map. The natural involution $i_a : C_a \longrightarrow C_a$ switching the sheets of the $2 : 1$ covering $\pi_a : C_a \longrightarrow \mathbb{P}^1$ induces the involution on $\mathrm{Pic}^{n-2}(C_a)$ mapping $\mathcal{L} \in \mathrm{Pic}^{n-2}(C_a)$ to $i_a^* \mathcal{L}$. We want to describe the corresponding involution on the fiber $H^{-1}(a) \subset \mathcal{M}_{\mathcal{A}_F}^{Higgs}$.

For this purpose, let us review the correspondence in Proposition 2.7(ii). Given a line bundle $\mathcal{L} \in \mathrm{Pic}^{n-2}(C_s)$, we consider the rank 2 vector bundle $E = (\pi_a)_* \mathcal{L}$ on $\mathbb{P}^1$. The parabolic points $p_1, \ldots, p_n \in \mathbb{P}^1$ are contained in the ramification locus of π_a. Therefore there is a distinguished 1-dimensional linear subspace V_i in the fiber E_{p_i}. The tautological section

$$s_a = s_{|C_a} \in \mathrm{H}^0\big(C_a, \pi_a^*(\omega_{\mathbb{P}^1}(S))\big)$$

induces a homomorphism $\theta = (s_a)_* : E \longrightarrow E \otimes \omega_{\mathbb{P}^1}(S)$. The parabolic Higgs bundle on $\big(\mathbb{P}^1, S\big)$ associated to the line bundle $\mathcal{L}$ is $(E, \mathbf{v}, \theta)$. The equation $E = (\pi_a)_* \mathcal{L}$ can be viewed as the eigenspace decomposition of θ on E.

Now notice that the parabolic Higgs bundle on $\big(\mathbb{P}^1, S\big)$ associated to the line bundle $i_a^* \mathcal{L}$ is $(E, \mathbf{v}, \theta')$, where θ' is obtained from θ by swapping the eigenspaces. Since θ is traceless, its eigenvalues λ_1, λ_2 satisfy $\lambda_1 = -\lambda_2$. Hence $\theta' = -\theta$.

We conclude that the involution on the fiber $H^{-1}(a) \subset \mathcal{M}_{\mathcal{A}_F}^{Higgs}$ induced by the natural involution $i_a : C_a \longrightarrow C_a$ maps $(E, \mathbf{v}, \theta)$ to $(E, \mathbf{v}, -\theta)$.

3. The automorphism group of the Fano model $\mathcal{M}_{\mathcal{A}_F}$

In this section we show that the automorphism group of the Fano model $\mathcal{M}_{\mathcal{A}_F}$ is the group of elementary transformations $\mathbf{El} \cong \big(\frac{\mathbb{Z}}{2\mathbb{Z}}\big)^{n-1}$ (Theorem 1.2).

Let $\varphi \in \mathrm{Aut}(\mathcal{M}_{\mathcal{A}_F})$ be an automorphism sending a general rank two parabolic vector bundle $(E, \mathbf{v})$ to $(E', \mathbf{v}')$. Since $\mathbf{El} \subset \mathrm{Aut}(\mathcal{M}_{\mathcal{A}_F})$ is finite, in order to prove that the groups coincide, it is enough to show that there is an elementary transformation $el_R \in \mathbf{El}$ as defined in Paragraph 1.1 sending $(E, \mathbf{v})$ to $(E', \mathbf{v}')$. This is equivalent to showing that the blowup of $\mathbb{P}(E)$ at the finite set of points $\{\mathbb{P}(V_i)\}_{i=1,\ldots,n}$ is isomorphic over $\mathbb{P}^1$ to the blowup of $\mathbb{P}(E')$ at $\{\mathbb{P}(V_i')\}_{i=1,\ldots,n}$. In order to prove this isomorphism, we first show how to recover the blowup of $\mathbb{P}(E)$ at $\{\mathbb{P}(V_i)\}_{i=1,\ldots,n}$ as the projectivization of the *nilpotent cone* associated to E. This construction works for any smooth projective curve C.

3.1. The nilpotent cone. Let C be a smooth projective curve and fix parabolic points $p_1, \ldots, p_n \in C$. Let $\big(E, \mathbf{v}\big)$ be a rank 2 quasi parabolic vector bundle on $\big(C, S\big)$. For any invertible sheaf $\mathcal{L}$ on C, we consider the locally free subsheaf of $\mathcal{SPE}nd(E) \otimes \mathcal{L}$ consisting of traceless endomorphisms. We denote this sheaf by $\mathcal{SPE}nd_0(E) \otimes \mathcal{L}$, and the corresponding vector bundle on C by $\mathrm{SPEnd}_0(E, \mathcal{L})$. Notice

that their rank is 3. We will define a codimension one quadratic cone bundle $N_E \subset \mathrm{SPEnd}_0(E, \mathcal{L})$, the *nilpotent cone of E*.

For any $p \in C \setminus \{p_1, \ldots, p_n\}$ consider the cone of $\mathrm{SPEnd}_0(E, \mathcal{L})_p \cong \mathbb{C}^3$ consiting of nilpotent elements:

$$N_{E,p} \;=\; \{\Phi \in \mathrm{SPEnd}_0(E, \mathcal{L})_p \mid \Phi^2 = 0\} \;\subseteq\; \mathrm{SPEnd}_0(E, \mathcal{L})_p.$$

Note that since the endomorphisms $\Phi \in \mathrm{SPEnd}_0(E, \mathcal{L})_p$ are traceless, the condition $\Phi^2 = 0$ is equivalent to $\det(\Phi) = 0$. By letting p vary in $C \setminus \{p_1, \ldots, p_n\}$, we get a cone bundle in $\mathrm{SPEnd}_0(E, \mathcal{L})$ over $C \setminus \{p_1, \ldots, p_n\}$. We define the nilpotent cone N_E as the closure of this cone bundle in $\mathrm{SPEnd}_0(E, \mathcal{L})$.

Proposition 3.2. *Let the notation be as above. Then the projectivized nilpotent cone $\mathbb{P}(N_E)$ is isomorphic over C to the blow-up of the ruled surface $\mathbb{P}(E)$ at the set of points $\{\mathbb{P}(V_i)\}_{i=1,\ldots,n}$.*

Proof. Let $U \subseteq C$ be a trivializing open subset for both E and $\mathrm{SPEnd}_0(E, \mathcal{L})$ containing only one of the parabolic points, $p_i \in U$. We fix an identification $E_{|U} \cong U \times \mathbb{C}^2$ and basis for $\mathbb{C}^2$ with respect to which the parabolic direction at p_i is $V_i = \langle(1, 0)\rangle$.

Write t for a local parameter for U at p_i. After shrinking U if necessary, we may assume that t is a regular function on U. Sections of $\mathrm{SPEnd}_0(E, \mathcal{L})$ over U are families of endomorphisms given by matrices of the form

$$M_t = \begin{pmatrix} t\alpha & \beta \\ t\gamma & -t\alpha \end{pmatrix},$$

with $\alpha, \beta, \gamma \in \Gamma(U, \mathcal{L})$. So we can fix an identification $\mathrm{SPEnd}_0(E, \mathcal{L})_{|U} \cong U \times \mathbb{C}^3$ and basis for $\mathbb{C}^3$ with respect to which the endomorphism of E_p corresponding to a point

$$\big(p, (a, b, c)\big) \in (U \setminus \{p_i\}) \times \mathbb{C}^3$$

is given by the matrix $\begin{pmatrix} t(p)a & b \\ t(p)c & -t(p)a \end{pmatrix}$. We have

$$\det(M_t) = -t(t\alpha^2 + \beta\gamma).$$

So we see that in $\mathrm{SPEnd}_0(E, \mathcal{L})_{|U} \cong U \times \mathbb{C}^3$, the nilpotent cone N_E is cut out by the equation

$$ta^2 + bc = 0.$$

This shows that $\mathbb{P}(N_E)_{|U} \subset U \times \mathbb{P}^2$ is a smooth surface, the fibers of $\mathbb{P}(N_E)_{|U} \longrightarrow U$ over $U \setminus \{p_i\}$ are smooth conics, and the fiber over p_i is the union of two intersecting lines

$$F_1 = \{b = 0\} \quad \text{and} \quad F_2 = \{c = 0\}.$$

From the defining equation of $\mathbb{P}(N_E)_{|U}$, we see that $(a : c) = (b : -ta)$, and so we have a morphism

$$f_U : \quad \begin{array}{ccc} \mathbb{P}(N_E)_{|U} & \longrightarrow & \mathbb{P}(E)_{|U} \\ (p, (a, b, c)) & \longmapsto & (p, (a : c)) \end{array}$$

mapping F_1 isomorphically onto the fiber of $\mathbb{P}(E)_{|U} \longrightarrow U$ over p_i, and contracting F_2 to the point $\mathbb{P}(V_i) \in \mathbb{P}(E)_{|U}$.

On $U \setminus \{p_i\}$, the vector (a, c) is precisely the eigenvector of the nilpotent matrix

$$\begin{pmatrix} ta & b \\ tc & -ta \end{pmatrix}.$$

Therefore the local morphisms f_U glue together to define a global birational morphism over C

$$f : \mathbb{P}(N_E) \longrightarrow \mathbb{P}(E).$$

It is an isomorphism away from n smooth rational curves, which get contracted to the points $\mathbb{P}(V_i) \in \mathbb{P}(E)$, $i = 1, \ldots, n$, and the result follows. $\square$

Now we go back to our original setting, with $C \cong \mathbb{P}^1$. In the proof of Theorem 1.2 we will apply Proposition 3.2 with $\mathcal{L} = \omega_{\mathbb{P}^1}(S)$. We will need the following result.

Lemma 3.3. *Suppose that $n \geq 6$, and let $(E, \mathbf{v}) \in \mathcal{M}_{\mathcal{A}_F}$ be a general parabolic vector bundle. Then $\mathcal{SPE}nd_0(E) \otimes \omega_{\mathbb{P}^1}(S)$ is globally generated.*

Proof. For any point $p \in \mathbb{P}^1$, evaluation at p yields an exact sequence:

$$0 \to \mathcal{SPEnd}_0(E) \otimes \omega_{\mathbb{P}^1}(S - p) \to \mathcal{SPEnd}_0(E) \otimes \omega_{\mathbb{P}^1}(S) \to \mathcal{SPEnd}_0(E) \otimes \omega_{\mathbb{P}^1}(S)_p \to 0.$$

By parabolic Serre duality,

$$\mathrm{H}^1(\mathbb{P}^1, \mathcal{SPEnd}_0(E) \otimes \omega_{\mathbb{P}^1}(S - p)) \cong \mathrm{H}^0(\mathbb{P}^1, \mathcal{PEnd}_0(E) \otimes \mathcal{O}_{\mathbb{P}^1}(p))^\vee.$$

So, in order to show that $\mathcal{SPEnd}_0(E) \otimes \omega_{\mathbb{P}^1}(S)$ is globally generated, it is enough to show that

$$(3.4) \qquad\qquad \mathrm{H}^0(\mathbb{P}^1, \mathcal{PEnd}_0(E) \otimes \mathcal{O}_{\mathbb{P}^1}(1)) = \{0\}.$$

Since $(E, \mathbf{v}) \in \mathcal{M}_{\mathcal{A}_F}$ is general, the underlying vector bundle E is free, and a global section in $\mathrm{H}^0(\mathbb{P}^1, \mathcal{End}_0(E) \otimes \mathcal{O}_{\mathbb{P}^1}(1))$ can be represented by a traceless 2×2 matrix of linear forms on $\mathbb{P}^1$. The vector space of such matrices has dimension 6. Each parabolic condition $\phi(V_i) \subseteq V_i$ imposes one linear condition. A straightforward computation shows that, since the parabolic directions are general, we get n linearly independent conditions. Therefore, (3.4) holds for $n \geq 6$. $\qquad\square$

Remark 3.5. It follows from Lemma 3.3 that there is a surjective map of vector bundles on $\mathbb{P}^1$

$$Higgs(E, \mathbf{v}) \times \mathbb{P}^1 \xrightarrow{\alpha} \mathrm{SPEnd}_0\big(E, \omega_{\mathbb{P}^1}(S)\big).$$

By identifying $Higgs(E, \mathbf{v})$ with the cotangent space $T^*_E \mathcal{M}^s_{\mathcal{A}_F}$, we describe the quadratic cone $\alpha^{-1}(N_E)$ in terms of the restriction of the Hitchin map

$$h_E = h_{|T^*_E \mathcal{M}^s_{\mathcal{A}}} : \ T^*_E \mathcal{M}^s_{\mathcal{A}} \ \longrightarrow \ \mathrm{H}^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes 2}(S)).$$

Given a point $p \in \mathbb{P}^1 \setminus \{p_1, \ldots, p_n\}$, let $V_p \subset \mathrm{H}^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes 2}(S))$ be the linear space consisting of sections vanishing at p. Then

$$\alpha^{-1}(N_E)_p \ = \ h_E^{-1}(V_p).$$

Working with a trivialization of $\mathrm{SPEnd}_0\big(E, \omega_{\mathbb{P}^1}(S)\big)$ in a neighborhood of p, as in the proof of Proposition 3.2 above, we see that the vertex of the cone $\alpha^{-1}(N_E)_p$ is a codimension 3 linear subspace of $Higgs(E, \mathbf{v})$ that coincides with the kernel of α_p.

Proof of Theorem 1.2. Let $\varphi \in \mathrm{Aut}(\mathcal{M}_{\mathcal{A}_F})$ be an automorphism, and consider the induced homomorphism on the cotangent bundle

$$d\varphi : \ T^* \mathcal{M}^s_{\mathcal{A}_F} \ \longrightarrow \ T^* \mathcal{M}^s_{\mathcal{A}_F}.$$

Recall from Remark 2.8 that the Hitchin map $h : T^* \mathcal{M}^s_{\mathcal{A}_F} \longrightarrow \mathrm{H}^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes 2}(S))$ is the affinization of $T^* \mathcal{M}^s_{\mathcal{A}_F}$. Therefore there is morphism of affine varieties

$$f : \ \mathrm{H}^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes 2}(S)) \ \longrightarrow \ \mathrm{H}^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes 2}(S))$$

making the following diagram commute:

$$
\begin{array}{ccc}
T^* \mathcal{M}^s_{\mathcal{A}_F} & \xrightarrow{\ \ d\varphi \ \ } & T^* \mathcal{M}^s_{\mathcal{A}_F} \\
{\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle h} \\
\mathrm{H}^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes 2}(S)) & \xrightarrow{\ \ f \ \ } & \mathrm{H}^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes 2}(S)).
\end{array}
$$

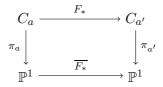We will show that the map f is multiplication by a nonzero constant.

The $\mathbb{C}^*$-action by dilations on the fibers of the map $T^* \mathcal{M}^s_{\mathcal{A}_F} \longrightarrow \mathcal{M}^s_{\mathcal{A}_F}$ induces the $\mathbb{C}^*$-action on $\mathrm{H}^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes 2}(S))$ given by $t \cdot a = t^2 a$. Since $d\varphi$ is $\mathbb{C}^*$-equivariant, so is f. This implies that f sends lines through the origin to lines through the origin, and hence f is linear.

Consider a general section $a \in \mathrm{H}^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes 2}(S))$, and set $a' = f(a)$. By Proposition 2.7, the spectral curves C_a and $C_{a'}$ are smooth, and the isomorphism $d\varphi_{|h^{-1}(a)} : h^{-1}(a) \longrightarrow h^{-1}(a')$ extends to an isomorphism of polarized abelian varieties

$$F : \ \mathrm{Pic}^{n-2}(C_a) \cong H^{-1}(a) \longrightarrow H^{-1}(a') \ \cong \ \mathrm{Pic}^{n-2}(C_{a'}).$$

By Paragraph 2.9, the isomorphism F commutes with the involutions on $\mathrm{Pic}^{n-2}(C_a)$ and $\mathrm{Pic}^{n-2}(C_{a'})$ induced by the natural involutions on C_a and $C_{a'}$.

Torelli theorem implies that F comes from an isomorphism $F_* : C_a \longrightarrow C_{a'}$ between the spectral curves. Moreover, since F commutes with the involutions on $\mathrm{Pic}^{n-2}(C_a)$ and $\mathrm{Pic}^{n-2}(C_{a'})$ induced by the natural involutions on C_a and $C_{a'}$, we have a commutative diagram

$$\begin{array}{ccc} C_a & \xrightarrow{\;\;F_*\;\;} & C_{a'} \\ \pi_a \downarrow & & \downarrow \pi_{a'} \\ \mathbb{P}^1 & \xrightarrow{\;\;\overline{F_*}\;\;} & \mathbb{P}^1 \end{array}$$
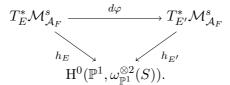
where $\overline{F_*}$ is an automorphism of $\mathbb{P}^1$ that sends the branch locus of π_a to the branch locus of $\pi_{a'}$. These branch loci are precisely the zero loci of a and a', and include the general points $p_1, \ldots, p_n$. We conclude that $\overline{F_*}$ is the identity, and a' is a nonzero multiple of a. Therefore the linear map

$$f : \mathrm{H}^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes 2}(S)) \longrightarrow \mathrm{H}^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes 2}(S))$$

is multiplication by a nonzero constant. After rescaling if necessary, we may assume that f is the identity.

Let $(E, \mathbf{v}) \in \mathcal{M}_{\mathcal{A}_F}^s$ be a general rank two parabolic vector bundle, and write $\varphi(E, \mathbf{v}) = (E', \mathbf{v}')$. As explained in the beginning of the section, and in view of Proposition 3.2, in order to prove the theorem, it suffices to show that the projectivized nilpotent cones $\mathbb{P}(N_E)$ and $\mathbb{P}(N_{E'})$ are isomorphic over $\mathbb{P}^1$.

From the above discussion, we have the following commutative diagram:

$$\begin{array}{ccc} T_E^* \mathcal{M}_{\mathcal{A}_F}^s & \xrightarrow{\;\;d\varphi\;\;} & T_{E'}^* \mathcal{M}_{\mathcal{A}_F}^s \\ & h_E \searrow \quad \swarrow h_{E'} & \\ & \mathrm{H}^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes 2}(S)). & \end{array}$$

If $n \geq 6$ then, by Lemma 3.3, there are surjective maps of vector bundles on $\mathbb{P}^1$

$$T_E^* \mathcal{M}_{\mathcal{A}_F}^s \times \mathbb{P}^1 \xrightarrow{\alpha} \mathrm{SPEnd}_0 \left(E, \omega_{\mathbb{P}^1}(S) \right), \text{ and}$$

$$T_{E'}^* \mathcal{M}_{\mathcal{A}_F}^s \times \mathbb{P}^1 \xrightarrow{\alpha'} \mathrm{SPEnd}_0 \left(E', \omega_{\mathbb{P}^1}(S) \right).$$

By Remark 3.5, the induced isomorphism

$$d\varphi : T_E^* \mathcal{M}_{\mathcal{A}_F}^s \times \mathbb{P}^1 \longrightarrow T_{E'}^* \mathcal{M}_{\mathcal{A}_F}^s \times \mathbb{P}^1$$

maps $\alpha^{-1}(N_E)$ to $(\alpha')^{-1}(N_{E'})$, and the kernel of α to the kernel of α'. Therefore it yields an isomorphism

$$\mathrm{SPEnd}_0 \left(E, \omega_{\mathbb{P}^1}(S) \right) \cong \mathrm{SPEnd}_0 \left(E', \omega_{\mathbb{P}^1}(S) \right)$$

over $\mathbb{P}^1$ mapping N_E to $N_{E'}$. We conclude that $\mathbb{P}(N_E)$ and $\mathbb{P}(N_{E'})$ are isomorphic over $\mathbb{P}^1$, as desired.

For $n = 5$ the result follows from [AC17, Proposition 1.9]. $\qquad\qquad\square$

4. Models $\mathcal{M}_{\mathcal{A}}$ that are small modifications of $\mathcal{M}_{\mathcal{A}_F}$

In this section we determine the automorphism group of moduli spaces $\mathcal{M}_{\mathcal{A}}$ that are small modifications of $\mathcal{M}_{\mathcal{A}_F}$. The weight polytope $\Pi \subset \Delta$ consisting of weights $\mathcal{A}$ for which this happens can be described after [Bau91] and [Muk05]. We note that [Bau91] and [Muk05] consider moduli spaces of rank 2 parabolic vector bundles on $\mathbb{P}^1$ of degree 1, while here we work with degree 0. So, in order to describe the weight polytopes that are relevant to our setting, we perform a reflection on the corresponding polytopes described in [Bau91]. This reflection corresponds to an elementary transformation centered at one parabolic point, as explained in Paragraph 1.1.

4.1. The polytopes Δ and Π. The vertices of the hypercube $[0,1]^n \subset \mathbb{R}^n$ are the points of the form $\xi_I = ((\xi_I)_1, \ldots, (\xi_I)_n)$, where $I \subset \{1, \ldots, n\}$, $(\xi_I)_i = 1$ if $i \in I$, and $(\xi_I)_i = 0$ otherwise. The parity of the subset I and the vertex ξ_I is the parity of $|I|$. For each subset $I \subset \{1, \ldots, n\}$, consider the degree one polynomial in the α_i's:

$$H_I := \sum_{j \notin I} \alpha_j + \sum_{i \in I}(1 - \alpha_i).$$

For any subset $J \subset \{1, \ldots, n\}$, we have:

$$(4.2) \qquad\qquad H_I(\xi_J) = \#(I^c \cap J) + \#(J^c \cap I).$$

Let Δ be the polytope generated by the even vertices of the hypercube. From (4.2) we see that Δ is defined by the following set of inequalities:

$$\Delta = \begin{cases} 0 \leq \alpha_i \leq 1, & i \in \{1, \ldots, n\} \\ H_I \geq 1, & I \subset \{1, \ldots, n\} \text{ odd.} \end{cases}$$

From (4.2) we also see that, for any vertex $\xi_I \in \Delta$, the hyperplane spanned by those vertices of Δ that are adjacent to ξ_I is $(H_I = 2)$. Hence, the polytope $\Pi \subset \Delta$ defined in the introduction can be defined by the following set of inequalities:

$$\Pi = \Delta \cap \left(H_I \geq 2 \mid I \subset \{1, \ldots, n\} \text{ even} \right).$$

More generally, we define a wall-and-chamber decomposition on Δ as follows. For each subset $I \subset \{1, \ldots, n\}$, and each integer k satisfying $2 \leq k \leq \frac{n}{2}$ and $|I| \equiv k \mod 2$, consider the hyperplane $(H_I = k)$. Now take the complement in the interior of Δ of the hyperplane arrangement

$$(4.3) \qquad\qquad \left(H_I = k \right)_{2 \leq k \leq \frac{n}{2}, \ |I| \equiv k \mod 2}$$

and consider its decomposition into connected components. Each connected component is called a *chamber* of Δ.

In [Bau91], Bauer proved that this wall-and-chamber decomposition on Δ corresponds to the variation of GIT for the moduli spaces $\mathcal{M}_{\mathcal{A}}$, and described the birational maps between models corresponding to different chambers. In particular, for $\frac{1}{n-2} < \varepsilon < \frac{1}{n-4}$ and $\mathcal{A}_\epsilon = (1 - \epsilon, \epsilon, \ldots, \varepsilon)$, the moduli space $\mathcal{M}_{\mathcal{A}_\varepsilon}$ is isomorphic to the blow-up X_n^{n-3} of $\mathbb{P}^{n-3}$ at n general points. This is known to be a *Mori dream space* ([CT06, Theorem 1.3]). In particular its effective cone $\mathrm{Eff}(X_n^{n-3})$ comes with a *Mori chamber decomposition*, and the chambers inside the movable cone $\mathrm{Mov}(X_n^{n-3}) \subset \mathrm{Eff}(X_n^{n-3})$ can be identified with the ample cones of small $\mathbb{Q}$-factorial modifications of X_n^{n-3} ([HK00]). These are $\mathbb{Q}$-factorial projective varieties which are isomorphic to X_n^{n-3} outside a subset of codimension at least two. Mukai realized in [Muk05] that there is a linear projection

$$\phi : \mathbb{R}^{n+1} \cong N^1(X_n^{n-3}) \longrightarrow \mathbb{R}^n$$

mapping the effective $\mathrm{Eff}(X_n^{n-3})$ onto Δ, so that the wall-and-chamber decomposition of Δ is induced by the Mori chamber decomposition of $\mathrm{Eff}(X_n^{n-3})$. More precisely, for an arbitrary weight $\mathcal{A} \in \Delta$, let $\mathcal{C}_{\mathcal{A}}$ denote the subset of Δ consisting of weight vectors defining the same stability condition as $\mathcal{A}$. Then the relative interior of $\mathcal{C}_{\mathcal{A}}$ is the image under ϕ of the ample cone of the moduli space $\mathcal{M}_{\mathcal{A}}$. In particular, ϕ maps the anti-canonical class $-K_{X_n^{n-3}}$ to the central weight $A_F = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$, and the movable cone $\mathrm{Mov}(X_n^{n-3})$ is mapped onto the polytope Π. The linear projection ϕ was made explicit in [AM16, Section 3].

Proof of Corollary 1.4. Let $\mathcal{A}$ be a weight vector in the interior of the polytope Π. Recall from the introduction that an elementary transformation $el_R \in \mathbf{El}$ defines an automorphism of $\mathcal{M}_{\mathcal{A}}^s$ if and only if it is $\mathcal{A}$-admissible, i.e., $\mathcal{A}^R \in \mathcal{C}_{\mathcal{A}}$. In particular, $\mathbf{El}_{\mathcal{A}} \subset \mathrm{Aut}\left(\mathcal{M}_{\mathcal{A}}\right)$.

As explained above, $\mathcal{M}_{\mathcal{A}}$ is a small modifications of $\mathcal{M}_{\mathcal{A}_F}$. So any automorphism $\varphi \in \mathrm{Aut}\left(\mathcal{M}_{\mathcal{A}}\right)$ induces a pseudo-automorphism φ_F of $\mathcal{M}_{\mathcal{A}_F}$. This means that φ_F is a birational automorphism of $\mathcal{M}_{\mathcal{A}_F}$ that restricts to an isomorphism on the complement of a subset of codimension ≥ 2. Since $\mathcal{M}_{\mathcal{A}_F}$ is a Fano variety, every pseudo-automorphism of $\mathcal{M}_{\mathcal{A}_F}$ is in fact an automorphism, and hence $\varphi_F \in \mathbf{El}$. We conclude that φ is induced by an elementary transformation, $\varphi \in \mathbf{El}_{\mathcal{A}}$. $\qquad\square$

Remark 4.4. If $n \geq 6$, it follows from Corollary 1.4 that $\mathrm{Aut}(X_n^{n-3}) = \{Id\}$. In fact, taking $\mathcal{A}_\epsilon$ with $\epsilon = \frac{1}{n-3}$ we can check that there is no $\mathcal{A}$-admissible elementary transformation other than identity. Besides that, for appropriate choices of weights, there are small modifications $\mathcal{M}_\mathcal{A}$ of X_n^{n-3} having intermediate automorphism group.

5. Moduli of involutional vector bundles

Let $n = 2g + 2 \geq 6$ be an even integer, and fix $p_1, \ldots, p_n \in \mathbb{P}^1$ general points. Let $\pi : C \longrightarrow \mathbb{P}^1$ be the $2 : 1$ cover branched over the points $p_1, \ldots, p_n$, set $c_i = \pi^{-1}(p_i) \in C$, and denote by $i : C \longrightarrow C$ the hyperelliptic involution. An involutional vector bundle (E, j) on C is an i-invariant vector bundle E, together with a lift $j : E \longrightarrow E$ of the involution i to E. We denote by $\mathcal{M}_{C/\mathbb{P}^1}^{inv}$ the moduli space of rank two semistable involutional vector bundles on C with trivial determinant, and such that $\mathrm{tr}(j_{c_i}) = 0$ for $1 \leq i \leq n$ (see for instance [Abe04, Section 2]). Forgetting the lift $j : E \longrightarrow E$ yields a $2 : 1$ morphism

$$\pi \ : \ \mathcal{M}_{C/\mathbb{P}^1}^{inv} \ \longrightarrow \ \mathcal{S}$$

onto an irreducible component $\mathcal{S}$ of the moduli space of i-invariant rank two semistable vector bundles on C with trivial determinant.

The Kummer variety of C is

$$\mathrm{Kum}(C) = \frac{\mathrm{Jac}(C)}{\iota},$$

where $\iota : \mathrm{Jac}(C) \longrightarrow \mathrm{Jac}(C)$ is the involution induced by $i : C \longrightarrow C$. It naturally embeds in $\mathcal{S}$ via the map

$$\begin{array}{ccc} \mathrm{Kum}(C) & \longrightarrow & \mathcal{S} \\ L & \longmapsto & L \oplus i^* L. \end{array}$$

By [Kum00, Theorem 2.1], the double cover $\pi : \mathcal{M}_{C/\mathbb{P}^1}^{inv} \longrightarrow \mathcal{S}$ is branched over the Kummer variety $\mathrm{Kum}(C) \subset \mathcal{S}$. We denote by

$$\eta \ : \ \mathcal{M}_{C/\mathbb{P}^1}^{inv} \ \longrightarrow \ \mathcal{M}_{C/\mathbb{P}^1}^{inv}$$

the involution of $\mathcal{M}_{C/\mathbb{P}^1}^{inv}$ induced by $\pi : \mathcal{M}_{C/\mathbb{P}^1}^{inv} \longrightarrow \mathcal{S}$.

As in the previous sections, we denote by $\mathcal{M}_{\mathcal{A}_F}$ the moduli space of rank two parabolic vector bundles with trivial determinant on $(\mathbb{P}^1, S)$ which are semistable with respect to the central weight $\mathcal{A}_F = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$. By [Bho84, Proposition 1.2] there is an isomorphism

$$(5.1) \qquad \qquad \mathcal{M}_{\mathcal{A}_F} \ \cong \ \mathcal{M}_{C/\mathbb{P}^1}^{inv}.$$

This map is obtained by pulling back parabolic vector bundles on $\mathbb{P}^1$ to C, performing an elementary transformation centered at the points $c_1, \ldots, c_n$, and then twisting by an appropriate line bundle.

Proposition 5.2. *Let the notation be as above. Then there is a splitting exact sequence*

$$0 \ \rightarrow \ \frac{\mathbb{Z}}{2\mathbb{Z}} \cong \{Id, \eta\} \ \rightarrow \ \mathrm{Aut}(\mathcal{M}_{C/\mathbb{P}^1}^{inv}) \ \rightarrow \ \mathrm{Aut}(\mathcal{S}, \mathrm{Kum}(C)) \rightarrow 0,$$

where $\mathrm{Aut}(\mathcal{S}, \mathrm{Kum}(C))$ denotes the group of automorphisms of $\mathcal{S}$ stabilizing $\mathrm{Kum}(C)$.

Proof. Let X be the blow-up of $\mathbb{P}^{2g-1}$ at $p_1, \ldots, p_{2g+2}$, and denote by E_i the exceptional divisor over p_i. By [Bau91] and [Muk05], there is a small birational modification
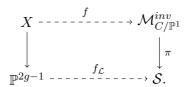
$$f \ : \ X \ \dashrightarrow \ \mathcal{M}_{\mathcal{A}_F} \ \cong \ \mathcal{M}_{C/\mathbb{P}^1}^{inv} \ ,$$

which is defined by the linear system $\left|m(-K_X)\right|$ for $m \gg 1$.

Let $\mathcal{L}$ be the linear system on $\mathbb{P}^{2g-1}$ of degree g hypersurfaces having multiplicity at least $g - 1$ at the p_i's, and denote by $\mathcal{L}_X$ the induced linear system on X. Then

$$(5.3) \qquad -K_X \sim 2gH - \sum_{i=1}^{2g+2} (2g-2)E_i \sim 2(gH - \sum_{i=1}^{2g+2} (g-1)E_i) \sim 2\mathcal{L}_X.$$

By [Kum00, Theorem 2.1] the rational map $f_{\mathcal{L}}$ induced by $\mathcal{L}$ is generically $2:1$, dominant onto $\mathcal{S}$, and makes the following diagram commute

$$
\begin{array}{ccc}
X & \dashrightarrow^{\ f\ } & \mathcal{M}^{inv}_{C/\mathbb{P}^1} \\
\downarrow & & \downarrow{\scriptstyle \pi} \\
\mathbb{P}^{2g-1} & \dashrightarrow_{\ f_{\mathcal{L}}\ } & \mathcal{S}.
\end{array}
$$

Since f is a small birational modification, (5.3) implies that $\pi : \mathcal{M}^{inv}_{C/\mathbb{P}^1} \longrightarrow \mathcal{S}$ is defined by the linear system $|\mathcal{L}'|$, where $2\mathcal{L}' \sim -K_{\mathcal{M}^{inv}_{C/\mathbb{P}^1}}$. In particular, any automorphism of $\mathcal{M}^{inv}_{C/\mathbb{P}^1}$ preserves the fibers of π, and hence descends to an automorphism of $\mathcal{S}$ stabilizing $\mathrm{Kum}(C)$, the branch locus of π. This gives a group homomorphism

$$
\mathrm{Aut}(\mathcal{M}^{inv}_{C/\mathbb{P}^1}) \ \longrightarrow \ \mathrm{Aut}(\mathcal{S}, \mathrm{Kum}(C)).
$$

Any automorphism in $\mathrm{Aut}(\mathcal{S}, \mathrm{Kum}(C))$ lifts to an automorphism of $\mathcal{M}^{inv}_{C/\mathbb{P}^1}$. Furthermore, if $\varphi \in \mathrm{Aut}(\mathcal{M}^{inv}_{C/\mathbb{P}^1})$ is a nontrivial automorphism that descends to the identity, then φ must switch the two points on a general fiber of π, i.e, $\varphi = \eta$, yielding the stated exact sequence. $\qquad\square$

Remark 5.4. By Theorem 1.2 and (5.1), $\mathcal{M}^{inv}_{C/\mathbb{P}^1} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{2g+1}$. Proposition 5.2 yields

$$
\mathrm{Aut}(\mathcal{S}, \mathrm{Kum}(C)) \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{2g}.
$$

On the other hand, tensoring by a 2-torsion line bundle on C induces an automorphism of $\mathcal{M}^{inv}_{C/\mathbb{P}^1}$. Therefore, $\mathrm{Aut}(\mathcal{S}, \mathrm{Kum}(C))$ can be naturally identified with the group of 2-torsion points of $\mathrm{Jac}(C)$.

Remark 5.5. When $n = 6$, the description of $\mathcal{M}^{inv}_{C/\mathbb{P}^1}$ is classical. In this case, C is a curve of genus 2, $\mathcal{S} \cong \mathbb{P}^3$, and $\mathrm{Kum}(C) \subset \mathbb{P}^3$ is the classical Kummer surface. It is a quartic surface whose singular locus consists of 16 singular points of type A_1. Remark 5.4 above recovers the group of automorphisms of $\mathbb{P}^3$ stabilizing $\mathrm{Kum}(C)$:

$$
\mathrm{Aut}(\mathbb{P}^3, \mathrm{Kum}(C)) \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^4.
$$

Remark 5.6. When $n = 2g + 1 \geq 5$ is odd, there is a similar isomorphism as in (5.1). In this case, in addition to the parabolic points $p_1, \ldots, p_n \in \mathbb{P}^1$, we pick an extra general point $p_{2g+2} \in \mathbb{P}^1$. As before, we let $\pi : C \longrightarrow \mathbb{P}^1$ be the $2:1$ cover branched over the points $p_1, \ldots, p_{2g+2}$, and set $c_i = \pi^{-1}(p_i) \in C$. We denote by $\mathcal{M}^{inv}_{C/\mathbb{P}^1}$ the coarse moduli space of rank two semistable involutional vector bundles on C with determinant $\mathcal{O}_C(c_{2g+2})$, such that $\mathrm{tr}(j_{c_i}) = 0$ for $1 \leq i \leq n$, and $j_{c_{2g+2}} = (-1)^{(g-1)}Id$. Then

$$
\mathcal{M}_{\mathcal{A}_F} \ \cong \ \mathcal{M}^{inv}_{C/\mathbb{P}^1}.
$$

By Theorem 1.2, their automorphism groups are isomorphic to $\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{2g}$. As before, they can be naturally identified with the group of 2-torsion points of $\mathrm{Jac}(C)$.

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