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Discrete velocity modeling of vehicular traffic and pedestrian dynamics

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Chapter 1

Introduction

Aim of the present thesis is to furnish rigorous results and corresponding qualitative analyses for a class of mathematical models of complex systems.

There is not an universally accepted definition of complexity or of what has to be meant by a complex system[1]. However, the different approaches to this question agree on the presence of a number of aspects that characterize complexity.

1. A complex system is composed by a large number of individuals (agents) that interact each other;

2. The interactions are not restricted to those of mechanical type, as the agents exhibit a strategy, that is, they have an objective, possibly different for different groups of agents, and they interact in such a way to reach it.

3. From a modeling point of view, the interactions are generally non-local and depend nonlinearly on their arguments.

4. Every complex system exhibits, in particular conditions, collective behaviors that are not easily related to the interactions at microscopic level.

The latter considerations need some explanation. It is indisputable that the great majority of real systems is indeed made of highly interconnected parts over many scales, whose interactions result in a complex behavior needing separate interpretation for each level. This realization leads us to appreciate that the new features emerge as one goes from one scale to another, so

[1]We refer here to *real world complexity*; in other words, we do not touch the concepts of *computational* or *algorithmic complexity*, whose study is much better set.

it follows that the study of complexity includes revealing the principles that govern the ways by which these new properties appear. In complex systems we recognize that processes occurring simultaneously on different scales or levels matter, and the intricate behavior of the whole system depends on its units in a non-trivial way. The description of the behavior of the whole systems requires a qualitatively new theory, because the laws describing its behavior are qualitatively different from those describing its units.

The most typical examples of complex systems, in some sense their paradigmatic representatives, are biological and social systems. When creating life, nature acknowledged the existence of the previously mentioned levels by spontaneously separating them as molecules, macromolecules, cells, organisms, species and societies. Of course, the big question is whether there is a unified theory for the ways elements of a system organize themselves to produce a behavior typical for wide classes of systems. Principles have been proposed, including self-organization, simultaneous existence of many degrees of freedom, self-adaption, rugged energy landscape and scaling of the parameters and the underlying network of connections. Presently, mathematicians and physicists are learning how to build relatively simple models producing complicated behavior, while those scientists working on inherently very complex systems (such as biologists and economists) are uncovering the ways the subjects they work on can be interpreted in terms of interacting units or subsystems.

The modeling of complex systems (biological, social) rests on the *a priori* assumption of a given mathematical framework, that one believes is able to capture the essence of the systems' behavior and that is suitably specialized to reproduce the specific features of the systems at study. There is a vast literature on mathematical and physical modeling of complex systems; we refer to the bibliography for a limited sample of significant papers. As is typical in the modeling, it does not exist a single framework capable to describe all the aspects of the systems' behavior which, on the other hand, exhibits a multi-scale character. Different scales are better modeled in different frameworks. Just as an example, if we focus on the evolution of gross quantities (i.e., densities, flows, etc.), the continuum mechanics furnishes methods to derive classes of equations which are in principle able to reproduce the behavior in space and time of macroscopic variables. On the other hand, if we are mainly interested on equilibrium properties of largely populated systems of interacting entities, the methods of statistical mechanics give a well-set pathway to reproduce macroscopic observed features, including the possible existence of phase transitions.

This dissertation is devoted to the mathematical study of a class of models of social systems, that include vehicular traffic flow and pedestrian dynamics, based on the generalized kinetic theory. The framework of kinetic theory is set up on the "mesoscopic" scale. Its fundamental quantity is a distribution function over the microscopic state, whose evolution equation is a partial differential equation that represents the balance between the spatial-temporal transport of the distribution function and the interaction terms among the individual entities (particles, agents). Introduced in the context of diluted gases, the kinetic theory revealed soon its ability to describe the non-equilibrium behavior not only of mechanical systems but also of more general ones. In this direction, in the last decade different generalizations of the theory have been proposed. In particular, the so-called kinetic theory of active particles (KTAP) introduces into the microscopic state balance ad additional variable, the activity, that models the ability of individuals, or groups of individuals, to express a strategy, a characteristic feature of complexity. The generalizations of kinetic theory, and in particular the KTAP, have been largely used to build mathematical models in biology and social sciences, often furnishing results in agreement with the modeling necessity.

In this thesis we focus on the mathematical properties of a class of models based on the discrete velocity kinetic theory. In this case, the velocity variable can assume only a discrete set of values, in this way reducing the computational complexity of the models. For some of the systems we treat, this simplification meets peculiar aspects of the systems' behavior, as in vehicular traffic flow that experimentally exhibits a granular nature.

In detail this thesis, in addition to this introduction, is composed by other seven chapters:

- Chapter 2 introduces the mathematical framework related to kinetic description of complex systems, the general kinetic equations governing them and the discretization of velocity variable, which permits to pass from a single kinetic equation to a system of first order semilinear hyperbolic equation.

- Chapter 3 deals with the study of the Cauchy problems related to discrete velocity models, when there is a one dimensional spatial variable. This framework is frequently used to describe traffic flow. First, we study a spatially homogeneous case, which permits to pass to a system of ordinary differential equations for which we analyze the long time behavior in a special case, useful to describe traffic flow. We then come to the non spatially homogeneous case for which, under weak hypotheses, we furnish a very general well-posedness result for the Cauchy problem, that includes global-in-time existence of solutions.

- Chapter 4 contains the well-posedness of Cauchy problems of discrete velocity models with a two dimensional spatial variable, used to describe pedestrian dynamics. We analyze relevant properties of solutions related to these problems, introducing a desired velocity in the set of achievable velocities.

- Chapter 5 deals with other three generalizations related to the Cauchy problems for discrete velocity models. We start studying initial-value problems with periodic initial data, then we give additional results of well-posedness under hypotheses weaker than the ones used in Chapters 3 and 4.

- Chapter 6 treats the initial-boundary value problems for discrete velocity models, paying attention to two different types of boundary conditions, one related to a complete absorption of mass, which models an exit on the boundary, and the other to a complete reflection, which describes the presence of a wall at the boundary.

- Chapter 7 contains a large study of systems with discretized spatial variable. Here the spatial variables are represented by cells, which are intervals in the one dimensional case or squares in two dimensional case. In each cell the problem is supposed spatially homogeneous and once we have fixed connections between neighbor cells we arrive to a system of ordinary differential equations for the systems' evolution. We then study the global-in-time behavior of solutions in special cases, once we have specialized the model.

- Chapter 8 summarizes the main results contained in the thesis and contains critical discussions and comments.

Chapter 2

Mathematical framework

In this chapter we present the mathematical framework of generalized kinetic theory on which this dissertation is based. The evolution equations that describe the collective behavior induced by interactions between agents will be introduced. In the following chapters these latter will be cars, in case of traffic flow, or people, in case of crowds dynamics. The present framework, in its version of kinetic theory of active particles, has also shown its ability to model various aspects of biological systems, like the competition among diseases and immune system. All the above mentioned systems share some of the issues characterizing the so-called *complexity*. In particular, their evolution is not based only on the mechanics of microscopic interactions, as social and biological systems have the ability to express strategies and to take decisions.

2.1 Preliminary definitions

Every system that will be considered in this thesis is composed by a population of agents, each agent having a typical way to interact with the others. The variable describing the state of an agent, the so called *microscopic state*, has in general the aspect:

$$\mathbf{w} = (\mathbf{x}, \mathbf{v}, \mathbf{u}) \in \Omega_{\mathbf{w}} = \Omega_{\mathbf{x}} \times \Omega_{\mathbf{v}} \times \Omega_{\mathbf{u}},$$

where $\mathbf{x}$ represents the microscopic geometrical state (the position), $\mathbf{v}$ the kinematic microscopic state (the velocity) and $\mathbf{u}$ is a variable, called *activity*, that represents the social or biological microscopic state. Again, Ω_w is the domain where microscopic state takes value. Once we defined a microscopic state, we have to introduce a distribution function over this state.

The microscopic state of the system is described by a *generalized* distribution function:

$$f : f(t, \mathbf{w}) \in [0, T] \times \Omega_{\mathbf{w}} \to \mathbb{R},$$

which is called generalized distribution function. In this scheme $f(t, \mathbf{w})d\ w$ denotes the number of agents having a microscopic state in the interval $[\mathbf{w}, \mathbf{w} + d\mathbf{w}]$, while $f(t, \mathbf{x}, \mathbf{v}, \mathbf{u})d\mathbf{x}$, $f(t, \mathbf{x}, \mathbf{v}, \mathbf{u})d\mathbf{v}$ and $f(t, \mathbf{x}, \mathbf{v}, \mathbf{u})d\mathbf{u}$ denote the number of agents having position in $[\mathbf{x}, \mathbf{x} + d\mathbf{x}]$, velocity in $[\mathbf{v}, \mathbf{v} + d\mathbf{v}]$ and activity in $[\mathbf{u}, \mathbf{u} + d\mathbf{u}]$, respectively. If f is known and it has suitable integrability properties, we can find macroscopic variables by integration over the microscopic state variables. For example

$$\rho(t, \mathbf{x}) := \int_{\Omega_{\mathbf{v}} \times \Omega_{\mathbf{u}}} f(t, \mathbf{x}, \mathbf{v}, \mathbf{u}) d\mathbf{v} d\mathbf{u},$$

is the macroscopic agents density, while

$$N(t) := \int_{\Omega_{\mathbf{x}}} \rho(t, x) d\mathbf{x} = \int_{\Omega_{\mathbf{w}}} f(t, \mathbf{w}) d\mathbf{w},$$

is the number of agents as function of time. In the same way, by requiring additional integrability properties, the moments of the distribution function provide macroscopic variables like the flux

$$\mathbf{q}(t, \mathbf{x}) := \int_{\Omega_{\mathbf{v}} \times \Omega_{\mathbf{u}}} \mathbf{v} f(t, \mathbf{x}, \mathbf{v}, \mathbf{u}) d\mathbf{v} d\mathbf{u},$$

and the local mean velocity

$$\xi(t, \mathbf{x}) := \frac{\mathbf{q}(t, \mathbf{x})}{\rho(t, \mathbf{x})}.$$

Related to second order moments we can define the energy

$$E(t, \mathbf{x}) := \int_{\Omega_{\mathbf{v}} \times \Omega_{\mathbf{u}}} |\mathbf{v}|^2 f(t, \mathbf{x}, \mathbf{v}, \mathbf{u}) d\mathbf{v} d\mathbf{u},$$

and the internal energy and speed variances

$$e(t, \mathbf{x}) := \int_{\Omega_{\mathbf{v}} \times \Omega_{\mathbf{u}}} |\mathbf{v} - \xi(t, x)|^2 f(t, \mathbf{x}, \mathbf{v}, \mathbf{u}) d\mathbf{v} d\mathbf{u}, \quad \sigma(t, \mathbf{x}) := \frac{e(t, \mathbf{x})}{\rho(t, \mathbf{x})}.$$

Functions defined previously are strictly related to the mechanical properties of system. On the other hand, we can define in the same way macroscopic variables related to activity $\mathbf{u}$, like activation

$$\mathbf{a}(t, \mathbf{x}) := \int_{\Omega_{\mathbf{v}} \times \Omega_{\mathbf{u}}} \mathbf{u} f(t, \mathbf{x}, \mathbf{v}, \mathbf{u}) d\mathbf{v} d\mathbf{u};$$

activation density and total activity which are respectively

$$\mathbf{a}^d(t,\mathbf{x}) := \frac{\mathbf{a}(t,\mathbf{x})}{\rho(t,\mathbf{x})}, \quad \mathbf{A}(t) := \int_{\Omega_\mathbf{x}} \mathbf{a}(t,\mathbf{x})d\mathbf{x}.$$

2.2 Interactions among agents

In this section we introduce a suitable framework in order to describe a large number of complex systems. We classify interactions between agents in the following ways.

- Local interactions: actions between a test (or candidate) and a field agent, happening when candidate and field agents are in the same spatial state.
- Non-local interactions: action over a test agent applied by all field agents which are into an interaction domain depending on the position of the test agent.

An example of local interaction is the one described by the interaction kernel of the Boltzmann equation. On the other hand we will not treat deeply systems with local interactions because they are pretty unrealistic. For example, the interaction between two pedestrians is not local but it is related to an interaction domain and to the orientation of their velocities. However we observe that it is more difficult to work with local interactions from a mathematical view-point, as will be clear in the sequel. Non-local interactions take place into an interaction domain $D_\mathbf{w}$ that can have different shape, related to the specific interaction

$$D_\mathbf{w} = D_\mathbf{x} \times D_\mathbf{v} \times D_\mathbf{u} \subseteq \Omega_\mathbf{x} \times \Omega_\mathbf{v} \times \Omega_\mathbf{u}.$$

The interactions are also characterized in the following way.

- Conservative interactions: interactions that do not change the number of agents but only their state.
- Non-conservative interactions: interactions that lead to proliferative o destructive phenomena and hence to a variation of agents' number.

In this dissertation we shall work only with conservative interactions. We will not treat the non-conservative case, that is on the other hand of fundamental relevance in the modeling of biological systems like tumor growth or immune competition between diseases and immune system. We now introduce the fundamental ingredients to describe interactions.

- Interaction rate, that could depend on the states interacting agents, which measures the frequency at which two agents may interact together.
- Transition probability density, which describes the probability that an agent, having a given state, suffers a transition to another state class due to an interaction with an agent located into the interaction domain.

Another element related to non-locality of interactions is the so called *interaction weight*, which gives more importance to closer interactions, since it is reasonable to suppose that a agents close to a candidate one are more influential then the distant ones.

2.3 Evolution equation

In this section we introduce the general kinetic equations describing the evolution of the distribution function. It is obtained by balancing variation of $f = f(t, \mathbf{w})$ in the elementary volume of the state space, with inlet and outlet fluxes due to microscopic interactions. The resulting balance equation, in absence of destructive and proliferative phenomena, is the following:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f = J(f) = \Gamma(f) - \Lambda(f), \tag{2.1}$$

where J is the interaction kernel that describes the interactions among agents, Γ is the gain term that describes the inlet flow while Λ is the loss term representing the outlet flow. In detail, the interaction kernel reads

$$\begin{aligned} J(f) = & \int_{D_{\mathbf{x}}} \int_{\Omega_{\mathbf{z}}^2} B(\mathbf{z}, \mathbf{z}^*, \mathbf{z}_*) \eta(\mathbf{z}^*, \mathbf{z}_*) f(t, \mathbf{x}, \mathbf{z}^*) f(t, \mathbf{y}, \mathbf{z}_*) w(\mathbf{x}, \mathbf{y}) d\mathbf{z}^* d\mathbf{z}_* d\mathbf{y} \\ & - f(t, \mathbf{x}, \mathbf{z}) \int_{D_{\mathbf{x}}} \int_{\Omega_{\mathbf{z}}} \eta(\mathbf{z}, \mathbf{z}^*) f(t, \mathbf{y}, \mathbf{z}^*) w(\mathbf{x}, \mathbf{y}) d\mathbf{z}^* d\mathbf{y}, \end{aligned} \tag{2.2}$$

where,

$$\mathbf{z} := (\mathbf{v}, \mathbf{u}), \quad \Omega_{\mathbf{z}} := \Omega_{\mathbf{v}} \times \Omega_{\mathbf{u}}.$$

Here we recognize the characters introduced in the previous section.

- The encounter rate η, that describes the frequency of interactions between a candidate agent with state $\mathbf{z}^*$ and a field agent with state $\mathbf{z}_*$.
- Transition probability density B gives us the probability that a candidate agent with state $\mathbf{z}^*$ changes his state to $\mathbf{z}$ after an interaction with a field agent having state $\mathbf{z}_*$.

- The interaction domain $D_{\mathbf{x}}$, the piece of spatial domain where interactions take place, plays the role of a visibility zone, as field agents outside from $D_{\mathbf{x}}$ don't interact with candidate agents.

- The function $w(\mathbf{x}, \mathbf{y})$, that weights the interactions between a candidate agent located in $\mathbf{x}$ and a field agent in $\mathbf{y}$.

The encounter rate and the transition probability density could depend not only on different state $\mathbf{u}^*$ and $\mathbf{u}_*$ but also on distribution function f or its moments, while the weight function w is intuitively related to distance between $\mathbf{x}$ and $\mathbf{y} \in D_{\mathbf{x}}$. We point out that the shape of $D_{\mathbf{x}}$ is strictly related to interaction described by system and it can possibly depend on additional variables like activity o velocity as well. An easy example is the interaction domain of a pedestrian, which is a half-cone oriented with velocity, as we'll see in the sequel. The transition probability density is also known in literature as the *table of games*, because it is related to a game played by two interacting agents. Moreover, when describing conservative interactions we bear in mind that a candidate agent with state $\mathbf{z}^*$, after an interaction with field agent having state $\mathbf{z}_*$, certainly achieves a state $\mathbf{z} \in \Omega_{\mathbf{z}}$ and this fact is summarized in the condition:

$$\int_{D_{\mathbf{z}}} B(\mathbf{z}, \mathbf{z}^*, \mathbf{z}_*) d\mathbf{z} = 1, \quad \forall \mathbf{z}^*, \mathbf{z}_* \in \Omega_{\mathbf{z}}, \tag{2.3}$$

We remark that interactions among agents produce only a change of velocity and activity of candidate agents, while it does not result in any variation of spatial variable x where candidate agents are placed.

Remark 2.3.1. *The interaction kernel defined in (2.2) is purely binary and this is reflected in the gain and loss terms. Both of them are quadratic with respect to the distribution function f. We don't study multiple interactions, that greatly increase the mathematical complexity, even if they play a role in living systems (see, e.g., [10] where triple interactions are treated in the kinetic framework).*

There is a ample literature concerning systems in which activity is more important then spatial variables. For these kind of systems the kinetic evolution equation is:

$$\begin{aligned} \frac{\partial f}{\partial t}(t, \mathbf{u}) &= J(f(u, \mathbf{u})) = \Gamma(f(t, \mathbf{u})) - \Lambda(f(t, \mathbf{u})) \\ &= \int_{\Omega_{\mathbf{u}}^2} \eta(\mathbf{u}^*, \mathbf{u}_*) B(\mathbf{u}, \mathbf{u}^*, \mathbf{u}_*) f(t, \mathbf{u}^*) f(t, \mathbf{u}_*) d\mathbf{u}^* d\mathbf{u}_* \\ &- f(t, \mathbf{u}) \int_{\Omega_{\mathbf{u}}} \eta(\mathbf{u}, \mathbf{u}_*) f(t, \mathbf{u}_*) d\mathbf{u}_*. \end{aligned} \tag{2.4}$$

The velocity does not have a strong physical meaning in system described by equation like (2.4). In [15] and [5] we find a more deep description of these models and also additional bibliography.

On the other hand, complex systems like vehicular traffic and pedestrian crowds are characterized by dominant mechanical interactions and consequently the most important role is played by purely mechanical variables $\mathbf{x}$ and $\mathbf{v}$. If we forget by the moment the activity $\mathbf{u}$, the equations (2.1) and (2.2) are written as:

$$\begin{aligned}
\frac{\partial f}{\partial t}(t,,x,\mathbf{v}) + \mathbf{v}\cdot f(t,,x,\mathbf{v}) &= J(f(u,\mathbf{x},\mathbf{v})) = \Gamma(f(t,\mathbf{x},\mathbf{v})) - \Lambda(f(t,\mathbf{x},\mathbf{v})) \\
&= \int_{D_\mathbf{x}}\int_{\Omega^2_\mathbf{v}} B(\mathbf{v},\mathbf{v}^*,\mathbf{v}_*)\eta(\mathbf{v}^*,\mathbf{v}_*)f(t,\mathbf{x},\mathbf{v}^*)f(t,\mathbf{y},\mathbf{v}_*)w(\mathbf{x},\mathbf{y})d\mathbf{v}^*d\mathbf{v}_*d\mathbf{y} \\
&- f(t,\mathbf{x},\mathbf{v})\int_{D_\mathbf{x}}\int_{\Omega_\mathbf{v}} \eta(\mathbf{v},\mathbf{v}^*)f(t,\mathbf{y},\mathbf{v}^*)w(\mathbf{x},\mathbf{y})d\mathbf{v}^*d\mathbf{y}.
\end{aligned} \tag{2.5}$$

2.4 Discretization of velocity

Classical models of the kinetic theory of gases come from the assumption that particles can achieve all values of velocity; in general, if spatial variable $\mathbf{x} \in \mathbb{R}^n$ then also velocity $\mathbf{v}$ takes value in the whole $\mathbb{R}^n$. A possible way to reduce the mathematical complexity of the model is to discretize the velocity variable. In [20] and [41] there is an introduction to discrete Boltzmann equation and a vast bibliography related to it. The fundamental idea is to assume that particles are allowed to move with a finite number of velocities $\Omega_\mathbf{v} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. In this scheme the distribution function becomes a vector function depending on time and space.

Some living systems can effectively assume only a finite number of velocities (or a finite number of values for the activity). Reminding that discretization of velocity leads also to a noteworthy simplification, many authors make use of this approach to describe complex systems, see [5] and [15].

Forgetting for a moment the activity variable and pointing out only mechanical variables, in the continuous case we have an equation like (2.5). As we have previously said talking about the discrete Boltzmann equation, we suppose that there is only a finite set of velocities:

$$\mathbf{v} \in \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}.$$

achievable by agents. The distribution function f then becomes a vector

$\mathbf{f} = (f_1, \ldots, f_n)$, where

$$f_i : [0, T] \times D_{\mathbf{x}} \to \mathbb{R}.$$

Each f_i is the distribution function for agents traveling with the i-th velocity, $f_i(t, \mathbf{x})d\mathbf{x}$ denotes the amount of agents in the infinitesimal volume $[\mathbf{x}, \mathbf{x}+d\mathbf{x}]$ with velocity $\mathbf{v}_i$. Once we have discretized the number of velocities we have to replace integral of $D_{\mathbf{v}}$ with a sum, and thus the local density and the linear momentum become:

$$\rho(t, \mathbf{x}) = \sum_{h=1}^{n} f_h(t, \mathbf{x}), \quad \mathbf{q}(t, \mathbf{x}) = \sum_{h=1}^{n} \mathbf{v} f_h(t, \mathbf{x}).$$

From a single kinetic evolution equation like (2.5), governing the behavior of complex systems, we arrive to a system of n equations:

$$\partial_t f_i(t, \mathbf{x}) + \mathbf{v}_i \cdot f_i(t, \mathbf{x}) = J_i(\mathbf{f}(t, \mathbf{x})), \quad \text{for all } i = 1, \ldots, n, \tag{2.6}$$

where each equation describes the evolution of the i-th agents' class having velocity $\mathbf{v}_i$. The interaction kernel becomes a vector of $\mathbb{R}^n$,

$$J_i(\mathbf{f}(t, \mathbf{x})) = \Gamma_i(\mathbf{f}(t, \mathbf{x})) - \Lambda_i(\mathbf{f}(t, \mathbf{x})),$$

where Γ_i is the gain term, describing inlet flow of agents that change their velocity in $\mathbf{v}_i$ and Λ_i is the loss term, that describes outlet flow from i-th class, amount of agents with velocity $\mathbf{v}_i$ that change their velocity after an interaction. We write extensively the interaction kernel:

$$\begin{aligned} J_i(\mathbf{f}(t, \mathbf{x})) = &\sum_{h,k=1}^{n} \int_{D_{\mathbf{x}}} \eta_{hk}(\mathbf{f}(t, \mathbf{y})) B_{hk}^{i}(\mathbf{f}(t, \mathbf{y})) f_h(t, \mathbf{x}) f_k(t, \mathbf{y}) w(\mathbf{x}, \mathbf{y}) d\mathbf{y} \\ &- f_i(t, \mathbf{x}) \sum_{h=1}^{n} \int_{D_{\mathbf{x}}} \eta_{hk}(\mathbf{f}(t, \mathbf{y})) f_h(t, \mathbf{y}) w(\mathbf{x}, \mathbf{y}) d\mathbf{y}. \end{aligned} \tag{2.7}$$

Once again we recognize the models' ingredients as in the continuous case:

- the transition probability density B_{hk}^{i}, that gives us the probability that a candidate agent, with velocity $\mathbf{v}_h$, changes his velocity in $\mathbf{v}_i$ after an interaction with an agent with velocity $\mathbf{v}_k$;
- the interaction rate η_{hk}, that gives the frequency of interactions between agents with velocity $\mathbf{v}_h$ and $\mathbf{v}_k$, respectively;
- the interaction weight w, depending on the distance from the position $\mathbf{x}$ of the candidate agents to that $\mathbf{y}$ of the field agent.

We have here stressed the dependence of η_{hk} and B^i_{hk} on distribution $\mathbf{f}$.

Finally, starting from an equation like (2.5), we arrive to a system of semilinear first order hyperbolic equations, whose solution is connected to the properties of the different functions present in the r.h.s. of (2.6).

Chapter 3

One dimensional Cauchy problem

In this chapter we prove global existence and uniqueness for the Cauchy problem related to systems describing traffic flow. There is a large literature concerning kinetic description of traffic flow that make use of discrete velocity models introduced in the previous chapter, [6], [7], [8], [14], [18], [22] and [26]. From the modeling point of view it is of crucial relevance the suitable choice of the functions that enter in the interaction rate, mainly the rate of interactions and the table of games, as they structure depicts the way in which the mathematical model reproduce the experimental observations. On the other hand, whatsoever form the interaction kernel may take, it is clear that in order the model to make sense we must give a positive answer to basic questions concerning the existence of solutions, to their uniqueness and to their continuous dependence on the initial data. In the first part of the present chapter we treat this problem from a general viewpoint, without fixing any particular form of the functions in the interaction kernel but only requiring this latter to verify suitable hypotheses. We give an existence and uniqueness theorem that generalizes previous results which were obtained under more restrictive conditions.

3.1 Statement of the problem

The Cauchy problem related to equations like (2.6), when space variable is a scalar $x \in \mathbb{R}$, is written as follows

$$\begin{cases} \partial_t f_i(t,x) + v_i \partial_x f_i(t,x) = J_i(\mathbf{f})(t,x), \\ f_i(0,x) = \overline{f}_i(x), \end{cases} \qquad i = 1, \dots, n. \tag{3.1}$$

In this chapter we forget by the moment the interaction weight w, that will be added in the sequel, and thus the gain and loss terms of interaction kernel are written as:

$$\Gamma_i(f)(t,x) := \sum_{h,k=1}^{n} \int_{D_x} \eta(\rho(t,y)) B_{hk}^i(\rho(t,y)) f_h(t,x) f_k(t,y) dy, \tag{3.2}$$

$$\Lambda_i(f)(t,x) := f_i(t,x) \sum_{h=1}^{n} \int_{D_x} \eta(\rho(t,y)) f_h(t,y) dy. \tag{3.3}$$

Here $\mathbf{f} = (f_i)$ is a vector such that $f_i(t,x)dx$ represents the number of agents traveling with velocity v_i. It is then clear that f_i has to attain only positive values:

$$f_i = f_i(t,x) : [0,T] \times \mathbb{R} \to \mathbb{R}_+, \quad i = 1 \ldots, n.$$

The velocity can assume only a finite set $\Omega_v = \{v_1, \ldots, v_n\}$ of values, with $v_i < v_{i+1}$ for $i = 1, \ldots, n-1$, while $\overline{f}_i(x)$ are the values of the distribution function components at $t = 0$. From the definition of B_{hk}^i itself, the following properties:

$$B_{hk}^i(\mathbf{f}) \geq 0, \quad \sum_{i=1}^{n} B_{hk}^i(\mathbf{f}) = 1, \quad \forall \rho \in \mathbb{R}_+^n, \forall h, k = 1, \ldots, n. \tag{3.4}$$

hold true. Thanks to (3.4) we have that:

$$\sum_{i=1}^{n} J_i(\mathbf{f}) = 0. \tag{3.5}$$

This relation reflects the absence in the system of proliferative and destructive effects, and will lead to the conservation of the total number of vehicles, as we shall see later on. Now we state the basic assumptions we need in order to get local and global existence of solutions for (3.1).

- The encounter rate η is Lipschitz continuous and bounded, i.e., there exist $L_\eta > 0$ and $C_\eta > 0$ such that:

$$\begin{aligned} |\eta(\mathbf{f}_1) - \eta(\mathbf{f}_2)| &\leq L_\eta |\mathbf{f}_1 - \mathbf{f}_2|, \quad \forall \mathbf{f}_1, \mathbf{f}_2 \in \mathbb{R}^n, \\ |\eta(\mathbf{f})| &\leq C_\eta, \quad \forall \mathbf{f} \in \mathbb{R}^n. \end{aligned} \tag{3.6}$$

- Transition probability density B_{hk}^i is locally Lipschitz for all $h, k, i = 1, \ldots, n$, i.e, for all $r > 0$ there exists a $C_{B_{hk}^i, r} > 0$ such that:

$$|B_{hk}^i(\rho_1) - B_{hk}^i(\rho_2)| \leq C_{B_{hk}^i, r} |\rho_1 - \rho_2|, \; \forall \rho_1, \rho_2 \in \mathbb{R} \text{ with } |\rho_1| < r, \; |\rho_2| < r, \tag{3.7}$$

  and verifies (3.4).

- Interaction domain is an interval written as follows

$$D_x = [x - \Delta^-, x + \Delta^+], \quad \text{with } \Delta^\pm \geq 0 \ (\Delta^-, \Delta^*) \neq (0, 0). \tag{3.8}$$

These hypotheses are crucial in order to have global solutions to system (3.1) in a suitable space.

3.2 Spatially homogeneous problem

In this section we treat spatially homogeneous problem, when the distribution function does not depend on the spatial variable, that is, $\mathbf{f} = \mathbf{f}(t)$. In this case (3.1) reduces to a system of ordinary differential equations:

$$\begin{cases} \dfrac{df_i}{dt} = J_i(\mathbf{f}) \\ f_i(0) = \overline{f}_i. \end{cases} \tag{3.9}$$

In this spatially homogeneous case conservative interactions among agents bring to the conservation of density:

$$\frac{d\rho}{dt} = \sum_{i=1}^{n} J_i(\mathbf{f}) = 0,$$

and thus $\rho(t) = \rho(0)$. We prove existence of solution $\mathbf{f} \in C([0, +\infty), \mathbb{R}^n)$ for (3.9).

Theorem 3.2.1. *Let η and B^i_{hk} be functions satisfying (3.4), (3.6) and (3.7). Then, given a vector $\overline{\mathbf{f}} \in \mathbb{R}^n$ such that $\overline{f}_i \geq 0$, there exists a unique $\mathbf{f} \in C([0, +\infty), \mathbb{R}^n)$ solution of (3.9). Moreover:*

$$f_i(t) \geq 0, \quad i = 1, \ldots, n, \quad t \geq 0,$$

$$\rho(t) = \sum_{i=1}^{n} f_i(t) = \rho(0) := \rho_0.$$

It is easy to prove existence and uniqueness of solution locally in time, while prolongability comes from the density conservation. If we suppose that the interaction rate is the same for all pairs of velocity distributions, and that it depends on the distribution $\mathbf{f}$ only through the density ρ:

$$\eta_{hk}(\mathbf{f}) = \eta\left(\sum_{h=1}^{n} f_h\right), h, k = 1, \ldots, n, \tag{3.10}$$

we can easily write the kernel $\mathbf{J}$ as:

$$J_i(\mathbf{f}) = K\left(\sum_{h,k=1}^{n} B^i_{hk}(\rho_0) f_h f_k - f_i \rho_0\right),$$

where $K = (\Delta^+ + \Delta^-)\eta(\rho_0)$. We fix also transition probability density. Assuming that the table of games depends on $\mathbf{f}$ through density ρ, we introduce a function $p = p(\rho)$ called *probability of passing* that describes the probability to increase velocity. If we work with normalized data, we assume following hypotheses

$$\begin{array}{c} \rho_1 < \rho_2 \Longrightarrow p(\rho_1) > p(\rho_2) \\ \rho(0) = 1, \quad p(1) = 0. \end{array} \tag{3.11}$$

We now fix table of games similarly to [30], where $p(\rho) = 1 - \rho$. There are three main types of interactions:

- a candidate agents, traveling with velocity v_h, interacts with a faster agent, i.e. $v_h \leq v_k$; in this case we assume

$$B^i_{hk}(\rho) = \begin{cases} p(\rho) & \text{if } i = h+1 \\ 1 - p(\rho) & \text{if } i = h \\ 0 & \text{otherwise,} \end{cases} \tag{3.12}$$

  that is, the candidate agent has the tendency to follow field agents, but this tendency decreases when density increases.

- Interaction with a slower agent, when candidate agent is faster then the field one, $v_h > v_k$

$$B^i_{hk}(\rho) = \begin{cases} p(\rho) & \text{if } i = h \\ 1 - p(\rho) & \text{if } i = k \\ 0 & \text{otherwise.} \end{cases} \tag{3.13}$$

  In this case we are basically saying that if density is large then the candidate agent is induced to stop and to follow slower agents.

- Interaction of a candidate with agents traveling with same velocity, and thus $v_h = v_k$. In this case we assume that the transition probability is equal to the one written in the case $v_h = v_k$ (3.12) and we have only to fix the case of agents traveling with the larger velocity $h = n$, when we suppose that the candidate agent maintains his velocity,

$$B^i_{nn}(\rho) = \begin{cases} 1 & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases} \tag{3.14}$$

Once we have focused transition probability density we are able to give some qualitative results, like existence of stable solutions. Equilibria are solutions of system $\mathbf{J}(\mathbf{f}) = 0$, and thus

$$\sum_{h,k=1}^{n} B_{hk}^{i}(\rho) f_h f_k - \rho f_i = 0, \quad i = 1, \ldots, n.$$

Following idea of [30] we find recursively equilibria $\hat{\mathbf{f}}$. If $\rho = 0$ then there exists only unique solution $\mathbf{f}(t) = 0$, while if $\rho > 0$ Starting from $i = 1$ and reminding that $\sum_{h=2}^{n} \hat{f}_h = \rho - \tilde{f}_1$ we find that $\hat{f}_1$ is a solution of

$$\hat{f}_1 \left[\rho(1 - 2p(\rho)) - (1 - p(\rho))\hat{f}_1 \right] = 0,$$

and thus we find

$$\hat{f}_1 = 0, \quad \hat{f}_1 = \rho \frac{1 - 2p(\rho)}{1 - p(\rho)}.$$

Fixed an initial density we find two stationary solutions related to first equation of system (3.9) and only one of them could be stable. In this special case it is easy to prove that the stable solution is the larger one; if we write the first equation of (3.9), which gives the evolution of f_1, we find that the time derivative of f_1 is equal to a polynomial with a negative coefficient of second degree term, this ensures that the larger solution is the stable one, and its value depends on density as follows

$$\hat{f}_1 = \begin{cases} 0 & \text{if } p(\rho) \geq 1/2, \\ \rho \frac{1-2p(\rho)}{1-p(\rho)} & \text{if } p(\rho) < 1/2. \end{cases}$$

Once we have found $\hat{f}_1$, we can find $\hat{f}_2$ as function of $\hat{f}_1$, hence recursively for $1 < i < n$ for all once we have found $\hat{f}_1, \ldots, \hat{f}_{i-1}$ we can find $\hat{f}_i$, in particular we can write equation $J_i(\mathbf{f}) = 0$ as follows,

$$(p(\rho) - 1)\, \hat{f}_i^2 + \left[b_{i-1} \left(3p(\rho) - 2 \right) + \rho \left(1 - 2p(\rho) \right) \right] \hat{f}_i + p(\rho) \left(\rho - b_{i-2} \right) \hat{f}_{i-1} = 0$$

where

$$b_i = \sum_{h=1}^{i} \hat{f}_h,$$

and thus we find a polynomial of second degree. Reminding that density is normalized and $\rho \leq 1$ then discriminant of equation is positive, and there are two real roots, moreover second degree polynomial has a negative leading coefficient and positive free term, and thus the first root is negative and the second one is positive, the only one admissible is the positive one, moreover is also stable. We have recursive formula to calculate stable equilibrium point,

$$\begin{cases} \begin{cases} 0 & \text{if } p(\rho) \geq 1/2, \\ \rho \dfrac{1-2p(\rho)}{1-p(\rho)} & \text{if } p(\rho) < 1/2. \end{cases} & \text{if } i = 1, \\ \dfrac{b_{i-1}\left(2-3p(\rho)\right)+\rho\left(2p(\rho)-1\right)-\sqrt{\Delta_i}}{2(p(\rho)-1)} & \text{if } 1 < i < n, \\ \rho - b_{n-1} & \text{if } i = n, \end{cases} \tag{3.15}$$

with Δ_i is the disciminant of i-th equation,

$$\Delta_i := \left[b_{i-1}\left(3p(\rho)-2\right)+\rho\left(1-2p(\rho)\right)\right]^2 + 4p(\rho)\left(\rho - b_{i-2}\right)\left(1-p(\rho)\right)\hat{f}_{i-1}.$$

We resume what we have found with the following theorem.

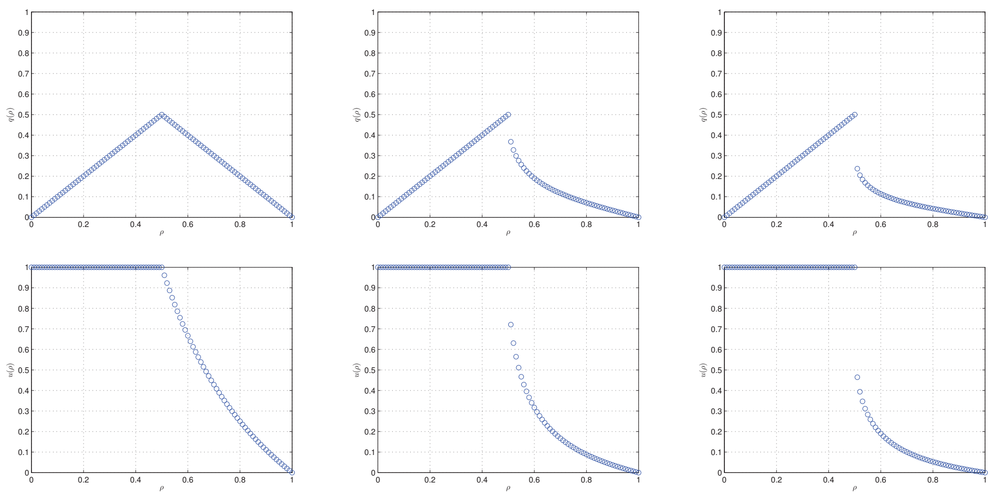

Figure 3.1: In the first line the flux of steady solutions, each steady state is dependent from $\mathbf{f}$ only through *density*, in each three cases $p(rho) = 1-\rho$; only number of velocities changes in examples, in the first $n = 2$, the second $n = 4$ and the third $n = 6$; in the second line mean velocity related $u(\rho) = q(\rho)/\rho$ to upper fluxes.

Theorem 3.2.2. *Let B^i_{hk} be a transition probability density like (3.12), (3.13) and (3.14), and let p be a function like (3.11), then for all $\rho \in [0,1]$ there exists a unique stable attractive point $\hat{\mathbf{f}}$, whose components verify (3.15) and*

$$\sum_{h=1}^{n} \hat{f}_h = \rho.$$

Previous Theorem encloses result of [30] choosing $p(\rho) = 1 - \rho$. We note that if $p(\rho) \geq 1/2$, or rather $\rho \leq p^{-1}(1/2)$, we have

$$\hat{\mathbf{f}} = \begin{cases} 0 & \text{if } i < n, \\ \rho & \text{if } i = n, \end{cases}$$

and $\hat{\mathbf{f}}$ is attractive, in other words when density is lower than a critical density, all agents achieve faster velocity. Critical density is given by equality $p(\rho) = 1/2$.

3.3 Global existence and uniqueness of solutions

Here we study the non-homogeneous problem, pointing out that (3.1) is a system of n semilinear first-order hyperbolic equations, whose characteristic curves in t-x plane are solution of ordinary differential equations,

$$\frac{dx}{dt} = v_i, \quad i = 1, \dots, n,$$

and thus if we fix the point (τ, ξ) in the $t - x$ plane the i-th characteristic passing through (τ, ξ) is:

$$x = \gamma_i(t) = \xi + v_i(t - \tau).$$

Along the characteristics we can rewrite the problem (3.1) in the following way:

$$\begin{cases} \dfrac{df_i}{dt}(t, \gamma_i(t, \tau, \xi)) = J_i(\mathbf{f}(t, \gamma_i(t, \tau, \xi))), \\ f_i(0) = \overline{f}_i(\gamma_i(0, \tau, \xi)), \end{cases} \qquad i = 1, \dots, n. \tag{3.16}$$

For the semilinear problem the value of the solution at any point (τ, ξ) can be determined by solving the Cauchy problem (3.16). Under suitable regularity conditions, once fixed (τ, ξ) we can integrate (3.16) from 0 to τ obtaining:

$$f_i(\tau, \xi) = \overline{f}_i(\gamma_i(0, \tau, \xi)) + \int_0^\tau J_i(\mathbf{f}(t, \gamma_i(t, \tau, \xi)))dt. \tag{3.17}$$

Definition 3.3.1. *Let $T > 0$ be a positive real number, then a mild solution of (3.1) is a function $\mathbf{f} : [0, T] \times \mathbb{R} \to \mathbb{R}$ verifying (3.17), that is:*

$$f_i(t, x) = \overline{f}_i(\gamma_i(0, t, x)) + \int_0^t J_i(\mathbf{f}(\tau, \gamma_i(\tau, t, x)))d\tau,$$

for all $(t, x) \in [0, T] \times \mathbb{R}$

It will be useful to write the solution as a function of the time only $\mathbf{f} : [0, T] \to X$, where X is a suitable functional space. We introduce

$$X := (L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))^n,$$

and define:

$$\|\mathbf{u}\|_X := \|\mathbf{u}\|_1 + \|\mathbf{u}\|_\infty, \qquad \mathbf{u} \in X,$$

where:

$$\|\mathbf{u}\|_\infty := \max_i \|u_i\|_{L^\infty}, \text{ and } \|\mathbf{u}\|_1 := \sum_{i=1}^{n} \|u_i\|_{L^1}.$$

It is easy to verify that $(X, \|\cdot\|_X)$ is a Banach space. With the norm defined previously we are able to control the number of agents $\|\cdot\|_1$ and the local density with $\|\cdot\|_\infty$. We give some properties related to interaction kernel $\mathbf{J}$ if we take X as domain.

Lemma 3.3.2. *Let η and B^i_{hk} be functions verifying (3.4), (3.6) and (3.7), and let D_x be an interval as in (3.8); then the function $\mathbf{J} = (J_i)$ maps X into itself.*

Proof. We have to prove that if $\mathbf{f} \in X$ then:

$$\|\mathbf{J}(\mathbf{f})\|_X = \|\mathbf{J}(\mathbf{f})\|_1 + \|\mathbf{J}(\mathbf{f})\|_\infty < +\infty.$$

We start analyzing the L^1 norm $\|\cdot\|_1$. We have:

$$\int_{\mathbb{R}} |J_i(\mathbf{f}(x))| dx \leq \int_{\mathbb{R}} |\Gamma_i(\mathbf{f}(x))| dx + \int_{\mathbb{R}} |\Lambda_i(\mathbf{f}(x))| dx.$$

Evaluating the gain term we find:

$$\int_{\mathbb{R}} |\Gamma_i(\mathbf{f}(x))| dx \leq \sum_{h,k=1}^{n} \int_{\mathbb{R}} |f_h(x)| \left(\int_{D_x} |\eta(\mathbf{f}(y)) B^i_{hk}(\mathbf{f}(y)) f_k(y)| dy \right) dx$$

$$\leq \sum_{h,k=1}^{n} \|f_h\|_{L^1} \int_{\mathbb{R}} |\eta(\mathbf{f}(y)) B^i_{hk}(\mathbf{f}(y)) f_k(y)| dy,$$

and using the boundedness of η and B^i_{hk} we get:

$$\int_{\mathbb{R}} |\Gamma_i(\mathbf{f}(x))| dx \leq \sum_{h,k=1}^{n} C_\eta \|f_h\|_{L^1} \int_{\mathbb{R}} |f_k(y)| dy = \sum_{h,k=1}^{n} C_\eta \|f_h\|_{L^1} \|f_k\|_{L^1} < +\infty.$$

Similarly, we have:

$$\int_{\mathbb{R}} |\Lambda_i(\mathbf{f}(x))|dx \leq \|f_i\|_{L^1} \sum_{h=1}^{n} \int_{\mathbb{R}} |\eta(\mathbf{f}(y))f_h(y)|dy$$

$$\leq C_\eta \|f_i\|_{L^1} \sum_{h=1}^{n} \int_{\mathbb{R}} |f_h(y)|dy \leq C_\eta \|f_i\|_{L^1} \sum_{h=1}^{n} \|f_h\|_{L^1} < +\infty.$$

Coming to analyze the L^∞ part, we find:

$$|\Gamma_i(\mathbf{f}(x))| \leq \sum_{h,k=1}^{n} \|f_h\|_{L^\infty} \int_{D_x} |\eta(\mathbf{f}(y))B^i_{hk}(\mathbf{f}(y))f_k(y)|dy$$

$$\leq C_\eta \sum_{h,k=1}^{n} \|f_h\|_{L^\infty} \|f_k\|_{L^1} < +\infty,$$

which gives the boundedness of $\|\Gamma_i(f)\|_{L^\infty}$. Furthermore:

$$|\Lambda_i(\mathbf{f}(x))| \leq C_\eta \|f_i\|_{L^\infty} \sum_{h,k=1}^{n} \|f_h\|_{L^1} < +\infty,$$

which ends the proof. □

Lemma 3.3.3. *Let η and B^i_{hk} be functions verifying (3.4), (3.6) and (3.7), then the function $\mathbf{J}$ is locally Lipschitz on X.*

Proof. Let $r > 0$ and $\mathbf{f}, \mathbf{g} \in X$ such that $\|\mathbf{f}\|_X, \|\mathbf{g}\|_X < r$. Then:

$$|J_i(\mathbf{f}(x)) - J_i(\mathbf{g}(x))| \leq |\Gamma_i(\mathbf{f}(x)) - \Gamma_i(\mathbf{g}(x))| + |\Lambda_i(\mathbf{f}(x)) - \Lambda_i(\mathbf{g}(x))|.$$

Now:

$$|\Gamma_i(\mathbf{f}(x)) - \Gamma_i(\mathbf{g}(x))| \leq \sum_{h,k=1}^{n} \Bigg| f_h(x) \int_{D_x} \eta(\mathbf{f}(y))B^i_{hk}(\mathbf{f}(y))f_k(y)dy$$

$$- g_h(x) \int_{D_x} \eta(\mathbf{g}(y))B^i_{hk}(\mathbf{g}(y))g_k(y)dy \Bigg|$$

$$\leq \sum_{h,k=1}^{n} |f_h(x) - g_h(x)| \int_{\mathbb{R}} |\eta(\mathbf{f}(y))B^i_{hk}(\mathbf{f}(y))f_k(y)|dy$$

$$+ \sum_{h,k=1}^{n} |g_h(x)| \int_{\mathbb{R}} |\eta(\mathbf{f}(y))B^i_{hk}(\mathbf{f}(y))f_k(y) - \eta(\mathbf{g}(y))B^i_{hk}(\mathbf{g}(y))g_k(y)|dy$$

$$= I_1(x) + I_2(x).$$

Reminding that, by hypothesis 3.7, the quantities B^i_{hk} are also locally Lipschitz for all i, h, k with Lipschitz constants $C_{B^i_{hk}r}$ we find the following estimates:

$$I_1(x) \leq C_\eta |f_h(x) - f_k(x)| \int_{\mathbb{R}} f_k(y)dy$$

$$\leq C_\eta \sum_{h,k=1}^{n} \|f_k\|_{L^1} |f_h(x) - g_h(x)| \leq C_\eta nr \sum_{h=1}^{n} |f_h(x) - g_h(x)|,$$

from which we have:

$$\|I_1\|_{L^1} \leq C_\eta nr \|\mathbf{f} - \mathbf{g}\|_1, \tag{3.18}$$

$$\|I_1\|_{L^\infty} \leq C_\eta n^2 r \|\mathbf{f} - \mathbf{g}\|_\infty. \tag{3.19}$$

Concerning I_2, thanks to the Lipschitzianity of η and B^i_{hk} we have:

$$I_2(x) \leq \sum_{h,k=1}^{n} |g_h(x)| \left(\int_{\mathbb{R}} \left| \eta(\mathbf{f}(y)) B^i_{hk}(\mathbf{f}(y)) - \eta(\mathbf{g}(y)) B^i_{hk}(\mathbf{g}(y)) \right| |f_k(y)| dy \right.$$

$$\left. + \int_{\mathbb{R}} |B^i_{hk}(\mathbf{g}(y)) \eta(\mathbf{g}(y)) (f_k(y) - g_k(y))| dy \right)$$

$$\leq \sum_{h,k=1}^{n} |g_h(x)| \left(\|f_k\|_{L^\infty} \int_{\mathbb{R}} \left(L_\eta + L_{B^i_{hk}} C_\eta \right) |\mathbf{f}(y) - \mathbf{g}(y)| \, dy \right.$$

$$\left. + C_\eta \int_{\mathbb{R}} |f_k(y) - g_k(y)| dy \right) \leq rn \left(L_\eta + (1 + L_{\beta r}) C_\eta \right) \|\mathbf{f} - \mathbf{g}\|_1 \sum_{h=1}^{n} |g_h(x)|,$$

with $C_{\beta r} = \max_{i,h,k} C_{B^i_{hk}r}$, from which the estimates:

$$\|I_2\|_{L^1} \leq r^2 n \left(L_\eta + (1 + L_{\beta r}) C_\eta \right) \|\mathbf{f} - \mathbf{g}\|_1, \tag{3.20}$$

$$\|I_2\|_{L^\infty} \leq r^2 n^2 \left(L_\eta + (1 + L_{\beta r}) C_\eta \right) \|\mathbf{f} - \mathbf{g}\|_1. \tag{3.21}$$

follow. In a similar way we have:

$$|\Lambda_i(\mathbf{f})(x) - \Lambda_i(\mathbf{g})(x)| \leq \sum_{h=1}^{n} |f_i(x) - g_i(x)| \int_{\mathbb{R}} |\eta(\mathbf{f}(y)) f_h(y)| dy$$

$$+ \sum_{h=1}^{n} |g_i(x)| \int_{\mathbb{R}} |\eta(\mathbf{f}(y)) f_h(y) - \eta(\mathbf{g}(y)) g_h(y)| dy,$$

which leads to the estimates:

$$\|\Lambda_i(\mathbf{f}) - \Lambda_i(\mathbf{g})\|_{L^\infty} \leq C_\eta r \|\mathbf{f} - \mathbf{g}\|_\infty + \left(C_\eta r + L_\eta r^2\right) \|\mathbf{f} - \mathbf{g}\|_1, \tag{3.22}$$

and:

$$\|\Lambda_i(\mathbf{f}) - g_i \Lambda_i(\mathbf{g})\|_{L^1} \leq \left(2C_\eta r + L_\eta r^2\right) \|\mathbf{f} - \mathbf{g}\|_1. \tag{3.23}$$

Combining the previous inequalities we conclude the proof. □

The previous lemma is crucial in order to use a fixed point argument in the following theorem.

Theorem 3.3.4. *Let η and B^i_{hk} be functions verifying (3.4), (3.6), (3.7) and let $\overline{\mathbf{f}} \in X$. Then, there exists $T > 0$ and an unique $\mathbf{f} \in C([0,T], X)$ mild solution of (3.1).*

Proof. Let $\delta > 0$ such that $\|\overline{\mathbf{f}}\|_X \leq \delta$. We take $r := 1 + \delta$ and $0 < a \leq 1$ and define:

$$D_r(a) := \left\{ \mathbf{u} \in C([0,a], X) \,\middle|\, \|\mathbf{u}\|_* := \sup_{t \in [0,a]} \|\mathbf{u}(t)\|_X \leq r \right\}.$$

Let $\mathbf{u} \in C([0,a], X)$, We introduce the map:

$$\Phi(\mathbf{u})_i(t,x) := \overline{f}_i(\gamma_i(0,t,x)) + \int_0^t J_i(\mathbf{u}(\tau, \gamma_i(\tau,t,x)))(\tau, \gamma_i(\tau,t,x)) d\tau. \tag{3.24}$$

Clearly $\Phi(\mathbf{u}) \in C([0,a], X)$. If we take $\mathbf{u}, \mathbf{v} \in C([0,a], X)$ we easily find the following estimate:

$$\|\Phi(\mathbf{u})(t)\|_X \leq \delta + \int_0^t (\|\mathbf{J}(\mathbf{u}(\tau)) - \mathbf{J}(0)\|_X + \|\mathbf{J}(0)\|_X) ds,$$

and thus, reminding that $\mathbf{J}(0) = 0$,

$$\|\Phi(\mathbf{u})(t)\|_X \leq \delta + aL_r,$$

with $L_r > 0$ the Lipschitz constant of $\mathbf{J}$. Moreover:

$$\|\Phi(\mathbf{u})(t) - \Phi(\mathbf{v})(t)\|_X \leq \int_0^t \|\mathbf{J}(\mathbf{u}(\tau)) - \mathbf{J}(\mathbf{v}(\tau))\|_X d\tau \leq aL_r \|\mathbf{u} - \mathbf{v}\|_*,$$

and thus:

$$\|\Phi(\mathbf{u}) - \Phi(\mathbf{v})\|_* \leq aL_r \|\mathbf{u} - \mathbf{v}\|_*. \tag{3.25}$$

Then, for every $a \in (0, L_r/2]$, we have that $\Phi(\mathbf{u}) \in D_r(a)$ and that Φ is Lipschitz in $D_r(a)$ with Lipschitz constant less then or equal to $1/2$. Since $D_r(a)$ is a complete metric space with respect to the metric induced by $\|\cdot\|_{L^\infty}$, we have the existence and uniqueness of a fixed point $\mathbf{f} = \Phi(\mathbf{f}) \in D_r(a)$, which is also solution of (3.1). □

Theorem 3.3.4 states that if the initial datum $\overline{\mathbf{f}}$ belongs to X, then there is a mild solution of (3.1), overlooking that we are interested to positive initial data. Now, let us suppose that the initial datum $\overline{\mathbf{f}} \in X$ is positive, i.e. $\overline{f}_i(x) \geq 0$ for all $x \in \mathbb{R}$ and for $i = 1, \ldots, n$. Reminding that the total number of agents at time t is:

$$N(t) := \int_{\mathbb{R}} \rho(t,x)dx = \int_{\mathbb{R}} \left(\sum_{i=1}^{n} f_i(t,x) \right) dx, \tag{3.26}$$

integrating on x the n equations of (3.1), summing over i and taking account of (3.5), we have:

$$\sum_{i=1}^{n} \int_{\mathbb{R}} (\partial_t f_i(t,x) + v_i \partial_x f_i(t,x))dx = 0, \tag{3.27}$$

from which:

$$\sum_{i=1}^{n} \left(\int_{\mathbb{R}} \partial_t f_i(t,x)dx + v_i[f_i(t,x))]_{-\infty}^{+\infty} \right) = \sum_{i=1}^{n} \int_{\mathbb{R}} \partial_t f_i(t,x)dx = 0, \tag{3.28}$$

and finally:

$$\frac{dN}{dt}(t) = 0, \tag{3.29}$$

that is, the total vehicles' number is conserved and is equal to its initial value N_0.
The following proposition shows that if the system has non-negative initial data then the solution remains non-negative.

Lemma 3.3.5. *Let $vf \in C([0,T], X)$ be a mild solution to problem (3.1) corresponding to non-negative initial data. Then, $\mathbf{f}$ remains non-negative for all $0 \leq t \leq T$.*

Proof. If $\overline{\mathbf{f}}$ is a non-negative initial datum, then $\overline{\mathbf{f}}(x) \geq 0$ for all $x \in \mathbb{R}$. Given a $x \in \mathbb{R}$ such that $\overline{f}_i(x) > 0$ for all indexes then the solution components along the characteristics remain positive for a certain time interval. If, on the contrary, $\overline{f}_i(x) = 0$ for $i \in L \subseteq \{1, \ldots, n\}$, then, for these indexes,

$$J_i(\overline{\mathbf{f}}(x)) = \Gamma_i(\overline{\mathbf{f}}(x)) \geq 0,$$

and thus along characteristics they are increasing functions, while the remaining components remains positive for a certain time interval, always along characteristics. □

The previous proposition, together with the conservation of the total number of vehicles, ensures that:

$$\|\mathbf{f}(t)\|_1 = \sum_{i=1}^{n} \int_{\mathbb{R}} |f_i(t,x)|dx = \sum_{i=1}^{n} \int_{\mathbb{R}} f_i(t,x)dx = N_0. \tag{3.30}$$

By Theorem 3.3.4 we established existence and uniqueness of local in-time solutions to (3.1). Putting:

$$T_{\bar{\mathbf{f}}} := \sup \{ T > 0 \,|\, \exists \mathbf{u} \in C([0,T], X) \text{ solution of (3.1)} \},$$

then, stitching together the solutions, we obtain the existence of a unique maximal mild solution $\mathbf{u} \in C([0, T_{\bar{\mathbf{f}}}), X)$. Our next goal is to prove that for system (3.1) with positive initial data we actually have $T_{\bar{\mathbf{f}}} = +\infty$.
Next proposition is easily derived from well known results of semigroups theory.

Lemma 3.3.6. *Let* $\mathbf{f} \in C([0, T_{\bar{\mathbf{f}}}), X)$ *the unique mild solution of (3.1). If* $T_{\bar{\mathbf{f}}} < +\infty$ *then:*

$$\lim_{t \to T_{\bar{\mathbf{f}}}} \|\mathbf{f}(t)\|_X = +\infty$$

Proof. First of all we observe that the length T of the existence time interval $[0, T]$ given by Theorem 3.3.4 is only related to the norm δ of initial data. Let $T_{\bar{\mathbf{f}}} < \infty$ and $\{t_n\}$ a sequence of instants such that $t_n < T_{\bar{\mathbf{f}}}$ and $t_n \to T_{\bar{\mathbf{f}}}$. Assume that $C := \sup_n \|\mathbf{u}(t_n)\|_X < \infty$. Then, from Theorem 3.3.4 there exists a $T_C > 0$ such that we have a unique solution $\mathbf{f} \in C([0, T_C], X)$ to (3.1) for all initial data with $\|\bar{\mathbf{f}}\| \leq C$. Now we fix an index $n \in \mathbb{N}$ such that $t_n + T_C > T_{\bar{\mathbf{f}}}$ and consider the solution $\mathbf{f} \in C([0, t_n], X)$ to (3.1) as well as the solution $\mathbf{f}_n \in C([0, T_C], X)$ corresponding to the initial value $\mathbf{f}(t_n)$. Pasting together these latter we obtain a solution defined in $[0, t_n + T_C]$, contradicting the property of $T_{\bar{\mathbf{f}}}$. Then we necessarily have that $\|\mathbf{f}(t)\|_X \to \infty$ when $t \to T_{\bar{\mathbf{f}}}$. □

Thanks to the previous Lemma we have only to verify that:

$$\lim_{t \to T_{\bar{\mathbf{f}}}} \|\mathbf{u}(t)\|_X < +\infty.$$

Moreover, since (3.30) says us that norm $\|\cdot\|_1$ is constant in time, we restrict our considerations to find a bound for the norm $\|\cdot\|_\infty$, reminding that η and

B^i_{hk} are both bounded functions. We have:

$$
\begin{aligned}
|f_i(t,x)| = f_i(t,x) &= \overline{f}_i(\gamma_i(0,t,x)) + \int_0^t J_i(\mathbf{f}(\tau,\gamma_i(\tau,t,x)))d\tau \\
&\leq \overline{f}_i(\gamma_i(0,t,x)) + \int_0^t \Gamma_i(\mathbf{f}(\tau,\gamma_i(\tau,t,x)))d\tau \\
&\leq \|\overline{f}_i\|_{L^\infty} + \sum_{h,k=1}^n C_\eta \int_0^t f_h(\tau,\gamma_i(\tau,t,x)) \int_{D_{\gamma_i(\tau,t,x)}} f_k(\tau,y)dyd\tau \\
&\leq \|\overline{f}_i\|_{L^\infty} + \sum_{h=1}^n C_\eta \int_0^t f_h(\tau,\gamma_i(\tau,t,x)) \int_{\mathbb{R}} \sum_{k=1}^n f_k(\tau,y)dyd\tau \\
&\leq \|\overline{\mathbf{f}}\|_\infty + C_\eta \int_0^t N(\tau) \sum_{h=1}^n f_h(\tau,\gamma_i(\tau,t,x))d\tau \\
&\leq \|\overline{\mathbf{f}}\|_\infty + C_\eta N_0 \int_0^t \sum_{h=1}^n \|f_h(\tau)\|_{L^\infty} d\tau \\
&\leq \|\overline{\mathbf{f}}\|_\infty + nC_\eta N_0 \int_0^t \|\mathbf{f}(\tau)\|_\infty d\tau,
\end{aligned}
$$

and thus:

$$\|\mathbf{f}(t)\|_\infty \leq \|\overline{\mathbf{f}}\|_\infty + nC_\eta N_0 \int_0^t \|\mathbf{f}(\tau)\|_\infty d\tau.$$

Using the Gronwall's lemma we find:

$$\|\mathbf{f}(t)\|_\infty \leq \|\overline{\mathbf{f}}\|_\infty \exp\left(nC_\eta N_0 t\right),$$

which gives us the bound for $\|\mathbf{f}\|_\infty$. This ends the proof of following theorem.

Theorem 3.3.7. *Let η and B^i_{hk} be functions verifying (3.4), (3.6), (3.7), D_x a subset of $\mathbb{R}$ verifying (3.8) and let $\overline{\mathbf{f}} \in X$ a positive initial data. Then, there exists a unique mild solution $\mathbf{f} \in C([0,+\infty), X)$ to (3.1).*

It is worth to remark that while in literature it is mainly assumed that the interaction rate and the transition probability density depend only on the density ρ, the previous result encompasses models in which they can depend on $\mathbf{f}$ in a more general way, for example through the flux q.

3.4 Remarks on the critical density

In this section we discuss a property strictly related to vehicular traffic flow. It is experimentally reported (se, e.g., [34]) the presence of a phase transition

between free and congested traffic flow, with the vehicular density playing the role of an order parameter. It is also well reported in [26, 30] and in section 3.2 that this phase transition is observed in spatially homogeneous systems like (3.1) as well.
Amazing thing seen in section 3.2 is that in spatially homogeneous case we find critical density ρ_c visible also in experimental data [34], if density is lower than critical one we have free flow and all agents achieve larger velocity, if density is larger than critical one flow is congested, and number of agents traveling with larger velocity decrease. When problem is not spatially homogeneous, this critical density is not evident, some authors impose existence of a critical density. In [14] authors impose a change in table of games, while in [6], authors study the initial value problem (3.1) where the interaction term $\mathbf{J}$ differs from that defined in (3.2), (3.3) for the presence of the function χ, defined as:

$$\chi(\rho \leq \rho_c) = \begin{cases} 1 \text{ if } \rho \leq \rho_c, \\ 0 \text{ if } \rho > \rho_c, \end{cases} \tag{3.31}$$

where ρ_c is the critical density. Authors of [6] justify the introduction of the cut-off function (3.31) observing that the vehicles are obliged to stop when the traffic density reaches a critical value, due to overcrowding, as is largely shown in the experimental fundamental diagrams speed-density. On the other hand, it is apparent that when $\rho > \rho_c$, if we have an interaction kernel like the one used in [6], all the vehicles in each velocity class continue to move at their own speed, i.e., each component $f_i(t, x)$ of the distribution function is transported along the corresponding characteristic, leading to the unrealistic situation in which at high density the vehicles are transparent each other.

Aim of this final part is to show, by a numerical example, that situations can occur in which the assumption that the interactions freeze at high density seems to be restrictive and that there isn't any threshold or critical density. In the sequel we furnish a toy model in which, given an arbitrary $\rho_c > 0$, we are able to find B^i_{hk} and η, an interaction interval $D_x = [x, x + \Delta]$ and initial data $\overline{f}$, with:

$$\|\overline{f}\|_\infty \leq \rho_c, \qquad \overline{\rho}(x) = \sum_{i=1}^{n} \overline{f}_i(x) \leq \rho_c, \quad \forall x \in \mathbb{R},$$

for which there exist $t^* > 0$ and $x^* \in \mathbb{R}$ such that $\rho(t^*, x^*) > \rho_c$. We consider a two velocities system, that is, we assume that every vehicle belongs to one of the two possible velocity classes f_1, corresponding to $v_1 = 0$, and f_2 that corresponds to $v_2 = 1$. The transition probability densities for this case model are assumed to be:

$$B^1_{hk} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \; B^2_{hk} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad h, k = 1, 2.$$

and $\eta(\rho) = 1$. Hence the first line of (3.1) specializes in:

$$\begin{cases} \partial_t f_1(t,x) = f_2(t,x) \displaystyle\int_{D_x} f_1(t,y)dy, \\ \partial_t f_2(t,x) + \partial_x f_2(t,x) = -f_2(t,x) \displaystyle\int_{D_x} f_1(t,y)dy. \end{cases} \tag{3.32}$$

We solve numerically the Cauchy problem for (3.32) relative to initial data $\overline{f}_1(x)$, $\overline{f}_2(x)$ having compact support and such that for any $y \in \mathrm{supp}(\overline{f}_1)$ and $x \in \mathrm{supp}(\overline{f}_2)$ then $x \leq y$. Specifically, we choose initial data as in Figure 3.2. A group of vehicles, corresponding to the component $\overline{f}_1(x)$ of

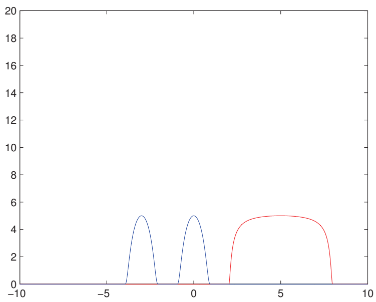

Figure 3.2: Initial data corresponding to $\rho_c = 5$. The class $\overline{f}_1$ of velocity $v_1 = 0$ is on the right in red color, the class $\overline{f}_2$ of velocity $v_2 = 1$ is represented by the bimodal distribution on the left in blue color.

the initial distribution function $\overline{f}$ is at rest ahead of a group $\overline{f}_2(x)$ of vehicles that is moving with velocity $v = 1$. The prescribed value of the "critical" density ρ_c is 5. In Table 1 we analyze the role of the size Δ of the interaction domain D_x. The maximum ρ_{t^*} of the equilibrium density increases as Δ decreases, and is reached in a time t^* that increases with increasing Δ.
In 3.3 final configurations for different values of Δ are plotted for the same initial data of Figure 3.2. Again, we stress that, if we take Δ small eough, the critical density is passed.
In summary, the well-posedness result furnishes global existence and uniqueness of solutions to (3.1) in its general form, in any traffic condition, existence of a critical density is clear where we have a spatially homogeneous case while in general case it is not so clear existence of critical density.

Δ	t^*	ρ_{t^*}
1.05	4.5425	5.1890
1.00	4.5885	5.2214
0.95	4.6365	5.2660
0.90	4.6855	5.3269
0.85	4.7365	5.4092
0.80	4.7925	5.5192
0.75	4.8555	5.6652
0.70	4.9255	5.8575
0.65	5.0025	6.1090
0.60	5.4595	6.4658

Δ	t^*	ρ_{t^*}
0.55	5.4985	6.9726
0.50	5.5406	7.5731
0.45	5.5786	8.3056
0.40	5.6106	9.2196
0.35	5.6346	10.3606
0.30	5.6536	11.7122
0.25	5.6706	13.0508
0.20	5.6846	13.7962
0.15	5.6926	13.3582

Table 3.1: Maximum ρ_{t^*} of the density reached at time t^* as function of the size Δ of the interaction domain D_x. Observe that in any case ρ_{t^*} is greater that ρ_c.

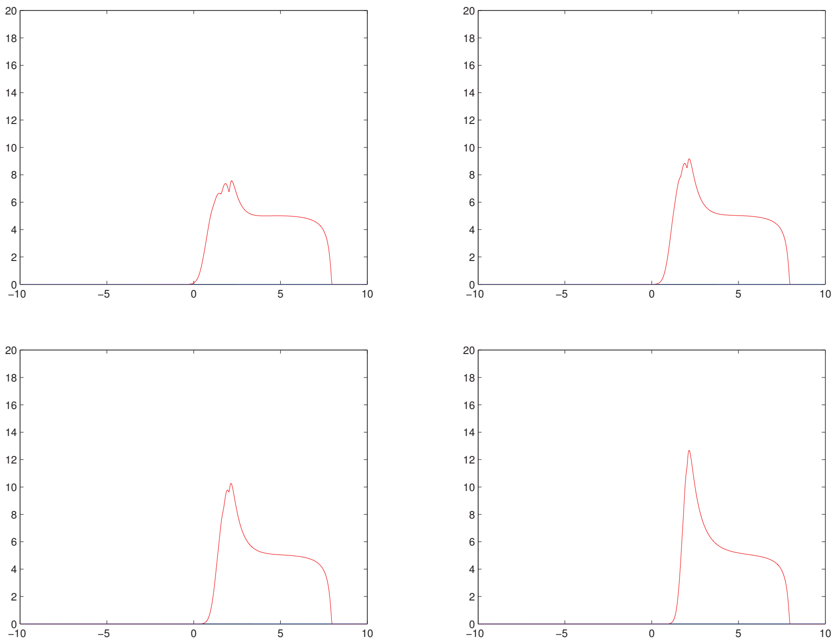

Figure 3.3: Final distribution functions corresponding to $\Delta = 0.5, 0.4, 0.35, 0.25$, and initial data as in Figure 3.2.

Chapter 4

Two dimensional problems

In this chapter we specialize system (2.6) to a two dimensional domain [9]. We introduce a discrete activity variable $u_\alpha \in \{u_1, \dots, u_m\} \subset \mathbb{R}$ and observe that in general the interaction domain is related not only to space variable $\mathbf{x}$ but also to the direction of velocity $\mathbf{v}_i$. In the present case the distribution function has the form:

$$\mathbf{f} = (f_{i\alpha}(t, \mathbf{x})) : [0, T] \times \mathbb{R}^2 \to \mathbb{R}^{nm},$$

where $i \in \{1, \dots, n\}$ is an index describing the velocity class, while the Greek index stays for the activity $\alpha \in \{1, \dots, m\}$. Hence, $f_{i\alpha}(t, x)d\mathbf{x}$ represents the number of agents traveling at velocity $\mathbf{v}_i$ with activity u_α in the infinitesimal volume $[\mathbf{x}, \mathbf{x} + d\mathbf{x}]$. The evolution equation (2.6) is written as:

$$\begin{aligned}
\partial_t f_{i\alpha}(t, \mathbf{x}) &+ \mathbf{v}_i \cdot \nabla_x f_{i\alpha}(t, \mathbf{x}) = J_{i\alpha}(\mathbf{f}(t, \mathbf{x})) \\
&= \Gamma_{i\alpha}(\mathbf{f}(t, \mathbf{x})) - \Lambda_{i\alpha}(\mathbf{f}(t, \mathbf{x})) \\
&= \sum_{h,k=1}^{n} \sum_{\alpha,\beta=1}^{m} \int_{D^i_{\mathbf{x}}} \eta(\mathbf{f}(t, \mathbf{y})) B^{i\alpha}_{h\beta,k\gamma}(\mathbf{f}(t, \mathbf{y})) f_{h\beta}(t, \mathbf{x}) f_{k\gamma}(t, \mathbf{y}) w(\mathbf{x}, \mathbf{y}) d\mathbf{y} \\
&\quad - f_{i\alpha}(t, \mathbf{x}) \sum_{h=1}^{n} \sum_{\alpha=1}^{m} \int_{D^i_{\mathbf{x}}} \eta(\mathbf{f}(t, \mathbf{y})) f_{h\beta}(t, \mathbf{y}) w(\mathbf{x}, \mathbf{y}) d\mathbf{y},
\end{aligned} \tag{4.1}$$

with obvious differences with respect to the previous chapter. Here:

- $B^{i\gamma}_{h\alpha,k\beta}$ is the probability density that an agents with velocity $\mathbf{v}_h$ and activity u_β falls into a new state $(\mathbf{v}_i, u_\alpha)$ after an interaction with an agents that has velocity $\mathbf{v}_k$ and activity u_γ. We suppose that $B^{i\gamma}_{h\alpha,k\beta}$ is locally Lipschitz for all $h, k, i = 1, \dots, n$ and $\alpha, \beta, \gamma = 1, \dots, m$, i.e, that for all $r > 0$ there exists a $C_{B^{i\gamma}_{h\alpha,k\beta} r} > 0$ such that:

$$\begin{gathered}
|B^{i\gamma}_{h\alpha,k\beta}(\mathbf{f}_1) - B^{i\gamma}_{h\alpha,k\beta}(\mathbf{f}_2)| \leq C_{B^{i\gamma}_{h\alpha,k\beta} r} |\mathbf{f}_1 - \mathbf{f}_2|, \\
\forall \mathbf{f}_1, \mathbf{f}_2 \in \mathbb{R}^{nm} \text{ with } |\rho_1 - \rho_2| < r,
\end{gathered} \tag{4.2}$$

and it verifies

$$B_{h\alpha,k\beta}^{i\gamma}(\rho) \geq 0, \quad \sum_{i=1}^{n}\sum_{\alpha=1}^{m} B_{h\alpha,k\beta}^{i\gamma}(\rho) = 1, \ \forall \mathbf{f} \in \mathbb{R}^{nm}, \tag{4.3}$$

which ensures the conservation of mass. As it will be useful in the sequel, we put

$$C_{Br} = \sup_{i,h,k,\alpha,\beta,\gamma} C_{B_{h\alpha,k\beta}^{i\gamma} r}.$$

- The interaction rate η is Lipschitz continuous and bounded, i.e., there exist $L_\eta > 0$ and $C_\eta > 0$ such that:

$$\begin{aligned} &|\eta(\mathbf{f}_1) - \eta(\mathbf{f}_2)| \leq L_\eta |\mathbf{f}_1 - \mathbf{f}_2|, \quad \forall \mathbf{f}_1, \mathbf{f}_2 \in \mathbb{R}^n, \\ &|\eta(\mathbf{f})| \leq C_\eta, \quad \forall \mathbf{f} \in \mathbb{R}^n. \end{aligned} \tag{4.4}$$

- $D_{\mathbf{x}}^i$ represents the spatial domain of interaction of an agents in $\mathbf{x} \in D_{\mathbf{x}}^i$ traveling with velocity $\mathbf{v}_i$. Such a domain is not restricted to be bounded.

- The weight function w has the following properties

$$\begin{gathered} 0 \leq w(\mathbf{x}, \mathbf{y}) \leq C_w, \\ w(\mathbf{x}, \mathbf{y}) = w(\mathbf{y}, \mathbf{x}), \\ \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^2. \end{gathered} \tag{4.5}$$

We start to study the existence of a local mild solution of following Cauchy problem:

$$\begin{cases} \partial_t f_{i\alpha}(t,\mathbf{x}) + \mathbf{v}_i \cdot \partial_{\mathbf{x}} f_{i\alpha}(t,\mathbf{x}) = J_{i\alpha}(\mathbf{f}(t,\mathbf{x})) & i = 1,\ldots,n, \\ f_{i\alpha}(0,\mathbf{x}) = \overline{f}_{i\alpha}(\mathbf{x}), & \alpha = 1,\ldots,m. \end{cases} \tag{4.6}$$

4.1 Existence and uniqueness of solution

Characteristic curves satisfy the following equations

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}_i, \quad i = 1,\ldots,n,$$

and thus the i-th characteristic passing though $(\tau, \mathbf{y})$ is

$$\mathbf{x} = \gamma_i(t,\tau,\mathbf{y}) = \mathbf{y} + \mathbf{v}_i(t-\tau),$$

There isn't any difference between the characteristic related to $i\alpha$-th class of agents and the one related to $i\beta$-th class. Along characteristics the solution is written as

$$f_{i\alpha}(t,\mathbf{x}) = \overline{f}_{i\alpha}(\gamma_i(0,t,\mathbf{x})) + \int_0^t J_{i\alpha}(\mathbf{f}(\tau,\gamma_i(\tau,t,\mathbf{x})))d\tau. \tag{4.7}$$

In the previous chapter we saw that in order to use fixed point theorem to prove existence and uniqueness, a crucial point consists in showing that the interaction kernel is a locally Lipschitz function in a functional space that in the present case becomes:

$$X = (L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))^{nm}. \tag{4.8}$$

For $f \in X$ we define

$$\|f\|_X := \|f\|_1 + \|f\|_\infty = \sum_{i=1}^{n}\sum_{j=1}^{m} \|f_{i\alpha}\|_{L^1} + \sup_{i\alpha} \|f_{i\alpha}\|_{L^\infty}.$$

Lemma 4.1.1. *Let η, $B^{i\gamma}_{h\alpha,k\beta}$ and w be functions verifying (4.2), (4.3), (4.4) and (4.5). Then, the function $\mathbf{J}$ defined in the r.h.s. of (4.1) maps X into itself. Moreover, it is locally Lipschitz in X.*

Proof. Let $f \in X$. We have to check that

$$\|\mathbf{J}(\mathbf{f})\|_X = \|\mathbf{J}(\mathbf{f})\|_1 + \|\mathbf{J}(\mathbf{f})\|_\infty < \infty.$$

We start to consider the first part $\|\cdot\|_1$

$$\int_{\mathbb{R}^2} |J_{i\alpha}(\mathbf{f})| \leq \int_{\mathbb{R}^2} |\Gamma_{i\alpha}(\mathbf{f})| + \int_{\mathbb{R}^2} |\Lambda_{i\alpha}(\mathbf{f})|,$$

$$\begin{aligned}\int_{\mathbb{R}^2} |\Gamma_{i\alpha}(\mathbf{f})|d\mathbf{x} &\leq C_w \sum_{h,k=1}^{n}\sum_{\beta,\gamma=1}^{m} \|f_{h\beta}\|_{L^1} \int_{\mathbb{R}^2} |\eta(\mathbf{f}(\mathbf{y}))A^{i\alpha}_{h\beta,k\gamma}(\mathbf{f}(\mathbf{y}))f_{k\gamma}(\mathbf{y})|d\mathbf{y} \\ &\leq C_\eta C_w \sum_{h,k}^{n}\sum_{\beta,\gamma=1}^{m} \|f_{h\beta}\|_{L^1}\|f_{k\gamma}\|_{L^1} < \infty.\end{aligned}$$

In the same way we find

$$\int_{\mathbb{R}^2} |\Lambda_{i\alpha}(\mathbf{f}(\mathbf{x}))|d\mathbf{x} \leq C_w C_\eta \|f_{i\alpha}\|_{L^1} \sum_{h=1}^{n}\sum_{\beta=1}^{m} \|f_{h\beta}\|_{L^1} < \infty,$$

which gives the boundedness of $\|\mathbf{J}(\mathbf{f})\|_1$. On the other hand

$$|\Gamma_{i\alpha}(\mathbf{f}(\mathbf{x}))| \leq C_w C_\eta \sum_{h,k=1}^{n} \sum_{\beta,\gamma=1}^{m} \|f_{h\beta}\|_{L^\infty} \|f_{k\gamma}\|_{L^1} < \infty,$$

and

$$|\Lambda_{i\alpha}(\mathbf{f}(\mathbf{x}))| \leq C_w C_\eta \|f_{i\alpha}\|_{L^\infty} \sum_{h=1}^{n} \sum_{\beta=1}^{m} \|f_{h\beta}\|_{L^1} < \infty,$$

which gives the boundedness of $\|\mathbf{J}(\mathbf{f})\|_\infty$. In order to check that $\mathbf{J}$ is locally Lipschitz, let $r > 0$ and $\mathbf{f}, \mathbf{g} \in X$ such that $\|\mathbf{f}\|_X, \|\mathbf{g}\|_X < r$,

$$|J_{i\alpha}(\mathbf{f})(\mathbf{x}) - J_{i\alpha}(\mathbf{g})(\mathbf{x})| \leq |\Gamma_{i\alpha}(\mathbf{f}(\mathbf{x})) - \Gamma_{i\alpha}(\mathbf{g}(\mathbf{x})| + |\Lambda_{i\alpha}(\mathbf{f}(\mathbf{x})) - \Lambda_{i\alpha}(\mathbf{g}(\mathbf{x}))|,$$

$$|\Gamma_{i\alpha}(\mathbf{f}(\mathbf{x})) - \Gamma_{i\alpha}(\mathbf{g}(\mathbf{x}))|$$

$$\leq \sum_{h,k=1}^{n} \sum_{\beta,\gamma=1}^{m} \left| f_{h\beta}(\mathbf{x}) \int_{D^i_{\mathbf{x}}} \eta(\mathbf{f}(\mathbf{y})) B^{i\gamma}_{h\alpha,k\beta}(\mathbf{f}(\mathbf{y})) f_{k\gamma}(\mathbf{y}) w(\mathbf{x}.\mathbf{y}) d\mathbf{y} \right.$$

$$\left. - g_{h\beta}(\mathbf{x}) \int_{D^i_{\mathbf{x}}} \eta(\mathbf{g}(y)) B^{i\gamma}_{h\alpha,k\beta}(\mathbf{g}(y)) g_{k\gamma}(\mathbf{y}) w(\mathbf{x},\mathbf{y}) d\mathbf{y} \right|$$

$$\leq C_w \sum_{h,k=1}^{n} \sum_{\beta,\gamma=1}^{m} \left(|f_{h\beta}(x) - g_{h\beta}(x)| \int_{\mathbb{R}^2} |\eta(\mathbf{f}(\mathbf{y})) B^{i\gamma}_{h\alpha,k\beta}(\mathbf{f}(\mathbf{y})) f_{k\gamma}(\mathbf{y})| d\mathbf{y} \right.$$

$$\left. + |g_{h\beta}(\mathbf{x})| \int_{\mathbb{R}^2} |\eta(\mathbf{f}(\mathbf{y})) B^{i\gamma}_{h\alpha,k\beta}(\mathbf{f}(\mathbf{y})) f_{k\gamma}(\mathbf{y}) - \eta(\mathbf{g}(y)) B^{i\gamma}_{h\alpha,k\beta}(\mathbf{g}(\mathbf{y})) g_{k\gamma}(\mathbf{y})| d\mathbf{y} \right),$$

and thus

$$|\Gamma_{i\alpha}(\mathbf{f}(\mathbf{x})) - \Gamma_{i\alpha}(\mathbf{g}(\mathbf{x}))| \leq C_w C_\eta r \sum_{h,k=1}^{n} \sum_{\beta,\gamma=1}^{m} |f_{h\beta}(\mathbf{x}) - g_{h\beta}(\mathbf{x})|$$

$$+ C_w C_\eta \sum_{h,k=1}^{n} \sum_{\beta,\gamma=1}^{m} |g_{h\beta}(\mathbf{x})| \int_{\mathbb{R}^2} |f_{k\gamma}(\mathbf{y}) - g_{k\gamma}(\mathbf{y})| d\mathbf{y}$$

$$+ C_w L_\eta r \sum_{h,k=1}^{n} \sum_{\beta,\gamma=1}^{m} |g_{h\beta}(x)| \int_{\mathbb{R}^2} |\mathbf{f}(\mathbf{y}) - \mathbf{g}(\mathbf{y})| d\mathbf{y}$$

$$+ C_{Br} C_\eta C_w r \sum_{h,k=1}^{n} \sum_{\beta,\gamma=1}^{m} |g_{h\beta}(\mathbf{x})| \int_{\mathbb{R}^2} |\mathbf{f}(\mathbf{y}) - \mathbf{g}(\mathbf{y})| d\mathbf{y}.$$

From this last inequality we find

$$\begin{aligned}\|\Gamma_{i\alpha}(\mathbf{f}) - \Gamma_{i\alpha}(\mathbf{g})\|_{L^\infty} \leq & 2mnC_\eta C_w r\|\mathbf{f}-\mathbf{g}\|_\infty \\ & + rmnC_w(2r(L_\eta + C_{Br}C_\eta)+1)\|\mathbf{f}-\mathbf{g}\|_1,\end{aligned} \tag{4.9}$$

$$\|\Gamma_{i\alpha}(\mathbf{f}) - \Gamma_{i\alpha}(\mathbf{g})\|_{L^1} \leq mnC_w[(mn+1)C_\eta + (L\eta + C_{Br}C_\eta)r]\|\mathbf{f}-\mathbf{g}\|_1. \tag{4.10}$$

In a similar way,

$$\begin{aligned}|\Lambda_{i\alpha}(\mathbf{f}(\mathbf{x})) - \Lambda_{i\alpha}(\mathbf{g}(\mathbf{x}))| \leq & C_\eta C_w r|f_{i\alpha}(x) - g_{i\alpha}(x)| \\ & + (L_\eta r + mnC_\eta)C_w|g_{i\alpha}(x)|\|\mathbf{f}-\mathbf{g}\|_1,\end{aligned}$$

and thus

$$\begin{aligned}\|\Lambda_{i\alpha}(\mathbf{f}) - \Lambda_{i\alpha}(\mathbf{g})\|_{L^\infty} \leq & mnC_\eta C_w r\|\mathbf{f}-\mathbf{g}\|_\infty \\ & + (L_\eta r + mnC_\eta)C_w r\|\mathbf{f}-\mathbf{g}\|_1,\end{aligned} \tag{4.11}$$

$$\|\Lambda_{i\alpha}(\mathbf{f}) - \Lambda_{i\alpha}(\mathbf{g})\|_{L^1} \leq +(L_\eta r + (mn+1)C_\eta)C_w r\|\mathbf{f}-\mathbf{g}\|_1. \tag{4.12}$$

From inequalities (4.9), (4.10), (4.11) and (4.12) we have local Lipschitzianity of $\mathbf{J}$, which ends the proof. □

We are now able to give an existence and uniqueness results.

Theorem 4.1.1. *Let* $\overline{\mathbf{f}} \in X$ *be an initial datum for (4.6). Then, there exist* $T > 0$ *and a unique* $\mathbf{f} \in C([0,T], X)$ *mild solution for (4.6). Moreover, if* $\overline{\mathbf{f}}$ *is a non-negative initial datum*

$$\overline{f}_{i\alpha}(\mathbf{x}) \geq 0, \quad \forall x \in \mathbb{R}^2, i = 1, \dots, n, \alpha, 1, \dots, m,$$

then the solution $\mathbf{f}$ *remains non-negative*

$$f_{i\alpha}(t, \mathbf{x}) \geq 0, \quad \forall (t, \mathbf{x}) \in [0,T] \times \mathbb{R}^2, i = 1, \dots, n, \alpha, 1, \dots, m.$$

The proof of the previous theorem is based on a fixed point argument and is completely similar to the one used for Theorem 3.3.4. Again, it isn't difficult to prolongate the solution up to $T = +\infty$.

Theorem 4.1.2. *Let* $\overline{\mathbf{f}} \in X$ *be a non-negative initial datum for (4.6), then there exists a unique non-negative mild solution* $\mathbf{f} \in C([0,\infty), X)$ *of the Cauchy problem (4.6).*

Proof. As in the proof of Theorem 3.3.7 we have only to verify that $\|\mathbf{f}(t)\|_X < \infty$ for all $t > 0$. Thanks to (4.3) we have that $\sum_i \mathbf{J}_i(\mathbf{f}) = 0$ and the conservation of agents' number

$$N(t) = \sum_{h=1}^{n} \int_{\mathbb{R}^2} f_h(t, \mathbf{x}) d\mathbf{x} = \sum_{h=1}^{n} \int_{\mathbb{R}^2} \overline{f}(\mathbf{x}) d\mathbf{x} = N(0).$$

Hence, due to non-negativity of the solution, the L^1-norm is not only bounded but also conserved,

$$\|\mathbf{f}(t)\|_1 = \sum_{h=1}^{n} \sum_{\beta=1}^{m} |f_{h\beta}(t, \mathbf{x})| d\mathbf{x} = N(0),$$

while for the L^∞-norm we use the integral equality (4.7),

$$\begin{aligned}
|f_{i\alpha}(t, \mathbf{x})| = f_{i\alpha}(t, \mathbf{x}) &= \overline{f}_{i\alpha}(\gamma_i(0, t, x)) + \int_0^t J_{i\alpha}(\mathbf{f}(\tau, \gamma_i(\tau, t, \mathbf{x}))) d\tau \\
&\leq \overline{f}_{i\alpha}(\gamma_i(0, t, \mathbf{x})) + \int_0^t \Gamma_{i\alpha}(\mathbf{f}(\tau, \gamma_i(\tau, t, \mathbf{x}))) d\tau \\
&\leq \|\overline{f}_{i\alpha}\|_{L^\infty} + C_\eta C_w \sum_{h,k=1}^{n} \sum_{\beta,\gamma=1}^{m} \int_0^t f_{h\beta}(\tau, \gamma_i(\tau, t, \mathbf{x})) \int_{D_{\gamma_i(\tau,t,\mathbf{x})}} f_{k\gamma}(\tau, \mathbf{y}) d\mathbf{y} d\tau \\
&\leq \|\overline{\mathbf{f}}\|_\infty + C_\eta C_w \int_0^t N(\tau) \sum_{h=1}^{n} \sum_{\beta=1}^{m} f_{h\beta}(\tau, \gamma_i(\tau, t, \mathbf{x})) d\tau \\
&\leq \|\overline{\mathbf{f}}\|_\infty + C_\eta C_w N_0 \int_0^t \sum_{h=1}^{n} \sum_{\beta=1}^{m} \|f_{h\beta}(\tau)\|_{L^\infty} d\tau \\
&\leq \|\overline{\mathbf{f}}\|_\infty + nm C_\eta C_w N_0 \int_0^t \|\mathbf{f}(\tau)\|_\infty d\tau,
\end{aligned}$$

and thus by using Gronwall's Lemma we find the inequality:

$$\|\mathbf{f}(t)\|_\infty \leq \|\overline{\mathbf{f}}\|_\infty \exp\left(nm C_\eta C_w N_0 t\right), \tag{4.13}$$

that ends the proof. □

The proof of previous theorem is basically equal to that of Theorem 3.3.7. In other words, the introduction of the weight function w and of an activity parameter does not create problem from the well-posedness viewpoint, except for a heavier notation.

4.2 Qualitative results

In this section we want to study system (4.6) in some easy cases. We are interested to solutions that show an emergent behavior induced by the presence of a desired velocity. In this section we once again forget the activity variable, while the set of velocities, fixed an $n \in \mathbb{N}$ with $n \leq 2$, is

$$v_i = (\cos \vartheta_i, \sin \vartheta_i),$$

with $\vartheta_i = \frac{2\pi(i-1)}{n}$. Agents can only change their directions, and for sake of simplicity the interaction rate is assumed to be constant $\eta = 1$. We study qualitatively the solutions with respect to changes in the interaction weight w.

Fixed an index $i^* \in \{1, \dots, n\}$, we start to study the simplest case, when the transition probability density is defined as:

$$B^i_{hk} = \begin{cases} 1 \text{ if } i = i^*, \\ 0 \text{ otherwise.} \end{cases} \tag{4.14}$$

In other words, there is a desired direction v_{i^*}, and agents try to achieve this latter unconditionally, taking no interest in other agents, while the change in direction is not gradual.

Theorem 4.2.1. *Let the transition probability density be as in (4.14). Then, for any initial datum $\overline{f} \in X^+$ we have*

$$\lim_{t \to \infty} \int_{\mathbb{R}^2} f_{i^*}(t,x) dx = m_0 =: \sum_{i=1}^{n} \int_{\mathbb{R}^2} \overline{f}_i(x) dx. \tag{4.15}$$

Proof. We remind that m_0 is the total mass, or number of agents, of the system and it conserved. If $m_0 = 0$ the limit (4.15) is obvious. If $m_0 > 0$ we start by evaluating the residual mass, that is, the number of agents that haven't reached v_{i^*}:

$$\int_{\mathbb{R}^2} f_{i^*}(t,x) dx = m_0 - \sum_{h \neq i^*} \int_{\mathbb{R}^2} f_h(t,x) dx.$$

Integrating the equality (4.7) over the whole spatial domain we find

$$\int_{\mathbb{R}^2} f_h(t,x) dx = \int_{\mathbb{R}^2} \overline{f}_h(x) dx - \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f_h(t,x) \sum_{k=1}^{n} f_k(t,y) w(w,y) dy dx d\tau.$$

The r.h.s. is a decreasing function of t, and thus it has to be

$$\lim_{t\to\infty} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f_h(t,x) \sum_{k=1}^{n} f_k(t,y) w(w,y) dy dx = 0,$$

otherwise it would be possible to have a negative mass. Reminding that the integrated function is positive, this implies

$$\lim_{t\to 0} f_h(t,x) \sum_{k=1}^{n} f_k(t,y) w(w,y) = 0, \tag{4.16}$$

a.e. $x \in \mathbb{R}^2, y \in \mathbb{R}^2$. Thanks to (4.16) with $y = x$ and to the obvious inequality

$$f_h^2(t,x) w(x,x) \le f_h(t,x) \sum_{k=1}^{n} f_k(t,x) w(x,x).$$

we have

$$\lim_{t\to\infty} f_h^2(t,x) w(x,x) = 0,$$

a.e. $x \in \mathbb{R}^2$, and thus,

$$\lim_{t\to\infty} f_h(t,x) = 0,$$

a.e. $x \in \mathbb{R}^2$ with $h \neq i^*$, that ends the proof. □

The previous theorem says us that if the transition probability density is as in (4.14), then there is a complete synchronization, i.e., all agents reach the velocity v_{i^*}.The behaviour doesn't change if we allow for a gradual velocity changing. Fixed $p \in]0,1]$, we modify the transition probability density as follows.
If $v_h \cdot v_{i^*} = -1$,

$$B_{hk}^i = \begin{cases} p/2 & \text{if } i = h+1, \\ p/2 & \text{if } i = h-1, \\ 1-p & \text{if } i = h, \\ 0 & \text{otherwise;} \end{cases} \tag{4.17}$$

if $v_h = v_{i^*}$,

$$B_{hk}^i = \begin{cases} 1 & \text{if } i = i^*, \\ 0 & \text{otherwise.} \end{cases} \tag{4.18}$$

These latter describe two opposite cases. In the first one is the candidate agent has the worst velocity, of opposite direction with respect to v_{i^*}, so he would absolutely change his velocity, while in the second one the candidate already moves at the desired velocity, so he doesn't change direction. In general, if $v_h \cdot v_{i^*} > v_k \cdot v_{i^*}$

$$B^i_{hk} = \begin{cases} 1 & \text{if } i = h, \\ 0 & \text{otherwise,} \end{cases} \tag{4.19}$$

in other words an interaction with a test agent that has the worse direction doesn't produce a change in the direction of candidate agents. If $v_h \cdot v_{i^*} \leq v_k \cdot v_{i^*}$,

$$B^i_{hk} = \begin{cases} 1-p & \text{if } i = h, \\ p & \text{if } i = h+1 \text{ and } v_h \cdot v_{i^*} \leq v_{h+1} \cdot v_{i^*}, \\ p & \text{if } i = h-1 \text{ and } v_h \cdot v_{i^*} \leq v_{h-1} \cdot v_{i^*}, \\ 0 & \text{otherwise,} \end{cases} \tag{4.20}$$

clearly, if $i = 1$ then $h - 1 := n$ and if $i = n$ then $h + 1 := 1$. This last includes also the case $v_h = v_k$, agents have the tendency to reach desired velocity and we describe this tendency by a gradual adaptation. Also in this case we have a qualitative result.

Theorem 4.2.2. *Let the transition probability density be as in (4.17),(4.18), (4.19) and (4.20). Then, for any initial datum $\overline{f} \in X^+$ we have*

$$\lim_{t\to\infty} \int_{\mathbb{R}^2} f_{i^*}(t,x)dx = m_0 =: \sum_{i=1}^{n} \int_{\mathbb{R}^2} \overline{f}_i(x)dx, \tag{4.21}$$

Proof. Thanks to (4.18) it is easy to verify that $\int_{\mathbb{R}^2} f_{i^*}(t,x)dx$ is an increasing function of time. First of all we prove that

$$\lim_{t\to\infty} \int_{\mathbb{R}^2} f_h(t,x)dx = 0.$$

If $h \neq i^*$, limit (4.21) follows from the conservation of agents' number. If there is an opposite velocity, and an associated index j such that $v_j \cdot v_{i^*} = -1$, then for (4.17)

$$\int_{\mathbb{R}^2} f_j(t,x)dx = \int_{\mathbb{R}^2} \overline{f}_j(x)dx - p\int_0^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f_j(\tau,x) \sum_{h=1}^{n} f_h(\tau,y)w(x,y)dxdtd\tau,$$

and using the same argument presented in the proof of Theorem 4.2.1 we have

$$\lim_{t\to\infty} f_j(t,x) = 0,$$

a.e. $x \in \mathbb{R}^2$. Forgetting the variables we have

$$\int f_{h-1} = \int \overline{f}_{j-1} + \frac{p}{2} \sum_h \iiint f_j f_h w - p \sum_{h \neq j} \iiint f_{i-1} f_h w,$$

and the same equality for f_{j+1}. Only agents having velocity v_j reach v_{j-1} (or v_{j+1}), on the other hand agents having v_{j-1} (or v_{j+1}) change their velocity with probability p reaching v_{j-2} (or v_{j+2}), the gain term goes to zero, so we can use the same argument of Theorem (4.2.1) to obtain,

$$\lim_{t\to\infty} f_{j+1}(t,x) = \lim_{t\to\infty} f_{j-1}(t,x) = 0,$$

a.e. $x \in \mathbb{R}^2$. We have found a recursion to obtain

$$\lim_{t\to\infty} f_h(t,x) = 0,$$

a.e. $x \in \mathbb{R}^2$, with $h \neq i^*$. This ends the proof. □

The crucial point in the previous two theorems is that an interaction between two agents having same velocity v_h gives rise to a possible change if $v_h \neq v_{i^*}$.

Theorem 4.2.3. *Let transition probability densities be as follows. Let j be an index such that*

$$B^i_{jj} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{for others } i \neq j. \end{cases} \tag{4.22}$$

Then, if the initial datum $\overline{f} \in X^+$ is such that

$$\overline{f}_h(x) = 0, \quad \forall x \in \mathbb{R}, \quad h \neq j,$$

the solution of the Cauchy problem (4.6) is a traveling wave

$$f_h(t,x) = \overline{f}_h(x - v_h t). \tag{4.23}$$

Proof. It is an easy computation to check that (4.23) is a solution of (4.7), and hence the unique solution of the Cauchy problem (4.6). □

The idea of previous theorem is quite simple: if all agents start to move with a velocity that in their opinion is good, they will not change it. The term B^i_{hh} describes also a self-interaction. If it is as in (4.22) then the velocity change is related only to the interaction between agents with different

velocities. On the other hand, if $B^h_{hh} < 1$, there is a self-interaction that produces an alteration of velocity. The previous two easy systems lead always to a synchronization independent from interaction weight w, produced by self-interactions of agents that haven't reached desired velocity. If we remove the self-interaction, then the shape of interaction function becomes important. From now on, fixed a desired velocity v_{i^*}, we work with the following transition probability density.
If $v_h = v_k$,

$$B^i_{hh} = \begin{cases} 1 & \text{if } i = h, \\ 0 & \text{otherwise,} \end{cases} \tag{4.24}$$

while if $v_h \cdot v_{i^*} > v_k \cdot v_{i^*}$ we maintain (4.19), and also if $v_h \cdot v_{i^*} < v_k \cdot v_{i^*}$ we maintain (4.20). Clearly, in this case Theorem 4.2.3 holds, and we have more information. In order to simplify the notation we call

$$m_h(t) := \int_{\mathbb{R}^2} f_h(t,x)dx, \quad \overline{m}_h := \int_{\mathbb{R}^2} \overline{f}_h(x)dx = m_h(0).$$

Thanks to positiveness of initial datum functions m_h is stricly related to L^1 norm of solution, in other words $m_h > 0$ say us that there are some agents travelling with velocity v_h.

Theorem 4.2.4. *Let B^i_{hk} be transition probability density like (4.24), (4.19) and 4.20 and let $w = c_w > 0$ be a constant weight function. Then, if $\overline{m}_{i^*} > 0$ the following limit*

$$\lim_{t\to\infty} m_{i^*}(t) = m_0 \tag{4.25}$$

holds.

Proof. If $\overline{m}_{i^*} = m_0$, then by Theorem 4.2.3 we have also the limit (4.25), while if $\overline{m}_h > 0$ for $h \neq i^*$, we start to estimate other densities from f_j with j such that $v_j \cdot v_{i^*} = -1$; if $\overline{m}_j = 0$ then

$$f_j(t,x) = 0, \quad \forall x \in \mathbb{R}^2, t > 0,$$

otherwise

$$m_j(t) \leq \overline{m}_j - c_w \frac{p}{2} \int_0^t m_j^2(\tau)d\tau,$$

and thus

$$\lim_{t\to\infty} m_j(t) = 0.$$

Now we are able to evaluate m_{j-1} and m_{j+1}.

$$m_{j-1}(t) \leq \overline{m}_{j-1} + c_w p \int_0^t \left(m_0 m_j(\tau) - \frac{1}{2} m_{j-1}^2(\tau) \right) d\tau,$$

and thus

$$\lim_{t\to\infty} m_{j-1}(t) = 0;$$

we can find the same limit for m_{j+1}, using the same iterative argument we have that,

$$\lim_{t\to\infty} m_h(t) = 0, \text{ if } h \neq i^*,$$

the agents' number ends the proof. □

A constant weight function doesn't take into account distance between agents. Reminding that it is unnatural, anyway the previous result gives us a convergence in case of $\overline{m}_{i^*} > 0$.

Theorem 4.2.5. *Let w be a weight function with bounded domain, fixed a pair of indexes such that $h \neq k$ there exists an initial datum $\overline{f}$ with $\overline{m}_h > 0$ and $\overline{m}_k > 0$ such that*

$$f_j(t,x) = \overline{f}_j(x - v_j t), \quad j = 1, \dots, n, \tag{4.26}$$

is a solution of the Cauchy problem (4.6).

Proof. Fixed two indexes we have to construct a suitable initial datum. We define

$$A(x) = \overline{\{y \in \mathbb{R}^2 | w(x,y) \neq 0\}} \subset \mathbb{R}^2,$$

the support of w once we have fixed x and

$$\delta(x) = \text{diam} A(x) < \infty,$$

clearly δ is a constant function $\delta(x) = \delta_0$. Fixed $\varepsilon > 0$, we define

$$\overline{f}_h(x) := \chi_{B_\varepsilon(x_h)}(x), \quad \overline{f}_k(x) := \chi_{B_\varepsilon(x_k)}(x),$$

and $\overline{f}_i(x) = 0$ if $i \neq h$, $i \neq k$, where $x_h = nv_h$ and $x_k = nv_k$ and $n \in \mathbb{N}$ such that $B_{\varepsilon+\delta}(x_h) \cap B_{\varepsilon+\delta}(x_k) = \emptyset$. Then it is easy to compute that:

$$\overline{m}_h = \overline{m}_k = \pi\varepsilon^2 > 0,$$

and (4.26) is a solution of (4.6). This concludes the proof. □

Theorem 4.2.6. *Let w be a weight function with bounded domain, then there exists an initial datum $\overline{f}$, with $\overline{m}_j > 0$ for all $j = 1, \dots, n$, such that*

$$f_j(t,x) = \overline{f}_j(x - v_j t), \quad j = 1, \dots, n, \tag{4.27}$$

is a solution of the Cauchy problem (4.6).

Proof. We generalize the same argument used in the proof of Theorem 4.2.5. □

These last two Theorems say us that if the support of the weight function is bounded, or rather if the interaction domain is bounded, then the agents do not interact with the whole population, it is possible to construct configurations of system in which the agents do not interact each other.

Chapter 5

Generalizations

In this chapter we make some remarks on results found in previous chapters, in particular on the functional spaces where solutions are searched, and on the interaction kernel of kinetic equations previously studied.

5.1 Periodic initial data

In relation to the study of vehicular traffic, especially in the one dimensional case, it is of great importance to work with a bounded spatial domain, even from an experimental viewpoint. In order to deal with this task we introduce a *periodic problem*. If the initial datum is periodic ,then there exists a $L > 0$ such that

$$\overline{\mathbf{f}}(x + L) = \overline{\mathbf{f}}(x), \quad \forall x \in \mathbb{R}.$$

Clearly the norm $\|\cdot\|_1$ related to the number of agents is in general unbounded if we consider the whole $\mathbb{R}$ as the integration domain. On the other hand, fixed a (t, x) in the $t - x$ space, with $t > 0$, we define the domain of influence of (t, x) as

$$\mathcal{D}_{t,x} = \{(\tau, \xi)|\tau \geq 0, \Pi_{ij}(\tau, \xi) \leq 0, i, j = 1, \ldots, n, i \neq j\},$$

where Π_{ij} is the hyperplane generated by i-th and j-th characteristics passing through (t, x),

$$\gamma_i(t, \tau, \xi) = \xi + v_i(t - \tau),\ \gamma_j(t, \tau, \xi) = \xi + v_j(t - \tau).$$

The solution in (t, x) depends only on the values attained into the domain of influence $\mathcal{D}_{t,x}$. Moreover, it depends only on initial data $\mathcal{D}_{t,x} \cap \{t = 0\}$. Thanks to this observation, it is obvious to see that if the initial data are independent of one of the space variables, then the solution of the problem

(3.1) is itself independent of this variable. Similarly, if the initial data are periodic functions, then the solution is also periodic with the same period. Hence, if we introduce the space X_L of L-periodic functions, with

$$\|\mathbf{u}\|_{X_L} = \|\mathbf{u}\|_{1,L} + \|\mathbf{u}\|_{\infty,L} = \max_i \sup_{x\in[0,L]} |u_i(x)| + \sum_{h=1}^{n} \int_0^L |u_i(x)|dx,$$

we have no problem in studying the Cauchy problem (3.1) with $\overline{\mathbf{f}} \in X_L$. We have only to do one remark: given the interaction domain $D_x = [x - \Delta^-, x + \Delta^+]$, then

$$\Delta^+ + \Delta^- \leq L. \tag{5.1}$$

Lemma 5.1.1. *Let η, B^i_{hk} and w be functions verifying (3.4), (3.6), (3.7) and (4.5), let D_x satisfying (5.1), then function $\mathbf{J}$ maps X_L into itself, moreover is locally Lipschitz on X_L.*

The proof of previous lemma is equal to the one given for lemma 3.3.3. We remind only that we have to replace integrals over entire $\mathbb{R}$ with integral over $[0, L]$. Once we have local Lipschitzianity, it easy to prove prove following theorem.

Theorem 5.1.1. *Let η, B^i_{hk} and w be functions verifying (3.4), (3.6), (3.7) and (4.5), let D_x satisfying (5.1). Then, given $\overline{\mathbf{f}} \in X_L$, there exists a $T > 0$ and a unique $\overline{f} \in C([0,T], X_L)$ mild solution of (3.1). Moreover, if $\overline{\mathbf{f}}$ is non-negative, i.e. $\overline{f}_i(x) \geq 0$ for all $i = 1, \ldots, n$ and for all $x \in \mathbb{R}$, then the mild solution is non-negative for all $t \geq 0$ and $T = +\infty$.*

5.2 Unbounded interaction rate

In this section we give an existence result related to problem (4.6), making different hypotheses on η. Previously, we have worked with Lipschitz and bounded interaction rate η,

$$\begin{gathered} |\eta(\mathbf{f}_1) - \eta(\mathbf{f}_2)| \leq L_\eta |\mathbf{f}_1 - \mathbf{f}_2|, \quad \forall \mathbf{f}_1, \mathbf{f}_2 \in \mathbb{R}^n, \\ |\eta(\mathbf{f})| \leq C_\eta, \quad \forall \mathbf{f} \in \mathbb{R}^n. \end{gathered} \tag{5.2}$$

Forgetting boundedness, in this section we suppose that interaction rate is only a Lipschitz function, and thus there exists a constant $L_\eta > 0$ such that

$$|\eta(\mathbf{f}_1) - \eta(\mathbf{f}_2)| \leq L_\eta |\mathbf{f}_1 - \mathbf{f}_2|, \quad \forall \mathbf{f}_1, \mathbf{f}_2 \in \mathbb{R}^n. \tag{5.3}$$

Though an unbounded interaction rate could seem not so physically consistent, on the other hand it is pretty interesting from a mathematical viewpoint. Lipschitzianity implies also that the interaction rate is sublinear

$$|\eta(\mathbf{f})| \leq L_\eta|\mathbf{f}| + \eta_0, \tag{5.4}$$

where $\eta_0 = \eta(\mathbf{0})$. Like in the previous chapter we look for solutions in $(X, \|\cdot\|_X)$.

Lemma 5.2.1. *Let B^i_{hk} be functions satisfying (3.4) and (3.7) for all indexes, let η and w be functions satisfying respectively (5.3) and (4.5). Then, the function $\mathbf{J} = (J_i(\mathbf{f}))$, defined as:*

$$\begin{aligned} J_i(\mathbf{f}) :=& \Gamma_i(\mathbf{f}) - \Lambda_i(\mathbf{f}) \\ =& \sum_{h,k=1}^{n} \int_{D^i_{\mathbf{x}}} \eta(\mathbf{f}(y)) B^i_{hk}(\mathbf{f}(\mathbf{y})) f_h(\mathbf{x}) f_k(\mathbf{y}) w(\mathbf{x},\mathbf{y}) d\mathbf{y} \\ &- f_i(\mathbf{x}) \sum_{h=1}^{n} \int_{D^i_{\mathbf{x}}} \eta(\mathbf{f}(\mathbf{y})) f_h(\mathbf{y}) w(\mathbf{x},\mathbf{y}) d\mathbf{y}, \end{aligned} \tag{5.5}$$

maps X into itself.

We notice that $\mathbf{J}$ defined in this lemma is different from the one defined in (4.1) due to the absence of the activity variable. On the other hand we have shown that adding a discrete activity does not introduce any problem concerning existence and uniqueness of solutions but only a heavier notation.

Proof. We have to prove that if $\mathbf{f} \in X$ then:

$$\|\mathbf{J}(\mathbf{f})\|_X = \|\mathbf{J}(\mathbf{f})\|_1 + \|\mathbf{J}(\mathbf{f})\|_\infty < +\infty.$$

We start analyzing the L^1 norm $\|\cdot\|_1$. We have:

$$\int_{\mathbb{R}^2} |J_i(\mathbf{f}(\mathbf{x}))| d\mathbf{x} \leq \int_{\mathbb{R}^2} |\Gamma_i(\mathbf{f}(\mathbf{x}))| d\mathbf{x} + \int_{\mathbb{R}^2} |\Lambda_i(\mathbf{f}(\mathbf{x}))| d\mathbf{x},$$

$$\int_{\mathbb{R}^2} |\Gamma_i(\mathbf{f}(\mathbf{x}))| d\mathbf{x} \leq \sum_{h,k=1}^{n} \|f_h\|_{L^1} \int_{\mathbb{R}^2} |\eta(\mathbf{f}(\mathbf{y})) B^i_{hk}(\mathbf{f}(\mathbf{y})) f_k(\mathbf{y}) w(\mathbf{x},\mathbf{y})| d\mathbf{y}$$

$$\leq \sum_{h,k=1}^{n} C_w L_\eta \|f_h\|_{L^1} \int_{\mathbb{R}^2} |f(\mathbf{y})||f_k(\mathbf{y})| d\mathbf{y} + \eta_0 C_w \sum_{h,k=1}^{n} \|f_h\|_{L^1} \int_{\mathbb{R}^2} |f_k(\mathbf{y})| d\mathbf{y}.$$

Reminding that $|\mathbf{f}| = \sum_{h=1}^{n} |f_h|$, we have

$$\int_{\mathbb{R}^2} |\Gamma_i(\mathbf{f}(\mathbf{x}))| d\mathbf{x} \leq \sum_{h,k,j=1}^{n} C_w L_\eta \|f_h\|_{L^1} \|f_k\|_{L^1} \|f_j\|_{L^\infty} + \eta_0 C_w \sum_{h,k=1}^{n} \|f_h\|_{L^1} \|f_k\|_{L^1}$$

$$\leq n C_w L_\eta \|\mathbf{f}\|_1^2 \|\mathbf{f}\|_\infty + \|\mathbf{f}\|_1^2 > +\infty.$$

Similarly, we have:

$$\int_{\mathbb{R}} |\Lambda_i(\mathbf{f}(\mathbf{x})| d\mathbf{x} \leq C_w L_\eta \|f_i\|_{L^1} \sum_{h,k=1}^{n} \|f_h\|_{L^1} \|f_k\|_{L^\infty} < +\infty.$$

Coming to analyze the L^∞ part, we find:

$$|\Gamma_i(\mathbf{f}(\mathbf{x}))| \leq C_w L_\eta \sum_{h,k,j=1}^{n} \|f_h\|_{L^\infty} \|f_j\|_{L^1} \|f_k\|_{L^\infty} < +\infty,$$

which gives the boundedness of $\|\Gamma_i(\mathbf{f})\|_{L^\infty}$. Furthermore:

$$|\Lambda_i(\mathbf{f}(\mathbf{x}))| \leq C_w L_\eta \|f_i\|_{L^\infty} \sum_{h,k=1}^{n} \|f_h\|_{L^1} \|f_k\|_{L^\infty} < +\infty,$$

which ends the proof. □

Lemma 5.2.2. *The function* $\mathbf{J}$ *is locally Lipschitz on* X.

Proof. Let $r > 0$ and $\mathbf{f}, \mathbf{g} \in X$ such that $\|\mathbf{f}\|_X, \|\mathbf{g}\|_X < r$. Then:

$$|J_i(\mathbf{f}(\mathbf{x}) - J_i(\mathbf{g}(\mathbf{x}))| \leq |\Gamma_i(\mathbf{f}(\mathbf{x})) - \Gamma_i(\mathbf{g}(\mathbf{x}))| + |\Lambda_i(\mathbf{f}(\mathbf{x})) - \Lambda_i(\mathbf{g}(\mathbf{x}))|.$$

After some inequalities we find,

$$|\Gamma_i(\mathbf{f}(\mathbf{x})) - \Gamma_i(\mathbf{g}(\mathbf{x})| \leq (nr^2 L_\eta + \eta_0 r) C_w \sum_{h=1}^{n} |f_h(\mathbf{x}) - g_h(\mathbf{x})|$$

$$+ (C_w L_\eta r (C_{Br} r + n + 1) + \eta_0 C_w (r + C_{Br})) \|\mathbf{f} - \mathbf{g}\|_1 \sum_{h=1}^{n} |g_h(\mathbf{x})|,$$

from which follows, taking the L^∞ norm,

$$\|\Gamma_i(\mathbf{f}) - \Gamma_i(\mathbf{g})\|_{L^\infty} \leq L_\Gamma(r) \|\mathbf{f} - \mathbf{g}\|_X; \tag{5.6}$$

while if we integrate over the whole spatial domain

$$\|\Gamma_i(\mathbf{f}) - \Gamma_i(\mathbf{g})\|_{L^1} \leq L_\Gamma(r) \|\mathbf{f} - \mathbf{g}\|_X \tag{5.7}$$

with a suitable Lipschitz constant

$$L_\Gamma(r) = \max\left\{(C_w L_\eta r(C_{Br} r + n + 1) + \eta_0 C_w (r + C_{Br}))r, (nr^2 L_\eta + \eta_0 r)C_w\right\}$$

In a similar way:

$$|\Lambda_i(\mathbf{f}(\mathbf{x})) - \Lambda_i(\mathbf{g}(\mathbf{x}))| \leq \sum_{h=1}^{n} |f_i(\mathbf{x}) - g_i(\mathbf{x})| \int_{\mathbb{R}^2} |\eta(\mathbf{f}(\mathbf{y})) f_h(\mathbf{y}) w(\mathbf{x}, \mathbf{y})| d\mathbf{y}$$

$$+ \sum_{h=1}^{n} |g_i(\mathbf{x})| \int_{\mathbb{R}^2} |(\eta(\mathbf{f}(\mathbf{y})) f_h(\mathbf{y}) - \eta(\mathbf{g}(\mathbf{y})) g_h(\mathbf{y})) w(\mathbf{x}, \mathbf{y})| d\mathbf{y},$$

$$\leq r C_w \left(n L_\eta r + \eta_0\right) |f_i(\mathbf{x}) - g_i(\mathbf{x})| + \left(n L_\eta (r+1) + \eta_0\right) C_w |g_i(\mathbf{x})| \|\mathbf{f} - \mathbf{g}\|_1,$$

which leads to the following inequalities:

$$\|\Lambda_i(\mathbf{f}) - \Lambda_i(\mathbf{g})\|_{L^\infty} \leq L_\Lambda(r) \|\mathbf{f} - \mathbf{g}\|_X, \tag{5.8}$$

and,

$$\|\Lambda_i(\mathbf{f}) - \Lambda_i(\mathbf{g})\|_{L^1} \leq L_\Lambda(r) \|\mathbf{f} - \mathbf{g}\|_X, \tag{5.9}$$

in this cases a suitable Lipschitz constant is

$$L_\Lambda(r) = r C_w \left(n L_\eta (r+1) + \eta_0\right),$$

Combining (5.6), (5.7), (5.8) and (5.9) we conclude the proof. □

The Lipschitz constant found in the previous proof is for sure not the optimal ones. Anyway, it is crucial in order to use a fixed point to ensure existence and uniqueness of solution related to the system

$$\begin{cases} \partial_t f_i(t, \mathbf{x}) + \mathbf{v}_i \cdot \partial_{\mathbf{x}} f_i(t, \mathbf{x}) = J_i(\mathbf{f}(t, \mathbf{x})) \\ f_i(0, \mathbf{x}) = \overline{f}_i(\mathbf{x}), \end{cases} \quad i = 1, \ldots, n. \tag{5.10}$$

Theorem 5.2.1. *Let B^i_{hk}, η and w be functions verifying (3.4), (3.6), (5.3) and (4.5); then, given $\overline{\mathbf{f}} \in X$, there exists a $T > 0$ and a unique $\overline{f} \in C([0, T], X)$ mild solution of (5.10), with interaction kernel $\mathbf{J}$ equal to (5.5). Moreover if $\overline{\mathbf{f}}$ is non-negative, i.e. $\overline{f}_i(\mathbf{x}) \geq 0$ for all $i = 1, \ldots, n$ and for all $\mathbf{x} \in \mathbb{R}^2$, then the mild solution is non-negative for all $t \in [0, T]$.*

Prologability of solution is an interesting and hard problem. Conservation of agents' number ensures also conservation, and thus boundedness, of the $\|\cdot\|_1$ norm. The problem is to find a bound for $\|\cdot\|_\infty$ norm. In the previous chapters we found prolongability results dominating the interaction kernel with the gain term, i.e. $J_i(\mathbf{f}) \leq \Gamma_i(\mathbf{f})$, and using Gronwall's lemma. Unfortunately, if we repeat a similar reasoning here we don't get the same result. Actually

$$
\begin{aligned}
|f_i(t,\mathbf{x})| = f_i(t,\mathbf{x}) &= \overline{f}_i(\gamma_i(0,t,\mathbf{x})) + \int_0^t J_i(\mathbf{f}(\tau,\gamma_i(\tau,t,\mathbf{x})))d\tau \\
&\leq \overline{f}_i(\gamma_i(0,t,\mathbf{x})) + \int_0^t \Gamma_i(\mathbf{f}(\tau,\gamma_i(\tau,t,\mathbf{x})))d\tau \\
&\leq \|\overline{f}_i\|_{L^\infty} + C_w \sum_{h,k=1}^n \int_0^t f_h(\tau,\gamma_i(\tau,t,\mathbf{x})) \\
&\qquad \times \int_{D^i_{\gamma_i(\tau,t,\mathbf{x})}} (L_\eta|\mathbf{f}(\tau,\mathbf{y})| + \eta_0) f_k(\tau,\mathbf{y}) d\mathbf{y} d\tau \\
&\leq \|\overline{f}_i\|_{L^\infty} + C_w\eta_0 \sum_{h=1}^n \int_0^t f_h(\tau,\gamma_i(\tau,t,x)) \int_{\mathbb{R}^2} \rho(\tau,\mathbf{y}) d\mathbf{y} d\tau \\
&\qquad + C_w L_\eta \sum_{h=1}^n \int_0^t f_h(\tau,\gamma_i(\tau,t,x)) \int_{\mathbb{R}^2} \left(\sum_{k=1}^n f_k(\tau,\mathbf{y}) \right)^2 d\mathbf{y} d\tau \\
&\leq \|\overline{\mathbf{f}}\|_\infty + C_w\eta_0 \sum_{h=1}^n \int_0^t N(\tau) f_h(\tau,\gamma_i(\tau,t,x)) d\tau \\
&\qquad + C_w L_\eta \sum_{h=1}^n \int_0^t f_h(\tau,\gamma_i(\tau,t,x)) \|\mathbf{f}(t)\|_{L^\infty} \|\mathbf{f}(t)\|_{L^1} \tau \\
&\leq \|\overline{\mathbf{f}}\|_\infty + C_w N_0 \sum_{h=1}^n \int_0^t (\eta_0 \|f_h(\tau)\|_{L^\infty} + L_\eta \|f_h(\tau)\|_\infty^2) d\tau,
\end{aligned}
$$

and thus we have

$$\|\mathbf{f}\|_\infty \leq \|\overline{\mathbf{f}}\|_\infty + nC_w N_0 \int_0^t \|\mathbf{f}(\tau)\|_\infty (\eta_0 + L_e ta \|\mathbf{f}(\tau)\|_\infty) d\tau.$$

In this case we can't use Gronwall's lemma because an unbounded interaction rate produces a quadratic term. However we are able to furnish other results.

Proposition 5.2.3. *Let $\overline{\mathbf{f}} \in X$ be a positive initial datum, let $\mathbf{f} \in C([0,T), X)$ a mild solution of Cauchy problem (5.10) related to $\overline{\mathbf{f}}$ and let $j \in \{1,\ldots,n\}$ and index such that*

$$\lim_{t\to T} \|f_i(t)\|_{L^\infty} < +\infty,$$

for all $i \neq j$. Then we have the boundedness of f_j:

$$\lim_{t\to T} \|f_i(t)\|_{L^\infty} < +\infty,$$

Proof. Thanks to positivity of the initial datum we have positivity of the solution and thus for $t \in [0,T)$ and $x \in \mathbb{R}^2$,

$$\begin{aligned}
|f_j(t,\mathbf{x})| = f_j(t,\mathbf{x}) &= \overline{f}_j(\gamma_j(0,t,\mathbf{x})) + \int_0^t J_j(\mathbf{f}(\tau,\gamma_j(\tau,t,\mathbf{x})))d\tau \\
&\leq \|\overline{f}_j\|_{L^\infty} + \sum_{h\neq j}\sum_{k=1}^n \int_0^t f_h(\tau,\gamma_j(\tau,t,\mathbf{x})) \\
&\quad \times \int_{D^j_{\gamma_j(\tau,t,\mathbf{x})}} \eta(\mathbf{f}(\tau,\mathbf{y}))f_k(\tau,\mathbf{y})w(\mathbf{x},\mathbf{y})d\mathbf{y}d\tau \\
&\leq \|\overline{f}_j\|_{L^\infty} + C_w \sum_{h\neq j}\sum_{k=1}^n \mu_h \int_0^t \int_{\mathbb{R}^2} \eta(\mathbf{f}(\tau,\mathbf{y}))f_k(\tau,\mathbf{y})d\mathbf{y}d\tau,
\end{aligned}$$

where μ_h is defined as

$$\mu_h := \sup_{t\in[0,T)}\sup_{\mathbf{x}\in\mathbb{R}^2} |f_h(t,\mathbf{x})| = \sup_{t\in[0,T)} \|f_h(t)\|_{L^\infty}, \tag{5.11}$$

which is bounded by hypothesis for all $h \neq j$. Taking

$$\mu := \sum_{h\neq j} \mu_h, \tag{5.12}$$

we find

$$\begin{aligned}
|f_j(t,\mathbf{x})| &\leq \|\overline{f}_j\|_{L^\infty} + \mu C_w \sum_{h=1}^n \int_0^t \int_{\mathbb{R}^2} (L_\eta|\mathbf{f}(\tau,\mathbf{y})| + \eta_0) f_k(\tau,\mathbf{y})d\mathbf{y}d\tau, \\
&\leq \|\overline{f}_j\|_{L^\infty} + \mu C_w \sum_{h,k=1}^n \int_0^t (L_\eta\|f_h(t)\|_{L^\infty} + \eta_0) \int_{\mathbb{R}^2} f_k(\tau,\mathbf{y})d\mathbf{y}d\tau.
\end{aligned}$$

Reminding that total number of agents is conserved, we get

$$\begin{aligned}
|f_j(t,\mathbf{x})| &\leq \|\overline{f}_j\|_{L^\infty} + \mu C_w\eta_0 N_0 \left(t + \sum_{h=1}^n \int_0^t \|f_h\|_{L^\infty}\right) \\
&\leq \|\overline{f}_j\|_{L^\infty} + \mu C_w\eta_0 N_0 \left((\mu+1)t + \int_0^t \|f_j(t)\|d\tau\right),
\end{aligned}$$

and thus

$$\|f_j(t)\|_{L^\infty} \leq A(t) + A_0 \int_0^t \|f_j(t)\|d\tau,$$

where

$$A(t) := \|\overline{f}_j\|_{L^\infty} + \mu C_w\eta_0 N_0(\mu+1)t,$$

$$A_0 := \mu C_w \eta_0 N_0.$$

In this case we can use Gronwall's lemma, finding

$$\|f_j(t)\|_{L^\infty} \leq A(t)e^{A_0 t},$$

and the proposition is proven. □

The previous property states that if we are able to control $n-1$ components of $\mathbf{f}$, then we can control also the last one. This is possible because we write the n-th component, along n-th characteristic, as function of $n-1$ remaining components. In other words, it is not possible that only one component blows up in a finite time. We are able to give more information about prolongability only in one special case. Let us fix $j \in \{1, \dots, n\}$ and consider the following table of games:

$$B^i_{hk}(\mathbf{f}) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \tag{5.13}$$

for all $h, k = 1, \dots, n$. In this particular case we can easily write the interaction kernel $\tilde{\mathbf{J}}$ as

$$\tilde{J}_i(\mathbf{f}) = \begin{cases} \displaystyle\sum_{h \neq j} \sum_{k=1}^{n} f_h(t, \mathbf{x}) \int_{D^i_{\mathbf{x}}} \eta(\mathbf{f}(t, \mathbf{y})) f_k(t, \mathbf{y}) w(\mathbf{x}, \mathbf{y}) d\mathbf{y} & \text{if } i = j, \\ \displaystyle -f_i(t, \mathbf{x}) \sum_{h=1}^{n} \int_{D^i_{\mathbf{x}}} \eta(\mathbf{f}(t, \mathbf{y})) f_h(t, \mathbf{y}) w(\mathbf{x}, \mathbf{y}) d\mathbf{y} & \text{otherwise.} \end{cases} \tag{5.14}$$

This kernel kernel represents the "worst" case of (5.10) for j-th solution's component, when every agent collapses in the j-th velocity class. The Cauchy problem (5.10) with an interaction kernel like (5.14) clearly posses a local solution $f \in C([0, T), X)$. Moreover, in order to prolongate it, we have only to give a bound of the norm $\|\mathbf{f}(t)\|_\infty$. If $i \neq j$

$$\|f_i(t)\|_{L^\infty} \leq \|\overline{f}_i\|_{L^\infty}, \tag{5.15}$$

while if $i = j$

$$|fj(t, \mathbf{x})| = f_j(t, \mathbf{x}) = \overline{f}_j(\gamma_j(0, t, \mathbf{x})) + \int_0^t \tilde{J}_j(\mathbf{f})(\tau, \gamma_j(\tau, t, \mathbf{x})) d\tau$$

$$\leq \|\overline{f}_j\|_{L^\infty} + \sum_{h \neq j} \int_0^t \left(\int_{\mathbb{R}^2} \eta(\mathbf{f}(\tau, \mathbf{y})) \sum_{k=1}^{n} f_k(\tau, \mathbf{y}) w(\mathbf{x}, \mathbf{y}) d\mathbf{y} \right) f_h(\tau, \gamma_j(\tau, t, \mathbf{x})) d\tau$$

$$\leq \|\overline{f}_j\|_{L^\infty} + L_\eta C_w \sum_{h\neq j}\int_0^t \left(\int_{\mathbb{R}^2}\sum_{k,i=1}^n f_i(\tau,\mathbf{y})f_k(\tau,\mathbf{y})d\mathbf{y}\right) f_h(\tau,\gamma_j(\tau,t,\mathbf{x}))d\tau$$

$$+C_w\eta_0\sum_{h\neq j}\int_0^t\left(\int_{\mathbb{R}^2}\sum_{k=1}^n f_k(\tau,\mathbf{y})d\mathbf{y}\right) f_h(\tau,\gamma_j(\tau,t,\mathbf{x}))d\tau$$

$$\leq \|\overline{f}_j\|_{L^\infty} + L_\eta C_w N_0 \sum_{k=1}^n\sum_{h\neq j}\int_0^t \|f_k(\tau)\|_{L^\infty}\|f_h(\tau)\|_{L^\infty}d\tau$$

$$+C_w\eta_0 N_0\sum_{h\neq j}\int_0^t \|f_h(\tau)\|_{L^\infty}d\tau$$

$$\leq \|\overline{f}_j\|_{L^\infty} + (L_\eta\alpha+\eta_0)C_w N_0\alpha t + L_\eta C_w N_0\alpha\int_0^t \|f_j(\tau)\|_{L^\infty}d\tau,$$

where $\alpha := \sum_{h\neq j}\|\overline{f}_h\|_{L^\infty}$, which gives:

$$\|f_h(t)\|_{L^\infty} \leq \|\overline{f}_j\|_{L^\infty} + (L_\eta\alpha+\eta_0)C_w N_0\alpha t + L_\eta C_w N_0\alpha\int_0^t \|f_j(\tau)\|_{L^\infty}d\tau. \tag{5.16}$$

Using Gronwall's inequality with this last inequality gives

$$\|\mathbf{f}_h(t)\| \leq \left(\|\overline{f}_j\|_{L^\infty} + (L_\eta\alpha+\eta_0)C_w N_0\alpha t\right)e^{L_\eta C_w N_0 t},$$

and the prolongability of solution.

Theorem 5.2.4. *Let η and w be functions verifying (5.3) and (4.5); then, given $\overline{\mathbf{f}} \in X$ a non-negative initial datum, there exists a unique $\overline{f} \in C([0,\infty), X)$ mild solution of (5.10), with interaction kernel $\mathbf{J}$ equal to (5.14), and it remains non-negative for all $t > 0$.*

In a certain sense we are not able to control solution of problem (5.10) with (5.5) as interaction kernel but we are able to control the collapse of all agents in the j-th class of velocity. On the other hand the following inequality

$$J_j(\mathbf{f}) \leq \tilde{J}_j(\mathbf{f}),$$

for all $\mathbf{f} \in X$, does not ensure the same inequality for two solutions

$$f_j \leq \tilde{f}_j,$$

where $\tilde{\mathbf{f}}$ is the solution related to interaction kernel (5.14), even if they start with the same initial datum.

5.3 Local interactions

In this section we study the well-posedness of a system different from (3.1). While in the class of models described by system (5.10) the interactions among agents are non-local, occurring in a spatial domain $D^i_{\mathbf{x}}$ around the actual position $\mathbf{x}$, in the present case we consider models in which different agents interact only if they occupy the same spatial position (local interactions).

Let us consider the initial value problem:

$$\begin{cases} \partial_t f_i(t,\mathbf{x}) + \mathbf{v}_i \cdot \nabla_{\mathbf{x}} f_i(t,\mathbf{x}) = \overline{J}_i(\mathbf{f})(t,\mathbf{x}), \\ f_i(0,\mathbf{x}) = \overline{f}_i(\mathbf{x}), \end{cases} \qquad i = 1,\ldots,n, \tag{5.17}$$

where $\overline{\mathbf{J}} = (\overline{J}_i)$ is written as:

$$\overline{J}_i(\mathbf{f}) = \overline{\Gamma}_i(\mathbf{f}) - \overline{\Lambda}_i(\mathbf{f}),$$

with:

$$\overline{\Gamma}_i(\mathbf{f}(t,\mathbf{x})) := \sum_{h,k=1}^{n} \eta(\mathbf{f}(t,\mathbf{x})) B^i_{hk}(\mathbf{f}(t,\mathbf{x})) f_h(t,\mathbf{x}) f_k(t,\mathbf{x}), \tag{5.18}$$

and:

$$\overline{\Lambda}_i(\mathbf{f}(t,\mathbf{x})) := f_i(t,\mathbf{x}) \sum_{h=1}^{n} \eta(\mathbf{f}(t,\mathbf{x})) f_h(t,\mathbf{x}). \tag{5.19}$$

Unlike ((5.5), in the latter expressions there is not any spatial integral, as two agents interact only when they share the same position. It is immediate to verify that, maintaining equality (3.4)

$$\sum_{i=1}^{n} \overline{J}_i(\mathbf{f}) = 0.$$

We can study Cauchy problem in the same functional space $(X, \|\cdot\|_X)$ of previous non-local system.

Lemma 5.3.1. *Let B^i_{hk} be functions satisfying (3.4) and (3.7) for all indexes, let η be a function satisfying (5.3), then function $\overline{\mathbf{J}} = (\overline{J}_i(\mathbf{f}))$, maps X into itself, and it is locally Lipschitz.*

Proof. Given $\mathbf{f} \in X$, we find,

$$\begin{aligned} |\overline{\Gamma}_i(\mathbf{f}(\mathbf{x}))| &\le \sum_{h,k=1}^{n} (L_\eta |\mathbf{f}(\mathbf{x})| + \eta_0) |f_h(\mathbf{x}) f_k(\mathbf{x})| \\ &\le \left(L_\eta \|\mathbf{f}\|_\infty^2 + \eta_0 \|vf\|_\infty\right) \sum_{h=1}^{n} |f_h(\mathbf{x})|, \end{aligned}$$

and thus

$$\|\overline{\Gamma}_i(\mathbf{f})\|_1 \leq \left(L_\eta n^2 \|\mathbf{f}\|_\infty^2 + \eta_0 n \|vf\|_\infty\right) \|\mathbf{f}\|_1,$$
$$\|\overline{\Gamma}_i(\mathbf{f})\|_\infty \leq n\left(L_\eta n^2 \|\mathbf{f}\|_\infty^2 + \eta_0 n \|vf\|_\infty\right) \|\mathbf{f}\|_\infty.$$

While for the loss term we have similarly

$$\|\overline{\Lambda}_i(\mathbf{f})\|_1 \leq (L_\eta n^2 \|\mathbf{f}\|_\infty^2 + \eta_0 n \|\mathbf{f}\|_\infty)\|\mathbf{f}\|_1,$$
$$\|\overline{\Lambda}_i(\mathbf{f})\|_\infty \leq (L_\eta n^2 \|\mathbf{f}\|_\infty^2 + \eta_0 n \|\mathbf{f}\|_\infty)\|\mathbf{f}\|_\infty,$$

hence we have that $\|\mathbf{f}\|_X < +\infty$. Let $r > 0$ be a positive number and $\mathbf{f}, \mathbf{g} \in X$ be functions such that $\|\mathbf{f}\|_X < r$ and $\|\mathbf{g}\|_X < r$. Then, we have:

$$|\overline{J}_i(\mathbf{f}) - \overline{J}_i(\mathbf{g})| \leq |\Gamma_i(\mathbf{f}) - \Gamma_i(\mathbf{g})| + |\Lambda_i(\mathbf{f}) - \Lambda_i(\mathbf{g})|,$$

where we have omitted the dependence on $\mathbf{x}$.

$$|\overline{\Gamma}_i(\mathbf{f}) - \overline{\Gamma}_i(\mathbf{g})| \leq [rnL_\eta(2 + (C_{Br} + 1)nr) + rn\eta_0(1 + rnC_{Br})]\, |\mathbf{f} - \mathbf{g}|$$

and thus:

$$\|\mathbf{\Gamma}(\mathbf{f}) - \mathbf{\Gamma}(\mathbf{g})\|_X \leq n\, [rnL_\eta(2 + (C_{Br} + 1)nr) + rn\eta_0(1 + rnC_{Br})]\, \|\mathbf{f} - \mathbf{g}\|_X, \tag{5.20}$$

for the loss term we have

$$|\Lambda_i(\mathbf{f}) - \Lambda_i(\mathbf{g})| \leq (L_\eta r + \eta_0)|f_i - g_i| + L_\eta r|\mathbf{f} - \mathbf{g}|$$

and thus,

$$\|\mathbf{\Lambda}(\mathbf{f}) - \mathbf{\Lambda}(\mathbf{g})\|_X \leq n(2L_\eta r + \eta_0)\|\mathbf{f} - \mathbf{g}\|_X, \tag{5.21}$$

Combining inequalities (5.20) and (5.21) we conclude the proof. □

Thanks to the last proposition, we immediately get the following result concerning local existence and uniqueness of solutions to (5.17).

Theorem 5.3.2. *Let η and B_{hk}^i be functions satisfying (3.4), (3.6), (5.3) and (4.5), let $\overline{\mathbf{f}} \in X$ ba an initial datum, then there exists $T > 0$ and a unique $\mathbf{f} \in C([0,T], X)$ mild solution to (5.17).*

Clearly if initial datum $\overline{\mathbf{f}}$ are non-negative, then the solution remains non-negative and the total number N of agents in the system is conserved. We do another hypothesis on interaction rate, supposing another time that it is bounded

$$|\eta(\mathbf{f})| \leq C_\eta, \quad \forall \mathbf{f} \in X.$$

Thanks to boundedness of interaction rate we can use same argument of previous section.

Proposition 5.3.3. *Let $\overline{\mathbf{f}} \in X$ be a positive initial datum, let $\mathbf{f} \in C([0,T), X)$ a mild solution of Cauchy problem (5.17) related to $\overline{\mathbf{f}}$ and let $j \in \{1, \dots, n\}$ and index such that*

$$\lim_{t \to T} \|f_i(t)\|_{L^\infty} < +\infty,$$

for all $i \neq j$. Then we have the boundedness of f_j,

$$\lim_{t \to T} \|f_i(t)\|_{L^\infty} < +\infty,$$

Proof. Thanks to positiveness of initial datum we have positiveness of solution and thus for $t \in [0,T)$ and $\mathbf{x} \in \mathbb{R}^2$,

$$\begin{aligned}
|f_j(t,\mathbf{x})| = f_j(t,\mathbf{x}) &= \overline{f}_j(\gamma_j(0,t,\mathbf{x})) + \int_0^t \overline{J}_j(\mathbf{f}(\tau, \gamma_j(\tau,t,\mathbf{x})))d\tau \\
&\leq \|\overline{f}_j\|_{L^\infty} + C_\eta \sum_{h \neq j} \sum_{k=1}^n \int_0^t f_h(\tau, \gamma_j(\tau,t,\mathbf{x})) f_k(\tau, \gamma_j(\tau,t,\mathbf{x}))d\tau \\
&\leq \|\overline{f}_j\|_{L^\infty} + C_\eta \sum_{h \neq j} \sum_{k=1}^n \mu_h \int_0^t \int_{\mathbb{R}^2} f_k(\tau, \mathbf{y}) d\mathbf{y} d\tau \\
&\leq\leq \|\overline{f}_j\|_{L^\infty} + \mu^2 C_\eta t + \mu C_\eta \int_0^t f_j(\tau, \gamma_j(\tau,t,\mathbf{x}))d\tau, \\
&\leq\leq \|\overline{f}_j\|_{L^\infty} + \mu^2 C_\eta t + \mu C_\eta \int_0^t \|f_j(\tau)\|_{L^\infty} d\tau,
\end{aligned}$$

where μ_h and μ is defined like in (5.11) and (5.12), bounded by hypothesis, using Gronwall's inequality we have

$$\|f_j(t)\|_{L^\infty} \leq A(t) e^{A_0 t},$$

where

$$\begin{aligned}
A(t) &:= \|\overline{f}_j\|_{L^\infty} + \mu C_w \eta_0 N_0 (\mu + 1)t, \\
A_0 &:= \mu C_w \eta_0 N_0,
\end{aligned}$$

this ends the proof. □

Like in previous section we fix an index $j \in \{1, \dots, n\}$ we start to study (5.17) with an interaction kernel similar to (5.14), even if with punctual interaction, using a table of games like in (5.13)

$$\hat{J}_i(\mathbf{f}) = \begin{cases} \sum_{h \neq j} \eta(\mathbf{f}(t,\mathbf{x})) f_h(t,\mathbf{x}) \rho(t,\mathbf{x}) & \text{if } i = j, \\ -\eta(\mathbf{f}(t,\mathbf{y})) f_i(t,\mathbf{x}) \rho(t,\mathbf{x}) & \text{otherwise.} \end{cases} \tag{5.22}$$

Existence and uniqueness of a local-in-time solution for (5.17), with (5.22) as interaction kernel, is granted. Concerning its prolongability, given $\mathbf{f} \in C([0,T), X)$ mild solution of Cauchy problem corresponding to a non-negative initial datum, we have:

$$f_i(t,x) \leq \|\overline{f}_i\|_{L^\infty},$$

if $i \neq j$. Otherwise, for the j-th class, supposing the boundedness of interaction rate, we have

$$|f_j(t,\mathbf{x})| = f_j(t,\mathbf{x}) = \overline{f}_j(\gamma_j(0,t,\mathbf{x})) + \int_0^t \hat{J}_j(\mathbf{f}(\tau,\gamma_j(\tau,t,\mathbf{x}))d\tau$$

$$\leq \|\overline{f}_j\|_{L^\infty} + C_\eta \sum_{h\neq j} \sum_{k=1}^{n} \int_0^t \|\overline{f}_h\|_{L^\infty} f_k(\tau,\gamma_j(\tau,t,\mathbf{x}))d\tau$$

$$\leq \|\overline{f}_j\|_{L^\infty} + C_e t a \alpha^2 t + C_\eta \alpha \int_0^t f_j(\tau,\gamma_j(\tau,t,\mathbf{x}))d\tau,$$

where:

$$\alpha := \sum_{h\neq j} \|f_h\|_{L^\infty}.$$

The last inequality gives us:

$$\|f_j(t)\|_{L^\infty} \leq \|\overline{f}_j\|_{L^\infty} + C_\eta \alpha^2 t + \alpha \int_0^t \|f_j(\tau)\|_{L^\infty} d\tau,$$

and thus, using Gronwall's lemma:

$$\|f_j(t)\|_{L^\infty} \leq (\|\overline{f}_j\|_{L^\infty} + C_\eta \alpha^2 t)e^{C_\eta \alpha t}.$$

The previous considerations are summarized in the following theorem.

Theorem 5.3.4. *Let η be a function verifying (4.4); then, given $\overline{\mathbf{f}} \in X$ a non-negative initial datum, there exists a unique $\overline{f} \in C([0,\infty), X)$ mild solution of (5.17), with interaction kernel $\hat{\mathbf{J}}$ equal to (5.22), and it remains non-negative for all $t > 0$.*

Summarizing the previous results, we have found estimates for the problem (5.17) in the case of a very particular interaction rate, like (5.22), that ensures prolongability of local solutions. We notice that this estimate is also quite similar to the one found for the Cauchy problem (5.10), with (5.14) as interaction kernel. On the other hand, the last result does not ensure prolongability for the general problem (5.17).

Chapter 6

Bounded domain

There is a vast literature concerning the study of complex systems like vehicular traffic, crowds dynamics and tumor growth by a kinetic point of view. On the other hand, there are few works that study these kinetic models in a bounded domain. In other words, while the Cauchy problem in an unbounded domain has been largely analyzed, a relatively small number of Authors addressed their attention to the richness of phenomena happening in a bounded domains $\Omega \subset \mathbb{R}^n$, with $n = 1, 2$, such that

$$\mathrm{diam}(\Omega) < \infty,$$

having a boundary $\partial\Omega$ that verifies suitable regularity properties, that we will discuss later.

The initial-boundary value problem for a system of first-order hyperbolic evolution equations is treated in [32, 36], while in [33, 38] the discrete Boltzmann equation in a bounded domain has been studied. Writing again the general equations analyzed in the previous chapters:

$$\partial_t f_i(t, \mathbf{x}) + \mathbf{v}_i \cdot \nabla_{\mathbf{x}} f_i(t, \mathbf{x}) = J_i(\mathbf{f}(t, \mathbf{x})). \tag{6.1}$$

we understand that the first problem is to give a different definition of the interaction kernel J_i in the r.h.s. of (6.1). Conversely from discrete Boltzmann equation, where the interaction kernel is punctual, in (6.1) the agents interact each other in an visibility domain $D^i_{\mathbf{x}}$. The presence of this non local interaction creates some troubles near the boundary. The simplest idea to avoid this problem it is to intersect the interaction domain with the whole domain

$$
\begin{aligned}
J_i(\mathbf{f}(t,x)) =&\Gamma_i(\mathbf{f}(t,\mathbf{x})) - \Lambda_i(\mathbf{f}(t,\mathbf{x})) \\
=& \sum_{h,k=1}^{n} \int_{D^i_{\mathbf{x}}\cap\Omega} B^i_{hk}(\mathbf{f}(t,\mathbf{y}))\eta(\mathbf{f}(t,\mathbf{y}))f_h(t,\mathbf{x})f_k(t,\mathbf{y})d\mathbf{y} \\
&- f_i(t,\mathbf{x})\sum_{h=1}^{n}\int_{D^i_{\mathbf{x}}\cap\Omega} \eta(\mathbf{f}(t,\mathbf{y}))f_h(t,\mathbf{y})d\mathbf{y}.
\end{aligned} \tag{6.2}
$$

With this choice the measure of interaction domain $D^i_{\mathbf{x}}$ decreases if $\mathbf{x}$ is close to the boundary and $\mathbf{v}_i$ points outside the boundary.

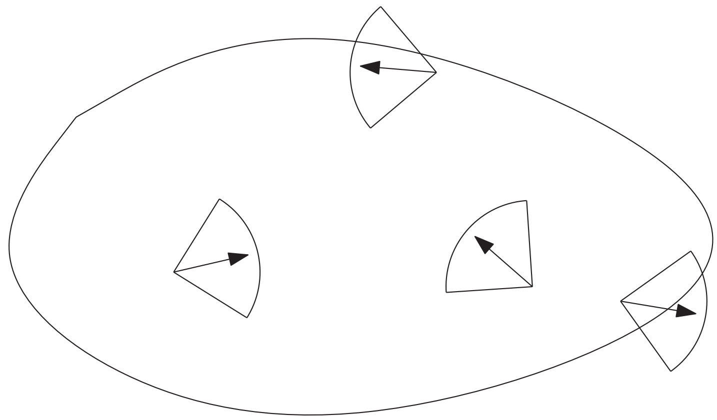

Figure 6.1: In this case interaction domain depends on velocity maintained by an agent, near boundary it is smaller then in other places.

6.1 One dimensional case

In this section we study the existence and uniqueness of solutions related to the one dimensional initial-boundary value problem in a bounded region $\Omega = [0, L]$:

$$
\begin{cases}
\partial_t \mathbf{f}(t,x) + V\partial_x \mathbf{f}(t,x) = \mathbf{J}(\mathbf{f}(t,x)) \\
\mathbf{f}(0,x) = \overline{\mathbf{f}}(x) \\
\mathbf{f}^+(t,0) = B^+\mathbf{f}^-(t,0) + \mathbf{b}^+(t) \\
\mathbf{f}^-(t,L) = B^-\mathbf{f}^+(t,L) + \mathbf{b}^-(t),
\end{cases} \tag{6.3}
$$

where

$$
\mathbf{f} = (f_1,\dots,f_n), \quad V = \mathrm{diag}(v_i),
$$

$$
\mathbf{J}(\mathbf{f}) = (J_1(\mathbf{f}),\dots,J_n(\mathbf{f})), \quad \overline{\mathbf{f}} = (\overline{f}_1,\dots,\overline{f}_n).
$$

Let

$$I = \{1, \dots, n\}, \quad I_+ = \{i \in I | v_i > 0\}, \quad I_- = \{i \in I | v_i < 0\},$$

then we can define

$$\mathbf{f}^\pm = (f_i)_{i \in I_\pm}, \; B^\pm = (B_{ij})_{i \in I \in I_\pm, j \in I_\mp}, \; \mathbf{b}^\pm = (b_i)_{i \in I_\pm}.$$

In other words $\mathbf{f}^+$ describes agents traveling with positive velocities, while $\mathbf{f}^-$ describes those traveling with negative velocities. The boundary conditions for (6.3) are written in the most general case and they are divided into two parts. The first term $B^\pm f^\mp$ describes the boundary conditions due to the collision on the boundary $x = 0, L$, while the term $\mathbf{b}^\pm$ describes the possible inner flow inside the domain. It is quite obvious to impose that

$$b_i^\pm \geq 0, \quad B_{ij}^\pm \geq 0.$$

We investigate briefly the case where there aren't any inner or outer flows, and there is a complete reflection on the boundary. The absence of inner flow implies that

$$\mathbf{b}^\pm(t) = 0,$$

for all $t \geq 0$. It is clear that a complete absorption is given by $B^\pm = (B_{ij}^\pm) = 0$; on the other hand a complete reflection has to give a mass conservation. Summing over i equations of (6.3) and integrating, we find

$$\sum_{i=1}^n \int_0^L (\partial_t f_i(t,x) + v_i \partial_x f_i(t,x)) dx = 0.$$

Under suitable conditions we have

$$\sum_{i=1}^n \left(\int_0^L \partial_t f_i(t,x) dx + [v_i f_i(t,x)]_{x=0}^{x=L} \right) = 0, \tag{6.4}$$

and thus

$$\sum_{i=1}^n v_i f_i(t,0) = \sum_{i=1}^n v_i f_i(t,L) = 0. \tag{6.5}$$

These two latter identities could seem quite artificial, since from (6.4) we expect only

$$\sum_{i=1}^n v_i \left(f_i(t,0) - f_i(t,L) \right) = 0.$$

On the other hand the identities (6.5) have a strong physical meaning, as the balance between agents traveling with a positive velocity and the other ones it is related to point where takes into account the reflection. If we use boundary conditions of (6.1) we find

$$\sum_{i\in I_+} v_i f_i(t,0) + \sum_{i\in I_-} v_i f_i(t,0) = \sum_{j\in I_-} f_j(t,0)\left(v_j + \sum_{i\in I_+} v_i B^+_{ij}\right) = 0,$$

and finally,

$$v_j + \sum_{i\in I_+} v_i B^+_{ij} = 0, \text{ for all } j \in I_-, \tag{6.6}$$

similarly we have

$$v_j + \sum_{i\in I_-} v_i B^-_{ij} = 0, \text{ for all } j \in I_+. \tag{6.7}$$

The previous two equalities are related to the case of a complete reflection when agents arrive to the boundary. On the other hand if we take into account also an absorption we find

$$v_j + \sum_{i\in I_+} v_i B^+_{ij} \le 0, \text{ for all } j \in I_-, \tag{6.8}$$

$$v_j + \sum_{i\in I_-} v_i B^-_{ij} \ge 0, \text{ for all } j \in I_+. \tag{6.9}$$

We have to remind that we are interested to describe complex system, and in the case of traffic flow and crowds dynamic on the boundary there are only two different cases, a complete reflection or a complete absorption. If we consider for example a one dimensional hallway which has only one exit, placed in $x = L$, and in the other extremity $x = 0$ there is a wall, we have different behaviors on the boundary. In $x = 0$ we have a complete reflection, agents traveling with negative velocities collide with the wall and change at least the sign of their velocities, while in $x = L$ there is a complete absorption, due to the exit, and thus $B^- = (B^-_{ij}) = 0$.

6.2 Two dimensional case

In the one dimensional case we have no problem about the domain regularity. On the other hand, this issue becomes crucial in the two dimensional case, when $\Omega \subset \mathbb{R}^2$. In order to understand more deeply this point we start to discuss an easy case. Let us consider a domain $\Omega \subset \mathbb{R}^2$ in which there are only two achievable velocities $\mathbf{v}$ and $-\mathbf{v}$. In this case the agents can move along the direction given by $\mathbf{v}$, see Figure 6.2. Thanks to this observation the problem can reduced to the one studied in the one dimensional case, when an agent arrives to the boundary $\mathbf{x} \in \partial\Omega$, with velocity $\mathbf{v}$, we require a collision and agent changes his velocity with $-\mathbf{v}$. Forgetting the possible inner flow,

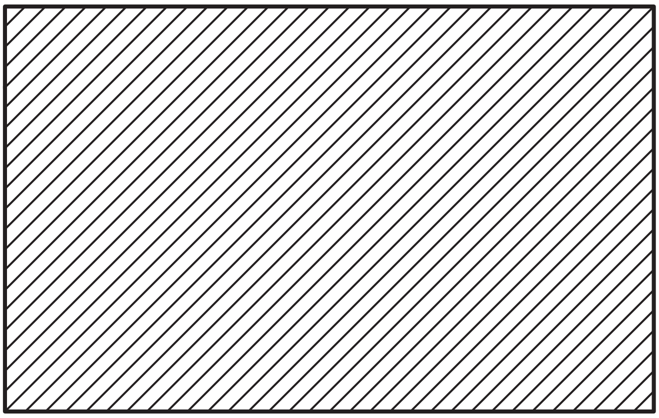

Figure 6.2: An idea of rectangular domain, where only two velocity are achievable, $\mathbf{v}$ and $-\mathbf{v}$. In this case agents can move along the straight lines, given by $\mathbf{v}$.

i.e. $\mathbf{b} = 0$, in the general case we can write the problem as follows:

$$\begin{cases} \partial_t f_i(t,\mathbf{x}) + \mathbf{v}_i \cdot \nabla_{\mathbf{x}} f_i(t,\mathbf{x}) = J_i(\mathbf{f}(t,\mathbf{x})) & \\ \mathbf{f}(0,\mathbf{x}) = \bar{\mathbf{f}}(\mathbf{x}), & \forall \mathbf{x} \in \Omega, \\ \mathbf{f}^-(t,\mathbf{x}) = B(\mathbf{x})\mathbf{f}^+(t,\mathbf{x}), & \forall \mathbf{x} \in \partial\Omega, \end{cases} \tag{6.10}$$

where $\mathbf{f}^{\pm}(\mathbf{x}) = (f_i(\mathbf{x}))$ is a vector such that,

$$\mathbf{f}^{\pm} = (f_i)_{i \in I_{\pm}(\mathbf{x})},$$

given $\mathbf{x} \in \partial\Omega$, $I_-(\mathbf{x})$ is the set of indexes satisfying the following property,

$$\exists \bar{\varepsilon} > 0, \text{ such that } \mathbf{x} + \varepsilon \mathbf{v}_i \in \Omega, \ \forall \varepsilon \in (0, \bar{\varepsilon}). \tag{6.11}$$

Conversely, given $\mathbf{x} \in \partial\Omega$, $I_+(\mathbf{x})$ si defined as the complement of $I_-(\mathbf{x})$, in the set of indexes,

$$I_+(\mathbf{x}) \cup I_-(\mathbf{x}) = \{1, \dots, n\}, \quad I_+(\mathbf{x}) \cap I_-(\mathbf{x}) = \emptyset.$$

Property (6.11) permits us to bypass the absence of a normal vector. The sets $I_{\pm}(\mathbf{x})$ are different from the ones defined in the previous section, as in this case $I_{\pm}(\mathbf{x})$ clearly depends on $\mathbf{x}$. Once we have defined $I_{\pm}(\mathbf{x})$ we understand the role of $\mathbf{f}^{\pm}$ in (6.10). $\mathbf{f}^{+}$ is the set of agents distributions whose velocities lead them outside the domain in $\mathbf{x}$; if we want to study problem (6.10), we have to impose boundary conditions related to $\mathbf{f}^{-}$, whose characteristics at the boundary go outside the domain Ω. We are particularly interested to two phenomena at the boundary:

- a complete absorption, which mimics the presence of an exit on the boundary. As in the one dimensional one, if there isn't any inner flow in the domain it is easy to understand that we have to require:

$$B_{ij} = 0, \ i \in I_-, j \in I_+. \tag{6.12}$$

- There is a complete reflection, which clearly mimics the presence of a wall on the boundary. In this case we require:

$$|\mathbf{v}_j| - \sum_{i \in I_+} |\mathbf{v}_i| B_{ij} = 0, \ \text{for all } j \in I_-. \tag{6.13}$$

We always suppose that $B_{ij} \geq 0$ for all pairs of indexes. If it is clear the meaning of (6.12) with respect to a complete absorption at the boundary, it could be less clear the meaning of (6.13). In equation (6.13) are enclosed equations (6.6) and (6.7), which give reflections on the boundary in the one dimensional case. The hypotheses are not stringent because only the conservation of mass is required, while we don't need nor the conservation of momentum or that of the kinetic energy as in the discrete Boltzmann equation models.

We don't spend any words on regularity of Ω, as we are not interested to point out which are the weakest hypotheses that ensure the existence of a solution. We will suppose that Ω is a non-empty connected set with a piecewise smooth boundary.

Once we have fixed boundary conditions we have only to define the functional space where the solutions of the initial-boundary problem live; after that we are in condition to give an existence and uniqueness result. Given the functional space:

$$X_\Omega = (L^1(\Omega) \cap L^\infty(\Omega))^n,$$

for $\mathbf{u} \in X_\Omega$ we define

$$\|\mathbf{u}\|_\Omega = \|\mathbf{u}\|_{1,\Omega} + \|\mathbf{u}\|_{\infty,\Omega} = \sum_{i=1}^n \|u_i\|_{L^1(\Omega)} + \max_i \|u_i\|_{L^\infty(\Omega)}.$$

Clearly X_Ω, endowed with the norm $\|\cdot\|_\Omega$, is a Banach space. We are then able to state the following theorem.

Theorem 6.2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded set, let B^i_{hk} and η be functions verifying (3.4), (3.6) and (3.7). Then, given $\overline{\mathbf{f}} \in X_\Omega$ a non-negative initial datum, there exists a unique $\mathbf{f} \in C([0,\infty), X_\Omega)$ solution of the following initial-boundary value problem*

$$\begin{cases} \partial_t f_i(t,\mathbf{x}) + \mathbf{v}_i \cdot \nabla_{\mathbf{x}} f_i(t,\mathbf{x}) = J_i(\mathbf{f}(t,\mathbf{x})), & \\ \mathbf{f}(0,\mathbf{x}) = \overline{\mathbf{f}}(\mathbf{x}), & \forall \mathbf{x} \in \Omega, \\ \mathbf{f}^-(t,\mathbf{x}) = B(\mathbf{x})\mathbf{f}^+(t,\mathbf{x}), & \forall \mathbf{x} \in \partial\Omega,\ t \geq 0, \end{cases} \tag{6.14}$$

with B_{ij} verifying (6.13). Moreover, we have

$$\frac{dN}{dt}(t) = \frac{d}{dt}\left(\sum_{i=1}^{n} \int_\Omega f_i(t,\mathbf{x})d\mathbf{x}\right) = 0. \tag{6.15}$$

We don't show the detailed proof of previous theorem as it is similar to existence and uniqueness results given in the previous chapters, limiting ourselves to sketch the main ideas. We remind that the interaction domain, defined in (6.2), is intersected with the entire domain Ω and thus the interaction kernel is always well defined in Ω. The hypotheses on interaction rate and transition probability density ensure that $\mathbf{J} = (J_i(\mathbf{f}))$ is a locally Lipschitz function, and thus we can use a fixed point argument to obtain existence an uniqueness of a solution, local in time, $\mathbf{f} \in C([0,T], X_\Omega)$ with $T > 0$. Thanks to Gronwall's inequality we can extend globally-in-time such a solution. The total number of agents is conserved, thanks to reflection on the boundary. If we suppose that there isn't any reflection on the boundary we have the following theorem.

Theorem 6.2.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded set, let B^i_{hk} and η be functions verifying (3.4), (3.6) and (3.7). Then, given $\overline{\mathbf{f}} \in X_\Omega$ a non-negative initial datum, there exists a unique $\mathbf{f} \in C([0,\infty), X_\Omega)$ solution of the initial-boundary value problem*

$$\begin{cases} \partial_t f_i(t,\mathbf{x}) + \mathbf{v}_i \cdot \nabla_{\mathbf{x}} f_i(t,\mathbf{x}) = J_i(\mathbf{f}(t,\mathbf{x})), & \\ \mathbf{f}(0,\mathbf{x}) = \overline{\mathbf{f}}(\mathbf{x}), & \forall \mathbf{x} \in \Omega, \\ \mathbf{f}^-(t,\mathbf{x}) = 0, & \forall \mathbf{x} \in \partial\Omega,\ t > 0. \end{cases} \tag{6.16}$$

Moreover:

$$\frac{dN}{dt}(t) \leq 0. \tag{6.17}$$

Last theorem asserts, roughly speaking, that if there is not any wall, hence we have complete absorption on the boundary, then when agents arrive on the boundary they exit from domain. We can intersect these two theorems as follows. First of all we divide the boundary in two subsets $W, E \subset \partial\Omega$ such that

$$W \cap E = \emptyset, \; W \cup E = \partial\Omega,$$

and $\mathcal{H}^1(E) > 0$, $\mathcal{H}^1(W) > 0$, where $\mathcal{H}$ stands for Hausdorff measure. The last two inequalities ensure, roughly speaking, that there is a large exit and that we want to avoid exits whose that reduce to single points.

Theorem 6.2.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded set, and let B^i_{hk} and η be functions verifying (3.4), (3.6) and (3.7). Then, given $\overline{\mathbf{f}} \in X_\Omega$ a non-negative initial datum, there exists a unique $\mathbf{f} \in C([0,\infty), X_\Omega)$ solution of following initial-boundary value problem*

$$\begin{cases} \partial_t f_i(t,\mathbf{x}) + \mathbf{v}_i \cdot \nabla_{\mathbf{x}} f_i(t,\mathbf{x}) = J_i(\mathbf{f}(t,\mathbf{x})), & \\ \mathbf{f}(0,\mathbf{x}) = \overline{\mathbf{f}}(\mathbf{x}), & \forall \mathbf{x} \in \Omega, \\ \mathbf{f}^-(t,\mathbf{x}) = 0, & \forall \mathbf{x} \in E, \; t > 0, \\ \mathbf{f}^-(t,\mathbf{x}) = B(\mathbf{x})\mathbf{f}^+(t,\mathbf{x}), & \forall \mathbf{x} \in W, \; t \geq 0, \end{cases} \tag{6.18}$$

with

$$\frac{dN}{dt}(t) \leq 0. \tag{6.19}$$

We don't specialize the system 6.18. On the other hand, if we want to describe an evacuating crowd, the transition probability density B^i_{hk} will describe the presence of a desired velocity inside the set of achievable velocities, which minimize the path which brings to exit.

The interaction kernel in the r.h.s. of (6.1) models interactions among agents. If there is a change of velocity induced only by the domain environment, like the shape of domain or the presence of obstacles, agents don't want to collide with walls and if there is any exit they want to reach rapidly it.

An idea, that has been suggested also in [1], is to introduce an extra term in the r.h.s. of (6.1):

$$J_i(\mathbf{f}, \mathbf{x}) = \Gamma_i(\mathbf{f}) - \Lambda_i(\mathbf{f}) + A_i(\mathbf{f}, \mathbf{x}),$$

such that

$$\sum_{i=1}^{n} A_i(\mathbf{f}, \mathbf{x}) = 0.$$

The function $\mathbf{A} = (A_i)$ describes the interaction between agents and domain. We don't discuss here this point but we are going to come back on it in the following chapter.

Chapter 7

Spatial discretization

Models arising by discretization of achievable velocity are based on semilinear hyperbolic equations. In the previous chapter we deeply studied existence and uniqueness of solutions for these systems and, under suitable regularity properties for the interaction kernel, we were also able to give existence for all positive time. On the other hand, it is well known that the study of long-time behavior for general Boltzmann-like systems is a hard issue that still presents aspects difficult to overcome. In order to treat such systems a number of authors [28, 29, 30] suggested to discretize the spatial variable together with the velocity one, to obtain a system of ordinary instead of partial differential equations.

7.1 One dimensional case

Inspired by [28], we start to recover a model with one spatial variable. The domain $\Omega = [0, L] \subset \mathbb{R}$ is divided in m cells, or intervals

$$[0, L] = \bigcup_{i=1}^{m} I_i,$$

with $I_i \cap I_j = \emptyset$ if $i \neq j$ and $x \leq y$,for all $x \in I_i, y \in I_{i+1}$, $i = 1, \dots, m-1$; moreover, the intervals have constant size $\ell = L/m$. The crucial assumption is that inside a given cell I_i the distribution function, which describes agents traveling with velocity v_j, does not depend on the spatial variable and thus the functions $f_j^i(t)$ represent the distribution functions of agents that have velocity v_j and are located in the i-th cell at time t:

$$f(t, x, v) = \sum_{j=1}^{n} \sum_{i=1}^{m} f_j^i(t) \chi_{I_i}(x) \delta_{v_j}(v).$$

The number of agents that have velocity v_j in the i-th cell is $N_j^i = f_j^i \ell$, the total number of agents in the i-th cell is

$$N_i = \sum_{j=1}^{n} N_j^i,$$

and the total number of agents in Ω

$$N = \sum_{i=1}^{m} N_i.$$

As usual, we can find macroscopic quantities like density and flux in the following way:

$$\rho(t,x) = \sum_{j=1}^{n} f_j^i(t)\chi_{I_i}(x), \;\; q(t,x) = \sum_{j=1}^{n} v_j f_j^i(t)\chi_{I_i}(x).$$

As in [28], in order to obtain equations describing the evolution of f_j^i, we start to evaluate the variation of N_j^i:

$$N_j^i(t+\Delta t) - N_j^i(t) = G_j^i([t,t+\Delta t]) - L_j^i([t,t+\Delta t]) + J_j^i(t)\ell\Delta t + o(\Delta t). \quad (7.1)$$

The term G_j^i describes the agents, with velocity v_j, that flow into i-th cell from near cells. If we suppose $v_j \geq 0$ and $v_j \Delta t \leq \ell$ we can assume that G_j^i depends only on N_j^{i-1}. Then, we define:

$$G_j^i([t,t+\Delta t]) = \frac{v_j \Delta t}{\ell} N_j^{i-1}(t) + o(\Delta t). \quad (7.2)$$

Making the same assumptions we define the outer flow:

$$L_j^i([t,t+\Delta t]) = \frac{v_j \Delta t}{\ell} N_j^{i}(t) + o(\Delta t). \quad (7.3)$$

By comparing with the previous chapter, the term J_j^i is like the usual interaction kernel J_j, apart from one point that we will remark later, and it describes the agents in th i-th cell that change velocity. If we divide equation (7.1) by $l\Delta t$ and take the limit $\Delta t \to 0$, we find:

$$\frac{df_j^i}{dt} + \frac{v_j}{\ell}(f_j^i - f_j^{i-1}) = J_j^i(\mathbf{f}), \quad (7.4)$$

where $\mathbf{f} \equiv (f_j^i) \in \mathbb{R}^{nm}$. We write explicitly the interaction kernel J_j^i as follows:

$$J_j^i = \sum_{h,k=1}^{n} \eta_{hk}(i) B_{hk}^j(i) f_h^i f_k^i - f_j^i \sum_{h=1}^{n} \eta_{jh}(i) f_j^i. \quad (7.5)$$

We notice that in the previous chapter we have studied a non local interaction kernel, described by an integral over a visibility zone D_x. In this chapter we forget this issue, focusing on the interactions that take place in the cells. The function $B^j_{hk}(i)$ in (7.5) describes the probability that an agent, with velocity v_h, changes his velocity in v_j after an interaction with an agent having velocity v_k. As in the previous chapters, it has to verify the following conditions:

$$\begin{aligned} &0 \leq B^j_{hk}(i) \leq 1, \quad h,k,j = 1,\dots,n, i = 1,\dots,m; \\ &\sum_{j=1}^{n} B^j_{hk}(i) = 1, \quad h,k = 1,\dots,n, i = 1,\dots,m. \end{aligned} \tag{7.6}$$

The function $\eta_{hk}(i)$ in (7.5) is the interaction rate which represents the number of interactions between f^i_h and f^i_k.

Equation (7.4) is well defined only for $i = 2,\dots,m$. In order to study a system of mn ordinary differential equations like (7.4) we have to impose boundary conditions $f^0_j = f^0_j(t)$ with $j = 1,\dots,n$. Moreover, we have to make the typical hypotheses on functions which are present in the interaction kernel J^i_j that guarantee the well-posedness.

7.2 Periodic problem

We start studying the periodic problem that corresponds, for example, to a circular hallway or a closed track. The appropriate boundary conditions for this situation are $f^0_j(t) = f^m_j(t)$. Clearly this is not the unique way to close (7.4), as we will see later. However, once we have fixed the initial datum, we have the following Cauchy problem:

$$\begin{cases} \dfrac{df^i_j}{dt} + \dfrac{v_j}{\ell}(f^i_j - f^{i-1}_j) = J_{ij}(\mathbf{f}) \quad i = 1,\dots,m, \; j = 1,\dots,n \\ f^i_j(0) = \overline{f}^i_j. \end{cases} \tag{7.7}$$

The previous system shows interesting properties induced by the periodic boundary conditions. If we sum equations (7.7) over j, reminding the properties of J_{ij} we have

$$\frac{d\rho_i}{dt}(t) + \sum_{j=1}^{n} \frac{v_j}{\ell}(f^i_j(t) - f^{i-1}_j(t)) = 0.$$

Now, if we sum over i we find

$$\frac{d}{dt}\left(\sum_{i=1}^{m} \rho_i(t)\right) = 0.$$

The last equality gives us the usual conservation of the total number of agents in the whole domain. In order to obtain a solution to the Cauchy problem (7.7) we have to impose some hypotheses on the interaction rate and on the transition probability density.
Following the ideas used in previous chapters we suppose that $\eta_{hk}(i)$ and $B^j_{hk}(i)$ depend on $\mathbf{f} = (f^i_j)$ and that they are Lipschitz functions, i.e., for all $h, k, j = 1, \dots, n$ and $i = 1, \dots, n$ there exist $\mu, \mu' \geq 0$ such that

$$|\eta_{hk}(i)(\mathbf{f}_1) - \eta_{hk}(i)(\mathbf{f}_2)| \leq \mu|\mathbf{f}_1 - \mathbf{f}_2|, \tag{7.8}$$

$$|B^j_{hk}(i)(\mathbf{f}_1) - B^j_{hk}(i)(\mathbf{f}_2)| \leq \mu'|\mathbf{f}_1 - \mathbf{f}_2|, \tag{7.9}$$

for all, $\mathbf{f}_1, \mathbf{f}_2 \in \mathbb{R}^{nm}$, $h, k, j = 1, \dots, n$ and $i = 1, \dots, n$. Clearly $B^j_{hk}(i)$ verifies (7.6), for all $\mathbf{f} \in \mathbb{R}^{nm}$ and for all indexes. We can easily prove the following theorem.

Theorem 7.2.1. *Let* $\overline{\mathbf{f}} = (\overline{f}^i_j) \in \mathbb{R}^{mn}$ *be a positive initial datum and let* $\eta_{hk}(i)$ *and* $B^j_{hk}(i)$ *be functions verifying (7.8), (7.9) and (7.6), then there exists a unique continuous function* $\mathbf{f} : [0, \infty) \to \mathbb{R}^{mn}$ *solution of (7.7). Moreover* $\mathbf{f}$ *has the following properties*

$$f^i_j(t) \geq 0, \forall j \in \{1, \dots, n\}, i \in \{1, \dots, m\}, t \geq 0, \tag{7.10}$$

$$\sum_{j=1}^{n}\sum_{i=1}^{m} f^i_j(t) = \sum_{j=1}^{n}\sum_{i=1}^{m} \overline{f}^i_j. \tag{7.11}$$

We omit the proof, as it is just an elementary application of a fixed point argument.

7.3 Desired velocity model

In this section we specialize the transition probability density B^j_{hk} and interaction rate $\eta_{hk}(i)$, in order to study more deeply the system obtained by the spatial discretization procedure. Suppose that $0 < v_1 < \ldots < v_n$, we define the table of games B^j_{hk} as follows.
If $h = k$

$$B^j_{hk} = \begin{cases} 1 & \text{if } j = h (= k) \\ 0 & \text{otherwise.} \end{cases} \tag{7.12}$$

If $v_h > v_k$

$$B^j_{hk} = \begin{cases} 1 & \text{if } j = h \\ 0 & \text{otherwise.} \end{cases} \tag{7.13}$$

Elsewhere, if $v_h < v_k$

$$B^j_{hk} = \begin{cases} 1-p & \text{if } j = h \\ p & \text{if } j = h+1 \\ 0 & \text{otherwise,} \end{cases} \tag{7.14}$$

with $p = (0, 1]$. The table of games, previously written, says us that agents have a desired velocity, which is the faster one v_n, and they increase their velocity only when they interact with agents that have a larger speed. We remark that from (7.12) there are not self-interactions in the system. We start to look for stationary solution. Hence we have to search solutions of following system:

$$\frac{v_j}{\ell}\left(f^i_j - f^{i-1}_j\right) = J^i_j(\mathbf{f})$$
$$i = 1, \ldots, m, \; j = 1, \ldots, n.$$

We expect that a stationary solution $\hat{\mathbf{f}}$ is at least a spatially homogeneous one, in other words that $f^h_j = f^k_j$, for all indexes $h, k = 1, \ldots, m$. If we write explicitly the equation for $j = 1$ and suppose that there is only one index i such that $\hat{f}^i_1 \neq \hat{f}^k_1$ and moreover $\hat{f}^k_1 = \hat{f}^{k'}_1$ for $k, k' \neq j$ then

$$\hat{f}^{i-1}_1 = \hat{f}^i_1 (1 + \frac{\ell p}{v_1} \sum_{h=2}^{n} \eta_{1h} \hat{f}^i_h) := \hat{f}^i_1 P_i.$$

and

$$\begin{aligned} \hat{f}^i_1 &= \hat{f}^{i+1}_1 (1 + \frac{\ell p}{v_1} \sum_{h=2}^{n} \eta_{1h} \hat{f}^{i+1}_h) = \hat{f}^{i+1}_1 P_{i+1}, \\ &= \hat{f}^{i-1}_1 P_{i+1} = \hat{f}^i_1 P_i P_{i+1}, \end{aligned}$$

and thus we find a contradiction. At the end we have that $\hat{f}^i_1 = \hat{f}^{i-1}_1$ for all i, from which it follows that if $\hat{f}^i_1 > 0$ then $\hat{f}^i_j = 0$ for $j = 1, \ldots, n$. If we repeat the same strategy for all j we have the complete proof of the following theorem.

Theorem 7.3.1. *For all $N > 0$ there exist n stationary solutions related to α for system (7.7) with table of games defined in (7.12), (7.13) and (7.14). Moreover, for all $j^* \in \{1, \ldots, n\}$*

$$\mathbf{E}_{j^*} = (\hat{f}^i_j) = \begin{cases} N/m & j = j^* \text{ and } i = 1, \ldots, m \\ 0 & \text{otherwise,} \end{cases}$$

is a stationary solution.

In the previous theorem N represents the total number of agents in the system, which is constant in time. Once we have fixed the number of agents we have n stationary solution related to (7.7), where n is the number of velocity classes; each stationary solution describes a state in which all agents travel with the same velocity and are homogeneously distributed in the m cells.

In order to study stability of stationary solutions, found in Theorem 7.3.1, we fix the number of velocities to be $n = 2$ and the number of cells at $m = 3$, and for simplicity we take a constant interaction rate $\eta_{hk} := \eta > 0$. Given a $N > 0$, thanks to the conservation of agents' number we study the evolution system on the hyperplane of equation:

$$\sum_{i=1}^{3}\sum_{j=1}^{2} \hat{f}_j^i = N = \sum_{i=1}^{3}\sum_{j=1}^{2} f_j^i = N.$$

In this way we can reduce the equations number, replacing $f_2^3(t)$ with

$$N - f_1^1(t) - f_2^1(t) - f_1^2(t) - f_2^2(t) - f_1^3(t),$$

so we write

$$\begin{cases} \dfrac{df_1^1}{dt} = -\alpha f_1^1 f_2^1 - \dfrac{v_1}{\ell}\left(f_1^1 - f_1^3\right) \\ \dfrac{df_2^1}{dt} = \alpha f_1^1 f_2^1 - \dfrac{v_2}{\ell}\left(f_1^1 + 2f_2^1 + f_1^2 + f_2^2 + f_1^3 - N)\right) \\ \dfrac{df_1^2}{dt} = -\alpha f_1^2 f_2^2 - \dfrac{v_1}{\ell}\left(f_1^2 - f_1^1\right) \\ \dfrac{df_2^2}{dt} = \alpha f_1^2 f_2^2 - \dfrac{v_2}{\ell}\left(f_2^2 - f_2^1\right) \\ \dfrac{df_1^3}{dt} = -\alpha f_1^3 (N - f_1^1 - f_2^1 - f_1^2 - f_2^2 - f_1^3) - \dfrac{v_1}{\ell}\left(f_1^3 - f_1^2\right) \end{cases} \tag{7.15}$$

where $\alpha = p\eta$. By means of Theorem 7.3.1 of the previous section we have that equilibrium points of the reduced system are:

$$\mathbf{E}_1 = (N/3, 0, N/3, 0, N/3),$$
$$\mathbf{E}_2 = (0, N/3, 0, N/3, 0).$$

In order to study stability of equilibria we evaluate the Jacobian matrix related to (7.15)

$$\mathcal{J} = \begin{pmatrix} -\alpha f_2^1 - \frac{v_1}{\ell} & -\alpha f_1^1 & 0 & 0 & \frac{v_1}{\ell} \\ \alpha f_2^1 + \frac{v_2}{\ell} & \alpha f_1^1 - 2\frac{v_2}{\ell} & -\frac{v_2}{\ell} & -\frac{v_2}{\ell} & -\frac{v_2}{\ell} \\ \frac{v_1}{\ell} & 0 & -\alpha f_2^2 - \frac{v_1}{\ell} & -\alpha f_1^2 & 0 \\ 0 & \frac{v_2}{\ell} & \alpha f_2^2 & \alpha f_1^2 - \frac{v_2}{\ell} & 0 \\ \alpha f_1^3 & \alpha f_1^3 & \alpha f_1^3 + \frac{v_1}{\ell} & \alpha f_1^3 & 2\alpha f_1^3 - \frac{v_1}{\ell} \end{pmatrix}.$$

After cumbersome calculations we find that the eigenvalues associated to $\mathcal{J}(\mathbf{E}_2)$ are roots of the polynomial:

$$P(\lambda) = (\lambda^2 + 3c\lambda + 3c^2)(b^3 + (b + \lambda)(b + a + \lambda)^2) = 0,$$

where $a = \frac{\alpha N}{3}$, $b = \frac{v_1}{l}$ and $c = \frac{v_2}{l}$. It is easy to see that all roots have strictly negative real part, so we have that $\mathbf{E}_2$ is asymptotically stable. Since we have only two stationary solutions and we know that one of them is asymptotically stable, we conclude that the other one, that is $\mathbf{E}_1$, has to be unstable.
The analysis just performed Previously concerns the stability of stationary solutions for the periodic case in a very particular case. On the other hand, m and n can be larger numbers, specially m. We know that 7.7 has n stationary solutions, but we would like to avoid to go through the eigenvalues' analysis of the Jacobian matrix.

We intuitively expect that only one stationary point, the one related to faster velocity v_n, is asymptotically stable, and the others are at least saddles. We start to show that $\mathbf{E}_1$:

$$\mathbf{E}_1 = \begin{cases} N/m & \text{if } j = 1, \text{ i=1,...,n} \\ 0 & \text{otherwise,} \end{cases}$$

is actually a saddle. For any $\varepsilon > 0$ we take $\mathbf{P}$ defined as:

$$\mathbf{P} = \begin{cases} \frac{N}{m} - \frac{\varepsilon}{3m} & \text{if } j = 1, \text{ i=1,...,n} \\ \frac{\varepsilon}{3m} & \text{if } j = 2, \text{ i=1,...,n} \\ 0 & \text{otherwise,} \end{cases}$$

as initial datum. First we remind that if the initial datum is spatially homogeneous, i.e., it does not depend on index i, $f_j^{i+1} = f_j^i$, then the solution is also spatially homogeneous. In addition, the solution $f(\mathbf{P}, t)$ of (7.7) with $\mathbf{P}$ as initial datum has the property:

$$f_j^i(\mathbf{P}, t) = 0, \quad \text{for } t \geq 0,\ i = 1, \dots, n \text{ and } j = 3, \dots, m.$$

In this case we can reduce (7.7) to:

$$\frac{df_1}{dt} = -\alpha f_1 f_2$$
$$\frac{df_2}{dt} = \alpha f_1 f_2$$

where we have omitted the dependence on i. We have that f_1 is a decreasing function and f_2 is increasing and

$$\frac{df_1}{dt} = -\alpha f_1 f_2 < -\frac{\alpha\varepsilon}{3m} f_1,$$

and thus

$$f_1(\mathbf{P},t) \le \left(\frac{N}{m} - \frac{\varepsilon}{3m}\right) e^{-\frac{\alpha\varepsilon}{3m}t}.$$

Using this last inequality we find that $f_1^i \to 0$ and $f_2^i \to \frac{N}{m}$ when $t \to 0$, from which follows that $\mathbf{E}_1$ is at least a saddle. Following the same steps we are able to prove the instability of $\mathbf{E}_2, \dots, \mathbf{E}_{n-1}$. Now we want to prove that $\mathbf{E}_n$ is asymptotically stable, and this is a task more difficult than the previous one.

Let us fix a point $\mathbf{P} = (P_j^i)$ with the property that

$$P_n^i > 0, \quad \text{i=1,...,m.} \tag{7.16}$$

and notice that every point in a sufficiently small neighborhood of $\mathbf{E}_n$ does have the previous property. Now we study the trend of the solution $f(\mathbf{P},t)$ of (7.7) starting from $\mathbf{P}$, and we'll prove that $f(\mathbf{P},t) \to \mathbf{E}_n$ when $t \to \infty$. Let

$$f_j := \sum_{i=1}^m f_j^i,$$

be the number of agents having velocity v_j. The idea is to prove that each f_j, with $j \neq n$ decays for $t \to \infty$. Let

$$P_j = \sum_{i=1}^m P_j^i = 0,$$

then, if $P_1 = 0$ from the nature of system (7.7) and of B_{hk}^j,

$$\frac{df_1}{dt}(t) = 0,$$

and thus $f_1(t) = 0$ for all $t \ge 0$. On the other hand, if $P_1 > 0$, that is, if there are agents with lower velocity, then $f_1(t) > 0$ at least in a interval $[0,T)$ and moreover it is easy to see that:

$$\begin{aligned} \frac{df_1}{dt}(t) &= -\alpha \sum_{i=1}^m \sum_{h=2}^n f_1^i(t) f_h^i(t) = -\alpha \sum_{i=1}^m f_n^i(t) f_1^i(t) \\ &\le -\alpha \min_i \left(f_n^i(t)\right) \sum_{i=1}^m f_1(t) = -\alpha \min_i \left(f_n^i(t)\right) f_1(t), \end{aligned}$$

Again:

$$\min_i \left(f_n^i(t)\right) \ge \min_i P_n^i = \gamma > 0,$$

thanks to (7.16). By using Gronwall's lemma we find

$$f_1(t) \le P_1 \exp(-\alpha\gamma t).$$

Thanks to this last inequality we are able to study the trend of f_2. Indeed:

$$\begin{aligned}\frac{df_2}{dt}(t) &= \alpha \sum_{i=1}^{m}\sum_{h=2}^{n} f_1^i(t) f_h^i(t) - \alpha \sum_{i=1}^{n}\sum_{h=3}^{n} f_2^i(t) f_h^i(t) \\ &\le \alpha N f_1(t) - \alpha\gamma f_2(t) \\ &\le \alpha N P_1 \exp(-\alpha\gamma t) - \alpha\gamma f_2(t),\end{aligned}$$

from which

$$f_2(t) \le (P_2 + \alpha N P_1 t) \exp(-\alpha\gamma t).$$

follows. For the sake of clearness we repeat the same calculations for $f_3(t)$:

$$\begin{aligned}\frac{df_3}{dt}(t) &= \alpha \sum_{i=1}^{m}\sum_{h=3}^{n} f_2^i(t) f_h^i(t) + \alpha \sum_{i=1}^{n}\sum_{h=4}^{n} f_3^i(t) f_h^i(t) \\ &\le \alpha N (P_2 + \alpha N P_1 t) \exp(-\alpha\gamma t) - \alpha\gamma f_2(t),\end{aligned}$$

from which it follows that:

$$\begin{aligned}f_3(t) &\le \left(P_3 + \alpha N \int_0^t (P_2 + \alpha N P_1 \tau d\tau)\right) \exp(-\alpha\gamma t) \\ &\le \left(P_3 + \alpha N P_2 t + \alpha^2 N^2 P_1 t^2\right) \exp(-\alpha\gamma t).\end{aligned}$$

Finally, we can state that:

$$f_j(t) \le p_{j-1}(t) \exp(-\alpha\gamma t),$$

for $j = 1, \ldots, n-1$ and p_j is polynomial of degree j with positive coefficients, and

$$\lim_{t\to\infty} f_j(t) = 0, \quad j = 1, \ldots, n-1.$$

Reminding that the total number of agents is conserved we have

$$\lim_{t\to\infty} f_n(t) = N.$$

Hence we have proved a large part of following theorem.

Theorem 7.3.2. *Let $N > 0$ and let*

$$C = \left\{ \mathbf{x} \in \mathbb{R}^{nm} \,\middle|\, x_j^i \ge 0, \sum_{i=1}^{m}\sum_{i^1}^{n} x_j^i = N \right\},$$

the nm-simplex related to N, and

$$D = \left\{ x \in C \,\middle|\, \forall i, x_n^i > 0 \right\} \subset C.$$

Then, if we take the initial datum $\overline{\mathbf{f}}$ of (7.7) such that $\overline{f} \in C$, then $\mathbf{f}(t) \in C$ for all t. Moreover, if $\overline{\mathbf{f}} \in D$ then the solution converges to $\mathbf{E}_n$, that is:

$$\lim_{t\to\infty} \mathbf{f}(t) = \mathbf{E}_n.$$

It is worth stressing that we have only proved that all agents achieve the maximal velocity, i.e.,

$$\lim_{t\to\infty} f_n(t) = N.$$

On the other hand we have not proved that the solution converges to the homogeneous state E_n. Actually, this assertion is strictly related to the properties of the flux term $\psi_j^i(f)$ and will be discussed in Section 7.9.

7.4 Non-dimensionalization

Though from a theoretical point of view it is not important, when one would like to perform some simulations it becomes convenient to work with dimensionless variables. We choose characteristic length, time and velocity as follows:

$$x_c := \ell, \; v_c := v_{max}, \; t_c := \frac{x_c}{v_c} = \frac{\ell}{v_{max}},$$

from which we find dimensionless variables and functions

$$x^* := \frac{x}{x_c}, \; t^* := \frac{t}{t_c}, \; v^* = \frac{v}{v_c}, \; g_j^i(t^*) := f_j^i(t_c t^*);$$

if we substitute these variables into the equations we find

$$\frac{dg_j^i}{dt^*} + v_j^*(g_j^i - g_j^{i-1}) = (J_j^i)^*(g)$$

where the new dimensionless interaction rate is

$$\eta_{hk}^*(i) := t_c \eta_{hk}(i).$$

and the new velocity lattice is such that $0 \le v_i \le 1$ for all $i = 1, \dots, n$. Omitting asterisks and substituting g with f we have new dimensionless equations

$$\frac{df_j^i}{dt}(t) + v_j(f_j^i - f_j^{i-1}) = J_j^i(f).$$

7.5 Transition probability depending on local density

In section 7.3 we have studied a model in which the transition probability does not depend on density. On the other hand we have an existence results that allows for a dependence on f through the local density ρ_i. We recall an idea introduced in [26] and we adjust it to the present framework. First of all we introduce the dimensionless density

$$\rho_i^*(t) := \frac{\rho_i(t)}{N_M},$$

where N_M is the maximum number of vehicles, corresponding, in the vehicular traffic context, to a bump-to-bump traffic jam. Once again, in the sequel we will omit asterisks. Thanks to normalization we have

$$N(t) = \sum_{i=1}^{m} \rho_i(t) \leq 1.$$

As in [26] we introduce the probability of passing depending on local density

$$p = p(\rho_i) = (1 - \rho_i)^q, \text{ with } q \geq 1, \tag{7.17}$$

and we fix the transition probability through p. When $h < k$ the candidate agents is interacting with a faster agent and he will change or maintain his velocity depending on surrounding environment:

$$A_{hk}^j(i) = A_{hk}^j(\rho_i) = \begin{cases} 1 - (1 - \rho_i)^q & \text{if } j = h, \\ (1 - \rho_i)^q & \text{if } j = h + 1, \\ 0 & \text{otherwise.} \end{cases} \tag{7.18}$$

On the other hand, if $h > k$ candidate agents will decelerate if there will be an high density inside cell, and thus:

$$A_{hk}^j(\rho_i) = \begin{cases} 1 - (1 - \rho_i)^q & \text{if } j = k, \\ (1 - \rho_i)^q & \text{if } j = h, \\ 0 & \text{otherwise.} \end{cases} \tag{7.19}$$

Finally, we fix the transition probability related to an encounter between two agents that have the same velocity, $h = k$, the easiest choice is the following:

$$A_{hh}^j(\rho_i) = \begin{cases} 1 & \text{if } j = h, \\ 0 & \text{otherwise.} \end{cases} \tag{7.20}$$

We spend few words about the exponent q in the probability of passing. If we have m cells the maximum density achievable inside each cell is $1/m$ and we have to choose q such that if $\rho_i \approx 1/m$ then $p(\rho_i) \approx 0$. Another possible choice could be a piecewise function

$$p(\rho_i) = \begin{cases} 1 - m\rho_i & \text{if } \rho_i \leq 1/m, \\ 0 & \text{otherwise.} \end{cases} \tag{7.21}$$

7.6 Elastic collisions at the boundary

We derived (7.4) supposing that $v_j \geq 0$. On the other hand we can suppose that $v_j \leq 0$ for all $j = 1, \ldots, n$. After similar steps we find

$$\frac{df_j^i}{dt} + \frac{v_j}{\ell}(f_j^{i+1} - f_j^i) = J_j^i(\mathbf{f}), \tag{7.22}$$

for all i, j. Comparing equation (7.4) with (7.22) we notice that the spatial derivative is replaced by different terms: in (7.4) there is a backward difference while in (7.22) there is a forward difference, and this is due to the different signs of velocities. Clearly if $v_j < 0$ the inflow is proportional to f_j^i, while the outflow is related to f_j^{i+1} and not to f_j^{i-1} as in (7.4).
We have to prescribe boundary conditions related $f_j^{m+1} = f_j^{m+1}(t)$ which describe the inner flow respect to negative velocities.
We mix equations (7.4) and (7.22) in order to allow for both positive and negative velocities. Let j be the index which labels velocity, $j \in \{-n, \ldots, 0, \ldots, n\}$. If we associate to a negative index $-j$ a negative velocity $-v_j$, we can continue to impose periodic boundary conditions, taking $f_{-j}^{m+1}(t) = f_{-j}^1(t)$. On the other hand it is also interesting to consider the possibility of collisions at the boundary, meaning that agents can't leave Ω. The simplest idea is to impose elastic collisions:

$$\begin{aligned} f_j^0(t) &= f_{-j}^1(t), \\ f_{-j}^{m+1}(t) &= f_j^m(t). \end{aligned}$$

With this choice we have only to impose initial conditions to study the following system:

$$\begin{cases} \dfrac{df_j^i}{dt}(t) + \dfrac{v_j}{\ell}(f_j^i(t) - f_j^{i-1}(t)) = J_j^i(\mathbf{f}(t)) \\ \dfrac{df_{-j}^i}{dt}(t) + \dfrac{v_{-j}}{\ell}(f_{-j}^{i+1}(t) - f_{-j}^i(t)) = J_{-j}^i(\mathbf{f}(t)) \\ \mathbf{f}(0) = \overline{\mathbf{f}}, \end{cases} \tag{7.23}$$

where the interaction kernel has the same properties as in the periodic case, and

$$\sum_{j=-n}^{n} J_j^i(\mathbf{f}) = 0.$$

Similarly to the periodic case we have, under the same hypotheses, existence and uniqueness of a solution and its prolongability in time.

7.7 Two dimensional case

The work done in the previous section permits to treat velocities with different directions, that is the first step to make possible studying problems that take place in more complicated spatial domains. For example, if we want to describe the vehicular traffic flow it is sufficient to work with a one dimensional spatial variable, i.e. $x \in \mathbb{R}$, while if we want to model a crowd exiting from a room it is necessary to account for $\mathbf{x} \in \mathbb{R}^2$. In this section we derive a spatially discrete model when the domain Ω is a square set

$$\Omega = [0, L] \times [0, L] = \bigcup_{i,p=1}^{m} I_{ip},$$

where $I_{i,p}$ are m^2 disjoint square sets of fixed area ℓ^2, such that $L/\ell = m \in \mathbb{N}$, ordered in the following sense: let $(x_1, y_1) \in I_{i_1,p_1}, (x_2, y_2) \in I_{i_2,p_2}$, then

$$x_1 < x_2 \Rightarrow i_1 < i_2,\ \ y_1 < y_2 \Rightarrow p_1 < p_2.$$

Inside $I_{i,p}$ the agents distribution function is spatially homogeneous. As in the previous section we suppose that the set of achievable velocities is finite. The first step is to take into account four velocities

$$\{\mathbf{v}_1 = (1, 0), \mathbf{v}_2 = (0, 1), \mathbf{v}_3 = (-1, 0), \mathbf{v}_4 = (0, -1)\}.$$

The agents distribution function, which describes the evolution of the system, depends again only on time t in $I_{i,p}$:

$$f(t, \mathbf{x}, \mathbf{v}) = \sum_{i,p=1}^{m} \sum_{j=1}^{4} f_j^{i,p}(t) \chi_{I_{i,p}}(\mathbf{x}) \delta_{v_j}(\mathbf{v}).$$

On the other hand, the flux term related to spatially discretization like the second term in the l.h.s. in (7.4) is strictly related to the orientation of velocity, see for example the difference between (7.4) and (7.22). For example,

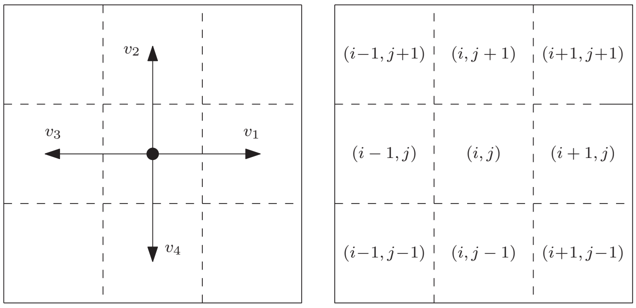

Figure 7.1: cells indexing in the two dimensional case and the four achievable velocities.

if we take v_2 and fix $I_{i,p}$ and we want to find the equation like (7.4) which describes the evolution of $f_2^{i,p}$, the flux will be connected to $f_2^{i,p-1}$. If we do same assumptions as in the previous section, we find

$$\frac{df_2^{i,p}}{dt} + \frac{\xi_j}{\ell}(f_2^{i,p} - f_2^{i,p-1}) = J_2^{i,p}(\mathbf{f}), \tag{7.24}$$

with $\mathbf{f} = (f_j^{i,p})$. The interaction kernel is equal to (7.5), except for the presence of another index p,

$$J_j^{i,p}(\mathbf{f}) = \sum_{h,k=1}^{4} \eta_{hk}(i,p) A_{hk}^j(i) f_h^{i,p} f_k^{i,p} - f_j^{i,p} \sum_{h=1}^{4} \eta_{jh}(i) f_j^{i,p}. \tag{7.25}$$

Making the same for all $j = 1, 2, 3, 4$, we have

$$\frac{df_j^{i,p}}{dt} + \ell^{-1}\psi_j^{i,p}(\mathbf{f}) = J_j^{i,p}(\mathbf{f}), \tag{7.26}$$

where

$$\psi_j^{i,p}(\mathbf{f}) = \begin{cases} f_1^{i,p} - f_1^{i-1,p}, \\ f_2^{i,p} - f_2^{i,p-1}, \\ f_3^{i-1,p} - f_3^{i,p}, \\ f_4^{i,p-1} - f_4^{i,p}. \end{cases} \tag{7.27}$$

Equations (7.26) are well defined for $i, p = 2, \ldots, m-1$. On the other hand we have to impose boundary conditions along walls, and make hypotheses on $\eta_{hk}(i)$ and $B_{hk}^j(i,p)$. In particular we here suppose that an agent, reaching a wall with normal velocity, has an elastic collision, that is:

$$\begin{aligned} &f_1^{0,p}(t) = f_3^{1,p}(t), \\ &f_2^{m+1,p}(t) = f_4^{m,p}(t), \\ &f_3^{i,m+1}(t) = f_1^{i,m}(t), \\ &f_4^{i,0}(t) = f_2^{2,1}(t). \end{aligned}$$

In addition, we assume that $\eta_{hk}(i,p)$ and $B^j_{hk}(i,p)$ are Lipschitz functions depending on $\mathbf{f} = (f^{i,p}_j)$, and thus for all $\mathbf{f}_1, \mathbf{f}_2 \in \mathbb{R}^{4m}$, $h,k,j = 0,\dots,4$ and $i,p = 1,\dots,n$ there exist $\lambda, \lambda' \geq 0$ such that

$$|\eta_{hk}(i,p)(\mathbf{f}_1) - \eta_{hk}(i,p)(\mathbf{f}_2)| \leq \lambda |\mathbf{f}_1 - \mathbf{f}_2|, \tag{7.28}$$

$$|B^j_{hk}(i,p)(\mathbf{f}_1) - B^j_{hk}(i,p)(\mathbf{f}_2)| \leq \lambda' |\mathbf{f}_1 - \mathbf{f}_2|. \tag{7.29}$$

Moreover,

$$B^j_{hk}(i,p)(\mathbf{f}) \geq 0, \ \sum_{j=1}^{4} B^j_{hk}(i,p)(\mathbf{f}) = 1, \tag{7.30}$$

for all $\mathbf{f} \in \mathbb{R}^{4m}$ and for all indexes. Now we are able to study the Cauchy problem:

$$\begin{cases} \dfrac{df^{i,p}_j}{dt} + \ell^{-1}\psi^{i,p}_j(\mathbf{f}) = J^{i,p}_j(\mathbf{f}) \quad i = 1,\dots,m, \ j = 1,\dots,n \\ f^{i,p}_j(0) = \overline{f}^{i,p}_j, \end{cases} \tag{7.31}$$

for which the following result holds.

Theorem 7.7.1. *Let $\overline{\mathbf{f}} = (\overline{f}^{i,p}_j) \in \mathbb{R}^{4m}$ be a positive initial data and let $\eta_{hk}(i,p)$ and $B^j_{hk}(i,p)$ be functions verifying (7.28), (7.29) and (7.30). Then, there exists a unique continuous function $f : [0,\infty) \to \mathbb{R}^{4m}$ solution of (7.31). Moreover, f has the following properties*

$$f^i_j(t) \geq 0, \forall j \in \{1,\dots,4\}, i \in \{1,\dots,m\}, t \geq 0, \tag{7.32}$$

$$\sum_{j=1}^{4}\sum_{i,p=1}^{m} f^i_j(t) = \sum_{j=1}^{4}\sum_{i,p=1}^{m} \overline{f}^i_j. \tag{7.33}$$

The proof of previous theorem is equal to the (7.2.1) one. It is possible to make some generalizations.

- We can increase the number of velocities, introducing velocities with different norms. In this case the equations are written as follows,

$$\frac{df^{i,p}_j}{dt} + |v_j|\ell^{-1}\psi^{i,p}_j(f) = J^{i,p}_j(f).$$

  In this case, in order to close the systems of ODEs we can impose again elastic conditions.

- We can introduce a zero velocity distribution, which describes agents that are not moving trough cells, i.e. $v = 0$. In this case the flux term is equat to 0,

$$\psi^i_0(\mathbf{f}) = 0,$$

  and there are no boundary conditions.

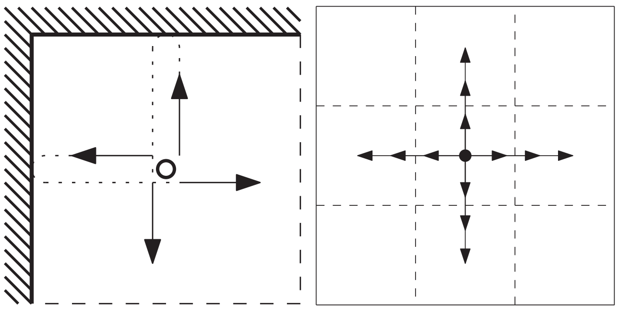

Figure 7.2: Boundary conditions in the four velocities case and an example of 12 velocities.

7.8 Eight velocities model

In view of applications, it is to increase the number of directions of velocities and not only their norms. in the scheme of Figure 7.3 we have eight velocities

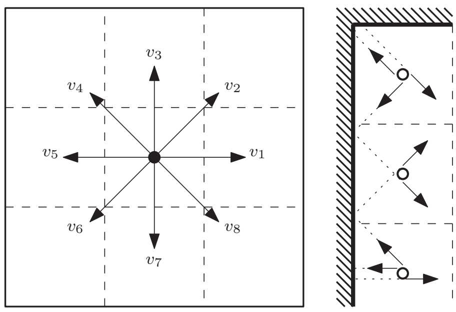

Figure 7.3: Eight velocities case and elastic boundary condition related to eight velocities case.

$$\mathbf{v}_1 = (1,0), \mathbf{v}_2 = (\sqrt{2}/2, \sqrt{2}/2), \mathbf{v}_3 = (0,1), \mathbf{v}_4 = (-\sqrt{2}/2, \sqrt{2}/2),$$

$$\mathbf{v}_5 = (-1,0), \mathbf{v}_6 = (-\sqrt{2}/2, -\sqrt{2}/2), \mathbf{v}_7 = (0,-1), \mathbf{v}_8 = (\sqrt{2}/2, -\sqrt{2}/2).$$

Each velocity is associated to a different flux. For $\mathbf{v}_1$, $\mathbf{v}_3$, $\mathbf{v}_5$, $\mathbf{v}_7$ the flux term is equal to the one in (7.27), while for the others

$$\begin{aligned}
\psi_2^i(\mathbf{f}) &= \ell^{-1}2^{-\frac{1}{2}}(f_2^{i,p} - f_2^{i+1,p+1}),\\
\psi_4^i(\mathbf{f}) &= \ell^{-1}2^{-\frac{1}{2}}(f_4^{i,p} - f_4^{i-1,p+1}),\\
\psi_6^i(\mathbf{f}) &= \ell^{-1}2^{-\frac{1}{2}}(f_6^{i,p} - f_6^{i-1,p-1}),\\
\psi_8^i(\mathbf{f}) &= \ell^{-1}2^{-\frac{1}{2}}(f_8^{i,p} - f_8^{i+1,p-1}).
\end{aligned}$$

In Figure 7.3 there is also an example of elastic boundary conditions related to the eight velocities case.
We want to describe an exiting crowd from a square room. In this case we have to impose different boundary condition in cells near exits. In order to explain this idea we make an example. Let us suppose that there exists one exit on the right wall and that it is large as one cell. We remind that each cell on the boundary, that are those of the type: $I_{1,i}$, $I_{i,1}$, $I_{m,i}$ and $I_{i,m}$ for $i = 1, \ldots, m$ show some problem with particular velocities. For example cells of the type $I_{1,i}$ have problem with $f_1^{i,i}$, $f_8^{i,i}$ and $f_2^{i,i}$ because we have to create ghost cells of $I^{0,i}$ to close their evolution equations. With elastic boundary condition, we join these distribution functions with the others of close cells

$$\psi_1^{1,i}(\mathbf{f}) = \ell^{-1}(f_1^{1,i} - f_5^{1,i});$$

if there is an exit we have no inner flow and in this case it is

$$\psi_1^{5,i}(\mathbf{f}) = \ell^{-1} f_1^{1,i}.$$

In the case of a closed bx we have the conservation of agents number but if

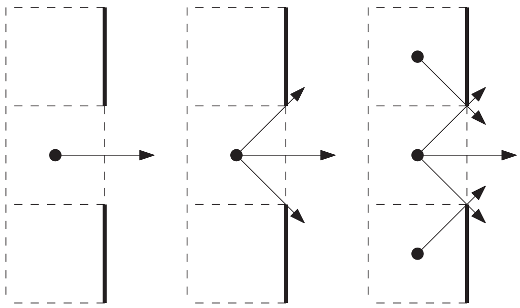

Figure 7.4: There are different ideas of modeling an exit.

we have an exit clearly we find

$$\frac{dN}{dt}(t) := \frac{d}{dt}\left(\sum_{i,p=1}^{m}\sum_{j=1}^{8} f_j^{i,p}(t)\right) \leq 0,$$

We describe formally the presence of exits in this frame. First of all the domain Ω

$$\Omega = \mathrm{int}\Omega \cup \partial\Omega,$$

is divided into square cells, that have different properties: the internal cells

$$\mathrm{int}\Omega = \{I_{i,p} | 1 < i < m \text{ and } 1 < p < m\},$$

and the boundary cells

$$\partial\Omega = \{I_{i,p} | i = 1, m \text{ or } p = 1, m\}$$

In presence of some exits the boundary is divided into wall cells and exit cells

$$\partial\Omega = E \cup W.$$

Now, for all $I_{i,p}$, there exists a desired direction that minimizes the distance between $I_{i,p}$ and the exit cell at the boundary:

$$\text{des}(I_{i,p}) := \min_{I_{h,q} \in E} \frac{I_{h,q} - I_{i,p}}{\|I_{h,q} - I_{i,p}\|},$$

where the difference between two cells is the difference between their centers. Finally, we are able to define the desired velocity labeled with an index $j^*(i,p)$,

$$j^*(i,p) := \arg\max_j \left(\mathbf{v}_j \cdot \text{des}(I_{i,p})\right).$$

Once we have defined the desired velocity we can choose the transition

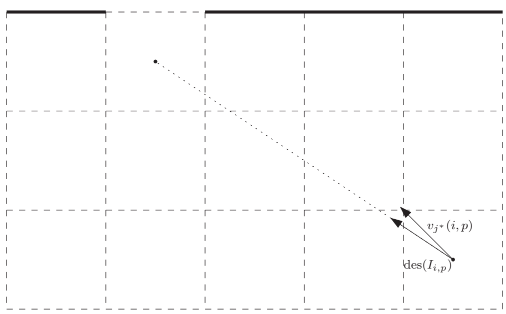

Figure 7.5: Idea of construction of desired velocity, It is strictly related to vector between two cells.

probability $B^j_{hk}(i,p)$. The simplest way is the following:

$$B^j_{hk}(i,p) = \begin{cases} 1 & \text{if } j = j^*(i,p) \\ 0 & \text{otherwise.} \end{cases} \tag{7.34}$$

This choice could seems too simple because it does not account for the local or neighbor density, but it is the first step in order to carefully describe this phenomenon. If we look at Figure 7.6 it is not clear if $N(t) \to 0$ when $t \to \infty$. First of all we study the case in which there isn't a wall and the desired velocities will be the ones pointing it. If we suppose that

N_e	$N(1)$	$N(4)$	$N(8)$	$N(12)$	$N(16)$	$N(20)$
1	0.9989	0.9956	0.9909	0.9861	0.9810	0.9759
3	0.9975	0.9901	0.9797	0.9688	0.9575	0.9460
5	0.9962	0.9845	0.9686	0.9519	0.9347	0.9171
7	0.9948	0.9790	0.9574	0.9352	0.9122	0.8889
9	0.9934	0.9735	0.9463	0.9185	0.8899	0.8610
11	0.9921	0.9680	0.9353	0.9018	0.8677	0.8332
13	0.9907	0.9625	0.9242	0.8852	0.8456	0.8056
15	0.9907	0.9625	0.9242	0.8852	0.8456	0.8056
17	0.9893	0.9570	0.9131	0.8686	0.8235	0.7780
19	0.9870	0.9476	0.8943	0.8404	0.7857	0.7305
21	0.9856	0.9421	0.8833	0.8238	0.7634	0.7025

Table 7.1: In the table we observe that a large exit enhances the escape. N_e is the number of exit cells, in this case $m = 21$ and each simulation starts from a homogeneous initial data such that $N(0) = 1$.

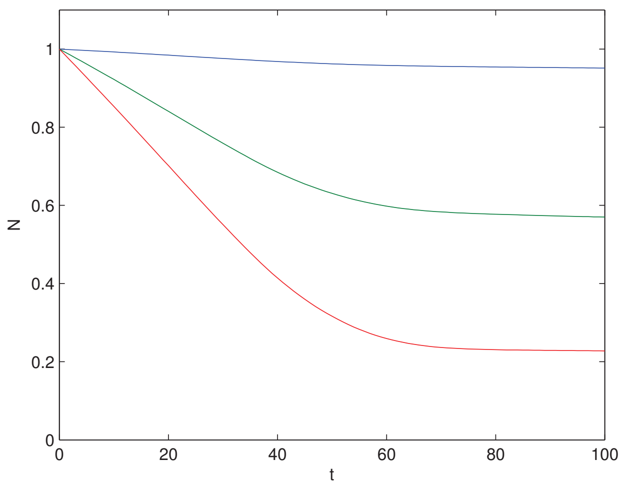

Figure 7.6: In this figure we report the percentage of agents N in function of time t, in three special case. The blue line is the case of $N_e = 1$, the green one $N_e = 11$ and the red one $N_e = 21$. In this last case there isn't any right wall.

there isn't any right wall, the desired velocity is $\mathbf{v}_1$. Moreover we suppose that interaction rate is constant and it is not restrictive to assume $\eta = 1$. Defining

$$\beta(t) = \sum_{p,i=1}^{m} \sum_{j \neq 1,5} f_j^{i,p}(t)$$

then, reminding that $f_j^{i,p}(t) \geq 0$ for all indexes and all time, we find

$$\begin{aligned}
\frac{d\beta}{dt} &= \sum_{i,p=1}^{m} \sum_{j \neq 1,5} J_j^{i,p} - \frac{1}{\sqrt{2}} \sum_{p=1}^{m} (f_2^{m,p} + f_8^{m,p}) \\
&\leq \sum_{i,p=1}^{m} \sum_{j \neq 1,5} J_j^{i,p} = - \sum_{i,p=1}^{m} \sum_{j \neq 1,5} f_j^{i,p} \rho_{i,p} \\
&\leq - \sum_{i,p=1}^{m} \sum_{j \neq 1,5} f_j^{i,p} \sum_{h=1}^{8} f_h^{i,p} \leq - \sum_{i,p=1}^{m} \left(\sum_{j \neq 1,5} f_j^{i,p} \right)^2 \leq -\beta^2
\end{aligned}$$

and thus,

$$\beta(t) \leq \frac{1}{t + k_1}, \tag{7.35}$$

with $k_1 > 0$ if $\beta(0) > 0$. On the other hand if $\beta(0) = 0$ then $\beta(t) = 0$ for all t. Resuming, the last inequalities say us that, not even if we forget the absorption phenomenon caused by the absence of right wall, all agents will change their velocities.
Now, defining

$$\lambda_i(t) = \sum_{p=1}^{m} f_5^{i,p}(t)$$

we find

$$\begin{aligned}
\frac{d\lambda_m}{dt} &= \sum_{p=1}^{m} (J_5^{m,p}(f) - f_5^{m,p}) \\
&\leq -\lambda_m(\lambda_m + 1) \leq -\lambda_m,
\end{aligned}$$

and thus

$$\lambda_m(t) \leq \lambda_m(0)e^{-t}.$$

The content of last inequality is quite obvious. Even if we forget the change of velocity caused by the interaction kernel, thanks to absorption boundary conditions we have all agents flowing out, and hence we can use it as follows

$$\frac{d\lambda_{m-1}}{dt} \leq -\lambda_{m-1} + \lambda_m \leq \lambda_{m-1} + \lambda_m(0)e^{-t}.$$

and thus

$$\lambda_{m-1}(t) \leq e^{-t}\left(\lambda_{m-1}(0) + \lambda_m t\right).$$

Recursively, we have that

$$\lambda_k(t) \leq e^{-t} \sum_{h=0}^{k} \lambda_{m-k+h}(0) t^h = e^{-t} p_k(t). \tag{7.36}$$

We can start to evaluate the number of agents having velocity v_1. Similarly we define

$$\gamma_i(t) = \sum_{p=1}^{m} f_1^{i,p}(t),$$

we have

$$\begin{aligned}
\frac{d\gamma_1}{dt} &= \sum_{p=1}^{m} \left(J_1^{1,p} - f_1^{1,p} + f_5^{1,p}\right) \\
&= \sum_{p=1}^{m} \sum_{h=1}^{8} \sum_{k\neq 1} \left(f_h^{1,p} f_k^{1,p} - f_1^{i,p} + f_5^{i,p}\right) \\
&\leq (N(0)+1)\lambda_1 - \gamma_1.
\end{aligned}$$

It is obvious, using (7.36) for $k = m$ and integrating

$$\gamma_1(t) \leq e^{-t}\left(\gamma_1(0) + P_m(t)\right)$$

where P_m is a polynomial of degree m given by integration of p_{m-1} present in (7.36). Finally we find the following estimate

$$\gamma_h \leq e^{-t} \overline{P}_{m-1+h}(t), \tag{7.37}$$

with $\overline{P}_{m-1+h}$ a polynomial of degree $m-1+h$ and with non-negative coefficients. At the end, reminding

$$N(t) = \beta(t) + \sum_{i=1}^{m} (\gamma_i(t) + \lambda_i(t)),$$

and using estimates (7.35), (7.36) and (7.37) we have

$$\lim_{t\to\infty} N(t) = 0.$$

What can we say about cases where exits have different shapes? We have just studied a special case, and we can pass to another one, in particular to the smallest possible, when there is only one exit cell, $N_e = 1$. It is not restrictive to suppose that the exit is on the right wall, having indexes $(m, \overline{p})$. It is obvious that cells with indexes of type $(i, \overline{p})$ for all $i = 1, \dots, m$ have v_1 as desired velocity. On the other hand there could be other cells having different desired velocities. We start reminding that

$$\frac{dN}{dt}(t) \leq -f_1^{m,\overline{p}}(t).$$

Let be $\overline{\ell} \in \mathbb{R}$ such that

$$\lim_{t\to\infty} f_1^{m,\overline{p}}(t) = \overline{\ell},$$

then $\overline{\ell} = 0$. Indeed, by non-negativity:

$$0 \leq f_1^{m,\overline{p}}(t) \leq N(0),$$

we find $\overline{\ell} \leq N(0)$; on the other hand if we suppose that $\overline{\ell} > 0$, then there exists a $\varepsilon > 0$ such that for all $t \in [\varepsilon, \infty[$

$$|f_1^{m,\overline{p}}(t) - \overline{\ell}| \leq \frac{\overline{\ell}}{2},$$

for all $t \geq \varepsilon$

$$\frac{dN}{dt} < -\frac{\overline{\ell}}{2},$$

from which it follows $N(t) \to -\infty$ when $t \to \infty$. Finally it must be $\overline{\ell} = 0$. Now we have to spend few words about regularity of solutions. The solution of the evolution equations is not only continuous but has also continuous time derivative, thanks to regularity of r.h.s. of system, so that:

$$\lim_{t\to\infty} \frac{df_1^{m,\overline{p}}}{dt}(t) = 0.$$

Reminding that v_1 is the desired velocity in cell $(m, \overline{p})$ and thus

$$\begin{aligned} \frac{df_1^{m,\overline{p}}}{dt}(t) + f_1^{m,\overline{p}}(t) &= J_1^{m,\overline{p}}(f(t)) + f_1^{m-1,\overline{p}}(t) \\ &\geq \sum_{h,k\neq 1} f_h^{m,\overline{p}}(t) f_k^{m,\overline{p}}(t) = \left(\sum_{h\neq 1} f_h^{m,\overline{p}}(t) \right)^2, \end{aligned} \tag{7.38}$$

Thanks to the last inequality, we can conclude for cell $(m, \overline{p})$ that

$$\lim_{t\to\infty} f_j^{m,\overline{p}}(t) = 0, \; j = 1, \dots, 8.$$

We have found information about exit cell. Now we are able to study neighbor cells, thanks to the fact that desired velocities of these latter point exit cell $(m, \overline{p})$, and they are connected by a suitable flux term. For example, the cell $(m, \overline{p}-1)$ has v_3 as desired velocity and we have

$$\begin{aligned}\frac{df_3^{m,\overline{p}}}{dt} &= -f_3^{m,\overline{p}} \sum_{h=1}^{8} f_h^{m,\overline{p}} - f_3^{m,\overline{p}} + f_3^{m,\overline{p}-1} \\ &\leq -f_3^{m,\overline{p}} + f_3^{m,\overline{p}-1},\end{aligned}$$

and thus

$$\lim_{t\to\infty} f_3^{m,\overline{p}-1}(t) = \lim_{t\to\infty} \frac{df_3^{m,\overline{p}-1}}{dt}(t) = 0,$$

where the limit of time derivative follows from a regularity argument. Once again we can write an inequality like (7.38)

$$\left(\sum_{h\neq 3} f_h^{m,\overline{p}-1}(t)\right)^2 \leq \frac{df_3^{m,\overline{p}-1}}{dt}(t) + f_3^{m,\overline{p}-1}(t),$$

and finally we obtain information also about cell $(m, \overline{p}-1)$

$$\lim_{t\to\infty} f_j^{m,\overline{p}-1}(t) = 0, \ j = 1, \ldots, 8.$$

The same argument can be used for other neighbor cells. In general, using the same reasoning in a recursive way we can prove that $f_j^{i,p}(t) \to 0$ when $t \to \infty$ for all indexes. At the end we emphasize that the method just presented can be replicated in the general case with different exits; on the other hand we have to remark that unfortunately it mainly consists in a theoretical result and it does not furnish any estimate about convergence. We can resume what we have just found in the following theorem.

Theorem 7.8.1. *Let $E \neq \emptyset$, let $\hat{f} = (\hat{f}_j^{i,p})$ be a non-negative initial datum related to system,*

$$\frac{f_j^{i,p}}{dt} + \psi_j^{i,f}(\mathbf{f}) = J_j^{i,p}(\mathbf{f}),$$

for $i, p = 1, \ldots, m$ and $j = 1 \ldots, 8$, let B_j^{hk} be a transition probability density defined as (7.34) and let $\eta_{ih} = 1$ be constant interaction rate. Then:

$$\lim_{t\to\infty} \mathbf{f}(t) = \mathbf{0}.$$

Roughly speaking, if there are some agents inside the room and the set of exit cells on the boundary is not empty, sooner or later they will leave the room.
The table of games (7.34) aims to describe a reaction of agents induced by the shape of the domain that is not a binary interaction among agents, and the model does not react so quickly to direction changing. In [1] the authors model this dynamics by a term different from the previous interaction kernel $J_j^i(\mathbf{f})$. We adjust their term to our framework

$$\frac{df_j^{i,p}}{dt} + \psi_j^{i,p}(\mathbf{f}) = J_j^{i,p}(\mathbf{f}) + W_j^{i,p}(\mathbf{f}), \tag{7.39}$$

where

$$W_j^{i,p}(\mathbf{f}) = \mu(\mathbf{f}) \left(\sum_{h=1}^{n} A_h^j(i,p) f_h^{i,p} - f_j^{i,p} \right). \tag{7.40}$$

We briefly describe terms present in (7.40),

- The function μ is an interaction rate and it weights the interaction.
- A_h^i is the transition probability, and it describes the transition caused by the environment, the presence of wall or exits near agent, and it has also the following normalization property:

$$\sum_{i=1}^{n} A_h^i = 1, \forall h = 1, \dots, n.$$

Thanks to the last equality we have

$$\sum_{j=1}^{n} W_j^{i,p}(\mathbf{f}) = 0,$$

and hence the conservation of mass. From an analytic viewpoint we first compare the new term $\mathbf{W} = (W_j^{i,p})$ with the classical interaction kernel,

$$\frac{df_j^{i,p}}{dt} + \psi_j^{i,p}(\mathbf{f}) = W_j^{i,p}(\mathbf{f}). \tag{7.41}$$

Then, in order to understand the different responses of the two terms we set

$$\mu(\mathbf{f}) = 1$$

N_e	$N(1)$	$N(4)$	$N(8)$	$N(12)$	$N(16)$	$N(20)$
1	0.9980	0.9713	0.8705	0.7025	0.5103	0.3303
3	0.9955	0.9524	0.8162	0.6184	0.4145	0.2381
5	0.9930	0.9380	0.7856	0.5844	0.3862	0.2166
7	0.9906	0.9238	0.7588	0.5584	0.3655	0.2006
9	0.9882	0.9099	0.7352	0.5376	0.3486	0.1871
11	0.9857	0.8963	0.7151	0.5210	0.3344	0.1756
13	0.9833	0.8835	0.6990	0.5077	0.3225	0.1657
15	0.9809	0.8720	0.6867	0.4970	0.3125	0.1575
17	0.9785	0.8627	0.6777	0.4885	0.3043	0.1509
19	0.9765	0.8562	0.6711	0.4819	0.2979	0.1458
21	0.9744	0.8512	0.6656	0.4762	0.2925	0.1415

Table 7.2: In the table the total number of agents N is reported as a function of time for different number of exits N_e, placed together on the right wall. In this case $m = 21$ and each simulation starts from an homogeneous initial data such that $N(0) = 1$ and there are only interactions with the environment $\mathbf{W} = (W_j^{i,p})$.

and the environmental interaction term similar to (7.34)

$$A_h^j(i,p) = \begin{cases} 1 & \text{if } j = j^*(i,p) \\ 0 & \text{otherwise.} \end{cases} \tag{7.42}$$

Let us compare the two tables 7.8 and 7.8. In the first one we have an exiting homogeneous crowds only with interaction kernel $\mathbf{J}$. In the second one there is not any interaction among agents and there are only interactions with the environment. We observe that in the second case we have a more quickly exit, as can be seen also in Figure 7.8.

We can also observe the difference between system (7.41) and the one with only interaction kernel J. Coming back to case with no wall studied before and defining

$$\mu(t) = \sum_{i,p=1}^{n} \sum_{l \neq 1} f_j^{i,p}(t),$$

it is the amount of agents that have not desired velocity if $\mu(0) > 0$ we have

$$\mu(t) \leq \mu(0)e^{-t},$$

taking into account only environmental interactions (7.41), while in the other we have

$$\mu(t) \leq \frac{1}{t + 1/\mu(0)}.$$

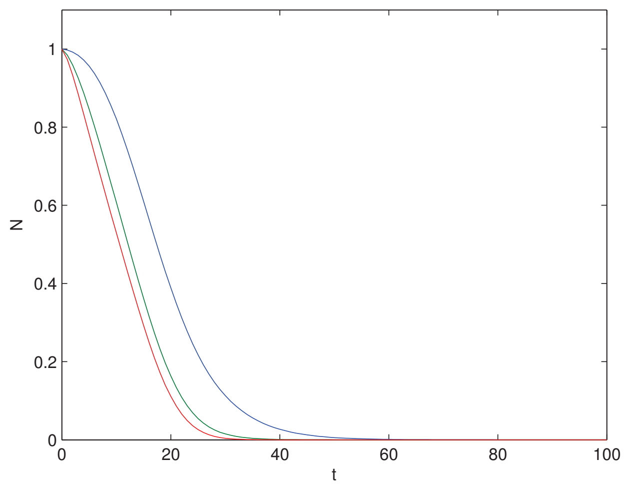

Figure 7.7: In this figure we can see the percentage of agents N as a function of time t, considering only environmental interaction $\mathbf{W}$, in three special cases: the blue line is the case of $N_e = 1$, the green one $N_e = 11$ and the red one $N_e = 21$. In this last case there is not any right wall.

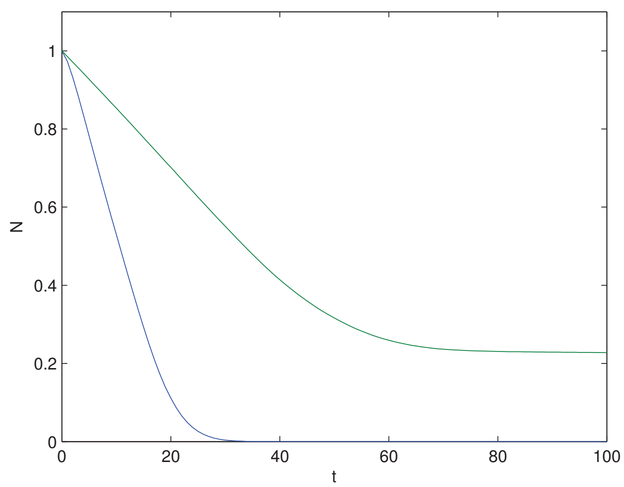

Figure 7.8: Comparison of N as a function of time, between system with interaction kernel among agents, green line, and system with environmental interaction, blue line, with homogeneous initial data with $m = 21$ and $N_e = 21$.

In other words agents in this second case change slowly their velocities. We conclude the discussion with the following theorem.

Theorem 7.8.2. *Let $E \neq \emptyset$, let $\hat{f} = (\hat{f}_j^{i,p})$ be a non-negative initial datum related to system of equations like (7.41), for $i, p = 1, \ldots, m$ and $j = 1 \ldots, 8$, let A_j^{hk} be an environmental inteaction like (7.42) and let $\mu(\mathbf{f}) = 1$ be constant interaction rate, then $\mathbf{f}(t)$ the solution related to $\hat{\mathbf{f}}$ has the following property*

$$\lim_{t \to \infty} \mathbf{f}(t) = \mathbf{0}.$$

Now we discuss an easy generalization related to the norm of velocities. Here we take into account not only velocities which differs only in direction but also in norm. To this end we introduce one more index

$$\mathbf{v}_{j,q}, \text{ with } j = 1, \ldots, 8, \; q = 1, \ldots, n,$$

such that $\mathbf{v}_{j,q} \neq \mathbf{0}$ for all indexes and $\mathbf{v}_{j,q} \cdot \mathbf{v}_{j,q'} = |\mathbf{v}_{j,q}||\mathbf{v}_{j,q'}|$ for $q, q' = 1, \ldots, n$, the first index is related to the direction and the second one to the norm. In this case system is described by the following functions

$$f_{j,q}^{i,p} : [0, T] \to \mathbb{R}^+, i, p = 1, \ldots, n, q = 1, \ldots, n, j = 1, \ldots, n,$$

whose evolution equations are

$$\frac{df_{j,q}^{i,p}}{dt} + \psi_{j,q}^{i,p}(\mathbf{f}) = J_{j,q}^{i,p}(\mathbf{f}) + W_{j,q}^{i,p}(\mathbf{f}).$$

The flux term is strictly related to direction and norm of velocity and differs from the one written in the eight velocities model only for the presence of $|\mathbf{v}_{j,q}|$. The elastic conditions can easily be generalized maintaining the norm. This generalization allows us to describe the decreasing of velocity caused by the presence of different densities. As suggested in [9] we split the change of norm and the change of angle as follows:

$$B_{hr,ks}^{jq}(\mathbf{f}) = C_{h,k}^{j}(\mathbf{f}) D_{r,s}^{q}(\mathbf{f}),$$

where:

- the angle transition $C_{h,k}^j$ describes the probability that an agent, moving with a velocity whose direction is summarized by the angle $\theta_h = \pi(h-1)/8$, changes his direction in θ_j after an interaction with an agents that travels in direction θ_k.
- the norm transition $D_{r,s}^q$ describes the change of velocity modulus.

The previously defined transition functions verify:

$$0 \leq C^{j}_{h,k} \leq 1, \quad \sum_{j=1}^{8} C^{j}_{h,k} = 1, h, k = 1, \ldots, 8,$$

$$0 \leq D^{q}_{r,s} \leq 1, \quad \sum_{q=1}^{8} C^{q}_{r,s} = 1, r, s = 1, \ldots, n,$$

and thus

$$\sum_{j=1}^{8} \sum_{q=1}^{n} B^{jq}_{hr,ks} = 1$$

for all indexes. In order to obtain some numerical results and a theorem concerning the long time behavior of solutions we fix all these functions in such a way that the angular transition is the same as in previous section (7.34), while the norm transition is equal to (7.18) and (7.19). We do make a little modification on transition probability density describing an encounter between two agents that have the same norm,

$$D^{q}_{r,r}(\rho_i) = \begin{cases} (1-\rho_i)^q & \text{if } q = r+1, \\ 1-(1-\rho_i)^q & \text{if } q = r, \\ 0 & \text{otherwise,} \end{cases} \tag{7.43}$$

if $r \neq n$, while for $q = n$

$$D^{q}_{n,n}(\rho_i) = \begin{cases} (1-\rho_i)^q & \text{if } q = n, \\ 1-(1-\rho_i)^q & \text{if } q = n-1, \\ 0 & \text{otherwise.} \end{cases} \tag{7.44}$$

Theorem 7.8.3. *Let $E \neq \emptyset$, let $\hat{\mathbf{f}} = (\hat{f}^{i,p}_j)$ be a non-negative initial datum related to Cauchy problem,*

$$\begin{cases} \dfrac{df^{i,p}_{j,q}}{dt} + \psi^{i,p}_{j,q}(\mathbf{f}) = J^{i,p}_{j,q}(\mathbf{f}) \\ \mathbf{f}(0) = \hat{\mathbf{f}} \\ i, p = 1, \ldots, m, j = 1, \ldots, 8, q = 1, \ldots, n. \end{cases} \tag{7.45}$$

for $i, p = 1, \ldots, m$ and $j = 1 \ldots, 8$, let $C^{j}_{h,k}$ be the angle transition like (7.34) and $D^{q}_{r,s}$ be the norm transition like (7.18), (7.19), (7.43) and (7.44); let $\mu(\mathbf{f}) = 1$ be constant interaction rate, then $\mathbf{f}(t)$ the solution related to $\hat{\mathbf{f}}$ has the following property

$$\lim_{t\to\infty} \mathbf{f}(t) = \mathbf{0}.$$

We omit the proof of the previous theorem as it is similar to that of 7.8.1. The crucial point is that there isn't any velocity $\mathbf{v}_{j,q} = 0$.

7.9 Some remarks on transport term

In this section we finally discuss a property of flux term. For the sake of simplicity we return to the one dimensional case. The crucial assumption is that there exists intervals inside the domain where we can suppose that densities are spatially homogeneous. After that we have to connect neighbor cells with an extra term, essentially replacing spatial derivative. The idea we used in the present work is to substitute it with an unbalanced difference. Forgetting the interaction kernel, this choice permits to rewrite the transport part of the evolution equation as:

$$\partial_t f(t,x) + \partial_x f(t,x) = 0 \longrightarrow \frac{df^i}{dt}(t) + \frac{f^i(t) - f^{i-1}(t)}{\ell} = 0.$$

A linear partial differential equations is substituted by a linear system of m ordinary differential equations. This system preserves both positiveness and mass conservation like a transport equation; on the other hand we get an homogenization of solution, in the sense that if we start from a given initial data $\mathbf{f}_0 = (f_0^i)$ then the solution converges to $\mathbf{f}_\infty = (f_\infty^i)$ when $t \to \infty$ in such a way that

$$\sum_{i=1}^{m} f_\infty^i = \sum_{i=1}^{m} f_0^i \quad f_\infty^i = f_\infty^j,$$

for all $i, j = 1, \ldots, m$. Indeed, if we denote by A the matrix related to system of ODEs

$$A = \begin{pmatrix} -a & 0 & 0 & \cdots & a \\ a & -a & 0 & \cdots & 0 \\ 0 & a & -a & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a & -a \end{pmatrix},$$

where $a = 1/\ell$, the solutions of $\dot{\mathbf{f}} = A\mathbf{f}$ are related to the eigenvalues of A which are solutions of

$$(x+a)^m - a^m = 0.$$

It easy to verify that the roots λ_i of the previous polynomial are such that $\mathrm{Re}\lambda_i \leq 0$. This observation sheds light on an important aspect: an unbalanced discretization is very natural by a physical viewpoint but we loose a transport property caused by the presence of a diffusive phenomenon. A different idea could be to substitute the spatial derivative with another approximation, but it does not work good. Actually, if we replace it with a centered difference $(f^{i+1} - f^{i-1})/\ell$, forgetting by the moment that it has no physical sense, we have that A, the matrix related to linear system of ODEs,

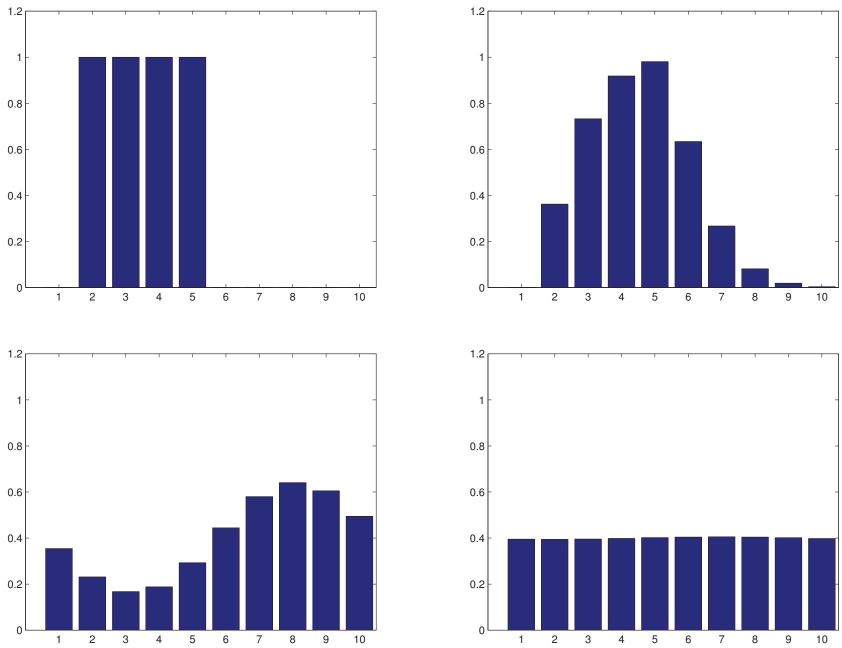

Figure 7.9: An example of diffusive phenomenon. In this particular case $m = 10$ and $\ell = 1$. In the first picture is the initial datum, then $\mathbf{f}(t) = (f^i(t))$ for $t = 1, 5, 25$, respectively. In the last picture we can observe that the solution is close to a homogeneous state.

is skew-symmetric and hence it has purely imaginary eigenvalues. This leads, in a certain sense, to the absence of diffusive phenomenon, while we loose the positiveness of solution. In conclusion, the diffusivity results to be strictly related to the discretization of spatial variables.

Chapter 8

Conclusions

In this thesis we mainly focus on two different kinds of complex systems, i.e., vehicular traffic and pedestrian dynamics. In order to describe the traffic flow, a single spatial variable is sufficient. On the other hand, if we want to obtain a reasonable modeling of pedestrian dynamics we cannot avoid increasing the number of spatial dimensions. Among the many different mathematical frameworks that can be adopted to model the systems at hand, in this dissertation we adopt the discrete velocity kinetic modeling. The models that arise in this way are characterized by a discretization of the velocity variable of the kinetic evolution equations and lead to a system of semilinear hyperbolic PDEs, whose general aspects are discussed in Chapter 2. One of the fundamental questions to which the thesis answers concerns the well-posedness of these models. In the first part we largely study this issue.

From a mathematical point of view, the simplification effect of velocity discretization related to the transformation of a single kinetic equation to a system of semilinear hyperbolic equation is evident. In Chapter 3 we analyze the Cauchy problem, in one spatial dimension, for a general discrete velocity problem related to vehicular traffic flow. Under weak analytic hypotheses and in a general modeling context, we furnish a result of existence and uniqueness of solutions, global in time, without assuming *a priori* restrictions on the model's macroscopic quantities, such as density. These results are the object of the paper [4].

The next step consists in analyzing problems in higher spatial dimensions. In the case of two dimensional spatial domains, in Chapter 4 we recover results similar to those found in Chapter 3. Moreover, we observe that the introduction of a discrete activity variable, that models specific attitudes of the agents involved in the systems at hand, in spite of increasing the number of semilinear equations, does not change the mathematical properties of models. In order to obtain global well-posedness in two spatial dimensions we don't need to change any hypotheses on functions that enter the interaction kernel.

On the other hand, if we try to weaken the hypotheses we use in Chapters 3 and 4 we show that, in general, it is not possible to find existence of solutions for all positive time. In particular in Chapter 5 we observe that if we do not suppose the boundedness of the interaction rate we cannot use Gronwall's lemma to prolongate the local solution, except for a particular case, we explicitly study therein, in which the transition probability density corresponds to the case when all agents try to adapt their velocity to an a priori prescribed *desired velocity* $\mathbf{v}_{i^*}$. In this latter case, we actually do prove global existence and uniqueness of a solution. An interesting question which we are not able to answer is the following: is there any connection between special results found in Chapter 5 and the general case when we require the interaction rate to be only Lipschitz continuous? Finally, we remark that unlike the large part of literature related to discrete velocity models, in which the interaction rate and the transition probability density are taken as functions of local density $\rho = \sum f_i$, in this thesis we study the problem of well-posedness assuming a more general dependence of these quantities on the vector distribution function $\mathbf{f} = (f_i)$.

Once established existence and uniqueness of solutions to the Cauchy problem and their prolongability, the next question we try to answer concerns their long-time behavior. Are we able to find out any eventual property of solutions, reflecting eventual collective behaviors of agents? We cannot avoid observing that it is very difficult to furnish a general, mathematically rigorous answer to this question. We give some qualitative result only in easy cases, with constant transition probability density B^i_{hk}, or studying spatially homogeneous problems as in Section 3.2, where we find a nice result, inspired by the paper [30], which emphasizes the presence of a phase transition in vehicular traffic flow, as is experimentally reported in [34].

In order to obtain further insight on long-time behavior results, in Chapter 7 we move to consider models which are also spatially discrete, in this way obtaining a system of ordinary differential equations in place of a system of hyperbolic partial differential equations. Unlike the approach taken by [28], [29], we choose to replace the continuous spatial derivative with forward or backward differences. In this framework existence and uniqueness of solutions and their prolongability is just an elementary exercise. Instead, we are able to give some results related to asymptotic time behavior, to existence of stationary solutions and to their stability, both for one and for two dimensional spatial domains.

Finally, in Chapter 6 we start to treat initial-boundary value problems for kinetic discrete velocity models, for which the literature is pretty limited [1], [11], [27]. While if we limit to consider vehicular traffic then the Cauchy problem or possibly a periodic initial-boundary problem can suffice, if we come to consider, for example, pedestrian dynamics we cannot avoid to pose the mathematical problem in a bounded domain with suitable boundary conditions. Though we are able to give results on the initial-boundary value problem only in simple cases, without specializing the interaction kernel, we anyway decide to include them in this dissertation and to put them in Chapter 6 for continuity reasons with respect to the whole content of the thesis.

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