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CICLO XXVII

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On the Transvection Group of a Rack

With an application to the classification of connected quandles of order a power of a prime

Settore Scientifico Disciplinare MAT/02

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In Memoriam
Alexandru Lascu
(1930-2012)
Mathematician, mentor and, above all, friend.

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List of Mathematical Notations

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x^G The orbit of x under the action of group G. 13

$\mathrm{Trans}_{\triangleright}(\pi)$ The group projection between the transvection groups induced by a rack projection π. 14

${}^{\pm\alpha}X$ The deformation of rack X by the automorphism α. 14

$\mathrm{Aut}(G)$ The automorphism group of the group G. 20

$\mathbb{S}_X$ The symmetric group on the set X. 22

$\mathrm{C}_G(B)$ The set of elements of a group G fixed by an the automorphisms in B. If $B \subseteq G$, the inner automorphism determined by the elements in B are meant. 23

$[G, B]$ The subgroup generated by the elements $g\alpha\left(g^{-1}\right)$ where $g \in G$ and $\alpha \in B \subseteq \mathrm{Aut}(G)$. If $B \subseteq G$, the inner automorphism determined by the elements in B are meant. 23

$\mathrm{Comm}(G, B)$ The set of the elements $g\alpha\left(g^{-1}\right)$ where $g \in G$ and $\alpha \in B \subseteq \mathrm{Aut}(G)$. If $B \subseteq G$, the inner automorphism determined by the elements in B are meant. 23

λ_g The function determined by left multiplication by $g \in G$ of (cosets of) elements of a group G 27

$\lambda_{G/H}$ The group morphism from G to $\mathbb{S}_{G/H}$ determined by $g \longmapsto \lambda_g$ 27

$\psi_{G_1/H}$ The function from G/H to $\dfrac{\psi(G)}{\psi(H)}$ determined by the group morphism ψ. 27

$\mathrm{Core}_G(H)$ The G-core of H, i.e. the intersection of all the conjugates of H in G. 28

$\dfrac{SN}{N}$ The set of cosets $\{gN \mid g \in S\}$ where G is a group $N \lhd G$, $S \subseteq G$. 40

$\dfrac{X}{N}$ The quotient quandle of a rack X by the subgroup $N \lhd \mathrm{Inn}_{\triangleright}(X)$. 41

K_π The kernel of $\mathrm{Trans}_{\triangleright}(\pi)$ for a rack morphism π. 42

K_N The kernel of $\mathrm{Trans}_{\triangleright}\left(\varphi_N\right)$ for an admissible subgroup N of $\mathrm{Trans}_{\triangleright}(X)$ for some rack X. 43

$S \div T$ The commutator quotient of S by T. 43

$\mathscr{O}_p(G)$ The p-core of a group G. 51

Z The centre of $\mathrm{Trans}_{\triangleright}(X)$ 51

$\Phi(G)$ The Frattini subgroup of G. 52

$\mathbb{F}_q$ The field of order q. 53

Contents

Introduction

Quandles are non-associative structures, which can be described as left-distributive idempotent letf quasigroups, whose algebraic structure encapsulates the Reidemeister moves and gives an important invariant of knots [Joy82a]. Besides, quandles have emerged also in the classification of pointed Hopf algebras [AG03], in the study of symmetric spaces [Loo67], of singularities of algebraic varieties [Bri88], of the solutions of the braid equation [Dri92], as important examples of quasigroups [Smi92].

Their classification up to isomorphism seems, at the moment of writing the present work, out of reach, so that, in the literature, it has been attacked by studying special classes of quandles, starting with finite quandles.

Every quandle naturally decomposes in a disjoint union of subquandles stable under left translations. In the case that no such proper subquandle exists the quandle is said to be connected. Connected quandles are important because of their application in knot theory [CESY14] and their classification is necessary for the more general classification task since they are somehow "building blocks" for general quandles.

In analogy with groups, connected quandles of order p and p^2, where p is a prime, have been studied and classified (by Etingof, Guralnick and Soloviev [ESG01] and Graña [Gra04] respectively). In this thesis we are concerned with the study of connected quandles of order p^3. Every connected quandle can be represented as cosets of a group while the rack operation depends on a choice of an automorphism of the group [Joy82a]. The chosen group can be the transvection group of the quandle, and in this case the coset representation enjoys a condition of minimality [HSV14].

The existence of this minimal coset representation gives us a general strategy in order to try to classify finite connected quandles. Given a quandle of order n find an upper bound, m, for the possible order of its transvection group. Then you can construct all the coset quandles $\mathscr{Q}(G, H, \alpha)$ where G is a group such that $n \leq |G| \leq m$, α is one of its automorphisms and $H \leq \mathsf{Fix}_G(\alpha)$ such that $|G/H| = n$. Since by Theorem 3.19 every connected quandle of order n has a coset representation with a group of order less or equal to m, then, by elimininating all the isomorphic ones, you obtain a list of all connected quandles of order n up to isomorphism.

The new results that the reader can find in the present work are the following.

In Chapter 2, it is shown that the transvection group does not change when deforming a rack and there is a criterion for the isomorphism of quandles (Proposition 2.18) based on the existence of an isomorphism of their transvection groups. Finally, the characterisation of the transvection group of a standard crossed set is given in terms of subgroups of the group generated by the quandle (Proposition 2.34).

In Chapter 3, the relationships between quandle theoretic properties of coset quandles and group theoretic properties of their defining groups and group automorphisms are described, in particular an isomorphism criterion for coset quandles is given. Analogously, in the connected case, quandle theoretic properties are characterised by means of properties of the transvection group, using the minimal coset representation. At the end of the chapter we give a new characterisation of the automorphism group of a connected quandle (Proposition 3.29) by means of the transvection group.

In Chapter 4, the correspondence between quandle quotients and subgroups of the transvection group which are normal in the inner automorphism group is explored. Various specific results are found when the normal subgroups are semiregular on the quandle, like when the subgroup under consideration is the centre of the transvection group.

In Chapter 5 a new classification strategy for connected quandles of prime power order is devised. In Proposition 5.6 a criterion is given to obtain information on the number of generators of the transvection group based on the number of generators of possible quotients and of its centre. Using this criterion, a new proof is given of the fact that connected quandles of order both p and p^2 are affine (5.13) and it is also proved that the transvection group of connected quandles of order p^3 has order at most p^4. In this way, using the minimal coset representation, we give a complete characterisation of connected quandles of order p^3 (5.17). In the Appendix A, two algorithms are given to construct all isomorphism classes of connected quandles of order p^2 and p^3.

All the results which are not attributed are claimed by the author as original. If a proof for an attributed result is given, this means that either the proof was not given in the original source or a certain degree of originality of the proof is claimed.

Chapter 1

Racks and Quandles

In this chapter, for the reader's convenience and ease of reference, we collect definitions, notation and some basic results.

Generalities on Racks

Since in the literature the notation for racks varies wildly let us start by establishing the notation we shall be using, mainly following the choices made by Andruskiewitsch and Graña [AG03].

1.1 Definition ([FR92, Definition 1.1])**.** A *rack* is a set endowed with two binary operations, $\triangleright$, $\triangleright^{-1}$, which satisfy the following equations:

R1) $\forall x, y, z, \in X$, $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$ *(left self-distributivity).*

R2) $\forall x, y, z, \in X$, $x \triangleright^{-1} (x \triangleright y) = x \triangleright \left(x \triangleright^{-1} y\right) = y$ *(left translations are bijections).*

A map ψ between two racks X and Y is a *rack morphism* if for every $x_1, x_2 \in X$ we have

$$\psi (x_1 \triangleright x_2) = \psi (x_1) \triangleright \psi (x_2) .$$

1.2 Remark. The rôle of the two operations of a rack, $\triangleright$ and $\triangleright^{-1}$, is completely symmetric. In particular, $\triangleright^{-1}$ is left-self distributive and is preserved by rack morphisms.

The name *(w)rack* was introduced by Conway and Wraith, in an unpublished private correspondence, to recall the meaning of "destruction" to refer to the algebraic structure you are left with when you discard the multiplication in a group and only retain the conjugation [FR92, Introduction]. And indeed, conjugation in a group gives the more important examples of racks and more precisely of a variety of racks, the *quandles*:

1.3 Definition ([Joy82a, Definition 1.1])**.** A *quandle* is a rack where every element is idempotent, i.e. for every element x we have:

(R3) $$x \triangleright x = x$$

1.4 Definition. Let G be a group. Consider the two binary operations $\triangleright$ and $\triangleright^{-1}$ defined by setting:

$$g \triangleright h := ghg^{-1} \quad , \quad g \triangleright^{-1} h := g^{-1}hg$$

G endowed with these two operations is a quandle which will be called the *conjugation quandle* on G.

More generally any subset of a group closed under conjugation is a quandle. In particular

1.5 Definition ([AG03, Definition 1.1])**.** A *standard crossed set* is a quandle isomorphic to a subrack of a group endowed with the structure of conjugation quandle.

Standard subsets satisfy an additional axiom. This prompted Andruskiewitsch and Graña to introduce the following definition:

1.6 Definition ([AG03, Definition 1.1])**.** A *crossed set* is a quandle for which the following condition holds true for all the couple of elements x, y:

$$\text{(R4)} \qquad x \triangleright y = y \iff y \triangleright x = x$$

1.7 Remark ([Ven15])**.** A rack is a conjugation quandle if and only if it is injective (see Definition 1.26), as can be deduced from the universal property of the enveloping group of a rack (Theorem 1.25). In view of this, using the [GAP13] package [RIG], an example of a crossed set that is not a conjugation quandle can be given. [RIG] maintains a database of connected quandles. Each quandle is indicated by two numbers, $Q(n, m)$, where n is the order of the quandle, m is the place in the [RIG] list on connected quandles of order n. The quandle $Q(8, 1)$ is a crossed set which is not injective and hence not a conjugation quandle. This example shows that the crossed set axiom (R4) does not characterise conjugation quandles.

The axioms imply that for every x in a rack X, the function $x \triangleright \bullet$ is a permutation on X (since it has a two sided inverse in $x \triangleright^{-1} \bullet$). Moreover, by the left self-distributivity axiom the resulting permutation is an automorphism of X and shall be denoted by φ_x.

1.8 Definition ([Joy82a, Section 6])**.** Let X be a rack, $x, y \in X$. If, for every $z \in X$, $x \triangleright z = y \triangleright z$ (i.e. if $\varphi_x = \varphi_y$), x and y are said to be *behaviourally equivalent*.

Given a rack X we denote $\mathrm{Aut}_\triangleright(X)$, the group of *rack automorphisms* on X. The assignment $x \mapsto \varphi_x$ defines a rack morphism $\varphi_X : X \longrightarrow \mathrm{Aut}_\triangleright(X)$ (or more simply φ) where $\mathrm{Aut}_\triangleright(X)$ is endowed with the structure of conjugation quandle. For every automorphism α on X the following equality holds [AG03, Equality 1.5], which expresses the fact that φ_X is an equivariant $\mathrm{Aut}_\triangleright(X)$-map when $\mathrm{Aut}_\triangleright(X)$ acts on X by automorphisms in the natural way and on itself by conjugation:

$$\varphi_{\alpha(x)} = \alpha\varphi_x\alpha^{-1} \tag{1.1}$$

1.9 Definition ([AG03, Definition 1.11])**.** A rack X is *faithful* if φ_X is injective.

1.10 Definition ([AG03, Definition 1.3])**.** In analogy with groups, the subgroup of the rack automorphism group generated by $\varphi_X(X)$ is called the *inner automorphism group* of the rack and is denoted by $\mathrm{Inn}_\triangleright(X)$.

1.11 Remark ([AG03, Remark 1.12])**.** Let X be a rack, $\mathrm{Inn}_\triangleright(X)$ its inner automorphism group. If X is faithful then $\mathrm{Inn}_\triangleright(X)$ is centreless.

The interaction between φ and any rack morphism ψ is described by the following lemma.

1.12 Lemma. *Let $\psi : X \longrightarrow Y$ be a rack morphism. Then for every x and y in X*

$$\psi\left(\varphi_x^{\pm 1}(y)\right) = \varphi_{\psi(x)}^{\pm 1}\psi(y) \tag{1.2}$$

Proof. We will prove the cases φ_x and φ_x^{-1} separately.
As for φ_x:

$$\psi(\varphi_x(y)) = \psi(x \triangleright y) = \psi(x) \triangleright \psi(y) = \varphi_{\psi(x)}\psi(y) \tag{1.3}$$

As for φ_x^{-1}:

$$\begin{aligned}
\psi\left(\varphi_x^{-1}(y)\right) = \varphi_{\psi(x)}^{-1}\psi(y) &\iff \varphi_{\psi(x)}\psi\left(\varphi_x^{-1}(y)\right) = \psi(y) \\
&\overset{(1.3)}{\iff} \psi\left(\varphi_x\varphi_x^{-1}(y)\right) = \psi(y) \\
&\iff \psi(y) = \psi(y)
\end{aligned}$$

□

Quandles are an important variety of racks not only for historical reasons and for their importance in applications. There is also a structural reason for their centrality in the theory of racks, which is illustrated by the next propositions. The first is a slight generalization of a remark made by Brieskorn in [Bri88] after Proposition 2.1.

1.13 Lemma. *Let X be a rack, $\alpha \in \mathrm{Aut}_\triangleright(X)$. Define a new operation, $\triangleright^{\pm\alpha}$, on X by $x \triangleright^{\pm\alpha} y = (\varphi_x\alpha)^{\pm 1}(y)$. $(X, \triangleright^{\pm\alpha})$ is a rack if and only if $\alpha \in \mathrm{C}_{\mathrm{Aut}_\triangleright(X)}(\mathrm{Inn}_\triangleright(X))$.*

Proof. Since α is a permutation on X, $(X, \triangleright^{\pm\alpha})$ being a rack is equivalent to the fact that for every $x, y, z \in X$

$$\begin{aligned}
x \triangleright^{\pm\alpha}\left(y \triangleright^{\pm\alpha} z\right) = \left(x \triangleright^{\pm\alpha} y\right) \triangleright^{\pm\alpha}\left(x \triangleright^{\pm\alpha} z\right) &\iff \\
(\varphi_x\alpha)^{\pm 1}(\varphi_y\alpha)^{\pm 1} = \left(\varphi_{(\varphi_x\alpha)^{\pm 1}(y)}\alpha\right)^{\pm 1}(\varphi_x\alpha)^{\pm 1} &
\end{aligned}$$

Let us consider first the case $\triangleright^\alpha$ and proceed by equivalences:

$$\begin{aligned}
\forall x, y \in X \qquad & \varphi_x\alpha\varphi_y\alpha = \varphi_{\varphi_x\alpha(y)}\alpha\varphi_x\alpha & \iff \\
\forall x, y \in X \qquad & \varphi_x\alpha\varphi_y = \varphi_x\alpha\varphi_y\alpha^{-1}\varphi_x^{-1}\alpha\varphi_x & \iff \\
\forall x \in X \qquad & 1 = \alpha^{-1}\varphi_x^{-1}\alpha\varphi_x & \iff \\
& \alpha \in \mathrm{C}_{\mathrm{Aut}_\triangleright(X)}(\mathrm{Inn}_\triangleright(X)) &
\end{aligned}$$

Analogously in the case $\triangleright^{-\alpha}$:

$$\begin{array}{rlr}
\forall x, y \in X & (\varphi_x\alpha)^{-1}(\varphi_y\alpha)^{-1} = \left(\varphi_{(\varphi_x\alpha)^{-1}(y)}\alpha\right)^{-1}(\varphi_x\alpha)^{-1} & \Longleftrightarrow \\
\forall x, y \in X & \alpha^{-1}\varphi_x^{-1}\alpha^{-1}\varphi_y^{-1} = \left(\alpha^{-1}\varphi_x^{-1}\varphi_y\varphi_x\alpha\alpha\right)^{-1}\alpha^{-1}\varphi_x^{-1} & \Longleftrightarrow \\
\forall x, y \in X & \alpha^{-1}\varphi_x^{-1}\alpha^{-1}\varphi_y^{-1} = \alpha^{-1}\alpha^{-1}\varphi_x^{-1}\varphi_y^{-1}\varphi_x\alpha\alpha^{-1}\varphi_x^{-1} & \Longleftrightarrow \\
\forall x \in X & \varphi_x^{-1}\alpha^{-1} = \alpha^{-1}\varphi_x^{-1} & \Longleftrightarrow \\
& \alpha \in \mathrm{C}_{\mathrm{Aut}_{\triangleright}(X)}\left(\mathrm{Inn}_{\triangleright}(X)\right) &
\end{array}$$

and we are done. □

1.14 Definition. Let X be a rack. The *association morphism* of X, ι_X, is defined by setting $\iota_X(x) = x \triangleright^{-1} x$. ι_X is bijective and the following equality holds:

$$x \triangleright \iota_X(x) = x \tag{1.4}$$

1.15 Proposition ([Bri88, Proposition 2.2])**.** *Let X be a rack, $x \in X$ and ι its association morphism. The following statements hold:*

i) $\iota \in \mathrm{Z}\left(\mathrm{Aut}_{\triangleright}(X)\right)$.

ii) $\iota^{-1}(x) = x \triangleright x$.

iii) $\varphi_{\iota(x)} = \varphi_x$.

1.16 Lemma ([Bri88, Section 2])**.** *For any rack X and $\alpha \in \mathrm{C}_{\mathrm{Aut}_{\triangleright}(X)}\left(\mathrm{Inn}_{\triangleright}(X)\right)$ the following statements hold:*

i) $X = \left(X, (\triangleright^{\alpha})^{\alpha^{-1}}\right)$

ii) $(X, \triangleright^{\iota})$ *is a quandle.*

1.17 Definition ([Bri88, Definition after Proposition 2.2])**.** For any rack X, the *associated quandle* to X, ${}^{\iota}X$, is $(X, \triangleright^{\iota})$.

The association from a rack X to the quandle ${}^{\iota}X$ is functorial as the following lemma shows.

1.18 Lemma ([AG03, Section 1.1.1])**.** *Let $\psi : X \longrightarrow Y$ be a morphism of racks. Then*

$$\psi\iota_X = \iota_Y\psi$$

In particular, $\mathrm{Aut}_{\triangleright}(X) \leq \mathrm{Aut}_{\triangleright}({}^{\iota}X)$

Proof. We want to show that for all $x \in X$, $\psi\iota_X(x) = \iota_Y\psi(x)$.
By definition of ι_Y this is equivalent to show that

$$\psi(x) \triangleright \psi\iota_X(x) = \psi(x)$$

but

$$\psi(x) \triangleright \psi\iota_X(x) = \psi\left(x \triangleright \iota_X(x)\right) = \psi(x)$$

Let now $\alpha \in \mathrm{Aut}_{\triangleright}(X)$, $x_1, x_2 \in X$. α is bijective and using the fact that $\alpha\iota_X = \iota_X\alpha$ we obtain

$$\begin{aligned}\alpha\left(x_1 \triangleright^{\iota_X} x_2\right) &= \alpha\left(x_1 \triangleright \iota_X(x_2)\right)\\ &= \alpha(x_1) \triangleright \alpha\iota_X(x_2)\\ &= \alpha(x_1) \triangleright \iota_X\alpha(x_2)\\ &= \alpha(x_1) \triangleright^{\iota_X} \alpha(x_2)\end{aligned}$$

so that $\alpha \in \mathrm{Aut}_{\triangleright}\left({}^{\iota_X}X\right)$. □

1.19 Remark. We note that the equality $\mathrm{Aut}_{\triangleright}(X) = \mathrm{Aut}_{\triangleright}({}^{\iota}X)$ claimed in [AG03, Section 1.1.1] is not true in general. Take X a permutation rack (i.e. a rack where $\varphi_x = \sigma$ for every $x \in X$ and for some $\sigma \in \mathbb{S}_X$) such that the defining permutation σ is a cycle of length $|X|$. In this case, $\iota = \sigma^{-1}$ and ${}^{\iota}X$ is the trivial quandle of order $|X|$. Now $\mathrm{Aut}_{\triangleright}({}^{\iota}X) = \mathbb{S}_X$ and $\mathrm{Aut}_{\triangleright}(X) = \langle\sigma\rangle$ and the inclusion is proper whenever $|X| > 2$.

1.20 Remark ([Bri88, Remark after Proposition 2.2])**.** The covariant functor that associates to every rack X the couple $({}^{\iota_X}X, \iota_X)$ and to every morphism of racks $\psi : X \longrightarrow Y$ the morphism $\psi\iota_X$, is an isomorphism of categories between the category of racks and the category of couples (Q, α) where Q is a quandle and $\alpha \in \mathrm{C}_{\mathrm{Aut}_{\triangleright}(Q)}(\mathrm{Inn}_{\triangleright}(Q))$. The inverse functor associates ${}^{\alpha}Q$ to (Q, α) as shown in Lemma 2.16.

The following lemma describes what is possibly the best approximation of the concept of a "normal" subrack.

1.21 Notation. From now on if $\pi : X \longrightarrow Y$ is a rack morphism and y is in Y we denote the fibre of y by $\pi^{-1}(y)$ and by *projection* we will mean a surjective morphism.

1.22 Lemma ([Ryd93, Proposition 2.31])**.** *Let $\pi : X \longrightarrow Y$ be a morphism of racks, $x \in X$. Then the fibre $\pi^{-1}(\pi(x))$ is a subquandle of ${}^{\iota}X$.*

Proof. Let $y, z \in \pi^{-1}(\pi(x))$ then

$$\begin{aligned}\pi\left(y \triangleright^{\iota} z\right) &= \pi\left(y \triangleright \iota(z)\right)\\ &= \pi(y) \triangleright \pi(\iota(z))\\ &\overset{1.18}{=} \pi(y) \triangleright \iota(\pi(z))\\ &= \pi(x) \triangleright \iota(\pi(x))\\ &= \pi(x)\end{aligned}$$

□

1.23 Definition ([Ryd93, Definition 2.30])**.** Let $\pi : X \longrightarrow Y$ be a morphism of racks, $x \in X$. Then *the associated quandle of* π *at* x, ${}^{\iota}\pi^{-1}(\pi(x))$, is the subquandle $\pi^{-1}(\pi(x))$ of ${}^{\iota}X$.

Notice that if X is a quandle the associated quandle of π at x is a subquandle while if $\pi(x)$ is idempotent then $\pi^{-1}(\pi(x))$ is also a subrack of X and the relation between $\pi^{-1}(\pi(x))$ and ${}^{\iota}\pi^{-1}(\pi(x))$ is the same as that between X and ${}^{\iota}X$.

Racks and Groups

Let us now recall some results on the interplay between racks and their automorphism groups. First of all, let us introduce two more groups relevant to rack theory.

1.24 Definition. The *enveloping group* of a rack X ([Joy82a, Section 6]) is

$$G_X := \frac{\mathrm{F}(X)}{\langle xyx^{-1} = x \triangleright y,\ \forall x, y \in X\rangle}$$

where $\mathrm{F}(X)$ is the free group generated by X.
If X is finite, the *finite enveloping group* ([GHV11, Definition before Lemma 2.19]) is

$$\overline{G}_X := \frac{G_X}{\left\langle x^{|\varphi_x|}, \forall x \in X\right\rangle}$$

where $|\varphi_x|$ is the order of φ_x.

In [GHV11, Lemma 2.19] Graña, Heckenberger and Vendramin prove that if X is finite, indeed $\overline{G}_X$ is finite as the name suggests.
The importance of the enveloping group of a rack is stressed by the following result.

1.25 Proposition (Universal Property of the Enveloping Group [FR92, Proposition 2.1])**.** *Let X be a rack and let G be a group. Given any rack homomorphism $\psi : X \longrightarrow G$, where G is endowed with the structure of conjugation quandle, there exists a unique group homomorphism $\psi' : G_X \longrightarrow G$ which makes the following diagram commute:*

$$\begin{array}{ccc} X & \xrightarrow{\ \eta\ } & G_X \\ {\scriptstyle\psi}\big\downarrow & \circlearrowleft & \big\downarrow{\scriptstyle\psi'} \\ G & \overset{id}{=\!=\!=} & G \end{array}$$

where η is the structure morphism $\eta(x) = x$. Moreover any group with the same universal property is isomorphic to G_X.

1.26 Definition. Let X be a rack. If the structure morphism $\eta : X \longrightarrow G_X$, defined by $\eta(x) = x$, is injective, X is said to be *injective.*

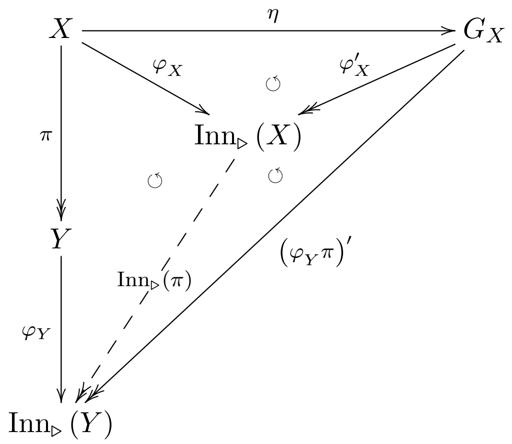
The following theorem is well known and fundamental for the developing of rack theory. Andruskiewitsch and Graña in [AG03, Lemma 1.8] give a (sketch of a) proof only for the finite case, Bunch et alii in [BLRY10, Theorem 3.1] give a proof only in the case of quandles and do not cite equality (1.5) of which we will make extensive use.

1.27 Theorem.
Let $\pi : X \twoheadrightarrow Y$ be a projection of racks. Then there is a unique surjective morphism of groups, $\mathrm{Inn}_{\triangleright}(\pi)$, from $\mathrm{Inn}_{\triangleright}(X)$ to $\mathrm{Inn}_{\triangleright}(Y)$ such that for every x in X

$$\mathrm{Inn}_{\triangleright}(\pi)(\varphi_x) = \varphi_{\pi(x)} \tag{1.5}$$

Moreover if π is an isomorphism so is $\mathrm{Inn}_{\triangleright}(\pi)$.

Proof. Applying Proposition 1.25 twice, to φ_X and to $\varphi_Y \pi$ respectively, we obtain the following commutative diagram:



We observe that both φ'_X and $(\varphi_Y\pi)'$ are surjective since

$$\mathrm{Inn}_{\triangleright}(X) = \langle \varphi_X(X) \rangle = \langle \varphi'_X \eta(X) \rangle \leq \varphi'_X(G_X)$$

and

$$\mathrm{Inn}_{\triangleright}(Y) = \langle \varphi_Y(Y) \rangle = \langle \varphi_Y \pi(X) \rangle = \left\langle (\varphi_Y\pi)' \eta(X) \right\rangle \leq (\varphi_Y\pi)'(G_X).$$

Given that φ'_X and $(\varphi_Y\pi)'$ are surjective, in order to obtain a surjective group morphism, $\mathrm{Inn}_{\triangleright}(\pi) : \mathrm{Inn}_{\triangleright}(X) \twoheadrightarrow \mathrm{Inn}_{\triangleright}(Y)$ which makes the diagram commute, it is necessary and sufficient (by the third isomorphism theorem for groups) that $\ker(\varphi'_X) \leq \ker\left((\varphi_Y\pi)'\right)$. Let $g \in \ker(\varphi'_X)$, i.e. there are $x_i \in G_X$ such that $g = \prod_i x_i$ and for every $x \in X$

$$\prod_i \varphi_{x_i}(x) = x \tag{1.6}$$

In order to show that $g \in \ker\left(\left(\varphi_Y \pi\right)'\right)$ we have to show that for every $y \in Y$, $\left(\left(\varphi_Y \pi\right)'(g)\right)(y) = y$. Since π is surjective there is $x_y \in X$ such that $\pi(x_y) = y$, and we can compute

$$\begin{aligned}
\left(\left(\varphi_Y \pi\right)'(g)\right)(y) &= \left(\left(\varphi_Y \pi\right)'\left(\prod_i x_i\right)\right) \pi(x_y) \\
&= \left(\prod_i \left(\varphi_Y \pi\right)'(x_i)\right) \pi(x_y) \\
&= \left(\prod_i \left(\varphi_Y \pi\right)' \eta(x_i)\right) \pi(x_y) \\
&= \left(\prod_i \varphi_Y \pi(x_i)\right) \pi(x_y) \\
&= \left(\prod_i \varphi_{\pi(x_i)}\right) \pi(x_y) \\
&\overset{(1.2)}{=} \pi\left(\left(\prod_i \varphi_{x_i}\right)(x_y)\right) \\
&\overset{(1.6)}{=} \pi(x_y) \\
&= y
\end{aligned}$$

Since, by construction, the diagram is commutative we have that for every $x \in X$

$$\mathrm{Inn}_{\triangleright}(\pi)(\varphi_x) = \mathrm{Inn}_{\triangleright}(\pi)\,\varphi_X(x) = \varphi_Y \pi(x) = \varphi_{\pi(x)}$$

Lastly suppose that π is an isomorphism. We can apply the previous part of the proof to π^{-1} and obtain a surjective group morphism $\mathrm{Inn}_{\triangleright}\left(\pi^{-1}\right) : \mathrm{Inn}_{\triangleright}(Y) \twoheadrightarrow \mathrm{Inn}_{\triangleright}(X)$ such that $\left(\mathrm{Inn}_{\triangleright}\left(\pi^{-1}\right)\right)(\varphi_y) = \varphi_{\pi^{-1}(y)}$ Now we have that

$$\left(\mathrm{Inn}_{\triangleright}\left(\pi^{-1}\right)\mathrm{Inn}_{\triangleright}(\pi)\right)(\varphi_x) = \varphi_{\pi^{-1}\pi(x)} = \varphi_x$$

hence $\mathrm{Inn}_{\triangleright}\left(\pi^{-1}\right)$ is a left inverse of $\mathrm{Inn}_{\triangleright}(\pi)$ on a set of generators of $\mathrm{Inn}_{\triangleright}(X)$ and then a left inverse tout court. Since we already know that $\mathrm{Inn}_{\triangleright}(\pi)$ is surjective we can conclude that if π is an isomorphism so is $\mathrm{Inn}_{\triangleright}(\pi)$. □

Let us state a couple of lemmas which clarify the interplay between rack epimorphisms and the automorphism groups action:

1.28 Lemma ([BLRY10, Lemma 5.3])**.** *Let $\pi : X \twoheadrightarrow Y$ be a projection of racks. Then for every α in $\mathrm{Inn}_{\triangleright}(X)$ and x in X*

$$\pi(\alpha(x)) = \left(\mathrm{Inn}_{\triangleright}(\pi)(\alpha)\right)(\pi(x)) \tag{1.7}$$

1.29 Notation. Throughout this work if $g \in G$ is an element of a group G, by $\widehat{g}$ we will mean the inner automorphism of G determined by the (left) conjugation by g, i.e. $\widehat{g}(h) = ghg^{-1}$ for every $h \in G$. When possible without causing ambiguity, if $H \leq G$ is a subgroup of G such that $\widehat{g}(H) = H$ we will continue to denote by $\widehat{g}$ the restriction and corestriction of the conjugation by g to a subgroup H.

1.30 Lemma. *Let $\pi : X \twoheadrightarrow Y$ be a projection of racks, $x \in X$. Then*

$$\mathrm{Inn}_{\triangleright}(\pi)\,\widehat{\varphi}_x = \widehat{\varphi}_{\pi(x)}\,\mathrm{Inn}_{\triangleright}(\pi) \tag{1.8}$$

where $\widehat{\varphi}_x$ and $\widehat{\varphi}_{\pi(x)}$ are the automorphism of $\mathrm{Inn}_{\triangleright}(X)$ induced by conjugation by φ_x and the automorphism of $\mathrm{Inn}_{\triangleright}(Y)$ induced by conjugation by $\varphi_{\pi(x)}$ respectively.

Proof. Let $\alpha \in \mathrm{Inn}_{\triangleright}(X)$ be an element. Let us proceed by equalities.

$$\begin{aligned}
\mathrm{Inn}_{\triangleright}(\pi)\,\widehat{\varphi}_x(\alpha) &= \mathrm{Inn}_{\triangleright}(\pi)\left(\varphi_x\alpha\varphi_x^{-1}\right)\\
&= \mathrm{Inn}_{\triangleright}(\pi)\left(\varphi_x\right)\mathrm{Inn}_{\triangleright}(\pi)(\alpha)\,\mathrm{Inn}_{\triangleright}(\pi)\left(\varphi_x^{-1}\right)\\
&\overset{(1.5)}{=} \varphi_{\pi(x)}\,\mathrm{Inn}_{\triangleright}(\pi)(\alpha)\,\varphi_{\pi(x)}^{-1}\\
&= \widehat{\varphi}_{\pi(x)}\;\mathrm{Inn}_{\triangleright}(\pi)(\alpha)
\end{aligned}$$

□

With the aid of Lemma 1.28 we can prove that automorphisms act on the fibres of any projection of racks:

1.31 Proposition. *Let $\pi : X \twoheadrightarrow Y$ be a projection of racks. Then for every α in $\mathrm{Inn}_{\triangleright}(X)$ and x in X we have*

$$\alpha\left(\pi^{-1}(\pi(x))\right) = \pi^{-1}(\pi(\alpha(x)))$$

Proof. Let $w \in \alpha\left(\pi^{-1}(\pi(x))\right)$. Then there is z in X such that $\pi(z) = \pi(x)$ and $\alpha(z) = w$. Using Lemma 1.28 we can compute

$$\pi(w) = \pi(\alpha(z)) = (\mathrm{Inn}_{\triangleright}(\pi)(\alpha))(\pi(z)) = (\mathrm{Inn}_{\triangleright}(\pi)(\alpha))(\pi(x)) = \pi(\alpha(x))$$

Hence w is in the fibre of $\pi(\alpha(x))$ and

$$\alpha\left(\pi^{-1}(\pi(x))\right) \subseteq \pi^{-1}(\pi(\alpha(x)))$$

If we now apply the same reasoning to the fibre of $\pi(\alpha(x))$ and to the inner automorphism α^{-1} we have

$$\alpha^{-1}\left(\pi^{-1}(\pi(\alpha(x)))\right) \subseteq \pi^{-1}\left(\pi\left(\alpha^{-1}(\alpha(x))\right)\right) = \pi^{-1}(\pi(x))$$

and applying α to both sides we obtain

$$\pi^{-1}(\pi(\alpha(x))) \subseteq \alpha\left(\pi^{-1}(\pi(x))\right)$$

and we are done. □

An immediate consequence of Proposition 1.31 is the following

1.32 Definition ([Joy82b, Beginning of Section 2])**.** A projection is *proper* if it is not an isomorphism nor a constant map. A rack without proper projections is called *simple.*

1.33 Corollary ([McC12])**.** *If a transitive subgroup of* $\mathrm{Inn}_{\triangleright}(X)$ *acts primitively on a rack* X*, then the rack is simple.*

Proof. Proposition 1.31 shows that the fibres of any proper quotient are proper blocks for the action of the inner automorphism group. □

Connected Quandles

1.34 Definition. A rack X is *homogeneous* if $\mathrm{Aut}_{\triangleright}(X)$ acts transitively on it [Joy82a, Section 7], and *(algebraically) connected* when $\mathrm{Inn}_{\triangleright}(X)$ acts transitively on it [Joy82a, Section 8].

Every connected rack is homogeneous. The image of a connected rack is still connected and the fact that fibres are blocks for $\mathrm{Inn}_{\triangleright}(X)$ (see Proposition 1.31) has the following useful (for inductive arguments) consequence:

1.35 Proposition ([AG03, Lemma 1.21])**.** *Let* $\pi : X \longrightarrow\!\!\!\!\!\rightarrow Y$ *be a projection of racks and let* x *be an element in* X*. If* X *is connected then the fibres of* π *have the same cardinality and*

$$|X| = \left|\pi^{-1}\left(\pi\left(x\right)\right)\right| |Y|$$

The following construction gives a way to obtain quandles starting with groups and their automorphisms.

1.36 Definition ([Loo67, Proof of Satz 1.5])**.** Let G be a group, α an automorphism on G and H a subgroup of the fixed points of α in G. If we consider the set of right cosets G/H and define:

$$gH \triangleright fH := g\alpha\left(g^{-1}f\right)H \quad , \quad gH \triangleright^{-1} fH := g\alpha^{-1}\left(g^{-1}f\right)H,$$

the resulting structure is a quandle which we shall call the *coset quandle* G with respect to H and α and we will denote it by $\mathscr{Q}(G, H, \alpha)$.Notice that the operation is well defined since if $g_1H = g_2H$ and $f_1H = f_2H$ we have that for some $h_g, h_f \in H$

$$\begin{aligned}
g_1H \triangleright f_1H &= g_1\alpha\left(g_1^{-1}f_1\right)H \\
&= g_2h_g\alpha\left(\left(g_2h_g\right)^{-1}f_2h_f\right)H \\
&= g_2\alpha\left(h_gh_g^{-1}g_2^{-1}f_2\right)h_fH \\
&= g_2\alpha\left(g_2^{-1}f_2\right)H \\
&= g_2H \triangleright f_2H
\end{aligned}$$

1.37 Example. Let $\mathbb{D}_8$ be the group of permutations generated by $\tau = (1,3)$ and $\rho = (1,2,3,4)$, the dihedral group of order eight.
Let σ be the outer automorphism defined by $\sigma(\rho) = \rho$, $\sigma(\tau) = \tau\rho$. We have that the subgroup of fixed elements by σ in $\mathbb{D}_8$ is $\langle \rho \rangle$. Take $H = \langle \rho^2 \rangle$.
Hence the set of elements of $\mathscr{Q}(\mathbb{D}_8, H, \sigma)$ is $\{H, \tau H, \rho H, \tau\rho H\}$.
Some examples of computations of the rack operation are:

$$H \triangleright \tau H = \sigma(\tau) H = \tau\rho H$$

$$\tau H \triangleright \rho H = \tau\sigma(\tau\rho) H = \tau\sigma(\tau)\sigma(\rho) H = \tau\tau\rho\rho H = H$$

1.38 Definition ([AG03, Section 1.3.6])**.** A quandle is *principal* if it isomorphic to a coset quandle

$$\mathscr{Q}(G, \alpha) := \mathscr{Q}(G, \{1\}, \alpha).$$

A particular case of coset quandles which is important for its properties and which is extensively studied in the literature, is when the group G is abelian. In this case the quandle is called *affine* or *Alexander*. Affine quandles enjoy some specific properties:

1.39 Proposition ([AG03, Section 1.3.8])**.** *A finite affine quandle is always principal and it is faithful if and only if it is connected.*

Every coset quandle is homogeneous (since left multiplication by an element of the defining group is an isomorphism for the quandle). The following characterisation due to Joyce tells us that the reverse is also true, i.e. that every homogeneous quandle (and hence every connected quandle) has a coset representation:

1.40 Proposition ([Joy82a, Theorem 7.1])**.** *A quandle Q is homogeneous if and only if it is isomorphic to the coset quandle $\mathscr{Q}\left(\mathrm{Aut}_{\triangleright}(Q), \mathsf{Stab}_{\mathrm{Aut}_{\triangleright}(Q)}(x), \hat{\varphi}_x\right)$ where x is any element of Q and $\hat{\varphi}_x$ the conjugation by φ_x.*

In this way any question about connected quandles translates naturally in questions about groups and their automorphisms.

Chapter 2

The Transvection Group

Generalities on the Transvection Group

Let us introduce another group of automorphisms associated to a rack. A group which will take centre stage in the course of the development of the present work.

2.1 Definition ([Joy82a])**.** The *transvection group*, $\mathrm{Trans}_{\triangleright}(X)$, of a rack X is the subgroup of $\mathrm{Inn}_{\triangleright}(X)$ generated by the elements of type $\varphi_y \varphi_x^{-1}$ with $x, y \in X$.

Using equality (1.1) one sees that for any rack X, both $\mathrm{Inn}_{\triangleright}(X)$ and $\mathrm{Trans}_{\triangleright}(X)$ are normal in $\mathrm{Aut}_{\triangleright}(X)$ and for every $x \in X$ we have:

$$\mathrm{Inn}_{\triangleright}(X) = \mathrm{Trans}_{\triangleright}(X) \langle \varphi_x \rangle \tag{2.1}$$

In general nothing can be said about $\mathrm{Trans}_{\triangleright}(X) \cap \langle \varphi_x \rangle$.

2.2 Definition. A rack X is *splitting* if there is $x \in X$ such that $\mathrm{Trans}_{\triangleright}(X) \cap \langle \varphi_x \rangle = 1$

2.3 Remark. A rack X is splitting if and only if there is $x \in X$ such that

$$\mathrm{Inn}_{\triangleright}(X) \cong \mathrm{Trans}_{\triangleright}(X) \dot{\rtimes} \langle \varphi_x \rangle$$

where the semidirect product is internal.

From equality 2.1 it follows that the quotient $\mathrm{Inn}_{\triangleright}(X) / \mathrm{Trans}_{\triangleright}(X)$ is always cyclic and $\mathrm{Trans}_{\triangleright}(X)$ always contains $[\mathrm{Inn}_{\triangleright}(X), \mathrm{Inn}_{\triangleright}(X)]$, the derived subgroup of $\mathrm{Inn}_{\triangleright}(X)$. If the rack is connected then the reverse inclusion is also true.

2.4 Proposition ([Joy82b, stated in Section 1])**.** *If X is a connected rack then*

$$\mathrm{Trans}_{\triangleright}(X) = [\mathrm{Inn}_{\triangleright}(X), \mathrm{Inn}_{\triangleright}(X)]$$

If the rack is a quandle something more can be said.

2.5 Proposition ([HSV14, Proposition 2.1])**.** *For every quandle Q,* $\mathrm{Inn}_{\triangleright}(Q)$ *and* $\mathrm{Trans}_{\triangleright}(Q)$ *have the same orbits in their natural action on Q. In particular, Q is connected if and only if* $\mathrm{Trans}_{\triangleright}(Q)$ *is transitive on Q.*

Proof. Let $y \in x^{\mathrm{Inn}_{\triangleright}(Q)}$, i.e. there is $\alpha \in \mathrm{Inn}_{\triangleright}(Q)$ and by equality (2.1) there is $\tau \in \mathrm{Trans}_{\triangleright}(Q)$ and $n \in \mathbb{Z}$ such that:

$$y = \alpha(x) = \tau\varphi_x^n(x) = \tau(x)$$

The other inclusion is trivial. □

In order to simplify our computations we observe that we have a smaller system of generators for $\mathrm{Trans}_{\triangleright}(X)$:

2.6 Lemma. *Let X be a rack, $x \in X$. Then $\left\{\varphi_y\varphi_x^{-1}\right\}_{y\in X}$ is a system of generators for* $\mathrm{Trans}_{\triangleright}(X)$.

Proof. $\mathrm{Trans}_{\triangleright}(X)$ is generated by the elements of the form $\varphi_y\varphi_z^{-1}$ with y and z in X. But

$$\varphi_y\varphi_z^{-1} = \varphi_y\varphi_x^{-1}\varphi_x\varphi_z^{-1} = \varphi_y\varphi_x^{-1}\left(\varphi_z\varphi_x^{-1}\right)^{-1} \in \left\langle\left\{\varphi_y\varphi_x^{-1}\right\}_{y\in X}\right\rangle.$$

□

These generators are distinct if and only if the rack is faithful. Every projection of racks induces a surjective morphism between the transvection groups as the following proposition shows.

2.7 Proposition. *Let $\pi : X \twoheadrightarrow Y$ be a projection of racks and* $\mathrm{Inn}_{\triangleright}(\pi)$ *the induced group morphism from* $\mathrm{Inn}_{\triangleright}(X)$ *to* $\mathrm{Inn}_{\triangleright}(Y)$. *Then* $\mathrm{Inn}_{\triangleright}(\pi)(\mathrm{Trans}_{\triangleright}(X)) = \mathrm{Trans}_{\triangleright}(Y)$.

Proof. Let y_1 and y_2 in Y. If π is a projection, $y_1 = \pi(x_1)$ and $y_2 = \pi(x_2)$ for some x_1 and x_2 in X. We have

$$\varphi_{y_1}\varphi_{y_2}^{-1} = \varphi_{\pi(x_1)}\varphi_{\pi(x_2)}^{-1} = \mathrm{Inn}_{\triangleright}(\pi)(\varphi_{x_1})\,\mathrm{Inn}_{\triangleright}(\pi)(\varphi_{x_2})^{-1} = \mathrm{Inn}_{\triangleright}(\pi)\left(\varphi_{x_1}\varphi_{x_2}^{-1}\right).$$

□

2.8 Notation. Let $\pi : X \twoheadrightarrow Y$ be a projection of racks and $\mathrm{Inn}_{\triangleright}(\pi)$ the induced group morphism from $\mathrm{Inn}_{\triangleright}(X)$ to $\mathrm{Inn}_{\triangleright}(Y)$. In view of Proposition 2.7, the restriction and corestriction of $\mathrm{Inn}_{\triangleright}(\pi)$ to the transvection groups will be denoted by $\mathrm{Trans}_{\triangleright}(\pi)$.

2.9 Corollary. *Let X and Y be racks. If $\pi : X \xrightarrow{\sim} Y$ is an isomorphism then* $\mathrm{Trans}_{\triangleright}(\pi) : \mathrm{Trans}_{\triangleright}(X) \xrightarrow{\sim} \mathrm{Trans}_{\triangleright}(Y)$ *is an isomorphism too.*

Proof. This follows from Proposition 2.7 and Proposition 1.27. □

On the Transvection group of Deformation Racks

As observed by Brieskorn in [Bri88] (see 1.16), every rack is associated, via its deformation morphism, to a quandle. Let us start by a slight generalisation of his point of view.

2.10 Definition. Let X be a rack and $\alpha \in \mathrm{C}_{\mathrm{Aut}_{\triangleright}(X)}(\mathrm{Inn}_{\triangleright}(X))$. The *deformation of* X *via* $\pm\alpha$, ${}^{\pm\alpha}X$, is the rack $(X, \triangleright^{\pm\alpha})$ (see Proposition 1.13 for the definition of $(X, \triangleright^{\pm\alpha})$).

2.11 Definition. Let X, Y be racks. We say that X is a *deformation* of Y, $Y \rightsquigarrow X$ if there is $\alpha \in \mathrm{C}_{\mathrm{Aut}_{\triangleright}(Y)}(\mathrm{Inn}_{\triangleright}(Y))$ such that $X \cong {}^{\pm\alpha}Y$

We now analyse the nature of the deformation relation between racks.

2.12 Lemma ([Bri88]). *The deformation relation is symmetric and reflexive (i.e. it is a* tolerance relation*).*

Proof. The deformation relation is reflexive since $X = {}^{id}X$ and it is symmetric since if $Y \cong {}^{\alpha}X$ then $X \cong {}^{\alpha^{-1}}Y$ by Lemma 1.16, while if $Y \cong {}^{-\alpha}X$ let us show that $X \cong {}^{-\alpha}Y$ by showing that $X = {}^{-\alpha-\alpha}X$. Let us start by showing that $\alpha \in \mathrm{C}_{\mathrm{Aut}_{\triangleright}({}^{-\alpha}X)}(\mathrm{Inn}_{\triangleright}({}^{-\alpha}X))$, in fact it is a bijection and a morphism as the following equalities show

$$\alpha\left(x \triangleright^{-\alpha} y\right) = \alpha\varphi_x^{-1}\alpha^{-1}(y) = \varphi_{\alpha(x)}^{-1}\alpha^{-1}\alpha(y) = \alpha(x) \triangleright^{-\alpha} \alpha(y)$$

and centralises $\mathrm{Inn}_{\triangleright}({}^{-\alpha}X)$ as the following equalities show

$$x \triangleright^{-\alpha} \alpha(y) = \varphi_x^{-1}\alpha^{-1}\alpha(y) = \alpha\varphi_x^{-1}\alpha^{-1}(y) = \alpha\left(x \triangleright^{-\alpha} y\right)$$

Lastly, let us show that $X = {}^{-\alpha-\alpha}X$.

$$x\left(\triangleright^{-\alpha}\right)^{-\alpha} y = \left(\left(\varphi_{{}^{-\alpha}X}(x)\right)\alpha\right)^{-1}(y) = \left(\varphi_x^{-1}\alpha^{-1}\alpha\right)^{-1}(y) = \varphi_x(y) = x \triangleright y$$

□

2.13 Lemma. *Let X, Y be racks. If X is a deformation of Y then* $\mathrm{Trans}_{\triangleright}(X) \cong \mathrm{Trans}_{\triangleright}(Y)$.

Proof. Since $\mathrm{Trans}_{\triangleright}(X)$ is an invariant for a rack up to isomorphism (by Proposition 2.7), it is not reductive to take $X = {}^{\pm\alpha}Y$. We have that

$$\mathrm{Trans}_{\triangleright}(Y) = \left\langle \varphi_y\varphi_x^{-1} \right\rangle_{y \in Y}$$

while

$$\mathrm{Trans}_{\triangleright}\left({}^{\pm\alpha}Y\right) = \left\langle \left(\varphi_y\alpha\right)^{\pm 1}\left(\varphi_x\alpha\right)^{\mp 1} \right\rangle_{y \in Y}$$

In the case $\triangleright^{\alpha}$ we have

$$\varphi_y\alpha\left(\varphi_x\alpha\right)^{-1} = \varphi_y\alpha\alpha^{-1}\varphi_x^{-1} = \varphi_y\varphi_x^{-1}$$

In the case $\triangleright^{-\alpha}$ we have

$$\left(\varphi_y\alpha\right)^{-1}\varphi_x\alpha = \alpha^{-1}\varphi_y^{-1}\varphi_x\alpha = \varphi_y^{-1}\varphi_x$$

where the last equality is true since $\alpha \in \mathrm{C}_{\mathrm{Aut}_{\triangleright}(Y)}(\mathrm{Inn}_{\triangleright}(Y))$. In both cases every generator of $\mathrm{Trans}_{\triangleright}(X)$ is in $\mathrm{Trans}_{\triangleright}(Y)$ and vice versa, hence $\mathrm{Trans}_{\triangleright}(X) = \mathrm{Trans}_{\triangleright}(Y)$.

□

The deformation relation is not an equivalence as the following example shows.

2.14 Example. Let X_1 be a permutation rack with 5 elements, $\sigma_1 = (1,2,3,4,5)$ its defining permutation, i.e. $x \triangleright y = \sigma(y)$ and X_2 another permutation rack with elements $\{1,2,3,4,5\}$ and $\sigma_2 = (2,1,3,4,5)$ its defining permutation. Now $\mathrm{Aut}_\triangleright(X_i) = \mathrm{Inn}_\triangleright(X_i) = \mathrm{C}_{\mathrm{Aut}_\triangleright(X_i)}(\mathrm{Inn}_\triangleright(X_i)) = \langle\sigma_i\rangle$. X_1 and X_2 are both deformations of the trivial rack of order 5, but are not one the deformation of the other since the intersection of their automorphism groups is trivial, while if it was $X_1 = {}^{\alpha}X_2$, α would be in the intersection of the automorphism groups of X_1 and X_2.

This prompts us to introduce the following definition:

2.15 Definition. Let X, Y be racks. We say that X and Y are *associated*, $X \leftrightsquigarrow Y$, if ${}^{\iota_X}X \cong {}^{\iota_Y}Y$.

Being associated is an equivalence relation. We see now that to every rack X is associated only one quandle (up to isomorphism) and the association morphism, ι_X is unique up to conjugation.

2.16 Lemma. *Let Q be a quandle and $\alpha \in \mathrm{C}_{\mathrm{Aut}_\triangleright(Q)}(\mathrm{Inn}_\triangleright(Q))$. If $X = {}^{\alpha}Q$ then $\iota_X = \alpha^{-1}$*

Proof. By definition of association morphism (Definition 1.14) we have to show that $x \triangleright^{\alpha} \alpha^{-1}(x) = x$. To this end we proceed by equalities.

$$x \triangleright^{\alpha} \alpha^{-1}(x) = x \triangleright \alpha\alpha^{-1}(x) = x \triangleright x = x$$

□

2.17 Lemma. *Let Q_1, Q_2 be quandles and $\alpha_i \in \mathrm{C}_{\mathrm{Aut}_\triangleright(Q_i)}(\mathrm{Inn}_\triangleright(Q_i))$ for $i = 1, 2$ and let $\beta : Q_1 \longrightarrow Q_2$ be a bijection. Then $\beta : {}^{\alpha_1}Q_1 \xrightarrow{\sim} {}^{\alpha_2}Q_2$ is an isomorphism if and only if $\beta : Q_1 \xrightarrow{\sim} Q_2$ is an isomorphism and $\beta\alpha_1 = \alpha_2\beta$.*

Proof. Suppose first that β is an isomorphism from ${}^{\alpha_1}Q_1$ to ${}^{\alpha_2}Q_2$. By Lemma 2.16 we have that $\iota^{-1}_{({}^{\alpha_1}Q_1)} = \alpha_1$ and $\iota^{-1}_{({}^{\alpha_2}Q_2)} = \alpha_2$ and by Lemma 1.18:

$$\beta\alpha_1^{-1} = \alpha_2^{-1}\beta$$

so for every $x, y \in Q_1$:

$$\begin{aligned}
\beta(x \triangleright y) &= \beta\left(x \triangleright \alpha_1\alpha_1^{-1}(y)\right) \\
&= \beta\left(x \triangleright^{\alpha_1} \alpha_1^{-1}(y)\right) \\
&= \beta(x) \triangleright^{\alpha_2} \beta\alpha_1^{-1}(y) \\
&= \beta(x) \triangleright \alpha_2\beta\alpha_1^{-1}(y) \\
&= \beta(x) \triangleright \alpha_2\alpha_2^{-1}\beta(y) \\
&= \beta(x) \triangleright \beta(y)
\end{aligned}$$

Suppose now that β is an isomorphism from Q_1 to Q_2 and that $\beta\alpha_1 = \alpha_2\beta$. Let us compute:

$$\begin{aligned}
\beta\left(x \triangleright^{\alpha_1} y\right) &= \beta\left(x \triangleright \alpha_1\left(y\right)\right) \\
&= \beta\left(x\right) \triangleright \beta\alpha_1\left(y\right) \\
&= \beta\left(x\right) \triangleright \alpha_2\beta\left(y\right) \\
&= \beta\left(x\right) \triangleright^{\alpha_2} \beta\left(y\right)
\end{aligned}$$

□

The Isomorphism Problem

We will now be concerned in establishing a criterion for two quandles to be isomorphic . We already know that every isomorphism of quandles induces an isomorphism between their transvection groups, let us see to what extent the reverse is also true.

2.18 Proposition. *Let Q_j, $j = 1, 2$ be quandles,* $\mathrm{Trans}_{\triangleright}\left(Q_j\right)$ *their transvection groups. Suppose moreover that $\left\{x_{i,j}\right\}_{i\in I}$ is a system of representatives for the orbits of Q_j under the action of* $\mathrm{Trans}_{\triangleright}\left(Q_j\right)$. *The following statements are equivalent:*

(i) There is an isomorphism of quandles, $\alpha : Q_1 \xrightarrow{\sim} Q_2$, such that $\alpha\left(x_{i,1}\right) = x_{i,2}$.

(ii) There is a group isomorphism, $\psi : \mathrm{Trans}_{\triangleright}\left(Q_1\right) \xrightarrow{\sim} \mathrm{Trans}_{\triangleright}\left(Q_2\right)$, such that

  a) There is $i_0 \in I$ such that $\psi\widehat{\varphi}_{x_{i_0,1}}\psi^{-1} = \widehat{\varphi}_{x_{i_0,2}}$.

  b) For every $i \in I$, $\psi\left(\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q_1)}\left(x_{i,1}\right)\right) = \mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q_2)}\left(x_{i,2}\right)$.

  c) For every $i \in I$, $\psi\left(\varphi_{x_{i,1}}\varphi^{-1}_{x_{i_0,1}}\right) = \varphi_{x_{i,2}}\varphi^{-1}_{x_{i_0,2}}$.

In particular, if the conditions are satisfied, $\psi = \mathrm{Trans}_{\triangleright}\left(\alpha\right)$.

Proof. Suppose first that there exists $\alpha : Q_1 \xrightarrow{\sim} Q_2$, an isomorphism of quandles such that $\alpha\left(x_{i,1}\right) = x_{i,2}$. Then by Corollary 2.9 $\mathrm{Trans}_{\triangleright}\left(\alpha\right)$ is an isomorphism from $\mathrm{Trans}_{\triangleright}\left(Q_1\right)$ to $\mathrm{Trans}_{\triangleright}\left(Q_2\right)$. Let us define $\psi := \mathrm{Trans}_{\triangleright}\left(\alpha\right)$. Let us verify that it satisfies the required conditions.

$$\begin{aligned}
\psi\widehat{\varphi}_{x_{i,1}} &= \mathrm{Trans}_{\triangleright}\left(\alpha\right)\widehat{\varphi}_{x_{i,1}} \\
&\overset{(1.8)}{=} \widehat{\varphi}_{\alpha\left(x_{i,1}\right)}\mathrm{Trans}_{\triangleright}\left(\alpha\right) \\
&= \widehat{\varphi}_{x_{i,2}}\psi
\end{aligned}$$

Let $\tau \in \mathrm{Trans}_{\triangleright}(Q_1)$. Then

$$\begin{aligned}
\tau \in \mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q_1)}(x_{i,1}) &\iff \tau(x_{i,1}) = x_{i,1} \\
&\iff \alpha\tau(x_{i,1}) = \alpha(x_{i,1}) \\
&\overset{(1.7)}{\iff} (\mathrm{Trans}_{\triangleright}(\alpha)(\tau))\,\alpha(x_{i,1}) = \alpha(x_{i,1}) \\
&\iff (\psi(\tau))(x_{i,2}) = x_{i,2} \\
&\iff \psi(\tau) \in \mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q_2)}(x_{i,2})
\end{aligned}$$

Since ψ is an isomorphism between $\mathrm{Trans}_{\triangleright}(Q_1)$ and $\mathrm{Trans}_{\triangleright}(Q_2)$, we have shown that $\psi\left(\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q_1)}(x_{i,1})\right) = \mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q_2)}(x_{i,2})$.

Lastly, let us verify the third set of conditions.

$$\begin{aligned}
\psi\left(\varphi_{x_{i,1}}\varphi^{-1}_{x_{i_0,1}}\right) &= \mathrm{Trans}_{\triangleright}(\alpha)\left(\varphi_{x_{i,1}}\varphi^{-1}_{x_{i_0,1}}\right) \\
&\overset{(1.7)}{=} \varphi_{\alpha\left(x_{i,1}\right)}\varphi^{-1}_{\alpha\left(x_{i_0,1}\right)} \\
&= \varphi_{x_{i,2}}\varphi^{-1}_{x_{i_0,2}}
\end{aligned}$$

Suppose now that there exists an isomorphism $\psi : \mathrm{Trans}_{\triangleright}(Q_1) \overset{\sim}{\longrightarrow} \mathrm{Trans}_{\triangleright}(Q_2)$ satisfying conditions a), b), c). Let us define $\alpha : Q_1 \longrightarrow Q_2$ by setting for every $\tau \in \mathrm{Trans}_{\triangleright}(Q_1)$

$$\alpha\tau(x_{i,1}) := \psi(\tau)(x_{i,2}) \tag{2.2}$$

Since $\{x_{i,1}\}_{i\in I}$ is a system of representatives for the orbits of Q_1 under the action of $\mathrm{Trans}_{\triangleright}(Q_1)$, γ is defined for every $z \in Q_1$. Let us see that the definition does not depend on the choice of τ and that it is injective. Let $\tau_1, \tau_2 \in \mathrm{Trans}_{\triangleright}(Q_1)$

$$\begin{aligned}
\tau_1^{-1}\tau_2 \in \mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q_1)}(x_{i,1}) &\iff \psi\left(\tau_1^{-1}\tau_2\right) \in \psi\left(\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q_1)}(x)\right) \\
&\overset{b)}{\iff} \psi(\tau_1)^{-1}\psi(\tau_2) \in \mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q_2)}(x_{i,2})
\end{aligned}$$

Since $\{x_{i,2}\}_{i\in I}$ is a system of representatives for the orbits of Q_2 under the action of $\mathrm{Trans}_{\triangleright}(Q_2)$, α is surjective. We are left to verify that α is a rack morphism.
Let us begin establishing that the hypotheses afford a more general set of relations between the orbit representatives, more precisely that for every $j, k \in I$ and $\tau \in \mathrm{Trans}_{\triangleright}(Q_1)$

$$\psi\left(\varphi_{x_{j,1}}\tau\varphi^{-1}_{x_{k,1}}\right) = \varphi_{x_{j,2}}\psi(\tau)\,\varphi^{-1}_{x_{j,2}} \tag{2.3}$$

Let us proceed by equalities.

$$\begin{aligned}
\psi\left(\varphi_{x_{j,1}}\tau\varphi_{x_{k,1}}^{-1}\right) &= \psi\left(\varphi_{x_{j,1}}\varphi_{x_{i_0,1}}^{-1}\varphi_{x_{i_0,1}}\tau\varphi_{x_{i_0,1}}^{-1}\varphi_{x_{i_0,1}}\varphi_{x_{k,1}}^{-1}\right)\\
&= \psi\left(\varphi_{x_{j,1}}\varphi_{x_{i_0,1}}^{-1}\right)\psi\left(\varphi_{x_{i_0,1}}\tau\varphi_{x_{i_0,1}}^{-1}\right)\psi\left(\varphi_{x_{i_0,1}}\varphi_{x_{k,1}}^{-1}\right)\\
&\overset{c)}{=} \varphi_{x_{j,2}}\varphi_{x_{i_0,2}}^{-1}\left(\psi\widehat{\varphi}_{x_{i_0,1}}(\tau)\right)\varphi_{x_{i_0,2}}\varphi_{x_{k,2}}^{-1}\\
&\overset{a)}{=} \varphi_{x_{j,2}}\varphi_{x_{i_0,2}}^{-1}\widehat{\varphi}_{x_{i_0,2}}\psi(\tau)\varphi_{x_{i_0,2}}\varphi_{x_{k,2}}^{-1}\\
&= \varphi_{x_{j,2}}\psi(\tau)\varphi_{x_{k,2}}^{-1}
\end{aligned}$$

Let now $y, z \in Q_1$ be elements. We want to establish the following equality.

$$\psi\left(\varphi_z\varphi_y^{-1}\right) = \varphi_{\alpha(z)}\varphi_{\alpha(y)}^{-1} \tag{2.4}$$

Since $\{x_{i,1}\}_{i\in I}$ is a set of representatives for the orbits of Q_1 under $\mathrm{Trans}_{\triangleright}(Q_1)$ for every $z \in Q_1$, there are $i_{z,1} \in I$ and $\tau_{z,1} \in \mathrm{Trans}_{\triangleright}(Q_1)$ such that $z = \tau_{z,1}(x_{i_z,1})$. Let us proceed by equalities to prove equality (2.4).

$$\begin{aligned}
\psi\left(\varphi_z\varphi_y^{-1}\right) &= \psi\left(\varphi_{\tau_z(x_{i_z,1})}\varphi_{\tau_y\left(x_{i_y,1}\right)}^{-1}\right)\\
&\overset{(1.1)}{=} \psi\left(\tau_z\varphi_{x_{i_z,1}}\tau_z^{-1}\tau_y\varphi_{x_{i_y,1}}\tau_y^{-1}\right)\\
&\overset{(2.3)}{=} \psi(\tau_z)\varphi_{x_{i_z,2}}\psi\left(\tau_z^{-1}\tau_y\right)\varphi_{x_{i_y,2}}^{-1}\psi\left(\tau_y^{-1}\right)\\
&\overset{(1.1)}{=} \varphi_{\psi(\tau_z)(x_{i_z,2})}\varphi_{\psi(\tau_y)\left(x_{i_y,2}\right)}\\
&\overset{(2.2)}{=} \varphi_{\alpha(z)}\varphi_{\alpha(y)}^{-1}
\end{aligned}$$

We are finally ready to show that α is a rack morphism. Let $y, z \in Q_1$ be elements.

$$\begin{aligned}
\alpha(y)\triangleright\alpha(z) &= \varphi_{\alpha(y)}\varphi_{\alpha(z)}^{-1}(\alpha(z))\\
&\overset{(2.4)}{=} \psi\left(\varphi_y\varphi_z^{-1}\right)\alpha(z)\\
&\overset{(2.2)}{=} \alpha\left(\varphi_y\varphi_z^{-1}(z)\right)\\
&= \alpha(y\triangleright z)
\end{aligned}$$

In particular for every $y, z \in Q_1$

$$\mathrm{Trans}_{\triangleright}(\alpha)\left(\varphi_z\varphi_y^{-1}\right) = \varphi_{\alpha(z)}\varphi_{\alpha(y)}^{-1} \overset{(2.4)}{=} \psi\left(\varphi_z\varphi_y^{-1}\right)$$

and since they coincide on the generators of $\mathrm{Trans}_{\triangleright}(Q_1)$, $\mathrm{Trans}_{\triangleright}(\alpha) = \psi$ □

If the quandle is connected, the preceding proposition simplifies as follows.

2.19 Corollary. *Let Q_j, $j = 1, 2$ be connected quandles,* $\mathrm{Trans}_\triangleright(Q_j)$ *their transvection groups, $x_j \in Q_j$. The following statements are equivalent:*

(i) There is an isomorphism of quandles, $\alpha : Q_1 \xrightarrow{\sim} Q_2$, such that $\alpha(x_1) = x_2$.

(ii) There is a group isomorphism, $\psi : \mathrm{Trans}_\triangleright(Q_1) \xrightarrow{\sim} \mathrm{Trans}_\triangleright(Q_2)$, such that

a) $\psi \widehat{\varphi}_{x_1} \psi^{-1} = \widehat{\varphi}_{x_2}$.

b) $\psi\left(\mathrm{Stab}_{\mathrm{Trans}_\triangleright(Q_1)}(x_1)\right) = \mathrm{Stab}_{\mathrm{Trans}_\triangleright(Q_2)}(x_2)$.

In particular, if the conditions are satisfied, we have $\psi = \mathrm{Trans}_\triangleright(\alpha)$.

Proof. Apply 2.18, where the sets of representatives of the orbits under the transvection groups reduce to x_j in Q_j. □

In Proposition 2.18 to every isomorphism between quandles an isomorphism between the corresponding transvection groups is associated. In general, this association is not injective. In the next results we will explore this correspondence between automorphisms of a quandle and group automorphisms of its transvection group.

Given a rack automorphism α, the induced group automorphism $\mathrm{Inn}_\triangleright(\alpha)$ which is an element of $\mathrm{Aut}(\mathrm{Inn}_\triangleright(X))$, not only restricts to a group automorphism of the tranvection group, but it also extends to a group automorphism of the automorphism group $\mathrm{Aut}_\triangleright(X)$. Let us see how.

2.20 Lemma. *Let X be a rack, $\alpha, \beta \in \mathrm{Aut}_\triangleright(X)$. Then*

$$\mathrm{Inn}_\triangleright(\alpha)\,\mathrm{Inn}_\triangleright(\beta) = \mathrm{Inn}_\triangleright(\alpha\beta) \tag{2.5}$$

Proof. Since $\mathrm{Inn}_\triangleright(\alpha), \mathrm{Inn}_\triangleright(\beta), \mathrm{Inn}_\triangleright(\alpha\beta) \in \mathrm{Aut}(\mathrm{Inn}_\triangleright(X))$ let us verify the equality on the generators of $\mathrm{Inn}_\triangleright(X)$.

$$\mathrm{Inn}_\triangleright(\alpha)\,\mathrm{Inn}_\triangleright(\beta)(\varphi_x) \overset{(1.5)}{=} \mathrm{Inn}_\triangleright(\alpha)\left(\varphi_{\beta(x)}\right) \overset{(1.5)}{=} \varphi_{\alpha\beta(x)} \overset{(1.5)}{=} \mathrm{Inn}_\triangleright(\alpha\beta)(\varphi_x)$$

□

The morphism described by Lemma 2.20 is nothing but the usual group morphism between a group and its inner automorphism group.

2.21 Lemma. *Let X be a rack, $\alpha \in \mathrm{Aut}_\triangleright(X)$ an automorphism on X. Then* $\mathrm{Inn}_\triangleright(\alpha)$ *(the automorphism on* $\mathrm{Inn}_\triangleright(X)$ *induced by α) is equal to $\widehat{\alpha}$ (the automorphism on* $\mathrm{Inn}_\triangleright(X)$ *induced by the conjugation by α).*

Proof. Let us show that the two automorphisms act in the same way on the generators of $\mathrm{Inn}_\triangleright(X)$.

$$\mathrm{Inn}_\triangleright(\alpha)(\varphi_x) \overset{(1.5)}{=} \varphi_{\alpha(x)} \overset{(1.1)}{=} \alpha\varphi_x\alpha^{-1} = \widehat{\alpha}(\varphi_x)$$

□

2.22 Proposition. *Let X be a rack, $\alpha \in \mathrm{Aut}_{\triangleright}(X)$ an automorphism on X. Then $\mathrm{Inn}_{\triangleright}(\alpha)$ extends to an internal automorphism on $\mathrm{Aut}_{\triangleright}(X)$, namely $\widehat{\alpha}$.*

Proof. By Lemma 2.21, $\widehat{\alpha}$ restricted to $\mathrm{Inn}_{\triangleright}(X)$ is equal to $\mathrm{Inn}_{\triangleright}(\alpha)$. □

Since $\mathrm{Trans}_{\triangleright}(X)$ is a normal subgroup of $\mathrm{Aut}_{\triangleright}(X)$, every rack automorphism induces by conjugation a group automorphism on $\mathrm{Trans}_{\triangleright}(X)$, which, by Lemma 2.21, corresponds to the restriction to $\mathrm{Trans}_{\triangleright}(X)$ of the induced automorphism on $\mathrm{Inn}_{\triangleright}(X)$ (and of $\mathrm{Aut}_{\triangleright}(X)$ by Proposition 2.22). Let us see which automorphisms of $\mathrm{Trans}_{\triangleright}(X)$ arise in this way.

2.23 Proposition. *Let Q be a quandle, $\mathrm{Trans}_{\triangleright}(Q)$ and $\mathrm{Aut}_{\triangleright}(Q)$ its transvection group and automorphism group respectively, $\psi \in \mathrm{Aut}(\mathrm{Trans}_{\triangleright}(Q))$. Suppose moreover that $\{x_i\}_{i\in I}$ is a system of representatives for the orbits of Q under the action of $\mathrm{Trans}_{\triangleright}(Q)$. The following statements are equivalent:*

(i) There is $\alpha \in \mathrm{Aut}_{\triangleright}(Q)$ such that $\psi = \widehat{\alpha}$ (the conjugation by α).

(ii) There is a system of representatives for the orbits of Q under the action of $\mathrm{Trans}_{\triangleright}(Q)$, $\{y_i\}_{i\in I}$, such that $\psi\left(\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x_i)\right) = \mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(y_i)$ and there is $i_0 \in I$ such that $\psi\widehat{\varphi}_{x_{i_0}}\psi^{-1} = \widehat{\varphi}_{y_{i_0}}$ and for every $i \in I$ $\psi\left(\varphi_{x_i}\varphi_{x_{i_0}}^{-1}\right) = \varphi_{y_i}\varphi_{y_{i_0}}^{-1}$.

Proof. *(i)* $\Rightarrow$ *(ii)* We apply Proposition 2.18 with $Q_1 = Q_2 = Q$, $\{x_{i,1}\}_{i\in I} = \{x_i\}_{i\in I}$, $\{x_{i,2}\}_{i\in I} = \{\alpha(x_i)\}_{i\in I}$ and *(ii)* follows immediately once we take $y_i = \alpha(x_i)$, observing that $\mathrm{Inn}_{\triangleright}(\alpha) = \widehat{\alpha}$ by Lemma 2.21. For the hypotheses of Proposition 2.18 to be satisfied we have to verify only that $\{\alpha(x_i)\}_{i\in I}$ is a system of representatives for the orbits of Q under the action of $\mathrm{Trans}_{\triangleright}(Q)$. Let $z \in Q$ be an element. Since $\{x_i\}_{i\in I}$ is a system of representatives for the orbits of Q under the action of $\mathrm{Trans}_{\triangleright}(Q)$, there are $\tau_z \in \mathrm{Trans}_{\triangleright}(Q)$ and $i_z \in I$ such that $\alpha^{-1}(z) = \tau_z(x_{i_z})$ and we can compute

$$\begin{aligned} z = \alpha(\tau_z(x_{i_z})) &= \alpha\tau_z\alpha^{-1}\alpha(x_{i_z}) \\ &= (\psi(\tau_z))(\alpha(x_{i_z})) \end{aligned}$$

So every element in Q is in $\alpha(x_i)^{\mathrm{Trans}_{\triangleright}(Q)}$ for some $i \in I$. Suppose now that

$$\alpha(x_j)^{\mathrm{Trans}_{\triangleright}(Q)} = \alpha(x_k)^{\mathrm{Trans}_{\triangleright}(Q)}$$

for some $k, j \in I$. Then there is $\tau \in \mathrm{Trans}_{\triangleright}(Q)$ such that $\alpha(x_k) = \tau\alpha(x_j)$ and

$$x_k = \alpha^{-1}(\tau\alpha(x_j)) = \psi^{-1}(\tau)(x_j)$$

and $j = k$.

(ii) $\Rightarrow$ *(i)* We apply Proposition 2.18 with $Q_1 = Q_2 = Q$, $\{x_{i,1}\}_{i\in I} = \{x_i\}_{i\in I}$, $\{x_{i,2}\}_{i\in I} = \{\alpha(y_i)\}_{i\in I}$, observing that $\mathrm{Inn}_{\triangleright}(\alpha) = \widehat{\alpha}$ by Lemma 2.21. □

If the quandle is connected the statement of Proposition 2.23 simplifies in the following way.

2.24 Corollary. *Let Q be a quandle,* $\mathrm{Trans}_{\triangleright}(Q)$ *and* $\mathrm{Aut}_{\triangleright}(Q)$ *its transvection group and automorphism group respectively,* $\psi \in \mathrm{Aut}(\mathrm{Trans}_{\triangleright}(Q))$, $x, y \in Q$. *If Q is connected the following statements are equivalent:*

(i) There is $\alpha \in \mathrm{Aut}_{\triangleright}(Q)$ such that $\psi = \widehat{\alpha}$ (the conjugation by α) and $\alpha(x) = y$.

(ii) $\psi\left(\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x)\right) = \mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(y)$ *and* $\psi\widehat{\varphi}_x\psi^{-1} = \widehat{\varphi}_y$.

Proof. Take $x_{i_0} = x$ and $y_{i_0} = y$ in Proposition 2.23 (both x and y are system of representatives for the orbits of $\mathrm{Trans}_{\triangleright}(Q)$ since by Proposition 2.5, $\mathrm{Trans}_{\triangleright}(Q)$ is transitive). Notice that the third set of conditions on the orbit representatives is void in the connected case. □

Proposition 2.23 gives a criterion to recognise which automorphisms of $\mathrm{Trans}_{\triangleright}(Q)$ are induced by conjugation by elements in $\mathrm{Aut}_{\triangleright}(Q)$. The next definition takes care of giving a name and a notation to this group of automorphisms.

2.25 Definition. Let Q be a quandle. The *induced automorphisms*, $\mathrm{Ind}_{\triangleright}(Q)$ of Q are the automorphisms ψ of $\mathrm{Trans}_{\triangleright}(Q)$ such that there is $\alpha \in \mathrm{Aut}_{\triangleright}(Q)$ such that $\psi = \widehat{\alpha}$.

Using Definition 2.25, Proposition 2.23 can be restated as follows.

2.26 Corollary. *Let Q be a quandle. Then*

$$\frac{\mathrm{Aut}_{\triangleright}(Q)}{\mathrm{C}_{\mathrm{Aut}_{\triangleright}(Q)}(\mathrm{Trans}_{\triangleright}(Q))} \cong \mathrm{Ind}_{\triangleright}(Q) \tag{2.6}$$

Proof. Since $\mathrm{Trans}_{\triangleright}(Q)$ is normal in $\mathrm{Aut}_{\triangleright}(Q)$ there is a natural group morphism from $\mathrm{Aut}_{\triangleright}(Q)$ to $\mathrm{Aut}(\mathrm{Trans}_{\triangleright}(Q))$ which associates to every quandle automorphism α the group automorphism $\widehat{\alpha} : \mathrm{Trans}_{\triangleright}(Q) \longrightarrow \mathrm{Trans}_{\triangleright}(Q)$, defined by $\widehat{\alpha}(\tau) = \alpha\tau\alpha^{-1}$. The kernel $\widehat{\alpha}$ is $\mathrm{C}_{\mathrm{Aut}_{\triangleright}(Q)}(\mathrm{Trans}_{\triangleright}(Q))$ while the image, by Proposition 2.23, is $\mathrm{Ind}_{\triangleright}(Q)$. □

2.27 Remark. In the more general situation of a rack X, the automorphisms of $\mathrm{Trans}_{\triangleright}(X)$ induced by conjugation by an element in $\mathrm{Aut}_{\triangleright}(X)$ are in the intersection of the induced automorphism of the associated quandle ${}^{\iota}X$ with the centraliser in $\mathrm{Aut}_{\triangleright}({}^{\iota}X)$ of ι:

$$\mathrm{Ind}_{\triangleright}(X) = \mathrm{Ind}_{\triangleright}({}^{\iota}X) \cap \mathrm{C}_{\mathrm{Aut}_{\triangleright}({}^{\iota}X)}(\iota) \tag{2.7}$$

Since our objective is to reconstruct information on the automorphism group of the quandle using information on the automorphisms of the transvection group, it is useful to observe that both extreme situations for the quotient $\dfrac{\mathrm{Aut}_{\triangleright}(Q)}{\mathrm{C}_{\mathrm{Aut}_{\triangleright}(Q)}(\mathrm{Trans}_{\triangleright}(Q))}$ can occur, i.e. it can happen that $\mathrm{C}_{\mathrm{Aut}_{\triangleright}(Q)}(\mathrm{Trans}_{\triangleright}(Q)) = 1$ and that $\mathrm{C}_{\mathrm{Aut}_{\triangleright}(Q)}(\mathrm{Trans}_{\triangleright}(Q)) = \mathrm{Aut}_{\triangleright}(Q)$, too.

2.28 Example. Let Q be a connected quandle of prime order. Even without making reference to the (known) classification of connected quandles of prime order, one knows that $\mathrm{Trans}_{\triangleright}(Q)$ is a transitive group of prime degree, hence primitive, hence $\mathrm{C}_{\mathbb{S}_Q}(\mathrm{Trans}_{\triangleright}(Q)) = 1$ and a fortiori $\mathrm{C}_{\mathrm{Aut}_{\triangleright}(Q)}(\mathrm{Trans}_{\triangleright}(Q)) = 1$.

2.29 Example. Let Q be a trivial quandle on order n, i.e. $x \triangleright y = y$ for every $x, y \in Q$. In this case $\mathrm{Trans}_{\triangleright}(Q) = 1$ and $\mathrm{Ind}_{\triangleright}(Q) = 1$. Notice that $\mathrm{Aut}_{\triangleright}(Q)$ (and hence $\mathrm{C}_{\mathrm{Aut}_{\triangleright}(Q)}(\mathrm{Trans}_{\triangleright}(Q))$) is the whole of $\mathbb{S}_Q$.

We conclude this section with a result on the centraliser of the transvection group in the automorphism group of a quandle.

2.30 Notation. Since it will be useful in the proof of the next lemma, and also later on, let us introduce a notation that generalises the usual one for centralisers and commutators in groups. Let G be a group, $B \subseteq \mathrm{Aut}(G)$ a set of automorphisms. From now on the *centraliser of a set of automorphisms* in a group, $\mathrm{C}_G(B)$, will be the subgroup of G of elements fixed by every $\alpha \in B$:

$$\mathrm{C}_G(B) := \{g \in G \mid \alpha(g) = g,\ \forall \alpha \in B\}$$

while *the subgroup of commutators of a set of automorphisms* in a group, $[G, B]$ will be the subgroup generated by elements of the form $g^{-1}\alpha(g)$ with $g \in G$ and $\alpha \in B$:

$$[G, B] := \left\langle \left\{ g^{-1}\alpha(g) \mid \alpha \in B, g \in G \right\} \right\rangle$$

Since we will be interested not only in the subgroup generated by the commutators with automorphisms but also in the set of commutators itself, let us introduce the following, less standard, notation:

$$\mathrm{Comm}(G, B) := \left\{ g^{-1}\alpha(g) \mid \alpha \in B, g \in G \right\}$$

2.31 Proposition. *Let Q be a quandle, $\mathrm{Aut}_{\triangleright}(Q)$ and $\mathrm{Trans}_{\triangleright}(Q)$ its automorphism group and transvection group respectively. Then*

$$\frac{\mathrm{C}_{\mathrm{Aut}_{\triangleright}(Q)}(\mathrm{Trans}_{\triangleright}(Q))}{\mathrm{C}_{\mathrm{Aut}_{\triangleright}(Q)}(\mathrm{Inn}_{\triangleright}(Q))} \hookrightarrow \mathrm{Z}(\mathrm{Trans}_{\triangleright}(Q)) \tag{2.8}$$

while if $\mathrm{Trans}_{\triangleright}(Q)$ *is centreless*

$$\mathrm{C}_{\mathrm{Aut}_{\triangleright}(Q)}(\mathrm{Trans}_{\triangleright}(Q)) = \mathrm{C}_{\mathrm{Aut}_{\triangleright}(Q)}(\mathrm{Inn}_{\triangleright}(Q)) \tag{2.9}$$

and if Q is faithful with centreless transvection group

$$\mathrm{C}_{\mathrm{Aut}_{\triangleright}(Q)}(\mathrm{Trans}_{\triangleright}(Q)) = 1 \tag{2.10}$$

Proof. $\mathrm{C}_{\mathrm{Aut}_{\triangleright}(Q)}(\mathrm{Trans}_{\triangleright}(Q))$ is the centraliser of a normal subgroup, hence it is normal itself (as shown for instance in [Hum96, Proposition 10.26]) and the function $\partial\widehat{\varphi}_x$ from $\mathrm{C}_{\mathrm{Aut}_{\triangleright}(Q)}(\mathrm{Trans}_{\triangleright}(Q))$ to itself defined by $\partial\widehat{\varphi}_x(\gamma) := [\gamma, \varphi_x] = \gamma\varphi_x\gamma^{-1}\varphi_x^{-1}$ is well defined.
We have that

$$\begin{aligned}
\partial\widehat{\varphi}_x\left(\mathrm{C}_{\mathrm{Aut}_{\triangleright}(Q)}(\mathrm{Trans}_{\triangleright}(Q))\right) &= \mathrm{Comm}\left(\mathrm{C}_{\mathrm{Aut}_{\triangleright}(Q)}(\mathrm{Trans}_{\triangleright}(Q)), \varphi_x\right) \\
&\subseteq \mathrm{C}_{\mathrm{Aut}_{\triangleright}(Q)}(\mathrm{Trans}_{\triangleright}(Q)) \cap [\mathrm{Aut}_{\triangleright}(Q), \varphi_x] \\
&\subseteq \mathrm{C}_{\mathrm{Aut}_{\triangleright}(Q)}(\mathrm{Trans}_{\triangleright}(Q)) \cap \mathrm{Trans}_{\triangleright}(Q) \\
&= \mathrm{Z}(\mathrm{Trans}_{\triangleright}(Q))
\end{aligned}$$

Let us show that $\partial\widehat{\varphi}_x$ is a morphism. Let $\sigma, \tau \in \mathrm{C}_{\mathrm{Aut}_\triangleright(Q)}\left(\mathrm{Trans}_\triangleright(Q)\right)$.

$$\begin{aligned}\partial\widehat{\varphi}_x(\sigma\tau) &= \sigma\tau\varphi_x\tau^{-1}\sigma^{-1}\varphi_x^{-1} \\ &= \sigma\varphi_{\tau(x)}\varphi_x^{-1}\varphi_x\sigma^{-1}\varphi_x^{-1} \\ &= \sigma\varphi_x\sigma^{-1}\varphi_x^{-1}\varphi_{\tau(x)}\varphi_x^{-1} \\ &= \partial\widehat{\varphi}_x(\sigma)\,\partial\widehat{\varphi}_x(\tau)\end{aligned}$$

The kernel of $\partial\widehat{\varphi}_x$ is $\mathrm{C}_{\mathrm{Aut}_\triangleright(Q)}\left(\mathrm{Trans}_\triangleright(Q)\right) \cap \mathrm{C}_{\mathrm{Aut}_\triangleright(Q)}\left(\varphi_x\right) = \mathrm{C}_{\mathrm{Aut}_\triangleright(Q)}\left(\mathrm{Inn}_\triangleright(Q)\right)$. The last statements follow immediately from the fact that if Q is faithful then $\mathrm{C}_{\mathrm{Aut}_\triangleright(Q)}\left(\mathrm{Inn}_\triangleright(Q)\right) = 1$. □

Transvection Group of a Standard Crossed Set

2.32 Remark. If Q is a standard crossed set (see Definition 1.5) it is not reductive to consider $Q \subseteq G$ for some group G. In this case $\varphi_x(y) = xyx^{-1}$ for every $x, y \in Q$ and φ can be extended from Q to G and it is both a rack and a group morphism from G to $\mathrm{Aut}(G)$ whose kernel is $\mathrm{Z}(G)$.

2.33 Lemma ([AG03, Lemma 1.9])**.** *If G is a group and $Q \subseteq G$ is a standard crossed set then*

$$\mathrm{Inn}_\triangleright(Q) \cong \frac{\langle Q\rangle}{\mathrm{Z}(\langle Q\rangle)}.$$

2.34 Proposition. *Let G be a group, $Q \subseteq G$ is a standard crossed set. Then*

$$\mathrm{Trans}_\triangleright(Q) \cong \frac{\left\langle QQ^{-1}\right\rangle}{\mathrm{C}_{\langle QQ^{-1}\rangle}(Q)}$$

where $\mathrm{C}_{\langle QQ^{-1}\rangle}(Q)$ is the centraliser of Q in $\left\langle QQ^{-1}\right\rangle$.

Proof. Let t be in $\left\langle QQ^{-1}\right\rangle$, then t is of the form $\prod_i x_i y_i^{-1}$ for some x_i, y_i in Q and

$$\varphi(t) = \varphi\left(\prod_i x_i y_i^{-1}\right) = \prod_i \varphi_{x_i}\varphi_{y_i}^{-1}$$

where φ associates to every element the automorphism determined by the conjugation by that element and coincides with φ_Q on the elements of Q (Remark 2.32). The second equality is justified by the fact that φ is a group morphism. This shows that $\varphi\left(\left\langle QQ^{-1}\right\rangle\right) \subseteq \mathrm{Trans}_\triangleright(Q)$. Conversely, if τ is in $\mathrm{Trans}_\triangleright(Q)$, it is of the form $\prod_i \varphi_{x_i}\varphi_{y_i}^{-1}$, hence

$$\tau = \varphi\left(\prod_i x_i y_i^{-1}\right)$$

where $\prod_i x_i y_i^{-1}$ is in $\left\langle QQ^{-1}\right\rangle$ and $\varphi\left(\left\langle QQ^{-1}\right\rangle\right) = \mathrm{Trans}_\triangleright(Q)$. Finally, $t \in \left\langle QQ^{-1}\right\rangle \cap \mathrm{Ker}(\varphi)$ if and only if t commutes with every x in Q, which are the generators of $\left\langle QQ^{-1}\right\rangle$, hence the conclusion. □

A special case of this situation occurs when G is $\mathrm{Inn}_{\triangleright}(X)$ for some rack X, and the standard subset taken into consideration is $\varphi(X)$. In this case we have

2.35 Corollary. *Let X be a rack. Then*

$$\mathrm{Trans}_{\triangleright}(\varphi(X)) \cong \frac{\mathrm{Trans}_{\triangleright}(X)}{\mathrm{Z}(\mathrm{Inn}_{\triangleright}(X)) \cap \mathrm{Trans}_{\triangleright}(X)}$$

where with $\mathrm{Z}(\mathrm{Inn}_{\triangleright}(X))$ we denote the centre of $\mathrm{Inn}_{\triangleright}(X)$. □

An application of this equality comes out when the rack is a connected quandle.

2.36 Proposition. *Let Q be a connected quandle, $x \in Q$. If $\left|\varphi^{-1}(\varphi_x)\right| = i$, $K = \mathrm{Z}(\mathrm{Inn}_{\triangleright}(Q)) \cap \mathrm{Trans}_{\triangleright}(Q)$ and $|K| = k$ then k divides i and*

$$\frac{i}{k}\left|\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x)\right| = \left|\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(\varphi(Q))}(\varphi_x)\right|.$$

In particular, if $\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(\varphi(Q))}(\varphi_x) = 1$ then $\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x) = 1$.

Proof. By Corollary 2.35, $\mathrm{Trans}_{\triangleright}(\varphi(Q))$ is isomorphic to

$$\frac{\mathrm{Trans}_{\triangleright}(Q)}{\mathrm{Z}(\mathrm{Inn}_{\triangleright}(Q)) \cap \mathrm{Trans}_{\triangleright}(Q)}$$

At the same time K is a subgroup of the centre of $\mathrm{Trans}_{\triangleright}(Q)$, and $\mathrm{Trans}_{\triangleright}(Q)$, being Q a connected quandle, is transitive. In this case K is semiregular, i.e. for every $x \in Q$ $\mathrm{Stab}_K(x) = \{1\}$,and the orbits of K, x^K, have all the cardinality of the group, k.
If we now take $y = \kappa(x)$ for some $\kappa \in K$ we have that

$$\varphi_y = \varphi_{\kappa(x)} = \kappa\varphi_x\kappa^{-1} = \varphi_x$$

so that $x^K \subseteq \varphi^{-1}(\varphi_x)$ and $\varphi^{-1}(\varphi_x)$ is partitioned in parts of size k, hence k divides i. Moreover, since both $\mathrm{Trans}_{\triangleright}(Q)$ and $\mathrm{Trans}_{\triangleright}(\varphi(Q))$ are transitive (on Q and φQ respectively) and because of Corollary 2.35 we have that

$$\frac{|Q|\left|\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x)\right|}{|K|} = |\varphi(Q)|\left|\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(\varphi(Q))}(\varphi_x)\right|$$

Using Proposition 1.35 the equality becomes

$$\frac{|\varphi(Q)|\left|\varphi^{-1}(\varphi_x)\right|\left|\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x)\right|}{|K|} = |\varphi(Q)|\left|\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(\varphi(Q))}(\varphi_x)\right|$$

from which we obtain the desired

$$\frac{i}{k}\left|\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x)\right| = \left|\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(\varphi(Q))}(\varphi_x)\right|.$$

In particular, if $\left|\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(\varphi(Q))}(\varphi_x)\right| = 1$ and since $\frac{i}{k}$ and $\left|\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x)\right|$ are positive integers both $k = i$ and $\left|\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x)\right| = 1$. □

The last proposition gives us a bound for the order of the transvection group of a finite connected quandle Q as function of the order of the transvection group of $\varphi(Q)$ and of the fibres $\varphi^{-1}(\varphi_x)$. When Q is not faithful, $\varphi(Q)$ is smaller than Q and induction reasoning can be applied.

2.37 Corollary. *Let Q be a connected quandle, $x \in Q$. Then the following inequality holds:*

$$|\mathrm{Trans}_{\triangleright}(Q)| \leq |\mathrm{Trans}_{\triangleright}(\varphi(Q))| \left|\varphi^{-1}(\varphi_x)\right|$$

□

Chapter 3

Homogeneous and Connected Quandles

Coset Quandles

Using the coset quandle construction (Definition 1.36), given any group G, automorphism $\alpha \in \mathrm{Aut}(G)$, and subgroup $H \leq G$ of elements fixed by α, we can construct a homogeneous quandle whose elements are the right cosets G/H and whose order is then the index of H in G. The first problem to address is that with many different choices we obtain isomorphic quandles. Since we will consider left multiplications of right cosets by elements of a group, let us introduce an appropriate notation.

3.1 Notation. Let G be a group, $H \leq G$ a subgroup, G/H the set of right cosets. For every $g \in G$ we can consider λ_g, the left multiplication of right cosets by g

$$\lambda_g(fH) := gfH \tag{3.1}$$

The map $g \longmapsto \lambda_g$ defines a group morphism from G to $\mathbb{S}_{G/H}$ which we denote by $\lambda_{G/H}$

$$\lambda_{G/H}(g) := \lambda_g \tag{3.2}$$

3.2 Notation. Let $\psi : G_1 \longrightarrow G_2$ a group morphism and $H \leq G_1$. By $\psi_{G_1/H}$ we mean the function from the set of right cosets G_1/H to the set of right cosets $G_2/\psi(H)$ determined by $\psi_{G/H}(gH) = \psi(g)\psi(H)$.
We observe that in the specific case that $\alpha \in \mathrm{Aut}(G_1)$ (and then $G_1 = G_2$) and H is α-admissible, i.e. $\alpha(H) = H$, we have that $\alpha_{G_1/H} \in \mathbb{S}_{G/H}$.

3.3 Lemma. *Let $Q = \mathcal{Q}(G, H, \alpha)$ be a coset quandle, and $N \leq H$ a subgroup of H. If $N \trianglelefteq G$ then $Q' = \mathcal{Q}\left(\frac{G}{N}, \frac{H}{N}, \alpha_{G/N}\right)$ is a quandle and $Q \cong Q'$.*

Proof. Since N is composed of elements fixed by α, $\alpha_{G/N}$ is an automorphism for $\frac{G}{N}$, and Q' is well defined. Consider then the function $\pi_N : Q \longrightarrow Q'$ defined by

$\pi_N(gH) = gNH/N$.
π_N is well defined and injective since

$$g_2^{-1}g_1 \in H \iff g_2^{-1}g_1N \subseteq HN \iff g_2^{-1}g_1N \in \frac{HN}{N} \iff g_2^{-1}g_1N \in \frac{H}{N}$$

π_N is also surjective. Let us show that it is a rack morphism.

$$\begin{aligned}\pi_N(g_1H \triangleright g_2H) = \pi_N\left(g_1\alpha\left(g_1^{-1}g_2H\right)\right) = g_1\alpha\left(g_1^{-1}g_2\right)N\frac{H}{N} = \\ = g_1N\alpha'\left(g_1^{-1}g_2N\right)\frac{H}{N} = g_1N\frac{H}{N} \triangleright g_2N\frac{H}{N} = \pi_N(g_1H) \triangleright \pi_N(g_2H)\end{aligned}$$

□

Hence every normal (in G) subgroup of H gives a different coset representation for $\mathscr{Q}(G,H,\alpha)$. Since all such normal subgroups are contained in the normal G-core of H, we are prompted to give the following definition.

3.4 Definition. A coset quandle $Q = \mathscr{Q}(G,H,\alpha)$ is *reduced* if H is core free.

Let us now examine the relationship between the group G which a coset quandle is constructed with and the automorphism group of the quandle.

3.5 Lemma. *Let $Q = \mathscr{Q}(G,H,\alpha)$ be a coset quandle and let $g, f \in G$. Then for each $g \in G$ the left translation $\lambda_g(fH) = gfH$ is an automorphism of Q.*

Proof. We already know that $\lambda_g \in \mathbb{S}_{G/H}$. Let us show that λ_g is a rack endomorphism. Let $g, f_1, f_2 \in G$.

$$\lambda_g(f_1H \triangleright f_2H) = gf_1\alpha\left(f_1^{-1}f_2\right)H = gf_1\alpha\left(f_1^{-1}g^{-1}gf_2\right)H = \lambda_g(f_1H) \triangleright \lambda_g(f_2H)$$

□

3.6 Lemma. *Let $Q = \mathscr{Q}(G,H,\alpha)$ be a coset quandle and let $g \in G$. Then the map $\lambda_{G/H}$ is a group morphism from G to $\mathrm{Aut}_\triangleright(Q)$.*

Proof. Lemma 3.5 proves that $\lambda_{G/H}$ can be corestricted to $\mathrm{Aut}_\triangleright(Q)$. □

3.7 Lemma. *Let $Q = \mathscr{Q}(G,H,\alpha)$ be a coset quandle. $\lambda_{G/H}$ embeds G in $\mathrm{Aut}_\triangleright(Q)$ if and only if Q is reduced.*

Proof. By Lemma 3.6 we have to show that $\mathrm{Ker}\left(\lambda_{G/H}\right) = \mathrm{Core}_G(H)$. We proceed by equivalences.

$$\begin{aligned} g \in \mathrm{Ker}\left(\lambda_{G/H}\right) &\iff \forall f \in G \; gfH = fH \\ &\iff \forall f \in G \quad g \in fHf^{-1} \\ &\iff \quad g \in \mathrm{Core}_G(H)\end{aligned}$$

□

By Lemma 3.3, every coset quandle has a coset representation as a reduced coset quandle, so it is appropriate, in order to study coset quandles, to consider only the reduced ones which, by Lemma 3.7, is the same as requiring that the group is a group of automorphisms for the quandle.

Reduced Coset Quandles

In this section we try to translate rack properties of reduced coset quandles to group properties.

3.8 Lemma. *Let $Q = \mathscr{Q}(G, H, \alpha)$ be a coset quandle. If $F \leq G$ is a subgroup then $\lambda_{G/H}(H \cap F) = \mathrm{Stab}_{\lambda_{G/H}(F)}(H)$. In particular, if Q is reduced, $H \cap F \cong \mathrm{Stab}_{\lambda_{G/H}(F)}(H)$.*

Proof. We proceed by equivalences. For every $f \in F$

$$\lambda_f(H) = H \iff fH = H \iff f \in H$$

□

3.9 Proposition. *Let $Q = \mathscr{Q}(G, H, \alpha)$ be a reduced coset quandle. Q is faithful if and only if $H = \mathrm{C}_G(\alpha)$.*

Proof. We proceed by equivalences.

$$\begin{aligned}
\varphi_{g_1H} = \varphi_{g_2H} &\iff \forall t \in G \quad g_1H \triangleright tH = g_2H \triangleright tH \\
&\iff \forall t \in G \quad g_1\alpha\left(g_1^{-1}t\right)H = g_2\alpha\left(g_2^{-1}t\right)H \\
&\iff \forall t \in G \alpha(t)^{-1}\alpha(g_2)\,g_2^{-1}g_1\alpha\left(g_1^{-1}\right)\alpha(t) \in H \\
&\iff \forall t \in G \quad t^{-1}\alpha(g_2)\,g_2^{-1}g_1\alpha\left(g_1^{-1}\right)t \in H \\
&\iff \alpha(g_2)\,g_2^{-1}g_1\alpha\left(g_1^{-1}\right) \in \mathrm{Core}_G(H) = \{1\} \\
&\iff g_2^{-1}g_1 = \alpha\left(g_2^{-1}g_1\right) \\
&\iff g_2^{-1}g_1 \in \mathrm{C}_G(\alpha)
\end{aligned}$$

Q is faithful if and only if $\forall g_1, g_2 \in G : \varphi_{g_1H} = \varphi_{g_2H} \iff g_2^{-1}g_1 \in H$. By the preceding equivalence $\forall g_1, g_2 \in G\ \varphi_{g_1H} = \varphi_{g_2H} \iff g_2^{-1}g_1 \in \mathrm{C}_G(\alpha)$. The conclusion follows. □

The following lemma shows that left multiplication permutation in a homogeneous quandle shares the permutation structure of a group automorphism acting on right cosets of (a subgroup of) its centraliser in the group. This fact puts a serious limitation on the possible permutation structures on left multiplications of homogeneous quandles.

3.10 Lemma. *Let $Q = \mathscr{Q}(G, H, \alpha)$ be a coset quandle. Then for all $g \in G$*

$$\varphi_H^{\pm 1}(gH) = \alpha^{\pm 1}(g)\,H \tag{3.3}$$

Proof. We proceed by equalities:

$$\varphi_H^{\pm 1}(gH) = H \triangleright^{\pm 1} gH = \alpha^{\pm 1}(g)\,H$$

□

We now give a characterisation of the transvection group of a reduced coset quandle.

3.11 Lemma. *Let $Q = \mathscr{Q}(G, H, \alpha)$ be a coset quandle, $g_0 \in G$. Then for all $g \in G$*

$$\varphi_{g_0 H}\varphi_H^{-1}(gH) = \left[g_0, \alpha\right] gH \tag{3.4}$$

i.e. $\lambda_{G/H}([g, \alpha]) = \varphi_{g_0 H}\varphi_H^{-1}$

Proof. We proceed by equalities:

$$\varphi_{g_0 H}\varphi_H^{-1}(gH) \overset{(3.3)}{=} g_0 H \triangleright \alpha^{-1}(g) H = g_0 \alpha\left(g_0^{-1}\alpha^{-1}(g)\right) H = \left[g_0, \alpha\right] gH \tag{3.5}$$

□

3.12 Proposition. *Let $Q = \mathscr{Q}(G, H, \alpha)$ be a reduced coset quandle. Then*

$$\mathrm{Trans}_{\triangleright}(Q) \cong [G, \alpha]$$

More precisely, $\lambda_{G/H}([G, \alpha]) = \mathrm{Trans}_{\triangleright}(Q)$

Proof. We already know (by Proposition 3.7) that $\lambda_{G/H}$ is an isomorphism on its image. But, by Lemma 3.11, we see that $\lambda_{G/H}$ is a bijection between the generators of $[G, \alpha]$ and those of $\mathrm{Trans}_{\triangleright}(Q)$, hence the conclusion. □

Minimal Coset Representation for Connected Quandles

By Joyce's result reported in Proposition 1.40, every homogeneous quandle has a representation as a coset quandle. Connected quandles admit a minimal such representation as cosets of the transvection group. The existence of this minimal coset representation is proved in [HSV14, Theorem 3.5]. We give here a slightly different proof.

3.13 Proposition. *Let Q be a homogeneous quandle, $x \in Q$, and let G be a transitive group of automorphisms of Q, invariant under $\hat{\varphi}_x$, the conjugation by φ_x. Then $Q \cong \mathscr{Q}(G, \mathrm{Stab}_G(x), \hat{\varphi}_x)$.*

Proof. Let us set $\mathrm{Stab}_G(x) = H$ and define $\pi : \mathscr{Q}(G, H, \hat{\varphi}_x) \longrightarrow Q$ by setting $gH \longmapsto g(x)$. The map is well defined and injective because $g_1 H = g_2 H \iff g_2^{-1} g_1 \in \mathrm{Stab}_G(x)$. It is surjective because G is transitive. Moreover, we have:

$$\begin{aligned}\pi(g_1 H \triangleright g_2 H) = \pi\left(g_1 \varphi_x g_1^{-1} g_2 \varphi_x^{-1} H\right) = g_1 \varphi_x g_1^{-1} g_2 \varphi_x^{-1}(x) = \\ = \varphi_{g_1(x)} g_2(x) = g_1(x) \triangleright g_2(x) = \pi(g_1) \triangleright \pi(g_2)\end{aligned}$$

and π is a rack morphism. □

3.14 Lemma. *Let Q be an homogeneous quandle, $x \in Q$, and let G be a transitive group of automorphisms of Q, invariant under $\hat{\varphi}_x$, the conjugation by φ_x. Then* $\mathrm{Trans}_{\triangleright}(Q) \leq G$.

Proof. Let $y \in Q$. Then there is $g \in G$ such that $y = g(x)$ and

$$\varphi_y \varphi_x^{-1} = g \varphi_x g^{-1} \varphi_x^{-1} \in G$$

Hence every generator of $\mathrm{Trans}_{\triangleright}(Q)$ is in G and the conclusion follows. □

3.15 Lemma. *Let $Q = \mathscr{Q}(G, H, \alpha)$ be a coset quandle. Then*

$$\hat{\varphi}_H \lambda_{G/H} = \lambda_{G/H} \alpha \tag{3.6}$$

where $\lambda_{G/H}$ is defined in Notation 3.1 and $\hat{\varphi}_H$ is the conjugation by φ_H.

Proof. For every $f, g \in G$

$$\varphi_H \lambda_g \varphi_H^{-1}(fH) = \alpha\left(g \alpha^{-1}(f)\right) H = \lambda_{\alpha(g)}(fH).$$

Hence, for every $g \in G$, $\hat{\varphi}_H \lambda_{G/H}(g) = \lambda_{G/H} \alpha(g)$ and we are done. □

3.16 Lemma. *Let $Q = \mathscr{Q}(G, H, \alpha)$ be a coset quandle. Then $\lambda_{G/H}(G)$ is transitive and invariant under $\hat{\varphi}_H$, the conjugation by φ_H.*

Proof. Let $f_1 H, f_2 H \in Q$ and $\lambda_{f_2 f_1^{-1}} \in \lambda_{G/H}(G)$ the left multiplication by $f_2 f_1^{-1}$. Then $f_2 H = \lambda_{f_2 f_1^{-1}}(f_1 H)$ hence $\lambda_{G/H}(G)$ is transitive. Moreover for every $g \in G$ by Lemma 3.15

$$\hat{\varphi}_H \lambda_{G/H}(g) = \lambda_{G/H} \alpha(g)$$

hence $\lambda_{G/H}(G)$ is invariant under $\hat{\varphi}_H$. □

3.17 Corollary. *Let $Q = \mathscr{Q}(G, H, \alpha)$ be a coset quandle. Then $\mathrm{Trans}_{\triangleright}(Q) \leq \lambda_{G/H}(G)$.*

Proof. Put together Lemma 3.16 and Lemma 3.14. □

3.18 Corollary. *Let G be a finite group and $Q := \mathscr{Q}(G, H, \alpha)$ a coset quandle. If $|G| = |\mathrm{Trans}_{\triangleright}(Q)|$ then $G \cong \mathrm{Trans}_{\triangleright}(Q)$ and Q is connected.*

Proof. By Corollary 3.17, $\mathrm{Trans}_{\triangleright}(Q)$ embeds in a quotient of G. But if they are both finite and of the same order they must be isomorphic. And since G is transitive on Q, so is $\mathrm{Trans}_{\triangleright}(Q)$, and Q is connected. □

We are now ready to prove the following

3.19 Theorem (Minimal coset representation of a connected quandle [HSV14, Theorem 3.10])**.**
Let Q be a connected quandle, $x \in Q$. Then

$$Q \cong \mathscr{Q}\left(\mathrm{Trans}_{\triangleright}(Q), \mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x), \hat{\varphi}_x\right)$$

where $\hat{\varphi}_x$ is the conjugation by φ_x. Moreover, if Q is isomorphic to some coset quandle $\mathscr{Q}(G, H, a)$ then $\mathrm{Trans}_{\triangleright}(Q)$ embeds in some quotient of G.

Proof. $\mathrm{Trans}_{\triangleright}(Q)$ is transitive on Q (by Proposition 2.5) and, since it is normal in $\mathrm{Inn}_{\triangleright}(Q)$, it is invariant under $\hat{\varphi}_x$, hence by Proposition 3.13

$$Q \cong \mathscr{Q}\left(\mathrm{Trans}_{\triangleright}(Q), \mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x), \hat{\varphi}_x\right)$$

Moreover, if we set $Q_1 = \mathscr{Q}(G,H,\alpha)$, by Corollary 3.17 and since the transvection group is an invariant for a rack (see Corollary 2.9), we have $\mathrm{Trans}_{\triangleright}(Q) \cong \mathrm{Trans}_{\triangleright}(Q_1) \leq \lambda_{\alpha,H}(G)$. □

Some consequences of the minimality of this coset representation are the following.

3.20 Corollary. *A finite connected quandle Q is principal if and only if*

$$|Q| = |\mathrm{Trans}_{\triangleright}(Q)|.$$

Proof. If Q is principal then there is a group G with an automorphism α such that $Q \cong \mathscr{Q}(G,\alpha)$, in particular $|Q| = |G|$. By Theorem 3.19, $|G| \geq |\mathrm{Trans}_{\triangleright}(Q)|$ and since $\mathrm{Trans}_{\triangleright}(Q)$ is transitive on Q, $|Q| \leq |\mathrm{Trans}_{\triangleright}(Q)|$, hence $|Q| = |\mathrm{Trans}_{\triangleright}(Q)|$.

Suppose now that $|Q| = |\mathrm{Trans}_{\triangleright}(Q)|$. For every $x \in Q$, by Theorem 3.19 $|Q| = \left|\mathrm{Trans}_{\triangleright}(Q) / \mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x)\right|$. If Q is finite we can conclude that $\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x) = \{1\}$ and Q is principal. □

3.21 Corollary ([HSV14, Theorem 5.3])**.** *A connected quandle Q is affine if and only if* $\mathrm{Trans}_{\triangleright}(Q)$ *is abelian.*

Proof. If Q is affine then $Q \cong \mathscr{Q}(A,\alpha)$ with A an abelian group. By Theorem 3.19 $\mathrm{Trans}_{\triangleright}(Q)$ embeds in a quotient of A and hence it is abelian.
Vice versa, again by Theorem 3.19, $Q \cong \mathscr{Q}\left(\mathrm{Trans}_{\triangleright}(Q), \mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x), \hat{\varphi}_x\right)$ and if $\mathrm{Trans}_{\triangleright}(Q)$ is abelian, by definition, Q is affine. □

The Isomorphism Problem for Connected Coset Quandles

We now study the isomorphism problem for connected coset quandles. The following Proposition is a generalisation of what Andruskiewitsch and Graña described in [AG03, Lemma 1.23] for connected affine quandles.

3.22 Proposition. *Let $Q_1 = \mathscr{Q}(G_1,H_1,\alpha_1)$ and $Q_2 = \mathscr{Q}(G_2,H_2,\alpha_2)$ be reduced connected coset quandles. Then $Q_1 \cong Q_2$ if and only if there is an isomorphism $\beta : [G_1,\alpha_1] \xrightarrow{\sim} [G_2,\alpha_2]$ such that $\beta(H_1 \cap [G_1,\alpha_1]) = H_2 \cap [G_2,\alpha_2]$ and $\beta\alpha_1 = \alpha_2\beta$.*

Proof. Suppose that there is an isomorphism $\gamma : Q_1 \xrightarrow{\sim} Q_2$. Without loss of generality we can suppose that $\gamma(H_1) = H_2$. Suppose in fact that $\gamma(H_1) \neq H_2$. Since Q_2 is connected, there is $\tau \in \mathrm{Trans}_{\triangleright}(Q_2)$ such that $\tau\gamma(H_1) = H_2$ and we can consider the isomorphism $\tau\gamma : Q_1 \xrightarrow{\sim} Q_2$ which has the required property. Then, by Proposition 2.19, there is an isomorphism $\psi : \mathrm{Trans}_{\triangleright}(Q_1) \xrightarrow{\sim} \mathrm{Trans}_{\triangleright}(Q_2)$ such that

a) $\psi\widehat{\varphi}_{H_1}a^{-1} = \widehat{\varphi}_{H_2}$.

b) $\psi\left(\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q_1)}(H_1)\right) = \mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q_2)}(H_2)$.

By Proposition 3.12, $\lambda_{G_j/H_j} : [G_j, \alpha_j] \xrightarrow{\sim} \mathrm{Trans}_{\triangleright}(Q_j)$, $j = 1, 2$ is an isomorphism and we can define $\beta : [G_1, \alpha_1] \xrightarrow{\sim} [G_2, \alpha_2]$ by $\beta := \lambda^{-1}_{G_2/H_2} \psi \lambda_{G_1/H_1}$. Let us verify that β satisfies the required properties.

$$\begin{aligned}
\beta\left(H_1 \cap [G_1, \alpha_1]\right) &= \lambda^{-1}_{G_2/H_2} a \lambda_{G_1/H_1}\left(H_1 \cap [G_1, \alpha_1]\right) \\
&\overset{3.8}{=} \lambda^{-1}_{G_2/H_2} \psi \left(\mathrm{Stab}_{\lambda_{G/H}([G_1,\alpha_1])}(H_1)\right) \\
&\overset{3.12}{=} \lambda^{-1}_{G_2/H_2} \psi \left(\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q_1)}(H_1)\right) \\
&\overset{b)}{=} \lambda^{-1}_{G_2/H_2} \left(\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q_2)}(H_2)\right) \\
&\overset{3.12}{=} \lambda^{-1}_{G_2/H_2} \left(\mathrm{Stab}_{\lambda_{G/H}([G_2,\alpha_2])}(H_2)\right) \\
&\overset{3.8}{=} H_2 \cap [G_2, \alpha_2]
\end{aligned}$$

and

$$\begin{aligned}
\beta\alpha_1 &= \lambda^{-1}_{G_2/H_2} \psi \lambda_{G_1/H_1} \alpha_1 \\
&\overset{(3.6)}{=} \lambda^{-1}_{G_2/H_2} a \widehat{\varphi}_{H_1} \lambda_{G_1/H_1} \\
&\overset{a)}{=} \lambda^{-1}_{G_2/H_2} \widehat{\varphi}_{H_2} \psi \lambda_{G_1/H_1} \\
&\overset{(3.6)}{=} \alpha_2 \lambda^{-1}_{G_2/H_2} a \lambda_{G_1/H_1} \\
&= \alpha_2 \beta
\end{aligned}$$

Suppose now that there is an isomorphism $\beta : [G_1, \alpha_1] \xrightarrow{\sim} [G_2, \alpha_2]$ such that $\beta(H_1 \cap [G_1, \alpha_1]) = H_2 \cap [G_2, \alpha_2]$ and $\beta\alpha_1 = \alpha_2\beta$. We can define $\psi : \mathrm{Trans}_{\triangleright}(Q_1) \xrightarrow{\sim} \mathrm{Trans}_{\triangleright}(Q_2)$ by $\psi := \lambda_{G_2/H_2} \beta \lambda^{-1}_{G_1/H_1}$. With computations analogous to those used for the other implication it can be shown that ψ satisfies the properties a) and b) and that, by Proposition 2.19, Q_1 and Q_2 are isomorphic. □

We restate the previous result adding, to the hypotheses, that we have two coset quandles built starting with the same group, isomorphic to the transvection group (i.e. two minimal coset representations). We obtain in this way an isomorphism criterion useful when constructing new connected quandles starting with groups.

3.23 Lemma. *Let $Q = \mathcal{Q}(G, H, \alpha)$ be a reduced coset quandle. If $G = [G, \alpha]$ then Q is connected.*

Proof. Since $G = [G, \alpha]$, we have that $\lambda_{G/H}([G, \alpha])$ is transitive and since Q is reduced, by Proposition 3.12 also $\mathrm{Trans}_{\triangleright}(Q)$ is transitive, hence Q is connected. □

3.24 Remark. Whilst, in general, $[G, \alpha]$ is a proper normal subgroup of G, it may well happen that $[G, \alpha] = G$. If G is simple, for instance, being $[G, \alpha]$ normal, the equality is necessary, for every automorphism α. Another source of examples are racks themselves. If Q is a connected finite quandle, by Theorem 3.19,

$$Q \cong \mathscr{Q}\left(\mathrm{Trans}_{\triangleright}(Q), \mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(\widehat{\varphi}_x)\right)$$

, and by Proposition 3.12, $\mathrm{Trans}_{\triangleright}(Q) \cong [\mathrm{Trans}_{\triangleright}(Q), \widehat{\varphi}_x]$, which in turn, in the finite context, is sufficient to prove equality.

3.25 Corollary. *Let $Q_1 = \mathscr{Q}(G, H_1, \alpha_1)$ and $Q_2 = \mathscr{Q}(G, H_2, \alpha_2)$ be two reduced coset quandles. If $G = [G, \alpha_1] = [G, \alpha_2]$ then $Q_1 \cong Q_2$ if and only if there is a group automorphism γ of G such that $\alpha_1 = \gamma\alpha_2\gamma^{-1}$ and $\gamma(H_2) = H_1$.*

Proof. Apply Proposition 3.22 with $G = G_1 = G_2$, knowing that by Lemma 3.23, Q_1 and Q_2 are connected. □

If we consider as subgroup of fixed points the whole group of fixed points the statement simplifies as follows.

3.26 Corollary. *Let $Q_1 = \mathscr{Q}(G, \mathrm{C}_G(\alpha_1), \alpha_1)$ and $Q_2 = \mathscr{Q}(G, \mathrm{C}_G(\alpha_2), \alpha_2)$ be two reduced coset quandles. If $G = [G, \alpha_1] = [G, \alpha_2]$ then $Q_1 \cong Q_2$ if and only if there is a group automorphism γ of G such that $\alpha_1 = \gamma\alpha_2\gamma^{-1}$.*

Proof. By Proposition 3.25, we have only to check that if $g \in \mathrm{C}_G(\alpha_2)$ then $\gamma(g) \in \mathrm{C}_G(\alpha_1)$. Let us compute

$$\alpha_1\gamma(g) = \gamma\alpha_2(g) = \gamma(g)$$

and we are done. □

The Automorphism Group of a Connected Quandle

In this section we aim at giving a description of the automorphism group of a connected quandle by means of its transvection group. This is done also in [HSV14, Proposition 4.8] where the rôle of the transvection group is played by the inner automorphism group.

Let us state what happens if we restrict the isomorphism (2.6) to the stabiliser of an element of the quandle in the automorphism group $\mathrm{Aut}_{\triangleright}(Q)$.

3.27 Proposition. *Let Q be a quandle. Then*

$$\frac{\mathrm{Stab}_{\mathrm{Aut}_{\triangleright}(Q)}(x)}{\mathrm{Stab}_{\mathrm{Aut}_{\triangleright}(Q)}(x) \cap \mathrm{C}_{\mathrm{Aut}_{\triangleright}(Q)}(\mathrm{Trans}_{\triangleright}(Q))}$$

is isomorphic to

$$C_{\mathrm{Aut}(\mathrm{Trans}_{\triangleright}(Q))}(\widehat{\varphi}_x) \cap \mathrm{Stab}_{\mathrm{Aut}(\mathrm{Trans}_{\triangleright}(Q))}\left(\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x)\right) \cap \mathrm{Ind}_{\triangleright}(Q)$$

where $\mathrm{Stab}_{\mathrm{Aut}(\mathrm{Trans}_{\triangleright}(Q))}\left(\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x)\right)$ *is the set-wise stabiliser, i.e.it is*

$$\left\{\beta \in \mathrm{Aut}(\mathrm{Trans}_{\triangleright}(Q)) \mid \beta\left(\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x)\right) = \mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x)\right\}$$

Proof. If we restrict the morphism from $\mathrm{Aut}_{\triangleright}(Q)$ to $\mathrm{Aut}(\mathrm{Trans}_{\triangleright}(Q))$, given by conjugation, to $\mathrm{Stab}_{\mathrm{Aut}_{\triangleright}(Q)}(x)$, under the conditions listed in Proposition 2.23, we can take $x_{i_0} = y_{i_0} = x$. Among the conditions for an automorphism ψ of $\mathrm{Trans}_{\triangleright}(Q)$ to be an image of an element of the stabiliser of x, there are

i) $\psi\left(\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x)\right) = \mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x)$,
i.e. $\psi \in \mathrm{Stab}_{\mathrm{Aut}(\mathrm{Trans}_{\triangleright}(Q))}\left(\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x)\right)$.

ii) $\psi\widehat{\varphi}_x\psi^{-1} = \widehat{\varphi}_x$ i.e. $\psi \in C_{\mathrm{Aut}(\mathrm{Trans}_{\triangleright}(Q))}(\widehat{\varphi}_x)$.

The conclusion follows from the isomorphism theorems. □

3.28 Proposition. *Let Q be a quandle. If Q is connected then*

$$\mathrm{Stab}_{\mathrm{Aut}_{\triangleright}(Q)}(x) \cong \mathrm{C}_{\mathrm{Aut}(\mathrm{Trans}_{\triangleright}(Q))}(\widehat{\varphi}_x) \cap \mathrm{Stab}_{\mathrm{Aut}(\mathrm{Trans}_{\triangleright}(Q))}\left(\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x)\right) \quad (3.7)$$

If moreover Q is faithful then

$$\mathrm{Stab}_{\mathrm{Aut}_{\triangleright}(Q)}(x) \cong \mathrm{C}_{\mathrm{Aut}(\mathrm{Trans}_{\triangleright}(Q))}(\widehat{\varphi}_x) \quad (3.8)$$

Proof. By Proposition 3.27, we need to show that, under the hypothesis that Q is connected,

$$\mathrm{Stab}_{\mathrm{Aut}_{\triangleright}(Q)}(x) \cap \mathrm{C}_{\mathrm{Aut}_{\triangleright}(Q)}(\mathrm{Trans}_{\triangleright}(Q)) = \{1\} \quad (3.9)$$

and that

$$\mathrm{C}_{\mathrm{Aut}(\mathrm{Trans}_{\triangleright}(Q))}(\widehat{\varphi}_x) \cap \mathrm{Stab}_{\mathrm{Aut}(\mathrm{Trans}_{\triangleright}(Q))}\left(\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x)\right) \subseteq \mathrm{Ind}_{\triangleright}(Q) \quad (3.10)$$

Since Q is connected, $\mathrm{Trans}_{\triangleright}(Q)$ is transitive (by 2.5), hence the centraliser of a transitive subgroup must be semiregular, which is the same as (3.9). To prove (3.10) we have to show that if $\psi \in \mathrm{C}_{\mathrm{Aut}(\mathrm{Trans}_{\triangleright}(Q))}(\widehat{\varphi}_x) \cap \mathrm{Stab}_{\mathrm{Aut}(\mathrm{Trans}_{\triangleright}(Q))}(x)$ it satisfies the conditions to be in $\mathrm{Ind}_{\triangleright}(Q)$. But in the connected case the set of representatives for the orbits of Q under $\mathrm{Trans}_{\triangleright}(Q)$ reduces to x, so all the conditions are satisfied by ψ.
If moreover Q is faithful then

$$\psi(x) = x \iff \psi\varphi_x\psi^{-1} = \varphi_x \iff \widehat{\varphi}_x(\psi) = \psi$$

hence $\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x) = \mathrm{C}_{\mathrm{Trans}_{\triangleright}(Q)}(\widehat{\varphi}_x)$.
If $\beta \in \mathrm{C}_{\mathrm{Aut}(\mathrm{Trans}_{\triangleright}(Q))}(\widehat{\varphi}_x)$ and $h \in \mathrm{C}_{\mathrm{Trans}_{\triangleright}(Q)}(\widehat{\varphi}_x)$

$$\widehat{\varphi}_x(\beta(h)) = \beta(\widehat{\varphi}_x(h)) = \beta(h)$$

hence $\beta\left(\mathrm{C}_{\mathrm{Trans}_{\triangleright}(Q)}(\widehat{\varphi}_x)\right) \subseteq \mathrm{C}_{\mathrm{Trans}_{\triangleright}(Q)}(\widehat{\varphi}_x)$ and since β is an automorphism

$$\beta\left(\mathrm{C}_{\mathrm{Trans}_{\triangleright}(Q)}(\widehat{\varphi}_x)\right) = \mathrm{C}_{\mathrm{Trans}_{\triangleright}(Q)}(\widehat{\varphi}_x)$$

hence $\mathrm{C}_{\mathrm{Aut}(\mathrm{Trans}_{\triangleright}(Q))}(\widehat{\varphi}_x) \subseteq \mathrm{Stab}_{\mathrm{Aut}(\mathrm{Trans}_{\triangleright}(Q))}\left(\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x)\right)$ and

$$\mathrm{C}_{\mathrm{Aut}(\mathrm{Trans}_{\triangleright}(Q))}(\widehat{\varphi}_x) \cap \mathrm{Stab}_{\mathrm{Aut}(\mathrm{Trans}_{\triangleright}(Q))}\left(\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x)\right) = \mathrm{C}_{\mathrm{Aut}(\mathrm{Trans}_{\triangleright}(Q))}(\widehat{\varphi}_x)$$

□

Proposition 3.28 gives us a way to describe the automorphism group of a connected quandle by means of its transvection group. This is done also in [HSV14, Proposition 4.8] where the rôle of the transvection group is played by the inner automorphism group.

3.29 Proposition. *Let Q be a connected quandle,* $\mathrm{Aut}_{\triangleright}(Q)$ *and* $\mathrm{Trans}_{\triangleright}(Q)$ *its automorphism group and transvection group respectively. Let moreover*

$$A := \mathrm{C}_{\mathrm{Aut}(\mathrm{Trans}_{\triangleright}(Q))}(\widehat{\varphi}_x) \cap \mathrm{Stab}_{\mathrm{Aut}(\mathrm{Trans}_{\triangleright}(Q))}\left(\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x)\right)$$

and

$$T := \{\tau \in \mathrm{Trans}_{\triangleright}(Q) \mid \widehat{\tau} \in A\}$$

Then

$$\mathrm{Aut}_{\triangleright}(Q) \cong \frac{\mathrm{Trans}_{\triangleright}(Q) \rtimes A}{\{(\tau, \widehat{\tau}) \mid \tau \in T\}} \tag{3.11}$$

If Q is faithful the isomorphism becomes

$$\mathrm{Aut}_{\triangleright}(Q) \cong \frac{\mathrm{Trans}_{\triangleright}(Q) \rtimes \mathrm{C}_{\mathrm{Aut}(\mathrm{Trans}_{\triangleright}(Q))}(\widehat{\varphi}_x)}{\{(\tau, \widehat{\tau}) \mid \tau \in T\}} \tag{3.12}$$

Proof. By Proposition 3.28, A is isomorphic to $\mathrm{Stab}_{\mathrm{Aut}_{\triangleright}(Q)}(x)$. If $\psi \in A$ let us call $\psi' \in \mathrm{Stab}_{\mathrm{Aut}_{\triangleright}(Q)}(x)$ such that $\psi = \hat{\psi}'$. Let us define $\Psi : \mathrm{Trans}_{\triangleright}(Q) \rtimes A \longrightarrow \mathrm{Aut}_{\triangleright}(Q)$ by $\Psi(\tau, \psi) = \tau\psi'$. Let us show that Ψ is a morphism.

$$\begin{aligned}
\Psi((\tau_1, \psi_1)(\tau_2, \psi_2)) &= \Psi(\tau_1\psi_1(\tau_2), \psi_1\psi_2) \\
&= \tau_1\psi_1(\tau_2)(\psi_1\psi_2)' \\
&= \tau_1\psi_1(\tau_2)\psi_1'\psi_2' \\
&= \tau_1\psi_1'\widehat{\psi}_1'^{-1}(\psi_1(\tau_2))\psi_2' \\
&= \tau_1\psi_1'\tau_2\psi_2' \\
&= \Psi(\tau_1, \psi_1)\Psi(\tau_2, \psi_2)
\end{aligned}$$

Since $\mathrm{Trans}_{\triangleright}(Q)$ is transitive, Ψ is surjective and

$$\begin{aligned}
\Psi(\tau, \psi) = 1 &\iff \tau = \psi'^{-1} \\
&\iff \widehat{\tau} = \psi^{-1} \\
&\iff \widehat{\tau} \in A
\end{aligned}$$

The second statement follows in the same way substituting everywhere A with $\mathrm{C}_{\mathrm{Aut}(\mathrm{Trans}_{\triangleright}(Q))}(\widehat{\varphi}_x)$. □

The Derived Subgroup Lemma

We end this chapter with a lemma which, under suitable hypotheses, gives us an upper bound for the stabiliser of a point in the transvection group of a connected quandle.

3.30 Lemma. *Let G be a finite group, $\alpha \in \operatorname{Aut}(G)$ one of its automorphisms. If $G = [G, \alpha]$ then $\mathrm{C}_G(\alpha) \leq [G, G]$.*

Proof. Let $A = G/[G, G]$ be the abelianisation of G. Since $[G, G]$ is characteristic in G, it is invariant under α so that α becomes an automorphism of A, which, by abuse of notation, we continue to denote with α. Let now us define

$$\begin{array}{rccc} \partial\alpha\colon & A & \longrightarrow & A \\ & a & \longmapsto & a\alpha\left(a^{-1}\right) \end{array}$$

Since A is commutative and α is an endomorphism of A, $\partial\alpha$ is an endomorphism of A too. Notice that if $a = g[G, G]$ for some $g \in G$

$$\begin{aligned} \partial\alpha(a) &= a\alpha\left(a^{-1}\right) \\ &= g[G, G]\,\alpha(g[G, G]) \\ &= g\alpha\left(g^{-1}\right)[G, G] \end{aligned}$$

where in the last term α is the automorphism of G and not its restriction to A as in the definition of $\partial\alpha$. Since $G = [G, \alpha]$ we have

$$\begin{aligned} A &= \left\langle g\alpha\left(g^{-1}\right)[G, G]\right\rangle_{g\in G} \\ &= \left\langle \partial\alpha(g[G, G])\right\rangle_{g\in G} \\ &= \partial\alpha(A) \end{aligned}$$

i.e. $\partial\alpha$ is surjective. In the finite setting this means that $\partial\alpha$ is also injective and then an automorphism.
Let now h be an element of $\mathrm{C}_G(\alpha)$. We have

$$\partial\alpha(h[G, G]) = h\alpha\left(h^{-1}\right)[G, G] = [G, G]$$

Thus $h[G, G] \in \operatorname{Ker}(\partial\alpha) = \{[G, G]\}$. We can then conclude that $h[G, G] = [G, G]$ and h is in $[G, G]$. □

3.31 Proposition. *If X is a finite rack with a transitive transvection group, $\mathrm{Trans}_\triangleright(Q)$, then, for each x in X, $\mathrm{Stab}_{\mathrm{Trans}_\triangleright(Q)}(x) \leq [\mathrm{Trans}_\triangleright(Q), \mathrm{Trans}_\triangleright(Q)]$.*

Proof. Let $\widehat{\varphi}_x$ be the automorphism of $\mathrm{Trans}_\triangleright(Q)$ induced by conjugation by φ_x. We have that

$$\begin{aligned} \left[\mathrm{Trans}_\triangleright(Q), \widehat{\varphi}_x\right] &= \left\langle \tau\varphi_x\tau^{-1}\varphi_x^{-1} | \tau \in \mathrm{Trans}_\triangleright(Q)\right\rangle \\ &= \left\langle \varphi_{\tau(x)}\varphi_x^{-1} | \tau \in \mathrm{Trans}_\triangleright(Q)\right\rangle \\ &= \left\langle \varphi_y\varphi_x^{-1} | y \in X\right\rangle \\ &= \mathrm{Trans}_\triangleright(Q) \end{aligned}$$

We also know that if $\xi \in \mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x)$ we have $\widehat{\varphi}_x(\xi) = \xi$, hence $\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x) \in \mathrm{C}_{\mathrm{Trans}_{\triangleright}(Q)}(\widehat{\varphi}_x)$.
We conclude using Lemma 3.30 with $G = \mathrm{Trans}_{\triangleright}(Q)$ and $\alpha = \widehat{\varphi}_x$. □

Chapter 4

Rack Quotients

A Connection for Rack Projections

Given a projection of racks $\pi : X \twoheadrightarrow Y$ there is an induced projection, $\mathrm{Inn}_{\triangleright}(\pi)$, between their inner automorphism groups and, by restriction and corestriction, between their transvection groups (see Proposition 2.7). In this section we investigate to what extent we are able to associate to a projection of the transvection group a projection of the rack. A first observation is that, for a normal subgroup of $\mathrm{Trans}_{\triangleright}(X)$ to be the kernel of an induced projection $\mathrm{Trans}_{\triangleright}(\pi)$, it is necessary that it is normal also in $\mathrm{Inn}_{\triangleright}(X)$, being $\mathrm{Trans}_{\triangleright}(\pi)$ a restriction of $\mathrm{Inn}_{\triangleright}(\pi)$. This fact prompts us to give the following definition.

4.1 Definition. Let X be a rack. A subgroup N of $\mathrm{Trans}_{\triangleright}(X)$ is *admissible* if N is normal in $\mathrm{Inn}_{\triangleright}(X)$.

Note that all characteristic subgroups of the transvection group are admissible, since a characteristic sugroup of a normal subgroup is itself normal. Let us give a characterisation of admissible subgroups of the transvection group which will let us simplify some proofs.

4.2 Lemma. *Let X be a rack, $x \in X$. A subgroup N of* $\mathrm{Trans}_{\triangleright}(X)$ *is normal in* $\mathrm{Inn}_{\triangleright}(X)$ *if and only if it is normal in* $\mathrm{Trans}_{\triangleright}(X)$ *and invariant under $\widehat{\varphi}_x$, i.e.*

$$\widehat{\varphi}_x(N) = \varphi_x N \varphi_x^{-1} = N$$

Proof. One implication is trivial. As for the other, let $\alpha \in \mathrm{Inn}_{\triangleright}(X)$. By definition of $\mathrm{Trans}_{\triangleright}(X)$, there are $\tau \in \mathrm{Trans}_{\triangleright}(X)$ and $n \in \mathbb{N}$ such that $\alpha = \tau\varphi_x^n$. Suppose then that N is normal in $\mathrm{Trans}_{\triangleright}(X)$ and invariant under $\widehat{\varphi}_x$. We have that

$$\alpha N \alpha^{-1} = \tau\varphi_x^n N \varphi_x^{-n}\tau^{-1} = \tau N \tau^{-1} = N$$

□

One may wonder if this condition (of being admissible) is also sufficient for a normal subgroup of the transvection group to be the kernel of an induced projection of racks, $\mathrm{Inn}_{\triangleright}(\pi)$. In general this is not the case, not even for connected quandles.

4.3 Example. Let us consider the small quandle $Q := Q_{27,7}$ (the numeration is as indicated in the database of connected quandles of [RIG], a package of the Computer Algebra System [GAP13], and is analogous to that one for small groups). Using the functions of RIG it can be computed that $|\mathrm{Trans}_{\triangleright}(Q)| = 81$ and that $|\mathrm{Z}(\mathrm{Trans}_{\triangleright}(Q))| = 3$. Since $\mathrm{Z}(\mathrm{Trans}_{\triangleright}(Q))$ is characteristic in $\mathrm{Trans}_{\triangleright}(Q)$, it is a normal subgroup of $\mathrm{Inn}_{\triangleright}(Q)$. Let $\pi : Q \twoheadrightarrow Q'$ be a projection of racks such that $\ker(\pi) = \mathrm{Z}(\mathrm{Trans}_{\triangleright}(Q))$. By Proposition 1.35, the order of Q' is either 27, 9, 3 or 1. Let us see that in each case $K_\pi := \ker(\pi) \cap \mathrm{Trans}_{\triangleright}(Q) \neq \mathrm{Z}(\mathrm{Trans}_{\triangleright}(Q))$.
If $|Q'| = 27$ then π is an isomorphism and $1 = K_\pi \neq \mathrm{Z}(\mathrm{Trans}_{\triangleright}(Q))$.
If $|Q'| = 1$ then $\mathrm{Trans}_{\triangleright}(Q) = K_\pi \neq \mathrm{Z}(\mathrm{Trans}_{\triangleright}(Q))$.
If $|Q'| = 9$ or 3 then, by the known classification of connected quandles of order p or p^2, for any prime p (see Theorem 5.11 and Theorem 5.13), in either case Q' is affine. This implies, by Proposition 1.39, that Q' is principal and that, by Corollary 3.20, that $|\mathrm{Trans}_{\triangleright}(Q')| = |Q'|$. This last fact implies that when $|Q'| = 9$ or 3, $|K_\pi| = 9$ or 27 respectively. In both case $K_\pi \neq \mathrm{Z}(\mathrm{Trans}_{\triangleright}(Q))$. Since for every projection π, $\mathrm{Z}(\mathrm{Trans}_{\triangleright}(Q)) \neq \ker(\pi) \cap \mathrm{Trans}_{\triangleright}(Q)$, we deduce that that there is no projection π such that $\ker(\pi) = \mathrm{Z}(\mathrm{Trans}_{\triangleright}(Q))$.

Let us give a characterisation of the elements in the kernel of an induced projection.

4.4 Lemma. *Let $\pi : X \twoheadrightarrow Y$ be a projection of racks, $\mathrm{Inn}_{\triangleright}(\pi) : \mathrm{Inn}_{\triangleright}(X) \twoheadrightarrow \mathrm{Inn}_{\triangleright}(Y)$ the induced projection between the inner automorphism groups. Then*

$$\alpha \in \mathrm{Ker}(\mathrm{Inn}_{\triangleright}(\pi)) \iff \forall x \in X : \ \pi\alpha(x) = \pi(x) \tag{4.1}$$

Proof. We proceed by equivalences.

$$\begin{aligned} \alpha \in \mathrm{Ker}(\mathrm{Inn}_{\triangleright}(\pi)) &\iff \forall x \in X : \ \mathrm{Inn}_{\triangleright}(\pi)(\alpha)(\pi(x)) = \pi(x) \\ &\overset{1.28}{\iff} \forall x \in X : \ \pi\alpha(x) = \pi(x) \end{aligned}$$

□

On the other hand, every admissible subgroup of the transvection group determines a quotient of racks that factorises through the structural morphism φ.

4.5 Notation. Let G be a group $N \trianglelefteq G$, $S \subseteq G$. From now on

$$\frac{SN}{N} := \{gN : g \in S\}$$

Note that if S is a group the notation coincides with the usual notation for quotient groups.

4.6 Proposition. *If X is a rack, and N is a normal subgroup of* $\mathrm{Inn}_{\triangleright}(X)$*, then* $\varphi(X)N/N$ *has a structure of rack given by*

$$\varphi_x N \triangleright \varphi_y N = \varphi_{x \triangleright y} N$$

Proof. $\varphi(X)N/N$ is the image of X under $\pi_N \varphi$ (where π_N is the canonical projection from $\mathrm{Inn}_{\triangleright}(X)$ to $\mathrm{Inn}_{\triangleright}(X)/N$) and every morphism of groups is a morphism between the corresponding conjugation quandles. □

4.7 Definition. Let X be a rack and $N \trianglelefteq \mathrm{Inn}_{\triangleright}(X)$. $\varphi(X)N/N$ endowed with the operation defined by

$$\varphi_x N \triangleright \varphi_y N = \varphi_{x \triangleright y} N$$

is *the quotient quandle of X by N* and will be denoted by X/N.

4.8 Notation. Let X be a rack, N a normal subgroup of $\mathrm{Inn}_{\triangleright}(X)$. From now on

$$\varphi_N := \pi_N \varphi$$

where π_N is the canonical projection of $\mathrm{Inn}_{\triangleright}(X)$ onto $\mathrm{Inn}_{\triangleright}(X)/N$. Note that, when $N = 1$, $\varphi_N = \varphi$, the structural morphism of X.

We verify next that the structure of the quotient quandle defined by a normal subgroup N of $\mathrm{Inn}_{\triangleright}(X)$ is completely determined by the intersection of N with the transvection subgroup $\mathrm{Trans}_{\triangleright}(X)$, so that in investigating the quotients determined by normal subgroups of $\mathrm{Inn}_{\triangleright}(X)$, we can limit ourselves to considering only the admissible subgroups of $\mathrm{Trans}_{\triangleright}(X)$.

4.9 Proposition. *Let X be a rack and M and N normal subgroups of* $\mathrm{Inn}_{\triangleright}(X)$*. If* $M \cap \mathrm{Trans}_{\triangleright}(X) = N \cap \mathrm{Trans}_{\triangleright}(X)$ *then $\varphi(X)M/M$ is isomorphic to $\varphi(X)N/N$.*

Proof. If $M \cap \mathrm{Trans}_{\triangleright}(X) = N \cap \mathrm{Trans}_{\triangleright}(X)$, define χ from $\varphi(X)M/M$ to $\varphi(X)N/N$ by $\chi(\varphi_x M) = \varphi_x N$. Since

$$\begin{aligned} \varphi_x M = \varphi_y M &\Leftrightarrow \varphi_x \varphi_y^{-1} \in M \cap \mathrm{Trans}_{\triangleright}(X) \\ &\Leftrightarrow \varphi_x \varphi_y^{-1} \in N \cap \mathrm{Trans}_{\triangleright}(X) \\ &\Leftrightarrow \varphi_x N = \varphi_y N \\ &\Leftrightarrow \chi(\varphi_x M) = \chi(\varphi_y M) \end{aligned}$$

we know that χ is well-defined and injective. Surjectivity and the fact that it is a quandle morphism are clear. □

We can then iterate the process (of going to and fro between projections and admissible subsets) and consider the induced projection $\mathrm{Inn}_{\triangleright}(\varphi_N)$ and its kernel. The next lemma gives a characterisation of the kernel of $\mathrm{Inn}_{\triangleright}(\varphi_N)$.

4.10 Lemma. *Let X be a rack, N an admissible subgroup of* $\mathrm{Trans}_{\triangleright}(X)$*. Then*

$$\alpha \in \mathrm{Ker}\left(\mathrm{Inn}_{\triangleright}(\varphi_N)\right) \iff \alpha N \in \mathrm{Z}\left(\frac{\mathrm{Inn}_{\triangleright}(X)}{N}\right) \tag{4.2}$$

In particular $N \leq \mathrm{Ker}\left(\mathrm{Inn}_{\triangleright}(\varphi_N)\right)$*.*

Proof. We proceed by equivalences.

$$\begin{aligned}
\alpha \in \mathrm{Ker}\left(\mathrm{Inn}_{\triangleright}(\varphi_N)\right) &\overset{(4.1)}{\iff} \forall x \in X: \quad \pi_N\varphi\alpha(x) = \pi_N\varphi(x) \\
&\overset{1.1}{\iff} \forall x \in X: \quad \alpha\varphi_x\alpha^{-1}N = \varphi_x N \\
&\iff \forall \beta \in \mathrm{Inn}_{\triangleright}(X): \ \alpha\beta\alpha^{-1}N = \beta N \\
&\iff \alpha N \in \mathrm{Z}\left(\frac{\mathrm{Inn}_{\triangleright}(X)}{N}\right)
\end{aligned}$$

□

4.11 Remark. Let us fix a rack X. The set of isomorphism classes ofrack projections originating from X is a poset (the order is given by $\pi|\pi' \iff \exists\psi: \ \pi' = \psi\pi$) and so is the set of admissible subgroups of $\mathrm{Trans}_{\triangleright}(X)$ (by inclusion). The maps $L(N) = \varphi_N$ and $R(\pi) = \mathrm{Ker}\left(\mathrm{Inn}_{\triangleright}(\pi)\right)$ are indeed order preserving maps between the two posets but the connection fails to be (monotonic) Galois because while $N \subset RL(N)$ we have $\pi|LR(\pi)$ and not $LR(\pi)|\pi$ as it would be required by the definition.

The fact that the connection between projections of a rack and the admissible subgroups of its transvection group is not Galois hampers the research of a closure (in the order-theoretic sense) for the order relation between admissible subgroups. It is nonetheless possible to find a group-theoretic criterion for it, based on Lemma 4.10.

4.12 Proposition. *Let X be a rack, N an admissible subgroup of* $\mathrm{Trans}_{\triangleright}(X)$*. Then*

$$\mathrm{Ker}\left(\mathrm{Inn}_{\triangleright}\left(\varphi_N\right)\right) = N \iff \mathrm{Z}\left(\frac{\mathrm{Inn}_{\triangleright}(X)}{N}\right) = \{N\}$$

In particular, if $K_\pi = \mathrm{Ker}\left(\mathrm{Inn}_{\triangleright}(\pi)\right) \cap \mathrm{Trans}_{\triangleright}(X)$*, for some rack projection* $\pi : X \twoheadrightarrow Y$*, and Y is faithful then* $\mathrm{Ker}\left(\mathrm{Inn}_{\triangleright}(\varphi_{K_\pi})\right) = K_\pi$*.*

Proof. The first statement is an immediate consequence of Lemma 4.10. The second follows from the fact that by Remark 1.11 if Y is faithful then

$$\mathrm{Inn}_{\triangleright}(Y) \cong \mathrm{Inn}_{\triangleright}(X) / \mathrm{Ker}\left(\mathrm{Inn}_{\triangleright}(\pi)\right)$$

is centreless. □

On the Kernel of Projections by Admissible Subgroups

We now turn our attention to $\mathrm{Trans}_{\triangleright}(\varphi_N)$ as a projection of $\mathrm{Trans}_{\triangleright}(X)$.

4.13 Notation. Let $\pi : X \twoheadrightarrow Y$ be a projection of racks and $\mathrm{Trans}_{\triangleright}(\pi) : \mathrm{Trans}_{\triangleright}(X) \twoheadrightarrow \mathrm{Trans}_{\triangleright}(Y)$ the induced projection between the transvection groups. From now on the kernel of $\mathrm{Trans}_{\triangleright}(\pi)$ will be denoted by K_π.

$$K_\pi := \mathrm{Ker}\left(\mathrm{Inn}_{\triangleright}(\pi)\right) \cap \mathrm{Trans}_{\triangleright}(X) \tag{4.3}$$

In the special case that $\pi = \varphi_N$, for some admissible subgroup of $\mathrm{Trans}_{\triangleright}(X)$, we will use $K_N := K_{\varphi_N}$ as an additional simplification of the notation.

We have that

$$\mathrm{Trans}_{\triangleright}\left(\frac{X}{N}\right) \cong \frac{\mathrm{Trans}_{\triangleright}(X)}{K_N} \tag{4.4}$$

4.14 Definition ([Bae45, Definition after Lemma 1])**.** Let S and T be subsets of a group G. The *commutator quotient* of S and T in G, $S \div T$ is the subset

$$S \div T = \{g \in G \mid [T, g] \subseteq S\}$$

4.15 Lemma ([Bae45, stated after Lemma 2])**.** *Let G be a group, N a normal subgroup of G. Then*

$$\frac{N \div G}{N} = \mathrm{Z}\left(\frac{G}{N}\right)$$

4.16 Lemma. *Let X be a rack and N an admissible subgroup of* $\mathrm{Trans}_{\triangleright}(X)$*. Then* $K_N = N \div \mathrm{Inn}_{\triangleright}(X)$*. In particular, for any $x \in X$, $[K_N, \widehat{\varphi}_x] \leq N$.*

Proof. By Lemma 4.10, K_N is the intersection of the preimage of the centre of $\dfrac{\mathrm{Inn}_{\triangleright}(X)}{N}$ with $\mathrm{Trans}_{\triangleright}(X)$. □

4.17 Lemma. *Let X be a rack and N an admissible subgroup of* $\mathrm{Trans}_{\triangleright}(X)$*. Then* $N \leq K_N$*.*

Proof. By Lemma 4.10 $N \leq \mathrm{Ker}\left(\mathrm{Inn}_{\triangleright}(\varphi_N)\right)$, at the same time N is in $\mathrm{Trans}_{\triangleright}(X)$ so that $N \leq \mathrm{Ker}\left(\mathrm{Inn}_{\triangleright}(\varphi_N)\right) \cap \mathrm{Trans}_{\triangleright}(X) = K_N$. □

4.18 Lemma. *Let X be a connected rack, $x \in X$ and N a normal subgroup of* $\mathrm{Inn}_{\triangleright}(X)$*. If* $\mathrm{Trans}_{\triangleright}(X)$ *is transitive (this hypothesis in verified, in particular, if X is a quandle) then*

$$\varphi_N{}^{-1}(\varphi_x N) = x^M \tag{4.5}$$

where

$$M = \pi_N^{-1}\left(\mathrm{C}_{\frac{\mathrm{Trans}_{\triangleright}(X)}{N}}(\widehat{\varphi}_x)\right) = \left\{\tau \in \mathrm{Trans}_{\triangleright}(X) \mid \tau\varphi_x\tau^{-1}\varphi_x^{-1} \in N\right\} = N \div \{\widehat{\varphi}_x\}$$

Proof. We proceed by equivalences. Let $y \in X$ be an element.

$$\begin{aligned} y \in {\varphi_N}^{-1}(\varphi_x N) &\iff \varphi_y \varphi_x^{-1} \in N \\ &\iff \exists \nu \in N\ \exists \tau \in \mathrm{Trans}_{\triangleright}(X): \quad \varphi_{\tau(x)}\varphi_x^{-1} = \nu \\ &\overset{1.1}{\iff} \exists \nu \in N\ \exists \tau \in \mathrm{Trans}_{\triangleright}(X): \quad \tau\varphi_x\tau^{-1}\varphi_x^{-1} = \nu \\ &\iff \exists \tau \in M: \quad y = \tau(x) \end{aligned}$$

□

4.19 Lemma. *Let X be a connected rack, $x \in X$ and N a normal subgroup of* $\mathrm{Inn}_{\triangleright}(X)$. *If* $\mathrm{Trans}_{\triangleright}(X)$ *is transitive (hence, in particular, if X is a quandle) then*

$$x^N \subseteq {\varphi_N}^{-1}(\varphi_x N) \tag{4.6}$$

and, in particular if X is finite then $\left|x^N\right|$ *divides* $\left|{\varphi_N}^{-1}(\varphi_x N)\right|$.

Proof. The first statement follows from Lemma 4.18 and the fact that $N \leq M$ where M is defined as in the abovementioned lemma. The second statement follows from the fact that N being normal in a transitive group is half-transitive, hence the orbits of N form a partition of the fibre in parts of equal size. □

The following proposition is essential for induction arguments based on the order of a rack and of its quotients.

4.20 Proposition. *Let X be a finite connected rack, $x \in X$ and N an admissible subgroup of* $\mathrm{Trans}_{\triangleright}(X)$. *If N is a non-trivial subgroup of* $\mathrm{Trans}_{\triangleright}(X)$ *then X/N is a non-trivial quotient of X.*

Proof. Suppose that X and X/N are isomorphic. By Proposition 2.9, $\mathrm{Trans}_{\triangleright}(X)$ is isomorphic to $\mathrm{Trans}_{\triangleright}(X/N)$ which is isomorphic to $\mathrm{Trans}_{\triangleright}(X)/K_N$ and, as observed in Lemma 4.17, K_N contains N. Hence it follows that $N \leq K_N = \{1\}$, contradiction.
Suppose that X/N is the trivial quandle. Since X is connected so must be X/N, so X/N must have only one element. Then $\varphi_x N = \varphi_y N$, i.e. $\varphi_y^{-1}\varphi_x \in N$ for every x and y in X, but this means that $N = \mathrm{Trans}_{\triangleright}(X)$, contradiction. □

A consequence of Proposition 4.20 is the following result.

4.21 Corollary. *Let X be a finite connected rack and N a normal subgroup of* $\mathrm{Inn}_{\triangleright}(X)$. *Then the intersection of N with* $\mathrm{Trans}_{\triangleright}(X)$ *is not trivial if and only if the intersection of N with $\varphi(X)\varphi(X)^{-1}$ is not trivial.*

Proof. Suppose that N has a non trivial intersection M with $\mathrm{Trans}_{\triangleright}(X)$. By Proposition 4.20, $\varphi(X)M/M$ is a proper projection of X and there are x and y in X, distinct, such that $\varphi_x M = \varphi_y M$, that is $\varphi_x\varphi_y^{-1}$ is not the unity and is in M and N has non-trivial intersection with $\varphi(X)\varphi(X)^{-1}$.
Since $\varphi(X)\varphi(X)^{-1}$, is included in $\mathrm{Trans}_{\triangleright}(X)$ the other implication is immediate. □

4.22 Lemma. *Let $\pi : X \twoheadrightarrow Y$ be a projection of racks, $x, y \in X$. If $\pi(x) = \pi(y)$ then $\varphi_y \varphi_x^{-1} \in K_\pi$*

Proof.

$$\begin{aligned}\pi(y) = \pi(x) &\Longrightarrow \varphi_{\pi(y)} = \varphi_{\pi(x)} \\ &\overset{2.20}{\Longleftrightarrow} \mathrm{Inn}_\triangleright(\pi)(\varphi_y) = \mathrm{Inn}_\triangleright(\pi)(\varphi_x) \\ &\Longleftrightarrow \varphi_y \varphi_x^{-1} \in K_\pi\end{aligned}$$

□

In order to gain information on the structure (and in particular on the order, in the finite case) of K_π we can consider its action on the elements of a fibre $\pi^{-1}(\pi(x))$.

4.23 Lemma. *Let $\pi : X \twoheadrightarrow Y$ be a projection of racks, $x \in X$. Then $K_\pi\left(\pi^{-1}(\pi(x))\right) = \pi^{-1}(\pi(x))$ and the restriction and corestriction of the elements of K_π to the elements of $\pi^{-1}(\pi(x))$ gives a group morphism from K_π to $\mathbb{S}_{\pi^{-1}(\pi(x))}$ whose kernel is*

$$\mathrm{C}_{K_\pi}\left(\pi^{-1}(\pi(x))\right) = \bigcap_{y \in \pi^{-1}(\pi(x))} \mathrm{Stab}_{K_\pi}(y)$$

Proof. Let $y \in \pi^{-1}(\pi(x))$ and $\kappa \in K_\pi \subseteq \mathrm{Ker}(\mathrm{Inn}_\triangleright(\pi))$

$$\pi\kappa(y) \overset{4.4}{=} \pi(y) = \pi(x)$$

hence $\kappa(y) \in \pi^{-1}(\pi(x))$. The bijectivity is assured by the fact that $\kappa^{-1} \in K_\pi$ also. □

The group morphism introduced in Lemma 4.23 can be better described as a group morphism between rack automorphisms as soon as we consider that the fibre of a rack morphism is endowed with a quandle structure, as described by Lemma 1.22.

4.24 Lemma. *Let $\pi : X \twoheadrightarrow Y$ be a projection of racks, $x \in X$. Then there is a group morphism $\sigma_{\pi(x)} : K_\pi \longrightarrow \mathrm{Aut}_\triangleright\left({}^\iota\pi^{-1}(\pi(x))\right)$ such that $\mathrm{Trans}_\triangleright\left({}^\iota\pi^{-1}(\pi(x))\right) \leq \sigma_{\pi(x)}(K_\pi)$*

Proof. Let us consider $\sigma_{\pi(x)}$ the group morphism defined in Lemma 4.23, which sends every $\kappa \in K_\pi$ to its restriction and corestriction on ${}^\iota\pi^{-1}(\pi(x))$. Since

$$\kappa \in \mathrm{Aut}_\triangleright(X) \overset{1.18}{\subseteq} \mathrm{Aut}_\triangleright({}^\iota X),$$

$\sigma_{\pi(x)}(\kappa)$ is an automorphism of ${}^\iota\pi^{-1}(\pi(x))$. As for the last statement, we have that, by Lemma 4.22, if $y, z \in {}^\iota\pi^{-1}(\pi(x))$ then $\varphi_y \varphi_z^{-1} \in K_\pi$ hence

$$\mathrm{Trans}_\triangleright\left({}^\iota\pi^{-1}(\pi(x))\right) = \left\langle \sigma\left(\varphi_y \varphi_z^{-1}\right) | y, z \in {}^\iota\pi^{-1}(\pi(x))\right\rangle \subseteq \sigma_{\pi(x)}(K_\pi)$$

□

The following lemma and proposition state a generalisation of ideas presented in the proof of Proposition 3.10 of [Gra04]. They give us an upper bound for the kernel of the projection between the transvection groups induced by a projection of racks. If we know a system of generators for the quotient of a rack we can construct a (usually larger) system of generators for the rack itself.

4.25 Lemma. *Let $\pi : X \twoheadrightarrow Y$ be a projection of racks. If $\{y_i\}_{i\in I}$ is a system of generators for Y, then $\bigcup_{i\in I} \pi^{-1}(y_i)$ is a system of generators for X.*

Proof. Let x be an element in X. There is y in Y such that $x \in \pi^{-1}(y)$. Since $\{y_i\}_{i\in I}$ is a system of generators for Y, we have $y = \prod_j \varphi_{y_{i_j}}(y_k)$ for some $i_j, k \in I$. Let x_i be in $\pi^{-1}(y_i)$.

$$\begin{aligned}
\pi^{-1}(y) = \pi^{-1}\left(\left(\prod_j \varphi_{y_{i_j}}\right)(y_k)\right) &= \pi^{-1}\left(\left(\prod_j \varphi_{\pi\left(x_{i_j}\right)}\right)\pi(x_k)\right)\\
&= \pi^{-1}\left(\left(\prod_j \mathrm{Inn}_{\triangleright}(\pi)\left(\varphi_{x_{i_j}}\right)\right)\pi(x_{i_k})\right)\\
&\overset{1.28}{=} \pi^{-1}\left(\pi\left(\prod_j \varphi_{x_{i_j}}(x_{i_k})\right)\right)\\
&\overset{1.31}{=} \prod_j \varphi_{x_{i_j}}\left(\pi^{-1}(\pi(x_{i_k}))\right)\\
&= \prod_j \varphi_{x_{i_j}}\left(\pi^{-1}(y_{i_k})\right)
\end{aligned}$$

Since $\prod_j \varphi_{x_{i_j}}$ is a bijection between $\pi^{-1}(y)$ and $\pi^{-1}(y_{i_k})$, there is $z \in \pi^{-1}(y_{i_k})$ such that $x = \prod_j \varphi_{x_{i_j}}(z)$ and $x \in \left\langle \bigcup_{i\in I} \pi^{-1}(y_i)\right\rangle$. □

4.26 Proposition. *Let $\pi : X \twoheadrightarrow Y$ be a projection of racks. If $\{y_i\}_{i\in I}$ is a system of generators for Y, and $\{x_i\}_{i\in I}$ is a system of representatives for the fibres $\pi^{-1}(y_i)$ then there is a group injection $\sigma : K_\pi \hookrightarrow \prod_i \mathrm{Aut}_{\triangleright}\left({}^{\iota}\pi^{-1}(\pi(x_i))\right)$.*

Proof. Take $\sigma = \Delta\sigma_{\pi(x_i)}$, the diagonal morphism of $\left\{\sigma_{\pi(x_i)}\right\}_{i\in I}$ where $\sigma_{\pi(x_i)} : K_\pi \longrightarrow \mathrm{Aut}_{\triangleright}\left({}^{\iota}\pi^{-1}(\pi(x_i))\right)$ is the group morphism defined in Lemma 4.24. If $\sigma(\kappa) = 1$ this means that $\sigma_{\pi(x_i)}(\kappa) = 1$ for every $i \in I$, i.e., by definition of $\sigma_{\pi(x_i)}$, $\kappa(x) = x$ for every $x \in \pi^{-1}(y_i)$ and $i \in I$. Hence κ fixes $\bigcup_{i\in I} \pi^{-1}(y_i)$ which, by Lemma 4.25, is a system of generators for X and then $\kappa = 1$ and σ is injective. □

Quotient by a Semiregular Normal Subgroup

Let us now see some properties of the quotient quandle X/N when N is a semiregular (i.e. all its elements apart from the identity are derangements) admissible subgroup of the transvection group. This special case is relevant when the transvection group is nilpotent, in which case its centre, which, being characteristic, is admissible, is always non-trivial. Let's start with a lemma.

4.27 Lemma. *Let G be a finite group and α a group automorphism of G. Then*

$$|\mathrm{Comm}\,(G,\alpha)| = \frac{|G|}{|\mathrm{C}_G\,(\alpha)|} \tag{4.7}$$

where the centaliser of α in G, $\mathrm{C}_G\,(\alpha)$, and the set of commutators of α in G, $\mathrm{Comm}\,(G,\alpha)$, are defined in 2.30. In particular, if α is fixed point free, i.e. if $\mathrm{C}_G\,(\alpha)=1$, then the map $\partial\alpha$ that sends an element g to $g\alpha\left(g^{-1}\right)$ is a permutation of G.

Proof. Suppose that $\partial\alpha\,(g) = \partial\alpha\,(h)$:

$$g\alpha\left(g^{-1}\right) = h\alpha\left(h^{-1}\right) \Leftrightarrow h^{-1}g = \alpha\left(h^{-1}g\right) \Leftrightarrow h^{-1}g = 1$$

Hence the image of $\partial\alpha$ is $\mathrm{Comm}\,(G,\alpha)$ and the fibre of $\partial\alpha$ is a coset of $\mathrm{C}_G\,(\alpha)$ for every element in the image. Since G is finite, the statements of the lemma follow. □

We now assume additionally that Q is a faithful quandle, a case that often requires to be treated in a special way.

4.28 Lemma. *Let Q be a faithful finite quandle, $x \in Q$, N a normal subgroup of $\mathrm{Inn}_{\triangleright}\,(Q)$. Assume N is semiregular on Q. Then for every y and x in Q, y is in the same orbit of x under N (i.e. $x^N = y^N$) if and only if φ_y is in the same coset of φ_x with respect to N (i.e. $\varphi_x N = \varphi_y N$).*

Proof.

$$\begin{aligned} x^N = y^N &\Leftrightarrow \exists \nu \in N: \quad \nu\,(x) = y \\ &\Leftrightarrow \exists \nu \in N: \quad \varphi_{\nu(x)} = \varphi_y \\ &\Leftrightarrow \exists \nu \in N: \quad \nu\varphi_x\nu^{-1} = \varphi_y \\ &\Leftrightarrow \exists \nu \in N: \nu\varphi_x\nu^{-1}\varphi_x^{-1} = \varphi_y\varphi_x^{-1} \quad (*) \end{aligned}$$

Now since N is normal

$$\exists \nu \in N:\ \nu\varphi_x\nu^{-1}\varphi_x^{-1} = \varphi_y\varphi_x^{-1} \Rightarrow \exists \nu \in N:\ \nu = \varphi_y\varphi_x^{-1}.$$

On the other hand, let $\widehat{\varphi}_x$ be the conjugation by φ_x. Since N is normal, $\widehat{\varphi}_x$ is an automorphism of N. Since N is semiregular, no element in N apart from the identity fixes x and then no element commutes with φ_x, which means that $\widehat{\varphi}_x$ is an isomorphism of N without fixed points. Applying Lemma 4.27 with $G = N$ and $\alpha = \widehat{\varphi}_x$, the conjugation by φ_x restricted to N, we have

$$\exists \nu \in N:\ \nu = \varphi_y\varphi_x^{-1} \Rightarrow \exists \nu \in N:\ \nu\varphi_x\nu^{-1}\varphi_x^{-1} = \varphi_y\varphi_x^{-1}.$$

We can then continue with the equivalences:

$$\begin{aligned}(*) \Leftrightarrow \exists \nu \in N: \quad & \nu = \varphi_y \varphi_x^{-1} \\ \Leftrightarrow \quad & \varphi_x N = \varphi_y N.\end{aligned}$$

□

A consequence of Lemma 4.28 is that

4.29 Corollary. *Let Q be a faithful finite quandle, N a semiregular subgroup normal in $\mathrm{Inn}_{\triangleright}(Q)$. Then x^N is the fibre of $\varphi_x N$ under φ_N.*

Proof.

$$\begin{aligned}\pi_N \varphi(y) = \varphi_x N &\iff \varphi_y N = \varphi_x N \\ &\overset{4.28}{\iff} x^N = y^N \\ &\iff y \in x^N\end{aligned}$$

□

4.30 Proposition. *Let Q be a faithful finite quandle, N a semiregular subgroup normal in $\mathrm{Inn}_{\triangleright}(Q)$. Then*

$$|Q| = |N| \cdot \left|\frac{Q}{N}\right| \tag{4.8}$$

Proof. We proceed by equalities

$$\begin{aligned}|Q| &\overset{1.35}{=} \left|\varphi_N{}^{-1}(\varphi_x N)\right| \cdot \left|\frac{Q}{N}\right| \\ &\overset{4.29}{=} \left|x^N\right| \cdot \left|\frac{Q}{N}\right| \\ &= |N| \cdot \left|\frac{Q}{N}\right|\end{aligned}$$

where the last equality is justified by the fact that N is semiregular. □

The next two propositions establish that if N is semiregular then K_N is contained in $\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x)\, N$, giving us an upper bound for K_N.

4.31 Proposition. *Let Q be a faithful finite rack, $x \in Q$, N an admissible subgroup of $\mathrm{Trans}_{\triangleright}(Q)$, K_N the kernel of the morphism induced by φ_N between the transvection groups. If N is semiregular then the orbit of x under K_N is equal to its orbit under N, i.e. $x^{K_N} = x^N$.*

Proof. N is a subgroup of K_N (see Lemma 4.17), so one inclusion is trivial. As for the other inclusion:

$$\begin{aligned} y \in x^{K_N} &\iff \exists \kappa \in K_N : \ y = \kappa(x) \\ &\implies \exists \kappa \in K_N : \ \pi_N \varphi(y) = \varphi_N(\kappa(x)) \\ &\overset{(1.7)}{\iff} \exists \kappa \in K_N : \ \pi_N \varphi(y) = (\mathrm{Trans}_\triangleright(\varphi_N)(\kappa))(\pi_N \varphi(x)) \\ &\iff \pi_N \varphi(y) = \pi_N \varphi(x) \\ &\iff \varphi_y N = \varphi_x N \\ &\overset{4.28}{\iff} y \in x^N \end{aligned}$$

□

4.32 Proposition. *Let Q be a faithful finite quandle,* $\mathrm{Trans}_\triangleright(Q)$ *its transvection group, N an admissible subgroup of* $\mathrm{Trans}_\triangleright(Q)$*. If N is semiregular then $K_N \leq$* $\mathrm{Stab}_{\mathrm{Trans}_\triangleright(Q)}(x)\, N$*.*

Proof. Let $\kappa \in K_N$. Then, by Proposition 4.31, there is $\nu \in N$ such that $\kappa(x) = \nu(x)$ and hence $\nu^{-1}\kappa \in \mathrm{Stab}_{\mathrm{Trans}_\triangleright(Q)}(x)$ and $\kappa \in \mathrm{Stab}_{\mathrm{Trans}_\triangleright(Q)}(x)\, N$. □

In the same setting x^N is a connected subquandle of Q and its transvection group is N.

4.33 Proposition. *Let Q be a faithful finite quandle, x any of its elements,* $\mathrm{Trans}_\triangleright(Q)$ *its transvection group, N an admissible subgroup of* $\mathrm{Trans}_\triangleright(Q)$*. If N is semiregular then the orbit of x in Q under N, x^N, is a connected subquandle whose transvection group is isomorphic to N.*

Proof. Since, by Proposition 4.29, x^N is the fibre of a projection on a quandle, by Lemma 1.22 it is also a subquandle (since for a quandle $\iota = 1$) and for every y in x^N, $\varphi_y\left(x^N\right) = x^N$ and φ_y can be restricted and corestricted to x^N. If we denote by $\sigma_{x^N}(\varphi_y)$ this restriction and corestriction to x^N, we have that $\mathrm{Inn}_\triangleright\left(x^N\right) = \langle \sigma_{x^N}(\varphi_y) \rangle_{y \in x^N}$ and $\mathrm{Trans}_\triangleright\left(x^N\right) = \left\langle \sigma_{x^N}\left(\varphi_y \varphi_x^{-1}\right) \right\rangle_{y \in x^N}$.
At the same time $\varphi_y \varphi_x^{-1}$ is in N (see Lemma 4.28) and, since Q is faithful, we have $\left|x^N\right|$ such elements in N. But N, being semiregular, has the same number of elements as any of its orbits, so it must be $N = \left\{\varphi_y \varphi_x^{-1} : y \in x^N\right\}$.
If we now consider the restriction morphism from N to $\mathrm{Trans}_\triangleright\left(x^N\right)$, an element ν is in the kernel if it fixes x, but this is true only for the unity, always since N is semiregular, so σ_{x^N} is an isomorphism.
Finally, since $x^{\mathrm{Trans}_\triangleright\left(x^N\right)} = x^N$, x^N is connected. □

Chapter 5

Connected Quandles of Prime Power Order

We now turn our attention to connected quandles Q of order a power of a prime. They have a minimal coset representation as $\mathscr{Q}\left(\mathrm{Trans}_{\triangleright}(Q), \mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x), \hat{\varphi_x}\right)$ (see Theorem 3.19). First we observe that, in this case, $\mathrm{Trans}_{\triangleright}(Q)$ is a p-group. In order to prove this, we need the following theorem due to Etingof, Soloviev and Guralnick [ESG01]. It is maybe worth underlining that their proof of this theorem is, in turn, dependent on the classification of finite simple groups. Recall that the p-core, $\mathscr{O}_p$, of a group is the intersection of its p-Sylow subgroups or, equivalently, the largest normal p-subgroup.

5.1 Theorem ([ESG01, Theorem A.2]).
Let G be a finite group and C a conjugacy class of G of order p^n with p prime. Let $N = \langle C \rangle$. Then $N/\mathscr{O}_p(N)$ is abelian, where $\mathscr{O}_p(N)$ is the p-core of N. In particular, if $G = N$, then $G/\mathscr{O}_p(G)$ is cyclic.

5.2 Corollary. *Let X be a connected rack. If X has order a power of a prime then* $\mathrm{Trans}_{\triangleright}(Q)$ *is a p-group.*

Proof. If X is connected and has order a power of a prime, by Proposition 2.4, $\mathrm{Trans}_{\triangleright}(X)$ is the derived subgroup of $\mathrm{Inn}_{\triangleright}(X)$ and $\varphi(X)$ is a conjugacy class of $\mathrm{Inn}_{\triangleright}(X)$ of order a power of a prime (apply Proposition 1.35 to $\varphi(X)$ as image of X trough φ_X) that generates $\mathrm{Inn}_{\triangleright}(X)$, so, by Theorem 5.1, $\mathrm{Inn}_{\triangleright}(X)/\mathscr{O}_p(\mathrm{Inn}_{\triangleright}(X))$ is cyclic and $\mathscr{O}_p(\mathrm{Inn}_{\triangleright}(X))$ must include $\mathrm{Trans}_{\triangleright}(X)$ which must then be a p-group. □

5.3 Remark. We will exploit the fact that every p-group has a non-trivial centre. Since it will be quoted many times in this section, we will indicate the centre of $\mathrm{Trans}_{\triangleright}(X)$ simply by Z. The key observation is that the centre of a transitive group is always semiregular and moreover that Z, being characteristic in $\mathrm{Trans}_{\triangleright}(X)$, is normal in $\mathrm{Inn}_{\triangleright}(X)$ and we can use all the results on semiregular normal subgroups of $\mathrm{Inn}_{\triangleright}(X)$ contained in $\mathrm{Trans}_{\triangleright}(X)$ proved in the last section of the previous chapter.

5.4 Remark (Classification strategy for connected quandles of order p^n)**.**
Let Q be a connected quandle of order p^n, p a prime. By Corollary 5.2, $\mathrm{Trans}_{\triangleright}(Q)$ is a p-group, its centre, Z, is a non-trivial semiregular admissible subgroup, hence its order must divide p^n and hence $|\mathrm{Z}| = p^i$ with $1 \leq i \leq n$. We can try to find the order of $\mathrm{Trans}_{\triangleright}(Q)$ discussing the possible orders of Z. To this end, we observe that if $\mathrm{Z} = \mathrm{Trans}_{\triangleright}(Q)$ then, by Corollary 3.21, Q is affine and, by Proposition 1.39 and Corollary 3.20, $|\mathrm{Trans}_{\triangleright}(Q)| = |Q|$. If Z is a proper subgroup of $\mathrm{Trans}_{\triangleright}(Q)$ then by Proposition 4.20, in the faithful case, the quotient quandle $\varphi(Q)\,\mathrm{Z}\,/\,\mathrm{Z}$ (see Definition 4.7) is a proper quotient of Q, hence, by Proposition 4.30, of order p^{n-i}. By an induction argument, we know the possible orders of $\mathrm{Trans}_{\triangleright}(Q/\mathrm{Z})$ and the problem is reduced to deducing the order of $\mathrm{Trans}_{\triangleright}(Q)$ from that of $\mathrm{Trans}_{\triangleright}(Q/\mathrm{Z})$ and of Z. In the non-faithful case we have a natural smaller quotient of order a power of a prime in $\varphi(Q)$, from which we can try to lift information to Q.

Let us see what happens when $|\mathrm{Z}| = |Q|$.

5.5 Proposition. *Let Q be a finite connected quandle and let $\mathrm{Z} = \mathrm{Z}(\mathrm{Trans}_{\triangleright}(Q))$. Then $|\mathrm{Z}| = |Q|$ if and only if Q is affine.*

Proof. Since $|\mathrm{Z}| = |Q|$ and Z is semiregular, we have that Z is transitive and normal in $\mathrm{Inn}_{\triangleright}(Q)$, hence, by Proposition 3.14, $\mathrm{Trans}_{\triangleright}(Q) \leq \mathrm{Z}$. From this follows $\mathrm{Z} = \mathrm{Trans}_{\triangleright}(Q)$ and $\mathrm{Trans}_{\triangleright}(Q)$ is abelian hence, by Corollary 3.21, Q is affine.
If Q is affine then, by Corollary 3.21, $\mathrm{Trans}_{\triangleright}(Q) = \mathrm{Z}$. By Proposition 1.39, Q is also principal and, by Corollary 3.20, we have $|Q| = |\mathrm{Trans}_{\triangleright}(Q)| = |\mathrm{Z}|$. □

The following proposition gives us, in the faithful case, a way to obtain information on the generators of the transvection group of a quandle, from information on the generators of the transvection group of one of its quotients.
Recall that the Frattini subgroup, Φ, is the intersection of all the maximal subgroup of a group G. It can be characterised as the subgroup whose elements are non-generators. A non-generator is an element g of G such that, if S is any system of generators of G, so is $S \setminus \{g\}$ [KS04, 5.2.3].

5.6 Proposition. *Let Q be a faithful, connected quandle of order a power of a prime,* $\mathrm{Trans}_{\triangleright}(Q)$ *its transvection group,* Z *the centre of its transvection group and K_{Z} the kernel of the morphism induced by φ_{Z} between the respective transvection groups. If $\psi_1 K_{\mathrm{Z}}, \ldots, \psi_n K_{\mathrm{Z}}$ is a system of generators for* $\mathrm{Trans}_{\triangleright}(Q)\,/K_{\mathrm{Z}}$ *and $\zeta_1, \ldots, \zeta_t$ is a system of generators for* Z *then $\psi_1, \ldots, \psi_n, \zeta_1, \ldots, \zeta_t$ is a system of generators for* $\mathrm{Trans}_{\triangleright}(Q)$*.*

Proof. Since Z is a semiregular admissible subgroup of $\mathrm{Trans}_{\triangleright}(Q)$, by Proposition 4.32 $K_{\mathrm{Z}} \leq \mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x)\,\mathrm{Z}$. If $\xi_1, \ldots, \xi_m$ is a system of generators for $\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x)$ then $\psi_1, \ldots, \psi_n, \zeta_1, \ldots, \zeta_t, \xi_1, \ldots, \xi_m$ is a system of generators for $\mathrm{Trans}_{\triangleright}(Q)$. On the other hand, Q, being a connected quandle, by Proposition 2.5, has a transitive transvection group which implies, (see Proposition 3.31) that $\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x) \leq [\mathrm{Trans}_{\triangleright}(Q), \mathrm{Trans}_{\triangleright}(Q)]$. Now $\mathrm{Trans}_{\triangleright}(Q)$ is a p-group by Corollary 5.2, and hence $[\mathrm{Trans}_{\triangleright}(Q), \mathrm{Trans}_{\triangleright}(Q)] \leq \Phi(\mathrm{Trans}_{\triangleright}(Q))$ (the Frattini subgroup of $\mathrm{Trans}_{\triangleright}(Q)$), so that $\xi_1, \ldots, \xi_m$ are non-generators for $\mathrm{Trans}_{\triangleright}(Q)$. The conclusion follows. □

5.7 Corollary. *Let Q be a faithful connected quandle of order a power of a prime,* $\mathrm{Trans}_{\triangleright}(Q)$ *its transvection group,* Z *the centre of* $\mathrm{Trans}_{\triangleright}(Q)$ *and* K_{Z} *the kernel of the morphism induced by* φ_{Z} *between the respective transvection groups. If* $\mathrm{Trans}_{\triangleright}(Q)/K_{\mathrm{Z}}$ *is cyclic then Q is affine.*

Proof. If $\mathrm{Trans}_{\triangleright}(Q)/K_{\mathrm{Z}}$ is cyclic, by Theorem 5.6, $\mathrm{Trans}_{\triangleright}(Q)$ is cyclic over its centre and hence abelian. By Corollary 3.21, Q is affine. □

Simple Quandles of Prime Power Order

Connected quandles of prime order have, by Proposition 2.5, transitive transvection groups of prime degree, hence the transvection group acts primitively on the quandle and, by Proposition 1.33, the quandle must be simple. Simple quandles are classified by Andruskiewitsch and Graña in [AG03]. Let us give here a self-contained treatment in the case of interest (i.e. simple quandles of order a power of a prime), using a different approach, based on the analysis of the subgroups of $\mathrm{Trans}_{\triangleright}(Q)$ normal in $\mathrm{Inn}_{\triangleright}(Q)$.

5.8 Lemma ([Joy82b, Lemma 1])**.** *A simple quandle is connected and faithful.*

If it has order a power of a prime it is affine:

5.9 Proposition. *Let Q be a simple quandle, $x \in Q$. If Q has order p^n, p a prime, then its transvection group is an elementary abelian p-group of rank n and the conjugation by φ_x, $\widehat{\varphi}_x$, has no invariant subgroups in* $\mathrm{Trans}_{\triangleright}(Q)$.

Proof. If Q is simple then $\mathrm{Trans}_{\triangleright}(Q)$ is a minimal normal subgroup of $\mathrm{Inn}_{\triangleright}(Q)$, hence it is elementary and, $\mathrm{Trans}_{\triangleright}(Q)$ being a p-group (by Corollary 5.2), must be elementary abelian. Since it is abelian and transitive, it is regular and has order p^n, and therefore has rank n. Finally, since all subgroups are normal in $\mathrm{Trans}_{\triangleright}(Q)$, no non-trivial subgroup can be invariant under $\widehat{\varphi}_x$, by Lemma 4.2. □

5.10 Theorem (Classification of simple quandles of prime power order [AG03, Theorem 3.9])**.**
Every simple quandle Q of order p^n, p a prime, is isomorphic to an affine quandle $\mathscr{Q}\left(\mathbb{F}_p^n, \alpha\right)$ *where α is an irreducible automorphism of* ${\mathbb{F}_p}^n$.

Proof. By Proposition 3.19, $Q \cong \mathscr{Q}\left(\mathrm{Trans}_{\triangleright}(Q), \mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x), \widehat{\varphi}_x\right)$ and, by Proposition 5.9, $\mathrm{Trans}_{\triangleright}(Q) \cong \mathbb{F}_p^n$ and $\mathrm{Trans}_{\triangleright}(Q)$ has no non-trivial subgroups invariant under $\widehat{\varphi}_x$. In particular by Corollary 3.21 they are affine and, by Proposition 1.39, principal, hence $\mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x) = 1$. □

The classification of simple quandles of order a power of a prime yields immediately a characterisation of connected quandles of prime order.

5.11 Theorem (Classification of connected quandles of prime order [ESG01, Lemma 3])**.**
Any connected quandle of prime order is affine. In particular, it is principal and faithful.

Proof. The transvection group $\mathrm{Trans}_{\triangleright}(Q)$ of such a quandle Q is a transitive p-group of prime degree hence it is primitive. By Corollary 1.33, Q is simple of order p and, by Theorem 5.10, it is affine and principal and, by Proposition 1.39, it is faithful. □

A consequence of this classification is the following proposition useful in a discussion over the possible orders of Z:

5.12 Proposition. *Let Q be a connected quandle of order p^n, p a prime,* $\mathrm{Trans}_{\triangleright}(Q)$ *its transvection group,* Z *the centre of* $\mathrm{Trans}_{\triangleright}(Q)$*. Then the order of* Z *is not* p^{n-1}*.*

Proof. Suppose that Z has order p^{n-1}. By Proposition 5.5, Q is not affine and Z is a proper subgroup of $\mathrm{Trans}_{\triangleright}(Q)$. By Lemma 4.19 and the fact that Z is semiregular, it follows that $|\mathrm{Z}|$ divides $|Q/\mathrm{Z}|$ which, in turn, by Proposition 1.35, divides $|Q| = p^n$ hence $|Q/\mathrm{Z}| = 1$ or p. Since Z is a proper subgroup of $\mathrm{Trans}_{\triangleright}(Q)$, by Proposition 4.20, it must be $|Q/\mathrm{Z}| = p$ and $x^{\mathrm{Z}} = {\varphi_{\mathrm{Z}}}^{-1}(\varphi_x \mathrm{Z})$ and in particular $x^{K_{\mathrm{Z}}} = x^{\mathrm{Z}}$. This last fact implies that $K_{\mathrm{Z}} \leq \mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x)\,\mathrm{Z}$. Reasoning as in Proposition 5.6, we conclude that $\mathrm{Trans}_{\triangleright}(Q)$ is generated by the generators of Z and any set $\{\psi_i\}_{i\in I}$ such that $\{\psi_i K_{\mathrm{Z}}\}_{i\in I}$ generates $\mathrm{Trans}_{\triangleright}(Q)(Q/\mathrm{Z})$. But, by Theorem 5.11, $\mathrm{Trans}_{\triangleright}(Q/\mathrm{Z})$ is cyclic, and hence $\mathrm{Trans}_{\triangleright}(Q)$ is cyclic over its centre and then abelian, which entails, by Corollary 3.21, that Q is affine, contradiction. □

Connected Quandles of Prime Square Order

The classification of quandles of prime square order is similar to that of quandles of prime order, in the sense that they are all affine.

5.13 Theorem (Classification of connected quandles of prime square order [Gra04, Proposition 3.10])**.**
Connected quandles of prime square order are affine. In particular, they are principal and faithful.

Proof. Let Q be a quandle of order p^2, p a prime. Suppose first that Q is faithful. By Proposition 4.30 and Proposition 5.12, Z has order p^2 and by Proposition 5.5 it is affine.
Suppose now that Q is not faithful. Then $\varphi(Q)$ has order p and by Theorem 5.11 must be principal and by Proposition 2.36 so must be Q. But then, by Corollary 3.20, $\mathrm{Trans}_{\triangleright}(Q)$ has order p^2 and hence it is abelian and, by Corollary 3.21, Q is affine and connected hence by Proposition 1.39 faithful, contradiction. □

Connected Quandles of Prime Cube Order

For connected quandles of prime cube order the situation gets slightly more complicated. Let Q be a faithful connected quandle of order a cube of a prime p. The centre Z of its transvection group $\mathrm{Trans}_{\triangleright}(Q)$, as seen in Remark 5.4, may be of order p, p^2 or p^3. By Proposition 5.12, it cannot be $|\mathrm{Z}| = p^2$. The case $|\mathrm{Z}| = p^3$ is clear.

5.14 Lemma. *Let Q be a connected quandle of order p^3. If* Z *has order p^3 then* $\mathrm{Trans}_{\triangleright}(Q) = \mathrm{Z}$ *and* $|\mathrm{Trans}_{\triangleright}(Q)| = p^3$.

Proof. If Z has order p^3, by Proposition 5.5, Q is affine and principal and, by Corollary 3.20, $\mathrm{Trans}_{\triangleright}(Q)$ has order p^3, too. □

In the analysis of the case $|Z| = p$ we shall make us of the following proposition:

5.15 Proposition ([Sim94, Proposition 9.2.5])**.** *If a group G is generated modulo G' by $x_1, ..., x_n$, then $\gamma_2(G)/\gamma_3(G)$ is generated by the images of $[x_j, x_i]$ with $1 \leq i < j \leq n$, where $\gamma_i(G)$ is the i-th member of the lower central series of G.*

5.16 Lemma. *Let Q be a faithful connected quandle of order p^3. If* Z *has order p then* $\mathrm{Trans}_{\triangleright}(Q)$ *has order p^3 or has order p^4 and is of maximal nilpotency class.*

Proof. If Z has order p by Proposition 4.30 Q/Z has order p^2 and by Theorem 5.13 Q/Z is affine and principal hence by Corollary 3.20 $\mathrm{Trans}_{\triangleright}(Q/\mathrm{Z})$ has order p^2 too.

In particular $\mathrm{Trans}_{\triangleright}(Q/\mathrm{Z})$ has one or two generators. We now show that $\mathrm{Trans}_{\triangleright}(Q/\mathrm{Z})$ must have two generators. If $\mathrm{Trans}_{\triangleright}(Q/\mathrm{Z})$ is cyclic by Corollary 5.7 then Q is affine and by Corollary 5.5 Z has order p^3, contradiction.

Since Z has order p it is included in the Frattini subgroup of $\mathrm{Trans}_{\triangleright}(Q)$ and its generator is a non generator for $\mathrm{Trans}_{\triangleright}(Q)$ hence by Proposition 5.6 $\mathrm{Trans}_{\triangleright}(Q)$ has two generators too .

Let now K_{Z} be the kernel of $\mathrm{Trans}_{\triangleright}(\varphi_{\mathrm{Z}})$. We have by Proposition 2.7 that $\mathrm{Trans}_{\triangleright}(Q/\mathrm{Z}) \cong \mathrm{Trans}_{\triangleright}(Q)/K_{\mathrm{Z}}$. Since $\mathrm{Trans}_{\triangleright}(Q)/K_{\mathrm{Z}}$ is abelian $[\mathrm{Trans}_{\triangleright}(Q), \mathrm{Trans}_{\triangleright}(Q)]$ is contained in K_Z. At the same time by Proposition 4.32 and Proposition 3.31

$$K_{\mathrm{Z}} \leq \mathrm{Stab}_{\mathrm{Trans}_{\triangleright}(Q)}(x)\,\mathrm{Z} \leq [\mathrm{Trans}_{\triangleright}(Q), \mathrm{Trans}_{\triangleright}(Q)]\,\mathrm{Z} \leq [\mathrm{Trans}_{\triangleright}(Q), \mathrm{Trans}_{\triangleright}(Q)]$$

The last inclusion is because Z having order p is included in every normal subgroup of $\mathrm{Trans}_{\triangleright}(Q)$. By the two inclusions we have that $\gamma_2(\mathrm{Trans}_{\triangleright}(Q)) = [\mathrm{Trans}_{\triangleright}(Q), \mathrm{Trans}_{\triangleright}(Q)] = K_Z$.

Let τ_1, τ_2 be generators of $\mathrm{Trans}_{\triangleright}(Q)$, let $\kappa = [\tau_1, \tau_2]$ and ζ a generator of Z. For every x in Q, there is i such that $\kappa(x) = \zeta^i(x)$ (see Proposition 4.31). By induction, using the fact that ζ is central, we have that $\kappa^m(x) = \zeta^{im}(x)$. In particular $\kappa^p(x) = \zeta^{ip}(x) = x$ for every $x \in Q$. Since $\kappa \neq 1$ we obtain that κ has order p.

By Remark 4.16 we have

$$\gamma_3(\mathrm{Trans}_{\triangleright}(Q)) = [\gamma_2(\mathrm{Trans}_{\triangleright}(Q)), \mathrm{Trans}_{\triangleright}(Q)] \leq [K_{\mathrm{Z}}, \mathrm{Inn}_{\triangleright}(Q)] \leq \mathrm{Z}$$

and since Z has order p we can conclude that $\gamma_3(\mathrm{Trans}_{\triangleright}(Q)) = \mathrm{Z}$ or $\gamma_3(\mathrm{Trans}_{\triangleright}(Q)) = \{1\}$.

We have now to distinguish two cases.

If $\gamma_3(\mathrm{Trans}_{\triangleright}(Q)) = \{1\}$ we have $[\mathrm{Trans}_{\triangleright}(Q), \mathrm{Trans}_{\triangleright}(Q)] = K_{\mathrm{Z}} = \mathrm{Z}$ hence $\mathrm{Trans}_{\triangleright}(Q)$ has order p^3 and it is non abelian, hence it is extraspecial and in particular of maximal nilpotency class.

If $\gamma_3(\mathrm{Trans}_\triangleright(Q)) = \mathrm{Z}$ by Proposition 5.15 we obtain that $[\mathrm{Trans}_\triangleright(Q), \mathrm{Trans}_\triangleright(Q)] = \langle \kappa, \zeta \rangle$. In particular $\kappa \notin \mathrm{Z}$ and $[\mathrm{Trans}_\triangleright(Q), \mathrm{Trans}_\triangleright(Q)] = K_{\mathrm{Z}} \cong \mathbb{Z}_p^2$ hence $\mathrm{Trans}_\triangleright(Q)$ has order p^4. Moreover $\mathrm{Trans}_\triangleright(Q) / [\mathrm{Trans}_\triangleright(Q), \mathrm{Trans}_\triangleright(Q)] = \mathrm{Trans}_\triangleright(Q) / K_{\mathrm{Z}}$ is elementary abelian we have $\Phi(\mathrm{Trans}_\triangleright(Q)) = [\mathrm{Trans}_\triangleright(Q), \mathrm{Trans}_\triangleright(Q)]$. This implies that, in the case $|\mathrm{Trans}_\triangleright(Q)| = p^4$, we have a strict chain of characteristic subgroups

$$1 < \mathrm{Z} < \Phi < \mathrm{Trans}_\triangleright(Q)$$

and $\mathrm{Trans}_\triangleright(Q)$ is again of maximal nilpotency class. □

We are ready to prove a description of the possible transvection groups for a connected quandle of order p^3.

5.17 Theorem (Characterisation of connected quandles of prime cube order)**.**
A connected quandle of order p^3, p a prime, is either principal or isomorphic to a minimal coset representation $\mathscr{Q}(G, H, \alpha)$ where G has order p^4 and is of maximal nilpotency class and $H = \mathrm{C}_G(\alpha)$ has order p.

Proof. Let $x \in Q$. Suppose first that Q is faithful.
By Theorem 3.19, $Q \cong \mathscr{Q}\left(\mathrm{Trans}_\triangleright(Q), \mathsf{Stab}_{\mathrm{Trans}_\triangleright(Q)}(x), \hat{\varphi}_x\right)$. By Remark 4.30 and Proposition 5.12, Z can be of order p^3 or p.

If Z is of order p^3, by Lemma 5.14 so has Z and we can apply Corollary 3.21 and Corollary 3.20 to deduce that Q is affine and principal.

If Z is of order p, by Lemma 5.16, $\mathrm{Trans}_\triangleright(Q)$ is either of order p^3 or p^4 of maximal nilpotency class. If $\mathrm{Trans}_\triangleright(Q)$ is of order p^3, by Corollary 3.20, Q is principal. If $\mathrm{Trans}_\triangleright(Q)$ is of order p^4, we have that, since Q is connected, by Proposition 2.5, $\mathrm{Trans}_\triangleright(Q)$ is transitive and

$$\left|\mathrm{C}_{\mathrm{Trans}_\triangleright(Q)}(\widehat{\varphi}_x)\right| \overset{3.9}{=} \left|\mathsf{Stab}_{\mathrm{Trans}_\triangleright(Q)}(x)\right| = \frac{|\mathrm{Trans}_\triangleright(Q)|}{|Q|} = p$$

Suppose now that Q is not faithful.
By Proposition 1.35, $\varphi(Q)$ has order p or p^2. By Theorem 5.11 and Theorem 5.13, in both cases $\varphi(Q)$ is principal and, by Proposition 2.36, so is Q. □

An inspection of the isomorphism classes of connected quandles of order 3^3 using the GAP [GAP13] package RIG [RIG] developed by Graña and Vendramin shows that all the cases foreseen by Theorem 5.17 do occur. Connected quandles are numbered in the same fashion as small groups in GAP. There are 65 isomorphism classes for connected quandles of order 27, which we will indicate as $Q_{27,i}$. All quandles from $Q_{27,7}$ to $Q_{27,13}$, $Q_{27,15}$, $Q_{27,16}$, from $Q_{27,35}$ to $Q_{27,46}$ and from $Q_{27,53}$ to $Q_{27,61}$ are faithful and not principal and their transvection group has order 81, quandles $Q_{27,1}$, $Q_{27,6}$, $Q_{27,14}$ are principal and not faithful while all the others are principal and faithful. We might underline that the isomorphism classes of connected quandles of order 27 have been computed with an algorithm based on the classification of transitive groups[Ven12, Algorithm 1] independent from the results of the present work.

Appendix A

Algorithmic Divertissement

The construction of new quandles of low order is a task well known in the literature (see [Ven12]) not only to produce examples or counter-examples of quandles with required properties, but also to be exploited for providing quandle colourings of knots (see [CESY14]).
We give two algorithms for computing all the isomorphism classes of quandles of order p^2 and p^3, based on the results of Graña [Gra04] and of the present work.

As for Algorithm 1, which computes all isomorphism classes of connected quandles of order p^2, by Theorem 5.13 we can limit ourselves to construct all coset quandles $Q = \mathscr{Q}(G, \{1\}, \alpha)$ where G has order p^2 to obtain all the isomorphism classes. By Proposition 3.25, it is sufficient to consider representatives of each conjugation class in $\mathsf{Aut}(G)$to be sure to obtain non-isomorphic quandles and check that $|\mathrm{Trans}_{\triangleright}(Q)| = |G|$ to be sure, by Proposition 3.18, that they are all connected.

Algorithm 1: Connected quandles of size p^2 up to isomorphism

Data: The list of all groups of order p^2
Result: The list L of all non-isomorphic connected quandles of order p^2
$L \longleftarrow \emptyset$;
for *all groups G of order p^2* **do**
  Compute $A = \mathrm{Aut}(G)$, the automorphism group of G;
  Compute $\mathrm{Rep}(A)$, a set of representatives of the conjugacy classes of $\mathrm{Aut}(G)$;
  for *all $a \in \mathrm{Rep}(A)$* **do**
    Compute the coset quandle $Q = \mathscr{Q}(G, \{1\}, a)$;
    if $|\mathrm{Trans}_{\triangleright}(Q)| = |G|$ **then**
      Add the quandle Q to L;
    end
  end
end
return L

As for Algorithm 2, which computes all isomorphism classes of connected quandles of order p^3, by Theorem 3.19 and Theorem 5.17, we have to construct all coset quandles $Q = \mathscr{Q}(G, \{1\}, \alpha)$ where G has order p^3 and all coset quandles $Q = \mathscr{Q}(G, \mathsf{Fix}_G(\alpha), \alpha)$ where G has order p^4 and is of maximal nilpotency class, $\mathrm{C}_G(\alpha)$ has order p, to obtain all the isomorphism classes. By Proposition 3.25 in the case $|G| = p^3$ and by Proposition 3.26 in the case $|G| = p^4$, it is sufficient to consider representatives of each conjugation class in $\mathsf{Aut}(G)$ to be sure to obtain non-isomorphic quandles and check that $|\mathrm{Trans}_{\triangleright}(Q)| = |G|$ to be sure, by Proposition 3.18, that they are all connected.

Algorithm 2: Connected quandles of size p^3

```
Data: The list of all groups of order p^3 and p^4
Result: The list L of all non-isomorphic connected quandles of order p^3
L ⟵ ∅;
for all groups G of order p^3 do
    Compute A = Aut(G), the automorphism group of G;
    Compute Rep(A), a set of representatives of the conjugacy classes of A;
    for all a ∈ Rep(A) do
        Compute the coset quandle Q = 𝒬(G, {1}, a);
        if |Trans_▷(Q)| = |G| then
            Add the quandle Q to L;
        end
    end
end
for all groups G of order p^4 of maximal nilpotency class do
    Compute A = Aut(G), the automorphism group of G;
    Compute Rep(A), a set of representatives of the conjugacy classes of A;
    for all a ∈ Rep(A) do
        Compute H := C_G(a) the subgroup of fixed points of a in G;
        if the order of H is p then
            Compute the coset quandle Q = 𝒬(G, H, a);
            if |Trans_▷(Q)| = |G| then
                Add the quandle Q to L;
            end
        end
    end
end
return L
```

We have implemented both algorithms using the Computer Algebra System [GAP13] and the package [RIG].

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