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PHD IN MATHEMATICS AND COMPUTER SCIENCE
CICLO XXIV

Some questions in algebraic geometry (vector bundles, normal bundles and fat points)

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Introduction

In the first chapter we deal with rank two globally generated vector bundles with $c_1 \leq 5$ on $\mathbb{P}^n$, where c_1 indicates the first Chern class of vector bundle. We classify this bundles through their Chern class. To do it, we use the lemma 1.1.6 and 1.1.8, in particulary on $\mathbb{P}^3$ we play on the following remark 1.1.9

REMARK 0.0.1 *If E is a globally generated rank two vector bundle on $\mathbb{P}^3$ we have an exact sequence:*

$$0 \to \mathcal{O} \to E \to \mathcal{I}_C(t) \to 0$$

where $t = c_1(E)$ and where C is a smooth curve. It follows that C is linked to a smooth curve, X, by a complete intersection (t, t). Moreover if $h^1(E(-t)) = 0$, then C is irreducible and then, if $h^1(\mathcal{I}_C(2t-4)) = 0$, X also is irreducible.

The last assertion follows from the liaison formula: if C, X are two curves in $\mathbb{P}^3$ linked by a complete intersection of type (a, b), then:

$$h^1(\mathcal{I}_X(k)) = h^1(\mathcal{I}_C(a+b-k-4)), \forall k \in \mathbb{Z}.$$

In the end we summarize our classification with the following theorem 1.6.30:

THEOREM 0.0.2 *Let E be a rank two vector bundle on $\mathbb{P}^n$, $n \geq 3$, generated by global sections with Chern classes c_1, c_2, $c_1 \leq 5$.*

1. *If $n \geq 4$, then E is the direct sum of two line bundles*

2. *If $n = 3$ and E is indecomposable, then*

$$(c_1, c_2) \in S = \{((2,2), (4,5), (4,6), (4,7), (4,8), (5,8), (5,10), (5,12)\}.$$

If E exists there is an exact sequence: $0 \to \mathcal{O} \to E \to \mathcal{I}_C(c_1) \to 0$ $(*)$*, where* $C \subset \mathbb{P}^3$ *is a smooth curve of degree* c_2 *with* $\omega_C(4 - c_1) \simeq \mathcal{O}_C$*. The curve C is* irreducible*, except maybe if* $(c_1, c_2) = (4, 8)$*: in this case C can be irreducible or the disjoint union of two smooth conics.*

3. *For every* $(c_1, c_2) \in S$*,* $(c_1, c_2) \neq (5, 12)$*, there exists a rank two vector bundle on* $\mathbb{P}^3$ *with Chern classes* (c_1, c_2) *which is globally generated (and with an exact sequence as in (2)).*

In the second chapter we study the normal bundle of projectively normal curves. More precisely there is a conjecture (see Conj.2.2.2), due to Hartshorne, which predicts when a "sufficiently" general projectively normal curve of invariants (d, g, s) should have a semi-stable normal bundle. We first reformulate in a more precise way this conjecture (see Conj. 2.2.3, Conj. 2.2.4).

Then we generalize a little bit the method of [8] (see Prop. 2.4.7, Corollary 2.4.12) and prove the conjecture in some special cases (see Theorem 2.6.3).

In the last chapter we deal with subschemes of $\mathbb{P}^2$ with fat points. In particulary given Z subscheme of $\mathbb{P}^2$, we want to calculate the dimension of linear system of plane curves of degree d that contained Z. This problem is connect to speciality of linear system. About this argument there exists a conjecture due to Harbourne-Hirschowitz (see Conjecture 3.2.4) which predicts that a linear system of plane curve $\mathcal{L}$, with general multiple base points is special if and only if there exists an exceptional curve with multiplicity at least two in the base locus. This conjecture is partial proved by Ciliberto-Miranda (see Theorem 3.2.9) and by S. Yang (see Theorem 3.2.10). We improved this results with the proposition 3.6.1:

PROPOSITION 0.0.3 *The conjecture of Harbourne-Hirschowitz holds in all the linear system* $\mathcal{L}_Z(v)$*, where* $Z = (b_1P_1 + ... + b_nP_n)$ *is a subscheme of* $\mathbb{P}^2$*, with* $n \leq 10$ *and* $b_i \leq 8$ *for all* $i = 1, ..., n$*, and v is its critical value.*

Moreover we prove that every analyzed subscheme is of maximum rank (see Theorem 3.6.4):

THEOREM 0.0.4 *Every subscheme of* $\mathbb{P}^2$ *in the form* $Z = (b_1P_1 + ... + b_nP_n)$ *with* $n \leq 10$ *and* $4 \leq b_i \leq 8$ *for all* $i = 1, ..., n$*, is of maximum rank.*

Chapter 1

Rank two globally generated vector bundles with $c_1 \leq 5$.

1.1 General facts.

We recall (without proofs) some definitions and some well-known general facts we will use in the sequel.

Definition 1.1.1 *A coherent sheaf, $\mathcal{F}$, on the projective scheme X is globally generated (or generated by global sections) if the natural morphism of evaluation: $ev : H^0(\mathcal{F}) \otimes \mathcal{O}_X \to \mathcal{F}$ is surjective.*

Remark 1.1.2 *In case $\mathcal{F}$ is locally free and globally generated, the kernel of ev is also locally free.*

The next lemma almost follows from the definition:

Lemma 1.1.3 *Let $\mathcal{F}, \mathcal{G}$ be two coherent sheaves on the projectve scheme X. If $\mathcal{F}$ is globally generated and if there exists a surjective morphism $\mathcal{F} \to \mathcal{G} \to 0$, then $\mathcal{G}$ is globally generated. In particular if $Y \subset X$ is a subscheme, $\mathcal{F}_{|Y}$ is globally generated if $\mathcal{F}$ is.*

Let E be a rank r vector bundle on $\mathbb{P}^n$. According to a famous theorem, for every line $L \subset \mathbb{P}^n$ there is an r-uple $a_E(L) = (a_1(L), ..., a_r(L)) \in \mathbb{Z}^r; a_1(L) \geq ... \geq a_r(L)$

such that

$$E_L \cong \mathcal{O}_L(a_1(L)) \oplus ... \oplus \mathcal{O}_L(a_r(L)).$$

We have $c_1(E) = \sum_{i=1}^{r} a_i(L)$.

Moreover if E is globally generated, then $a_i(L) \geq 0, \forall i$ (apply Lemma 1.1.3). In particular $c_1(E) \geq 0$ for every globally generated vector bundle on $\mathbb{P}^n$.

The r-uple $a_E(L) \in \mathbb{Z}^r$ is called *the splitting type* of E on L.

Let $G(1,n)$ be the Grassmannian of lines in $\mathbb{P}^n$ and define a map:

$$a_E : G(1,n) \to \mathbb{Z}^r : L \to a_E(L)$$

by semi-continuity there is a dense open subset $U \subset G(1,n)$ such that a_E is constant on U, the corresponding splitting type (i.e. the one of $L \in U$) is called the *generic* splitting type of E. If $U = G(1,n)$ (i.e. if a_E is constant), the vector bundle E is said to be *uniform.*

LEMMA 1.1.4 *Let E be a globally generated rank r vector bundle on $\mathbb{P}^n$. If $r > n$ there is an exact sequence:*

$$0 \to (r-n).\mathcal{O} \to E \to F \to 0$$

where F is a rank n vector bundle generated by global sections.

PROOF 1.1.5
See [18] °4, Lemma 4.3.1.

LEMMA 1.1.6 *Let E be a rank two vector bundle on $\mathbb{P}^n$. If E is globally generated then a general section of E vanishes along a smooth codimension two subscheme.*

PROOF 1.1.7
See [14].

We shall also need the following:

LEMMA 1.1.8 *Let $C \subset \mathbb{P}^3$ be a smooth curve. If $\mathcal{I}_C(k)$ is globally generated and if S and S' are sufficiently general in $H^0(\mathcal{I}_C(k))$, the complete intersection $S \cap S'$ links C to a smooth curve.*

Remark 1.1.9 *If E is a globally generated rank two vector bundle on $\mathbb{P}^3$ we have an exact sequence:*

$$0 \to \mathcal{O} \to E \to \mathcal{I}_C(t) \to 0$$

where $t = c_1(E)$ and where C is a smooth curve. It follows that C is linked to a smooth curve, X, by a complete intersection (t,t). Moreover if $h^1(E(-t)) = 0$, then C is irreducible and then, if $h^1(\mathcal{I}_C(2t-4)) = 0$, X also is irreducible.

The last assertion follows from the liaison formula: if C, X are two curves in $\mathbb{P}^3$ linked by a complete intersection of type (a,b), then:

$$h^1(\mathcal{I}_X(k)) = h^1(\mathcal{I}_C(a+b-k-4)), \forall k \in \mathbb{Z}.$$

Finally we recall Horrocks criterion for a vector bundle to split:

Theorem 1.1.10 (Horrocks)
A rank r vector bundle E on $\mathbb{P}^n$ is a direct sum of line bundles ($E \simeq \bigoplus_{i=1}^{r} \mathcal{O}(a_i)$) if and only if $h^i(E(m)) = 0$, $1 \le i \le n-1$ and all $m \in \mathbb{Z}$.

As a corollary we have:

Proposition 1.1.11 *Let E be a rank r vector bundle on $\mathbb{P}^n$, $n > 2$. Then $E \simeq \bigoplus_{i=1}^{r} \mathcal{O}(a_i)$ if and only if there exists a plane $\Pi \subset \mathbb{P}^n$ such that: $E_\Pi \simeq \bigoplus_{i=1}^{r} \mathcal{O}_\Pi(a_i)$.*

Proof 1.1.12
See [18] Chap. I, 2.3.

1.2 Globally generated vector bundle on $\mathbb{P}^n$, with $c_1 = 0$

In this section we recall the classification of globally generated vector bundles with $c_1 = 0$. This is well known and follows directly from a more general result of Van de Ven on uniform vector bundles, here we provide an elementary direct proof.

LEMMA 1.2.1 *Let E be a rank r vector bundle on $\mathbb{P}^2$, with $c_1(E) = 0$. If E is generated by global sections then E is trivial ($E \simeq r.\mathcal{O}$).*

PROOF 1.2.2
Let's first proof that E has a nowhere vanishing section.
Let $L \subset \mathbb{P}^2$ be a line, then by Grothendieck theorem E_L splits. We have $E_L = \mathcal{O}_L(a_1) \oplus ...\mathcal{O}_L(a_r)$; since E_L (being a quotient of E) is generated by global sections, $a_i \geq 0$, $\forall i$. Since $c_1(E) = 0 = \sum a_i$, it follows that $a_i = 0, \forall i$ i.e. $E_L = r\mathcal{O}_L$. We note that the bundle E splits in the same way for all lines of $\mathbb{P}^2$, hence E is a uniform bundle. Since E is generated by global sections, there exists a non zero section s and if L is general, $s_L \neq 0$. We have $s_L = (\lambda_1^L, ..., \lambda_r^L)$, where the λ_i^L are constants. Since $s_L \neq 0$, $(\lambda_1^L, ..., \lambda_r^L) \neq (0, ..., 0)$. Suppose that s vanishes at a point $p \in \mathbb{P}^2$, consider a line D through p. As before $s_D = (\lambda_1^D, ..., \lambda_r^D)$, but since $s_D(p) = 0$, $\lambda_i^D = 0$, $\forall i$. Now consider the point $q = L \cap D$, we must have $(\lambda_1 L, ..., \lambda_r^L) = (\lambda_1^D, ..., \lambda_r^D)$ and we get a contradiction. We conclude that s is nowhere vanishing hence yields an injective morphism of vector bundles: $0 \to \mathcal{O} \to E$, it follows that we have an exact sequence:

$$(**)\ 0 \to \mathcal{O} \to E \to F \to 0$$

where F is a rank $(r-1)$ vector bundle. Moreover $c_1(F) = c_1(E) = 0$ and F, being a quotient of E, is globally generated. Let's conclude the proof by induction on r. If $r = 1$ there is nothing to prove. Assume the statement holds for $r-1$. It follows that $F \simeq (r-1).\mathcal{O}$. The exact sequence $(**)$ belongs to
$Ext^1(F, \mathcal{O}) \simeq (r-1).H^1(\mathcal{O})$, since $h^1(\mathcal{O}) = 0$, the exact sequence splits. So $E = \mathcal{O} \oplus F = r.\mathcal{O}$.

PROPOSITION 1.2.3 *Let E be a vector bundle of rank r on $\mathbb{P}^n$, generated by global sections and with $c_1(E) = 0$. Then E is trivial.*

PROOF 1.2.4
We prove the prop by induction on n. The case $n = 1$ is clear and the case $n = 2$ is Lemma 1.2.1. Let's assume $n > 2$. Let $\Pi \subset \mathbb{P}^n$ be a plane. We have $c_1(E_\Pi) = 0$ and E_Π is generated by global sections. By Lemma 1.2.1, $E_\Pi \simeq r.\mathcal{O}_\Pi$. It follows (cf Proposition 1.1.11) that $E \simeq r.\mathcal{O}$.

REMARK 1.2.5 *The following result is due to Van de Ven ([23]):*
Let E be a uniform vector bundle of rank r on $\mathbb{P}^n$, if the splitting type of E is $(a, ..., a)$, then $E \simeq r.\mathcal{O}(a)$. *As we have seen it is fairly immediate that a globally generated vector bundle, E, with $c_1(E) = 0$ is uniform of splitting type $(0, ..., 0)$. From Van de Ven's result it follows that $E \simeq r.\mathcal{O}$. Observe that in Van de Ven's theorem no assumption is made on the existence of a non-zero section of E.*

1.3 Globally generated vector bundles on $\mathbb{P}^n$, with $c_1 = 1$.

Goal of this section is to prove the following:

PROPOSITION 1.3.1 *Ler E be a rank r vector bundle on $\mathbb{P}^n$ with $c_1(E) = 1$. If E is globally generated then:*

1. $E \simeq \mathcal{O}(1) \oplus (r-1).\mathcal{O}$, *or:*
2. $E \simeq T(-1) \oplus (r-n).\mathcal{O}$.

This result is not new and follows from a (less known, but) more general result on uniform vector bundles (see 1.3.11). Here we will give a different and more elementary proof using the extra asumption that E is globally generated.

To start with let us observe the following:

LEMMA 1.3.2 *Let E be a rank r vector bundle on $\mathbb{P}^n$ with $c_1(E) = 1$. If E is globally generated, then E is uniform of splitting type $(1, 0, ..., 0)$.*

PROOF 1.3.3
Il $L \subset \mathbb{P}^n$ is a line, then $E_L \simeq \oplus \mathcal{O}_L(a_i(L))$ with $a_i(L) \geq 0, \forall i$ and $\sum_{i=1}^{r} a_i(L) = 1$. It follows that $(a_1(L), ..., a_r(L)) = (1, 0, ...0)$, which is independant of L.

This being said we will distinguish two cases:

1. $h^0(E(-1)) \neq 0$

2. $h^0(E(-1)) = 0$.

For case (1) we have:

PROPOSITION 1.3.4 *Let E be a globally generated vector bundle of rank r on $\mathbb{P}^n$ with $c_1(E) = 1$. If $h^0(E(-1)) \neq 0$, then $E \simeq \mathcal{O}(1) \oplus (r-1).\mathcal{O}$.*

PROOF 1.3.5
(a) The result is clear if $n = 1$. Let's assume $n = 2$.

If $L \subset \mathbb{P}^2$ is a line we have an exact sequence:

$$0 \to E(-m-1) \to E(-m) \to E_L(-m) \to 0 \ \ (*)$$

From Lemma 1.3.2 it follows that $h^0(E_L(-m)) = 0$ if $m \geq 2$. It follows that $h^0(E(-m-1)) = h^0(E(-m))$ for $m \geq 2$. Since $h^0(E(-m)) = 0$ if $m >> 0$, we get $h^0(E(-m)) = 0$ if $m \geq 2$, in particular $h^0(E(-2)) = 0$ and the map, r_L, induced by $(*)$ for $m = 1$:

$$0 \to H^0(E(-1)) \xrightarrow{r_L} H^0(E_L(-1)) \to ...$$

is an isomorphism (because $h^0(E(-1)) \neq 0$ and $h^0(E_L(-1)) = 1$). Let $0 \neq s \in H^0(E(-1))$, then $s_L := r_L(s)$ is a non zero section of $E_L(-1) \simeq \mathcal{O}_L(-1) \oplus (r-1).\mathcal{O}_L$, in particular s_L doesn't vanish at any point of L. Since this holds for every line L, we conclude that s is a nowhere vanishing section, hence we have:

$$0 \to \mathcal{O} \to E(-1) \to F \to 0$$

where F is a vector bundle. Twisting by $\mathcal{O}(1)$ we get:

$$0 \to \mathcal{O}(1) \to E \to F(1) \to 0$$

The vector bundle $F(1)$ is globally generated with $c_1(F(1)) = c_1(E) - c_1(\mathcal{O}(1)) = 0$. By Proposition 1.2.3, $F(1) \simeq (r-1).\mathcal{O}$. Since $h^1(\mathcal{O}(1)) = 0$, the exact sequence splits and $E \simeq \mathcal{O}(1) \oplus (r-1).\mathcal{O}$.

(b) Now assume $n > 2$. Take $0 \neq s \in H^0(E(-1))$. There exists a plane $\Pi \subset \mathbb{P}^n$ such that $s_\Pi \neq 0$. Then E_Π is a globally generated vector bundle with $c_1(E_\Pi) = 1$ and $h^0(E_\Pi(-1)) \neq 0$. From (a): $E_\Pi \simeq \mathcal{O}_\Pi(1) \oplus (r-1).\mathcal{O}_\Pi$. We conclude with Proposition 1.1.11.

Now we turn to case (2):

LEMMA 1.3.6 *Let E be a rank r globally generated vector bundle on $\mathbb{P}^n$ with $c_1(E) = 1$. If $h^0(E(-1)) = 0$, then $h^0(E) \leq r+1$.*

PROOF 1.3.7
Observe that the condition $h^0(E(-1)) = 0$ implies $n \geq 2$. We will prove the statement by induction on n.

Assume $n = 2$. Let $L \subset \mathbb{P}^2$ be a line. We have an exact sequence:

$$0 \to E(-1) \to E \to E_L \to 0$$

Since $h^0(E(-1)) = 0$, $h^0(E) \leq h^0(E_L)$. We have (see Lemma 1.3.2): $E_L \simeq \mathcal{O}_L(1) \oplus (r-1).\mathcal{O}_L$. Hence $h^0(E) \leq 2 + (r-1) = r+1$.

Now assume $n > 2$ and that the statement holds for $n-1$. Let $H \subset \mathbb{P}^n$ be an hyperplane and consider:

$$0 \to E(-1) \to E \to E_H \to 0$$

We have $h^0(E) \leq h^0(E_H)$. The vector bundle E_H is globally generated with $c_1(E_H) = 1$. If $h^0(E_H(-1)) \neq 0$, then (cf Proposition 1.3.4) $E_H \simeq \mathcal{O}_H(1) \oplus (r-1).\mathcal{O}_H$. This in turn implies (cf Proposition 1.1.11) $E \simeq \mathcal{O}(1) \oplus (r-1).\mathcal{O}$, in contradiction with the assumption $h^0(E(-1)) = 0$. We conclude that $h^0(E_H(-1)) = 0$. By inductive assumption $h^0(E_H) \leq r+1$ and we are done.

PROPOSITION 1.3.8 *Let E be a rank r globally generated vector bundle on $\mathbb{P}^n$ with $c_1 = 1$. If $h^0(E(-1)) = 0$, then $E \simeq T(-1) \oplus (r-n).\mathcal{O}$. (In particular $n \geq 2$ and $r \geq n$.)*

PROOF 1.3.9
From Lemma 1.3.6, $h^0(E) \leq r+1$. Since E is globally generated we have a surjective morphism:

$$ev : H^0(E) \otimes \mathcal{O} \to E$$

This implies $h^0(E) \geq r$. Moreover if $h^0(E) = r$, ev is a surjective morphism between two vector bundles of the same rank hence it is an isomorphism. This is impossible

($c_1(H^0(E)\otimes\mathcal{O})=0$ while $c_1(E)=1$). We conclude that $h^0(E)=r+1$ and that we have an exact sequence:

$$0\to\mathcal{O}(-1)\to H^0(E)\otimes\mathcal{O}\to E\to 0$$

Dualizing, we get:

$$0\to E^*\to H^0(E)^*\otimes\mathcal{O}\to\mathcal{O}(1)\to 0$$

This exact sequence expresses that $\mathcal{O}(1)$ is globally generated, so it is:

$$0\to(r-n).\mathcal{O}\oplus\Omega(1)\to(r-n).\mathcal{O}\oplus(H^0(\mathcal{O}(1))\otimes\mathcal{O})\to\mathcal{O}(1)\to 0$$

and the statement follows.

Gathering everything together:

Proof 1.3.10 (Proof of Proposition 1.3.1)
It follows from Proposition 1.3.4 and Proposition 1.3.8.

Remark 1.3.11 *In [6] (IV. Prop. 2.2) the following is proved:*

Let E be a uniform vector bundle on $\mathbb{P}^n$, $n\geq 2$ with splitting type $(1,0...,0)$, then $E\simeq\mathcal{O}(1)\oplus(r-1).\mathcal{O}$ or $E\simeq T(-1)\oplus(r-n)\mathcal{O}$.

So Proposition 1.3.1 readily follows from this result and Lemma 1.3.2, but the proof of Ellia's result (which makes no assumption on the existence of global sections of E) is much more involved.

1.4 A general result.

As a first general result we have:

Lemma 1.4.1 *Let E be a rank r globally generated vector bundle on $\mathbb{P}^n$, $n\geq 2$, with $c_1(E)=c$.*
(i) $h^0(E(-c-1))=0$
(ii) If $h^0(E(-c))\neq 0$, then $E\simeq\mathcal{O}(c)\oplus(r-1).\mathcal{O}$

PROOF 1.4.2
(i) This is clear if $n = 1$ ($E \simeq \oplus\mathcal{O}(a_i)$ with $0 \leq a_i \leq c, \forall i$). Let's prove the lemma for $n = 2$.
Let $L \subset \mathbb{P}^2$ be a line. Consider the exact sequence:

$$0 \to E(-m-1) \to E(-m) \to E_L(-m) \to 0$$

We have $E_L(-m) \simeq \oplus\mathcal{O}(a_L(i) - m)$ with $0 \leq a_L(i) \leq c, \forall i$. It follows that $h^0(E_L(-m)) = 0$ if $m \geq c+1$. So $h^0(E(-m-1)) = h^0(E(-m))$ for $m \geq c+1$. Since $h^0(E(-m)) = 0$ if $m >> 0$, the result follows.

Now assume $n > 2$. Let $\Pi \subset \mathbb{P}^n$ be a plane. The vector bundle E_Π is globally generated with $c_1(E_\Pi) = c$. By the previous step: $h^0(E_\Pi(-c-1)) = 0$. Since ths is true for any plane $\Pi \subset \mathbb{P}^n$, we get $h^0(E(-c-1)) = 0$.

(ii) Let's prove it for $n = 2$ (the result clearly holds for $n = 1$). Let $L \subset \mathbb{P}^2$ be a line. From the exact sequence:

$$0 \to E(-c-1) \to E(-c) \to E_L(-c) \to 0$$

we get (since $h^0(E(-c-1)) = 0$ by (i)), that $h^0(E_L(-c)) \neq 0$. Since $E_L \simeq \oplus\mathcal{O}a_i(L)$ with $\sum a_i(L) = c$, $0 \leq a_i(L), \forall i$, the only possibility is $(a_i(L)) = (c, 0, ..., 0)$. Since this is true for any *line $L \subset \mathbb{P}^2$, the vector bundle E is uniform of splitting type $(c, 0, ..., 0)$. If $c = 0$ or $c \geq 2$ it follows that $E \simeq \mathcal{O}(c) \oplus (r-1).\mathcal{O}$ If $c = 1$, then $E \simeq T(-1) \oplus (r-2).\mathcal{O}$ or $E \simeq \mathcal{O}(1) \oplus (r-1).\mathcal{O}$ (see 1.3.11). In the first case $h^0(E(-1)) = 0$, so if $c = 1$, under our assumptions, $E \simeq \mathcal{O}(1) \oplus (r-1).\mathcal{O}$ and the lemma is proved for $n = 2$.*

Now assume $n > 2$. If $h^0(E(-c)) \neq 0$, take $0 \neq s \in H^0(E(-c))$. Then there exists a plane Π such that $s_\Pi \neq 0$. The vector bundle E_Π is globally generated with $h^0(E_\Pi(-c)) \neq 0$. By the first part of the proof $E_\Pi \simeq \mathcal{O}_\Pi(c) \oplus (r-1).\mathcal{O}_\Pi$. We conclude with Lemma 1.1.11.

REMARK 1.4.3 *This lemma is already proved in [20].*

Without the assumption that E is globally generated the lemma is not true. For instance if E is a rank two vector bundle on $\mathbb{P}^3$ with $c_1 = 0$, the assumption $h^0(E(-c-1)) = 0$ just means that E is semi-stable. There are many, indecomposable, non semi-stable, rank two vector bundles with $c_1 = 0$ on $\mathbb{P}^3$.

There are also many semi-stable but non stable ($h^0(E) \neq 0$) indecomposable rank two vector bundles with $c_1 = 0$ on $\mathbb{P}^3$.

The next result, which is new, goes one step further:

THEOREM 1.4.4 *Let E be a rank r globally generated vector bundle on $\mathbb{P}^n$, $n \geq 2$, with $c_1(E) = c$. If $h^0(E(-c)) = 0$ and $h^0(E(-c+1)) \neq 0$, then: either $E \simeq \mathcal{O}(c-1) \oplus \mathcal{O}(1) \oplus (r-2).\mathcal{O}$ or there is an exact sequence:*

$$0 \to \mathcal{O} \to \mathcal{O}(c-1) \oplus T(-1) \oplus (r-n).\mathcal{O} \to E \to 0.$$

First we have:

LEMMA 1.4.5 *Let E be a rank r globally generated vector bundle on $\mathbb{P}^n$, $n \geq 2$, with $c_1(E) = c \geq 2$. Assume $h^0(E(-c)) = 0$ and $h^0(E(-c+1)) \neq 0$ and E not a direct sum of line bundles. If $s \in H^0(E(-c+1))$, $s \neq 0$, then $(s)_0 = \{p\}$ (the scheme of zeroes of s is a single point), $h^0(E(-c+1)) = 1$ and the jumping lines of E are precisely the lines through p. If H is an hyperplane with $p \notin H$, then $E_H \simeq T_H(-1) \oplus \mathcal{O}_H(c-1) \oplus (r-n)\mathcal{O}_H$; in particular $r \geq n$.*

PROOF 1.4.6
Let $s \in H^0(E(-c+1)), s \neq 0$.

(a) If $L \subset \mathbb{P}^n$ is a line we denote by s_L the restriction of s to L. Let's show that $\forall L \subset \mathbb{P}^n$, $s_L \neq 0$.
Suppose $s_L = 0$ for some line L. Let $L = K_1 \subset K_2 \subset ... \subset K_{n-1} \subset K_n = \mathbb{P}^n$ be a flag of linear spaces ($\dim K_i = i$). We claim that if $s_L = 0$, then $s_{K_i} = 0, 1 \leq i \leq n$, which is of course a contradiction.

We prove the claim by induction on i. The initial case holds by assumption. Assume $s_{K_{i-1}} = 0$. Consider:

$$0 \to E_{K_i}(-c) \to E_{K_i}(-c+1) \xrightarrow{r} E_{K_{i-1}}(-c+1) \to 0$$

If $s_{K_i} \neq 0$, then $r(s_{K_i}) = s_{K_{i-1}} = 0$, so $h^0(E_{K_i}(-c)) \neq 0$. By Lemma 1.4.1, E_{K_i} is a direct sum of line bundles. By Lemma 1.1.11, E also is a direct sum of line bundles. This contradicts our assumption, so $s_{K_i} = 0$ and the claim is proved.

(b) If $s \in H^0(E(-c+1)), s \neq 0$, then s vanishes precisely at one *point, more precisely $(s)_0 = \{p\}$ (as schemes).*

Observe that the possible splitting types for $E(-c+1)$ are: $(0, -c+2, -c+1, ..., -c+1)$, $(1, -c+1, ..., -c+1)$. Indeed $E_L \simeq \bigoplus \mathcal{O}_L(a_i(L))$ with $a_1(L) \geq ... \geq a_r(L) \geq 0$ and $\sum a_i(L) = c$. Since $h^0(E_L(-c+1)) \neq 0$ by (a), we have $a_1(L) - c + 1 \geq 0$ hence: $(a_i(L)) = (c-1, 1, 0, ..., 0)$ or $(c, 0, ..., 0)$. It follows that $\forall L \subset \mathbb{P}^n$, s_L (which is non zero by (a)) can vanish at at most one point of L (with multiplicity one). If $s(p) = s(q) = 0$ with $p \neq q$, then $s_D = 0$ where $D = \langle p, q \rangle$, in contradiction with (a). This shows that $(s)_0 = \{p\}$ (as a scheme, the zero locus being defined by linear forms).

We see that the jumping lines (where the splitting is $(1, -c+1, ..., -c+1)$) are precisely the lines through p, moreover if $p \notin H$ then s_H doesn't vanish and we have an exact sequence:

$$0 \to \mathcal{O}_H \to E_H(-c+1) \to F_H(-c+1) \to 0 \ \ (*)$$

where F_H is a vector bundle. The vector bundle F_H has $c_1(F_H) = 1$ and is globally generated. By Proposition 1.3.1 either F_H is a direct sum of line bundles or $F_H \simeq T_H(-1) \oplus t.\mathcal{O}_H$. In the first case we will have that E_H also is a direct sum of line bundles, which in turn implies that E itself is a direct sum of line bundles, since this is excluded by assumption, we have $F_H \simeq T_H(-1) \oplus t.\mathcal{O}_H$. Since $h^1(\Omega(c)) = 0$, the exact sequence $(*)$ splits and $E_H \simeq T_H(-1) \oplus \mathcal{O}_H(c-1) \oplus (r-n)\mathcal{O}_H$.

If $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n$ we denote by $Sing(\mathcal{F})$ its *singular set*: $Sing(\mathcal{F}) = \{x \in \mathbb{P}^n \mid \mathcal{F}_x \text{ is not a free } \mathcal{O}_{\mathbb{P}^n,x}\text{-module}\}$.

We also recall that given a point $p \in \mathbb{P}^n$, there exists $\sigma \in H^0(T(-1))$ such that $(\sigma)_0 = \{p\}$. This can be seen as follows: if $\mathbb{P}^n = \mathbb{P}(V)$ (projective space of lines), a point $x \in \mathbb{P}^n$ corresponds to a line $l_x \subset V$ and the exact sequence:

$$0 \to \mathcal{O}(-1)(x) \to V \otimes \mathcal{O}(x) \to T(-1)(x) \to 0$$

is:

$$0 \to l_x \to V \to V/l_x \to 0$$

so that we may identify the (vector bundle) fiber $T(-1)(x) := T(-1) \otimes k(x)$ with V/l_x. With this identification any $v \in V$ yields a section, s_v, of $T(-1)$ with $s_v(x) = \overline{v} \in V/l_x$. Clearly $s_v(x) = 0 \Leftrightarrow l_x = \langle v \rangle$.

LEMMA 1.4.7 *Let $\mathcal{G}$ be a rank $n-1$, globally generated torsion free sheaf on $\mathbb{P}^n$, $n \geq 2$, with $Sing(\mathcal{G}) = \{p\}$, $c_1(\mathcal{G}) = 1$ and $h^0(\mathcal{G}) = n$. If $n = 2$ then $\mathcal{G} \simeq \mathcal{I}_p(1)$; if $n > 2$, $\mathcal{G}$ is reflexive. In any case there is an exact sequence:*

$$0 \to \mathcal{O} \to T(-1) \to \mathcal{G} \to 0$$

PROOF 1.4.8
Since $\mathcal{G}$ is globally generated with $h^0(\mathcal{G}) = n$ there is an exact sequence:

$$0 \to \mathcal{L} \to n.\mathcal{O} \to \mathcal{G} \to 0$$

Since $\mathcal{G}$ is torsion free and $n.\mathcal{O}$ is reflexive, $\mathcal{L}$ is normal ([18], Chap. II, Lemma 1.1.16). Since $\mathcal{L}$ is torsion free and normal, $\mathcal{L}$ is reflexive ([18], Chap. II, Lemma 1.1.12). Finally since $\mathcal{L}$ has rank one and is reflexive, it is locally free ([18], Chap. II, Lemma 1.1.15). Looking at Chern classes we get that $\mathcal{L} \simeq \mathcal{O}(-1)$. In conclusion we have:

$$0 \to \mathcal{O}(-1) \xrightarrow{f} n.\mathcal{O} \to \mathcal{G} \to 0 \ \ (+)$$

The map f is given by n linear forms $\varphi_1, ..., \varphi_n$. Since $Sing(\mathcal{G}) = \{p\}$, $\varphi_1, ..., \varphi_n$ are linearly independent and define the point p.

If $n = 2$ we clearly have $\mathcal{G} \simeq \mathcal{I}_p(1)$; moreover if $n > 2$, $\mathcal{G}$ is reflexive (just take the bidual of $(+)$ and take into account that $\mathcal{E}xt^1(\mathcal{O}, \mathcal{I}_p) = 0$ if $n > 2$).

After a change of coordinates we may assume $p = (1 : 0 : ... : 0)$ and f given by

$(x_1, ..., x_n)$. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O} & = & \mathcal{O} & & \\
 & & & & \downarrow j & & \downarrow \sigma & & \\
0 & \to & \mathcal{O}(-1) & \stackrel{(x_0,\ldots,x_n)}{\longrightarrow} & (n+1).\mathcal{O} & \to & T(-1) & \to & 0 \\
 & & || & & \downarrow p & & \downarrow g & & \\
0 & \to & \mathcal{O}(-1) & \stackrel{(x_1,\ldots,x_n)}{\longrightarrow} & n.\mathcal{O} & \to & Q & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where σ is the section of $T(-1)$ vanishing at p.

The rightmost vertical exact sequence yields the result.

We can now prove the main result of this section:

Proof 1.4.9 (Proof of Theorem 1.4.4)

First of all we observe that the statement makes sense only for $c > 0$; moreover the case $c = 1$ follows from Proposition 1.3.1. So we may assume $c > 1$.

(a) We observe that E is a direct sum of line bundles if and only if $E \simeq \mathcal{O}(c-1) \oplus \mathcal{O}(1) \oplus (r-2).\mathcal{O}$, (*). Indeed if $E \simeq \bigoplus \mathcal{O}(a_i)$, with $a_1 \geq ... \geq a_r \geq 0$, $\sum a_i = c$, the condition $h^0(E(-c)) = 0$ implies $a_i < c, \forall i$ and then the condition $h^0(E(-c+1)) \neq 0$ implies $a_1 = c-1$. Finally we observe that if $E(-c+1)$ has a nowhere vanishing global section, then E is a direct sum of line bundles or $E \simeq \mathcal{O}(c-1) \oplus T(-1) \oplus (r-n-1).\mathcal{O}$. Indeed if $E(-c+1)$ has a nowhere vanishing section we have:

$$0 \to \mathcal{O}(c-1) \to E \to F \to 0$$

where F is a globally generated vector bundle with $c_1(F) = 1$. By 1.3.1, $h^1(F^*(c-1)) = 0$, the exact sequence splits and the result follows.

(b) From (a) we may assume that E is not a direct sum of line bundles and that (any) non zero section of $E(-c+1)$ has a non empty zero locus. By Lemma 1.4.5 if $0 \neq s \in H^0(E(-c+1))$ then $(s)_0 = \{p\}$, $h^0(E(-c+1)) = 1$ and if $p \notin H$, $E_H \simeq T_H(-1) \oplus \mathcal{O}(c-1) \oplus (r-n).\mathcal{O}_H$. In particular $r \geq n$.

(b.1) Assume first $r = n$. The section s yields:

$$0 \to \mathcal{O} \to E(-c+1) \to Q(-c+1) \to 0 \ \ (+)$$

We have $Sing(Q) = \{p\}$ (because $(s)_0 = \{p\}$). Moreover if $p \notin H$, Q_H is locally free (s has constant rank one on H), in particular Q is torsion free and we have:

$$0 \to \mathcal{O}_H(c-1) \to T_H(-1) \oplus \mathcal{O}_H(c-1) \to Q_H \to 0$$

It follows that $h^0(Q_H(-m)) = 0$ if $m > 0$. This implies $h^0(Q(-m)) = 0$ if $m > 0$ and $h^0(Q) \leq h^0(Q_H) = n$. Now Q, as a quotient of E is globally generated, since $rk(Q) = n-1$ and Q is not trivial, $h^0(Q) = n$. Finally $c_1(Q) = 1$ (exact sequence $(+)$ twisted by $\mathcal{O}(c-1)$). By Lemma 1.4.7 there is an exact sequence:

$$0 \to \mathcal{O} \to T(-1) \xrightarrow{g} Q \to 0 \ \ (+)$$

From this we get:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \mathcal{O} & & \\
 & & & & & & \downarrow \sigma & & \\
 & & & & & & T(-1) & & \\
 & & & & & & \downarrow g & & \\
0 & \to & \mathcal{O}(c-1) & \to & E & \xrightarrow{\pi} & Q & \to & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

Applying $Hom(T(-1), -)$ to the bottom row, since $Ext^1(T(-1), \mathcal{O}(1)) = H^1(\Omega(2)) = 0$, we get that the morphism:

$$Hom(T(-1), E) \to Hom(T(-1), Q) : f \to \pi \circ f$$

is surjective, so there exists $\varphi : T(-1) \to E$ such that $\pi \circ \varphi = g$. Finally we get:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O} & = & \mathcal{O} & & \\
 & & & & \downarrow & & \downarrow \sigma & & \\
0 & \to & \mathcal{O}(c-1) & \xrightarrow{s} & \mathcal{O}(c-1) \oplus T(-1) & \to & T(-1) & \to & 0 \\
 & & || & & \downarrow (s,\varphi) & & \downarrow g & & \\
0 & \to & \mathcal{O}(c-1) & \xrightarrow{s} & E & \to & Q & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

In conclusion:

$$0 \to \mathcal{O} \to \mathcal{O}(c-1) \oplus T(-1) \to E \to 0$$

(b.2) Assume now $r > n$.
By Lemma 1.1.4 we have an exact sequence:

$$0 \to (r-n).\mathcal{O} \to E \to \mathcal{E} \to 0$$

where $\mathcal{E}$ is a rank n, globally generated vector bundle with $h^0(E(-t)) = h^0(\mathcal{E}(-t)), \forall t > 0$ and $c_1(E) = c_1(\mathcal{E})$. If $\mathcal{E}$ is a direct sum of line bundles, then E also is, in contrast with our assumption. So $\mathcal{E}$ is as in (b.1), i.e there is an exact sequence:

$$0 \to \mathcal{O} \to \mathcal{O}(c-1) \oplus T(-1) \to \mathcal{E} \to 0$$

We have:

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \mathcal{O} & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \mathcal{O}(c-1) \oplus T(-1) & & \\
 & & & & & & \downarrow g & & \\
0 & \to & (r-n).\mathcal{O} & \to & E & \xrightarrow{\pi} & \mathcal{E} & \to & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}
$$

Applying $Hom(\mathcal{O}(c-1) \oplus T(-1), -)$ *to the bottom row, since* $Ext^1(\mathcal{O}(c-1) \oplus T(-1), \mathcal{O}) = H^1(\Omega(1) \oplus \mathcal{O}(-c+1)) = 0$, *we get that the morphism:*

$$Hom(\mathcal{O}(c-1) \oplus T(-1), E) \to Hom(\mathcal{O}(c-1) \oplus T(-1), Q) : f \to \pi \circ f$$

is surjective, so there exists $\varphi : \mathcal{O}(c-1) \oplus T(-1) \to E$ *such that* $\pi \circ \varphi = g$. *Finally we get:*

$$0 \to \mathcal{O} \to \mathcal{O}(c-1) \oplus T(-1) \oplus (r-n).\mathcal{O} \to E \to 0$$

and the proof is over.

1.5 Globally generated vector bundles on $\mathbb{P}^n$ with $c_1 = 2$.

The classification of globally generated vector bundles on $\mathbb{P}^n$ with $c_1 = 2$ has been achieved recently by Sierra-Ugaglia ([21]). Their result is:

THEOREM 1.5.1 *Let* E *be a globally generated, rank* r, *vector bundle on* $\mathbb{P}^n$, $n \geq 2$, *with* $c_1 = 2$, *then one of the following holds:*

1. $E \simeq \mathcal{O}(2) \oplus (r-1).\mathcal{O}$
2. $E \simeq 2.\mathcal{O}(1) \oplus (r-2).\mathcal{O}$
3. *there is an exact sequence:*

$$0 \to \mathcal{O} \to \mathcal{O}(1) \oplus T(-1) \oplus (r-n).\mathcal{O} \to E \to 0$$

4. *there is an exact sequence:*

$$0 \to 2.\mathcal{O}(-1) \to (r+2).\mathcal{O} \to E \to 0$$

5. *there is an exact sequence:*

$$0 \to \mathcal{O}(-2) \to (r+1).\mathcal{O} \to E \to 0$$

6. *$n = 3$ and $E \simeq \Omega(2) \oplus (r-3).\mathcal{O}$*

7. *$n = 3$ and $E \simeq \mathcal{N}(1) \oplus (r-2).\mathcal{O}$, where $\mathcal{N}$ is a normalized null-correlation bundle.*

As we can see the number of possibilities increases and also some "exceptional" cases appear (for $n = 3$). We observe that $h^0(E(-2)) \neq 0$ in case (1), $h^0(E(-2)) = 0$ but $h^0(E(-1)) \neq 0$ in cases (2), (3) and $h^0(E(-1)) = 0$ in the remaining cases.

Let's sketch the proof of the theorem.

LEMMA 1.5.2 *Let E be a rank r globally generated vector bundle on $\mathbb{P}^n$, $n \geq 2$ with $c_1 = 2$. If $h^0(E(-1)) \neq 0$, then one of the cases (1), (2), (3) of Theorem 1.5.1 occurs.*

PROOF 1.5.3
Follows from Lemma 1.4.1 and Theorem 1.4.4.

From now on we may assume $h^0(E(-1)) = 0$. Let's prove the Theorem for $n = 2$:

LEMMA 1.5.4 *Let E be a rank r globally generated vector bundle on $\mathbb{P}^2$ with $c_1 = 2$. If $h^0(E(-1)) = 0$ then one of the following occurs:*

1. $$0 \to \mathcal{O}(-2) \to (r+1).\mathcal{O} \to E \to 0$$

2. $$0 \to 2.\mathcal{O}(-1) \to (r+2).\mathcal{O} \to E \to 0$$

PROOF 1.5.5
Let $L \subset \mathbb{P}^2$ be a line, the exact sequence:

$$0 \to E(-1) \to E \to E_L \to 0$$

shows, since $h^0(E(-1)) = 0$, that $h^0(E) \leq h^0(E_L) = r+2$. So $r+1 \leq h^0(E) \leq r+2$ and if $h^0(E) = r+1$, we are in case (1). If $h^0(E) = r+2$, we have:

$$0 \to \mathcal{E} \to (r+2).\mathcal{O} \to E \to 0$$

where $\mathcal{E}$ has rank two and $c_1 = -2$. We have $h^1(\mathcal{E}(-m)) = 0$ if $m \geq 0$. Since $h^1(\mathcal{E}(m)) = h^1(\mathcal{E}^*(-m-3)) = h^1(\mathcal{E}(-m-1))$, we conclude that $H^1_*(\mathcal{E}) = 0$ and (by Theorem 1.1.10), $\mathcal{E}$ is a direct sum of line bundles. Since $\mathcal{E}^*$ is globally generated and $h^0(E) = r+2$ we get $\mathcal{E} \simeq 2.\mathcal{O}(-1)$.

So far the Theorem is proved for $n = 2$. The following lemma shows that the case $n = 3$ is special.

LEMMA 1.5.6 *Let E be a globally generated rank r vector bundle on $\mathbb{P}^n$, $n \geq 3$, with $c_1(E) = 2$. If $h^0(E(-1)) = 0$ but $h^0(E_H(-1)) \neq 0$ for some hyperplane H, then $n = 3$.*

PROOF 1.5.7
If E_H is a direct sum of line bundles, then E also is a direct sum of line bundles, but this is impossible since $h^0(E(-1)) = 0$. By 1.4.4 we conclude that:

$$0 \to \mathcal{O}_H \to T_H(-1) \oplus \mathcal{O}_H(1) \oplus (r-n+1).\mathcal{O}_H \to E_H \to 0$$

In particular $r \geq n-1$. If $n > 3$, $h^i(E_H(-m)) = 0$ for $i = 0, 1$ and $m \geq 2$. It follows that $h^0(E(-2)) = h^1(E(-2)) = 0$. From the exact sequence:

$$0 \to E(-2) \to E(-1) \to E_H(-1) \to 0$$

we get $h^0(E(-1)) = h^0(E_H(-1)) > 0$: a contradiction if $n > 3$, hence $n = 3$.

PROPOSITION 1.5.8 *Let E be a globally generated vector bundle of rank r on $\mathbb{P}^3$ with $c_1(E) = 2$. Assume $h^0(E(-1)) = 0$ but $h^0(E_H(-1)) \neq 0$ for some hyperplane. Then:*

$$E \simeq \Omega(2) \oplus (r-3).\mathcal{O} \text{ or } E \simeq \mathcal{N}(1) \oplus (r-2).\mathcal{O}$$

where $\mathcal{N}$ is a normalized null-correlation bundle.

PROOF 1.5.9
We have

$$0 \to \mathcal{O}_H \to T_H(-1) \oplus \mathcal{O}_H(1) \oplus (r-n+1).\mathcal{O}_H \to E_H \to 0 \quad (+)$$

In particular $r \geq 2$, $c_1(E) = c_2(E) = 2$.

(a) If $r = 2$, $E(-1)$ *is stable with* $c_1 = 0$, $c_2 = 1$, *hence* $E(-1)$ *is a null-correlation bundle. This can be seen also as follows: a general section yields:* $0 \to \mathcal{O} \to E \to \mathcal{I}_C(2) \to 0$, *where* C *is a smooth curve with* $\omega_C(2) \simeq \mathcal{O}_C$. *It follows that* C *is a disjoint union of lines. Since* $\mathcal{I}_C(2)$ *is globally generated,* C *has degree* ≤ 2; *it can't be* 1 *because* E *doesn't split, so* C *is the union of two skew lines (or* $deg(C) = c_2(E) = 2$*) and* $E = \mathcal{N}(1)$.

(b) Assume $r = 3$. *We have* $h^0(E) \leq h^0(E_H) = 6$. *By Riemann-Roch, since* $c_1 = c_2 = 2$, $\chi(E) = \frac{c_3}{2} + 6$. *Using the dual of (+) we have* $h^i(E_H^*(-m)) = 0$ *if* $m \geq 4$, $i = 0, 1$. *From* $0 \to E^*(-m-1) \to E^*(-m) \to E_H^*(-m) \to 0$, *we get* $h^1(E^*(-4)) = h^2(E) = 0$. *So* $\chi(E) = h^0(E) - h^1(E) - h^3(E) = \frac{c_3}{2} + 6$. *Since* $h^0(E) \leq 6$ *and* $c_3 \geq 0$ *(because* E *is globally generated), it follows that* $c_3 = 0$. *This implies (**??**) that a general section of* E *doesn't vanish, hence we have:*

$$0 \to \mathcal{O} \to E \to F \to 0$$

where F *is a rank two globally generated vector bundle with* $c_1(F) = 2$. *Since* $h^0(F(-1)) = 0$ *and* $h^0(F_H(-1)) \neq 0$, $F \simeq \mathcal{N}(1)$. *We have* $\dim Ext^1(\mathcal{O}, \mathcal{N}(1)) = h^1(\mathcal{N}^*(-1)) = 1$ *and it is well known that there are two extensions: the trivial one and*

$$0 \to \mathcal{O} \to \Omega(2) \to \mathcal{N}(1) \to 0$$

We conclude that $E \simeq \Omega(2)$ *or* $E \simeq \mathcal{N}(1) \oplus \mathcal{O}$.

(c) Finally if $r > 3$ *there is an exact sequence:*

$$0 \to (r-3).\mathcal{O} \to E \to \mathcal{E} \to 0$$

where $\mathcal{E}$ *is globally generated of rank three, with* $c_1(\mathcal{E}) = 2$, $h^0(\mathcal{E}(-1)) = 0$ *and* $h^0(\mathcal{E}_H(-1)) \neq 0$. *Then* $\mathcal{E}$ *is as in (b) and we easily conclude that* $E \simeq \Omega(2) \oplus (r-3).\mathcal{O}$ *or* $E \simeq \mathcal{N}(1) \oplus (r-2).\mathcal{O}$.

We can now conclude the proof of the Theorem for $n = 3$:

Lemma 1.5.10 *Let* E *be a globally generated vector bundle of rank* r *on* $\mathbb{P}^3$ *with* $c_1 = 2$. *If* $h^0(E(-1)) = 0$ *and* $h^0(E_H(-1)) = 0$ *for all (some) hyperplane, then* *either:*

$$0 \to \mathcal{O}(-2) \to (r+1).\mathcal{O} \to E \to 0$$

or

$$0 \to 2.\mathcal{O}(-1) \to (r+2).\mathcal{O} \to E \to 0$$

PROOF 1.5.11
Since $h^0(E(-1)) = h^0(E_H(-1)) = 0$, we have $h^0(E) \leq h^0(E_H) \leq h^0(E_L) = r+2$. If $h^0(E) = r+1$ we are in the first case. If $h^0(E) = r+2$ there is an exact sequence:

$$0 \to \mathcal{E} \to (r+2).\mathcal{O} \to E \to 0$$

restricting to H we get (see 1.5.4) $\mathcal{E} \simeq 2.\mathcal{O}(-1)$.

The final touch:

PROOF 1.5.12 (PROOF OF THEOREM 1.5.1)
According to the previous results we may assume $n \geq 4$ and $h^0(E(-1)) = 0$. By Lemma 1.5.6, $h^0(E_H(-1)) = 0$ for some (in fact all) hyperplane. Let's prove by induction on n that under these assumptions either case (4) or case (5) of 1.5.1 holds.

Assume first $n = 4$. From Proposition 1.5.8 and Lemma 1.5.10, either $h^0(E_H) \leq r+2$ or $E_H \simeq \Omega_H(2) \oplus (r-3)\mathcal{O}_H$ or $E_H \simeq \mathcal{N}(1) \oplus (r-2).\mathcal{O}_H$. The last two cases are impossible. Indeed there is no globally generated vector bundle E on $\mathbb{P}^4$ with $E_H \simeq \Omega_H(2) \oplus (r-3).\mathcal{O}_H$ or $\mathcal{N}(1) \oplus (r-2).\mathcal{O}_H$: such a vector bundle would have $c_1 = c_2 = 2$ and $c_3(E) = 0$; $r-2$ general sections would give a morphism of constant rank (because $c_3 = 0$) hence a rank two vector bundle as quotient with $c_1 = c_2 = 2$, but this contradicts Schwarzenberger's conditions. So $h^0(E_H) \leq r+2$, hence, since $h^0(E(-1)) = 0$, $h^0(E) \leq r+2$ and we easily conclude (see proof of 1.5.10).

Assume the result for $n-1$. In particular this means $h^0(E_H) \leq r+2$. It follows that $h^0(E) \leq r+2$ and we are home.

1.6 Globally generated rank two vector bundles on $\mathbb{P}^n$, $n \geq 3$, with $c_1 \leq 5$.

In this section we consider *rank two* globally generated vector bundles on $\mathbb{P}^n$ with $c_1 \leq 5$. The final result is:

THEOREM 1.6.1 *Let E be a rank two vector bundle on $\mathbb{P}^n$, $n \geq 3$, generated by global sections with Chern classes c_1, c_2, $c_1 \leq 5$.*

1. *If $n \geq 4$, then E is the direct sum of two line bundles*

2. *If $n = 3$ and E is indecomposable, then*

$$(c_1, c_2) \in S = \{((2,2), (4,5), (4,6), (4,7), (4,8), (5,8), (5,10), (5,12)\}.$$

   If E exists there is an exact sequence: $0 \to \mathcal{O} \to E \to \mathcal{I}_C(c_1) \to 0$ $(*)$, where $C \subset \mathbb{P}^3$ is a smooth curve of degree c_2 with $\omega_C(4 - c_1) \simeq \mathcal{O}_C$. The curve C is irreducible, *except maybe if $(c_1, c_2) = (4, 8)$: in this case C can be irreducible or the disjoint union of two smooth conics.*

3. *For every $(c_1, c_2) \in S$, $(c_1, c_2) \neq (5, 12)$, there exists a rank two vector bundle on $\mathbb{P}^3$ with Chern classes (c_1, c_2) which is globally generated (and with an exact sequence as in (2)).*

REMARK 1.6.2 *The case $n = 3$, $(c_1, c_2) = (5, 12)$ remains open: we are unable to prove or disprove the existence of such bundles.*

1.6.1 Globally generated rank two vector bundles on $\mathbb{P}^3$ with $c_1 = 3$.

The following result has been proved in [22] (with a different and longer proof).

PROPOSITION 1.6.3 *Let E be a rank two globally generated vector bundle on $\mathbb{P}^3$. If $c_1(E) = 3$ then E splits.*

Proof 1.6.4 *Assume a general section vanishes in codimension two, then it vanishes along a smooth curve C such that $\omega_C \simeq \mathcal{O}_C(-1)$. Moreover $\mathcal{I}_C(3)$ is generated by global sections. We have $C = \cup_{i=1}^r C_i$ (disjoint union) where each C_i is smooth irreducible with $\omega_{C_i} \simeq \mathcal{O}_{C_i}(-1)$. It follows that each C_i is a smooth conic. If $r \geq 2$ let $L = \langle C_1 \rangle \cap \langle C_2 \rangle$ ($\langle C_i \rangle$ is the plane spanned by C_i). Every cubic containing C contains L (because it contains the four points $C_1 \cap L$, $C_2 \cap L$). This contradicts the fact that $\mathcal{I}_C(3)$ is globally generated. Hence $r = 1$ and $E = \mathcal{O}(1) \oplus \mathcal{O}(2)$.*

1.6.2 Globally generated rank two vector bundles on $\mathbb{P}^3$ with $c_1 = 4$.

Let's start with a general result:

Lemma 1.6.5 *Let E be a non split rank two vector bundle on $\mathbb{P}^3$ with Chern classes c_1, c_2. If E is globally generated and if $c_1 \geq 4$ then:*

$$c_2 \leq \frac{2c_1^3 - 4c_1^2 + 2}{3c_1 - 4}.$$

Proof 1.6.6 *By our assumptions a general section of E vanishes along a smooth curve, C, such that $\mathcal{I}_C(c_1)$ is generated by global sections. Let U be the complete intersections of two general surfaces containing C. Then U links C to a smooth curve, Y. We have $Y \neq \varnothing$ since E doesn't split. The exact sequence of liaison: $0 \to \mathcal{I}_U(c_1) \to \mathcal{I}_C(c_1) \to \omega_Y(4-c_1) \to 0$ shows that $\omega_Y(4-c_1)$ is generated by global sections. Hence $\deg(\omega_Y(4-c_1)) \geq 0$. We have $\deg(\omega_Y(4-c_1)) = 2g' - 2 + d'(4-c_1)$ ($g' = p_a(Y)$, $d' = \deg(Y)$). So $g' \geq \frac{d'(c_1-4)+2}{2} \geq 0$ (because $c_1 \geq 4$). On the other hand, always by liaison, we have: $g' - g = \frac{1}{2}(d'-d)(2c_1-4)$ ($g = p_a(C)$, $d = \deg(C)$). Since $d' = c_1^2 - d$ and $g = \frac{d(c_1-4)}{2} + 1$ (because $\omega_C(4-c_1) \simeq \mathcal{O}_C$), we get: $g' = 1 + \frac{d(c_1-4)}{2} + \frac{1}{2}(c_1^2 - 2d)(2c_1 - 4) \geq 0$ and the result follows.*

Now we have:

Proposition 1.6.7 *Let E be a rank two globally generated vector bundle on $\mathbb{P}^3$. If $c_1(E) = 4$ and if E doesn't split, then $5 \leq c_2 \leq 8$ and there is an exact sequence:*

$0 \to \mathcal{O} \to E \to \mathcal{I}_C(4) \to 0$, *where C is a smooth irreducible elliptic curve of degree c_2 or, if $c_2 = 8$, C is the disjoint union of two smooth elliptic quartic curves.*

PROOF 1.6.8 *A general section of E vanishes along C where C is a smooth curve with $\omega_C = \mathcal{O}_C$ and where $\mathcal{I}_C(4)$ is generated by global sections. Let $C = C_1 \cup ... \cup C_r$ be the decomposition into irreducible components: the union is disjoint, each C_i is a smooth elliptic curve hence has degree at least three.*
By Lemma 1.6.5 $d = \deg(C) \leq 8$. If $d \leq 4$ then C is irreducible and is a complete intersection which is impossible since E doesn't split. If $d = 5$, C is smooth irreducible.
Claim: If $8 \geq d \geq 6$, C cannot contain a plane cubic curve.
Assume $C = P \cup X$ where P is a plane cubic and where X is a smooth elliptic curve of degree $d-3$. If $d = 6$, X is also a plane cubic and every quartic containing C contains the line $\langle P \rangle \cap \langle X \rangle$. If $deg(X) \geq 4$ then every quartic, F, containing C contains the plane $\langle P \rangle$. Indeed $F|H$ vanishes on P and on the $deg(X) \geq 4$ points of $X \cap \langle P \rangle$, but these points are not on a line so $F|H = 0$. In both cases we get a contradiction with the fact that $\mathcal{I}_C(4)$ is generated by global sections. The claim is proved.
It follows that, if $8 \geq d \geq 6$, then C is irreducible except if $C = X \cup Y$ is the disjoint union of two elliptic quartic curves.

Now let's show that all possibilities of Proposition 1.6.7 do actually occur. For this we have to show the existence of a smooth irreducible elliptic curve of degree d, $5 \leq d \leq 8$ with $\mathcal{I}_C(4)$ generated by global sections (and also that the disjoint union of two elliptic quartc curves is cut off by quartics).

LEMMA 1.6.9 *There exist rank two vector bundles with $c_1 = 4, c_2 = 5$ which are globally generated. More precisely any such bundle is of the form $\mathcal{N}(2)$, where $\mathcal{N}$ is a null-correlation bundle (a stable bundle with $c_1 = 0, c_2 = 1$).*

PROOF 1.6.10 *The existence is clear (if $\mathcal{N}$ is a null-correlation bundle then it is well known that $\mathcal{N}(k)$ is globally generated if $k \geq 1$). Conversely if E has $c_1 = 4, c_2 = 5$ and is globally generated, then E has a section vanishing along a smooth, irreducible quintic elliptic curve (cf 1.6.7). Since $h^0(\mathcal{I}_C(2)) = 0$, E is stable, hence $E = \mathcal{N}(2)$.*

Lemma 1.6.11 *There exist smooth, irreducible elliptic curves, C, of degree 6 with $\mathcal{I}_C(4)$ generated by global sections.*

Proof 1.6.12 *Let X be the union of three skew lines. The curve X lies on a smooth quadric surface, Q, and has $\mathcal{I}_X(3)$ globally generated (indeed the exact sequence $0 \to \mathcal{I}_Q \to \mathcal{I}_X \to \mathcal{I}_{X,Q} \to 0$ twisted by $\mathcal{O}(3)$ reads like: $0 \to \mathcal{O}(1) \to \mathcal{I}_C(3) \to \mathcal{O}_Q(3,0) \to 0$). The complete intersection, U, of two general cubics containing X links X to a smooth curve, C, of degree 6 and arithmetic genus 1. Since, by liaison, $h^1(\mathcal{I}_C) = h^1(\mathcal{I}_X(-2)) = 0$, C is irreducible. The exact sequence of liaison: $0 \to \mathcal{I}_U(4) \to \mathcal{I}_C(4) \to \omega_X(2) \to 0$ shows that $\mathcal{I}_C(4)$ is globally generated.*

In order to prove the existence of smooth, irreducible elliptic curves, C, of degree $d = 7, 8$, with $\mathcal{I}_C(4)$ globally generated, we have to recall some results due to Mori ([16]).

According to [16] Remark 4, Prop. 6, there exists a smooth quartic surface $S \subset \mathbb{P}^3$ such that $Pic(S) = \mathbb{Z}H \oplus \mathbb{Z}X$ where X is a smooth elliptic curve of degree d ($7 \leq d \leq 8$). The intersection pairing is given by: $H^2 = 4$, $X^2 = 0$, $H.X = d$. Such a surface doesn't contain any smooth rational curve ([16] p.130). In particular: $(*)$ every integral curve, Z, on S has degree ≥ 4 with equality if and only if Z is a planar quartic curve or an elliptic quartic curve.

Lemma 1.6.13 *With notations as above, $h^0(\mathcal{I}_X(3)) = 0$.*

Proof 1.6.14 *A curve $Z \in |3H - X|$ has invariants $(d_Z, g_Z) = (5, -2)$ (if $d = 7$) or $(4, -5)$ (if $d = 8$), so Z is not integral. It follows that Z must contain an integral curve of degree < 4, but this is impossible.*

Lemma 1.6.15 *With notations as above $|4H - X|$ is base point free, hence there exist smooth, irreducible elliptic curves, X, of degree d, $7 \leq d \leq 8$, such that $\mathcal{I}_X(4)$ is globally generated.*

Proof 1.6.16 *Let's first prove the following: Claim: Every curve in $|4H - X|$ is integral.*

If $Y \in |4H - X|$ is not integral then $Y = Y_1 + Y_2$ where Y_1 is integral with $\deg(Y_1) = 4$ *(observe that* $\deg(Y) = 9$ *or* 8*).*

If Y_1 is planar then $Y_1 \sim H$, so $4H - X \sim H + Y_2$ and it follows that $3H \sim X + Y_2$, in contradiction with $h^0(\mathcal{I}_X(3)) = 0$ (cf 1.6.13).

So we may assume that Y_1 is a quartic elliptic curve, i.e. (i) $Y_1^2 = 0$ and (ii) $Y_1.H = 4$. Setting $Y_1 = aH + bX$, we get from (i): $2a(2a + bd) = 0$. Hence (α) $a = 0$, or (β) $2a + bd = 0$.

(α) In this case $Y_1 = bX$, hence (for degree reasons and since S doesn't contain curves of degree < 4), $Y_2 = \varnothing$ and $Y = X$, which is integral.

(β) Since $Y_1.H = 4$, we get $2a + (2a + bd) = 2a = 4$, hence $a = 2$ and $bd = -4$ which is impossible ($d = 7$ or 8 and $b \in \mathbb{Z}$).

This concludes the proof of the claim.

Since $(4H - X)^2 \geq 0$, the claim implies that $4H - X$ is numerically effective. Now we conclude by a result of Saint-Donat (cf [16], Theorem 5) that $|4H - X|$ is base point free, i.e. $\mathcal{I}_{X,S}(4)$ is globally generated. By the exact sequence: $0 \to \mathcal{O} \to \mathcal{I}_X(4) \to \mathcal{I}_{X,S}(4) \to 0$ we get that $\mathcal{I}_X(4)$ is globally generated.

REMARK 1.6.17 *If $d = 8$, a general element $Y \in |4H - X|$ is a smooth elliptic curve of degree 8. By the way $Y \neq X$ (see [2]). The exact sequence of liaison: $0 \to \mathcal{I}_U(4) \to \mathcal{I}_X(4) \to \omega_Y \to 0$ shows that $h^0(\mathcal{I}_X(4)) = 3$ (i.e. X is of maximal rank). In case $d = 8$ Lemma 1.6.15 is stated in [5], however the proof there is incomplete, indeed in order to apply the enumerative formula of [12] one has to know that X is a* connected *component of $\bigcap_{i=1}^{3} F_i$; this amounts to say that the base locus of $|4H - X|$ on F_1 has dimension ≤ 0.*

To conclude we have:

LEMMA 1.6.18 *Let X be the disjoint union of two smooth, irreducible quartic elliptic cuvres, then $\mathcal{I}_X(4)$ is generated by global sections.*

PROOF 1.6.19 *Let $X = C_1 \sqcup C_2$. We have: $0 \to \mathcal{O}(-4) \to 2.\mathcal{O}(-2) \to \mathcal{I}_{C_1} \to 0$, twisting by $\mathcal{I}_{C_2}$, since $C_1 \cap C_2 = \varnothing$, we get:*
$0 \to \mathcal{I}_{C_2}(-4) \to 2.\mathcal{I}_{C_2}(-2) \to \mathcal{I}_X \to 0$ *and the result follows.*

Summarizing:

Proposition 1.6.20 *There exists an indecomposable rank two vector bundle, E, on $\mathbb{P}^3$, generated by global sections and with $c_1(E) = 4$ if and only if $5 \leq c_2(E) \leq 8$ and in these cases there is an exact sequence:*

$$0 \to \mathcal{O} \to E \to \mathcal{I}_C(4) \to 0$$

where C is a smooth irreducible elliptic curve of degree $c_2(E)$ or, if $c_2(E) = 8$, the disjoint union of two smooth elliptic quartic curves.

1.6.3 Globally generated rank two vector bundles on $\mathbb{P}^3$ with $c_1 = 5$.

We start by listing the possible cases:

Proposition 1.6.21 *If E is an indecomposable, globally generated, rank two vector bundle on $\mathbb{P}^3$ with $c_1(E) = 5$, then $c_2(E) \in \{8, 10, 12\}$ and there is an exact sequence:*

$$0 \to \mathcal{O} \to E \to \mathcal{I}_C(5) \to 0$$

where C is a smooth, irreducible curve of degree $d = c_2(E)$, with $\omega_C \simeq \mathcal{O}_C(1)$.

In any case E is stable.

Proof 1.6.22 *A general section of E vanishes along a smooth curve, C, of degree $d = c_2(E)$ with $\omega_C \simeq \mathcal{O}_C(1)$. Hence every irreducible component, Y, of C is a smooth, irreducible curve with $\omega_Y \simeq \mathcal{O}_Y(1)$. In particular* $\deg(Y) = 2g(Y) - 2$ *is even and* $\deg(Y) \geq 4$.

1. *If $d = 4$, then C is a planar curve and E splits.*

2. *If $d = 6$, C is necessarily irreducible (of genus 4). It is well known that any such curve is a complete intersection $(2, 3)$, hence E splits.*

3. *If $d = 8$ and C is not irreducible, then $C = P_1 \sqcup P_2$, the disjoint union of two planar quartic curves. If $L = \langle P_1 \rangle \cap \langle P_2 \rangle$, then every quintic containing C contains L in contradiction with the fact that $\mathcal{I}_C(5)$ is generated by global sections. Hence C is irreducible.*

4. *If $d = 10$ and C is not irreducible, then $C = P \sqcup X$, where P is a planar curve of degree 4 and where X is a degree 6 curve (X is a complete intersection $(2, 3)$). Every quintic containing C vanishes on P and on the 8 points of $X \cap \langle P \rangle$, since these 8 points are not on a line, the quintic vanishes on the plane $\langle P \rangle$. This contradicts the fact that $\mathcal{I}_C(5)$ is globally generated.*

5. *If $d = 12$ and C is not irreducible we have three possibilities:*

   (a) $C = P_1 \sqcup P_2 \sqcup P_3$, P_i planar quartic curves

   (b) $C = X_1 \sqcup X_2$, X_i complete intersection curves of types $(2, 3)$

   (c) $C = Y \sqcup P$, Y a canonical curve of degree 8, P a planar curve of degree 4.

   (a) This case is impossible (consider the line $\langle P_1 \rangle \cap \langle P_2 \rangle$).
   (b) We have $X_i = Q_i \cap F_i$. Let Z be the quartic curve $Q_1 \cap Q_2$. Then $X_i \cap Z = F_i \cap Z$, i.e. X_i meets Z in 12 points. It follows that every quintic containing C meets Z in 24 points, hence such a quintic contains Z. Again this contradicts the fact that $\mathcal{I}_C(5)$ is globally generated.
   (c) This case too is impossible: every quintic containing C vanishes on P and on the points $\langle P \rangle \cap Y$, hence on $\langle P \rangle$.

   We conclude that if $d = 12$, C is irreducible.

The normalized bundle is $E(-3)$, since in any case $h^0(\mathcal{I}_C(2)) = 0$ (every smooth irreducible subcanonical curve on a quadric surface is a complete intersection), E is stable.

Now we turn to the existence part.

Lemma 1.6.23 *There exist indecomposable rank two vector bundles on $\mathbb{P}^3$ with Chern classes $c_1 = 5$ and $c_2 \in \{8, 10\}$ which are globally generated.*

Proof 1.6.24 *Let $R = \sqcup_{i=1}^{s} L_i$ be the union of s disjoint lines, $2 \leq s \leq 3$. We may perform a liaison $(s, 3)$ and link R to $K = \sqcup_{i=1}^{s} K_i$, the union of s disjoint conics. The exact sequence of liaison: $0 \to \mathcal{I}_U(4) \to \mathcal{I}_K(4) \to \omega_R(5-s) \to 0$ shows that $\mathcal{I}_K(4)$ is globally generated (n.b. $5 - s \geq 2$).*
Since $\omega_K(1) \simeq \mathcal{O}_K$ we have an exact sequence: $0 \to \mathcal{O} \to \mathcal{E}(2) \to \mathcal{I}_K(3) \to 0$, where $\mathcal{E}$ is a rank two vector bundle with Chern classes $c_1 = -1, c_2 = 2s - 2$. Twisting by $\mathcal{O}(1)$ we get: $0 \to \mathcal{O}(1) \to \mathcal{E}(3) \to \mathcal{I}_K(4) \to 0$ $(*)$. The Chern classes of $\mathcal{E}(3)$ are $c_1 = 5$, $c_2 = 2s + 4$ (i.e. $c_2 = 8, 10$). Since $\mathcal{I}_K(4)$ is globally generated, it follows from $(*)$ that $\mathcal{E}(3)$ too, is generated by global sections.

Remark 1.6.25

1. *If $\mathcal{E}$ is as in the proof of Lemma 1.6.23 a general section of $\mathcal{E}(3)$ vanishes along a smooth, irreducible (because $h^1(\mathcal{E}(-2)) = 0$) canonical curve, C, of genus $1 + c_2/2$ $(g = 5, 6)$ such that $\mathcal{I}_C(5)$ is globally generated. By construction these curves are not of maximal rank ($h^0(\mathcal{I}_C(3)) = 1$ if $g = 5$, $h^0(\mathcal{I}_C(4)) = 2$ if $g = 6$). As explained in [13] °4 this is a general fact: no canonical curve of genus $g, 5 \leq g \leq 6$ in $\mathbb{P}^3$ is of maximal rank. We don't know if this is still true for $g = 7$.*

2. *According to [13] the general canonical curve of genus 6 lies on a unique quartic surface.*

3. *The proof of 1.6.23 breaks down with four conics: $\mathcal{I}_K(4)$ is no longer globally generated, every quartic containing K vanishes along the lines L_i $(5 - s = 1)$. Observe also that four disjoint lines always have a quadrisecant and hence are an exception to the* normal generation conjecture*(the omogeneous ideal is not generated in degree three as it should be).*

Remark 1.6.26 *The case $(c_1, c_2) = (5, 12)$ remains open. It can be shown that if E exists, a general section of E is linked, by a complete intersections of two quintics, to a smooth, irreducible curve, X, of degree 13, genus 10 having $\omega_X(-1)$ as a base point free g_5^1. One can prove the existence of curves $X \subset \mathbb{P}^3$, smooth, irreducible, of degree 13, genus 10, with $\omega_X(-1)$ a base point free pencil and lying on* one *quintic* *surface. But we are unable to show the existence of such a curve with $h^0(\mathcal{I}_X(5)) \geq 2$ (which will imply the existence of our vector bundle).*

1.6.4 Globally generated rank two vector bundles on $\mathbb{P}^n$, $n \geq 4$ with $c_1 \leq 5$.

For $n \geq 4$ and $c_1 \leq 5$ there is no surprise:

PROPOSITION 1.6.27 *Let E be a globally generated rank two vector bundle on $\mathbb{P}^n$, $n \geq 4$. If $c_1(E) \leq 5$, then E splits.*

PROOF 1.6.28 *It is enough to treat the case $n = 4$. A general section of E vanishes along a smooth (irreducible) subcanonical surface, S: $0 \to \mathcal{O} \to E \to \mathcal{I}_S(c_1) \to 0$. By [10], if $c_1 \leq 4$, then S is a complete intersection and E splits. Assume now $c_1 = 5$. Consider the restriction of E to a general hyperplane H. If E doesn't split, by 1.6.21 we get that the normalized Chern classes of E are: $c_1 = -1$, $c_2 \in \{2, 4, 6\}$. By Schwarzenberger condition: $c_2(c_2 + 2) \equiv 0 \pmod{12}$. The only possibilities are $c_2 = 4$ or $c_2 = 6$. If $c_2 = 4$, since E is stable (because $E|H$ is, see 1.6.21), we have ([?]) that E is a Horrocks-Mumford bundle. But the Horrocks-Mumford bundle (with $c_1 = 5$) is not globally generated.*

The case $c_2 = 6$ is impossible: such a bundle would yield a smooth surface $S \subset \mathbb{P}^4$, of degree 12 with $\omega_S \simeq \mathcal{O}_S$, but the only smooth surface with $\omega_S \simeq \mathcal{O}_S$ in $\mathbb{P}^4$ is the abelian surface of degree 10 of Horrocks-Mumford.

REMARK 1.6.29
For $n > 4$ the results in [?] give stronger and stronger (as n increases) conditions for the existence of indecomposable rank two vector bundles generated by global sections.

Putting everything together we have:

THEOREM 1.6.30 *Let E be a rank two vector bundle on $\mathbb{P}^n$, $n \geq 3$, generated by global sections with Chern classes c_1, c_2, $c_1 \leq 5$.*

1. *If $n \geq 4$, then E is the direct sum of two line bundles*

2. *If $n = 3$ and E is indecomposable, then*

$$(c_1, c_2) \in S = \{((2,2),(4,5),(4,6),(4,7),(4,8),(5,8),(5,10),(5,12)\}.$$

If E exists there is an exact sequence: $0 \to \mathcal{O} \to E \to \mathcal{I}_C(c_1) \to 0$ $(*)$, where $C \subset \mathbb{P}^3$ is a smooth curve of degree c_2 with $\omega_C(4 - c_1) \simeq \mathcal{O}_C$. The curve C is irreducible*, except maybe if $(c_1, c_2) = (4,8)$: in this case C can be irreducible or the disjoint union of two smooth conics.*

3. *For every $(c_1, c_2) \in S$, $(c_1, c_2) \neq (5,12)$, there exists a rank two vector bundle on $\mathbb{P}^3$ with Chern classes (c_1, c_2) which is globally generated (and with an exact sequence as in (2)).*

Remark 1.6.31 *As already said the case $n = 3$, $(c_1, c_2) = (5,12)$ remains open: we are unable to prove or disprove the existence of such bundles.*

Chapter 2

On the normal bundle of projectively normal space curves.

2.1 Basic facts on a.C.M. curves.

Throughout this chapter a curve $C \subset \mathbb{P}^3$ is a one-dimensional, equidimensional, closed subscheme which is locally Cohen-Macaulay.

DEFINITION 2.1.1 *A curve $C \subset \mathbb{P}^3$ is arithmetically Cohen-Macaulay (a.C.M.) if $H^1_*(\mathcal{I}_C) = 0$.*
A curve is projectively normal (p.n) if it is a.C.M. and smooth (hence irreducible).

It turns out that $C \subset \mathbb{P}^3$ is a.C.M. if and only if its graded ideal, $I(C) := H^0_*(\mathcal{I}_C)$, has a length one minimal resolution:

$$0 \to L_1 \to L_0 \to \mathcal{I}_C \to 0 \quad (*)$$

($L_1 = \bigoplus_{j=1}^{k} \mathcal{O}(-b_j)$, $L_0 = \bigoplus_{i=1}^{k+1} \mathcal{O}(-a_i)$). If H is *any* plane such that $\Gamma := C \cap H$ is zero-dimensional, then $(*)$ yields a free resolution of $I(\Gamma)$:

$$0 \to L_1 \to L_0 \to \mathcal{I}_\Gamma \to 0 \quad (*H)$$

Now $(*H)$ determines the Hilbert function of $\Gamma \subset \mathbb{P}^3 w$. There are several ways to encode the Hilbert function of a zero-dimensional subscheme of $\mathbb{P}^2$, here we will use the *numerical character*, $\chi(\Gamma)$ (cf [13]). We recall that $\chi(\Gamma)$ is a sequence of integers $(n_0, ..., n_{s-1})$ such that:

1. $n_0 \geq n_1 \geq ... \geq n_{s-1} \geq s$
2. $s = min\{k \mid h^0(\mathcal{I}_\Gamma(k)) \neq 0\}$
3. $h^1(\mathcal{I}_\Gamma(n) = \sum_{i=0}^{s-1}([n_i - n - 1]_+ - [i - n - 1]_+)$
4. In particular $\deg(\Gamma) = \sum_{i=0}^{s-1}(n_i - i)$.

More generally:

DEFINITION 2.1.2 *A numerical character χ, of degree d, length s is a sequence of integers: $\chi = (n_0, ..., n_{s-1})$, $n_0 \geq ... \geq n_{s-1} \geq s$, with $\sum_{i=0}^{s-1}(n_i - i) = d$. The* genus *of χ is: $g(\chi) = \sum_{n\geq 1} h_\chi(n)$, where:*

$$h_\chi(n) = \sum_{i=0}^{s-1}([n_i - n - 1]_+ - [i - n - 1]_+).$$

The numerical character χ is said to be connected *if $n_i \leq n_{i+1} + 1$, $0 \leq i \leq s - 2$.*

DEFINITION 2.1.3 *If $C \subset \mathbb{P}^3$ is a curve (one dimensional, equidimensional, locally Cohen-Macaulay subscheme) its* numerical character, *$\chi(C)$, is the numerical character of its general plane section.*

If $C \subset \mathbb{P}^3$ is integral then the basic observation of Castelnuovo's theory is: $p_a(C) \leq g(\chi(C))$. From this we get another characterization of a.C.M. curves:

PROPOSITION 2.1.4 *Let $C \subset \mathbb{P}^3$ be an integral curve. Then: C is a.C.M. if and only if $p_a(C) = g(\chi(C))$.*

In fact it turns out that a.C.M. curves (and in particular projectively normal curves) are classified by their numerical character. More precisely we have:

THEOREM 2.1.5

1. *An a.C.M. curve* $C \subset \mathbb{P}^3$ *corresponds to a smooth point of* $Hilb(\mathbb{P}^3)$

2. *Two a.C.M. curves* $C, X \subset \mathbb{P}^3$ *are in the same irreducible component of the Hilbert scheme if and only* $\chi(C) = \chi(X)$

3. *The numerical character of an integral curve is connected. Moreover every* connected *numerical character is realized by a projectively normal curve*

4. *Let* H_χ *denote the irreducible component of* $Hilb(\mathbb{P}^3)$ *parametrizing a.C.M. curves with numerical character* χ*. The general curve of* H_χ *is smooth (hence irreducible) if and only if* χ *is connected.*

5. *Let* χ *be a connected character of length* s*, then the general curve of* H_χ *is projectively normal and lies on a smooth surface of degree* s*.*

PROOF 2.1.6 *For the convenience of the reader we include a proof of (5) (cf [7]), which will be used frequently in the sequel.*

We take over the proof of Thm. 2.5 in [13]. The argument is by induction. Assume C *is a p.n. curve with* $\chi(C) = (n_0, ..., n_{s-1})$ *and that* C *lies on,* S*, a smooth surface of degree* s*. (If* $s = 1$ *this is clearly satisfied). Following [13] we show the existence of a p.n. curve* C_1 *(resp.* C_2*) with* $\chi(C_1) = (n_0+2, n_0+1, n_1+1, ..., n_{s-1}+1)$ *(resp.* $\chi(C_2) = (n_0+1, n_0+1, n_1+1, ..., n_{s-1}+1)$*), lying on a smooth surface of degree* $s+1$*. Moreover, as in [13], the following condition is also part of the induction:*

$$\omega_C(-e(C)) \text{ has a section } \sigma, \text{with smooth zero} - \text{locus } (\sigma)_0 \ (+)$$

The curve C_1 *(resp.* C_2*) is constructed as follows: there exists a smooth surface,* F*, of degree* n_0+2 *(resp.* n_0+1*) containing* C *(observe that* $n_0 = e+3$*). If* $L = \mathcal{O}_F(C)$*, then* C_1 *(resp.* C_2*) is a general section of* $L(1)$*. This means that* C_i *is obtained by double linkage from* C*:* $U := F \cap G_a = X \cup C$ *and* $F \cap T_{a+1} = X \cup C_i$*. If we take* $G_a = S$*, it is enough to show that* X *lies on a smooth surface of degree* $s+1$*,* T*. The exact sequence of liaison:*

$$0 \to \mathcal{I}_U(s+1) \to \mathcal{I}_X(s+1) \xrightarrow{r} \omega_C(-e+i-1) \to 0$$

shows that $H^0(\mathcal{I}_X(s+1))$ *contains (at least)* $V = \langle H_i S, T' \rangle$*, where* $r(T') = (\sigma)_0$ *(see (+)). Hence* T' *is not a multiple of* S*. The base locus of* V *is* $B = T' \cap S$*. By*

Bertini's theorem the general surface in $H^0(\mathcal{I}_X(s+1))$ is smooth out of B. Since S is smooth, the general surface $T = HS + T' \in V$ is smooth with $r(T) = (\sigma)_0$. For general $T \in V$ the linked curve C_i will be smooth and will satisfy $(+)$ ($\omega_{C_i}(-e-3+i)$ has a section with zero-locus $(\sigma)_0$ cut out by S residually to $X \cap C_i$). We conclude as in [13].

For parts (1),...,(4), see [6], [13]. For the computation of the dimension of H_χ (that we won't need here), see [6].

It turns out that projectively normal curves are classified by the *connected* numerical character of length s, $s \geq 1$.

Let's see now how to compute the genus of a p.n. (projectively normal) curve by means of its character:

Lemma 2.1.7 *Let $\chi = (n_0, ..., n_{s-1})$ be a numerical character of degree d, length s. Then: $g(\chi) = g_- + g_+$, where:*

$$g_- = 1 + d(s-1) - \binom{s+2}{3}$$

$$g_+ = \sum_{i=0}^{s-1} \frac{(n_i - s - 1)(n_i - s)}{2}$$

Proof 2.1.8 *We may assume that χ is the numerical character of a zero-dimensional subscheme $Z \subset \mathbb{P}^2$ (see [7]). Then $g(\chi) = \sum_{n \geq 1} h^1(\mathcal{I}_Z(n))$. If $n \leq s-1$, then $h^1(\mathcal{I}_Z(n)) = d - h^0(\mathcal{O}_{\mathbb{P}^2}(n))$, so:*

$$\sum_{n=1}^{s-1} h^1(\mathcal{I}_Z(n)) = \sum_{n=1}^{s-1} (d - h^0(\mathcal{O}_{\mathbb{P}^2}(n))$$

$$= d(s-1) + 1 - h^0(\mathcal{O}(s-1)) =: g_-$$

For $n \geq s$ we have $h^1(\mathcal{I}_Z(n)) = \sum_{i=0}^{s-1} [n_i - n - 1]_+$ (where $[x]_+ = max\{0, x\}$). It follows that $\sum_{n \geq s} h^1(\mathcal{I}_Z(n)) = g_+$.

Finally we will also need the following:

LEMMA 2.1.9 *Let $C \subset \mathbb{P}^3$ be a p.n. curve of degree d with $\chi(C) = (n_0, ..., n_{s-1})$. Define $s(C) = min\{n \mid h^0(\mathcal{I}_C(n)) \neq 0\}$, $e(C) = max\{n \mid h^1(\mathcal{O}_C(n)) \neq 0\}$ and $\tau(C) = max\{n \mid h^1(\mathcal{I}_{C\cap H}(n)) \neq 0\}$ ($H \subset \mathbb{P}^3$ a general plane). Then: $s(C) = s$, $e(C) = n_0 - 3 = \tau(C) - 1$ and $\mathcal{I}_C(n)$ is generated by global sections for $n \geq n_0$.*

PROOF 2.1.10 *It is clear that $s(C) = s$ (because $H^1_*(\mathcal{I}_C) = 0$) and that $\tau = n_0 - 2$ (because $h^1(\mathcal{I}_{C\cap H}(n)) = \sum_{i=0}^{s-1}[n_i - n - 1]_+ - [i - n - 1]_+$). From the exact sequence:*

$$0 \to \mathcal{I}_C(n-1) \to \mathcal{I}_C(n) \to \mathcal{I}_{C\cap H}(n) \to 0$$

we easily get that $e = \tau - 1$. Finally the last statement follows from Castelnuovo-Mumford's lemma.

2.2 A conjecture on the normal bundle.

Let $C \subset \mathbb{P}^3$ be a smooth irreducible curve, of degree d, genus g. The normal bundle $\mathcal{N}_C$ is defined by the exact sequence:

$$0 \to T_C \to T\mathbb{P}^3 \mid C \to \mathcal{N}_C \to 0$$

From this it follows that:

$$det(\mathcal{N}_C) \simeq \omega_C(4);\ hence:\ \ \deg \mathcal{N}_C = 2g - 2 + 4d$$

We recall the following:

DEFINITION 2.2.1 *Let C be a smooth irreducible curve. A rank two vector bundle E on C is semi-stable (resp. stable) if for any sub-line bundle $L \subset E$, we have $\deg L \leq \mu(E)$ (resp. $\deg L < \mu(E)$), where $\mu(E) := \deg(E)/2$.*

This is the definition of Mumford-Takemoto of stability. In fact for the general definition one considers torsion-free subsheaves $\mathcal{L} \subset E$, but on a nonsingular curve a torsion-free sheaf is locally free (structure of modules over a P.I.D.). Also if the quotient, Q, of $0 \to L \to E$ has torsion (so that L is not a sub-bundle) then $Q \simeq M \oplus \mathcal{T}$ where M is locally free and where $\mathcal{T}$ is a torsion sheaf (again structure of modules over a P.I.D.). By composing the surjection $E \to Q \simeq M \oplus \mathcal{T} \to 0$, with $M \oplus \mathcal{T} \to M \to 0$, we get: $E \to M \to 0$, whose kernel, $\hat{L}$, is locally free (because torsion-free). In conclusion we have:

$$0 \to \hat{L} \to E \to M \to 0$$

and:

$$0 \to L \to \hat{L} \to \mathcal{T} \to 0$$

Since $\deg \hat{L} = \deg L + \deg \mathcal{T}$, we see that it is enough to test (semi)-stability with sub-line bundles.

Going back to our normal bundle we see that $\mathcal{N}_C$ is semi-stable (resp. stable) if and only if for every sub-line bundle $L \hookrightarrow \mathcal{N}_C$ we have: $\deg L \leq 2d + g - 1$ (resp. $\deg L < 2d + g - 1$).

Concerning projectively normal curves we have the following (see [15], Conj. 4.2):

Conjecture 2.2.2 (Hartshorne)
Let $C \subset \mathbb{P}^3$ be a sufficiently general projectively normal curve of degree d, genus g with $s(C) = s$. If

$$g \leq d(s-2) + 1 \quad (*s)$$

then $\mathcal{N}_C$ is semi-stable.

Let's explain where does the inequality of the conjecture come from. If C is sufficiently general it is reasonable to think that C will lie on a *smooth* surface, S, of degree s. And, indeed, this is true (see Thm. 2.1.5 (5)). The inclusion $C \subset S$ yields the exact sequence:

$$0 \to \mathcal{N}_{C,S} \to \mathcal{N}_C \to \mathcal{N}_S \mid C \to 0$$

using the adjunction formula and the fact that S is a divisor, this exact sequence reads like:

$$0 \to \omega_C(4-s) \to \mathcal{N}_C \to \mathcal{O}_C(s) \to 0$$

If $\mathcal{N}_C$ is semi-stable then $\omega_C(4-s)$ doesn't destabilize, i.e. $\deg \omega_C(4-s) \leq 2d-g+1$. Since $\deg \omega_C(4-s) = 2g-2+d(4-s)$, we get: $2g-2+d(4-s) \leq 2d+g-1$, which is precisely the inequality $(*s)$.

The conjecture then means that if $\mathcal{N}_C$ is not destabilized for evident numerical reasons, then it is semi-stable, if C is sufficiently general. Observe that a smooth degree $k > s$ surface containing C will yield a subbundle, $\omega_C(4-k) \hookrightarrow \mathcal{N}_C$, of smaller degree. The bet of the conjecture is that *singular* surfaces containing C won't give subbundles $L \hookrightarrow \mathcal{N}_C$ of too high degree (always if C is sufficiently general).

Indeed let $C \subset F$ where F is a surface of degree k such that $\dim(C \cap Sing(F)) = 0$. The inclusion $C \subset F$ corresponds to a section $\mathcal{O} \to \mathcal{I}_C(k)$. Twisting by $\mathcal{O}_C$ and using the isomorphism $\mathcal{I}_C \otimes \mathcal{O}_C \simeq \mathcal{I}_C/\mathcal{I}_C^2 \simeq \mathcal{N}_C^*$, we get a section, σ, of $\mathcal{N}_C^*(k) \simeq \mathcal{I}_C/\mathcal{I}_C^2(k)$ which vanishes (to some order) on $Sing(F) \cap C$. Let Δ be the divisorial part of $(\sigma)_0$, then dividing by (the equation of) Δ, we get a non-vanishing section: $\mathcal{O}_C \hookrightarrow \mathcal{N}_C^*(k) \otimes \mathcal{O}_C(-\Delta)$, hence an exact sequence:

$$0 \to \mathcal{O}_C \to \mathcal{N}_C^*(k) \otimes \mathcal{O}_C(-\Delta) \to M \to 0$$

where M is a line-bundle. By dualizing, twisting by $\mathcal{O}_C(k)$ and looking at determinants we get:

$$0 \to \omega_C(4-k) \otimes \mathcal{O}_C(\Delta) \to \mathcal{N}_C \to \mathcal{O}_C(k) \otimes \mathcal{O}_C(-\Delta) \to 0$$

The line bundle $L = \omega_C(4-k) \otimes \mathcal{O}_C(\Delta)$ is the "normal bundle of C in F", the divisor Δ has support on $Sing(F) \cap C$ and it (or its degree) is called the contribution of the singularities of F. If $Sing(F) \cap C = \{P_1, ..., P_r\}$ where any P_i is an ordinary double point of F, then $\Delta = P_1 + ... + P_r$ ([19]). For other singularities the behaviour of Δ is quite mysterious.

Of course if $C \subset Sing(F)$, the corresponding section of $\mathcal{N}_C^*(k)$ is identically zero and F doesn't define any subbundle of $\mathcal{N}_C$.

In conclusion surfaces containing C with isolated singularities along C contribute to subbundles of $\mathcal{N}_C$ of higher degrees than smooth surfaces. The bet in the conjecture is that surfaces containing C don't have too many singularities on C, if C is sufficiently general.

This being said let's notice that Conjecture 2.2.2 is not very precise since it envolves just the invariants d, g, s whereas p.n. curves are classified by numerical characters. Now it may well happen that two different characters (hence corresponding to different irreducible components of the Hilbert scheme) have the same invariants. For example

$$\chi_1 = (s+3, s+2, s+1, s+1, s+1, s+1, s^{s-6})$$

and

$$\chi_2 = (s+2, s+2, s+2, s+2, s+1, s, s^{s-6})$$

have the same d, g, s but $\chi_1 \neq \chi_2$.

For $s = 6$, $\chi_1 = (9, 8, 7, 7, 7, 7)$, $\chi_2 = (8, 8, 8, 8, 7, 6)$ both have $d = 30$, $g = 99$ and $s = 6$.

So we may rephrase Conjecture 2.2.2 in two different ways:

Conjecture 2.2.3 (Strong)
Let χ be a connected character of length s, degree d and genus g. Assume:

$$g \leq d(s-2) + 1 \quad (*s)$$

then the general curve of H_χ has a semi-stable normal bundle.

Or:

Conjecture 2.2.4 (Weak)
Let d, g, s be integers such that there exist p.n. curves with these invariants. If

$$g \leq d(s-2) + 1 \quad (*s)$$

then there exist a connected numerical character, χ, of length s, degree d, genus g such that the general curve of H_χ has a semi-stable normal bundle.

Remark 2.2.5 *Finally it is worth observing that if the inequality $(*s)$ is not satisfied, then every curve of degree d, genus g lying on a surface of degree s has a non semi-stable normal bundle.*

2.3 Numerical characters and the inequality $(*s)$.

Our first task here is to try to understand, for fixed s, which numerical characters are concerned with the inequality

$$g \leq d(s-2)+1 \;\; (*s)$$

of the conjecture.

First let's observe an equivalent formulation of $(*s)$:

LEMMA 2.3.1 *Let $\chi = (n_0, ..., n_{s-1})$ be a length s character of degree d, genus g. Then*

$$d + g_+ \leq \binom{s+2}{3} \Leftrightarrow g \leq 1 + d(s-2) \;\; (*s)$$

where $g_+ = \sum_{i=0}^{s-1} (n_i - s - 1)(n_i - s)/2.$

PROOF 2.3.2 *Follows from the fact that $g = g_- + g_+$ where $g_- = 1 + d(s-1) - \binom{s+2}{3}$ (see lemma 2.1.7).*

Now let's consider a complete intersection of type (s, s). The degree is s^2 and the genus is $g = 1 + d(s-2)$ (recall that for, C, a complete intersection (a, b), $\omega_C \simeq \mathcal{O}_C(a+b-4)$). So inequality $(*s)$ is an equality in this case.

By the way $\mathcal{N}_C \simeq 2.\mathcal{O}_C(s)$ is (properly) semi-stable.

The numerical character of a complete intersection (s, s) is:

$$\Phi = (2s-1, 2s-2, ..., s+1, s)$$

We claim that this is the *biggest* (for lexicographical order) character of length s satisfying $(*s)$.

LEMMA 2.3.3 *Let $\chi = (n_0, ..., n_{s-1})$ be a connected character of length s, degree d and genus g. If $g \leq d(s-2)+1$, then:*

1. *$\chi = \Phi$, or*

2. *$n_0 \leq 2s-2$.*

Proof 2.3.4 *First let's show that* $n_0 \leq 2s-1$. *If* $n_0 > 2s-1$, *then* $n_i > m_i, \forall i$ *(*$\Phi = (m_i)$*). In particular* $d_\chi > d_\Phi$ *and also* $g_+(\chi) \geq g_+(\Phi)$. *Set* $d_\chi = d_\Phi + r$. *Then* $g_-(\chi) = 1 + (d_\Phi + r)(s-1) - \binom{s+2}{3}$, *it follows that:* $g_-(\chi) - g_-(\Phi) = r(s-1)$. *Since* $g_+(\chi) \geq g_+(\Phi)$, *we get* $g(\chi) > g(\Phi) + r(s-1)$. *It follows that:*

$$g(\chi) > g(\Phi) + r(s-1) = d_\Phi(s-2) + 1 + r(s-1)$$

$$= (d_\Phi + r)(s-2) + 1 + r = d_\chi(s-2) + 1 + r$$

This shows $n_0 \leq 2s-1$.

Now if $n_0 = 2s-1$ *and* $\chi \neq \Phi$, *we have* $n_i \geq m_i, \forall i$ *and* $n_i > m_i$ *if* $i \geq i_0$, *for some* i_0, $2 \leq i_0 \leq s-1$ *and the same argument applies.*

Corollary 2.3.5 *Let* C *be a p.n. curve with invariants* (d, g, s). *If* $g \leq d(s-2)+1$, *then* $e(C) \leq 2s-4$, *with equality if and only if* C *is a complete intersection* (s, s).

Proof 2.3.6 *Since* $e = n_0 - 3$ *(see 2.1.9) the inequality follows from Lemma 2.3.3, moreover if* $e = 2s-4$, *then* $\chi(C) = \Phi$. *Since* $\deg C = \deg \Phi = s^2$ *and since* $h^0(\mathcal{I}_C(s)) = 2$, *it follows (*$C$ *is integral) that* C *is a complete intersection* (s, s).

We also have:

Lemma 2.3.7 *Let* $\chi = (n_0, ..., n_{s-1})$ *be a connected character of length* s, *degree* d, *genus* g.

1. *If* $n_{s-1} = s$, *then* χ *satisfies* $(*s)$ *(i.e.* $g \leq 1 + d(s-2)$*)*

2. *If* χ *satisfies* $(*s)$, *then:*

$$n_{s-1} \leq \frac{-3 + \sqrt{12s^2 - 3}}{6} + s < \frac{5s}{3}.$$

Proof 2.3.8 *(1) Let* $\Phi = (m_0, ..., m_{s-1})$. *If* $n_{s-1} = s$, *then* $m_i \geq n_i$, $\forall i$. *It follows that* $d \leq d_\Phi$ *and* $g_+ \leq g_+(\Phi)$. *Since* $d_\Phi + g_+(\Phi) = \binom{s+2}{3}$, *we conclude by Lemma 2.3.1.*

(2) Let $\chi_a = (s+a, ..., s+a)$. *We have* $\deg \chi_a = s(s+1)/2 + as$, $g_+(\chi_a) = sa(a-1)/2$. *A short computation shows that* $(*s)$ *is equivalent to:* $3a^2 + 3a - s^2 + 1 \leq 0$, *and*

this implies $a \leq \frac{-3+\sqrt{12s^2-3}}{6} =: s(a)$. *If* $n_{s-1} > s(a)+s$, *then* $\chi_{\overline{a}}$, *with* $\overline{a} = n_{s-1} - s$, *doesn't verify* $(*s)$. *Since* $n_i \geq \overline{a} + s = n_{s-1}$, χ *also doesn't verify* $(*s)$. *The last inequality is a simple computation.*

The "smallest" character of length s is $(s, ..., s)$ (the *flat* character), since $n_{s-1} = s$ it satisfies $(*s)$, then we have:

LEMMA 2.3.9 *(1) If* $s \geq 3$, *then* $\chi = ((s+1)^s)$ *satisfies* $(*s)$.
(2) Let $\chi = ((s+2)^a, (s+1)^b)$, $a + b = s$, *be a connected character of length* s. *If* $s \geq 5$, *then* χ *satisfies* $(*s)$.

PROOF 2.3.10 *(1) We have* $d = \frac{s(s+3)}{2}$ *and* $g_+ = 0$ *and we easily conclude.*

(2)This time $d = s(s+1)/2 + 2a + b$, $g_+ = a$. *It follows that* $(*s)$ *is equivalent to:* $3a + b \leq s(s^2-1)/6$. *Since* $3a + b \leq 3s$, *it is enough to check that:* $3s \leq s(s^2-1)/6$, *which holds for* $s \geq 5$.

REMARK 2.3.11 *In particular if* $s = 4$ *the character* $(6, 6, 6, 6)$ *doesn't satisfy* $(*s)$. *(More generally* $\chi = ((2s-2)^s)$ *never satisfies* $(*s)$ *for* $s \geq 4$.*) This shows that* <u>*not*</u> *every character* $\chi \leq \Phi$ *(lexicographical order) satisfies* $(*s)$.

Projectively normal curves with $h^1(\mathcal{O}_C(s-1)) = 0$ *are precisely the curves with* $\chi = ((s+1)^a, s^b)$*: for* $s \geq 3$, *they all satisfy* $(*s)$.

It seems quite tricky to determine exactly all the characters satisfying $(*s)$. For more on this topic see Section 2.6 and the Appendix.

2.4 Double structures and normal bundle.

Let $C \subset \mathbb{P}^3$ be a smooth irreducible curve. If L is a sub-bundle of $\mathcal{N}_C$:

$$0 \to L \to \mathcal{N}_C \to M \to 0$$

dualizing we get:

$$0 \to M^* \to \mathcal{N}_C^* \to L^* \to 0$$

Using the defining sequence of the conormal bundle we get a commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & M^* & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \to & \mathcal{I}_C^2 & \to & \mathcal{I}_C & \to & \mathcal{N}_C^* & \to & 0 \\
 & & \downarrow & & || & & \downarrow & & \\
0 & \to & \mathcal{I}_X & \to & \mathcal{I}_C & \to & L^* & \to & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & M^* & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

The ideal $\mathcal{I}_X$ defines a double structure on C, i.e. a locally complete intersection subscheme, with support C and degree $2d$ ($d = \deg C$). From the exact sequence:

$$0 \to \mathcal{I}_X \to \mathcal{I}_C \to L^* \to 0$$

we see that $L^* \simeq \mathcal{I}_{C,X}$ hence we have:

$$0 \to L^* \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

So $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_C) + \chi(L^*)$ and it follows that:

$$p_a(X) = l + 2g - 1$$

where $g = g(C)$, $l := \deg L$. So to bound the degree of L is equivalent to bound the (arithmetic) genus of X.

Although double structures seem awful, they are not really:

Proposition 2.4.1 *A double structure on a smooth irreducible curve has a connected character.*

Proof 2.4.2 *See [8], Thm. 10.*

This prop means that we can use Castelnuovo's method of studying the general plane section to bound the genus of double structures. For example:

THEOREM 2.4.3 *Let X be a double structure on a smooth irreducible curve C of genus g and degree $d \geq 3$. Let $t \geq 1$ be an integer such that $2d > t^2 + 1$ and $h^0(\mathcal{I}_C(t)) = 0$. Then $p_a(X) \leq G_{CM}(2d, t+1)$ with equality if and only if X is a.C.M.*

In particular if $G_{CM}(2d, t+1) - 2d - 3g + 2 < 0$ (resp. ≤ 0), then $\mathcal{N}_C$ is stable (resp. semi-stable).

PROOF 2.4.4 *See [8], Thm. 13.*

Here $G_{CM}(d, s) = max\{g(C) \mid \exists C$ an a.C.M. curve of degree d with $h^0(\mathcal{I}_C(s-1)) = 0\}$. We recall (see [13]) that:

- if $2d > s(s-1)$ and if $2d + r = st$, $0 \leq r < s$, then:

$$G_{CM}(2d, s) = 1 + d(s^2 - 4s + 2d)/s - r(s-r)(s-1)/2s$$

Moreover, under these assumptions, $G_{CM}(2d, s)$ is the genus of a curve linked to a plane curve of degree r by a complete intersection (s, t) (cf [13]).

COROLLARY 2.4.5 *Let C be a projectively normal curve with invariants (d, g, s). If*

$$G_{CM}(2d, s) - 2d - 3g + 2 \leq 0$$

then $\mathcal{N}_C$ is semi-stable (and if the inequality is strict then $\mathcal{N}_C$ is stable).

PROOF 2.4.6 *Apply 2.4.3 with $t = s - 1$ (observe that $d \geq \frac{s(s+1)}{2}$).*

This yields a purely numerical criteria for testing (semi)-stability. Before to start tricky computations, let's try to improve our numerical criteria.

PROPOSITION 2.4.7 *Let $\chi = (n_0, ..., n_{s-1})$ be a connected character of degree $d \geq 3$, genus g, length s. Assume (d, g, s) satisfies $(*s)$ (i.e. $g \leq d(s-2) + 1$). Set $t := n_{s-1} - 1$. If:*

1. $2d > t^2 + 1$
2. *and* $G_{CM}(2d, n_{s-1}) - 2d - 3g + 2 \leq 0$ *(resp. < 0)*

then the general curve of H_χ has a semi-stable (resp. stable) normal bundle.

PROOF 2.4.8 *Let C denote a sufficiently general curve in H_χ. By Thm. 2.1.5 we may assume that C lies on a* smooth *surface of degree s, S. Let $L \subset \mathcal{N}_C$ be a sub-line bundle. If $\deg L \leq \deg \omega_C(4-s)$, then L doesn't destabilize (because (d,g,s) satisfies $(*s)$). Assume $\deg L > \deg \omega_C(4-s)$ and let X denote the corresponding double structure. The curve X is not contained in S (because S being smooth, $\mathcal{N}_{C,S} \simeq \omega_C(4-s)$). The next minimal generator of $I(C)$ has degree n_{s-1}. Since any surface containing X also contains C, we conclude that $h^0(\mathcal{I}_X(t)) = 0$ ($t = n_{s-1}-1$).*

If $2d > t^2+1$, we claim that $p_a(X) \leq G_{CM}(2d, t+1)$.

Indeed let χ' denote the character of X and let σ denote its length. If $p_a(X) > G_{CM}(2d,t+1)$, then:

$$G_{CM}(2d,\sigma) > g(\chi') \geq p_a(X) > G_{CM}(2d,t+1)$$

and this implies $\sigma \leq t$. Since by assumption $2d > t^2+1 \geq \sigma^2+1$, it follows from [8] Lemma 12, that $h^0(\mathcal{I}_X(\sigma)) \neq 0$, in contradiction with: $h^0(\mathcal{I}_X(t)) = 0$ (note that case (ii) of Lemma 12 loc. cit. cannot occur because C is integral of degree ≥ 3).

Since $p_a(X) = \deg L + 2g - 1$, the inequality $G(2d,t+1) - 2d - 3g + 2 \leq 0$ (resp. < 0) implies $\deg L \leq \mu(\mathcal{N}_C)$ (resp. $<$), so $\mathcal{N}_C$ is semi-stable (resp. stable).

REMARK 2.4.9 *Since $G_{CM}(2d,k)$ is a decreasing function of k for $2d$ fixed, Prop. 2.4.7 improves Corollary 2.4.5 if $n_{s-1} > s$ i.e. if $h^0(\mathcal{I}_C(s)) = 1$.*

If we try to improve these results for curves with $h^0(\mathcal{I}_C(s)) > 1$ we are faced with the following difficulty: there could exist surfaces $S' \in H^0(\mathcal{I}_C(s))$ having singularities on C, to use our method we have to control the *contribution* of such singularities to the normal bundle. In general this is an almost impossible task. However if $h^0(\mathcal{I}_C(s)) = 2$, something can be done.

LEMMA 2.4.10 *Let $\chi = (n_0, ..., s+1, s)$ be a connected numerical character of length s, degree d, genus g. If $C \in H_\chi$ is sufficiently general, then* any *surface of degree s containing C has at most one ordinary double point as singularity (the general degree s surface being smooth).*

Proof 2.4.11 *If $C \in H_\chi$, C is linked by a complete intersection, Y, of type (s,s) to an a.C.M. curve Γ. We observe that $\chi' = \chi(\Gamma)$ is completely determined by χ. This follows from the fact that χ determines the minimal free resolution of C (up to repeated terms), by liaison this determines a free resolution of $\mathcal{I}_\Gamma$, hence the Hilbert function of Γ, i.e. its numerical character. More precisely if:*

$$0 \to \bigoplus \mathcal{O}(-b_j) \to \bigoplus_{a_i > s} \mathcal{O}(-a_i) \oplus 2.\mathcal{O}(-s) \to \mathcal{I}_C \to 0$$

is the minimal free resolution of $\mathcal{I}_C$, then, by mapping cone, we obtain:

$$0 \to \bigoplus \mathcal{O}(-(2s-a_i)) \to \bigoplus \mathcal{O}(-(2s-b_j)) \to \mathcal{I}_\Gamma \to 0$$

which is a (minimal in fact) free resolution of $\mathcal{I}_\Gamma$. In particular we see that $s(\Gamma) = 2s - b^+ = 2s - n_0 - 1 \leq s-2$ since $n_0 \geq s+1$.

This establishes a correspondence between H_χ and $H_{\chi'}$. The linked curve satisfies $h^1(\mathcal{O}_\Gamma(s-4)) = 0$. This follows from the exact sequence of liaison:

$$0 \to \mathcal{I}_Y(s) \to \mathcal{I}_C(s) \to \omega_\Gamma(s-4) \to 0.$$

It follows from Castelnuovo-Mumford's lemma that $\mathcal{I}_\Gamma(n)$ is globally generated if $n \geq s-2$.

Now take $\Gamma \in H_{\chi'}$ sufficiently general (hence smooth). It follows from [19] Prop. 4.8, Corollary 4.10, that if $\mathcal{P} \subset \mathbb{P}(H^0(\mathcal{I}_\Gamma(s))$ is a sufficiently general pencil, then any surface in $\mathcal{P}$ has at most an ordinary double point as singularity (the general one being smooth). (Observe that the proof of [19] loc. cit. applies for $s > s(\Gamma)$.) If C is linked to Γ by two general elements of $\mathcal{P}$, then C is smooth and $\mathbb{P}(H^0(\mathcal{I}_C(s)) = \mathcal{P}$: we are done.

Corollary 2.4.12 *Let $\chi = (n_0, ..., s+1, s)$ be a connected numerical character of length s, degree d, genus g. If $C \in H_\chi$ is sufficiently general and if:*

$$G_{CM}(2d, s+1) - 2d - 3g + 2 \leq 0 \ \ (*)$$

then $\mathcal{N}_C$ is semi-stable (and stable if the inequality is strict).

Proof 2.4.13 *As in the proof of Prop. 2.4.7 it is enough to show that* $h^0(\mathcal{I}_X(s)) = 0$ *where* X *is the double structure given by a quotient* $\mathcal{N}_C^* \to L^* \to 0$ *, where* $\deg L > \mu = 2d+g-1$. *Observe that since* $n_{s-1} = s$, *we already know that* $g \leq 1+d(s-2)$ $(*s)$ *(see Lemma 2.3.7).*

Assume S *is a degree* s *surface containing* X. *If* S *is smooth then* X *is* $2C$ *on* S *i.e.* $L \simeq \omega_C(4-s)$. *This is impossible since* $(*s)$ *says that* $\omega_C(4-s)$ *doesn't destabilize* $\mathcal{N}_C$. *If* S *is singular then, by 2.4.10* S *has at most one ordinary double point on* C *and* $N_{C,S} = \omega_C(4-s) \otimes \mathcal{O}_C(P)$. *It follows that* $L \simeq \omega_C(4-s) \otimes \mathcal{O}_C(P)$ *which has degree* $\deg L = 2g-2+d(4-s)+1$. *If* L *destabilizes:*

$$\deg L = 2g-2+d(4-s)+1 > \mu \geq 2g-2+d(4-s)$$

It follows that $\mu = 2g-2+d(4-s)$ *i.e.* $g = 1+d(s-2)$ *or, equivalently* $d+g_+ = \binom{s+2}{3}$ *(see Lemma 2.3.1). If* $\Phi = (2s-1, ..., s)$ *is the character of a complete intersection* (s,s), *then* $n_i \leq m_i$, $\forall i$ *(because* $n_{s-1} = s$*) and this implies* $d \leq d(\Phi)$, $g_+ \leq g_+(\Phi)$. *It follows that:*

$$\binom{s+2}{3} = d + g_+ \leq d(\Phi) + g_+(\Phi) = \binom{s+2}{3}$$

This implies $d = d(\Phi) = s^2$, $g_+ = g_+(\Phi)$. *From* $d = s^2$ *and* $h^0(\mathcal{I}_C(s)) \geq 2$ *we get that* C *is a complete intersection* (s,s), *hence* $\mathcal{N}_C$ *is semi-stable: a contradiction.*

This proves $h^0(\mathcal{I}_X(s)) = 0$ *if* $\deg L > \mu$. *Since* $\chi(X)$ *is connected,* $p_a(X) = \deg L + 2g - 1 \leq G_{CM}(2d, s+1)$ *and we conclude as usual.*

2.5 Some general results.

We now apply the results of the previous section.

Proposition 2.5.1 *Let* $\chi = ((s+1)^a, s^b)$, $0 \leq a \leq s$, $a+b = s$, $s \geq 3$. *Then the general curve of* H_χ *has a semi-stable normal bundle.*

Proof 2.5.2 *The invariants* d, g *of* χ *are given by:* $d = s(s+1)/2 + a$, $g = g_- = 1 + d(s-1) - \binom{s+2}{3}$. *In particular:* $2d = s(s+1) + 2a$. *The cases* $a \leq s/2$ *follow from [8] Remark 20 (v). So we may assume* $a > s/2$. *We will apply Cor. 2.4.5. We* *have* $2d+2(s-a)=s(s+3)$ *and* $0 \leq 2(s-a) < s$, *so* $r=2(s-a)$ *in the formula of* $G_{CM}(2d,s)$ *and:*

$$G_{CM}(2d,s) = 1 + \frac{d(2s^2-3s+2a}{s} - \frac{(s-a)(2a-s)(s-1)}{2s}$$

After some computations:

$$G_{CM}(2d,s) = 1 + d(2s-3) + s(s-1) + 2a(a+2) - 2as$$

and it follows (after some other computations) that:

$$G_{CM}(2d,s) - 2d - 3g + 2 = 2a^2 + a(2-3s) + s^2 - s \ \ (*)$$

So the inequality $G_{CM}(2d,s) - 2d - 3g + 2 \leq 0 \ \ (+)$ *is satisfied if:*

$$\frac{3s-2-\sqrt{s^2-4s+4}}{4} \leq a \leq \frac{3s-2+\sqrt{s^2-4s+4}}{4}$$

Since $\frac{s}{2} \geq \frac{3s-2-\sqrt{s^2-4s+4}}{4}$ *and* $s-1 \leq \frac{3s-2+\sqrt{s^2-4s+4}}{4}$, *we conclude that* $(+)$ *holds for* $s/2 \leq a \leq s-1$. *For* $a=s$ *(i.e.* $\chi=(s+1,...,s+1)$*), the inequality doesn't hold (in this case* $G_{CM}(2d,s)-2d-3g+2=s$*). However in this case we may use Prop. 2.4.7. Indeed* $n_{s-1} > s$ *and* χ *satisfies* $(*s)$ *(see Lemma 2.3.9), also* $2d > s^2+1$. *We have* $2d+2=(s+1)(s+2)$, *so* $r=2$ *and* $G := G_{CM}(2d,s+1)$ *is the genus of a curve linked to a conic in a complete intersection* $(s+1,s+2)$. *It follows that:* $G=(d-1)(2s-1)$ *and with some computations we get:* $G-2d-3g+2=-s<0$.

Remark 2.5.3

1. *The p.n. curves with character* $\chi=((s+1)^a,s^b)$ *are precisely the p.n. curves with* $s(C)=s$ *and* $h^1(\mathcal{O}_C(s-1))=0$.

2. *The proof shows that if* $a \neq s$, *then* <u>*any*</u> *p.n. curve with character* χ *has a semi-stable normal bundle.*

3. *If* $a=s$ *the genericity assumption is necessary. Indeed if* $\chi=(4,4,4)$, *then* $d=9$ *and* $g=9$. *If the cubic containing* C *is smooth, then* $\mathcal{N}_C$ *is stable ([17]), but if the cubic is singular* $\mathcal{N}_C$ *can be not semi-stable ([9], Remarque 11).*

The characters considered so far are the "smallest" ones (for lexicographical order), now we turn to the greatest ones (under the assumption $n_{s-1} = s$).

LEMMA 2.5.4 *Let C be a projectively normal curve with $\chi(C) = (n_0, ..., n_{s-1})$. The following are equivalent:*

1. *$n_0 = 2s - 2$ and $n_{s-1} = s$*

2. *C is linked to a plane curve of degree r, $1 \leq r \leq s-1$ by a complete intersection (s, s)*

If the conditions (1), (2) are satisfied then $\chi = (2s-2, ..., (2s-1-r)^2, ..., s)$ for some r, $1 \leq r \leq s-1$. In particular the invariants (d, g, s) of C satisfy $(*s)$.

PROOF 2.5.5 *(1) $\Rightarrow$ (2): If $n_0 = 2s-2$ and $n_{s-1} = s$, then χ contains $(2s-2, ..., s)$: these are $s-1$ terms, so only one is missing. It follows that $\chi = (2s-2, ..., (2s-1-r)^2, ..., s)$ with $1 \leq r \leq s-1$. The degree is $s^2 - r$. Since $n_{s-1} = s$, $h^0(\mathcal{I}_C(s)) \geq 2$ and we can make a liaison (s, s). The exact sequence of liaison:*

$$0 \to \mathcal{I}_U(1) \to \mathcal{I}_P(1) \to \omega_C(5-2s) \to 0 \quad (*)$$

shows that $h^0(\mathcal{I}_P(1)) \neq 0$ (indeed $n_0 = 2s-2$ implies $e(C) = 2s-5$).

(2) $\Rightarrow$ (1): If C is linked to a plane curve P of degree r, $1 \leq r \leq s-1$ by (s, s), then the exact sequence of liaison $(*)$ shows that $s(C) = s$, $e(C) = 2s-5$. So χ has length s and $n_0 = 2s-2$. Since $h^0(\mathcal{I}_C(s)) \geq 2$, $n_{s-1} = s$.

From the first part of the proof it follows that χ is as claimed. Since $n_{s-1} = s$, the inequality $g \leq 1 + d(s-2)$ $(*s)$ is satisfied (cf Lemma 2.3.7).

Instead of running into specific computations in order to apply 2.4.5 or 2.4.12, we will set a general framework.

Let's set $2d + r = k\sigma$, $0 \leq r < \sigma$. Then, if $2d > \sigma(\sigma-1)$

$$G_{CM}(2d, \sigma) = 1 + \frac{k\sigma^2 + \sigma(-4k + k^2 - 2r) - 2kr + r^2 + 5r}{2}$$

Now with $g = 1 + d(s-1) - h^0(\mathcal{O}(s-1)) + g_+$, we have:

$$G_{CM}(2d, \sigma) - 2d - 3g + 2 =$$

$$\frac{1}{2}[k\sigma^2 + \sigma(-3k + k^2 - 2r) - 2kr + r^2 + 4r - 3ks\sigma + 3rs + s(s+1)(s+2) - 6g_+]$$

DEFINITION 2.5.6 *We define* $C(\sigma) := G_{CM}(2d, \sigma) - 2d - 3g + 2$

LEMMA 2.5.7

1. *We have*

$$C(s) = [s^3 + s^2(-2k + 3) + s(k^2 - 3k + \alpha + 2) - 2\alpha k + 4\alpha + \alpha^2 - 6g_+]/2$$

(where $2d + \alpha = ks$, $0 \leq \alpha < s$*)*

2. *We have:*

$$C(s+1) = \frac{1}{2}[s(s-k)^2 + (k^2 - 4ks + 3s^2) + (s\beta + \beta^2 - 2k\beta) + 2(s + \beta - k) - 6g_+]$$

(where $2d + \beta = (s + 1)k$, $0 \leq \beta < s + 1$*)*

PROOF 2.5.8 *Just replace* σ *with* s *(resp.* $s + 1$*) in the formula above.*

PROPOSITION 2.5.9 *Let* C *be a p.n. curve linked to a plane curve of degree* r, $1 \leq r \leq s - 1$, *by a complete intersection* (s, s), $s \geq 3$.

1. *If* $r = 1$, $\mathcal{N}_C$ *is semi-stable*
2. *If* $r = s - 1$, $\mathcal{N}_C$ *is stable*
3. *If* $1 < r < s - 1$ *and if* C *is sufficiently general, then* $\mathcal{N}_C$ *is stable.*

PROOF 2.5.10 *(1) If* $r = 1$ *then* C *is linked to a line by a complete intersection* (s, s), *so* $d = s^2 - 1$ *and by liaison* $g = (s^2 - 2)(s - 2)$. *We have* $2d + 2 = 2s^2$, *so* $G_{CM}(2d, s)$ *is the genus of a curve linked to a conic by a complete intersection* $(2s, s)$, *it follows that* $G_{CM}(2d, s) = (s^2 - 2)(3s - 4)$. *A short computation shows that:*

$$G_{CM}(2d, s) - 2d - 3g + 2 = 0.$$

We conclude with 2.4.5.

(2) As above if $r = s - 1$, *we get* $d = s^2 - s + 1$, $g = (s - 2)(s - 1)(2s - 1)/2$ *and* $G_{CM}(2d, s) = (s - 1)(3s^2 - 6s + 4)$. *It follows that:*

$$G_{CM}(2d, s) - 2d - 3g + 2 = -(s - 1)(s - 2)/2 < 0.$$

We conclude with 2.4.5.

(3) Here we will use Cor. 2.4.12, hence $C(s+1)$*. We have to compute* g_+*. For a complete intersection* (s,s)*, we have* $G_+ + s^2 = \binom{s+2}{3}$*. We have* $g_+ = G_+ - \frac{r(2s-3-r)}{2}$*, indeed* $\chi(C)$ *is obtained from the character of a complete intersection* (s,s) *by replacing* $m_0 = 2s-1$ *by* $2s-1-r$*. Finally* $6g_+ = s(s-1)(s-2) - 3r(2s-3-r)$*. If* $r \leq s/2+1$ *we write:* $2d + (2r-2) = (2s-2)(s+1)$ *and apply 2.5.7 with* $\beta = 2r-2$*,* $k = 2s-2$*. We find that:* $C(s+1) \leq 0 \Leftrightarrow r^2 - 5r - 2s^2 + 6s \leq 0$*, which is satisfied (with strict inequality) for* $s \geq 3$*. We conclude that, for general* C*,* $\mathcal{N}_C$ *is stable.*

If $r > s/2+1$*, we write:* $2d + (2r-s-3) = (2s-3)(s+1)$ *and apply 2.5.7 with* $\beta = 2r-s-3$*,* $k = 2s-3$*. We find that:* $C(s+1) \leq 0 \Leftrightarrow r^2 - r(4s+5) + 12s \leq 0$*. Since* $(s+3)/2 > \frac{4s+5-\sqrt{16s^2-8s+25}}{2}$ *and* $s-2 < \frac{4s+5+\sqrt{16s^2-8s+25}}{2}$ *for* $s \geq 2$*, we get* $C(s+1) < 0$*.*

2.6 Small values of s and conclusion.

In this section we investigate the conjecture for small values of s. For this we will make use of a computer. The strategy is as follows:

- For given s we list (by decreasing lexicographical order), all connected characters between $(2s-2, ..., 2s-2)$ (see Lemma 2.3.3) and $(s, ..., s)$. The algorithm for this is quite simple: given $\chi = (n_0, ..., n_{s-1})$ we look for the greatest i such that $n_i = n_{i-1} > s$. If we find it we set: $\chi_- = (m_j)$ with $m_j = n_j$, $0 \leq j < i$, $m_j = n_i - 1$ for $j \geq i$. If we don't find such a i, we set $m_j = n_1, \forall j$. Then $\chi_- < \chi$ and if $\chi' < \chi$, then $\chi_- \geq \chi'$. We make the program run while $n_0 > s$. The program will list all the characters up to the flat one (s^s).

  Here is the code in Java language:

  ```
  public int[] lexico(int[] T){
  int t=0;
  int s = T.length;
  for(int i=0; i < s-1;i++){
  ```

```
if(t==0){
if(T[s-1-i-1]==T[s-1-i] && T[s-1-i] > s){
for(int j=s-1-i;j <s;j++){
T[j] = T[s-1-i-1]-1;}
t=1;
}
}
}
if(t==0){
for(int i=0; i<s; i++){
T[i]=T[1];}
}
return T;
}
```

- For any listed character we compute its degree and genus and $(*s) := g-d(s-2)-1$, if $(*s) > 0$, the character is not in the range of the conjecture and is no longer considered.

- For the remaining characters, χ, we compute $C(s) = G_{CM}(2d, s) - 2d - 3g + 2$. If $C(s) \leq 0$ (resp. < 0), then *any* p.n. curve with character χ has a semi-stable (resp. stable) normal bundle (cf 2.4.5).

- For the remaining characters (those with $(*) \leq 0$ and $C(s) > 0$, we compute $Lft := 2d-(n_{s-1}-1)^2-1$. If $Lft > 0$, the lifting condition of 2.4.7 is satisfied and we compute $C(n_{s-1}) = G_{CM}(2d, n_{s-1}) - 2d - 3g + 2$. If $C(n_{s-1}) \leq 0$ (resp. < 0) then the general curve of H_χ has a semi-stable (resp. stable) normal bundle (2.4.7).

- The remaining characters with $n_{s-1} = s$ have: $(*) \leq 0$, $C(s) > 0$. If $n_{s-2} = s+1$, we compute $C(s+1)$. If $C(s+1) \leq 0$, the general curve of H_χ has a semi-stable normal bundle (2.4.12).

- If no character remains, the strong conjecture (2.2.3) is proved for s, otherwise we get the list of the character of length s that the methods developed so far are unable to handle.

In this way we get:

PROPOSITION 2.6.1 *The strong conjecture 2.2.3 holds for $s \leq 6$.*

The strong conjecture holds for $s = 7$ but for one case: $\chi = (9,9,8,8,8,7,7)$, $d = 35$, $g = 129$.

PROOF 2.6.2 *See the complete listings in the appendix.*

Summarizing our results are:

THEOREM 2.6.3 *The strong conjecture 2.2.3 holds in the following cases:*

1. *if $n_i \leq s+1$, $\forall i$ (curves with $h^1(\mathcal{O}_C(s-1)) = 0$)*
2. *$n_0 = 2s-2$ and $n_{s-1} = s$ (curves linked to a plane curve by a complete intersection (s,s))*
3. *$s \leq 6$*
4. *$s = 7$ and $\chi \neq (9,9,8,8,8,7,7)$*

Actually our results are more precise since in some cases we are able to state them for <u>every</u> curve (and also sometime we get stability).

For $s > 7$ we also get some results but it is difficult to make a precise statement. The advantage (or disadvantage, depending on the point of view) of our approach is to reduce the problem to a purely numerical check.

So the first case the methods developed so far can't handle is a case with $h^0(\mathcal{I}_C(s)) = 3$. However inspections with the computer show that, as s get bigger, also other cases "at the top" of the list (i.e. with $n_{s-1} > s$) will not be covered.

For $s = 8$, the conjecture is proved except for two cases:

$\chi_1 = (11,10,10,10,10,9,8,8)$ ($d = 48g_{=}217g+ = 7, g = 224$) and:

$\chi_2 = (10, 10, 9, 9, 9, 9, 8, 8)$ $(d = 44, g_= 189, g+ = 2, g = 191)$. These are still cases with $h^0(\mathcal{I}_C(s)) > 2$.

For $s = 11$ there are plenty of cases which are not proved and among them two cases with $n_{s-1} > s$:

$\chi_1 = (17, 17, 17, 16, 16, 16, 16, 15, 14, 13, 12)$, $(d = 114, g_= 855, g+ = 95, g = 950)$ and:

$\chi_2 = (17, 16, 16, 15, 15, 15, 15, 15, 14, 13, 12)$,$(d = 108, g_= 795, g+ = 69, g = 864)$.

The situation gets worst as s increases...

We have refrained from using the "good old time" method of degeneration to construct examples in the cases not proved.

2.7 Appendix

The complete listing for $s = 3$:

If C(s)<=0, any p.n. has a semi-stable character. If Lft >0, the lifting cdt is verified and if C(n_{s-1})<=0, the general curve has a semi-stable normal bundle. If $n_{s-1} = s, n_{s-2} = s + 1$ and C(s+1)<=0, the general curve has a semi stable normal bundle.

(4,4,4) [degree = 9 g- = 9 g+ = 0 genus = 9] (C(s) =3 is > 0, but C(n_{s-1})= -3 the general curve has a semi-stable n.bdle)

(4,4,3) [degree = 8 g- = 7 g+ = 0 genus = 7] (C(s)= 0 every curve has a semi-stable n.bdle)

(4,3,3) [degree = 7 g- = 5 g+ = 0 genus = 5] (C(s)= -1 every curve has a semi-stable n.bdle)

(3,3,3) [degree = 6 g- = 3 g+ = 0 genus = 3] (C(s)= 0 every curve has a semi-stable n.bdle)

The complete listing for $s = 4$:

If C(s)<=0, any p.n. has a semi-stable character. If Lft >0, the lifting cdt is verified and if C($n_s - 1$)<=0, the general curve has a semi-stable normal bundle. If $n_{s-1} = s, n_{s-2} = s + 1$ and C(s+1)<=0, the general curve has a semi stable normal bundle.

(6,6,6,6) [degree = 18 g- = 35 g+ = 4 genus = 39] ((*s)= 2 every curve has a NON semi-stable normal bdle)

(6,6,6,5) [degree = 17 g- = 32 g+ = 3 genus = 35] (C(s) =7 is > 0, but C(n_{s-1})= -5 the general curve has a semi-stable n.bdle)

(6,6,5,5) [degree = 16 g- = 29 g+ = 2 genus = 31] (C(s) =6 is > 0, but C(n_{s-1})= -6 the general curve has a semi-stable n.bdle)

(6,6,5,4) [degree = 15 g- = 26 g+ = 2 genus = 28] (C(s)= 0 every curve has a semi-stable n.bdle)

(6,5,5,5) [degree = 15 g- = 26 g+ = 1 genus = 27] (C(s) =3 is > 0, but C(n_{s-1})= -3 the general curve has a semi-stable n.bdle)

(6,5,5,4) [degree = 14 g- = 23 g+ = 1 genus = 24] (dim(I(s))=2, C(s+1)= -7 the general curve has a semi-stable n.bdle)

(6,5,4,4) [degree = 13 g- = 20 g+ = 1 genus = 21] (C(s)= -3 every curve has a semi-stable n.bdle)

(5,5,5,5) [degree = 14 g- = 23 g+ = 0 genus = 23] (C(s) =4 is > 0, but C(n_{s-1})= -4 the general curve has a semi-stable n.bdle)

(5,5,5,4) [degree = 13 g- = 20 g+ = 0 genus = 20] (C(s)= 0 every curve has a semi-stable n.bdle)

(5,5,4,4) [degree = 12 g- = 17 g+ = 0 genus = 17] (C(s)= 0 every curve has a semi-stable n.bdle)

(5,4,4,4) [degree = 11 g- = 14 g+ = 0 genus = 14] (C(s)= -2 every curve has a semi-stable n.bdle)

(4,4,4,4) [degree = 10 g- = 11 g+ = 0 genus = 11] (C(s)= 0 every curve has a semi-stable n.bdle)

The complete listing for $s = 5$:

If C(s)<=0, any p.n. has a semi-stable character. If Lft >0, the lifting cdt is verified and if C(n_{s-1})<=0, the general curve has a semi-stable normal bundle. If $n_{s-1} = s, n_{s-2} = s + 1$ and C(s+1)<=0, the general curve has a semi stable normal bundle.

(8,8,8,8,8) [degree = 30 g- = 86 g+ = 15 genus = 101] ((*s)= 10 every curve has a NON semi-stable normal bdle)

(8,8,8,8,7) [degree = 29 g- = 82 g+ = 13 genus = 95] ((*s)= 7 every curve has a NON semi-stable normal bdle)

(8,8,8,7,7) [degree = 28 g- = 78 g+ = 11 genus = 89] ((*s)= 4 every curve has a NON semi-stable normal bdle)

(8,8,8,7,6) [degree = 27 g- = 74 g+ = 10 genus = 84] ((*s)= 2 every curve has a NON semi-stable normal bdle)

(8,8,7,7,7) [degree = 27 g- = 74 g+ = 9 genus = 83] ((*s)= 1 every curve has a NON semi-stable normal bdle)

(8,8,7,7,6) [degree = 26 g- = 70 g+ = 8 genus = 78] (C(s) =11 is > 0, but C(n_{s-1})= -9 the general curve has a semi-stable n.bdle)

(8,8,7,6,6) [degree = 25 g- = 66 g+ = 7 genus = 73] (C(s) =9 is > 0, but C(n_{s-1})= -11 the general curve has a semi-stable n.bdle)

(8,8,7,6,5) [degree = 24 g- = 62 g+ = 7 genus = 69] (C(s)= 0 every curve has a semi-stable n.bdle)

(8,7,7,7,7) [degree = 26 g- = 70 g+ = 7 genus = 77] (C(s) =14 is > 0, but C(n_{s-1})= -14 the general curve has a semi-stable n.bdle)

(8,7,7,7,6) [degree = 25 g- = 66 g+ = 6 genus = 72] (C(s) =12 is > 0, but C(n_{s-1})= -8 the general curve has a semi-stable n.bdle)

(8,7,7,6,6) [degree = 24 g- = 62 g+ = 5 genus = 67] (C(s) =6 is > 0, but C(n_{s-1})= -6 the general curve has a semi-stable n.bdle)

(8,7,7,6,5) [degree = 23 g- = 58 g+ = 5 genus = 63] (dim(I(s))=2, C(s+1)= -13 the general curve has a semi-stable n.bdle)

(8,7,6,6,6) [degree = 23 g- = 58 g+ = 4 genus = 62] (C(s) =4 is > 0, but C(n_{s-1})= -10 the general curve has a semi-stable n.bdle)

(8,7,6,6,5) [degree = 22 g- = 54 g+ = 4 genus = 58] (C(s)= -1 every curve has a semi-stable n.bdle)

(8,7,6,5,5) [degree = 21 g- = 50 g+ = 4 genus = 54] (C(s)= -6 every curve has a semi-stable n.bdle)

(7,7,7,7,7) [degree = 25 g- = 66 g+ = 5 genus = 71] (C(s) =15 is > 0, but C(n_{s-1})= -9 the general curve has a semi-stable n.bdle)

(7,7,7,7,6) [degree = 24 g- = 62 g+ = 4 genus = 66] (C(s) =9 is > 0, but C(n_{s-1})= -3 the general curve has a semi-stable n.bdle)

(7,7,7,6,6) [degree = 23 g- = 58 g+ = 3 genus = 61] (C(s) =7 is > 0, but C(n_{s-1})= -7 the general curve has a semi-stable n.bdle)

(7,7,7,6,5) [degree = 22 g- = 54 g+ = 3 genus = 57] (dim(I(s))=2, C(s+1)= -10 the general curve has a semi-stable n.bdle)

(7,7,6,6,6) [degree = 22 g- = 54 g+ = 2 genus = 56] (C(s) =5 is > 0, but C(n_{s-1})= -7 the general curve has a semi-stable n.bdle)

(7,7,6,6,5) [degree = 21 g- = 50 g+ = 2 genus = 52] (C(s)= 0 every curve has a semi-stable n.bdle)

(7,7,6,5,5) [degree = 20 g- = 46 g+ = 2 genus = 48] (C(s)= -1 every curve has a semi-stable n.bdle)

(7,6,6,6,6) [degree = 21 g- = 50 g+ = 1 genus = 51] (C(s) =3 is > 0, but C(n_{s-1})= -3 the general curve has a semi-stable n.bdle)

(7,6,6,6,5) [degree = 20 g- = 46 g+ = 1 genus = 47] (dim(I(s))=2, C(s+1)= -8 the general curve has a semi-stable n.bdle)

(7,6,6,5,5) [degree = 19 g- = 42 g+ = 1 genus = 43] (C(s)= -3 every curve has a semi-stable n.bdle)

(7,6,5,5,5) [degree = 18 g- = 38 g+ = 1 genus = 39] (C(s)= -4 every curve has a semi-stable n.bdle)

(6,6,6,6,6) [degree = 20 g- = 46 g+ = 0 genus = 46] (C(s) =5 is > 0, but C(n_{s-1})= -5 the general curve has a semi-stable n.bdle)

(6,6,6,6,5) [degree = 19 g- = 42 g+ = 0 genus = 42] (C(s)= 0 every curve has a semi-stable n.bdle)

(6,6,6,5,5) [degree = 18 g- = 38 g+ = 0 genus = 38] (C(s)= -1 every curve has a semi-stable n.bdle)

(6,6,5,5,5) [degree = 17 g- = 34 g+ = 0 genus = 34] (C(s)= -2 every curve has a semi-stable n.bdle)

(6,5,5,5,5) [degree = 16 g- = 30 g+ = 0 genus = 30] (C(s)= -3 every curve has a semi-stable n.bdle)

(5,5,5,5,5) [degree = 15 g- = 26 g+ = 0 genus = 26] (C(s)= 0 every curve has a semi-stable n.bdle)

The complete listing for $s = 6$:

If C(s)<=0, any p.n. has a semi-stable character. If Lft >0, the lifting cdt is verified and if C(n_{s-1})<=0, the general curve has a semi-stable normal bundle. If $n_{s-1} = s, n_{s-2} = s + 1$ and C(s+1)<=0, the general curve has a semi stable normal bundle.

(10,10,10,10,10,10) [degree = 45 g- = 170 g+ = 36 genus = 206] ((*s)= 25 every curve has a NON semi-stable normal bdle)

(10,10,10,10,10,9) [degree = 44 g- = 165 g+ = 33 genus = 198] ((*s)= 21 every curve has a NON semi-stable normal bdle)

(10,10,10,10,9,9) [degree = 43 g- = 160 g+ = 30 genus = 190] ((*s)= 17 every curve has a NON semi-stable normal bdle)

(10,10,10,10,9,8) [degree = 42 g- = 155 g+ = 28 genus = 183] ((*s)= 14 every curve has a NON semi-stable normal bdle)

(10,10,10,9,9,9) [degree = 42 g- = 155 g+ = 27 genus = 182] ((*s)= 13 every curve has a NON semi-stable normal bdle)

(10,10,10,9,9,8) [degree = 41 g- = 150 g+ = 25 genus = 175] ((*s)= 10 every curve has a NON semi-stable normal bdle)

(10,10,10,9,8,8) [degree = 40 g- = 145 g+ = 23 genus = 168] ((*s)= 7 every curve has a NON semi-stable normal bdle)

(10,10,10,9,8,7) [degree = 39 g- = 140 g+ = 22 genus = 162] ((*s)= 5 every curve has a NON semi-stable normal bdle)

(10,10,9,9,9,9) [degree = 41 g- = 150 g+ = 24 genus = 174] ((*s)= 9 every curve has a NON semi-stable normal bdle)

(10,10,9,9,9,8) [degree = 40 g- = 145 g+ = 22 genus = 167] ((*s)= 6 every curve has a NON semi-stable normal bdle)

(10,10,9,9,8,8) [degree = 39 g- = 140 g+ = 20 genus = 160] ((*s)= 3 every curve has a NON semi-stable normal bdle)

(10,10,9,9,8,7) [degree = 38 g- = 135 g+ = 19 genus = 154] ((*s)= 1 every curve has a NON semi-stable normal bdle)

(10,10,9,8,8,8) [degree = 38 g- = 135 g+ = 18 genus = 153] (C(s) =22 is > 0, but C(n_{s-1})= -26 the general curve has a semi-stable n.bdle)

(10,10,9,8,8,7) [degree = 37 g- = 130 g+ = 17 genus = 147] (C(s) =15 is > 0, but C(n_{s-1})= -15 the general curve has a semi-stable n.bdle)

(10,10,9,8,7,7) [degree = 36 g- = 125 g+ = 16 genus = 141] (C(s) =12 is > 0, but C(n_{s-1})= -18 the general curve has a semi-stable n.bdle)

(10,10,9,8,7,6) [degree = 35 g- = 120 g+ = 16 genus = 136] (C(s)= 0 every curve has a semi-stable n.bdle)

(10,9,9,9,9,9) [degree = 40 g- = 145 g+ = 21 genus = 166] ((*s)= 5 every curve has a NON semi-stable normal bdle)

(10,9,9,9,9,8) [degree = 39 g- = 140 g+ = 19 genus = 159] ((*s)= 2 every curve has a NON semi-stable normal bdle)

(10,9,9,9,8,8) [degree = 38 g- = 135 g+ = 17 genus = 152] (C(s) =25 is > 0, but C(n_{s-1})= -23 the general curve has a semi-stable n.bdle)

(10,9,9,9,8,7) [degree = 37 g- = 130 g+ = 16 genus = 146] (C(s) =18 is > 0, but C(n_{s-1})= -12 the general curve has a semi-stable n.bdle)

(10,9,9,8,8,8) [degree = 37 g- = 130 g+ = 15 genus = 145] (C(s) =21 is > 0, but C(n_{s-1})= -21 the general curve has a semi-stable n.bdle)

(10,9,9,8,8,7) [degree = 36 g- = 125 g+ = 14 genus = 139] (C(s) =18 is > 0, but C(n_{s-1})= -12 the general curve has a semi-stable n.bdle)

(10,9,9,8,7,7) [degree = 35 g- = 120 g+ = 13 genus = 133] (C(s) =9 is > 0, but C(n_{s-1})= -11 the general curve has a semi-stable n.bdle)

(10,9,9,8,7,6) [degree = 34 g- = 115 g+ = 13 genus = 128] (dim(I(s))=2, C(s+1)= -21 the general curve has a semi-stable n.bdle)

(10,9,8,8,8,8) [degree = 36 g- = 125 g+ = 13 genus = 138] (C(s) =21 is > 0, but C(n_{s-1})= -15 the general curve has a semi-stable n.bdle)

(10,9,8,8,8,7) [degree = 35 g- = 120 g+ = 12 genus = 132] (C(s) =12 is > 0, but C(n_{s-1})= -8 the general curve has a semi-stable n.bdle)

(10,9,8,8,7,7) [degree = 34 g- = 115 g+ = 11 genus = 126] (C(s) =7 is > 0, but C(n_{s-1})= -15 the general curve has a semi-stable n.bdle)

(10,9,8,8,7,6) [degree = 33 g- = 110 g+ = 11 genus = 121] (dim(I(s))=2, C(s+1)= -21 the general curve has a semi-stable n.bdle)

(10,9,8,7,7,7) [degree = 33 g- = 110 g+ = 10 genus = 120] (C(s) =6 is > 0, but C(n_{s-1})= -18 the general curve has a semi-stable n.bdle)

(10,9,8,7,7,6) [degree = 32 g- = 105 g+ = 10 genus = 115] (C(s)= -4 every curve has a semi-stable n.bdle)

(10,9,8,7,6,6) [degree = 31 g- = 100 g+ = 10 genus = 110] (C(s)= -10 every curve has a semi-stable n.bdle)

(9,9,9,9,9,9) [degree = 39 g- = 140 g+ = 18 genus = 158] ((*s)= 1 every curve has a NON semi-stable normal bdle)

(9,9,9,9,9,8) [degree = 38 g- = 135 g+ = 16 genus = 151] (C(s) =28 is > 0, but C(n_{s-1})= -20 the general curve has a semi-stable n.bdle)

(9,9,9,9,8,8) [degree = 37 g- = 130 g+ = 14 genus = 144] (C(s) =24 is > 0, but C(n_{s-1})= -18 the general curve has a semi-stable n.bdle)

(9,9,9,9,8,7) [degree = 36 g- = 125 g+ = 13 genus = 138] (C(s) =21 is > 0, but C(n_{s-1})= -9 the general curve has a semi-stable n.bdle)

(9,9,9,8,8,8) [degree = 36 g- = 125 g+ = 12 genus = 137] (C(s) =24 is > 0, but C(n_{s-1})= -12 the general curve has a semi-stable n.bdle)

(9,9,9,8,8,7) [degree = 35 g- = 120 g+ = 11 genus = 131] (C(s) =15 is > 0, but C(n_{s-1})= -5 the general curve has a semi-stable n.bdle)

(9,9,9,8,7,7) [degree = 34 g- = 115 g+ = 10 genus = 125] (C(s) =10 is > 0, but C(n_{s-1})= -12 the general curve has a semi-stable n.bdle)

(9,9,9,8,7,6) [degree = 33 g- = 110 g+ = 10 genus = 120] (dim(I(s))=2, C(s+1)= -18 the general curve has a semi-stable n.bdle)

(9,9,8,8,8,8) [degree = 35 g- = 120 g+ = 10 genus = 130] (C(s) =18 is > 0, but C(n_{s-1})= -16 the general curve has a semi-stable n.bdle)

(9,9,8,8,8,7) [degree = 34 g- = 115 g+ = 9 genus = 124] (C(s) =13 is > 0, but C(n_{s-1})= -9 the general curve has a semi-stable n.bdle)

(9,9,8,8,7,7) [degree = 33 g- = 110 g+ = 8 genus = 118] (C(s) =12 is > 0, but C(n_{s-1})= -12 the general curve has a semi-stable n.bdle)

(9,9,8,8,7,6) [degree = 32 g- = 105 g+ = 8 genus = 113] (dim(I(s))=2, C(s+1)= -14 the general curve has a semi-stable n.bdle)

(9,9,8,7,7,7) [degree = 32 g- = 105 g+ = 7 genus = 112] (C(s) =5 is > 0, but C(n_{s-1})= -11 the general curve has a semi-stable n.bdle)

(9,9,8,7,7,6) [degree = 31 g- = 100 g+ = 7 genus = 107] (C(s)= -1 every curve has a semi-stable n.bdle)

(9,9,8,7,6,6) [degree = 30 g- = 95 g+ = 7 genus = 102] (C(s)= -3 every curve has a semi-stable n.bdle)

(9,8,8,8,8,8) [degree = 34 g- = 115 g+ = 8 genus = 123] (C(s) =16 is > 0, but C(n_{s-1})= -16 the general curve has a semi-stable n.bdle)

(9,8,8,8,8,7) [degree = 33 g- = 110 g+ = 7 genus = 117] (C(s) =15 is > 0, but C(n_{s-1})= -9 the general curve has a semi-stable n.bdle)

(9,8,8,8,7,7) [degree = 32 g- = 105 g+ = 6 genus = 111] (C(s) =8 is > 0, but C(n_{s-1})= -8 the general curve has a semi-stable n.bdle)

(9,8,8,8,7,6) [degree = 31 g- = 100 g+ = 6 genus = 106] (dim(I(s))=2, C(s+1)= -12 the general curve has a semi-stable n.bdle)

(9,8,8,7,7,7) [degree = 31 g- = 100 g+ = 5 genus = 105] (C(s) =5 is > 0, but C(n_{s-1})= -9 the general curve has a semi-stable n.bdle)

(9,8,8,7,7,6) [degree = 30 g- = 95 g+ = 5 genus = 100] (dim(I(s))=2, C(s+1)= -15 the general curve has a semi-stable n.bdle)

(9,8,8,7,6,6) [degree = 29 g- = 90 g+ = 5 genus = 95] (C(s)= -5 every curve has a semi-stable n.bdle)

(9,8,7,7,7,7) [degree = 30 g- = 95 g+ = 4 genus = 99] (C(s) =6 is > 0, but C(n_{s-1})= -12 the general curve has a semi-stable n.bdle)

(9,8,7,7,7,6) [degree = 29 g- = 90 g+ = 4 genus = 94] (C(s)= -2 every curve has a semi-stable n.bdle)

(9,8,7,7,6,6) [degree = 28 g- = 85 g+ = 4 genus = 89] (C(s)= -6 every curve has a semi-stable n.bdle)

(9,8,7,6,6,6) [degree = 27 g- = 80 g+ = 4 genus = 84] (C(s)= -6 every curve has a semi-stable n.bdle)

(8,8,8,8,8,8) [degree = 33 g- = 110 g+ = 6 genus = 116] (C(s) =18 is > 0, but C(n_{s-1})= -12 the general curve has a semi-stable n.bdle)

(8,8,8,8,8,7) [degree = 32 g- = 105 g+ = 5 genus = 110] (C(s) =11 is > 0, but C(n_{s-1})= -5 the general curve has a semi-stable n.bdle)

(8,8,8,8,7,7) [degree = 31 g- = 100 g+ = 4 genus = 104] (C(s) =8 is > 0, but C(n_{s-1})= -6 the general curve has a semi-stable n.bdle)

(8,8,8,8,7,6) [degree = 30 g- = 95 g+ = 4 genus = 99] (dim(I(s))=2, C(s+1)= -12 the general curve has a semi-stable n.bdle)

(8,8,8,7,7,7) [degree = 30 g- = 95 g+ = 3 genus = 98] (C(s) =9 is > 0, but C(n_{s-1})= -9 the general curve has a semi-stable n.bdle)

(8,8,8,7,7,6) [degree = 29 g- = 90 g+ = 3 genus = 93] (dim(I(s))=2, C(s+1)= -11 the general curve has a semi-stable n.bdle)

(8,8,8,7,6,6) [degree = 28 g- = 85 g+ = 3 genus = 88] (C(s)= -3 every curve has a semi-stable n.bdle)

(8,8,7,7,7,7) [degree = 29 g- = 90 g+ = 2 genus = 92] (C(s) =4 is > 0, but C(n_{s-1})= -8 the general curve has a semi-stable n.bdle)

(8,8,7,7,7,6) [degree = 28 g- = 85 g+ = 2 genus = 87] (C(s)= 0 every curve has a semi-stable n.bdle)

(8,8,7,7,6,6) [degree = 27 g- = 80 g+ = 2 genus = 82] (C(s)= 0 every curve has a semi-stable n.bdle)

(8,8,7,6,6,6) [degree = 26 g- = 75 g+ = 2 genus = 77] (C(s)= -6 every curve has a semi-stable n.bdle)

(8,7,7,7,7,7) [degree = 28 g- = 85 g+ = 1 genus = 86] (C(s) =3 is > 0, but C(n_{s-1})= -3 the general curve has a semi-stable n.bdle)

(8,7,7,7,7,6) [degree = 27 g- = 80 g+ = 1 genus = 81] (dim(I(s))=2, C(s+1)= -9 the general curve has a semi-stable n.bdle)

(8,7,7,7,6,6) [degree = 26 g- = 75 g+ = 1 genus = 76] (C(s)= -3 every curve has a semi-stable n.bdle)

(8,7,7,6,6,6) [degree = 25 g- = 70 g+ = 1 genus = 71] (C(s)= -5 every curve has a semi-stable n.bdle)

(8,7,6,6,6,6) [degree = 24 g- = 65 g+ = 1 genus = 66] (C(s)= -3 every curve has a semi-stable n.bdle)

(7,7,7,7,7,7) [degree = 27 g- = 80 g+ = 0 genus = 80] (C(s) =6 is > 0, but C(n_{s-1})= -6 the general curve has a semi-stable n.bdle)

(7,7,7,7,7,6) [degree = 26 g- = 75 g+ = 0 genus = 75] (C(s)= 0 every curve has a semi-stable n.bdle)

(7,7,7,7,6,6) [degree = 25 g- = 70 g+ = 0 genus = 70] (C(s)= -2 every curve has a semi-stable n.bdle)

(7,7,7,6,6,6) [degree = 24 g- = 65 g+ = 0 genus = 65] (C(s)= 0 every curve has a semi-stable n.bdle)

(7,7,6,6,6,6) [degree = 23 g- = 60 g+ = 0 genus = 60] (C(s)= -4 every curve has a semi-stable n.bdle)

(7,6,6,6,6,6) [degree = 22 g- = 55 g+ = 0 genus = 55] (C(s)= -4 every curve has a semi-stable n.bdle)

(6,6,6,6,6,6) [degree = 21 g- = 50 g+ = 0 genus = 50] (C(s)= 0 every curve has a semi-stable n.bdle)

The complete listing for $s = 7$:

If C(s)<=0, any p.n. has a semi-stable character. If Lft >0, the lifting cdt is verified and if C(n_{s-1})<=0, the general curve has a semi-stable normal bundle. If $n_{s-1} = s, n_{s-2} = s + 1$ and C(s+1)<=0, the general curve has a semi stable normal bundle.

(12,12,12,12,12,12,12) [degree = 63 g- = 295 g+ = 70 genus = 365] ((*s)= 49 every curve has a NON semi-stable normal bdle)

(12,12,12,12,12,12,11) [degree = 62 g- = 289 g+ = 66 genus = 355] ((*s)= 44 every curve has a NON semi-stable normal bdle)

(12,12,12,12,12,11,11) [degree = 61 g- = 283 g+ = 62 genus = 345] ((*s)= 39 every curve has a NON semi-stable normal bdle)

(12,12,12,12,12,11,10) [degree = 60 g- = 277 g+ = 59 genus = 336] ((*s)= 35 every curve has a NON semi-stable normal bdle)

(12,12,12,12,11,11,11) [degree = 60 g- = 277 g+ = 58 genus = 335] ((*s)= 34 every curve has a NON semi-stable normal bdle)

(12,12,12,12,11,11,10) [degree = 59 g- = 271 g+ = 55 genus = 326] ((*s)= 30 every curve has a NON semi-stable normal bdle)

(12,12,12,12,11,10,10) [degree = 58 g- = 265 g+ = 52 genus = 317] ((*s)= 26 every curve has a NON semi-stable normal bdle)

(12,12,12,12,11,10,9) [degree = 57 g- = 259 g+ = 50 genus = 309] ((*s)= 23 every curve has a NON semi-stable normal bdle)

(12,12,12,11,11,11,11) [degree = 59 g- = 271 g+ = 54 genus = 325] ((*s)= 29 every curve has a NON semi-stable normal bdle)

(12,12,12,11,11,11,10) [degree = 58 g- = 265 g+ = 51 genus = 316] ((*s)= 25 every curve has a NON semi-stable normal bdle)

(12,12,12,11,11,10,10) [degree = 57 g- = 259 g+ = 48 genus = 307] ((*s)= 21 every curve has a NON semi-stable normal bdle)

(12,12,12,11,11,10,9) [degree = 56 g- = 253 g+ = 46 genus = 299] ((*s)= 18 every curve has a NON semi-stable normal bdle)

(12,12,12,11,10,10,10) [degree = 56 g- = 253 g+ = 45 genus = 298] ((*s)= 17 every curve has a NON semi-stable normal bdle)

(12,12,12,11,10,10,9) [degree = 55 g- = 247 g+ = 43 genus = 290] ((*s)= 14 every curve has a NON semi-stable normal bdle)

(12,12,12,11,10,9,9) [degree = 54 g- = 241 g+ = 41 genus = 282] ((*s)= 11 every curve has a NON semi-stable normal bdle)

(12,12,12,11,10,9,8) [degree = 53 g- = 235 g+ = 40 genus = 275] ((*s)= 9 every curve has a NON semi-stable normal bdle)

(12,12,11,11,11,11,11) [degree = 58 g- = 265 g+ = 50 genus = 315] ((*s)= 24 every curve has a NON semi-stable normal bdle)

(12,12,11,11,11,11,10) [degree = 57 g- = 259 g+ = 47 genus = 306] ((*s)= 20 every curve has a NON semi-stable normal bdle)

(12,12,11,11,11,10,10) [degree = 56 g- = 253 g+ = 44 genus = 297] ((*s)= 16 every curve has a NON semi-stable normal bdle)

(12,12,11,11,11,10,9) [degree = 55 g- = 247 g+ = 42 genus = 289] ((*s)= 13 every curve has a NON semi-stable normal bdle)

(12,12,11,11,10,10,10) [degree = 55 g- = 247 g+ = 41 genus = 288] ((*s)= 12 every curve has a NON semi-stable normal bdle)

(12,12,11,11,10,10,9) [degree = 54 g- = 241 g+ = 39 genus = 280] ((*s)= 9 every curve has a NON semi-stable normal bdle)

(12,12,11,11,10,9,9) [degree = 53 g- = 235 g+ = 37 genus = 272] ((*s)= 6 every curve has a NON semi-stable normal bdle)

(12,12,11,11,10,9,8) [degree = 52 g- = 229 g+ = 36 genus = 265] ((*s)= 4 every curve has a NON semi-stable normal bdle)

(12,12,11,10,10,10,10) [degree = 54 g- = 241 g+ = 38 genus = 279] ((*s)= 8 every curve has a NON semi-stable normal bdle)

(12,12,11,10,10,10,9) [degree = 53 g- = 235 g+ = 36 genus = 271] ((*s)= 5 every curve has a NON semi-stable normal bdle)

(12,12,11,10,10,9,9) [degree = 52 g- = 229 g+ = 34 genus = 263] ((*s)= 2 every curve has a NON semi-stable normal bdle)

(12,12,11,10,10,9,8) [degree = 51 g- = 223 g+ = 33 genus = 256] (C(s) =24 is > 0, but C(n_{s-1})= -18 the general curve has a semi-stable n.bdle)

(12,12,11,10,9,9,9) [degree = 51 g- = 223 g+ = 32 genus = 255] (C(s) =27 is > 0, but C(n_{s-1})= -39 the general curve has a semi-stable n.bdle)

(12,12,11,10,9,9,8) [degree = 50 g- = 217 g+ = 31 genus = 248] (C(s) =19 is > 0, but C(n_{s-1})= -23 the general curve has a semi-stable n.bdle)

(12,12,11,10,9,8,8) [degree = 49 g- = 211 g+ = 30 genus = 241] (C(s) =15 is > 0, but C(n_{s-1})= -27 the general curve has a semi-stable n.bdle)

(12,12,11,10,9,8,7) [degree = 48 g- = 205 g+ = 30 genus = 235] (C(s)= 0 every curve has a semi-stable n.bdle)

(12,11,11,11,11,11,11) [degree = 57 g- = 259 g+ = 46 genus = 305] ((*s)= 19 every curve has a NON semi-stable normal bdle)

(12,11,11,11,11,11,10) [degree = 56 g- = 253 g+ = 43 genus = 296] ((*s)= 15 every curve has a NON semi-stable normal bdle)

(12,11,11,11,11,10,10) [degree = 55 g- = 247 g+ = 40 genus = 287] ((*s)= 11 every curve has a NON semi-stable normal bdle)

(12,11,11,11,11,10,9) [degree = 54 g- = 241 g+ = 38 genus = 279] ((*s)= 8 every curve has a NON semi-stable normal bdle)

(12,11,11,11,10,10,10) [degree = 54 g- = 241 g+ = 37 genus = 278] ((*s)= 7 every curve has a NON semi-stable normal bdle)

(12,11,11,11,10,10,9) [degree = 53 g- = 235 g+ = 35 genus = 270] ((*s)= 4 every curve has a NON semi-stable normal bdle)

(12,11,11,11,10,9,9) [degree = 52 g- = 229 g+ = 33 genus = 262] ((*s)= 1 every curve has a NON semi-stable normal bdle)

(12,11,11,11,10,9,8) [degree = 51 g- = 223 g+ = 32 genus = 255] (C(s) =27 is > 0, but C(n_{s-1})= -15 the general curve has a semi-stable n.bdle)

(12,11,11,10,10,10,10) [degree = 53 g- = 235 g+ = 34 genus = 269] ((*s)= 3 every curve has a NON semi-stable normal bdle)

(12,11,11,10,10,10,9) [degree = 52 g- = 229 g+ = 32 genus = 261] (C(s) =42 is > 0, but C(n_{s-1})= -32 the general curve has a semi-stable n.bdle)

(12,11,11,10,10,9,9) [degree = 51 g- = 223 g+ = 30 genus = 253] (C(s) =33 is > 0, but C(n_{s-1})= -33 the general curve has a semi-stable n.bdle)

(12,11,11,10,10,9,8) [degree = 50 g- = 217 g+ = 29 genus = 246] (C(s) =25 is > 0, but C(n_{s-1})= -17 the general curve has a semi-stable n.bdle)

(12,11,11,10,9,9,9) [degree = 50 g- = 217 g+ = 28 genus = 245] (C(s) =28 is > 0, but C(n_{s-1})= -30 the general curve has a semi-stable n.bdle)

(12,11,11,10,9,9,8) [degree = 49 g- = 211 g+ = 27 genus = 238] (C(s) =24 is > 0, but C(n_{s-1})= -18 the general curve has a semi-stable n.bdle)

(12,11,11,10,9,8,8) [degree = 48 g- = 205 g+ = 26 genus = 231] (C(s) =12 is > 0, but C(n_{s-1})= -18 the general curve has a semi-stable n.bdle)

(12,11,11,10,9,8,7) [degree = 47 g- = 199 g+ = 26 genus = 225] (dim(I(s))=2, C(s+1)= -31 the general curve has a semi-stable n.bdle)

(12,11,10,10,10,10,10) [degree = 52 g- = 229 g+ = 31 genus = 260] (C(s) =45 is > 0, but C(n_{s-1})= -39 the general curve has a semi-stable n.bdle)

(12,11,10,10,10,10,9) [degree = 51 g- = 223 g+ = 29 genus = 252] (C(s) =36 is > 0, but C(n_{s-1})= -30 the general curve has a semi-stable n.bdle)

(12,11,10,10,10,9,9) [degree = 50 g- = 217 g+ = 27 genus = 244] (C(s) =31 is > 0, but C(n_{s-1})= -27 the general curve has a semi-stable n.bdle)

(12,11,10,10,10,9,8) [degree = 49 g- = 211 g+ = 26 genus = 237] (C(s) =27 is > 0, but C(n_{s-1})= -15 the general curve has a semi-stable n.bdle)

(12,11,10,10,9,9,9) [degree = 49 g- = 211 g+ = 25 genus = 236] (C(s) =30 is > 0, but C(n_{s-1})= -28 the general curve has a semi-stable n.bdle)

(12,11,10,10,9,9,8) [degree = 48 g- = 205 g+ = 24 genus = 229] (C(s) =18 is > 0, but C(n_{s-1})= -12 the general curve has a semi-stable n.bdle)

(12,11,10,10,9,8,8) [degree = 47 g- = 199 g+ = 23 genus = 222] (C(s) =10 is > 0, but C(n_{s-1})= -22 the general curve has a semi-stable n.bdle)

(12,11,10,10,9,8,7) [degree = 46 g- = 193 g+ = 23 genus = 216] (dim(I(s))=2, C(s+1)= -31 the general curve has a semi-stable n.bdle)

(12,11,10,9,9,9,9) [degree = 48 g- = 205 g+ = 23 genus = 228] (C(s) =21 is > 0, but C(n_{s-1})= -33 the general curve has a semi-stable n.bdle)

(12,11,10,9,9,9,8) [degree = 47 g- = 199 g+ = 22 genus = 221] (C(s) =13 is > 0, but C(n_{s-1})= -19 the general curve has a semi-stable n.bdle)

(12,11,10,9,9,8,8) [degree = 46 g- = 193 g+ = 21 genus = 214] (C(s) =9 is > 0, but C(n_{s-1})= -25 the general curve has a semi-stable n.bdle)

(12,11,10,9,9,8,7) [degree = 45 g- = 187 g+ = 21 genus = 208] (C(s)= 0 every curve has a semi-stable n.bdle)

(12,11,10,9,8,8,8) [degree = 45 g- = 187 g+ = 20 genus = 207] (C(s) =3 is > 0, but C(n_{s-1})= -27 the general curve has a semi-stable n.bdle)

(12,11,10,9,8,8,7) [degree = 44 g- = 181 g+ = 20 genus = 201] (C(s)= -8 every curve has a semi-stable n.bdle)

(12,11,10,9,8,7,7) [degree = 43 g- = 175 g+ = 20 genus = 195] (C(s)= -15 every curve has a semi-stable n.bdle)

(11,11,11,11,11,11,11) [degree = 56 g- = 253 g+ = 42 genus = 295] ((*s)= 14 every curve has a NON semi-stable normal bdle)

(11,11,11,11,11,11,10) [degree = 55 g- = 247 g+ = 39 genus = 286] ((*s)= 10 every curve has a NON semi-stable normal bdle)

(11,11,11,11,11,10,10) [degree = 54 g- = 241 g+ = 36 genus = 277] ((*s)= 6 every curve has a NON semi-stable normal bdle)

(11,11,11,11,11,10,9) [degree = 53 g- = 235 g+ = 34 genus = 269] ((*s)= 3 every curve has a NON semi-stable normal bdle)

(11,11,11,11,10,10,10) [degree = 53 g- = 235 g+ = 33 genus = 268] ((*s)= 2 every curve has a NON semi-stable normal bdle)

(11,11,11,11,10,10,9) [degree = 52 g- = 229 g+ = 31 genus = 260] (C(s) =45 is > 0, but C(n_{s-1})= -29 the general curve has a semi-stable n.bdle)

(11,11,11,11,10,9,9) [degree = 51 g- = 223 g+ = 29 genus = 252] (C(s) =36 is > 0, but C(n_{s-1})= -30 the general curve has a semi-stable n.bdle)

(11,11,11,11,10,9,8) [degree = 50 g- = 217 g+ = 28 genus = 245] (C(s) =28 is > 0, but C(n_{s-1})= -14 the general curve has a semi-stable n.bdle)

(11,11,11,10,10,10,10) [degree = 52 g- = 229 g+ = 30 genus = 259] (C(s) =48 is > 0, but C(n_{s-1})= -36 the general curve has a semi-stable n.bdle)

(11,11,11,10,10,10,9) [degree = 51 g- = 223 g+ = 28 genus = 251] (C(s) =39 is > 0, but C(n_{s-1})= -27 the general curve has a semi-stable n.bdle)

(11,11,11,10,10,9,9) [degree = 50 g- = 217 g+ = 26 genus = 243] (C(s) =34 is > 0, but C(n_{s-1})= -24 the general curve has a semi-stable n.bdle)

(11,11,11,10,10,9,8) [degree = 49 g- = 211 g+ = 25 genus = 236] (C(s) =30 is > 0, but C(n_{s-1})= -12 the general curve has a semi-stable n.bdle)

(11,11,11,10,9,9,9) [degree = 49 g- = 211 g+ = 24 genus = 235] (C(s) =33 is > 0, but C(n_{s-1})= -25 the general curve has a semi-stable n.bdle)

(11,11,11,10,9,9,8) [degree = 48 g- = 205 g+ = 23 genus = 228] (C(s) =21 is > 0, but C(n_{s-1})= -9 the general curve has a semi-stable n.bdle)

(11,11,11,10,9,8,8) [degree = 47 g- = 199 g+ = 22 genus = 221] (C(s) =13 is > 0, but C(n_{s-1})= -19 the general curve has a semi-stable n.bdle)

(11,11,11,10,9,8,7) [degree = 46 g- = 193 g+ = 22 genus = 215] (dim(I(s))=2, C(s+1)= -28 the general curve has a semi-stable n.bdle)

(11,11,10,10,10,10,10) [degree = 51 g- = 223 g+ = 27 genus = 250] (C(s) =42 is > 0, but C(n_{s-1})= -30 the general curve has a semi-stable n.bdle)

(11,11,10,10,10,10,9) [degree = 50 g- = 217 g+ = 25 genus = 242] (C(s) =37 is > 0, but C(n_{s-1})= -21 the general curve has a semi-stable n.bdle)

(11,11,10,10,10,9,9) [degree = 49 g- = 211 g+ = 23 genus = 234] (C(s) =36 is > 0, but C(n_{s-1})= -22 the general curve has a semi-stable n.bdle)

(11,11,10,10,10,9,8) [degree = 48 g- = 205 g+ = 22 genus = 227] (C(s) =24 is > 0, but C(n_{s-1})= -6 the general curve has a semi-stable n.bdle)

(11,11,10,10,9,9,9) [degree = 48 g- = 205 g+ = 21 genus = 226] (C(s) =27 is > 0, but C(n_{s-1})= -27 the general curve has a semi-stable n.bdle)

(11,11,10,10,9,9,8) [degree = 47 g- = 199 g+ = 20 genus = 219] (C(s) =19 is > 0, but C(n_{s-1})= -13 the general curve has a semi-stable n.bdle)

(11,11,10,10,9,8,8) [degree = 46 g- = 193 g+ = 19 genus = 212] (C(s) =15 is > 0, but C(n_{s-1})= -19 the general curve has a semi-stable n.bdle)

(11,11,10,10,9,8,7) [degree = 45 g- = 187 g+ = 19 genus = 206] (dim(I(s))=2, C(s+1)= -24 the general curve has a semi-stable n.bdle)

(11,11,10,9,9,9,9) [degree = 47 g- = 199 g+ = 19 genus = 218] (C(s) =22 is > 0, but C(n_{s-1})= -28 the general curve has a semi-stable n.bdle)

(11,11,10,9,9,9,8) [degree = 46 g- = 193 g+ = 18 genus = 211] (C(s) =18 is > 0, but C(n_{s-1})= -16 the general curve has a semi-stable n.bdle)

(11,11,10,9,9,8,8) [degree = 45 g- = 187 g+ = 17 genus = 204] (C(s) =12 is > 0, but C(n_{s-1})= -18 the general curve has a semi-stable n.bdle)

(11,11,10,9,9,8,7) [degree = 44 g- = 181 g+ = 17 genus = 198] (dim(I(s))=2, C(s+1)= -19 the general curve has a semi-stable n.bdle)

(11,11,10,9,8,8,8) [degree = 44 g- = 181 g+ = 16 genus = 197] (C(s) =4 is > 0, but C(n_{s-1})= -16 the general curve has a semi-stable n.bdle)

(11,11,10,9,8,8,7) [degree = 43 g- = 175 g+ = 16 genus = 191] (C(s)= -3 every curve has a semi-stable n.bdle)

(11,11,10,9,8,7,7) [degree = 42 g- = 169 g+ = 16 genus = 185] (C(s)= -6 every curve has a semi-stable n.bdle)

(11,10,10,10,10,10,10) [degree = 50 g- = 217 g+ = 24 genus = 241] (C(s) =40 is > 0, but C(n_{s-1})= -20 the general curve has a semi-stable n.bdle)

(11,10,10,10,10,10,9) [degree = 49 g- = 211 g+ = 22 genus = 233] (C(s) =39 is > 0, but C(n_{s-1})= -19 the general curve has a semi-stable n.bdle)

(11,10,10,10,10,9,9) [degree = 48 g- = 205 g+ = 20 genus = 225] (C(s) =30 is > 0, but C(n_{s-1})= -24 the general curve has a semi-stable n.bdle)

(11,10,10,10,10,9,8) [degree = 47 g- = 199 g+ = 19 genus = 218] (C(s) =22 is > 0, but C(n_{s-1})= -10 the general curve has a semi-stable n.bdle)

(11,10,10,10,9,9,9) [degree = 47 g- = 199 g+ = 18 genus = 217] (C(s) =25 is > 0, but C(n_{s-1})= -25 the general curve has a semi-stable n.bdle)

(11,10,10,10,9,9,8) [degree = 46 g- = 193 g+ = 17 genus = 210] (C(s) =21 is > 0, but C(n_{s-1})= -13 the general curve has a semi-stable n.bdle)

(11,10,10,10,9,8,8) [degree = 45 g- = 187 g+ = 16 genus = 203] (C(s) =15 is > 0, but C(n_{s-1})= -15 the general curve has a semi-stable n.bdle)

(11,10,10,10,9,8,7) [degree = 44 g- = 181 g+ = 16 genus = 197] (dim(I(s))=2, C(s+1)= -16 the general curve has a semi-stable n.bdle)

(11,10,10,9,9,9,9) [degree = 46 g- = 193 g+ = 16 genus = 209] (C(s) =24 is > 0, but C(n_{s-1})= -22 the general curve has a semi-stable n.bdle)

(11,10,10,9,9,9,8) [degree = 45 g- = 187 g+ = 15 genus = 202] (C(s) =18 is > 0, but C(n_{s-1})= -12 the general curve has a semi-stable n.bdle)

(11,10,10,9,9,8,8) [degree = 44 g- = 181 g+ = 14 genus = 195] (C(s) =10 is > 0, but C(n_{s-1})= -10 the general curve has a semi-stable n.bdle)

(11,10,10,9,9,8,7) [degree = 43 g- = 175 g+ = 14 genus = 189] (dim(I(s))=2, C(s+1)= -21 the general curve has a semi-stable n.bdle)

(11,10,10,9,8,8,8) [degree = 43 g- = 175 g+ = 13 genus = 188] (C(s) =6 is > 0, but C(n_{s-1})= -18 the general curve has a semi-stable n.bdle)

(11,10,10,9,8,8,7) [degree = 42 g- = 169 g+ = 13 genus = 182] (dim(I(s))=2, C(s+1)= -25 the general curve has a semi-stable n.bdle)

(11,10,10,9,8,7,7) [degree = 41 g- = 163 g+ = 13 genus = 176] (C(s)= -8 every curve has a semi-stable n.bdle)

(11,10,9,9,9,9,9) [degree = 45 g- = 187 g+ = 14 genus = 201] (C(s) =21 is > 0, but C(n_{s-1})= -15 the general curve has a semi-stable n.bdle)

(11,10,9,9,9,9,8) [degree = 44 g- = 181 g+ = 13 genus = 194] (C(s) =13 is > 0, but C(n_{s-1})= -7 the general curve has a semi-stable n.bdle)

(11,10,9,9,9,8,8) [degree = 43 g- = 175 g+ = 12 genus = 187] (C(s) =9 is > 0, but C(n_{s-1})= -15 the general curve has a semi-stable n.bdle)

(11,10,9,9,9,8,7) [degree = 42 g- = 169 g+ = 12 genus = 181] (dim(I(s))=2, C(s+1)= -22 the general curve has a semi-stable n.bdle)

(11,10,9,9,8,8,8) [degree = 42 g- = 169 g+ = 11 genus = 180] (C(s) =9 is > 0, but C(n_{s-1})= -19 the general curve has a semi-stable n.bdle)

(11,10,9,9,8,8,7) [degree = 41 g- = 163 g+ = 11 genus = 174] (C(s)= -2 every curve has a semi-stable n.bdle)

(11,10,9,9,8,7,7) [degree = 40 g- = 157 g+ = 11 genus = 168] (C(s)= -9 every curve has a semi-stable n.bdle)

(11,10,9,8,8,8,8) [degree = 41 g- = 163 g+ = 10 genus = 173] (C(s) =1 is > 0, but C(n_{s-1})= -19 the general curve has a semi-stable n.bdle)

(11,10,9,8,8,8,7) [degree = 40 g- = 157 g+ = 10 genus = 167] (C(s)= -6 every curve has a semi-stable n.bdle)

(11,10,9,8,8,7,7) [degree = 39 g- = 151 g+ = 10 genus = 161] (C(s)= -9 every curve has a semi-stable n.bdle)

(11,10,9,8,7,7,7) [degree = 38 g- = 145 g+ = 10 genus = 155] (C(s)= -14 every curve has a semi-stable n.bdle)

(10,10,10,10,10,10,10) [degree = 49 g- = 211 g+ = 21 genus = 232] (C(s) =42 is > 0, but C(n_{s-1})= -24 the general curve has a semi-stable n.bdle)

(10,10,10,10,10,10,9) [degree = 48 g- = 205 g+ = 19 genus = 224] (C(s) =33 is > 0, but C(n_{s-1})= -21 the general curve has a semi-stable n.bdle)

(10,10,10,10,10,9,9) [degree = 47 g- = 199 g+ = 17 genus = 216] (C(s) =28 is > 0, but C(n_{s-1})= -22 the general curve has a semi-stable n.bdle)

(10,10,10,10,10,9,8) [degree = 46 g- = 193 g+ = 16 genus = 209] (C(s) =24 is > 0, but C(n_{s-1})= -10 the general curve has a semi-stable n.bdle)

(10,10,10,10,9,9,9) [degree = 46 g- = 193 g+ = 15 genus = 208] (C(s) =27 is > 0, but C(n_{s-1})= -19 the general curve has a semi-stable n.bdle)

(10,10,10,10,9,9,8) [degree = 45 g- = 187 g+ = 14 genus = 201] (C(s) =21 is > 0, but C(n_{s-1})= -9 the general curve has a semi-stable n.bdle)

(10,10,10,10,9,8,8) [degree = 44 g- = 181 g+ = 13 genus = 194] (C(s) =13 is > 0, but C(n_{s-1})= -7 the general curve has a semi-stable n.bdle)

(10,10,10,10,9,8,7) [degree = 43 g- = 175 g+ = 13 genus = 188] (dim(I(s))=2, C(s+1)= -18 the general curve has a semi-stable n.bdle)

(10,10,10,9,9,9,9) [degree = 45 g- = 187 g+ = 13 genus = 200] (C(s) =24 is > 0, but C(n_{s-1})= -12 the general curve has a semi-stable n.bdle)

(10,10,10,9,9,9,8) [degree = 44 g- = 181 g+ = 12 genus = 193] (C(s) =16 is > 0, but C(n_{s-1})= -4 the general curve has a semi-stable n.bdle)

(10,10,10,9,9,8,8) [degree = 43 g- = 175 g+ = 11 genus = 186] (C(s) =12 is > 0, but C(n_{s-1})= -12 the general curve has a semi-stable n.bdle)

(10,10,10,9,9,8,7) [degree = 42 g- = 169 g+ = 11 genus = 180] (dim(I(s))=2, C(s+1)= -19 the general curve has a semi-stable n.bdle)

(10,10,10,9,8,8,8) [degree = 42 g- = 169 g+ = 10 genus = 179] (C(s) =12 is > 0, but C(n_{s-1})= -16 the general curve has a semi-stable n.bdle)

(10,10,10,9,8,8,7) [degree = 41 g- = 163 g+ = 10 genus = 173] (dim(I(s))=2, C(s+1)= -19 the general curve has a semi-stable n.bdle)

(10,10,10,9,8,7,7) [degree = 40 g- = 157 g+ = 10 genus = 167] (C(s)= -6 every curve has a semi-stable n.bdle)

(10,10,9,9,9,9,9) [degree = 44 g- = 181 g+ = 11 genus = 192] (C(s) =19 is > 0, but C(n_{s-1})= -17 the general curve has a semi-stable n.bdle)

(10,10,9,9,9,9,8) [degree = 43 g- = 175 g+ = 10 genus = 185] (C(s) =15 is > 0, but C(n_{s-1})= -9 the general curve has a semi-stable n.bdle)

(10,10,9,9,9,8,8) [degree = 42 g- = 169 g+ = 9 genus = 178] (C(s) =15 is > 0, but C(n_{s-1})= -13 the general curve has a semi-stable n.bdle)

(10,10,9,9,9,8,7) [degree = 41 g- = 163 g+ = 9 genus = 172] (dim(I(s))=2, C(s+1)= -16 the general curve has a semi-stable n.bdle)

(10,10,9,9,8,8,8) [degree = 41 g- = 163 g+ = 8 genus = 171] (C(s) =7 is > 0, but C(n_{s-1})= -13 the general curve has a semi-stable n.bdle)

(10,10,9,9,8,8,7) [degree = 40 g- = 157 g+ = 8 genus = 165] (C(s)= 0 every curve has a semi-stable n.bdle)

(10,10,9,9,8,7,7) [degree = 39 g- = 151 g+ = 8 genus = 159] (C(s)= -3 every curve has a semi-stable n.bdle)

(10,10,9,8,8,8,8) [degree = 40 g- = 157 g+ = 7 genus = 164] (C(s) =3 is > 0, but C(n_{s-1})= -9 the general curve has a semi-stable n.bdle)

(10,10,9,8,8,8,7) [degree = 39 g- = 151 g+ = 7 genus = 158] (C(s)= 0 every curve has a semi-stable n.bdle)

(10,10,9,8,8,7,7) [degree = 38 g- = 145 g+ = 7 genus = 152] (C(s)= -5 every curve has a semi-stable n.bdle)

(10,10,9,8,7,7,7) [degree = 37 g- = 139 g+ = 7 genus = 146] (C(s)= -12 every curve has a semi-stable n.bdle)

(10,9,9,9,9,9,9) [degree = 43 g- = 175 g+ = 9 genus = 184] (C(s) =18 is > 0, but C(n_{s-1})= -18 the general curve has a semi-stable n.bdle)

(10,9,9,9,9,9,8) [degree = 42 g- = 169 g+ = 8 genus = 177] (C(s) =18 is > 0, but C(n_{s-1})= -10 the general curve has a semi-stable n.bdle)

(10,9,9,9,9,8,8) [degree = 41 g- = 163 g+ = 7 genus = 170] (C(s) =10 is > 0, but C(n_{s-1})= -10 the general curve has a semi-stable n.bdle)

(10,9,9,9,9,8,7) [degree = 40 g- = 157 g+ = 7 genus = 164] (dim(I(s))=2, C(s+1)= -9 the general curve has a semi-stable n.bdle)

(10,9,9,9,8,8,8) [degree = 40 g- = 157 g+ = 6 genus = 163] (C(s) =6 is > 0, but C(n_{s-1})= -6 the general curve has a semi-stable n.bdle)

(10,9,9,9,8,8,7) [degree = 39 g- = 151 g+ = 6 genus = 157] (dim(I(s))=2, C(s+1)= -15 the general curve has a semi-stable n.bdle)

(10,9,9,9,8,7,7) [degree = 38 g- = 145 g+ = 6 genus = 151] (C(s)= -2 every curve has a semi-stable n.bdle)

(10,9,9,8,8,8,8) [degree = 39 g- = 151 g+ = 5 genus = 156] (C(s) =6 is > 0, but C(n_{s-1})= -12 the general curve has a semi-stable n.bdle)

(10,9,9,8,8,8,7) [degree = 38 g- = 145 g+ = 5 genus = 150] (dim(I(s))=2, C(s+1)= -17 the general curve has a semi-stable n.bdle)

(10,9,9,8,8,7,7) [degree = 37 g- = 139 g+ = 5 genus = 144] (C(s)= -6 every curve has a semi-stable n.bdle)

(10,9,9,8,7,7,7) [degree = 36 g- = 133 g+ = 5 genus = 138] (C(s)= -9 every curve has a semi-stable n.bdle)

(10,9,8,8,8,8,8) [degree = 38 g- = 145 g+ = 4 genus = 149] (C(s) =4 is > 0, but C(n_{s-1})= -14 the general curve has a semi-stable n.bdle)

(10,9,8,8,8,8,7) [degree = 37 g- = 139 g+ = 4 genus = 143] (C(s)= -3 every curve has a semi-stable n.bdle)

(10,9,8,8,8,7,7) [degree = 36 g- = 133 g+ = 4 genus = 137] (C(s)= -6 every curve has a semi-stable n.bdle)

(10,9,8,8,7,7,7) [degree = 35 g- = 127 g+ = 4 genus = 131] (C(s)= -5 every curve has a semi-stable n.bdle)

(10,9,8,7,7,7,7) [degree = 34 g- = 121 g+ = 4 genus = 125] (C(s)= -12 every curve has a semi-stable n.bdle)

(9,9,9,9,9,9,9) [degree = 42 g- = 169 g+ = 7 genus = 176] (C(s) =21 is > 0, but C(n_{s-1})= -15 the general curve has a semi-stable n.bdle)

(9,9,9,9,9,9,8) [degree = 41 g- = 163 g+ = 6 genus = 169] (C(s) =13 is > 0, but C(n_{s-1})= -7 the general curve has a semi-stable n.bdle)

(9,9,9,9,9,8,8) [degree = 40 g- = 157 g+ = 5 genus = 162] (C(s) =9 is > 0, but C(n_{s-1})= -3 the general curve has a semi-stable n.bdle)

(9,9,9,9,9,8,7) [degree = 39 g- = 151 g+ = 5 genus = 156] (dim(I(s))=2, C(s+1)= -12 the general curve has a semi-stable n.bdle)

(9,9,9,9,8,8,8) [degree = 39 g- = 151 g+ = 4 genus = 155] (C(s) =9 is > 0, but C(n_{s-1})= -9 the general curve has a semi-stable n.bdle)

(9,9,9,9,8,8,7) [degree = 38 g- = 145 g+ = 4 genus = 149] (dim(I(s))=2, C(s+1)= -14 the general curve has a semi-stable n.bdle)

(9,9,9,9,8,7,7) [degree = 37 g- = 139 g+ = 4 genus = 143] (C(s)= -3 every curve has a semi-stable n.bdle)

(9,9,9,8,8,8,8) [degree = 38 g- = 145 g+ = 3 genus = 148] (C(s) =7 is > 0, but C(n_{s-1})= -11 the general curve has a semi-stable n.bdle)

(9,9,9,8,8,8,7) [degree = 37 g- = 139 g+ = 3 genus = 142] (C(s)= 0 every curve has a semi-stable n.bdle)

(9,9,9,8,8,7,7) [degree = 36 g- = 133 g+ = 3 genus = 136] (C(s)= -3 every curve has a semi-stable n.bdle)

(9,9,9,8,7,7,7) [degree = 35 g- = 127 g+ = 3 genus = 130] (C(s)= -2 every curve has a semi-stable n.bdle)

(9,9,8,8,8,8,8) [degree = 37 g- = 139 g+ = 2 genus = 141] (C(s) =3 is $>$ 0, but C(n_{s-1})= -9 the general curve has a semi-stable n.bdle)

(9,9,8,8,8,8,7) [degree = 36 g- = 133 g+ = 2 genus = 135] (C(s)= 0 every curve has a semi-stable n.bdle)

(9,9,8,8,8,7,7) [degree = 35 g- = 127 g+ = 2 genus = 129] (We are in the range of the conjecture BUT CAN'T HANDLE THIS CASE)

(9,9,8,8,7,7,7) [degree = 34 g- = 121 g+ = 2 genus = 123] (C(s)= -6 every curve has a semi-stable n.bdle)

(9,9,8,7,7,7,7) [degree = 33 g- = 115 g+ = 2 genus = 117] (C(s)= -9 every curve has a semi-stable n.bdle)

(9,8,8,8,8,8,8) [degree = 36 g- = 133 g+ = 1 genus = 134] (C(s) =3 is $>$ 0, but C(n_{s-1})= -3 the general curve has a semi-stable n.bdle)

(9,8,8,8,8,8,7) [degree = 35 g- = 127 g+ = 1 genus = 128] (dim(I(s))=2, C(s+1)= -10 the general curve has a semi-stable n.bdle)

(9,8,8,8,8,7,7) [degree = 34 g- = 121 g+ = 1 genus = 122] (C(s)= -3 every curve has a semi-stable n.bdle)

(9,8,8,8,7,7,7) [degree = 33 g- = 115 g+ = 1 genus = 116] (C(s)= -6 every curve has a semi-stable n.bdle)

(9,8,8,7,7,7,7) [degree = 32 g- = 109 g+ = 1 genus = 110] (C(s)= -5 every curve has a semi-stable n.bdle)

(9,8,7,7,7,7,7) [degree = 31 g- = 103 g+ = 1 genus = 104] (C(s)= -6 every curve has a semi-stable n.bdle)

(8,8,8,8,8,8,8) [degree = 35 g- = 127 g+ = 0 genus = 127] (C(s) =7 is $>$ 0, but C(n_{s-1})= -7 the general curve has a semi-stable n.bdle)

(8,8,8,8,8,8,7) [degree = 34 g- = 121 g+ = 0 genus = 121] (C(s)= 0 every curve has a semi-stable n.bdle)

(8,8,8,8,8,7,7) [degree = 33 g- = 115 g+ = 0 genus = 115] (C(s)= -3 every curve has a semi-stable n.bdle)

(8,8,8,8,7,7,7) [degree = 32 g- = 109 g+ = 0 genus = 109] (C(s)= -2 every curve has a semi-stable n.bdle)

(8,8,8,7,7,7,7) [degree = 31 g- = 103 g+ = 0 genus = 103] (C(s)= -3 every curve has a semi-stable n.bdle)

(8,8,7,7,7,7,7) [degree = 30 g- = 97 g+ = 0 genus = 97] (C(s)= -6 every curve has a semi-stable n.bdle)

(8,7,7,7,7,7,7) [degree = 29 g- = 91 g+ = 0 genus = 91] (C(s)= -5 every curve has a semi-stable n.bdle)

(7,7,7,7,7,7,7) [degree = 28 g- = 85 g+ = 0 genus = 85] (C(s)= 0 every curve has a semi-stable n.bdle)

Chapter 3

Subschemes of $\mathbb{P}^2$ with ten fat point of maximum rank.

3.1 Subschemes of $\mathbb{P}^2$ with fat point.

In this section we deal with linear systems of curves of degree d containing zero-dimensional subschemes of $\mathbb{P}^2$. The aim is to understand how determining the dimension of such systems; we will desume it is not just a mere sum up of conditions.

Let Z be a zero-dimensional subscheme of $\mathbb{P}^2$,then we can write $Z = b_1P_1 + ... + b_kP_k$ where P_i's are general points of $\mathbb{P}^2$ and where $b_i \geq 1$ for every i. We simply denote by bP the point $P \in \mathbb{P}^2$ with multiplicity b. In other words, bP denotes the (b-1)-th infinitesimal neighborhood of the point P; let C be a curve, if C has multiplicity b in P then the curve must contain bP. That is to say, when a curve C of equation $f(x_0, x_1, x_2) = 0$ in the local coordinates x_0, x_1, x_2 of $\mathbb{P}^2$, contains a point P of multiplicity b then, any $b-1$-th derivative annihilates in P. We deduce that a point of multiplicity b imposes $\frac{b(b+1)}{2}$ conditions. Let us denote by $\mathcal{L}_Z(d)$ the linear system of the curves of degree d containing Z, i.e. all the curves of degree d in $\mathbb{P}^2$ with multiplicity in P_i at least b_i. If it is a non-empty system, then we can compute its dimension as follows:

$$expdim(\mathcal{L}_Z(d) := \frac{d(d+3)}{2} - \sum_{i=1}^{k} \frac{b_i(b_i+1)}{2}$$

where $\frac{d(d+3)}{2}$ is the dimension of the linear system of all the curves of degree d in $\mathbb{P}^2$, whereas $\frac{b_i(b_i+1)}{2}$ is given by the number of conditions imposed by the points P_i's. In particular, we try to understand which subschemes $Z \subset \mathbb{P}^2$ are such that, for every $d \in \mathbb{N}$, the linear system $\mathcal{L}_Z(d)$ has the expected dimension. Clearly, this depends on the choice of the points in $\mathbb{P}^2$ and that's why we always work with general points. Let us set some useful definitions.

Definition 3.1.1 *Let Z be a subscheme of $\mathbb{P}^2$, Z has maximum rank if and only if for all $k \in \mathbb{Z}$; $h^0(\mathcal{I}_Z(k)) * h^1(\mathcal{I}_Z(k)) = 0$, if and only if, for all $k \in \mathbb{Z}$ the function $H^0(\mathcal{O}_{\mathbb{P}^2}(k)) \to H^0(\mathcal{O}_Z(k))$ is injective or surjective.*

Definition 3.1.2 *Let $Z = b_1P_1, ..., b_kP_k$ a subscheme of $\mathbb{P}^2$, of degree $z = \sum_{i=1}^{k} \frac{b_i(b_i+1)}{2}$, then there exist a unique value v such that: $h^0(\mathcal{O}_{\mathbb{P}^2}(v)) \geq z$, $h^0(\mathcal{O}_{\mathbb{P}^2}(v-1)) < z$. This value v is called critical value of Z.*

We observe that $v = min\{n \in \mathbb{N} | \binom{n+2}{2} > z\}$, therefore we expect that every curve containing Z has degree greater or equal to v. In this setting we can prove the following equivalence.

Proposition 3.1.3 *Let Z be a zero-dimensional subscheme of $\mathbb{P}^2$, and let v be its critical value, then Z has maximum rank if and only if $h^1(\mathcal{I}_Z(v)) = 0$ and $h^0(\mathcal{I}_Z(v-1)) = 0$.*

Proof:

$\Leftarrow$) Let v be the critical value of Z and let L be a line such that $L \cap Z = \emptyset$. We consider the exact sequence:

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_L \to 0$$

twisting by $\mathcal{I}_Z(k)$ we obtain

$$0 \to \mathcal{I}_Z(k-1) \to \mathcal{I}_Z(k) \to \mathcal{O}_Z(k) \to 0.$$

We note that if $k \geq -1$ and if $h^1(\mathcal{I}_Z(k-1)) = 0$ then $h^1(\mathcal{I}_Z(k)) = 0$. Since by hypothesis $h^1(\mathcal{I}_Z(v)) = 0$, we have $h^1(\mathcal{I}_Z(k)) = 0$ for all $k \geq v$. Moreover if

$h^0(\mathcal{I}_Z(v-1)) = 0$ then $h^0(\mathcal{I}_Z(k)) = 0$ for all $k \le v-1$. In conclusion we have $h^1(\mathcal{I}_Z(k)) = 0$ for all $k \ge v-1$, $h^0(\mathcal{I}_Z(k)) = 0$ for all $k \le v-1$ and so Z has maximum rank. □

REMARK 3.1.4 *Let us consider a subscheme of $\mathbb{P}^2$ of maximum rank. Given $d \in \mathbb{N}$ such that $\mathcal{L}_Z(d) \neq \emptyset$, then $h^0(\mathcal{I}_Z(d)) \neq 0$ and thus $h^1(\mathcal{I}_Z(d)) = 0$. This means that:*

$$dim(\mathcal{L}_Z(d)) = h^0(\mathcal{I}_Z(d)) = h^0(\mathcal{O}_{\mathbb{P}^2}(d)) - h^0(\mathcal{O}_Z(d)) = \frac{d(d+3)}{2} - \sum_{i=1}^{k} \frac{b_i(b_i+1)}{2} =$$

$$= expdim(\mathcal{L}_Z(d)$$

Finally, if a zero-dimensional subscheme Z of $\mathbb{P}^2$ has maximum rank, then, for every $d \in \mathbb{N}$, or $\mathcal{L}_Z(d) = 0$ or $dim(\mathcal{L}_Z(d)) = expdim(\mathcal{L}_Z(d))$.

Subschemes of $\mathbb{P}^2$ has not maximum rank are very well-known. For instance, given the following subscheme $Z = 2P_1 + 2P_2$ of $\mathbb{P}^2$, we note that it has degree $z = 2\frac{2\cdot(2+1)}{2} = 6$ and its critical value is $v = 2$ since $h^0(\mathcal{O}_{\mathbb{P}^2}(1)) = 3$ and $h^0(\mathcal{O}_{\mathbb{P}^2}(2)) = 6$. In order Z to have maximum rank, by Proposition 3.1.3 we have that $h^1(\mathcal{I}_Z(2)) = 0$. We observe that $h^1(\mathcal{I}_Z(2)) = h^0(\mathcal{I}_Z(2))$ and since there exists a conic in $\mathbb{P}^2$ containing Z, given by the double line passing through P_1 and P_2, we have $h^0(\mathcal{I}_Z(2)) \neq 0$.

Such example guarantees the existence of subschemes in $\mathbb{P}^2$ that has not maximum rank but we can't describe their properties.

3.2 Special linear systems of plane curves.

In this section we deal with special linear systems, $\mathcal{L}$, of plane curves, i.e. those systems with $h^1(\mathcal{L}) \neq 0$ in cohomology. Concerning the dimensional problem of such systems, there exists a particular conjecture called Harbourne-Hirschowitz's conjecture.

Let us introduce some definitions:

DEFINITION 3.2.1 *Let $Z = b_1P_1 + \ldots + b_nP_n$ be a subscheme of $\mathbb{P}^2$. The linear system of plane curves of degree d containing Z, $\mathcal{L}_Z(d)$ is said non-special if $h^1(\mathcal{L}_Z(d)) = 0$. Otherwise the linear system is said special.*

Going back to the previous section, given a zero-dimensional subscheme Z in $\mathbb{P}^2$, if $\mathcal{L}_Z(d) \neq 0$ is a non-special linear system, then $h^1(\mathcal{L}_Z(d)) = 0$ and thus $dim(\mathcal{L}_Z(d)) = expdim(\mathcal{L}_Z(d))$. We deduce that if Z has maximum rank, the linear system $\mathcal{L}_Z(d)$ is zero or non-special.

In the literature, the first conjecture about special linear systems goes back to B. Segre. It states the following:

CONJECTURE 3.2.2 *If a linear system of plane curves with general multiple base points $\mathcal{L}_Z(d)$ is special, then one of its general member is non-reduced, namely the linear system has, according to Bertini's theorem, some multiple fixed component.*

Such a conjecture deduce the existence of a double curve in the special linear system of curves in $\mathbb{P}^2$. Later on, Hirschowitz and Harbourne determine a new property of the double curve: it is, in fact, an exceptional curve of the system in his locus.

Let us define an exceptional curve.

Let us consider the blow-up $\pi : \tilde{\mathbb{P}^2} \to \mathbb{P}^2$ of the plane $\mathbb{P}^2$ at $P_1, ..., P_n$. Let $E_1, ..., E_n$ be the exceptional divisors corresponding to the blow-up points $P_1, ..., P_n$ and let H be the pull-back of a general line of $\mathbb{P}^2$ via π. The strict transform of the linear system $\mathcal{L}_Z(d)$, where $Z = b_1P_1 + ... + b_nP_n$, is
$\tilde{\mathcal{L}}_Z(d) := |dH - b_1E_1 - ... - b_nE_n|$. In the system $\tilde{\mathcal{L}}_Z(d)$, the rules intersection are the following:

$H^2 = 1$;

$E_i \cdot E_j = \delta_{ij}$.

DEFINITION 3.2.3 *A curve $C \subset \mathbb{P}^2$ is said exceptional curve if it is a rational curve and with self-intersection -1.*

For instance, every exceptional divisor is an exceptional curve, such as any line passing through two blow-up points.

Now, let us state Harbourne-Hirschowitz's conjecture:

CONJECTURE 3.2.4 *A linear system of plane curve $\mathcal{L}$, with general multiple base points is special if and only if there exists an exceptional curve with multiplicity at least two in the base locus.*

Let us go back to the example in the first section, the linear system $\mathcal{L}_Z(2)$ with $Z = 2P_1 + 2P_2$. Since $h^0(\mathcal{I}_Z(2)) \neq 0$, the sequence

$$0 \to H^0(\mathcal{I}_Z(2)) \to H^0(\mathcal{O}_{\mathbb{P}^2}) \to H^0(\mathcal{O}_Z(2)) \to ...$$

is not short exact, therefore $h^1(\mathcal{I}_Z(2)) \neq 0$ and the system is special. Now, let us consider the line $R = H - E_1 - E_2$ passing through P_1, P_2. We note that $R^2 = -1$ and, since $2R$ is the only curve in $\mathcal{L}_Z(2)$, such a line is in the locus of the system with multiplicity two.

In order to see more examples, let us consider a subscheme Z in $\mathbb{P}^2$ of the form $Z = 2P_1 + ... + 2P_{\frac{d(d+3)}{2}}$ and the linear system $\mathcal{L}_Z(2d)$, with $d < 4$. It is easy to prove that the curve of degree d passing through P_i is an exceptional curve and it is in the locus of $\mathcal{L}_Z(2d)$ with multiplicity two.

More generally, one has special linear systems in the following situation. Let $\mathcal{L}$ be a linear system on $\mathbb{P}^2$ which is, non-empty and let C be an exceptional curve on $\tilde{\mathbb{P}}^2$ corresponding to a curve Γ on $\mathbb{P}^2$, such that $\tilde{\mathcal{L}} \cdots C = -N < 0$. Then C (respectively Γ) splits off with multiplicity N as a fixed component from all curves of $\tilde{\mathcal{L}}$ (respectively $\mathcal{L}$) and one has:

$$\tilde{\mathcal{L}} = NC + \tilde{\mathcal{M}}(respectively \mathcal{L} = N\Gamma + \mathcal{M})$$

where $\tilde{\mathcal{M}}$ (resp. $\mathcal{M}$) is the residual linear system. We can see that if $N \geq 2$, then $\mathcal{L}$ is special.

Even more generally, consider a linear system $\mathcal{L}$ on $\mathbb{P}^2$ non-empty , $C_1, ...C_k$ exceptional curves on $\tilde{\mathbb{P}}^2$ corresponding to curves $\Gamma_1, ..., \Gamma_k$ on $\mathbb{P}^2$, such that $\tilde{\mathcal{L}} \cdot C_i = -N_i < 0$, $i = 1, .., k$. Then:

$$\tilde{\mathcal{L}} = \sum_{i=1}^{k} N_i C_i + \tilde{\mathcal{M}}, \mathcal{L} = \sum_{i=1}^{k} N_i \Gamma_i + \mathcal{M}$$

and $\tilde{\mathcal{M}} \cdot C_i = 0$ for $i = 1, .., k$. As before, $\mathcal{L}$ is special as soon as there exists an $i = 1, ..., k$ such that $N_i \geq 2$. Furthermore $C_i C_j = -\delta_{ij}$ and $C := \sum_{i=1}^{k} C_i$ is called (-1)-configuration on $\tilde{\mathbb{P}}^2$. Now we give another definition.

DEFINITION 3.2.5 *A linear system $\mathcal{L}$ on $\mathbb{P}^2$ is called (-1)-special if, $\tilde{\mathcal{L}} = \sum_{i=1}^{k} N_i C_i + \tilde{\mathcal{M}}$ where $C = \sum_{i=1}^{k} C_i$ is a (-1)-configuration, $\tilde{\mathcal{M}} C_i = 0$, for all $i = 1,..,k$, $dim(\mathcal{M}) \geq 0$ and there exists an $i = 1,...,k$ such that $N_i > 1$.*

We want to remember that in the work of Ciliberto, they were classified every homogeneous linear system which are (-1)-special. Until now, we can find only these linear systems as examples of special linear systems. The theorem is the following:

THEOREM 3.2.6 *(C.Ciliberto) The only homogeneous linear systems $\mathcal{L}_Z(d)$, where $Z = mP_1 + ... + mP_h$ is a subscheme of $\mathbb{P}^2$, which are (-1)-special are:*

$\mathcal{L}_Z(d)$ with $Z = mP_1 + mP_2$ and with $m \leq d \leq 2m-2$ $\mathcal{L}_Z(d)$ with $Z = mP_1 + mP_2 + mP_3$ and with $\frac{3}{2}m \leq d \leq 2m-2$ $\mathcal{L}_Z(d)$ with $Z = mP_1 + mP_2 + mP_3 + mP_4 + mP_5$ and with $2m \leq d \leq \frac{5m-2}{2}$ $\mathcal{L}_Z(d)$ with $Z = mP_1 + mP_2 + mP_3 + mP_4 + mP_5 + mP_6$ and with $\frac{12}{5}m \leq d \leq \frac{5m-2}{2}$ $\mathcal{L}_Z(d)$ with $Z = mP_1 + mP_2 + mP_3 + mP_4 + mP_5 + mP_6 + mP_7$ and with $\frac{21}{8}m \leq d \leq \frac{8m-2}{3}$ $\mathcal{L}_Z(d)$ with $Z = mP_1 + mP_2 + mP_3 + mP_4 + mP_5 + mP_6 + mP_7 + mP_8$ and with $\frac{48}{17}m \leq d \leq \frac{17m-2}{6}$

As a remarkable consequence we have that the Harbourne-Hirschowitz conjecture for homogeneous system take the form:

CONJECTURE 3.2.7 *Every homogeneous system of the form $\mathcal{L}_Z(d)$, where $Z = mP_1 + ... + mP_h$ is a subscheme of $\mathbb{P}^2$, with $h \geq 10$, is non-special.*

Moreover we can prove that if the Harbourne-Hirschowitz conjecture hold, then the system $\mathcal{L}$ on $\mathbb{P}^2$ is special if and only if it is (-1)-special, and so the linear systems of theorem 3.2.6 could be the only special systems.

Many other people studied the Harbourne-Hirschowitz conjecture. Now we want to enunciate the most important result of this study.

Hirschowitz worked on this conjecture and formulated the following theorem:

THEOREM 3.2.8 *(Hirschowitz) The Harbourne-Hirschowitz conjecture holds in the homogeneous case $\mathcal{L}_Z(d)$ with $Z = mP_1 + ... + mP_h$ subscheme of $\mathbb{P}^2$, with $m \leq 3$.*

To prove this theorem he uses a degeneration technique called the Horace's method, which we will deal with in the next section. The application of Horace's method usually requires a deep geometric understanding of the problem and a special capability of guessing the right specializations to be performed.

Afterward Ciliberto and Miranda proved the following theorem:

THEOREM 3.2.9 *The Harbourne-Hirschowitz conjecture holds in the quasi-homogeneous cases, $\mathcal{L}_Z$, where $Z = nP_1 + mP_2 + ... + mP_h$ is a subscheme of $\mathbb{P}^2$ with $m \leq 3$ and in the homogeneous case, $\mathcal{L}_Z$, where $Z = +mP_1 + ... + mP_h$ is a subscheme of $\mathbb{P}^2$ with $m \leq 12$.*

Despite of Hirschowitz technique, the idea of Ciriberto and Miranda which I will explain in some detail, consists in using a degeneration technique worked out by Z. Ran manly for studying enumerative problems of families of plane nodal curves. It consists in degenerating the plane to a reducible surface and in following the linear system in the degeneration. The restriction of the limit linear system to the component of the reducible limit surface are easier than the system one starts with, so that one can hope to successfully use induction.

Another important result is due to S. Yang:

THEOREM 3.2.10 *The Harbourne-Hirschowitz conjecture holds for all linear systems of plane curves with at most ten points having multiplicity up to seven.*

S. Yang approach to this problem has a simple geometric description. Suppose we are given a linear system $\mathcal{L}_Z$ of a plane curve with multiple base point. Choose a triangle of three lines in $\mathbb{P}^2$ that meet in three distinct points. We specialize the base points by moving them onto these points and sliding the multiple points along the three lines to collide them. Each collision creates a larger singularity in the base locus of the limiting linear system, and the class of singularities that arise can be completely described via a combinatorial game involving checkers on a triangular board.
The second technique, used by S. Yang is a modification of a well-known degeneration, first exploited by Ciliberto and Miranda.

I want to start just from this result. As a matter of fact we will prove with this thesis that such result is also valid, even if we have 10 points of multiplicity at most eight. Actually, we will be able to do better, as we will prove that every subscheme

Z that we are going to calculate have maximum rank, that means that the linear system $\mathcal{L}_Z(v)$ has to be non-special and $H^0(\mathcal{L}_Z(v-1)) = 0$.

3.3 The Horace's method.

In this section we are going to explain the Horace's method, the instrument we use to prove whether a particular subscheme of $\mathbb{P}^2$ has maximum rank. This method was found by Hirschowitz, who shows us the truthfulness of it. There is no time here to enter in any detail about this idea, which is the one elaborated by A. Hirschowitz in his paper. The approach of Hirschowitz is to argue by degeneration, meaning with this that one specializes the base points of the linear system in order to be able to better compute the dimension of the linear system. Recall that the dimension of $\mathcal{L}_Z$ (where $Z = m_1P_1 + ... + m_hP_h$) is upper-semicontinuous in the position of the points $P_1, ..., P_h$. Therefore if one finds a particular set of points $Q_1, ..., Q_h$ such that $Z_1 = m_1Q_1 + ... + m_hQ_h$ and $\mathcal{L}_{Z_1}$ is non-special, then also $\mathcal{L}_Z$ is non-special. Unfortunately, this is often too naive: as soon as one puts the points $P_1, ..., P_h$ in a particular position, e.g one puts them on some curves on which they should not lie, then the dimension of $\mathcal{L}_Z$ tends to increase, and the method, in this crude form, does not work. However, there is still something which one can do: even if the dimension of $\mathcal{L}_Z$ increases, one can actually compute the limit of $\mathcal{L}_Z$ when $P_1, ..., P_h$ approach $Q_1, ...Q_h$.

Concretely, this method consists of the following specializations of the base point of the system on a particular curve C, so that the whole curve results in the base locus of the linear system and we can divide it by the equation C. This way we can reduce the degree of the curves of the linear system getting easier to deal with and collecting more notions of it. We would like to remind that we want to understand whether a zero-dimensional subscheme Z of $\mathbb{P}^2$ has maximum rank.

In order to achieve it, we have to prove that $h^0(\mathcal{I}_Z(v-1)) = h^1(\mathcal{I}_Z(v)) = 0$, such connections will be demonstrated through the Horace's method. First of all we concentrate our attention verifying that $h^1(\mathcal{I}_Z(v)) = 0$, that is the linear system $\mathcal{L}_Z(v)$ is a non-special one; in a second time we will also prove that $h^0(\mathcal{I}_Z(v-1)) = 0$.

Before to show any example we need to prove this proposition:

PROPOSITION 3.3.1 *Let C, D be two subscheme of $\mathbb{P}^2$ such that $C \subset D$. If $\mathcal{L}_D(d)$ is non-special, then $\mathcal{L}_C(d)$ is non-special too, where with $\mathcal{L}_C(d)$ we denote the linear system $\mathcal{I}_Z(d)$.*

<u>Dimostrazione:</u>
As $C \subset D$, the exact sequence is true:

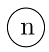
$$0 \to \mathcal{I}_D(d) \to \mathcal{I}_C(d) \to \mathcal{I}_{C,D} \to 0$$

In cohomology we obtain:

$$... \to H^1(\mathcal{I}_D(d)) \to H^1(\mathcal{I}_C(d)) \to 0$$

Since the support of $\mathcal{I}_{C,D}$ is zero-dimensional, then $h^1(\mathcal{I}_{C,D}) = 0$ and it's clear that if $\mathcal{I}_D(d) = 0$ then $\mathcal{I}_C(d) = 0$ too. □

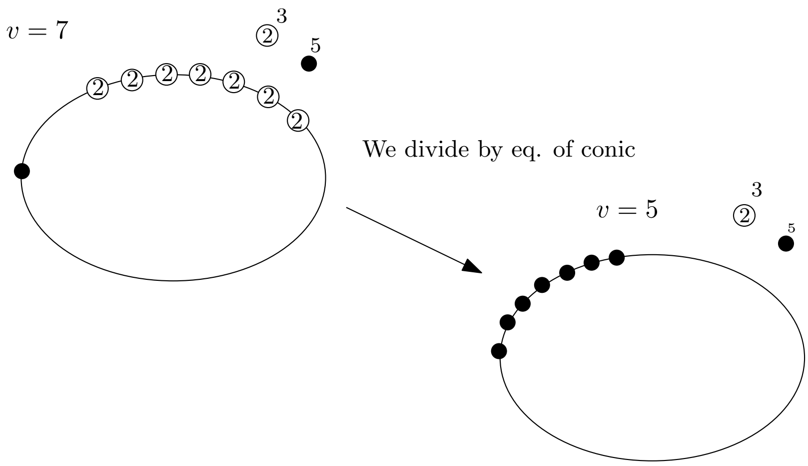
We give some notations. We denote by (n) a fat point of multiplicity n and we represent it as follows



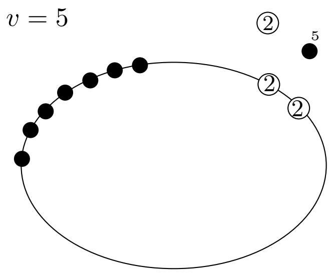
while we will represent a simple point as a little black disk. We bring a simple example about the use of the Horace's method.

EXAMPLE 3.3.2 *Let $Z = 2P_1 + 2P_2 + ... + 2P_{10}$ be a subscheme of $\mathbb{P}^2$, Z has degree equal to $10 \cdot 3 = 30$ and critical value 7 because $h^0(\mathcal{O}_{\mathbb{P}^2}(6)) = 28$ and $h^0(\mathcal{O}_{\mathbb{P}^2}(7)) = 36$. We must prove that $h^1(\mathcal{I}_Z(7)) = 0$ thus we need that $h^0(\mathcal{I}_Z(7)) = h^0(\mathcal{O}_{\mathbb{P}^2}(7)) - h^0(\mathcal{O}_Z(7)) = 36 - 30 = 6$ and so it is sufficient to show that $h^0(\mathcal{I}_Z(7)) = 6$. Now, let us take a new subscheme $Z_1 = Z + P_{11} + ... + P_{16}$, composed by Z and by other six points in general position. This subscheme has degree 36 and so the linear system $\mathcal{I}_{Z_1}(7)$ is non-special if and only if $h^0(\mathcal{I}_{Z_1}) = 0$. Specializing the points of Z_1 we show that there doesn't exist any curve of degree 7 containing Z_1, therefore $\mathcal{I}_{Z_1}(7)$ is non-special and by proposition 3.3.1, $\mathcal{I}_Z(7)$ is non-special too.*

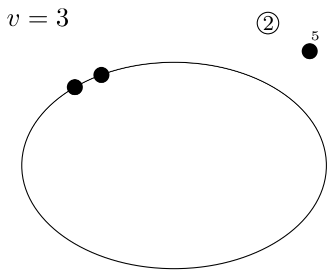
With notations as above, we represent the subscheme Z_1 by 10 double points, (2) and by 6 simple points. We start specializing seven double points and a normal point *in a conic curve C, outside we have three double points and five points. On C we have $7 * 2 + 1 = 15 = 2 * 7 + 1$ conditions. So we can divide by the equation of the conic and reduce to the following situation for the curves of degree five (i.e.$v = 5$).*



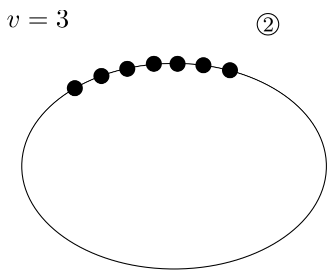
We have on C seven simple points, outside of C there are three (2) and five simple points. Now we specialize other two (2) on C.



On conic we have $2 * 2 + 7 = 11 = 5 * 2 + 1$ conditions and so we can divide by the equations of conic and we obtain the following situation for curves of degree three.



We specialize five simple points in the conic C, so we have the right number of the conditions $1 * 7 = 7 = 3 * 2 + 1$ and we can divide by the equation of C.



Since the remaining double point is clearly not contained in a line, we conclude that $h^0(\mathcal{I}_{Z_1}(7)) = 0$. By **??** *$h^0(\mathcal{I}_Z(7)) = 0$, i.e. $\mathcal{L}_Z(7)$ is non-special.*

$v = 1$

②

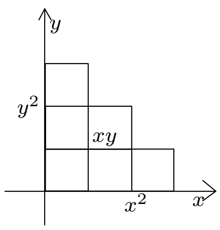
A double point can't lie in a line.

For simplicity's sake, we specialize the points of Z only on curves with genus equal to zero or one, with some singularity of order two. Moreover if we put a fat point bP in a node, this fact imposes $2b$ conditions on curve. Through this method we notice that we can't do general proves, but the only way to use it is analyzing case by case, which is what we are going to do with this paper. In order to achieve our goal, we have to consider every zero-dimensional subscheme of $\mathbb{P}^2$ composed by 10 points with multiplicity minor or equal than 8, apply the Horace's method to everyone of them in order to check $h^1(\mathcal{I}_Z(v)) = 0$ and $h^0(\mathcal{I}_Z(v-1)) = 0$. This way the work could get very demanding, because the cases are more than thousand. Fortunately, thank proposition 3.3.1 and some transformations, called Cremona's transformations, we can just take into consideration only hundred cases. In addition, further studies improved the Horace's method through good observations concerning the geometry of every linear system.

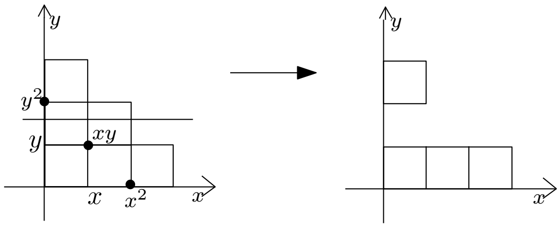
3.4 Horace differential's method and collision of infinitesimal neighborhood.

Horace differential

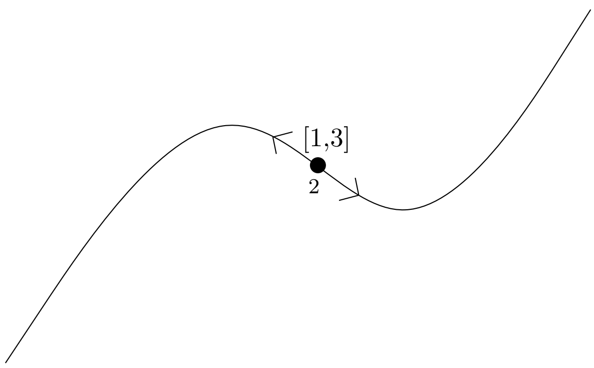
The first stratagem to use, in order to make our job easier, is called Horace differential's method. It consists of seeing a fat point of size $n, (n)$, made up by slices. For instance, a triple point is made up by the slices (123), (see illustration).



In this illustration, every small square corresponds to one condition and the sum of every small square corresponds to the number of conditions imposed by the fat point. In particular, specializing the point on a curve, you can see only the conditions that are on the axis x. Horace differential allows us, specializing a multiplicity point n on a divisor C, we can choose a particular slice, not necessarily the one on the axis x, but the one offering us the most convenient number of conditions. We can adopt such procedure only once for a particular point, afterwards we have to begin again counting the number of conditions as usual (that is starting from the ones on the axis x). In the illustration below, we see what happens choosing the slice giving us only two conditions from a triple point (3).

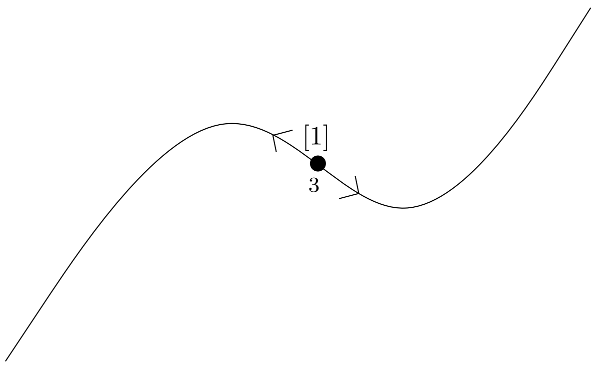


To represent this situation on a curve, we use a more convenient notation to our aim as in figure.

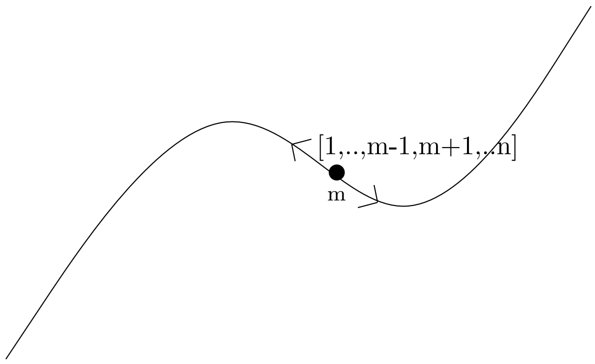


where the notation $[1, 3]$ represents the remaining slices.

Dividing by the equation of C, it occurs.



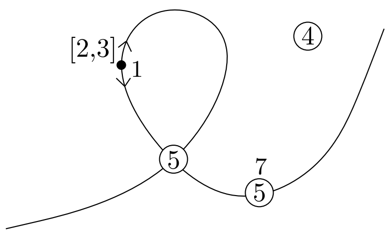
We rest with three conditions on C and, dividing by equation of the curve, we remain with only one condition on C.
In general we represent the slice m of a fat point (n) as in the following picture:



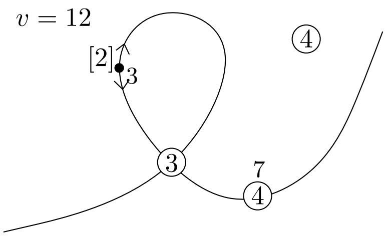
When we apply the Horace differential to a point P, we can't specialize P on another curve. Moreover it's not easy to remove the rest slices. Thus we will use this method with moderation.

We show an example.

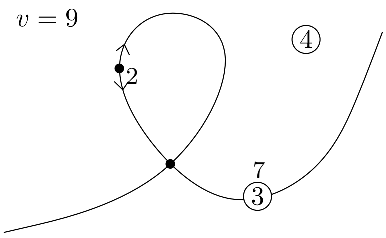
EXAMPLE 3.4.1 *We consider the subscheme* $Z = 5P_1 + 5P_2 + 5P_3 + 5P_4 + 5P_5 + 5P_6 + 5P_7 + 5P_8 + 4P_) + 3P_{10}$ *of* $\mathbb{P}^2$. *The critical value of* Z *is* 15 *and* $\mathcal{L}_Z(15)$ *is non-special if and only if* $h^=(\mathcal{I}_Z(15)) = 0$. *This happens only when* $h^0(\mathcal{I}_Z(15)) = h^0(\mathcal{O}_{\mathbb{P}^2}(15)) - h^0(\mathcal{O}_Z(15)) = 136 - 136 = 0$. *In this case we don't need to add any simple point because the conditions are right to start. We begin with a nodal cubic* C *and specialize one* (5) *at the node, seven* (5) *on* C *and, by differential Horace, the slice* 1 *of the* (3) *on* C*; outside of* C *we have the* (4).



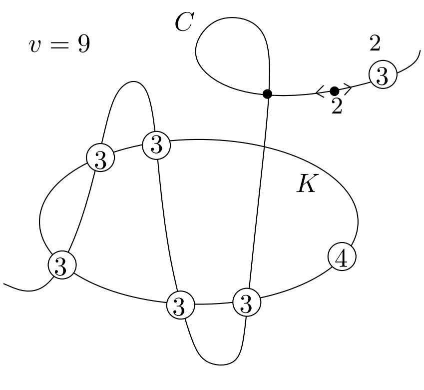
On the cubic we have: $2 * 5 + 7 * 5 + 1 = 46 = 3 * 15 + 1$ *conditions and so we can divide by the equation of the cubic and we get the following situation:*



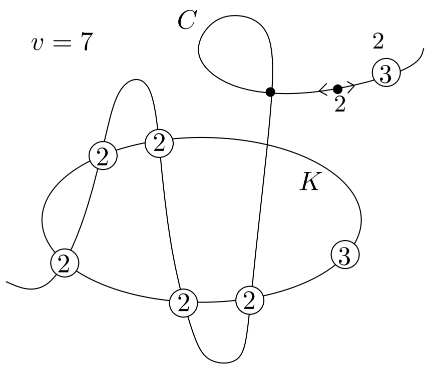
On the cubic we have $2 * 3 + 7 * 4 + 3 = 37 = 3 * 12 + 1$ *conditions, so we can divide by the equation of* C *and we reach the following situation for curves of degree* 9*:*



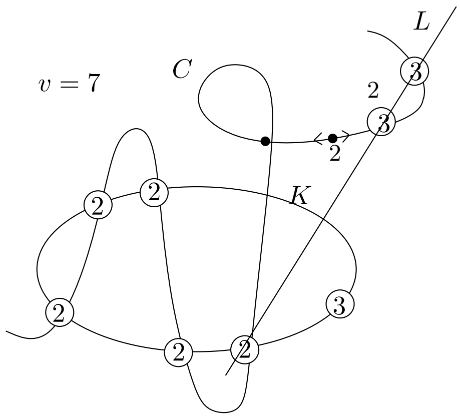
So we have on C*, one simple point at the node, one double point and* 7 (3)*; outside of* C *there is a* (4)*. It's time to give up with the cubic (because we have only a simple point at the node) and look for another curve to exploit. Let* K *be a general conic through the support of* (4)*. The conic* K *intersects* C *at six points (different from the node). We specialize five of the seven* (3) *to the points of* $C \cap K$ *and get:*



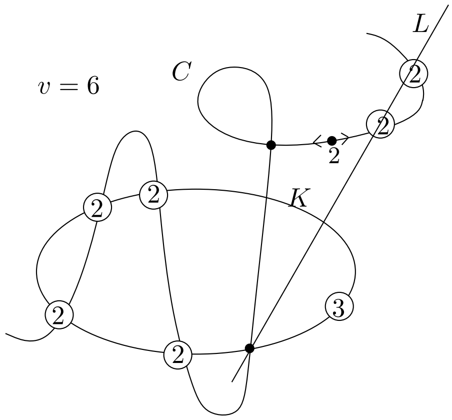
On K *we have:* $5 * 3 + 4 = 19 = 2 * 9 + 1$ *conditions. So we can divide by the equation of the conic and reduce to the following situation for the curves of degree* *seven:*



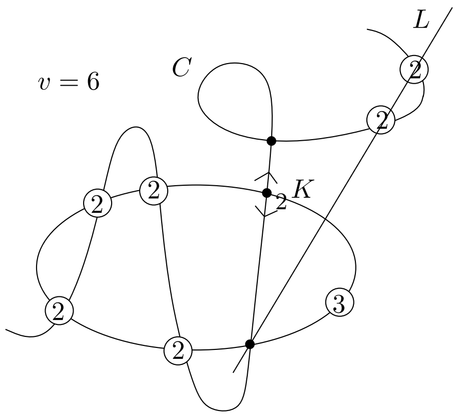
The line L through p, q intersects the cubic C in three points. We specialize the (3) with the support r at this third point. After the specialization we have on the line L: $2 * 3 + 2 = 8$ conditions.



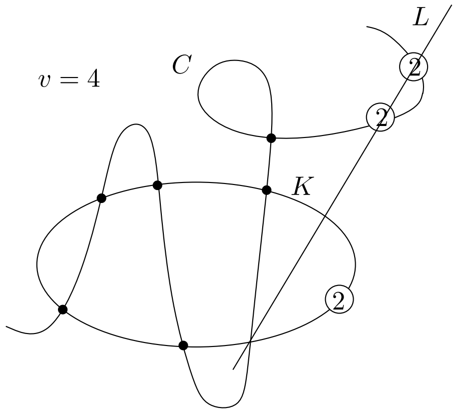
Since we are dealing with degree 7 curves we can divide by the equation of line L. After this reduction we remain with the following situation for degree 6 curves.



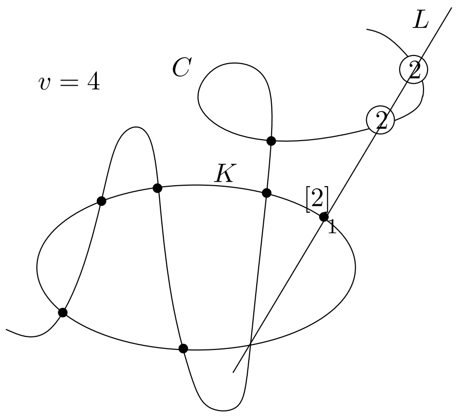
On the conic K *we have* $4 * 2 + 1 + 3 = 12$ *conditions, so we are missing one condition and we specialize the double point on* C *to the six points of* $C \cap K$



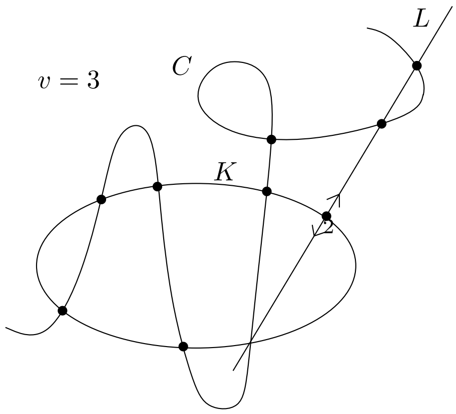
Now, on K *we have the right number of conditions, i.e.* $4*2+1+3+1 = 13 = 2*6+1$ *and we can divide by the equation of* K *and reduce to the following situation of degree 4 curves (note that the residual of the double point with respect to* K *is just a simple point).*



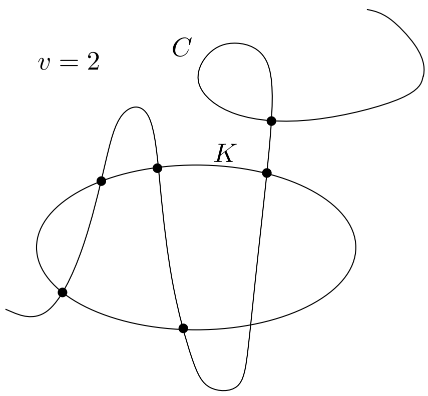
One may think that, since we are left with few nilpotents and simple points in relatively general position, we are done. Actually, the situation is quite tied and we have to be very careful in order to conclude. We move the (2) *on* K *till it arrives to the second point of* $L \cap K$ *and, by differential Horace, we take only its* $1-$*slice (we are going to exploit the divisor* L*)*



On L *we have* $2 * 2 + 1 = 5 = 4 + 1$ *conditions and we can divide by the equation of* L *and reduce to the following situation for curves of degree* 3*:*



Now on L we have $2 * 2 * 1 = 4$ conditions and we can divide again by the equation of L and finally we get:



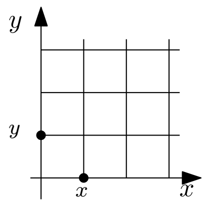
Since the remaining six points are clearly not contained in a conic we conclude that $h^0(\mathcal{I}_Z(15)) = 0$, i.e. $\mathcal{L}_Z(15)$ is non-special.

In this example, we move a double point in a different curve, but we count only one condition for this point and not two.

From the example we can see how this instrument makes the process of specializations of points of subscheme easier. At the same time it emphasizes the way the point, to which was applied the Horace differential, is tied to the curve. Let's see now which is the collision of 2 fat points. Also in this case, we don't show every step, but we want to explain what it is concretely and we want to describe an example in order to understand the utility.

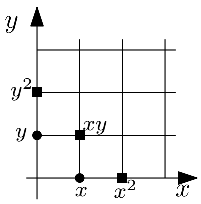
Collision of infinitesimal neighborhood.

Such system consists of letting collide two fat points in the same node in order to increase the number of conditions set out by the points on the curve. We just want to explain the way the collision happens and how to count the conditions. Before starting, we give piece of useful advice. Every infinitesimal neighborhood of a fat point (n) can be represented by an escalier, that is a step structure, where $x+y<n$, and every point represents the order of derivation at which it vanishes. For instance a fat point of size (2) represents the first infinitesimal neighborhood.



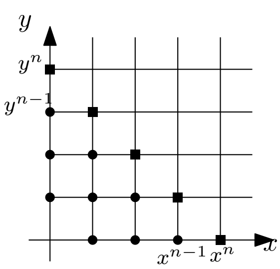
First infinitesimal neighbour $x+y<2$

While the ideal of fat point (2) is generated by the powers outside of the escalier and therefore $\mathcal{I}_{2P}=(x^2,xy,y^2)$.



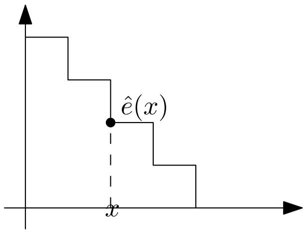
The ideal of (2): $\mathcal{I}_{2P}=(x^2,xy,y^2)$

In general every fat point of size n can be represented as follows:

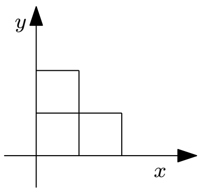


The ideal of (n), $\mathcal{I}_{nP}=(x^n,x^{n-1}y,...,y^n)$, $x+y<n$.

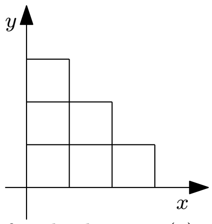
In order to proceed with the collision, we have to privilege an axis. We choose the one of the y, that we indicate as the collision axis. In view of that, we notice that an escalier e determines a function $\hat{e} : \mathbb{N} \to \mathbb{N} : x \to \hat{e}(x)$ so that $e = \{(x, y) | y < \hat{e}(x)\}$



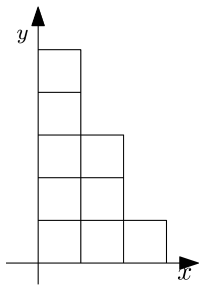
Given two escaliers, we can define the operation of the sum in this way. Let e_1, e_2 be two escalier, then the sum $e_1 + e_2$ is an escalier too, such that $e_1 \hat{+} e_2 = \hat{e_1} + \hat{e_2}$. In practice, we are placing an escalier upon another one, letting slide the several small squares along the axis y.(see illustration)



Double point (2)



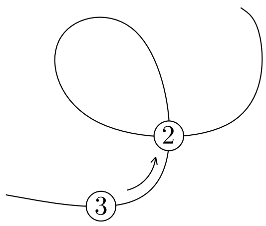
Point of multiplicity 3, (3)



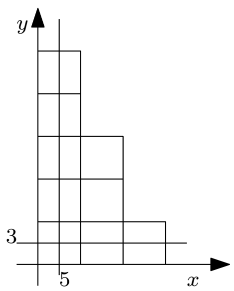
Collision in the direction y of the fat point (2) e (3)

This is the sum along the vertical axis y, that we have considered as collision axis.

Now we have to try to join what we have said so far with the Horace's method. Therefore, we presume to have a double point on a node of a nodal cubic and a point of multiplicity (3), on a branch of the same cubic, (see illustration):

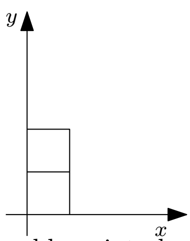


Doing the collision according the direction $\rightarrow$, we get:



Whose trace on the curve has $5 + 3 = 8$ degrees, because, being on a node, one has to count as the conditions on the x axis, the ones on the y axis. For such point we have to use the numbering [32211].

Considering this sequence of numbers as $32, 21, 1$, these indicate the number of conditions which there are in the first, second and third column of the illustration shown above(i.e. after the collision of (3) with (2)). As matter of fact, in the first column there are $3 + 2 = 5$, in the second $2 + 1 = 3$ and in the third one only a small square is left. Before the collision, the number of the conditions was given by $2 \cdot 2 + 3 = 7$. Therefore we increase by one the number of the conditions. Dividing by equation of cubic, the rest of collision results:



Double point along the direction $\uparrow$

a double point in the direction $\uparrow$, which we denote by [11], starting from [32211] one removes the first digit, 3, and decreases by one the other digits.

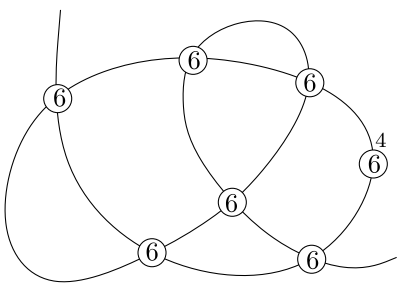
More generally, if we consider a point with multiplicity n on a node of a curve and a point of multiplicity $n+r$ on a branch of the curve, the collision of the points in the node determine $n+n+r+n+r = 3n+2r$ conditions, where $n+n+r$ are in the direction of y and $n+r$ in the direction of x. Note that, before the collision, the points determined $2n+n+r$ conditions, r less that after the collision.

There are disadvantages in using collision: in fact, after that, we cannot move anymore the point from the node of the collision and the remaining collision is not so easy to solve. That's why we will be careful in using the process.

Let us consider an example of collision of infinitesimal neighborhood. Let us begin with a curve of degree five with 6 nodes. Obviously, after a few steps, it is quite difficult to treat with it, so that we specialize it in the union of a quartic with a line. This makes sense since the dimension of the linear system of the quintc curves in $\mathbb{P}^2$ is bigger than the dimension of the linear system of quartic curves with a line. This process is true in a more general setting and it can apply to different curves.

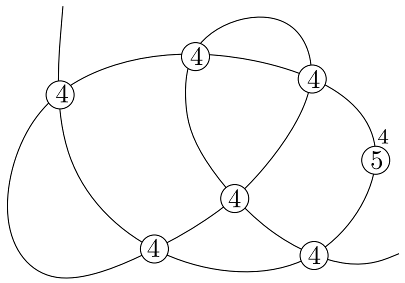
Example 3.4.2 *We consider the subscheme* $Z = 6P_1+6P_2+6P_3+6P_4+6P_5+6P_6+6P_7+6P_8+6P_9+6P_{10}$ *of* $\mathbb{P}^2$. *We define* Z *a homogeneous subscheme because the multiplicity of points of* Z *is constant. The critical value of* Z *is* 19 *and* $\mathcal{L}_Z(19)$ *is non-special if and only if* $h^{-}(\mathcal{I}_Z(19)) = 0$. *As above, in this case we don't add simple point. We start specializing all the points in a quintic curve,* C, *with six nodes in this way: six* (6) *on the nodes and the other four on the curve.*

$v = 19$



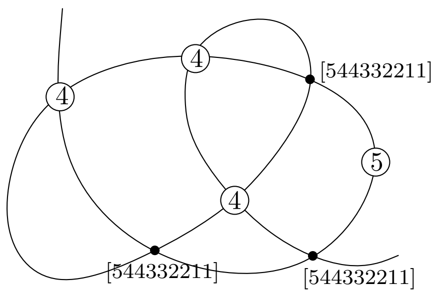
On C *we have:* $6 \cdot 12 + 24 = 96 = 5 \cdot 19 + 1$ *conditions, so we can divide by the equation of the quintic and we obtain the following situation*

$v = 14$



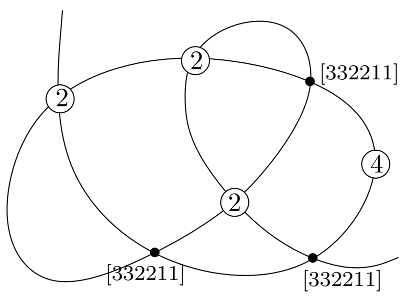
We do a collision of three (5) *on three nodes.*

$v = 14$

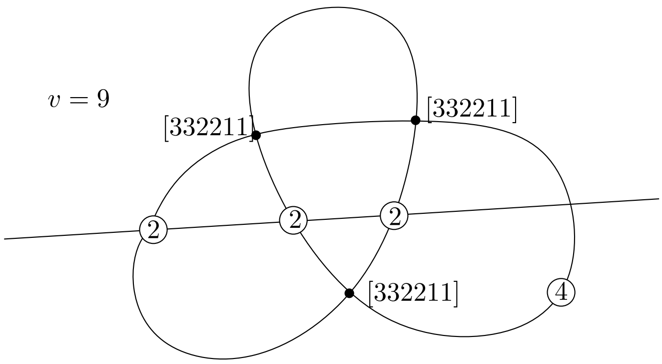


In these nodes we have $[544332211]$ *that imposes* $9+5=14$ *conditions. On the quintic we have* $3\cdot 14+3\cdot 8+5=71=5\cdot 14+1$ *conditions and so we can divide by the equation of* C *and reduce to the following situation for the curve of degree nine.*

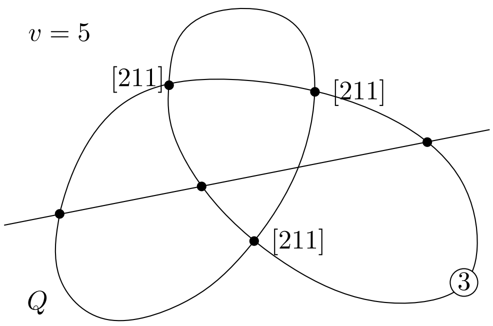
$v = 9$



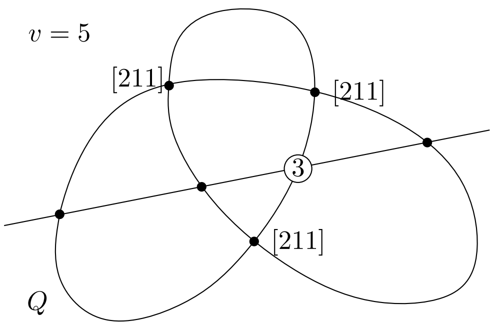
Now we specialize the quintic C *in a union of a quartic with three nodes* Q*, and a line* L*, in this way:*



Any $[332211]$ *gives* $6+3=9$ *conditions and on* Q *we have* $3\cdot 9+6+4=37=4\cdot 9+1$ *conditions, so we can divide by equation of quartic and we get the following situation for the curves of degree* 5.

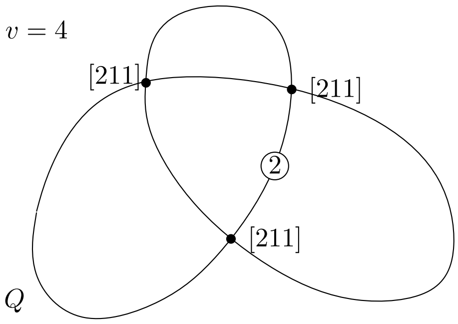


We move the fat point (3) *on the line* L



On L *we have* $3+3=6=5\cdot 1+1$ *conditions and so we can divide by the equation of the line, and we obtain the following situation for the degree* 4 *curves.*

$v = 4$



Any $[211]$ *gives* 5 *conditions so that, on a quartic, we have* $3 \cdot 5 + 2 = 17 = 4 \cdot 4 + 1$ *conditions and we can divide by the equation of* Q. *In the end we obtain*

$v = 0$

Since the remaining point is clearly not contained in a curve of degree zero, we conclude that $h^0(\mathcal{I}_Z(19)) = 0$ *and thus* $\mathcal{L}_Z(19)$ *is non-special.*

3.5 Cremona transformations.

In this section we deal with Cremona transformations. We will use them in this work in order to reduce more linear systems to only one, decreasing the number of cases to study.
Cremona transformations act on the blow-up points. In other words, applying a Cremona transformation is the same as changing the basis of $Pic(S)$, where we denote by S the $\mathbb{P}^2$ blown-up in the base points of the considered system. After a Cremona transformation, the system seems to be different from the starting one, even if it just seen in two different basis.

We recall some notation. Let $Z = b_1P_1 + ...b_kP_k$ be a subscheme of $\mathbb{P}^2$ and let $p : S \to \mathbb{P}^2$ denote the blowing-up at the P_i's. Let E_i be the corresponding exceptional curves. A bases of $Pic(S)$ is given by $\varepsilon = (l - E_1 - ... - E_k)$ where l is the class of line in $\mathbb{P}^2$; a such basis is also called an exceptional configuration.

If $D \in \mathcal{I}_Z(v)$ where v is the critical value of Z then we can write $D = (v; b_1, ..., b_k)$

in $Pic(S)$.

DEFINITION 3.5.1 *As usual an exceptional curve (of the first kind), E, is a curve isomorphic to $\mathbb{P}^1$ with $E^2 = -1$.*
We will say that $\Sigma = \sum n_i E_i$ is a bunch of disjoint exceptional curves if $n_i \geq 0$ for all i, each E_i is an exceptional curve and $E_i \cdot E_j = 0$ if $i \neq j$.

We have $Pic(S) \simeq \mathbb{Z}^{k+1}$ with the basis $(l; -E_1, ..., -E_K)$ given by exceptional configuration ε. The intersection product defines a bilinear form on $Pic(S)$. Let $r_0 = l - E_1 - E_2 - E_3, r_i = E_i - E_{i+1}$. Moreover associated to the r_i's we have the reflections $s_i : Pic(S) \to Pic(S)$ defined by $s_i(x) = x + (x \cdot r_i)r_i$. the s_i's preserve the intersection product and fix the canonical bundle of $\mathbb{P}^2$, K.

We are interested in s_0 (Cremona transformation). We have:
$s_0(l) = 2l - E_1 - E_2 - E_3$,
$s_0(-E_1) = E_2 + E_3 - l$,
$s_0(-E_2) = E_1 + E_3 - l$,
$s_0(-E_3) = E_1 + E_2 - l$,
$s_0(-E_i) = -E_i$ for $i > 3$.

If $(v; b_1, ..., b_k)$ are the coordinates of D in the basis ε, then $B' = (s_0(l); s_0(-E_i))$ is a basis of $Pic(S)$ and the coordinates of D in this basis are $(v'; c_i)$ where:
$v' = 2v - b_1 - b_2 - b_3'$,
$c_1 = v - b_2 - b_3$,
$c_2 = v - b_1 - b_3$,
$c_3 = v - b_1 - b_2$,
$c_i = b_i$ for $i > 3$.

If the points P_i are distinct and in general position, the basis B' corresponds to an exceptional configuration $\varepsilon' = (l'; -E_1', ..., -E_k')$; moreover the corresponding points, obtained by blowing down, P_i are again distinct and in general position.

Finally we observe that : $v' < v$ if and only if $a < b_1 + b_2 + b_3$.
Of course if $b_1 \geq b_2 \geq ... \geq b_9$ then $c_1 \geq c_2 \geq c_3$ but it could be that $c_3 < c_4$.

We show an example

EXAMPLE 3.5.2 *Let $D = (17; 8, 6, 6, 5, 5, 5, 5, 4, 4, 4)$ be a divisor in $Pic(S)$, we apply to D the Cremona transformation. The new coordinates of D are (v', c_i) where:*

$v' = 34 - 8 - 6 - 6 = 14$,
$c_1 = 17 - 6 - 6 = 5$,
$c_2 = 17 - 8 - 6 = 3$,
$c_3 = 17 - 8 - 6 = 3$,
$c_i = b_i$ *for* $i > 3$.

After the transformations, it occurs that $D' = (14; 5, 5, 5, 5, 5, 4, 4, 4, 3, 3)$, *since* $2v - c_1 - c_2 - c_3 = 28 - 15 = 13 < 14$, *we can apply again the Cremona transformation to* D'. *We obtain* $D'' = (13; 5, 5, 4, 4, 4, 4, 4, 4, 3, 3)$ *and another time we apply the transformation. In the end we have*
$D''' = (12; 4, 4, 4, 4, 4, 4, 4, 3, 3, 3)$ *and we stop the transformation process because* $2v - c_1 - c_2 - c_3 = 12 = v$.

3.6 Subschemes of $\mathbb{P}^2$ with ten fat points of multiplicity less than nine.

In this section we get to the heart of the matter focusing our attention on particular subschemes $Z = b_1P_1 + ... + b_{10}P_{10}$ of $\mathbb{P}^2$ for which exists almost a point of multiplicity like 8 so that $4 \leq b_i \leq 8$for all i.

We don't consider the subscheme with a point of multiplicity minor than 4 because the associated linear system verifies the conjecture of Harbourne-Hirschowitz.

Let's begin with the studying of he first cases. Taken ten general points $P_1, ..., P_{10}$ we consider the subscheme $Z = 8P_1 + 4P_2 + 4P_3 + ... + 4P_{10}$, In order to make it easy, we indicate Z with the list $(8, 4, 4, 4, 4, 4, 4, 4, 4, 4,)$. The critical value of Z is 15 and we denote with $x := h^0(\mathcal{O}_{\mathbb{P}^2}(v)) - h^0(\mathcal{O}_Z(v))$, this number indicates the dimension which $\mathcal{L}_Z(v)$ need to have in order to be non-special and it also represents the number of the simple points to have add to subscheme Z in order to start with Horace's method. In this case $x = 10$. Considering now $Z_1 = (8, 5, 4, 4, 4, 4, 4, 4, 4, 4)$ and $Z_2 = (8, 5, 5, 4, 4, 4, 4, 4, 4, 4)$, they have critical value equal 15 and respectively x equal 5 and 0. By the proposition 3.3.1, if Z_2 is non-special then Z_1 e Z are non-special too, afterwards lets analyze the case Z_2. Before starting with the Horace's method, we apply the Cremona's transformation s_0 to Z_2, we get

$s_0((15;8,5,5,4,4,4,4,4,4,4)) = (6;2,2,2,2,2,2,2,2,2,1)$ which is a system, where the conjecture of Harbourne-Hirschowitz holds.

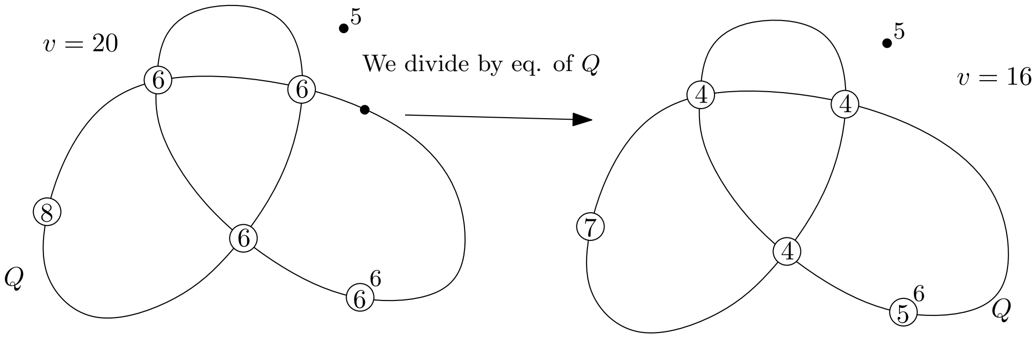
This proceeding bring us to a reduction of several cases, that we list below:

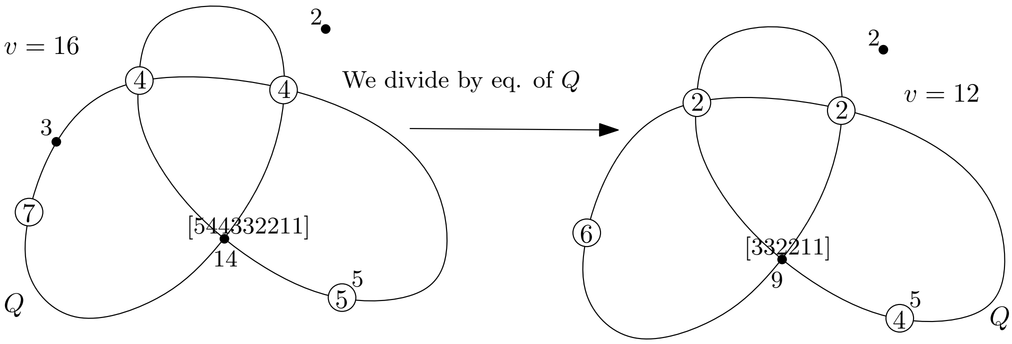
$Z_1 = (2,2,2,2,2,2,2,2,2,1)$ $v = 6$ $x = 0$
$Z_2 = (4,3,3,3,3,3,3,3,3,3)$ $v = 10$ $x = 2$
$Z_3 = (5,4,4,4,4,4,4,4,4,4)$ $v = 13$ $x = 0$
$Z_4 = (3,3,3,3,3,3,3,3,2,2)$ $v = 9$ $x = 1$
$Z_5 = (5,4,4,4,4,4,4,4,4,3)$ $v = 13$ $x = 4$
$Z_6 = (4,4,4,4,4,4,4,3,3,3)$ $v = 12$ $x = 3$
$Z_7 = (5,5,5,5,5,5,5,5,5,5)$ $v = 16$ $x = 0$
$Z_8 = (6,5,5,5,5,5,5,5,5,4)$ $v = 16$ $x = 2$
$Z_9 = (6,6,6,6,6,6,6,6,6,6)$ $v = 19$ $x = 0$
$Z_{10} = (4,4,4,4,4,4,4,4,3,1)$ $v = 12$ $x = 4$
$Z_{11} = (4,4,4,4,4,4,4,4,3,2)$ $v = 12$ $x = 2$
$Z_{12} = (5,5,5,5,5,5,5,5,4,3)$ $v = 15$ $x = 0$
$Z_{13} = (7,6,6,6,6,6,6,6,5,5)$ $v = 19$ $x = 5$
$Z_{14} = (3,3,3,3,3,3,3,2,2,2)$ $v = 9$ $x = 4$
$Z_{15} = (6,6,6,6,6,6,5,5,5,5)$ $v = 18$ $x = 4$
$Z_{16} = (6,6,6,6,6,6,6,5,5,4)$ $v = 18$ $x = 3$
$Z_{17} = (7,6,6,6,6,6,6,6,6,4)$ $v = 19$ $x = 4$
$Z_{18} = (8,7,7,7,7,7,7,7,6,6)$ $v = 22$ $x = 2$
$Z_{19} = (8,7,7,7,7,7,7,7,7,5)$ $v = 22$ $x = 1$
$Z_{20} = (4,4,4,4,4,4,4,4,4,0)$ $v = 12$ $x = 1$
$Z_{21} = (5,5,5,5,5,5,5,5,4,2)$ $v = 15$ $x = 3$
$Z_{22} = (3,3,3,3,3,3,3,3,2,1)$ $v = 9$ $x = 3$
$Z_{23} = (7,7,7,7,7,7,6,6,6,6)$ $v = 21$ $x = 1$
$Z_{24} = (7,7,7,7,7,7,7,6,6,5)$ $v = 21$ $x = 0$
$Z_{25} = (2,2,2,2,2,2,2,2,2,0)$ $v = 6$ $x = 1$
$Z_{26} = (4,4,4,4,4,4,4,4,4,4)$ $v = 13$ $x = 5$
$Z_{27} = (6,6,6,6,6,6,6,6,4,4)$ $v = 18$ $x = 2$
$Z_{28} = (1,1,1,1,1,1,1,1,1,0)$ $v = 3$ $x = 1$

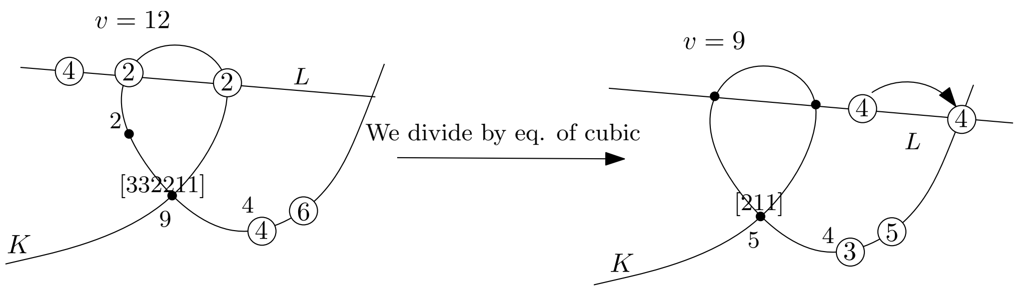
$Z_{29} = (2, 2, 2, 2, 2, 2, 2, 2, 1, 1)$ $v = 6$ $x = 2$
$Z_{30} = (5, 5, 5, 5, 5, 5, 5, 4, 4, 4)$ $v = 15$ $x = 1$
$Z_{31} = (5, 5, 5, 5, 5, 5, 5, 4, 4, 3)$ $v = 15$ $x = 5$
$Z_{32} = (7, 7, 7, 7, 7, 7, 7, 7, 7, 6)$ $v = 22$ $x = 3$
$Z_{33} = (8, 6, 6, 6, 6, 6, 6, 6, 6, 6)$ $v = 20$ $x = 6$
$Z_{34} = (8, 8, 7, 7, 7, 7, 7, 7, 7, 7)$ $v = 23$ $x = 4$
$Z_{35} = (8, 8, 8, 8, 8, 7, 7, 7, 7, 7)$ $v = 24$ $x = 5$
$Z_{36} = (8, 8, 8, 8, 8, 8, 7, 7, 7, 6)$ $v = 24$ $x = 1$
$Z_{37} = (8, 8, 8, 8, 8, 8, 8, 7, 6, 6)$ $v = 24$ $x = 3$
$Z_{38} = (8, 8, 8, 8, 8, 8, 8, 7, 7, 5)$ $v = 24$ $x = 2$
$Z_{39} = (8, 8, 8, 8, 8, 8, 8, 8, 6, 5)$ $v = 24$ $x = 1$
$Z_{40} = (8, 8, 8, 8, 8, 8, 8, 8, 7, 7)$ $v = 25$ $x = 7$
$Z_{41} = (8, 8, 8, 8, 8, 8, 8, 8, 8, 6)$ $v = 25$ $x = 6$

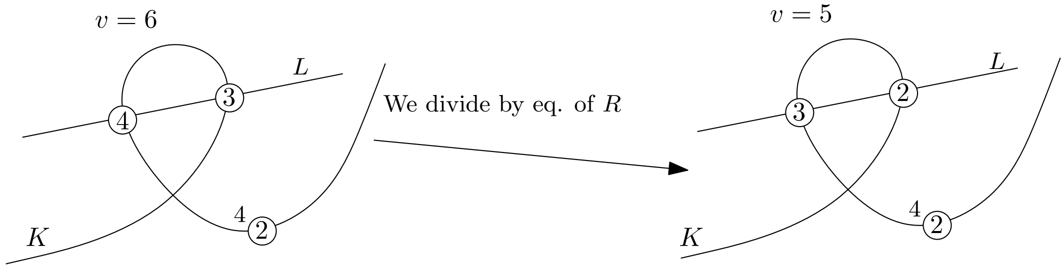
We note that the case Z_{10} is contained in Z_{11}; Z_{21} is contained in Z_{12}; Z_{31} is contained in Z_{30}; the cases Z_{25} and Z_{29} are contained in Z_1; Z_{22} and Z_{14} are contained in Z_4 and Z_{26},Z_5 are contained in Z_3; proposition 3.3.1 leave us with only 32 cases to analyze. In 11 cases we can find points of multiplicity eight and in these cases we will apply the Horace's Method to prove that the conjecture of Harbourne-Hirschowitz holds. Before proceeding with the method is better to write down some notations . We indicate the conic curves with C, the cubic curves with K, the quartic curves with Q, the quintic curves with Y and the lines are recognizable. As there is an only line containing two points, you will notice, that the lines containing two points will be not drown anymore between two passages. In order to avoid repetitions, we are going to present just pictures in our demonstrations, because all the passages, we are illustrating, are very similar to the ones already explained in the previous examples and moreover they are all easy to understand.

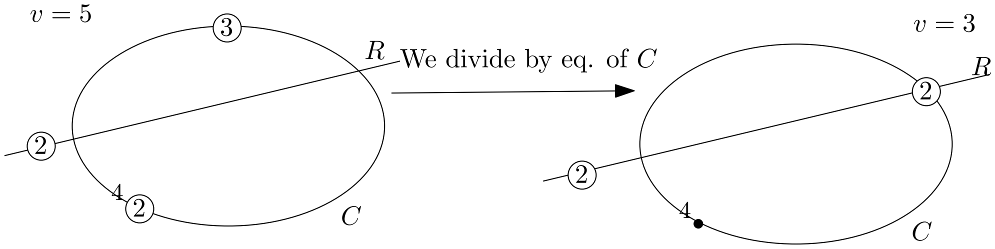
Case $(8, 6, 6, 6, 6, 6, 6, 6, 6, 6)$ $v = 20$ $x = 6$.


v = 20
We divide by eq. of Q
v = 16
Q
Q

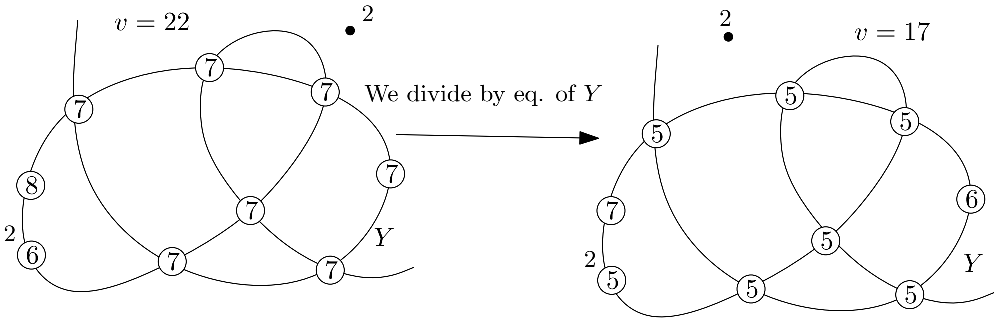

v = 16
We divide by eq. of Q
v = 12
[544332211]
[332211]
Q
Q

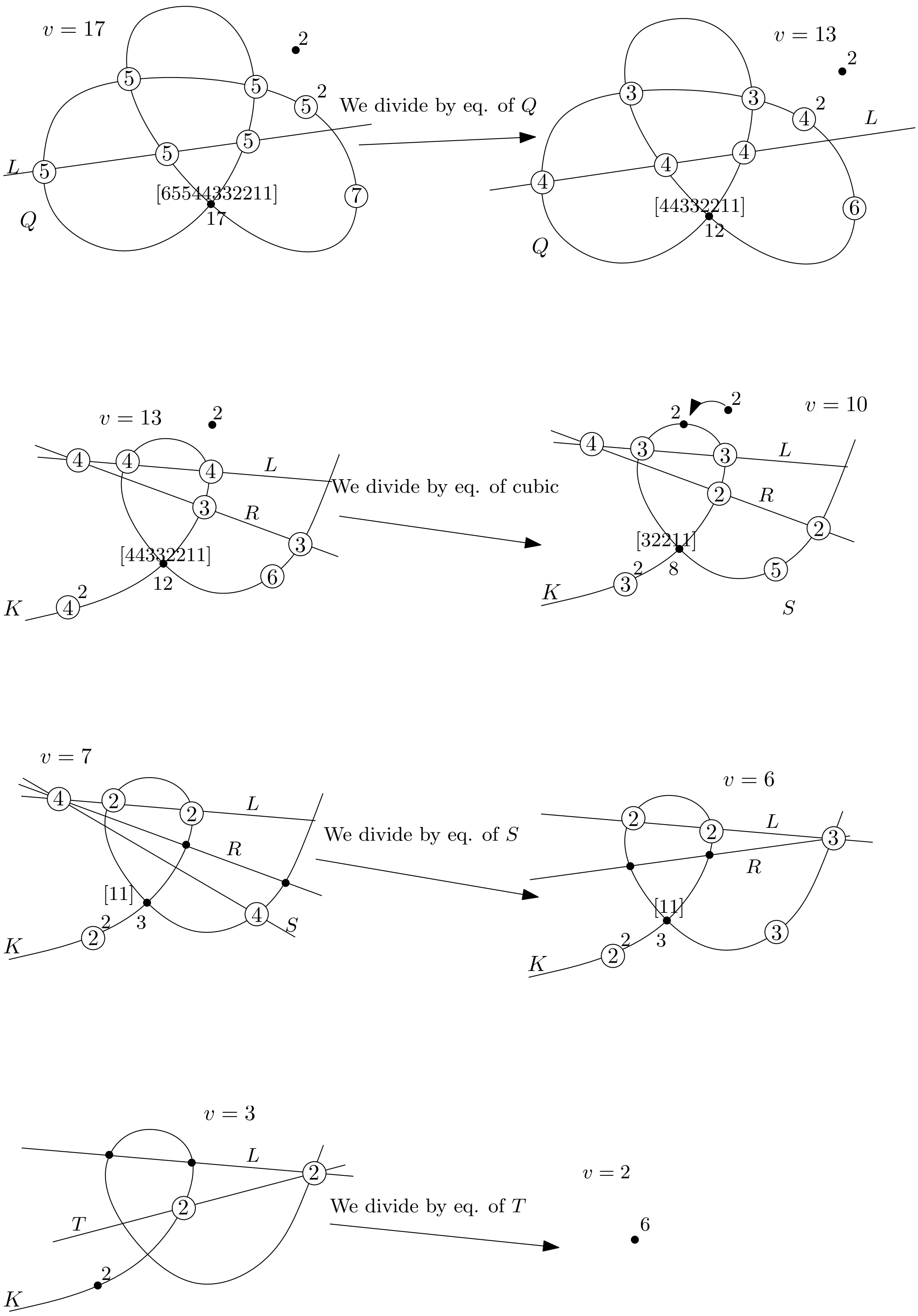

v = 12
We divide by eq. of cubic
v = 9
L
L
K
K
[332211]
[211]



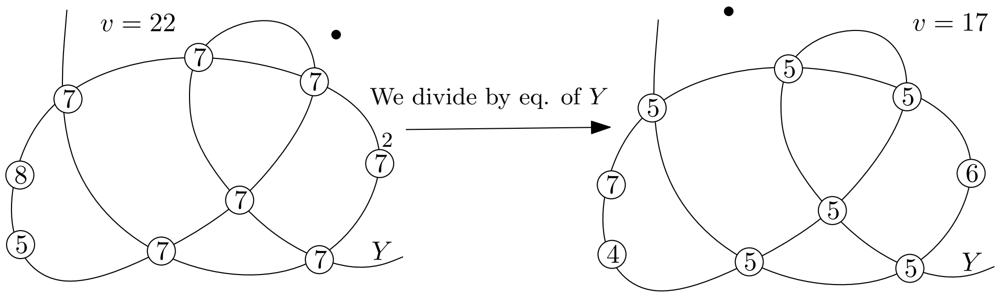


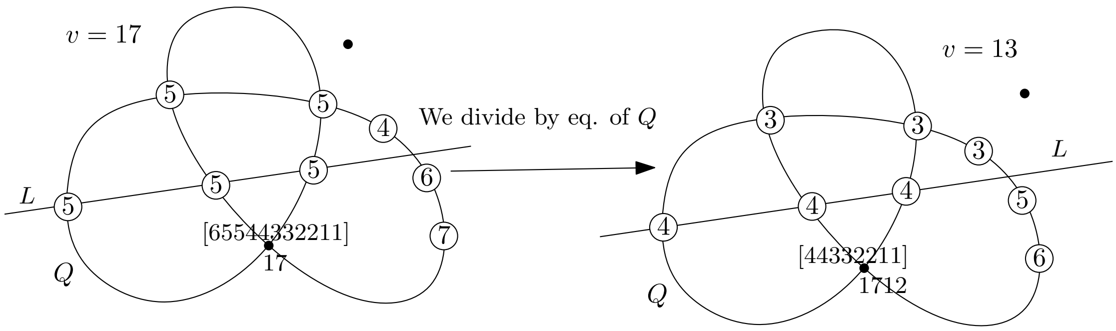
Case $(8,7,7,7,7,7,7,7,6,6)$ $v = 22$ $x = 2$

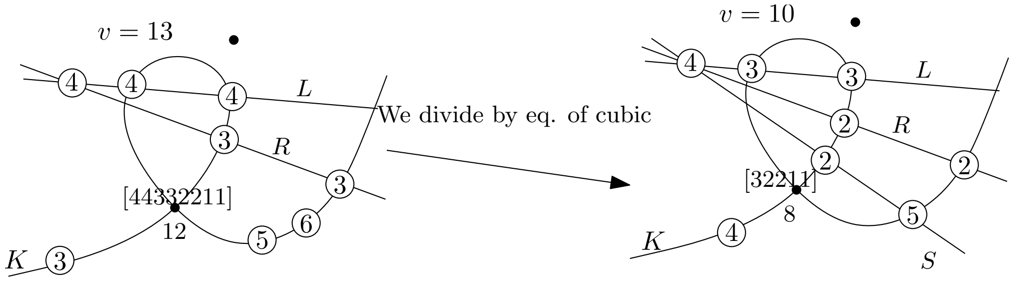


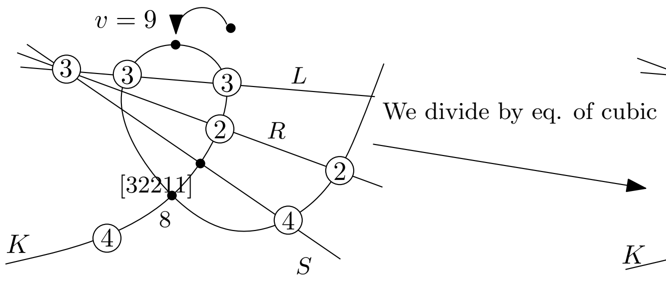

v = 17
L
Q
[65544332211]
17
We divide by eq. of Q
v = 13
L
Q
[44332211]
12
v = 13
L
R
K
[44332211]
12
We divide by eq. of cubic
v = 10
L
R
K
S
[32211]
8
v = 7
L
R
K
S
[11]
3
We divide by eq. of S
v = 6
L
R
K
[11]
3
v = 3
L
T
K
We divide by eq. of T
v = 2
6

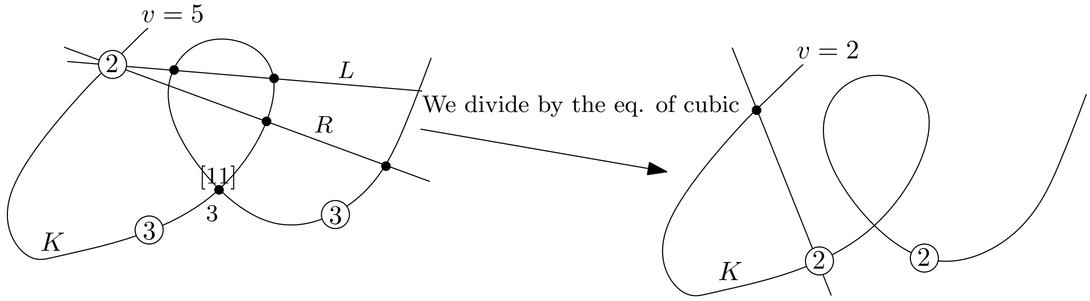
Case $(8,7,7,7,7,7,7,7,7,7,5)$ $v=22$ $x=1$

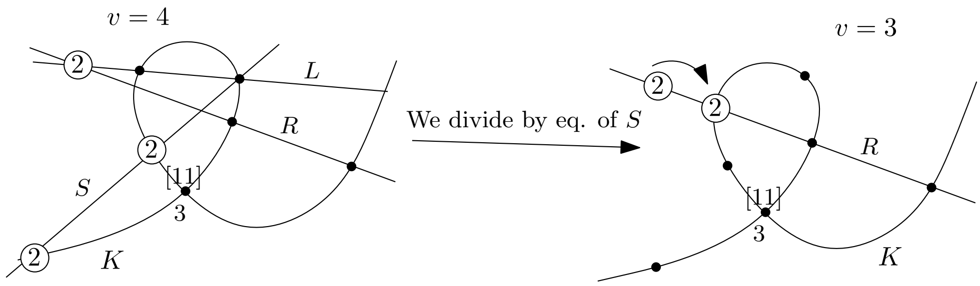




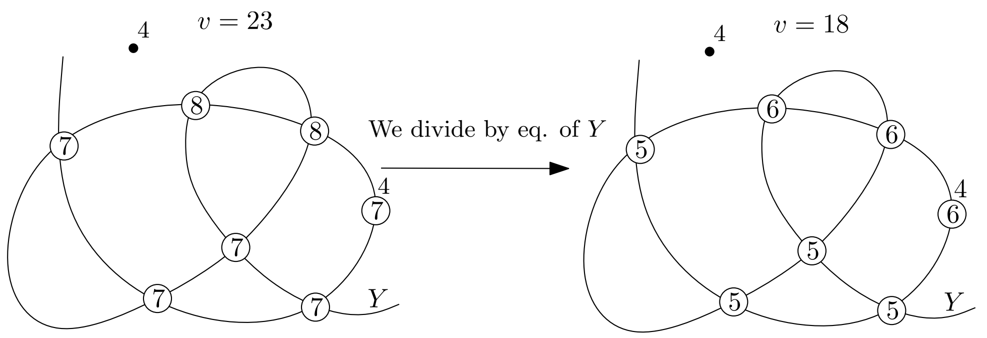




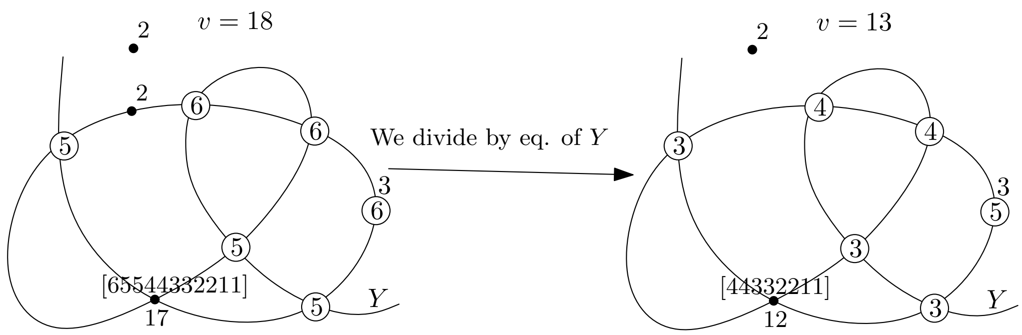




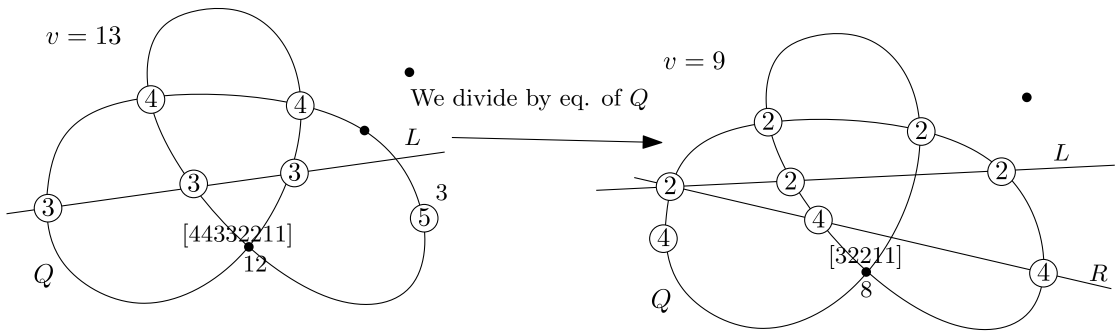
Case $(8, 8, 7, 7, 7, 7, 7, 7, 7, 7)$ $v = 23$ $x = 4$



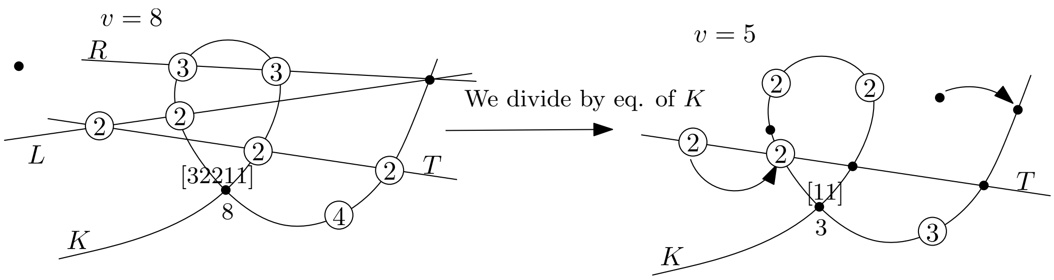
v = 23
We divide by eq. of Y
v = 18
Y
Y

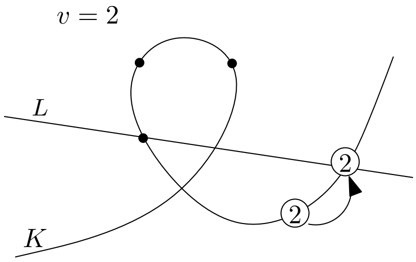


v = 18
We divide by eq. of Y
v = 13
[65544332211]
17
[44332211]
12
Y
Y



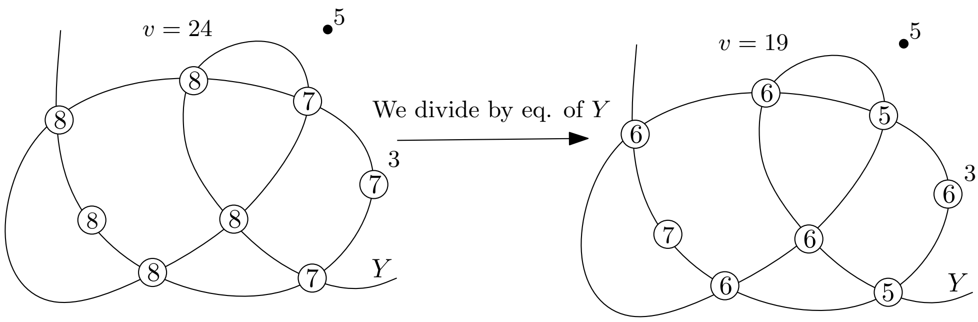
v = 13
We divide by eq. of Q
v = 9
L
L
R
Q
Q
[44332211]
12
[32211]
8

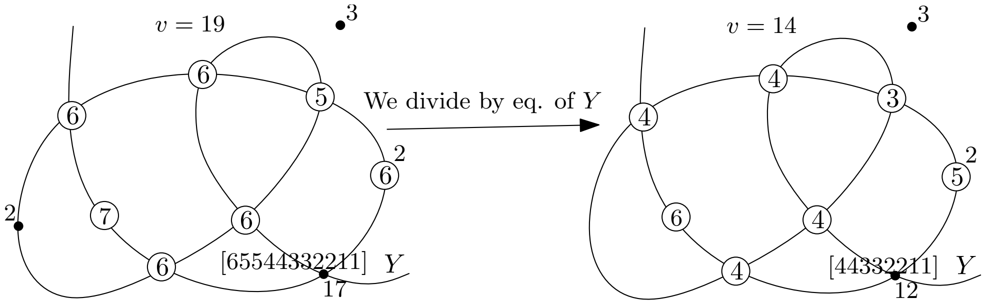


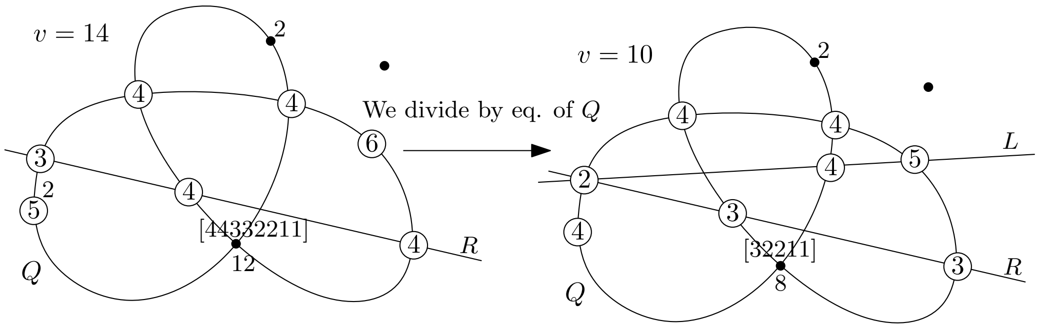


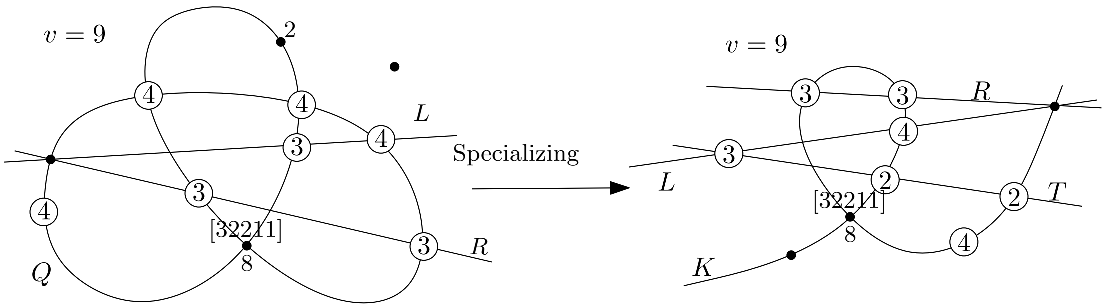
In the end, dividing by the equation of the line L, we rest with $v = 1$ and three simple point. Absurd.

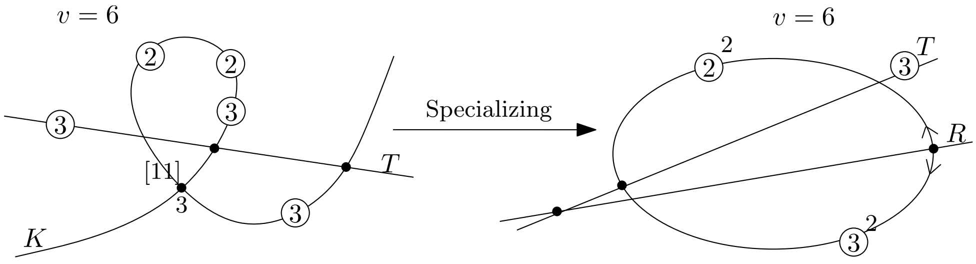
Case $(8, 8, 8, 8, 8, 7, 7, 7, 7, 7)$ $v = 24$ $x = 5$

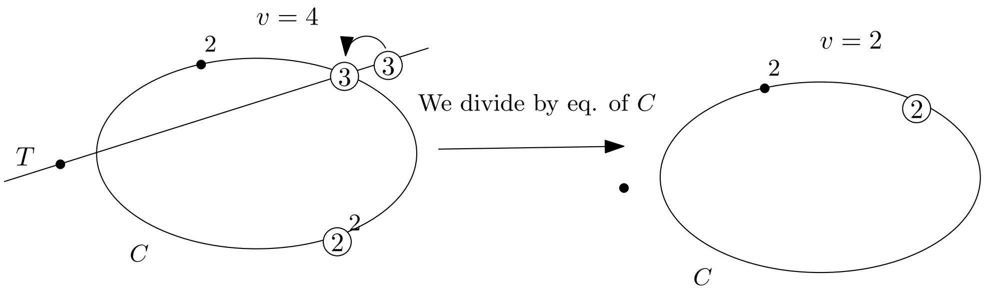



v = 19
We divide by eq. of Y
v = 14
[65544332211]
17
Y
[44332211]
12

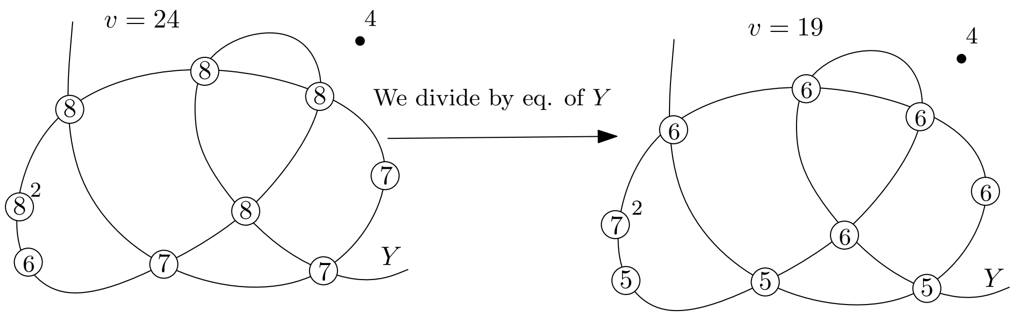

v = 14
We divide by eq. of Q
v = 10
[44332211]
12
[32211]
8
Q
R
L

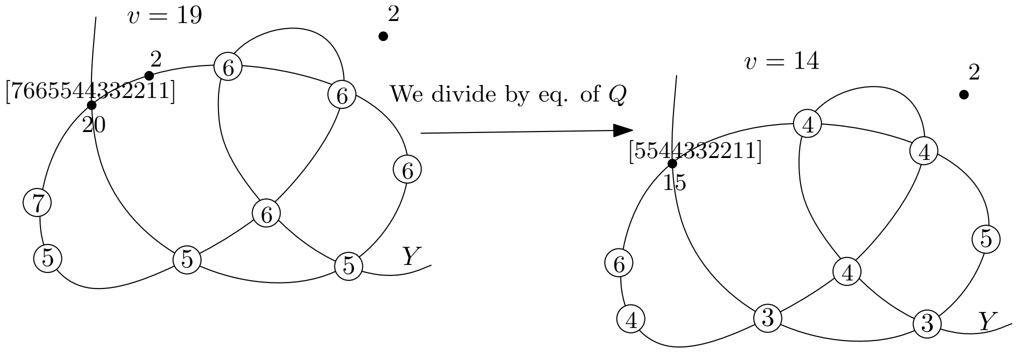

v = 9
Specializing
v = 9
[32211]
8
Q
R
L
T
K

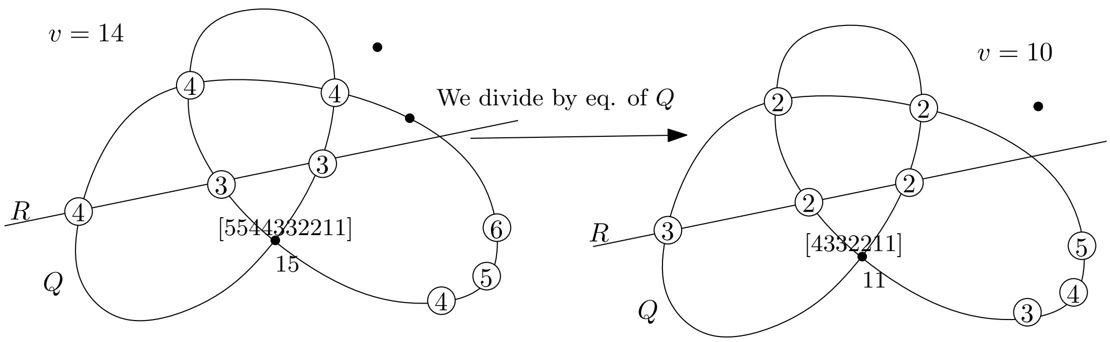


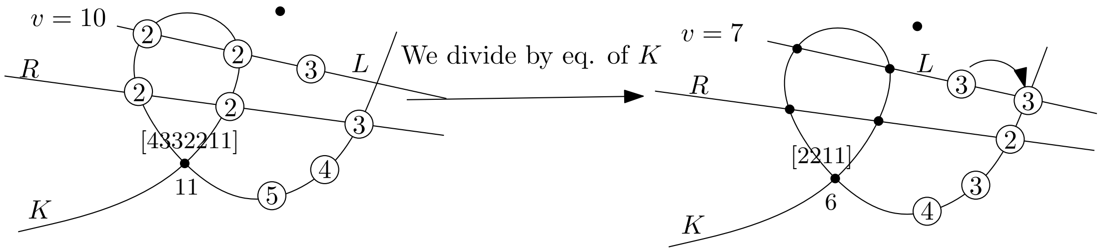


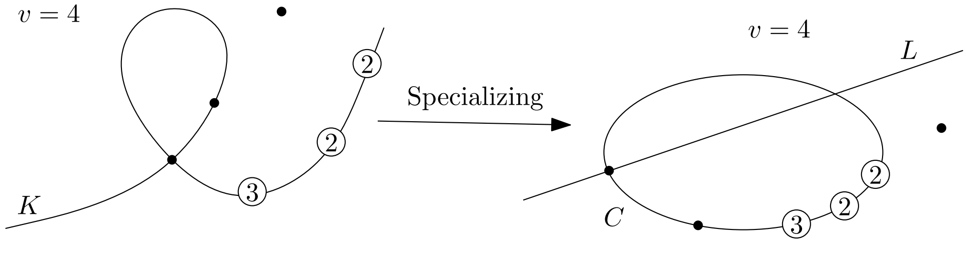
Case $(8,8,8,8,8,8,7,7,7,6)$ $v = 24$ $x = 1$

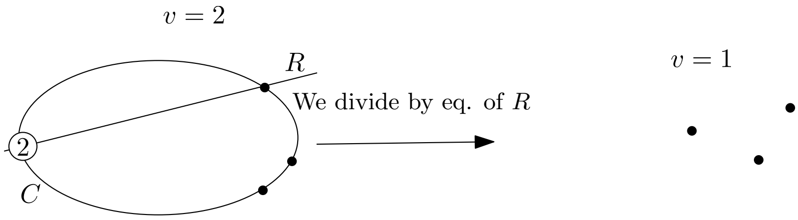



v = 19
2
2
[7665544332211]
20
We divide by eq. of Q
Y
v = 14
2
[5544332211]
15
Y

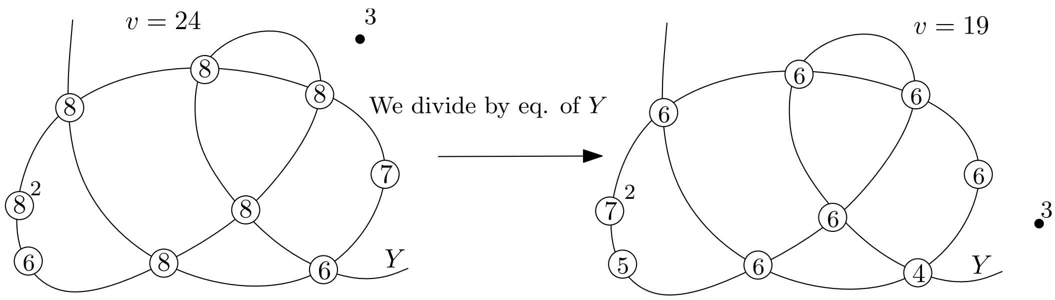

v = 14
We divide by eq. of Q
R
[5544332211]
15
Q
v = 10
R
[4332211]
11
Q

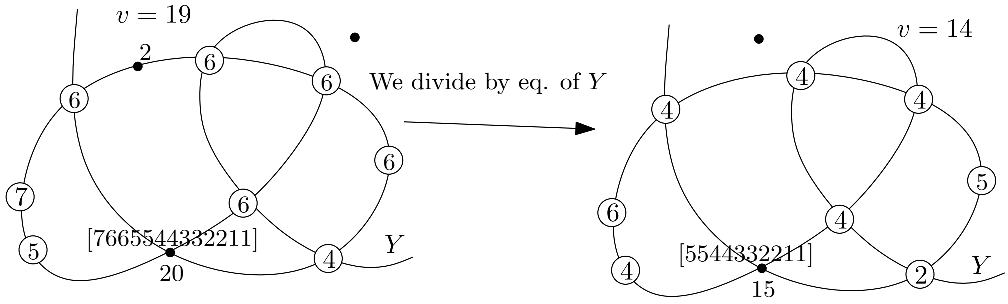

v = 10
R
L
We divide by eq. of K
[4332211]
11
K
v = 7
R
L
[2211]
6
K



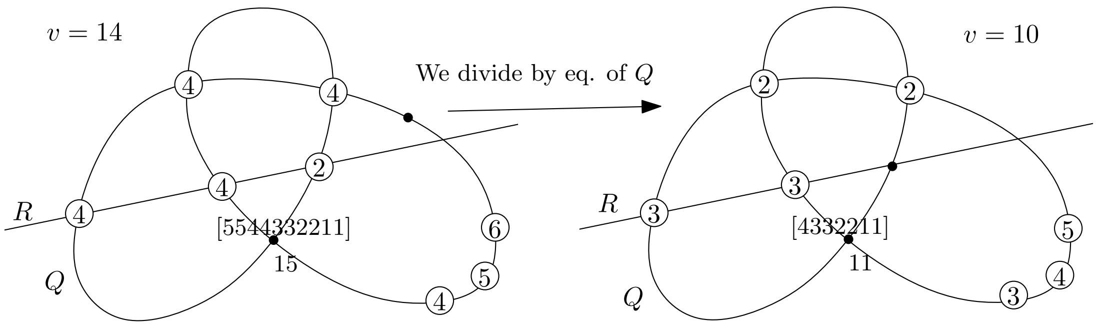


Case $(8, 8, 8, 8, 8, 8, 8, 7, 6, 6)$ $v = 24$ $x = 3$

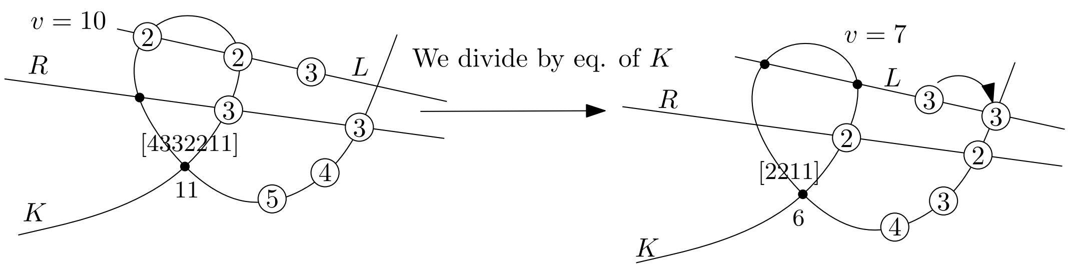




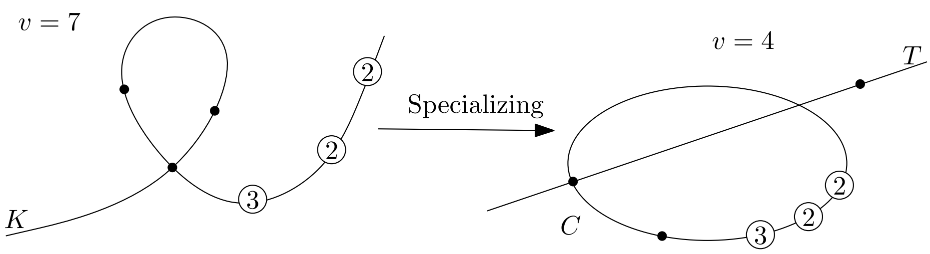
v = 19
2
6
6
6
6
7
6
5
[7665544332211]
4
20
Y
We divide by eq. of Y
v = 14
4
4
4
5
6
4
[5544332211]
4
2
15
Y



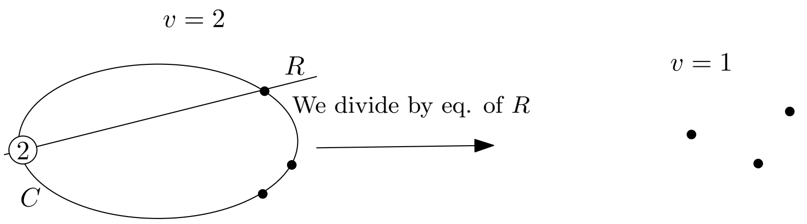
v = 14
4
4
4
2
R
4
[5544332211]
6
15
5
Q
4
We divide by eq. of Q
v = 10
2
2
3
R
3
[4332211]
5
11
4
Q
3



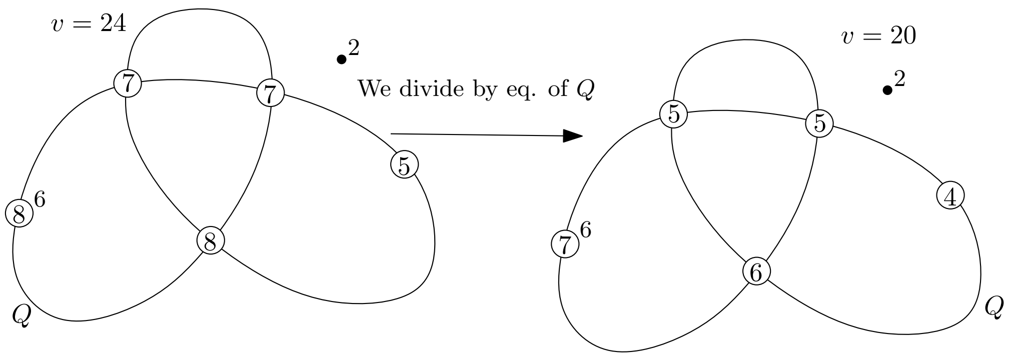
v = 10
2
2
3
L
R
3
3
[4332211]
4
11
5
K
We divide by eq. of K
v = 7
L
3
3
R
2
2
[2211]
3
6
4
K

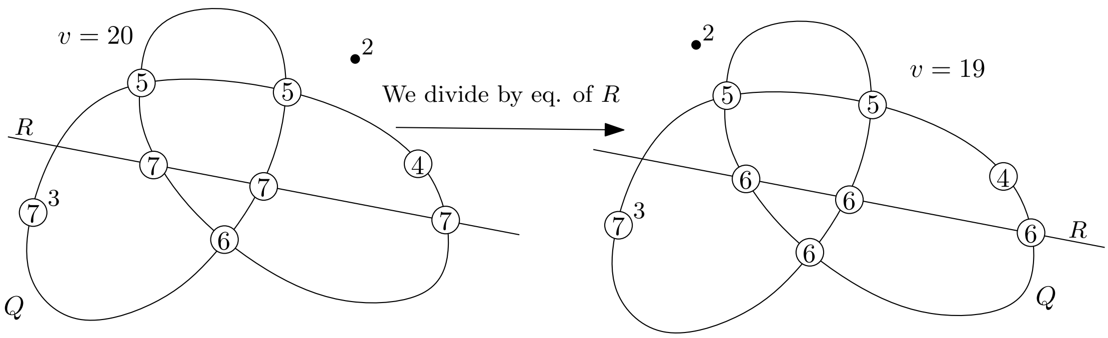


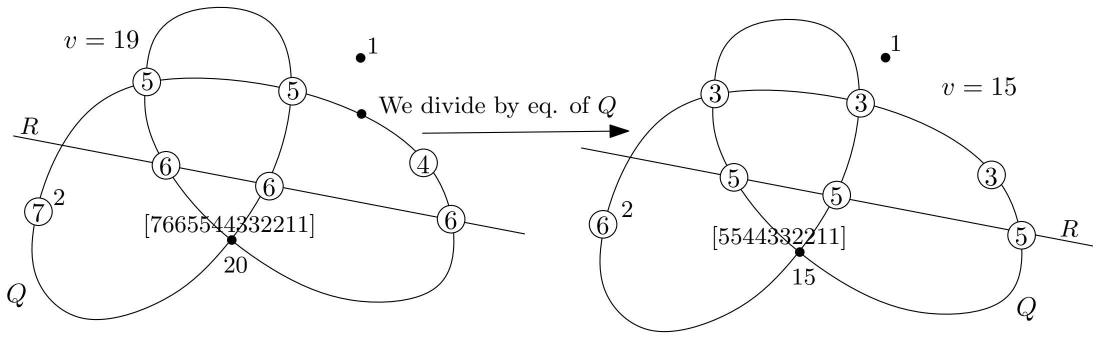
v = 7
2
2
3
K
Specializing
v = 4
T
2
2
3
C

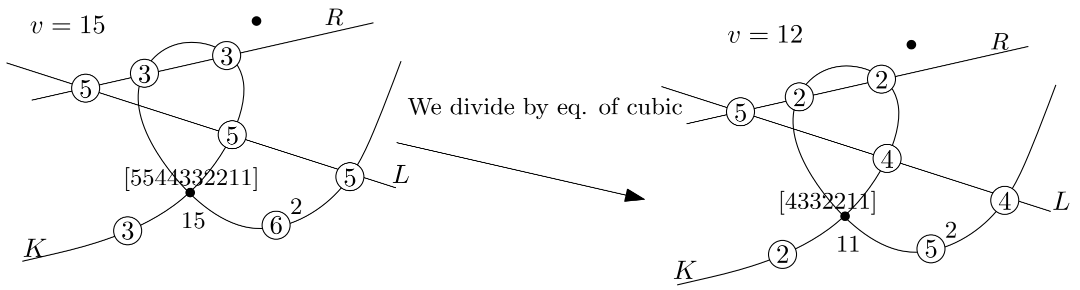


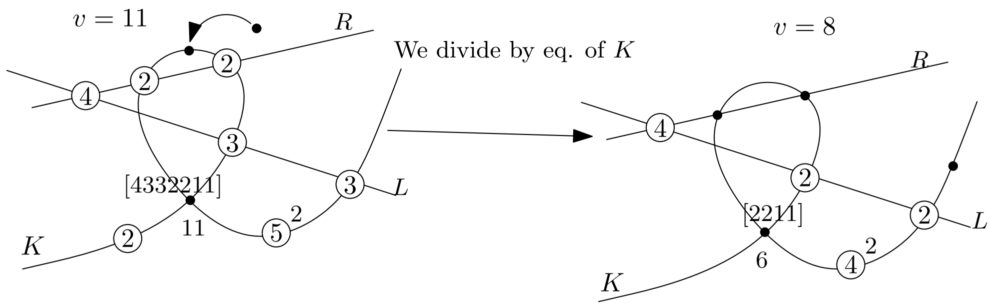
Case $(8, 8, 8, 8, 8, 8, 8, 7, 7, 5)$ $v = 24$ $x = 2$

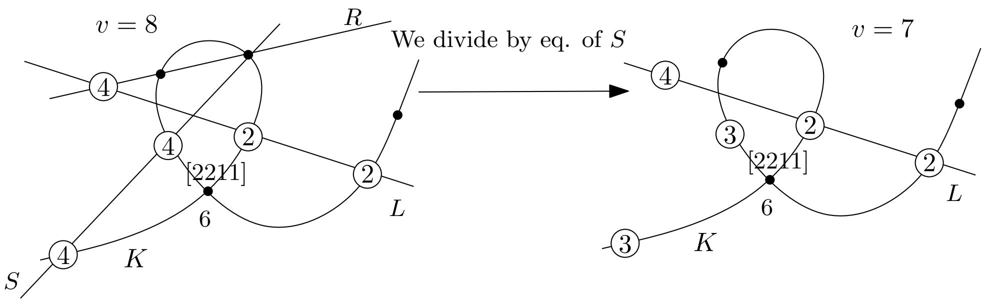


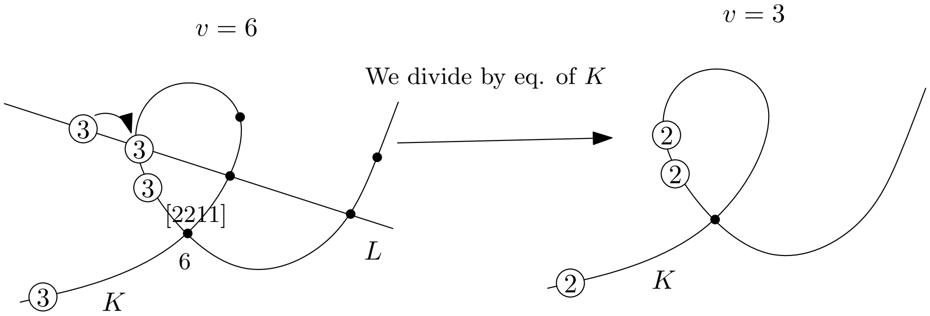


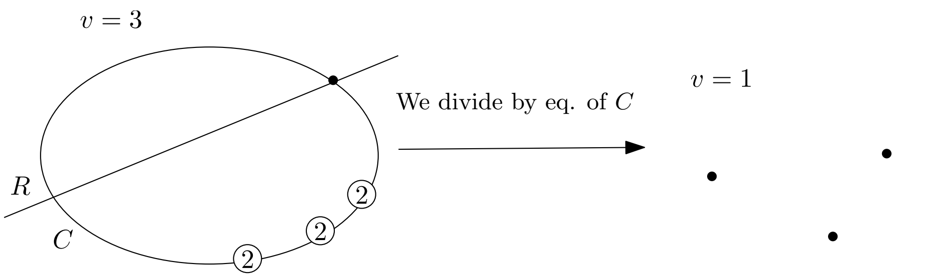

v = 19
1
R
5
5
6
6
4
7
2
6
[7665544332211]
20
Q
We divide by eq. of Q
1
v = 15
3
3
5
5
3
6
2
5
R
[5544332211]
15
Q


v = 15
R
3
3
5
5
5
[5544332211]
L
15
6
2
3
K
We divide by eq. of cubic
v = 12
R
2
2
5
4
4
[4332211]
L
11
5
2
2
K

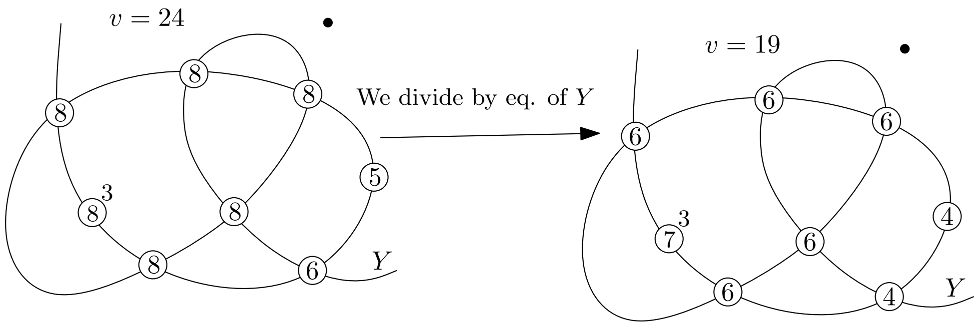

v = 11
R
2
2
4
3
3
[4332211]
L
11
5
2
2
K
We divide by eq. of K
v = 8
R
4
2
2
[2211]
L
6
4
2
K

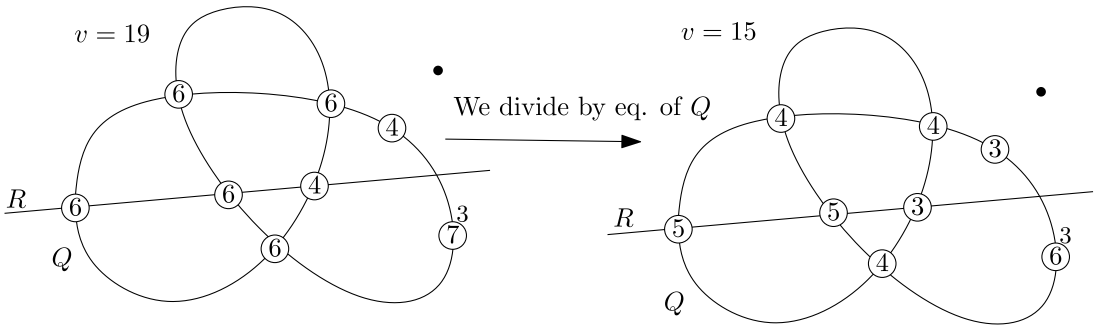


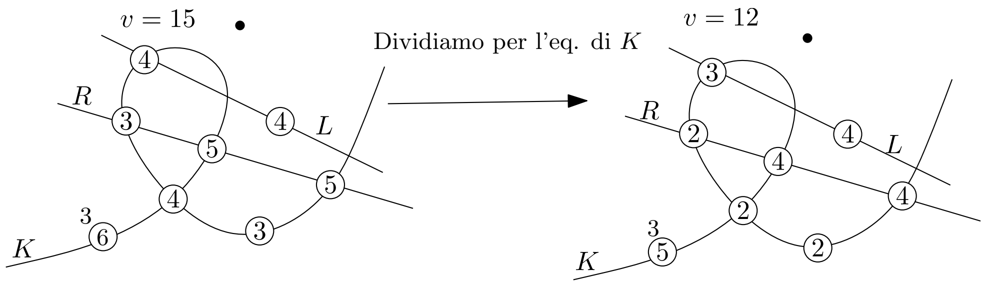


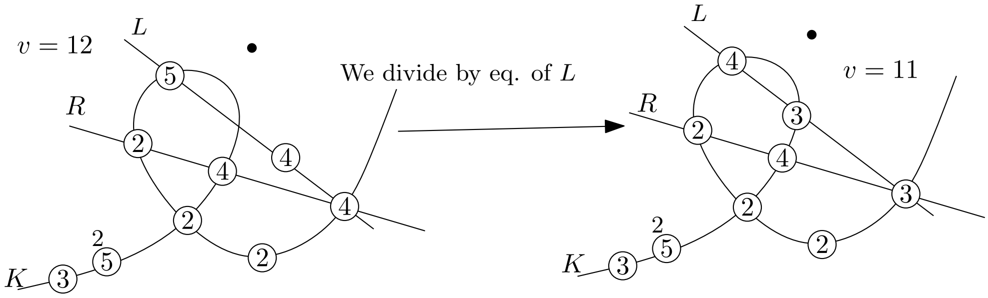


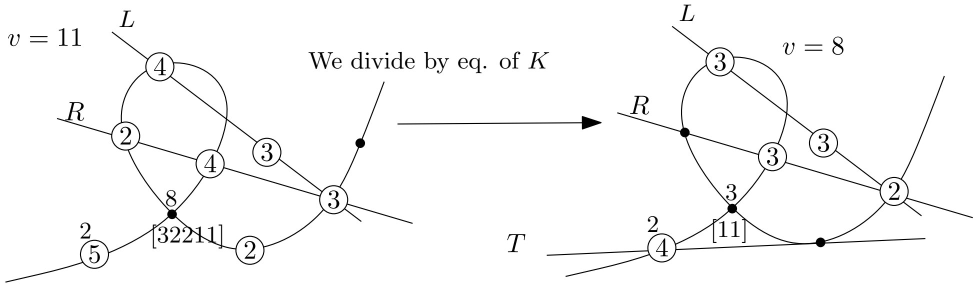
Case $(8, 8, 8, 8, 8, 8, 8, 8, 6, 5)$ $v = 24$ $x = 1$

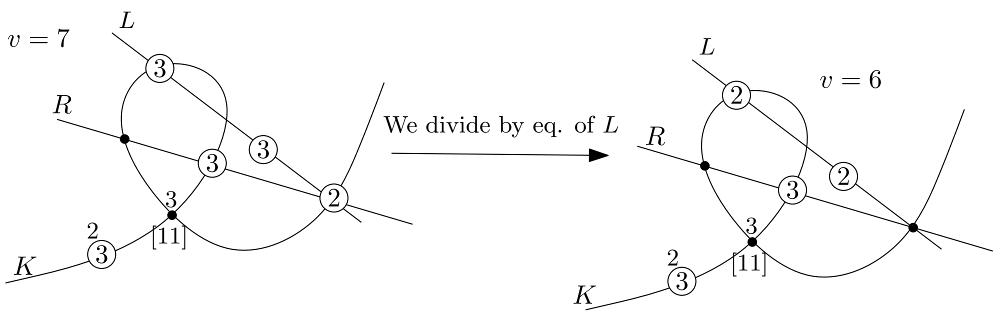

v = 24
We divide by eq. of Y
v = 19
Y
Y

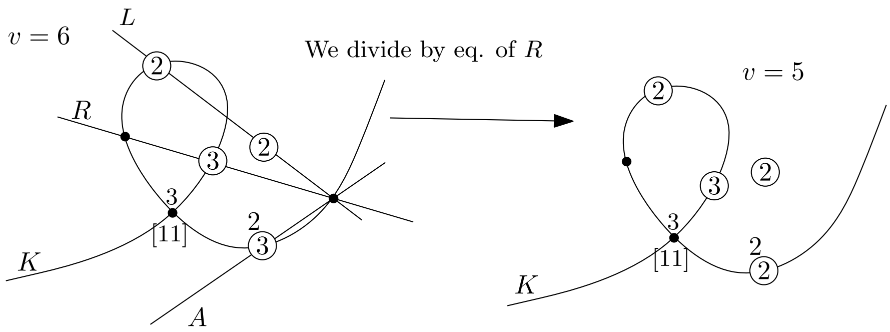

v = 19
We divide by eq. of Q
v = 15
R
Q
R
Q

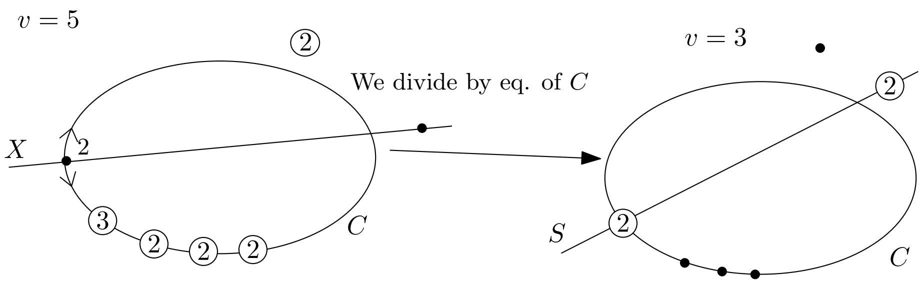

v = 15
Dividiamo per l'eq. di K
v = 12
R
L
K
R
L
K


v = 12
L
R
K
5
2
4
4
4
2
2
2
5
3
We divide by eq. of L
v = 11
L
R
K
4
2
3
4
3
2
2
2
5
3


v = 11
L
R
4
2
3
4
3
8
[32211]
2
5
2
We divide by eq. of K
v = 8
L
R
T
3
3
3
2
3
[11]
2
4

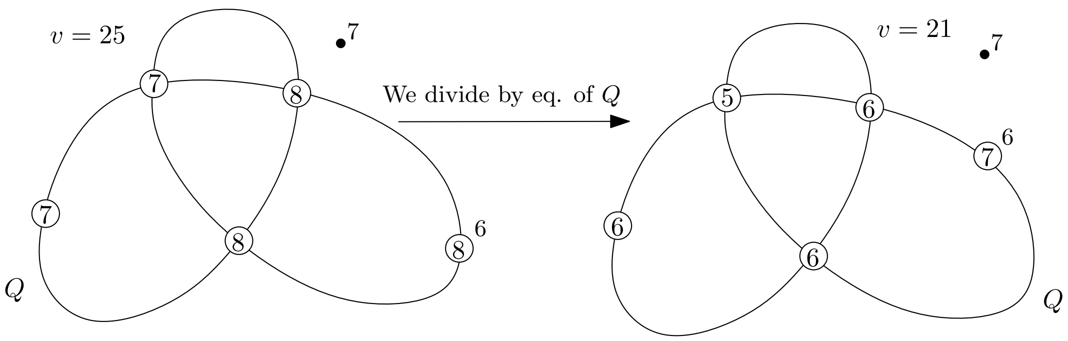

v = 7
L
R
K
3
3
3
2
3
[11]
2
3
We divide by eq. of L
v = 6
L
R
K
2
2
3
3
[11]
2
3

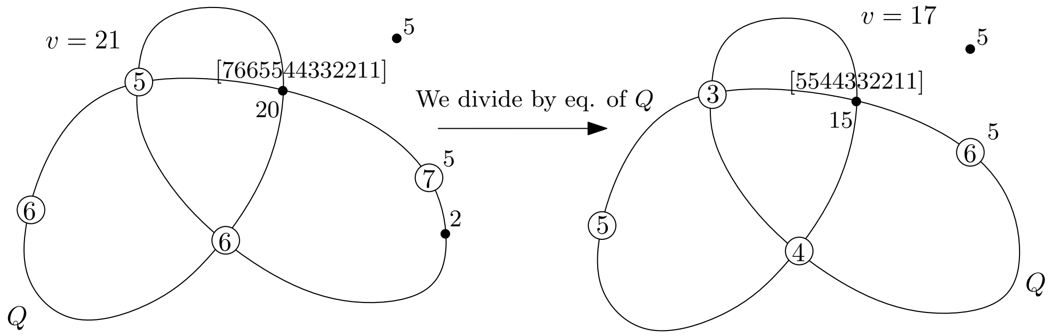




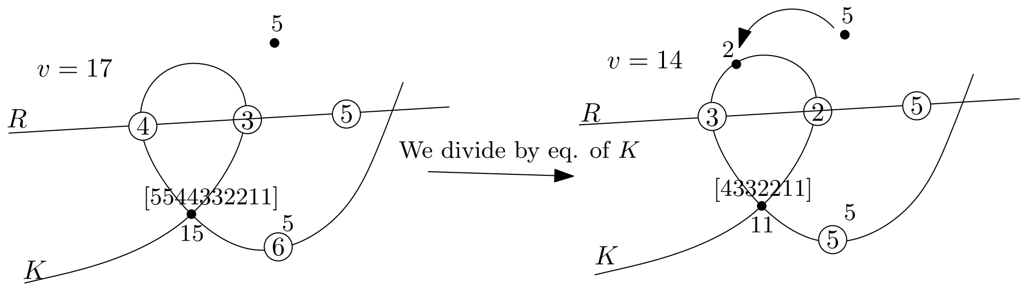
In the end, dividing by the equation of the line L, we rest with $v = 2$ and six simple point. Absurd.

Case $(8, 8, 8, 8, 8, 8, 8, 8, 8, 7, 7)$ $v = 25$ $x = 7$

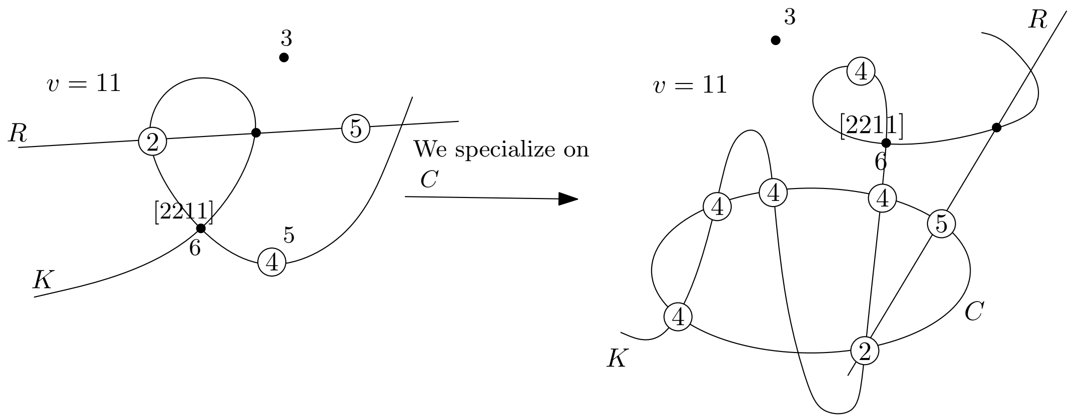




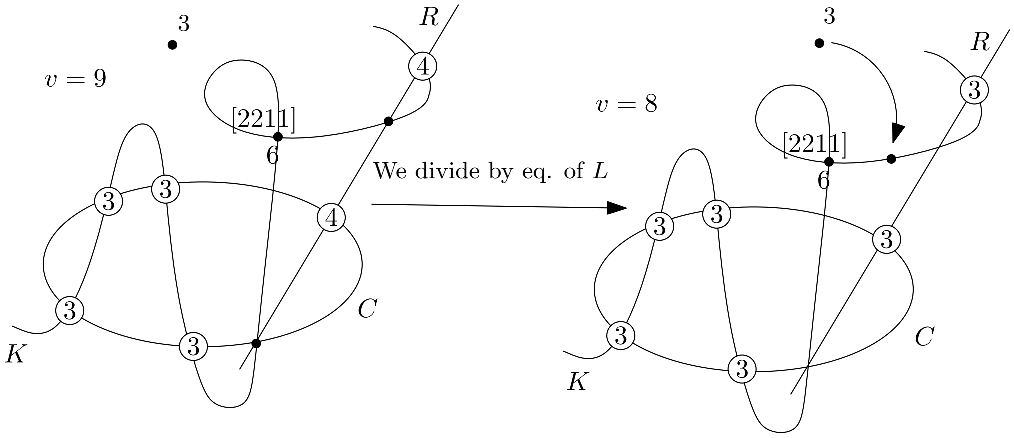
v = 21
[7665544332211]
20
Q
We divide by eq. of Q
v = 17
[5544332211]
15
Q

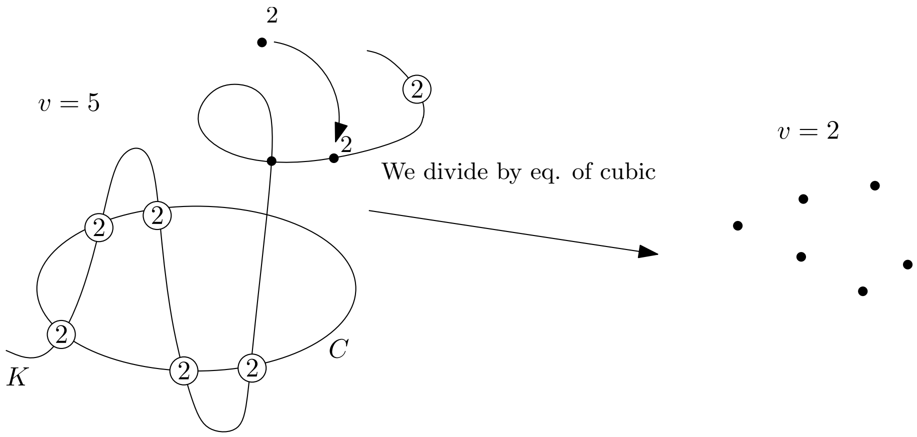


v = 17
R
K
[5544332211]
15
We divide by eq. of K
v = 14
R
K
[4332211]
11

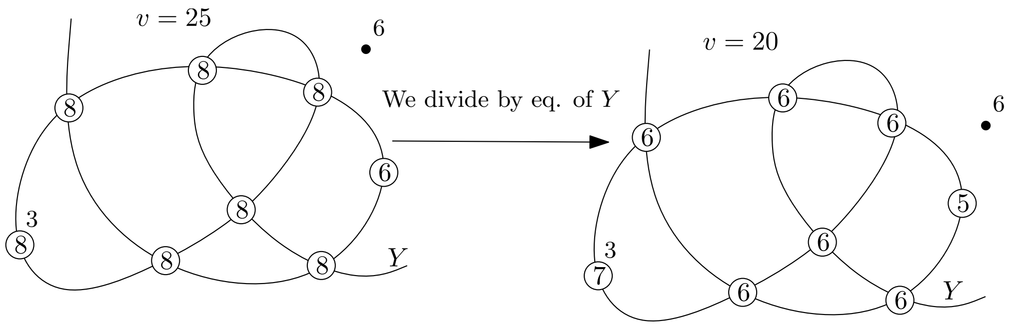


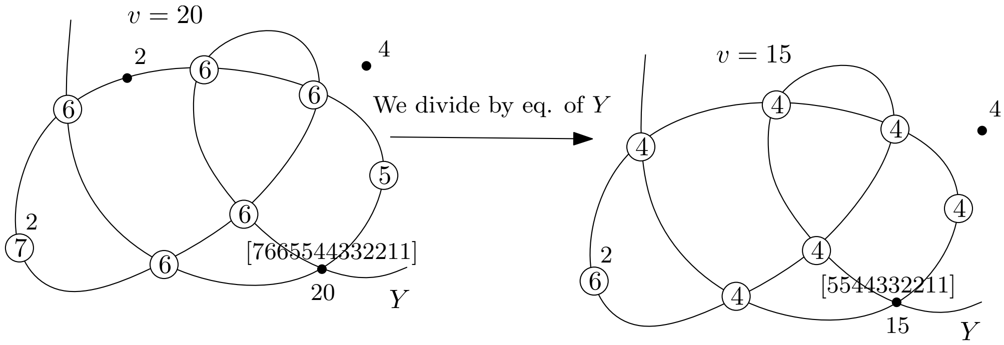
v = 11
R
K
[2211]
6
We specialize on C
v = 11
[2211]
6
R
C
K



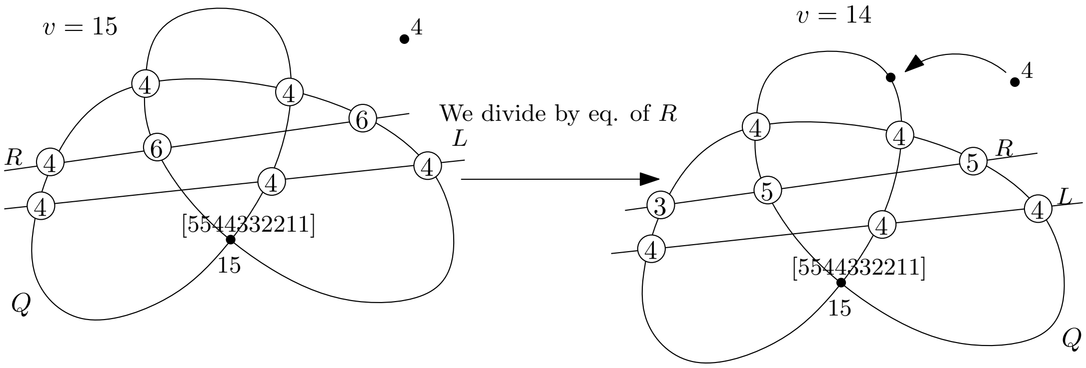


Case $(8, 8, 8, 8, 8, 8, 8, 8, 8, 6)$ $v = 25$ $x = 6$

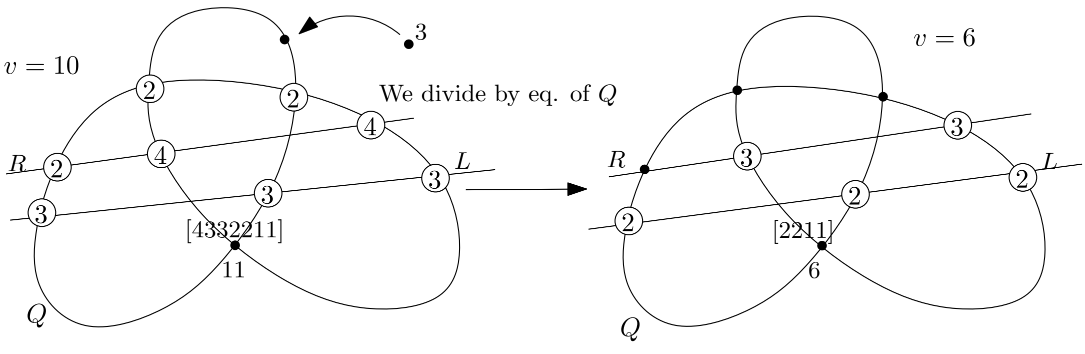




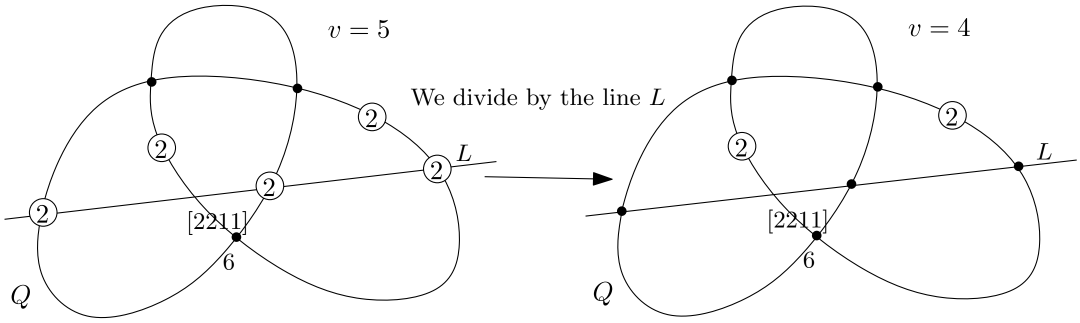
v = 20
2
4
6
6
6
5
6
2
7
6
[7665544332211]
20
Y
We divide by eq. of Y
v = 15
4
4
4
4
4
2
6
4
[5544332211]
15
Y

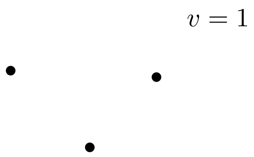


v = 15
4
4
4
6
R
4
6
L
4
4
4
[5544332211]
15
Q
We divide by eq. of R
v = 14
4
4
4
R
5
3
5
L
4
4
4
[5544332211]
15
Q



v = 10
3
2
2
4
R
2
4
L
3
3
3
[4332211]
11
Q
We divide by eq. of Q
v = 6
R
3
3
L
2
2
2
[2211]
6
Q



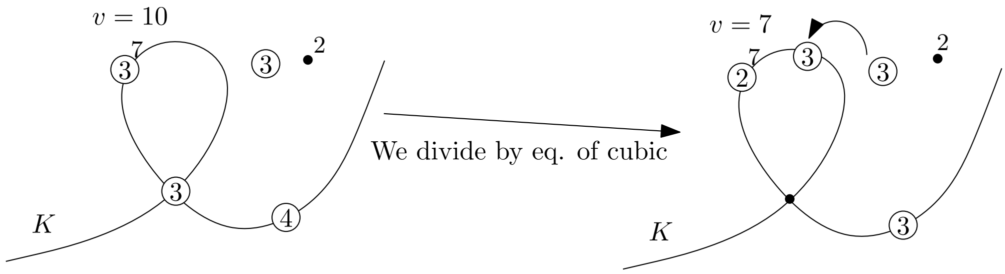


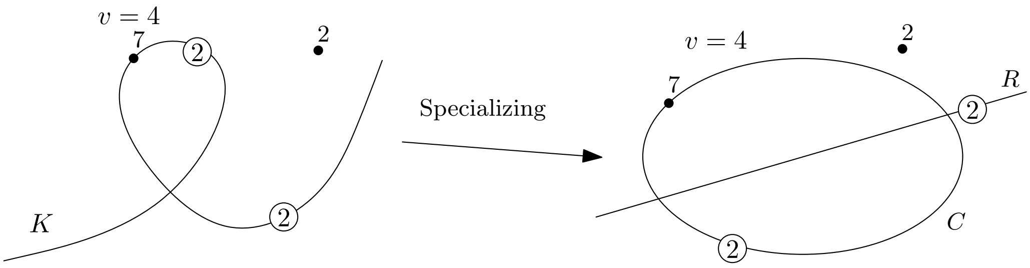
In conclusion we have just prove

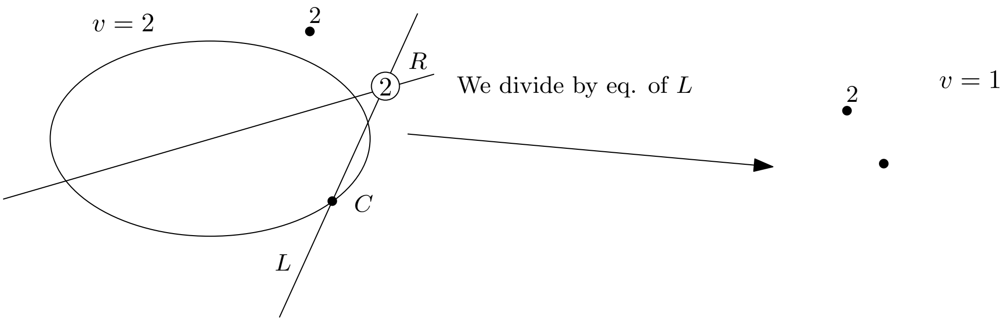
PROPOSITION 3.6.1 *The conjecture of Harbourne-Hirschowitz holds in all the linear system $\mathcal{L}_Z(v)$, where $Z = (b_1P_1 + ... + b_nP_n)$ is a subscheme of $\mathbb{P}^2$, with $n \leq 10$ and $b_i \leq 8$ for all $i = 1, ..., n$, and v is its critical value.*

Our aim is to prove that these subschemes of $\mathbb{P}^2$ have maximum rank, thus we have to show that $h^1(\mathcal{I}_Z(v)) = 0$ and $h^0(\mathcal{I}_Z(v-1)) = 0$, where v is the critical value of Z. Until now, we have proved only in eleven cases that the linear system $\mathcal{L}_Z(v)$ is non- special. Next step is to show that the other linear system are non-special too. We use again the Horace's method.

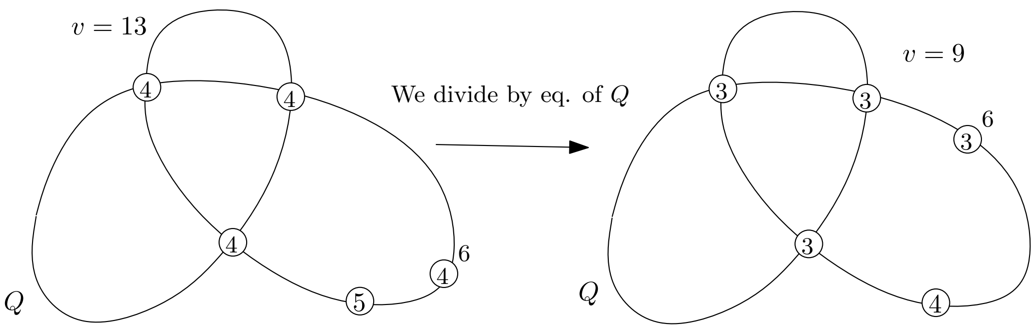
Case $(4, 3, 3, 3, 3, 3, 3, 3, 3, 3)$ $v = 10$ $x = 2$

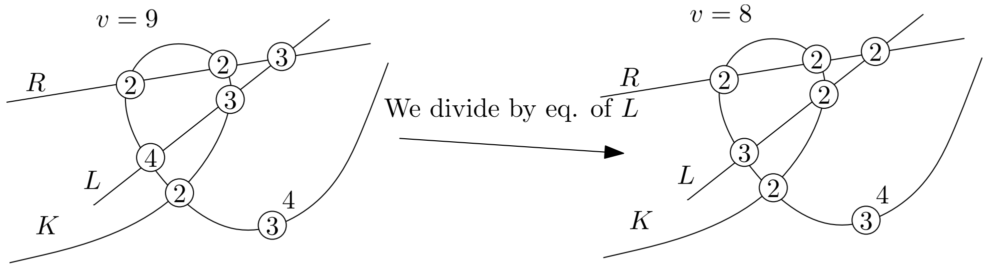


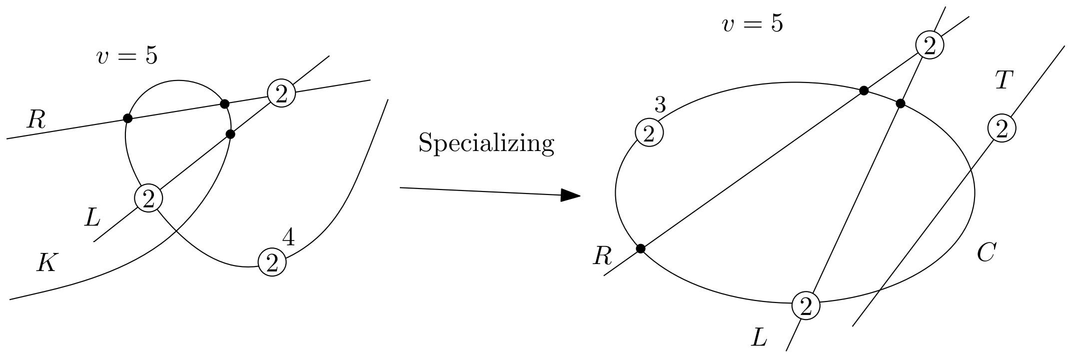


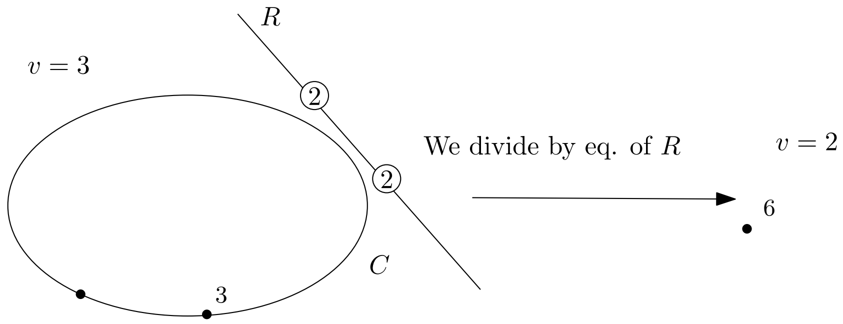


Case $(5, 4, 4, 4, 4, 4, 4, 4, 4, 4)$ $v = 13$ $x = 0$

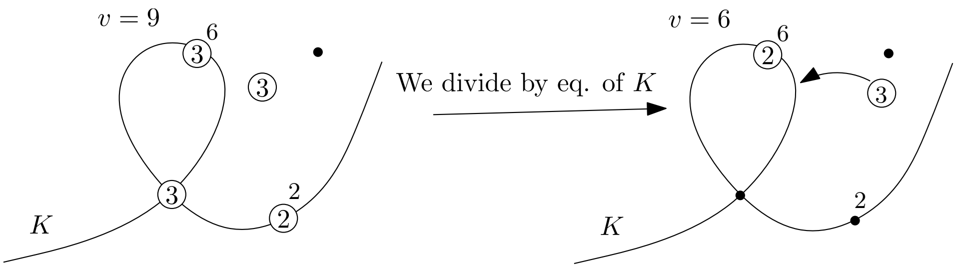


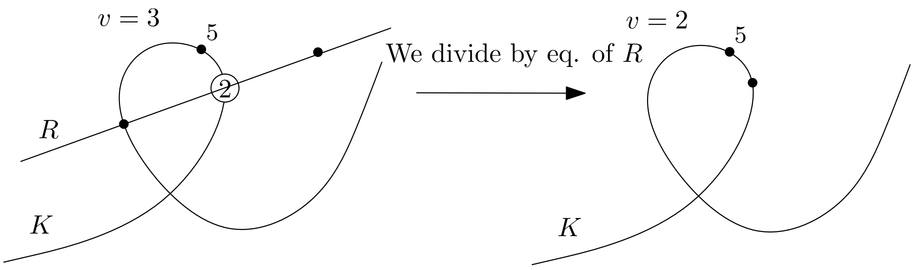




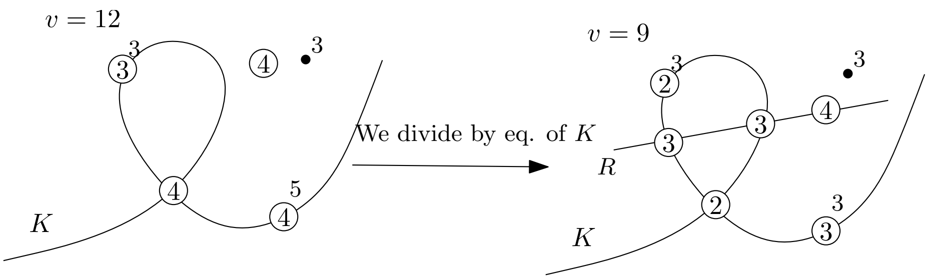


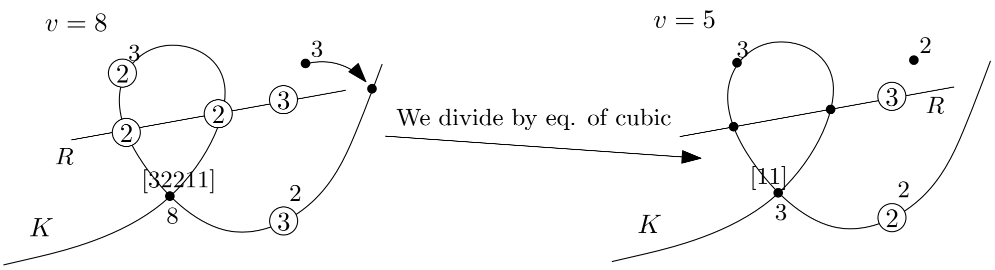
Case $(3,3,3,3,3,3,3,3,3,2,2)$ $v=9$ $x=1$

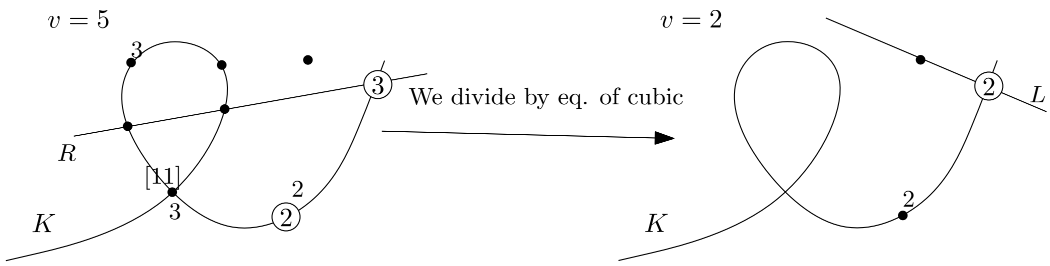




Case $(4, 4, 4, 4, 4, 4, 4, 3, 3, 3)$ $v = 12$ $x = 3$

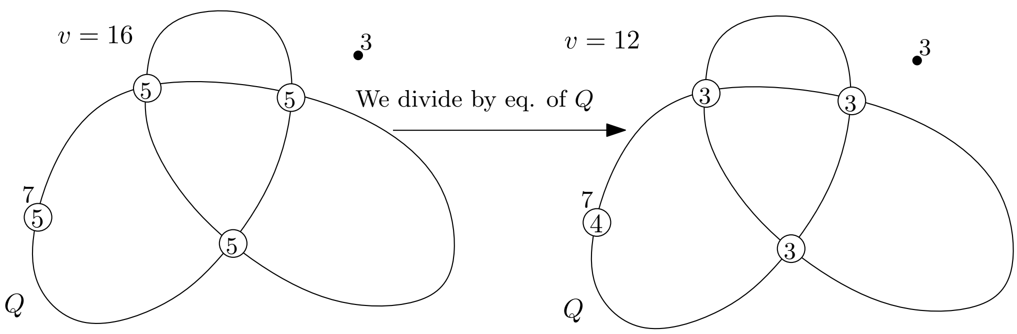


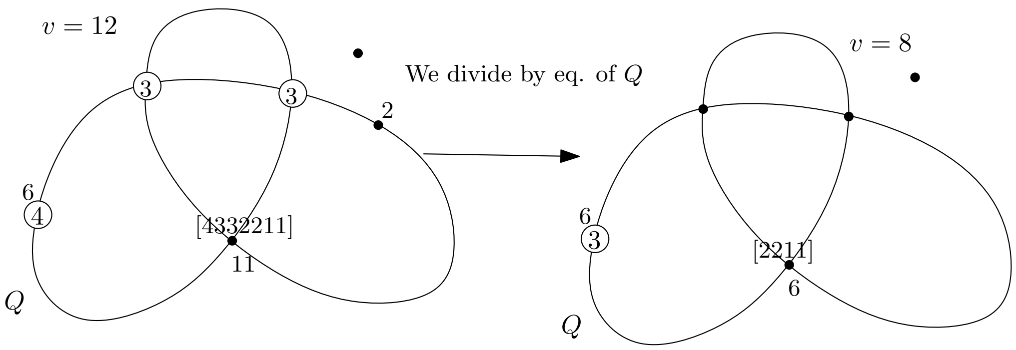


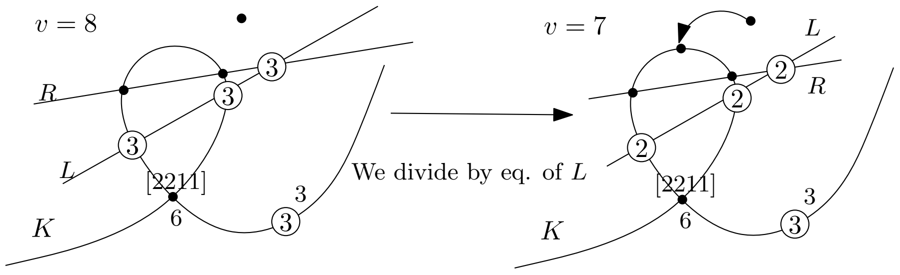


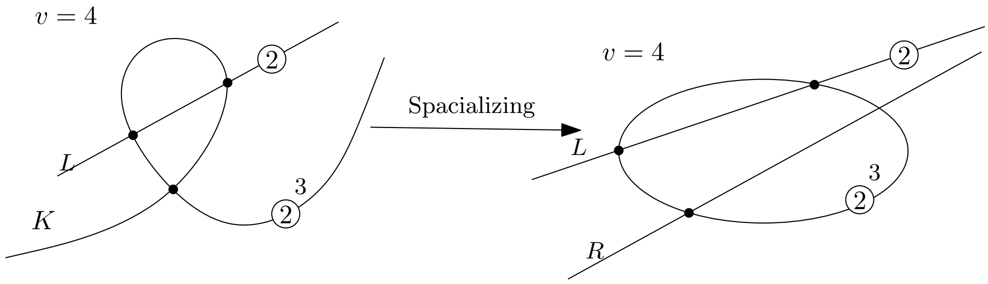
In the end, dividing by the equation of the line L, we rest with $v = 1$ and three simple point. Absurd.

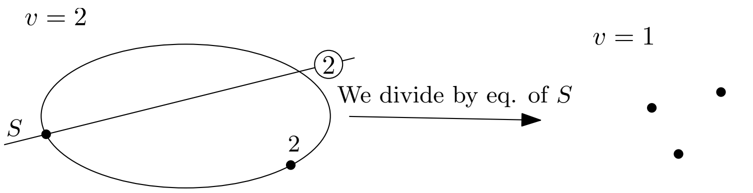
Case $(5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5)$ $v = 16$ $x = 0$



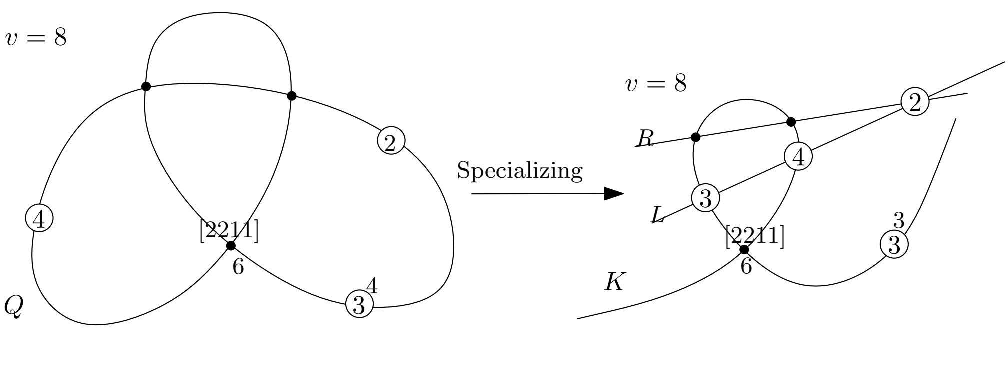


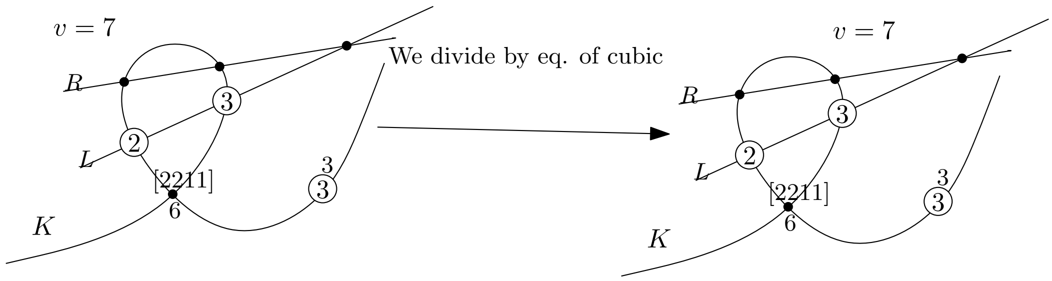


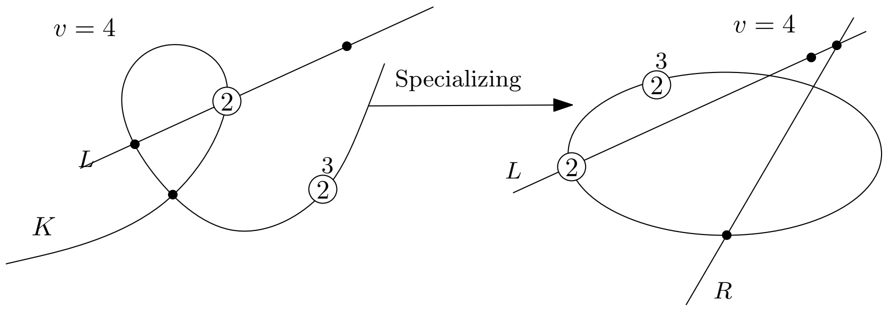




Case $(6, 5, 5, 5, 5, 5, 5, 5, 5, 4)$ $v = 16$ $x = 2$

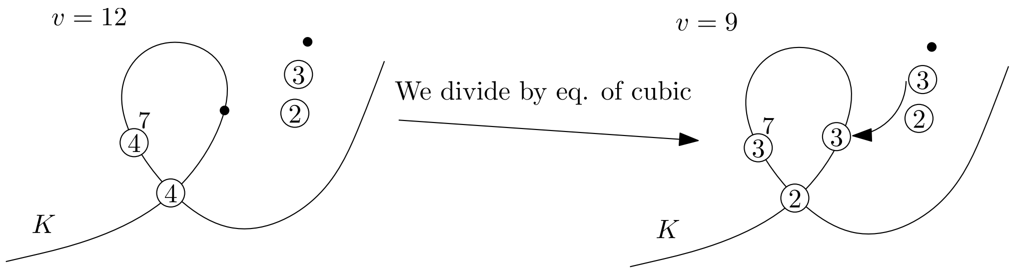


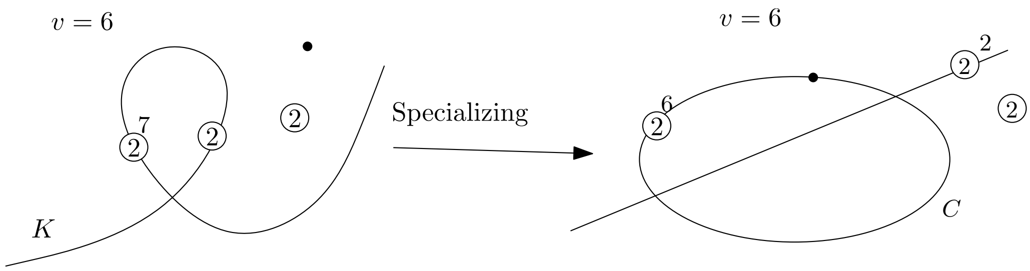


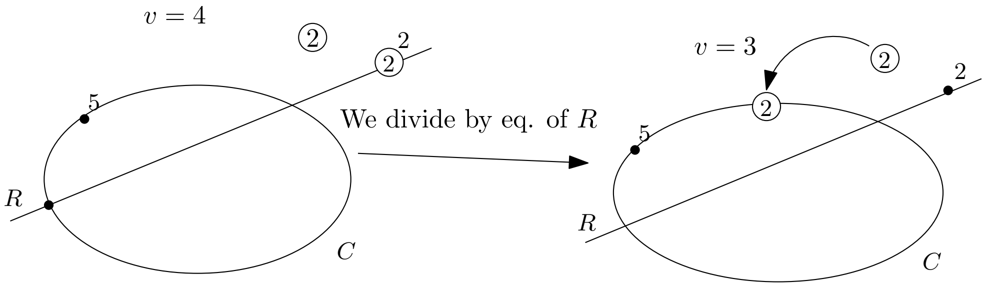


In the end, dividing by the equation of the conic, we rest with $v = 2$ and six simple point. Absurd.

Case $(4, 4, 4, 4, 4, 4, 4, 4, 4, 3, 2)$ $v = 12$ $x = 2$

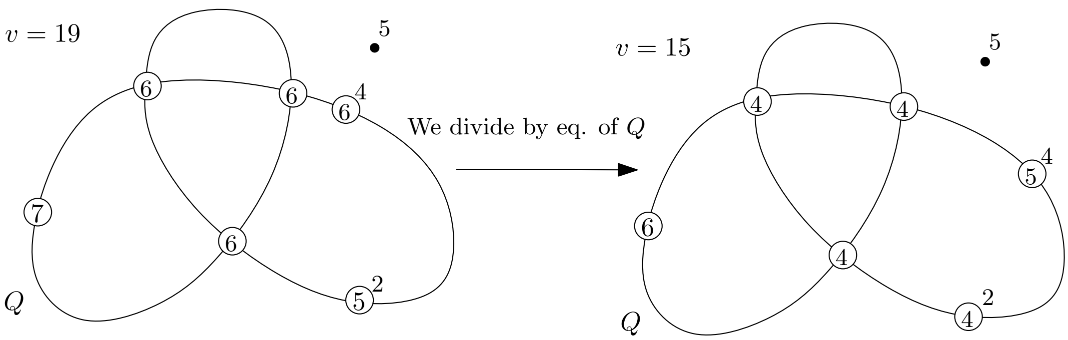


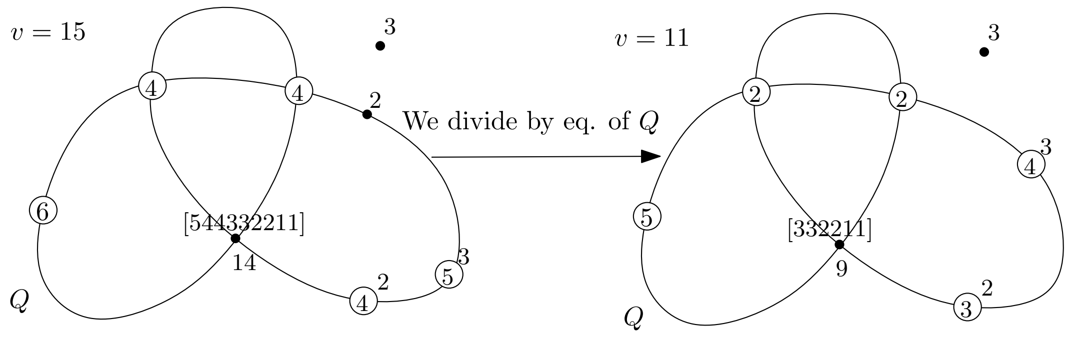


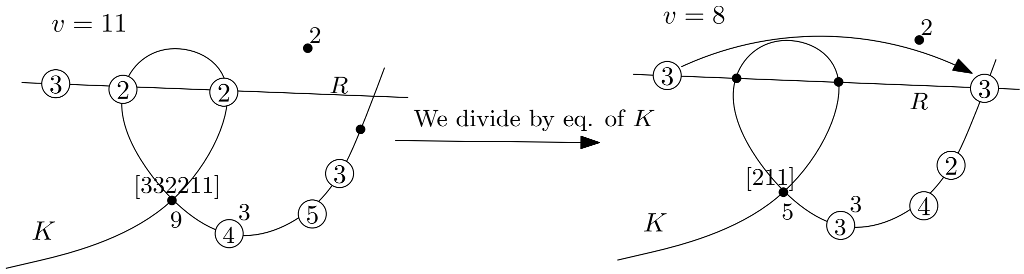


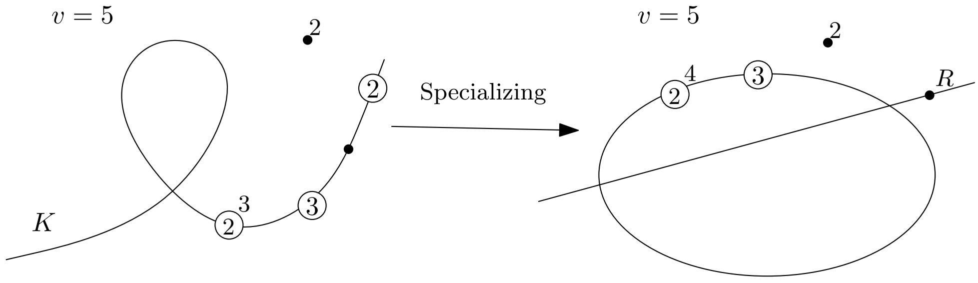
Dividing by the equation of the conic, we rest with $v = 2$ and six simple point. Absurd.

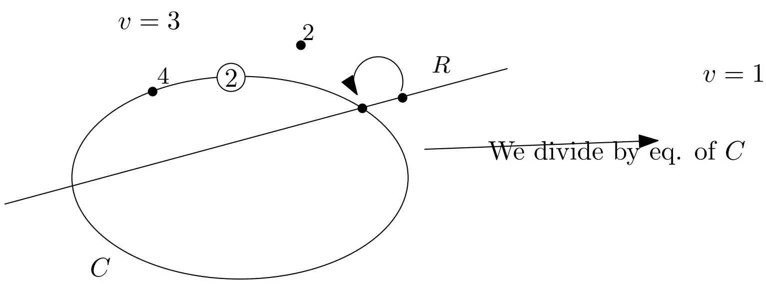
Case $(7, 6, 6, 6, 6, 6, 6, 6, 5, 5)$ $v = 19$ $x = 5$



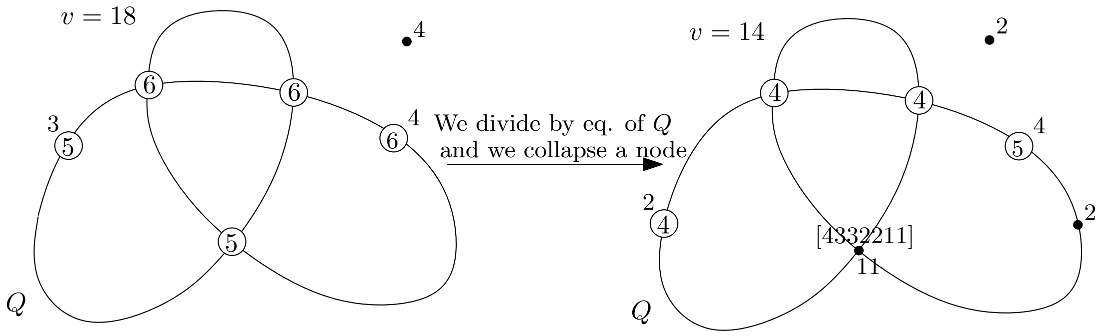

$v = 15$
3
4
4
2
6
[544332211]
14
Q
4
2
5
3
We divide by eq. of Q
$v = 11$
3
2
2
4
3
5
[332211]
9
Q
3
2

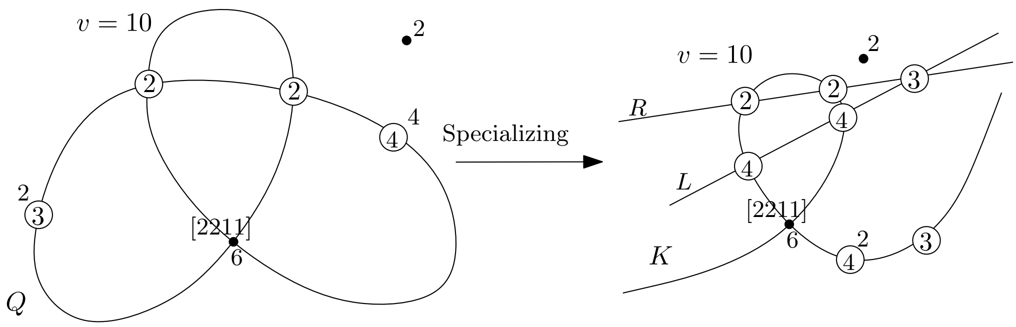

$v = 11$
2
3
2
2
R
[332211]
9
K
4
3
5
3
We divide by eq. of K
$v = 8$
2
3
3
R
2
[211]
5
K
3
3
4

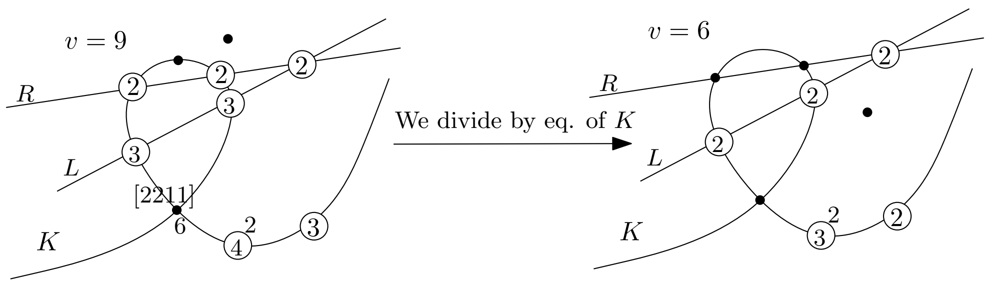

$v = 5$
2
2
K
2
3
3
Specializing
$v = 5$
2
4
2
3
R

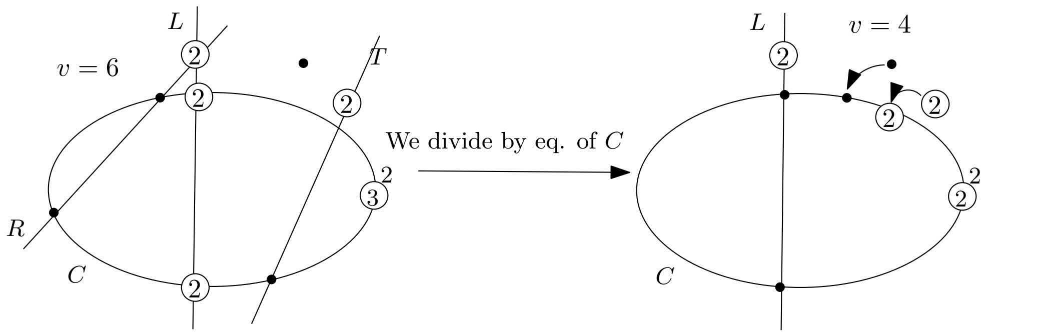

$v = 3$
2
4
2
R
C
We divide by eq. of C
$v = 1$
2

Case $(6, 6, 6, 6, 6, 6, 5, 5, 5, 5)$ $v = 18$ $x = 4$



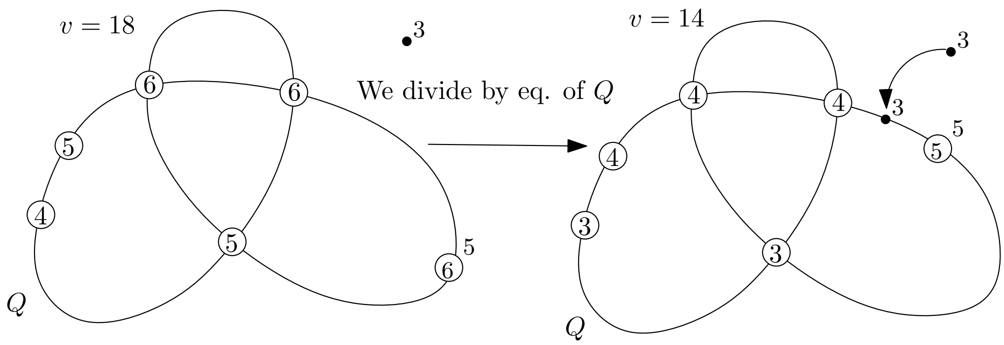


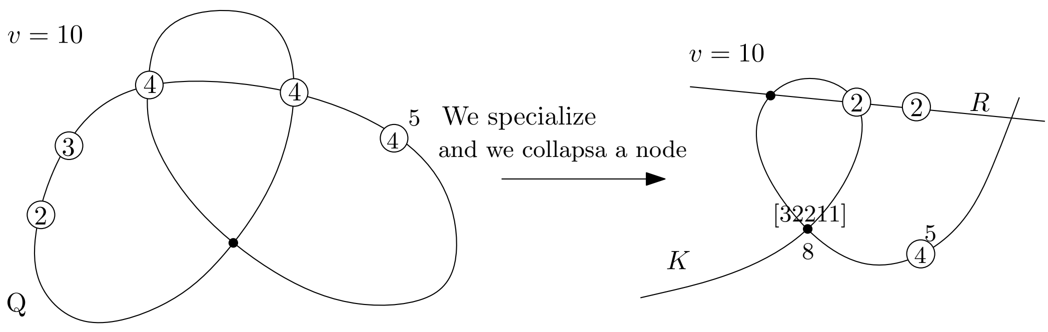


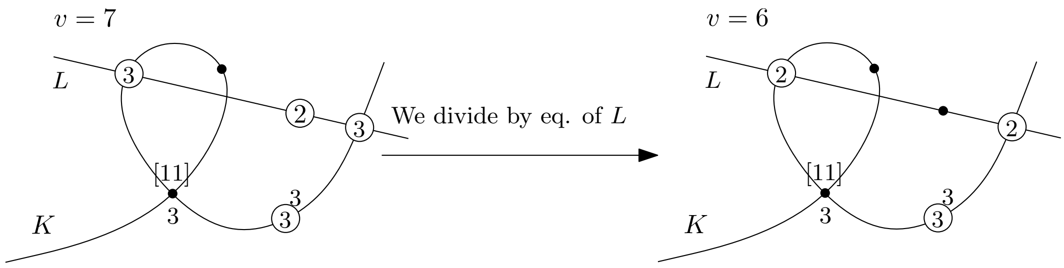


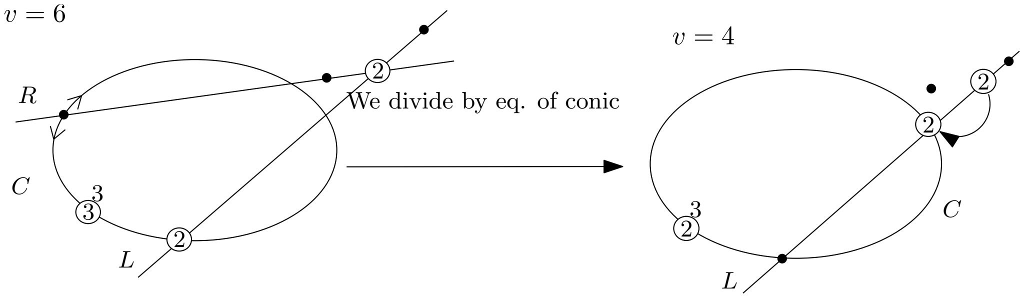
Dividing by the equation of the conic, we can go back to the last passage of case $(5, 5, 5, 5, 5, 5, 5, 5, 5, 5)$.

Case $(6, 6, 6, 6, 6, 6, 6, 5, 5, 4)$ $v = 18$ $x = 3$



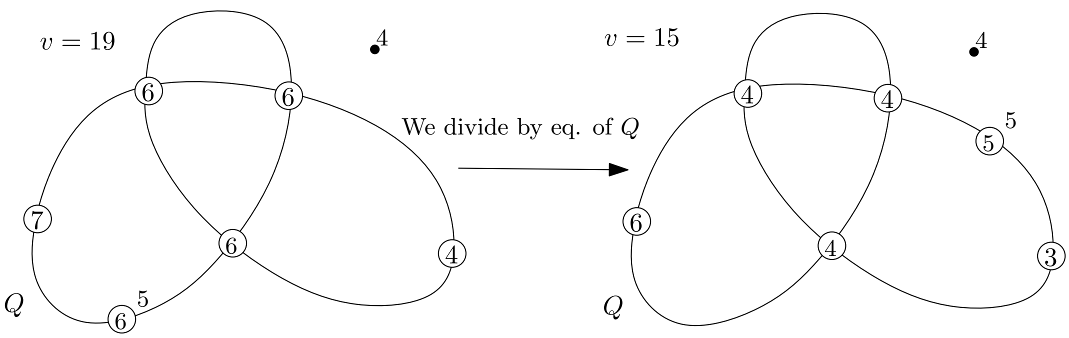


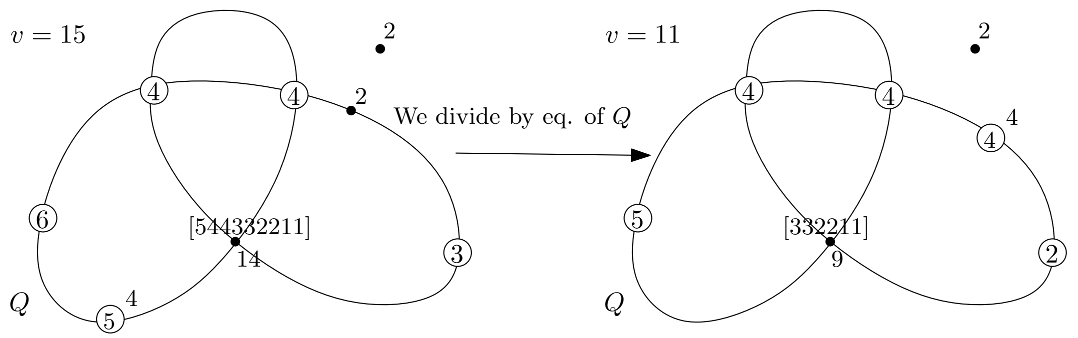


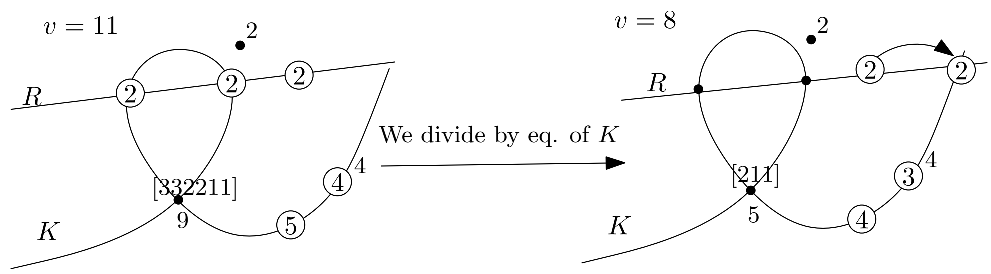


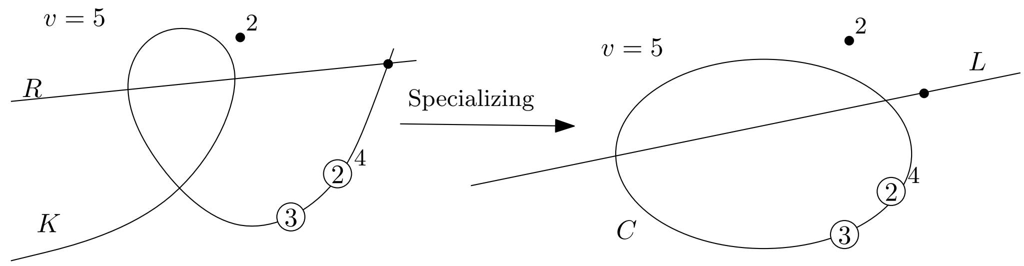
Dividing by the equation of the conic, we rest with $v = 2$ and six simple point. Absurd.

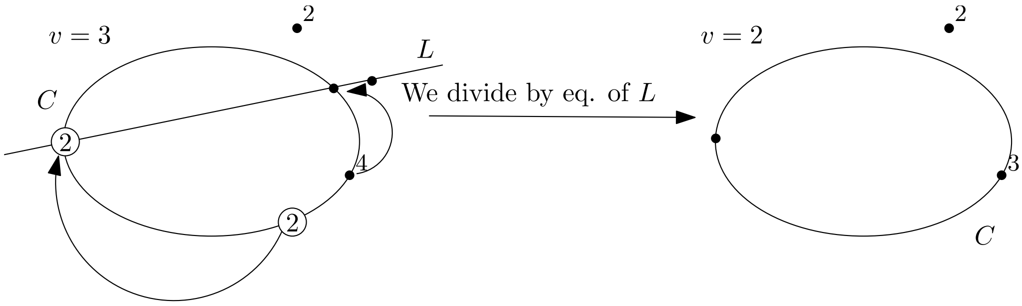
Case $(7, 6, 6, 6, 6, 6, 6, 6, 6, 4)$ $v = 19$ $x = 4$



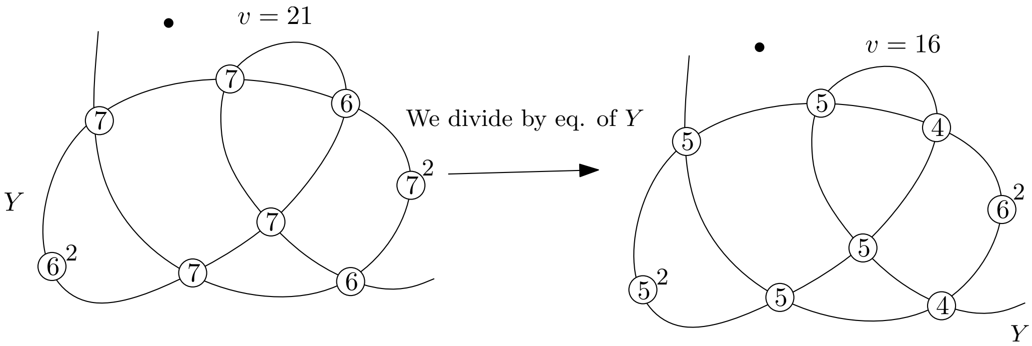

v = 15
2
2
We divide by eq. of Q
[544332211]
14
4
Q
v = 11
2
4
[332211]
9
Q

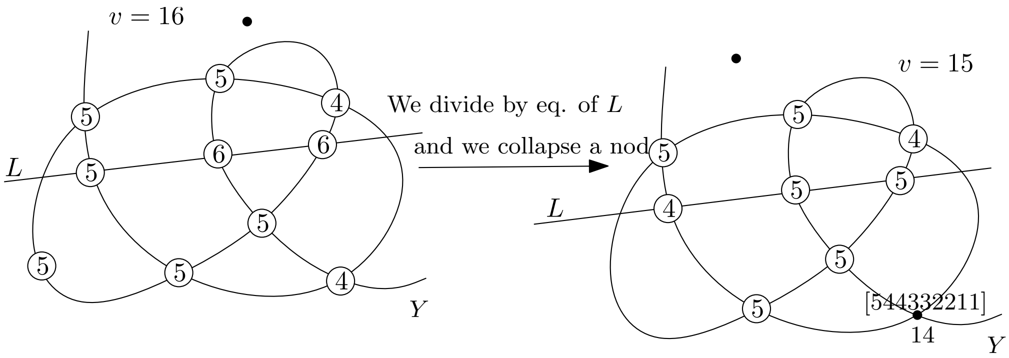

v = 11
2
R
K
[332211]
9
4
We divide by eq. of K
v = 8
2
R
[211]
5
4
K

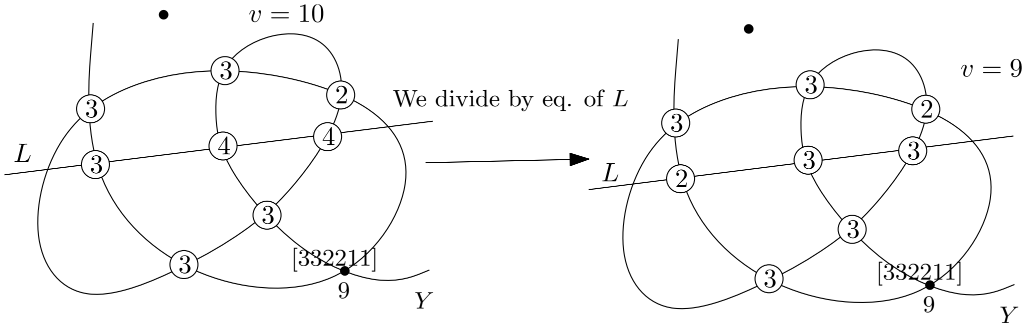

v = 5
2
R
K
4
Specializing
v = 5
2
L
4
C

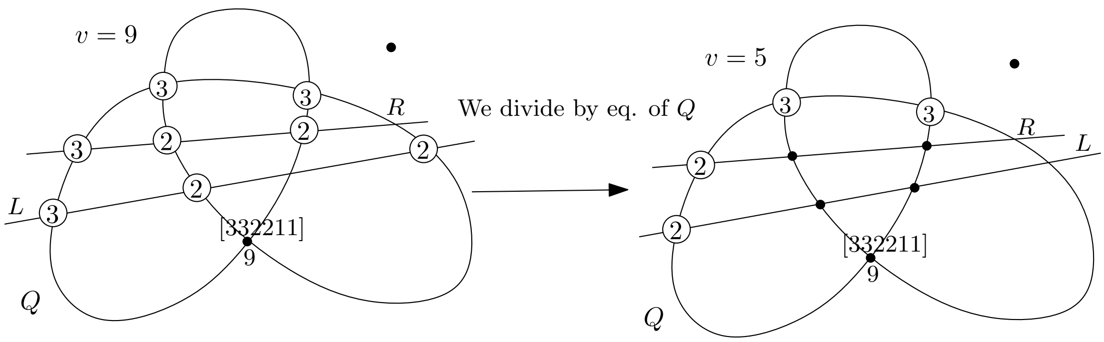


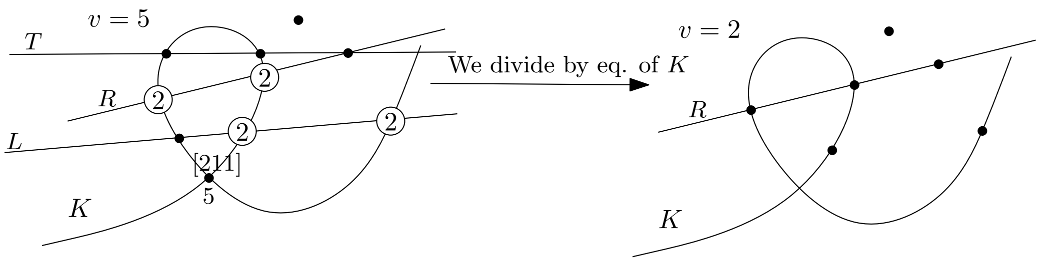
Case $(7, 7, 7, 7, 7, 7, 6, 6, 6, 6)$ $v = 21$ $x = 1$



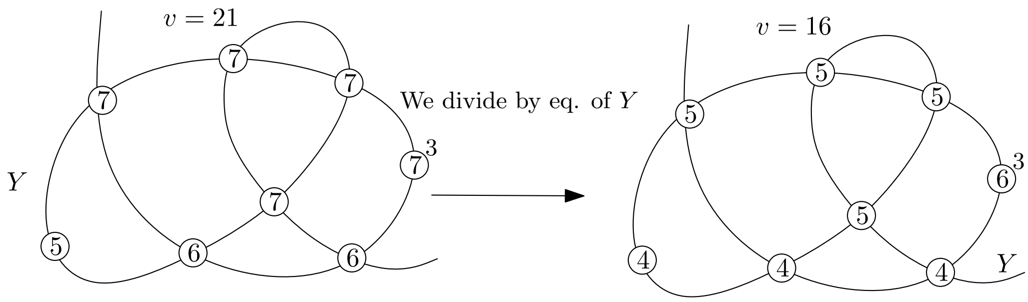


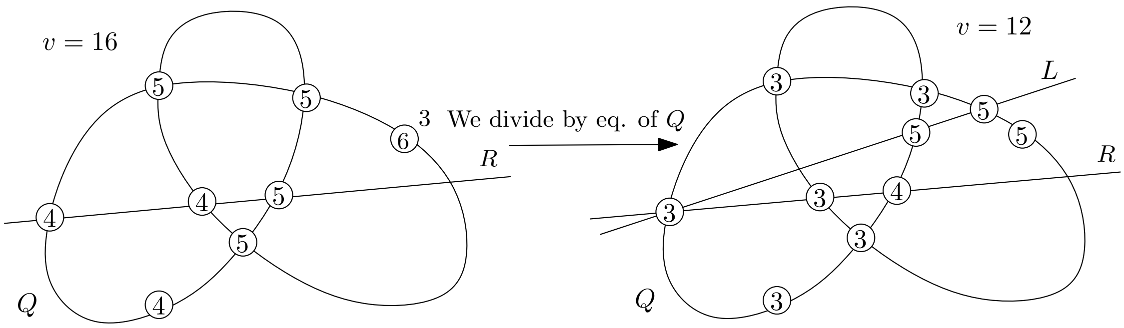


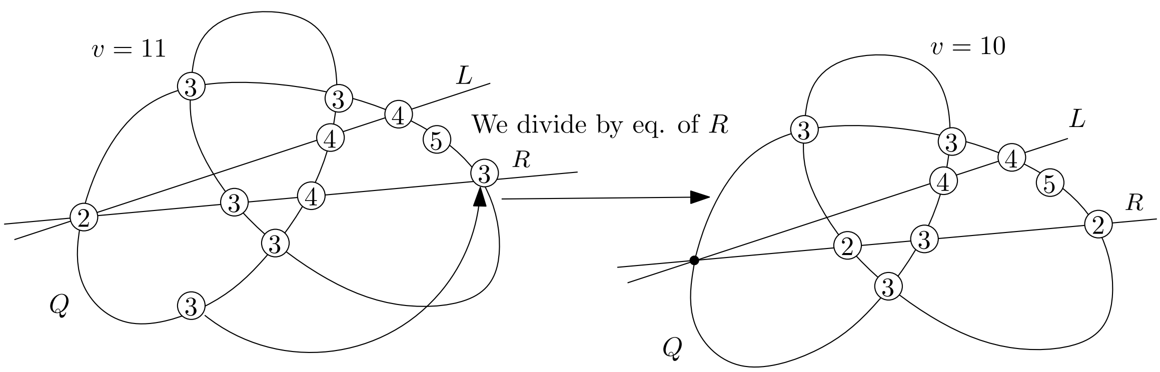


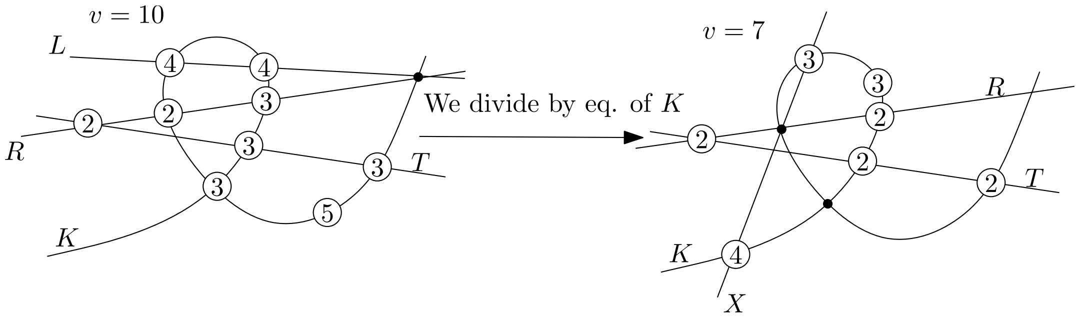


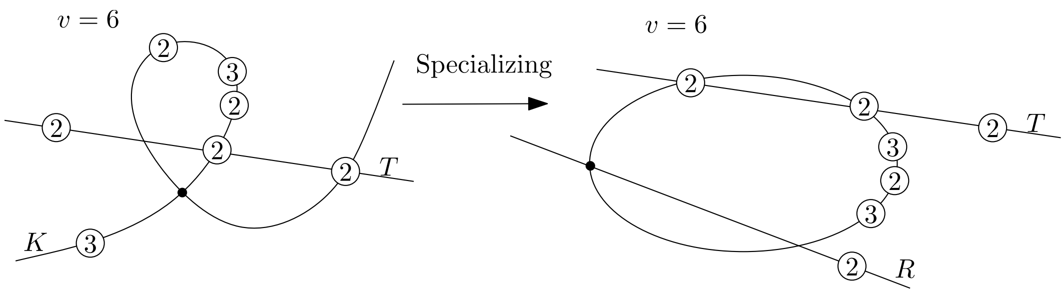
Case $(7, 7, 7, 7, 7, 7, 7, 7, 6, 6, 5)$ $v = 21$ $x = 0$

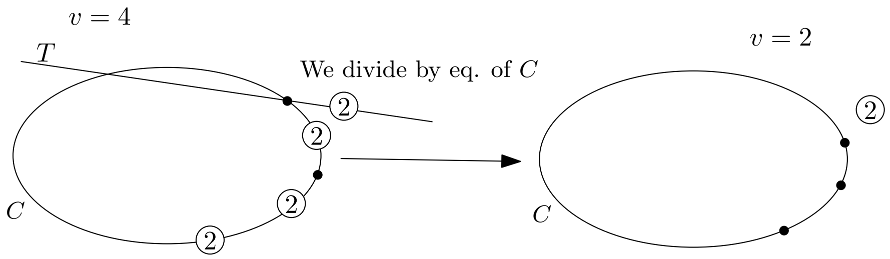

v = 21
Y
We divide by eq. of Y
v = 16
Y


v = 16
R
Q
We divide by eq. of Q
v = 12
L
R
Q

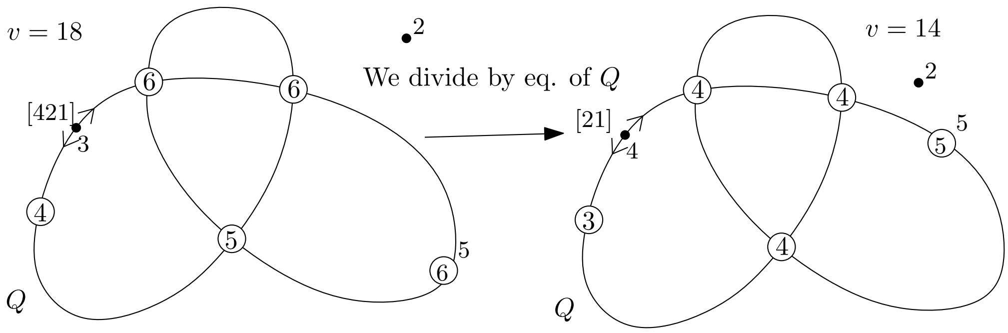

v = 11
L
R
Q
We divide by eq. of R
v = 10
L
R
Q



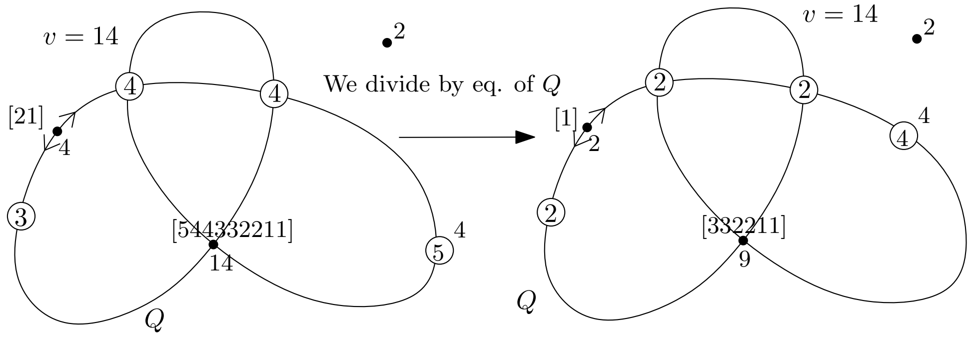




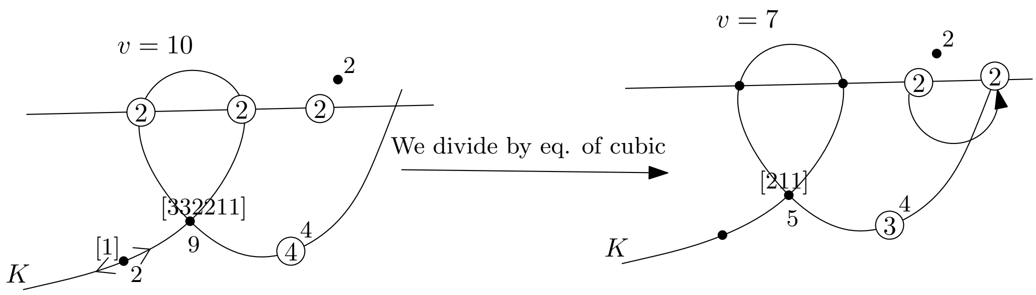
Case $(6, 6, 6, 6, 6, 6, 6, 6, 6, 4, 4)$ $v = 18$ $x = 2$



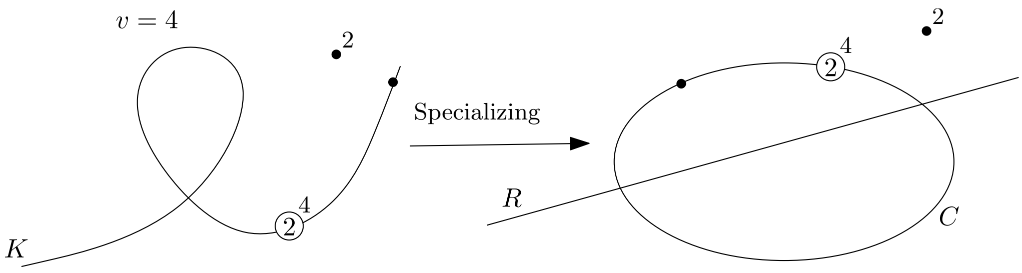
v = 18
2
We divide by eq. of Q
v = 14
[421]
3
6
6
4
5
6
5
Q
[21]
4
4
4
5
5
3
4
2
Q



v = 14
2
We divide by eq. of Q
v = 14
2
[21]
4
4
4
3
[544332211]
14
5
4
Q
[1]
2
2
2
4
4
2
[332211]
9
Q

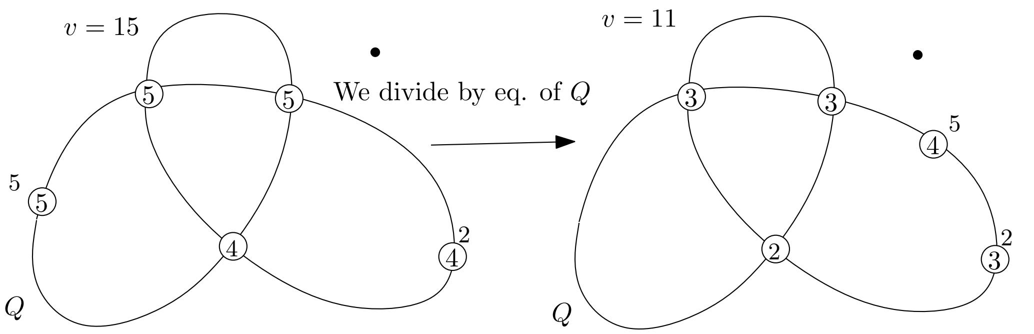


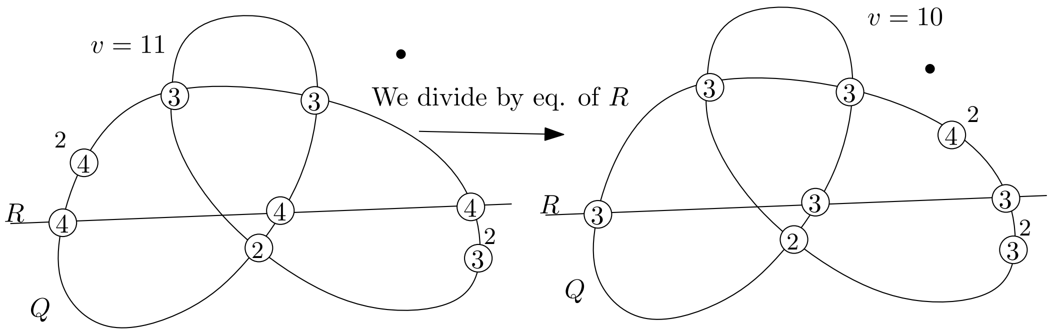
v = 10
2
We divide by eq. of cubic
v = 7
2
2
2
2
[332211]
9
[1]
2
4
4
K
2
2
[211]
5
3
4
K

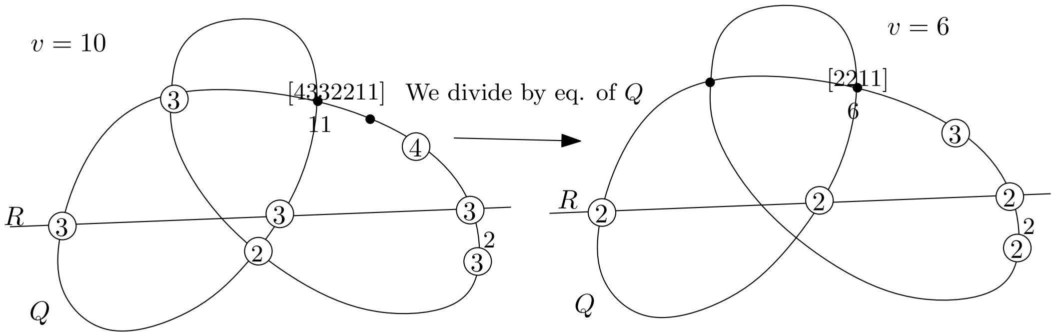


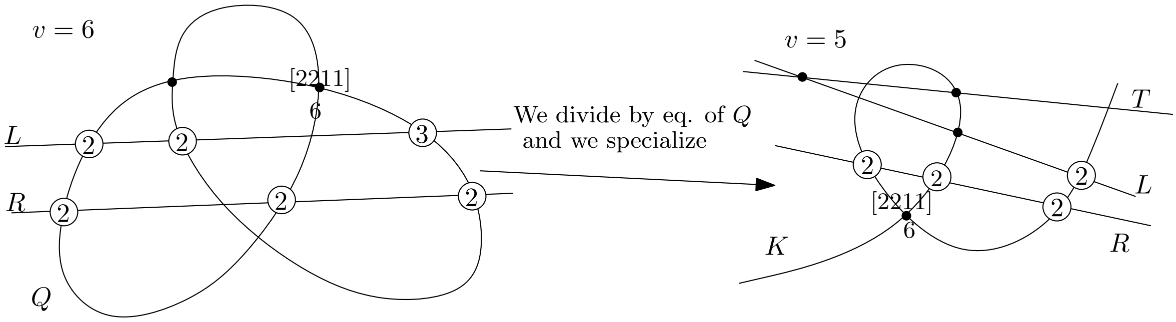
Dividing by the equation of the conic, we rest with $v = 2$ and six simple point. Absurd.

Case $(5, 5, 5, 5, 5, 5, 5, 4, 4, 4)$ $v = 15$ $x = 1$



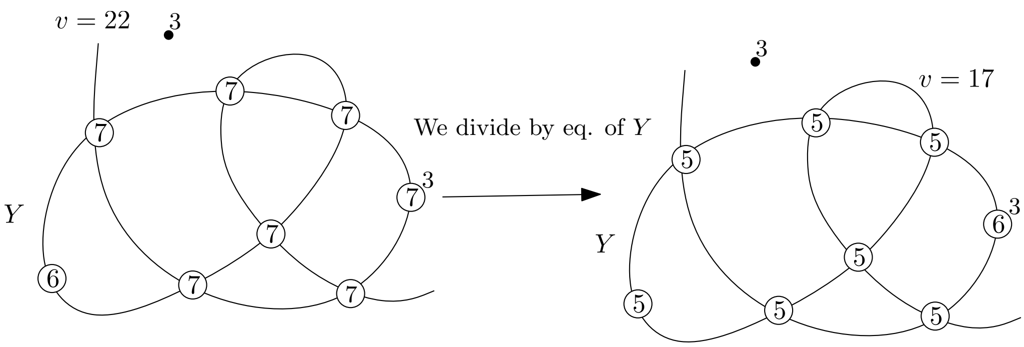


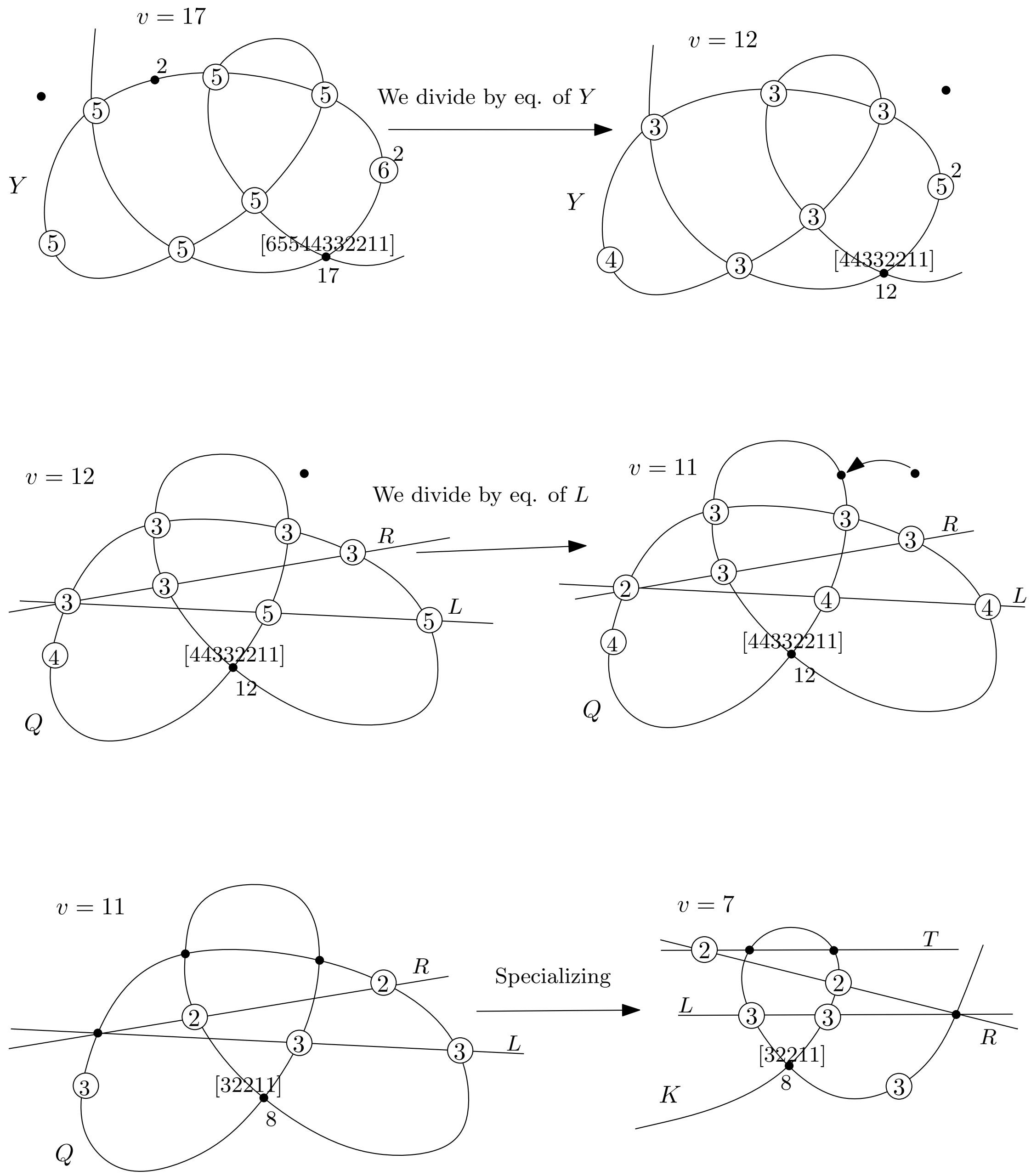


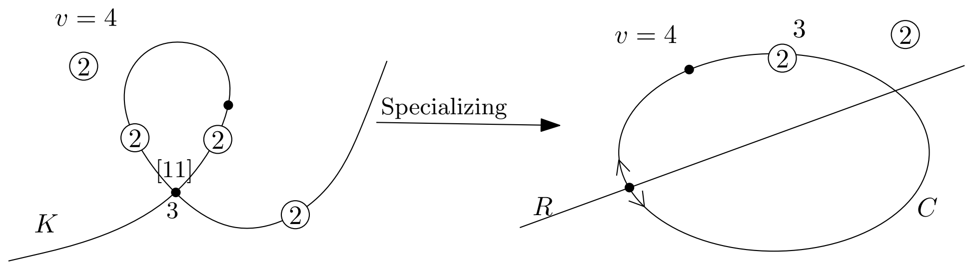


Dividing by the equation of the conic, we rest with $v = 2$ and six simple point. Absurd.

Case $(7, 7, 7, 7, 7, 7, 7, 7, 7, 6)$ $v = 22$ $x = 3$




v = 17
We divide by eq. of Y
v = 12
Y
[65544332211]
17
[44332211]
12
v = 12
We divide by eq. of L
v = 11
R
L
Q
[44332211]
12
v = 11
Specializing
v = 7
T
K
[32211]
8



Dividing by the equation of the conic, we can go back to the last passage of case $(5,5,5,5,5,5,5,5,5,5)$.
By this proves we can improve the proposition 3.6.1 with the follow:

PROPOSITION 3.6.2 *Let $Z = (b_1P_1 + ... + b_nP_n)$ be a subscheme of $\mathbb{P}^2$, with $n \leq 10$ and $b_i \leq 8$ for all $i = 1, ..., n$, the respective linear system $\mathcal{L}_Z(v)$ is non-special, where v is the critical value of Z.*

In the end we have to prove that even $h^0(\mathcal{I}_Z(v-1)) = 0$. We use again the Horace's method. As above we will deal with case by case. By definition of v, we immediately note that we don't need to add simple points to Z, because $deg(Z) > h^0(\mathcal{O}_{\mathbb{P}^2}(v-1))$. To reduce the number of case we apply the Cremona transformations, and then the Horace's method. We indicate with $a = v - 1$ and with $r := deg(Z) - h^0(\mathcal{O}_{\mathbb{P}^2}(v-1))$, r represent the rest of conditions, i.e. the remain conditions when $a = 0$.
After the Cremona transformations we get the following cases:

$Z_1 = (2,2,2,2)$ $a = 2$ $r = 6$
$Z_2 = (2,2,2,2,1,1,1)$ $a = 3$ $r = 5$
$Z_3 = (2,1,1,1,1,1,1)$ $a = 2$ $r = 3$
$Z_4 = (2,2,2,1,1,1,1,1)$ $a = 3$ $r = 4$
$Z_5 = (4,4,4,4,4,4,4,4,2,2)$ $a = 12$ $r = 1$
$Z_6 = (3,3,3,3,3,3,3,3,3,2)$ $a = 9$ $r = 2$
$Z_7 = (4,4,4,4,4,4,4,4,4,3)$ $a = 12$ $r = 5$
$Z_8 = (4,4,4,4,4,4,4,4,4,2)$ $a = 12$ $r = 2$
$Z_9 = (6,5,5,5,5,5,5,5,5,5)$ $a = 16$ $r = 3$
$Z_{10} = (7,6,6,6,6,6,6,6,6,5)$ $a = 19$ $r = 1$

$Z_{11} = (1,1,1,1)$ $a = 1$ $r = 1$
$Z_{12} = (2,1,1,1,1,1)$ $a = 2$ $r = 2$
$Z_{13} = (3,3,2,2,2,2,1,1,1)$ $a = 5$ $r = 6$
$Z_{14} = (5,5,5,5,5,5,5,5,4,4)$ $a = 15$ $r = 4$
$Z_{15} = (6,6,6,6,6,6,6,5,5,5)$ $a = 18$ $r = 2$
$Z_{16} = (2,2,1,1,1,1)$ $a = 2$ $r = 4$
$Z_{17} = (6,6,6,6,6,6,6,6,5,4)$ $a = 18$ $r = 3$
$Z_{18} = (8,7,7,7,7,7,7,7,7,6)$ $a = 22$ $r = 5$
$Z_{19} = (2,2,2,2,1,1)$ $a = 3$ $r = 4$
$Z_{20} = (5,5,5,5,5,5,5,5,5,2)$ $a = 15$ $r = 2$
$Z_{21} = (2,2,2,1,1,1,1)$ $a = 3$ $r = 4$
$Z_{22} = (2,2,2,2,2,1,1)$ $a = 3$ $r = 7$
$Z_{23} = (7,7,7,7,7,7,7,6,6,6)$ $a = 21$ $r = 6$
$Z_{24} = (7,7,7,7,7,7,7,7,6,4)$ $a = 21$ $r = 2$
$Z_{25} = (2,2,1,1,1,1,1)$ $a = 2$ $r = 6$
$Z_{26} = (7,7,7,7,7,7,7,7,7,7)$ $a = 22$ $r = 4$
$Z_{27} = (2,2,2,1,1)$ $a = 2$ $r = 5$
$Z_{28} = (2,2,2,2,1,1,1,1)$ $a = 3$ $r = 6$
$Z_{29} = (3,2,2,2,2,1,1,1)$ $a = 4$ $r = 6$
$Z_{30} = (2,2,1,1,1)$ $a = 2$ $r = 3$
$Z_{31} = (3,2,2,2,1,1,1)$ $a = 3$ $r = 8$
$Z_{32} = (8,8,8,8,8,8,8,7,7,6)$ $a = 24$ $r = 4$
$Z_{33} = (8,8,8,8,8,8,7,7,7,7)$ $a = 24$ $r = 3$
$Z_{34} = (8,8,8,8,8,8,8,8,6,6)$ $a = 24$ $r = 5$
$Z_{35} = (8,8,8,8,8,8,8,8,6,4)$ $a = 24$ $r = 1$
$Z_{36} = (8,8,8,8,8,8,8,8,8,7)$ $a = 25$ $r = 1$

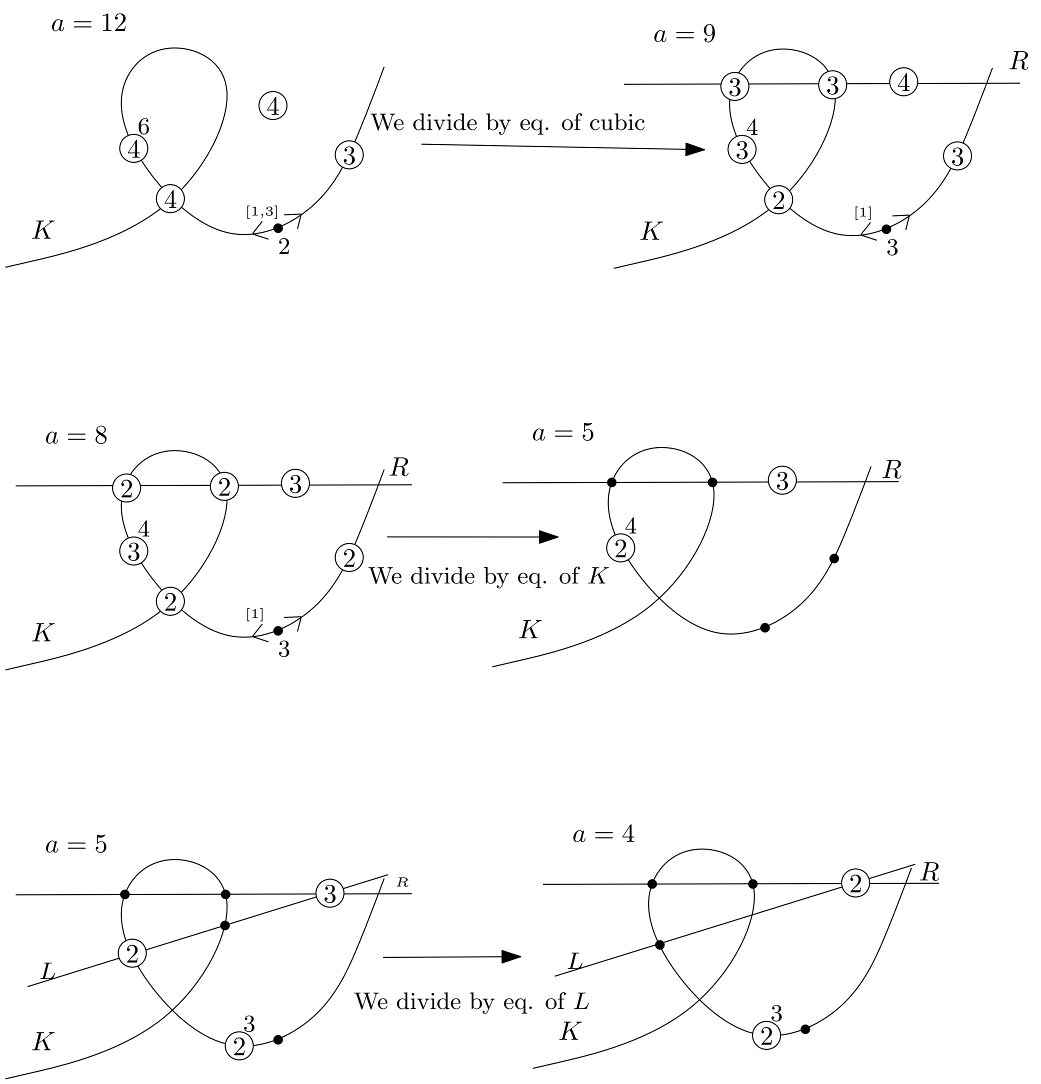
We note that these cases are more easier and less than precedent cases.

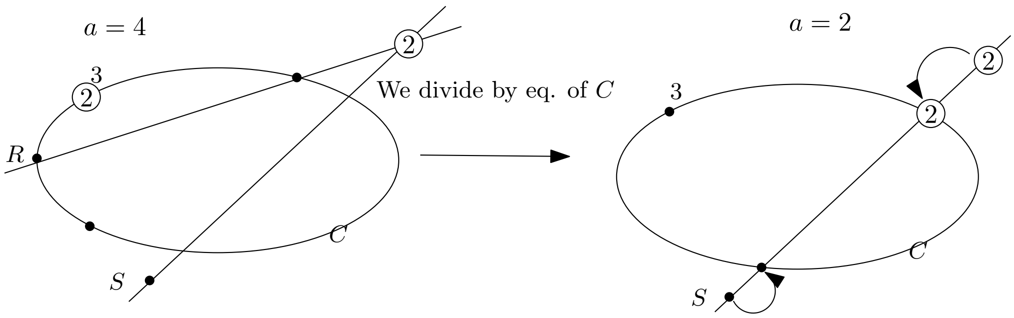
Remark 3.6.3 *Let $Z \subset X$ be subschemes of $\mathbb{P}^2$; it easy to show that if $h^0(\mathcal{L}_Z(d)) = 0$ then $h^0(\mathcal{L}_X(d)) = 0$ too.*

By this remark we can reduce the listed above cases because : Z_{19} is contained in

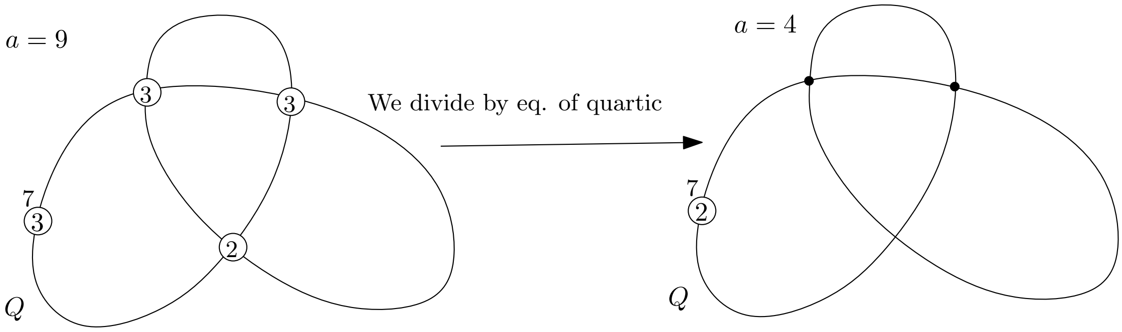
Z_2, Z_{28}, Z_{25} and Z_{31}; Z_{21} is contained in Z_4 and Z_5 is contained in Z_7. The cases with $a = 2, 3$ are easy to show, and we don't give the proof here.

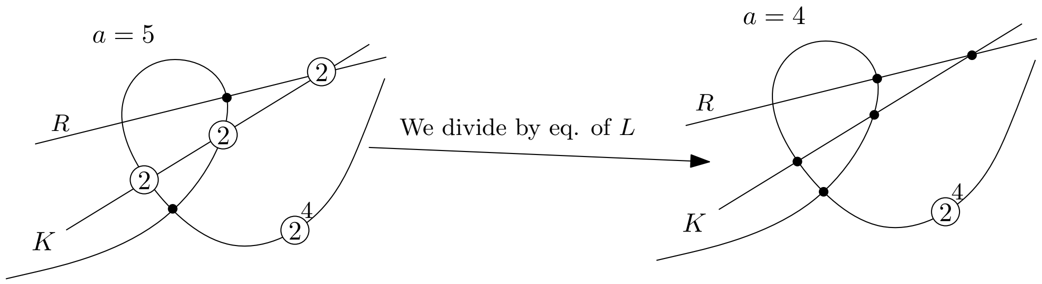
Case $Z_5 = (4,4,4,4,4,4,4,4,3,3)$ $a = 12$ $r = 1$

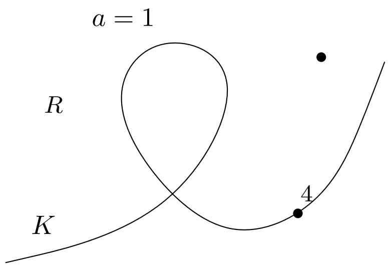




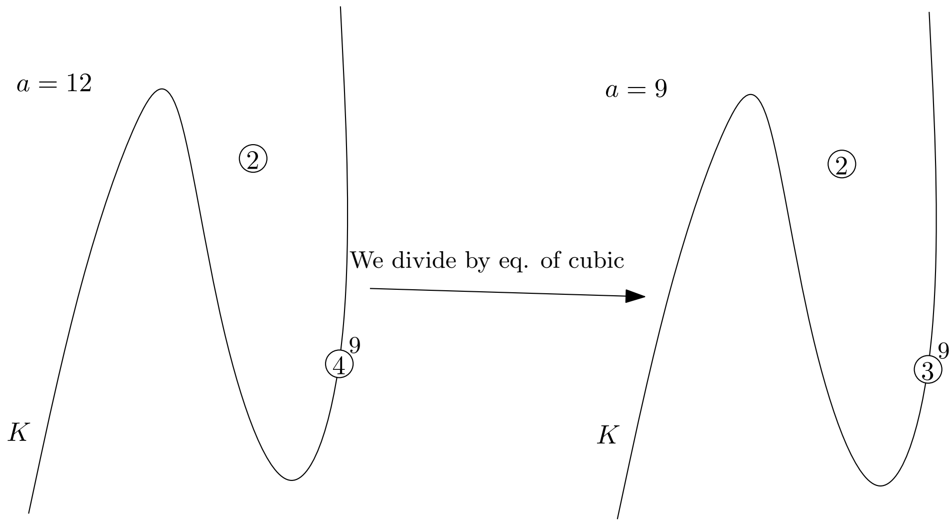
Case $Z_6 = (3,3,3,3,3,3,3,3,3,2)$ $a = 9$ $r = 2$

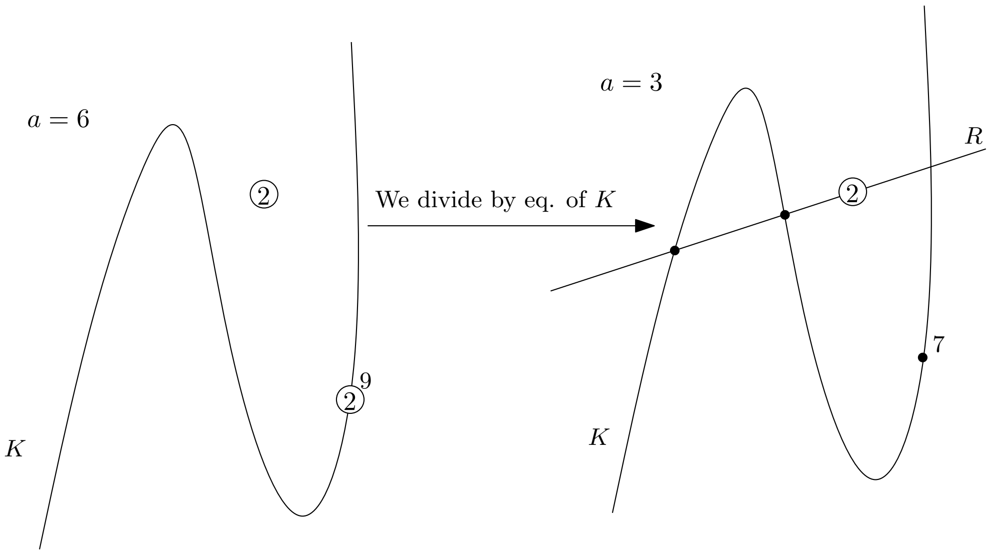


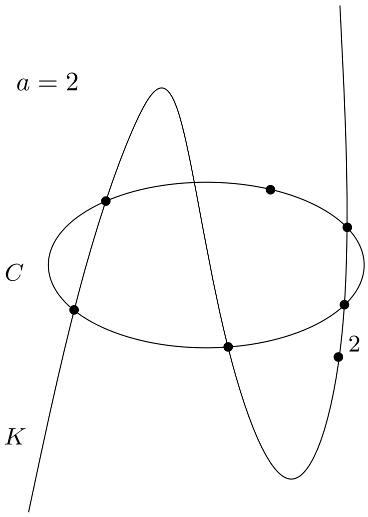




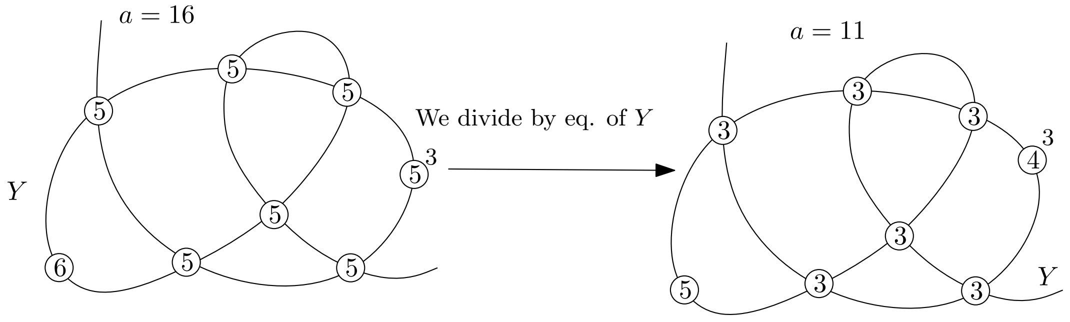
Case $Z_8 = (4, 4, 4, 4, 4, 4, 4, 4, 4, 2)$ $a = 12$ $r = 2$

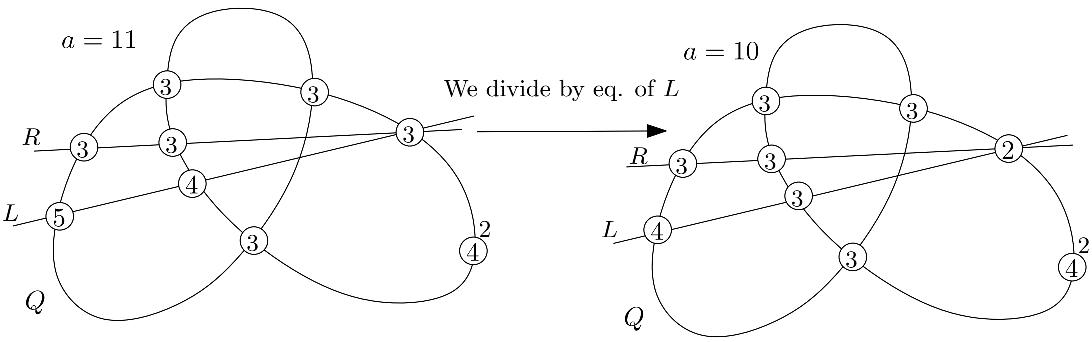


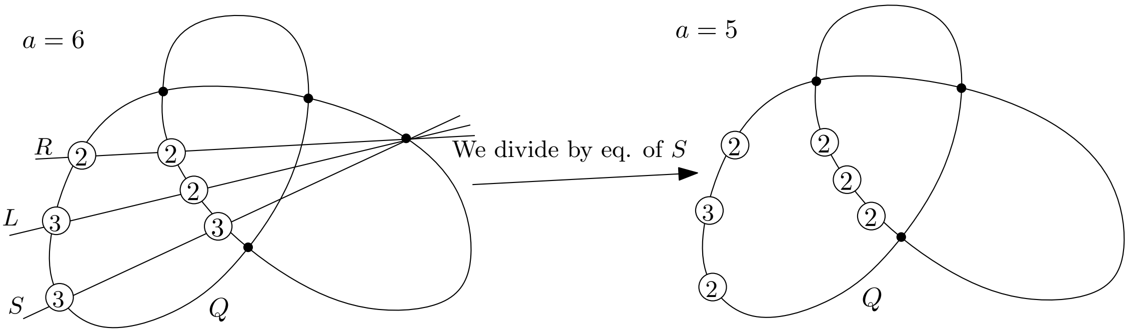


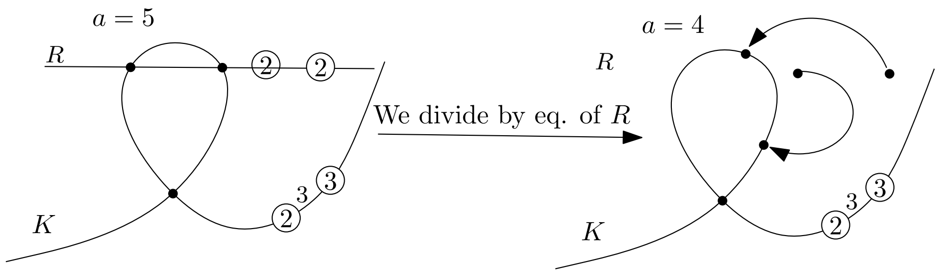


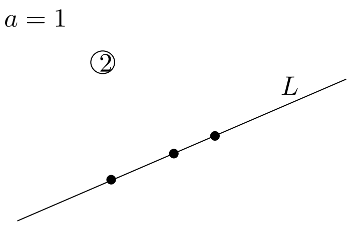
Case $Z_9 = (6, 5, 5, 5, 5, 5, 5, 5, 5, 5)$ $a = 16$ $r = 3$


a = 16
Y
We divide by eq. of Y
a = 11
Y

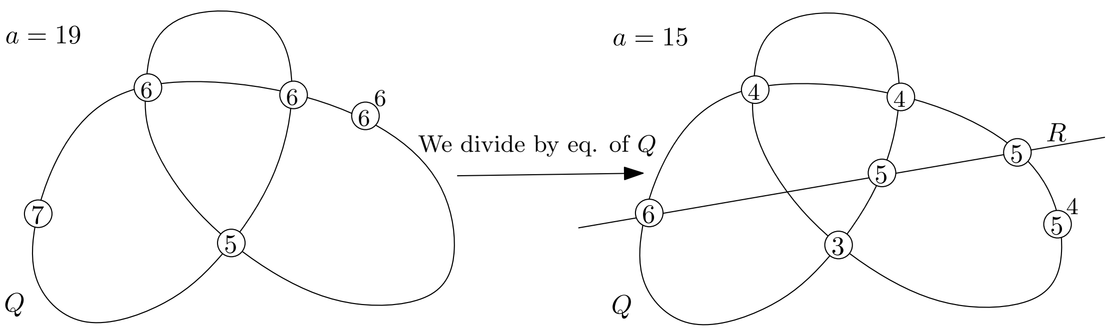

a = 11
R
L
Q
We divide by eq. of L
a = 10
R
L
Q

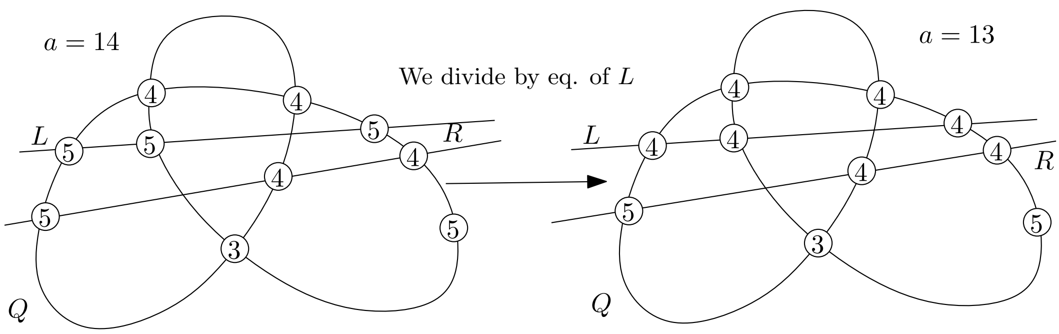

a = 6
R
L
S
Q
We divide by eq. of S
a = 5
Q

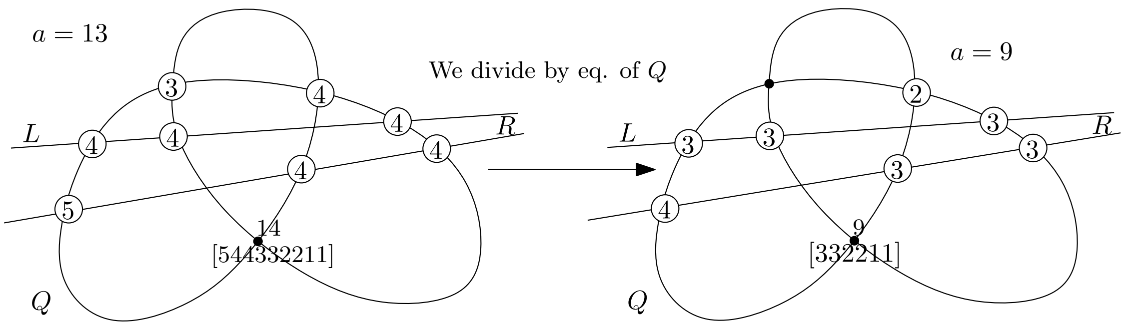


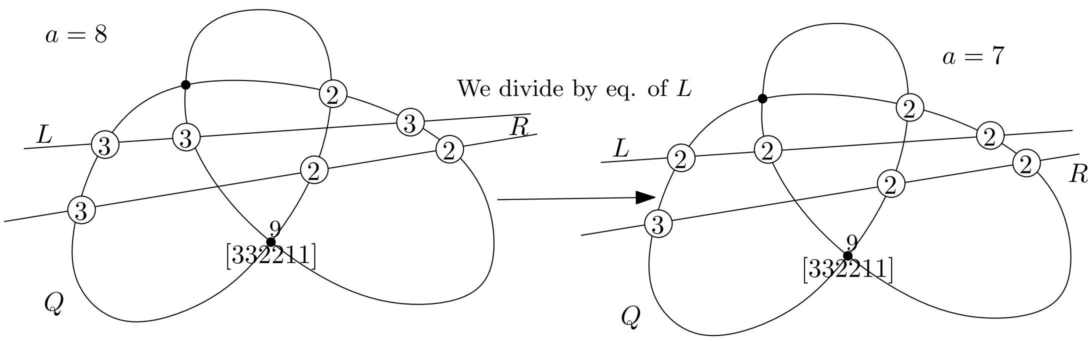


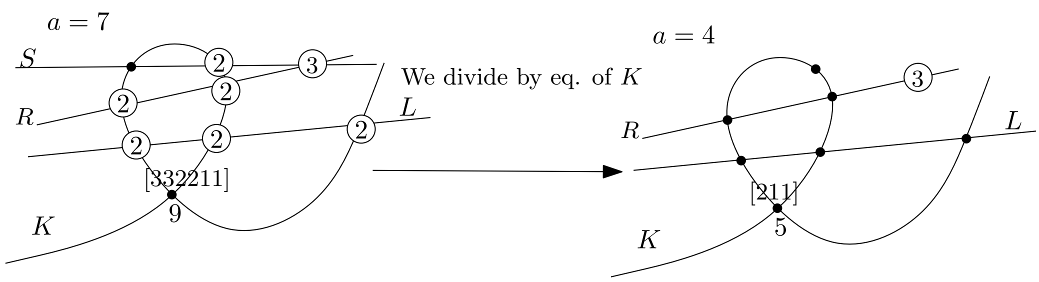
Case $Z_{10} = (7, 6, 6, 6, 6, 6, 6, 6, 6, 5)$ $a = 19$ $r = 1$

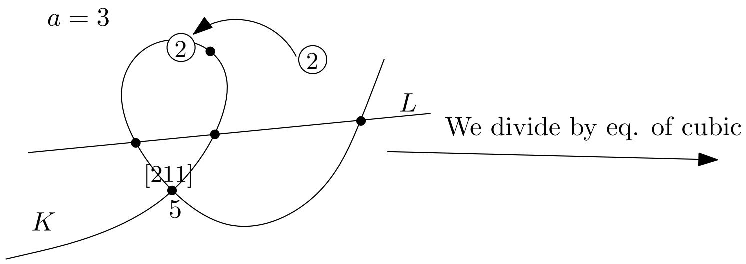




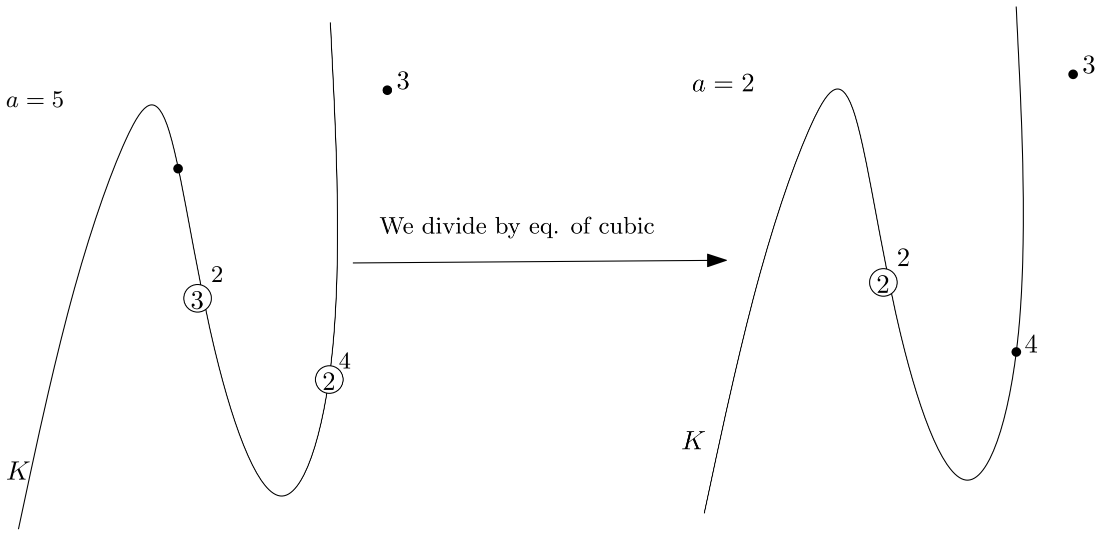

$a = 13$
We divide by eq. of Q
$a = 9$
L
R
Q
14
[544332211]
9
[332211]

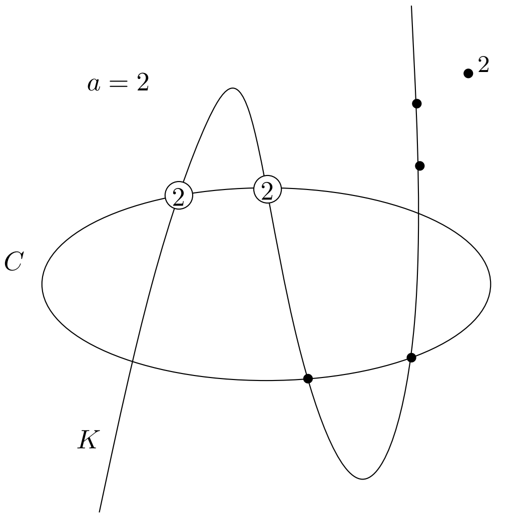

$a = 8$
We divide by eq. of L
$a = 7$
L
R
Q
9
[332211]


$a = 7$
We divide by eq. of K
$a = 4$
S
R
L
K
[332211]
9
[211]
5

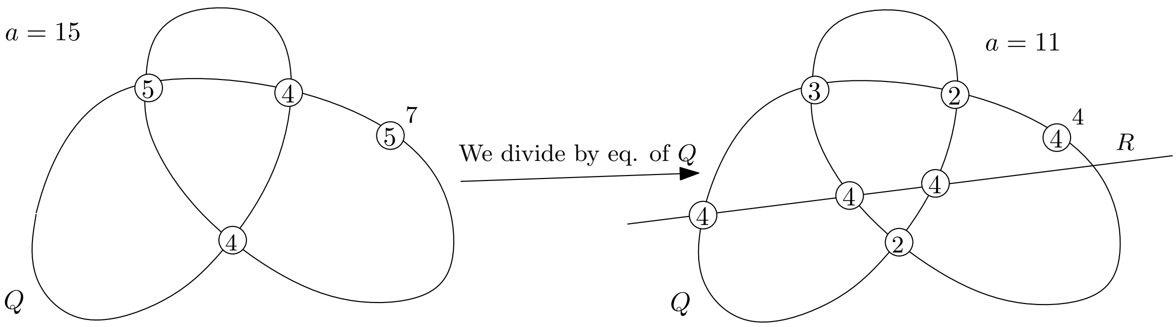


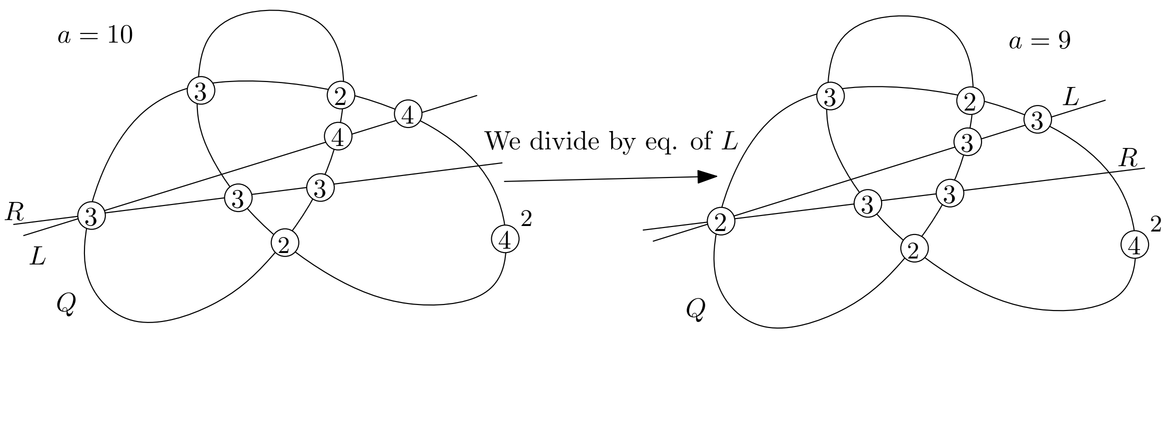
Case $Z_{13} = (3, 3, 2, 2, 2, 2, 2, 1, 1, 1)$ $a = 5$ $r = 6$

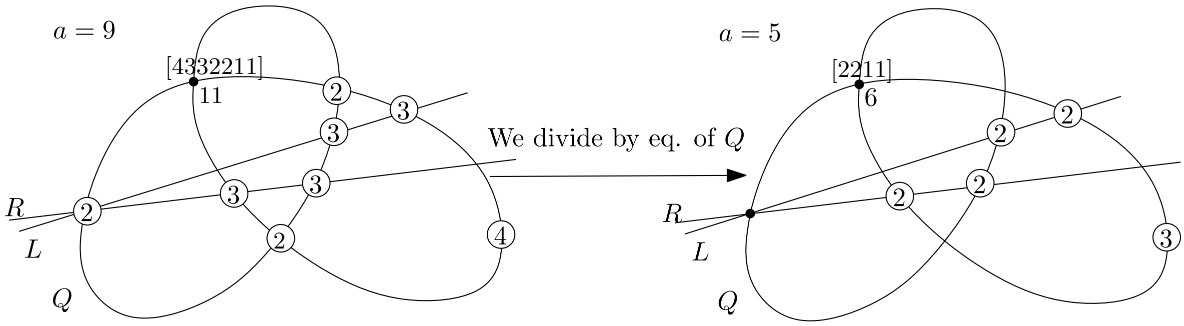


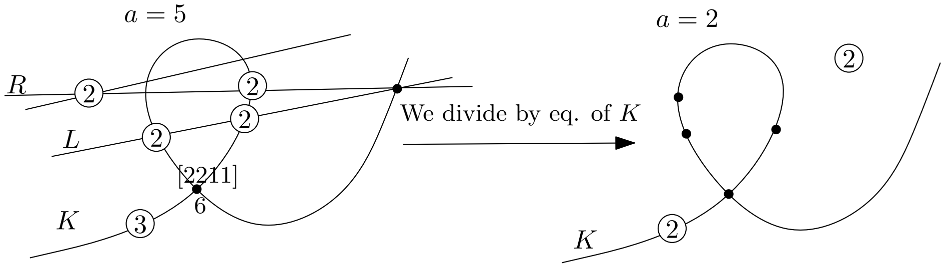


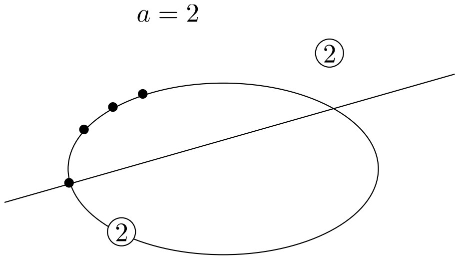
Case $Z_{14} = (5, 5, 5, 5, 5, 5, 5, 5, 4, 4)$ $a = 15$ $r = 4$



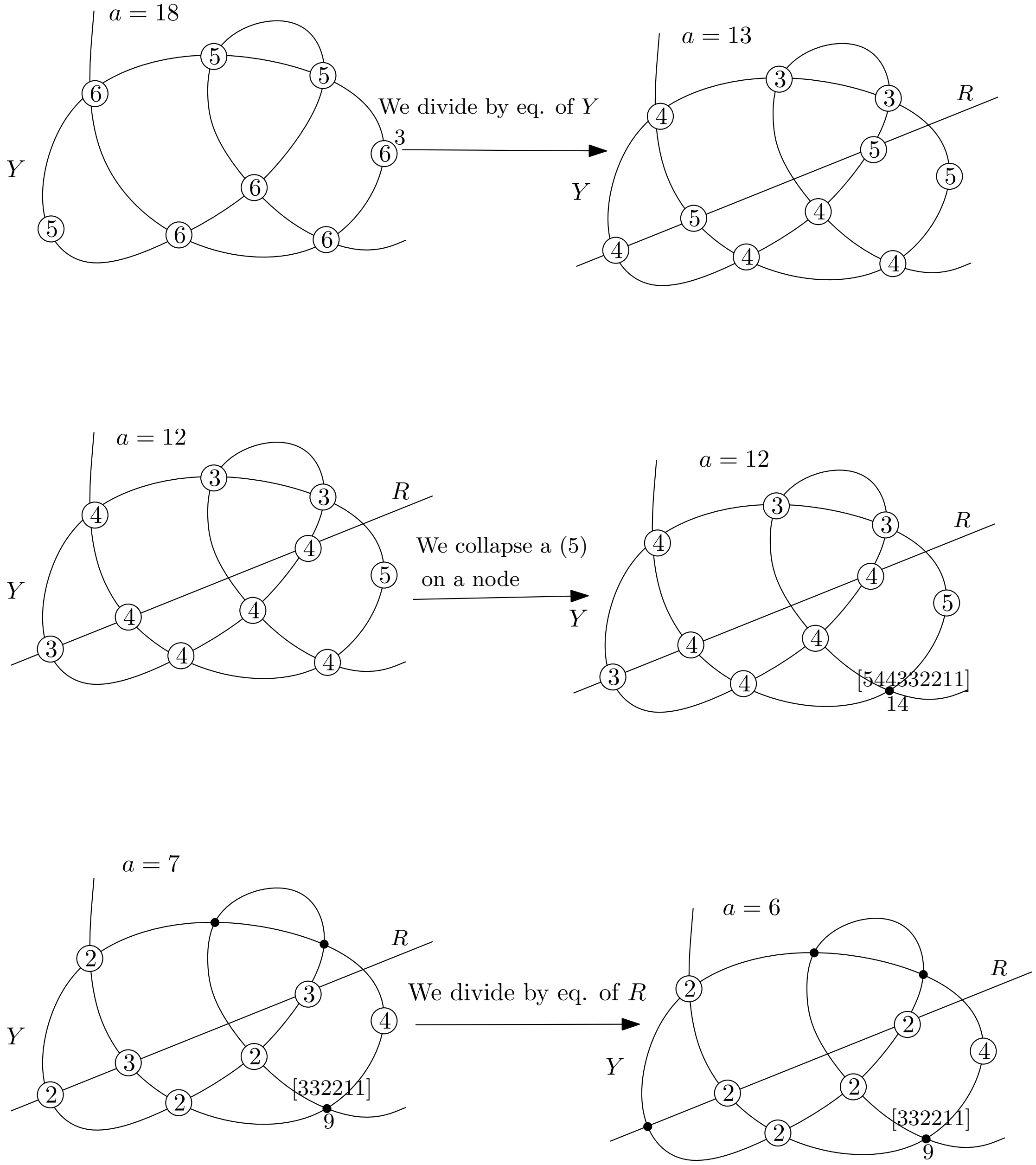


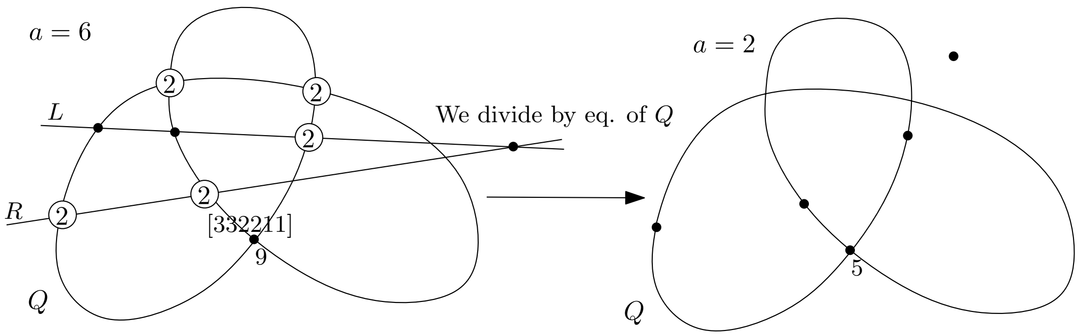


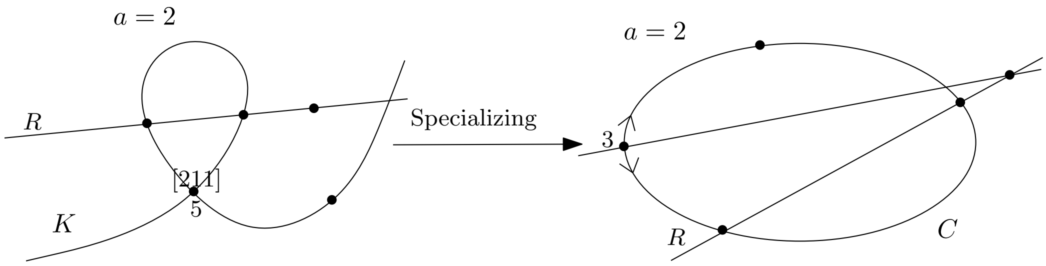




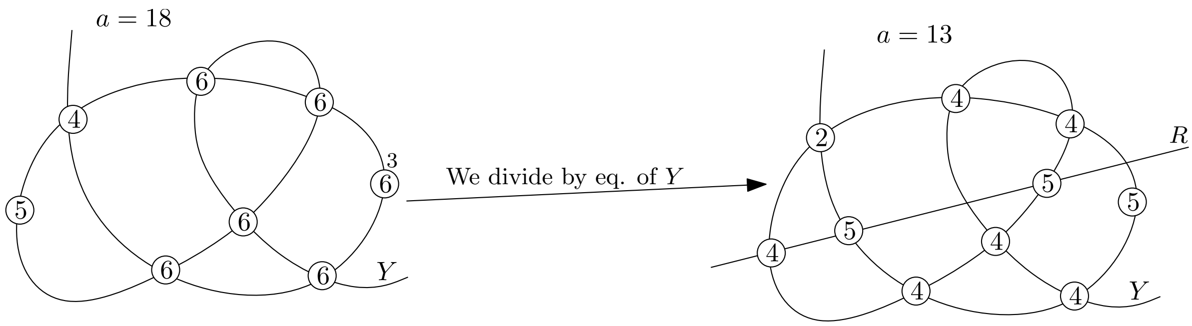
Case $Z_{15} = (6, 6, 6, 6, 6, 6, 6, 5, 5, 5)$ $a = 18$ $r = 2$

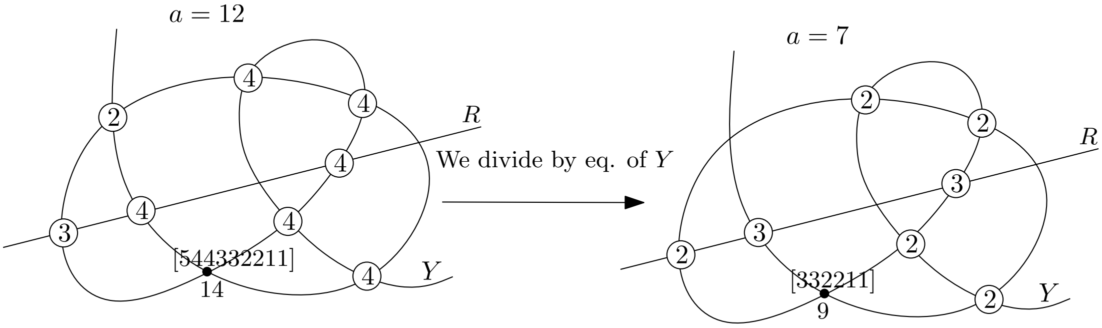

a = 18
Y
We divide by eq. of Y
3
a = 13
R
a = 12
We collapse a (5)
on a node
[544332211]
14
a = 7
We divide by eq. of R
[332211]
9
a = 6

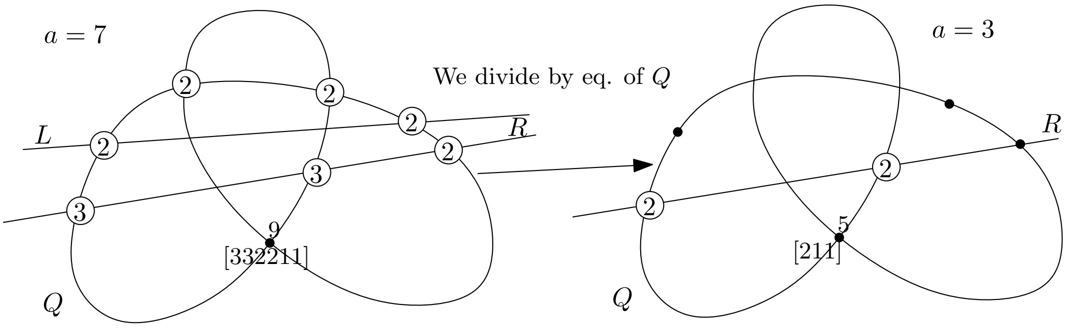


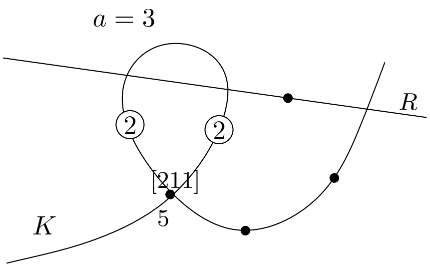


Case $Z_{17} = (6, 6, 6, 6, 6, 6, 6, 6, 5, 4)$ $a = 18$ $r = 3$

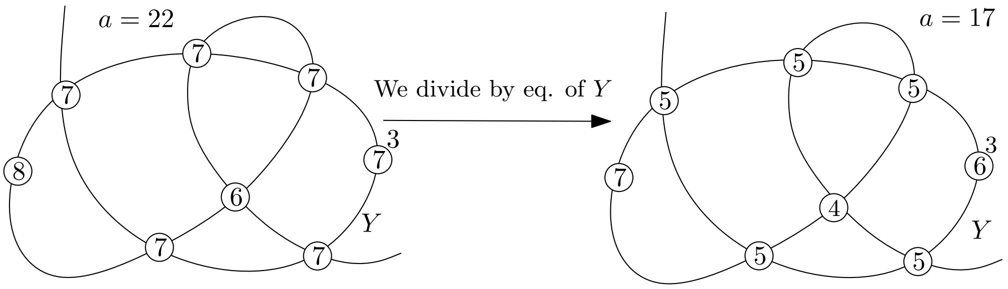


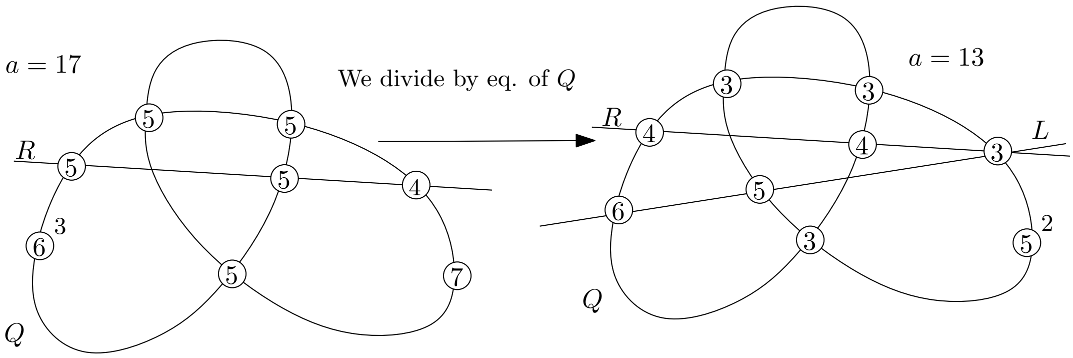


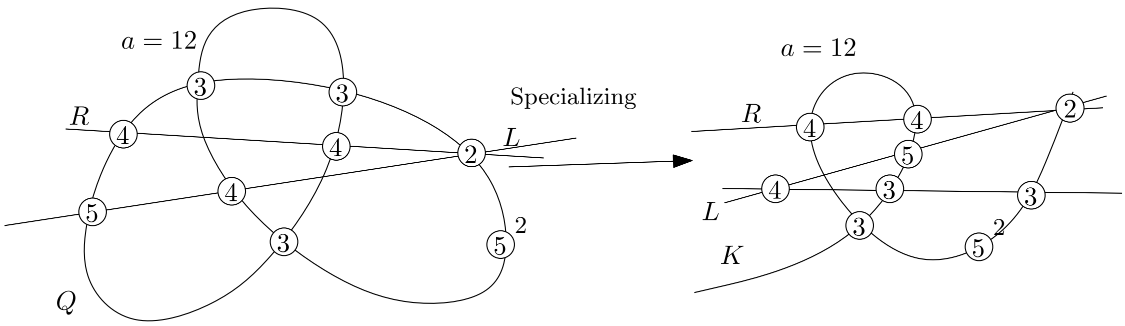


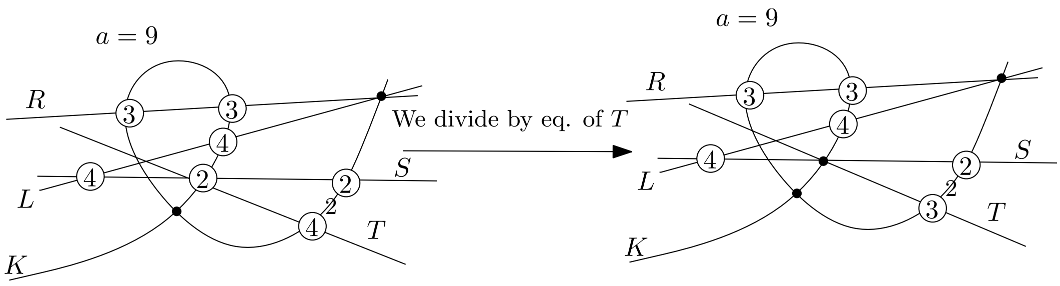


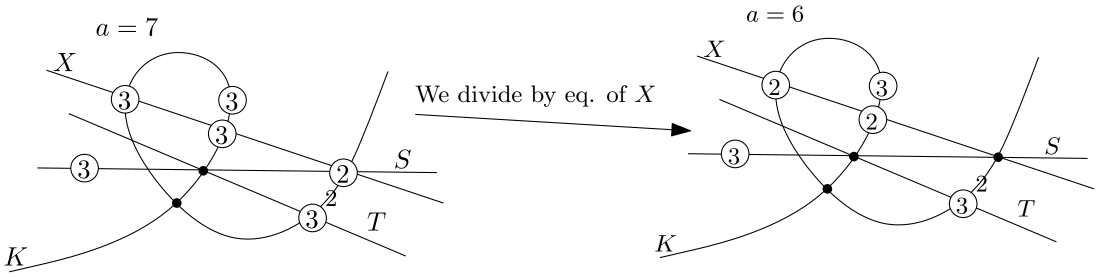
Case $Z_{18} = (8, 7, 7, 7, 7, 7, 7, 7, 7, 6)$ $a = 22$ $r = 5$

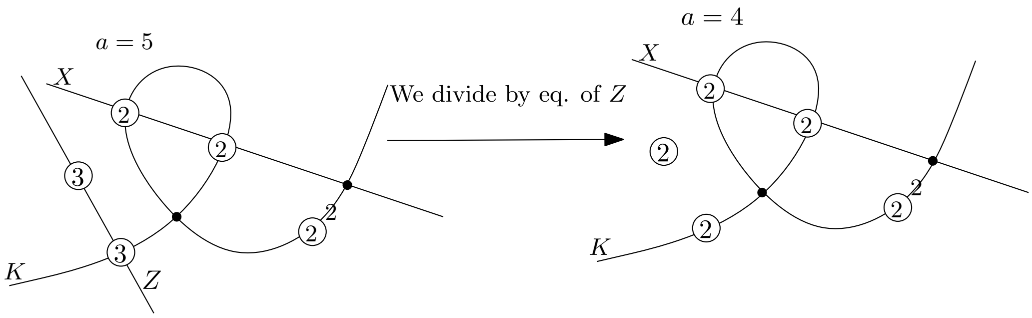

a = 22
We divide by eq. of Y
a = 17
Y

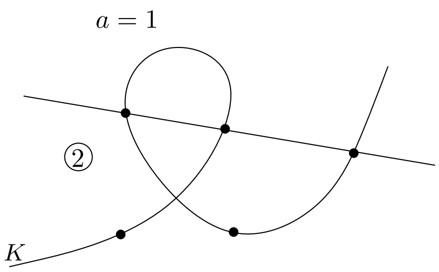

a = 17
R
Q
We divide by eq. of Q
a = 13
L


a = 12
R
L
Q
Specializing
a = 12
K

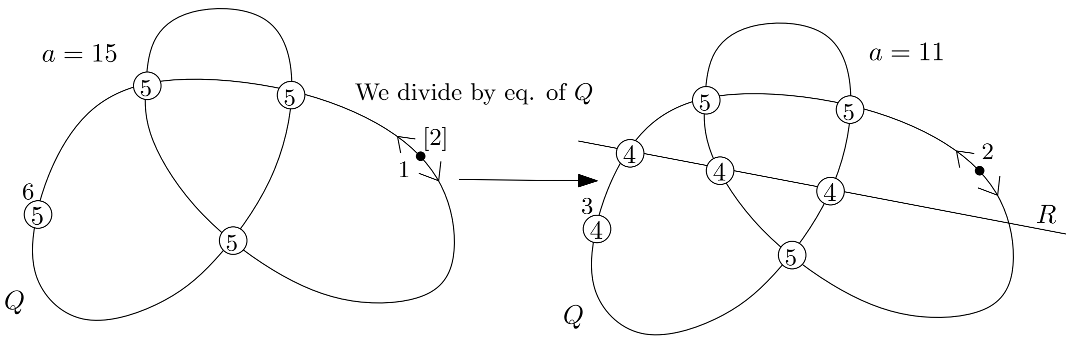

a = 9
We divide by eq. of T
a = 9

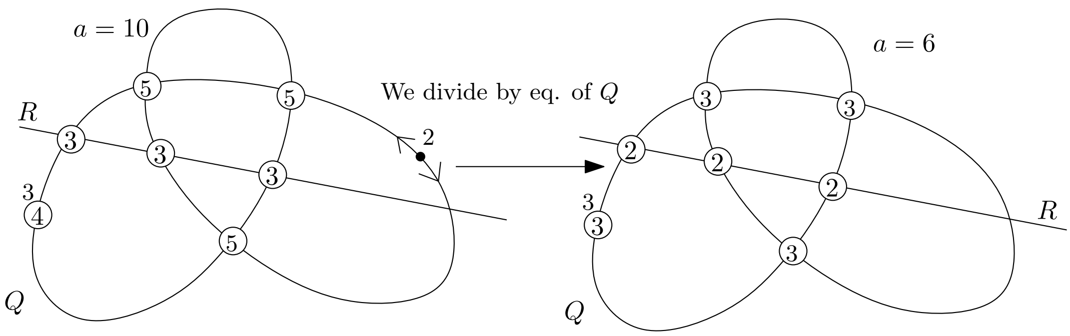

a = 7
We divide by eq. of X
a = 6

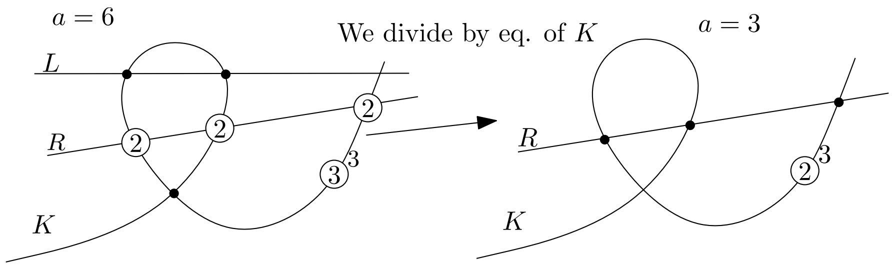

a = 5
We divide by eq. of Z
a = 4

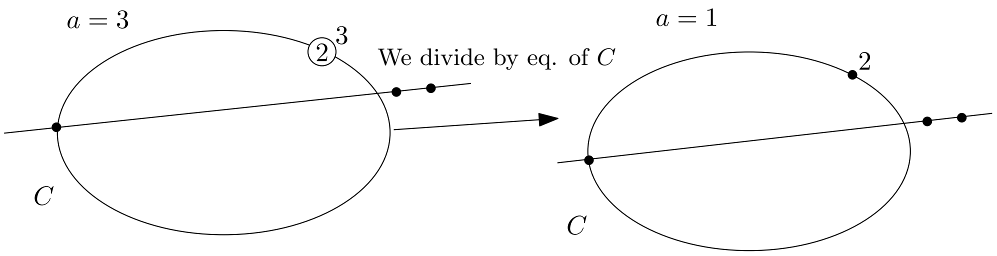

a = 1

Case $Z_{20} = (5, 5, 5, 5, 5, 5, 5, 5, 5, 2)$ $a = 15$ $r = 2$

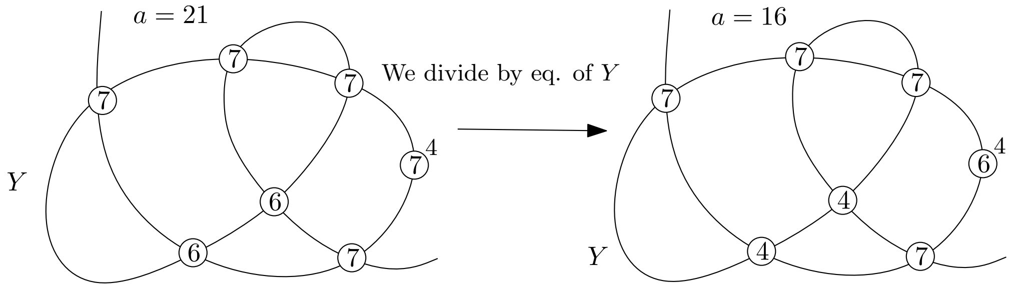


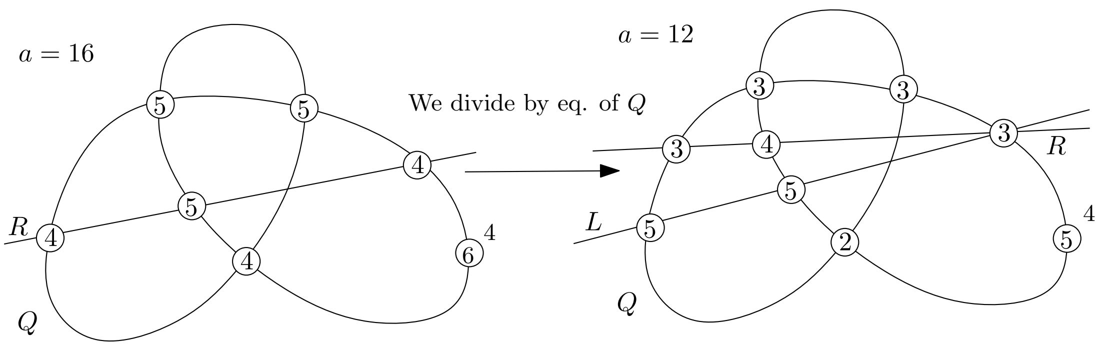


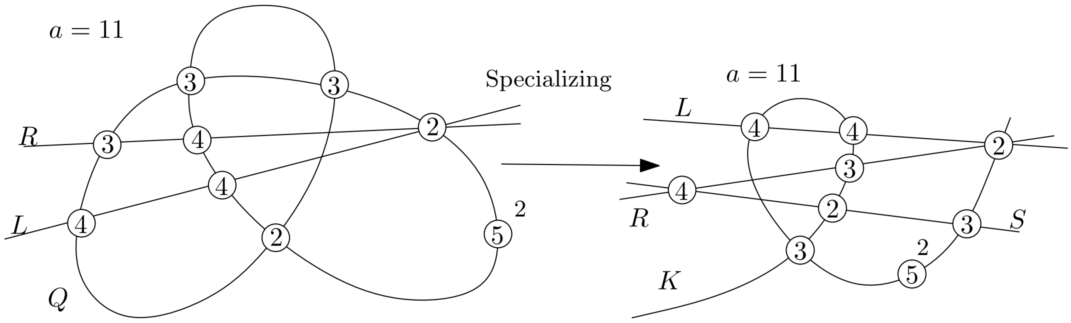


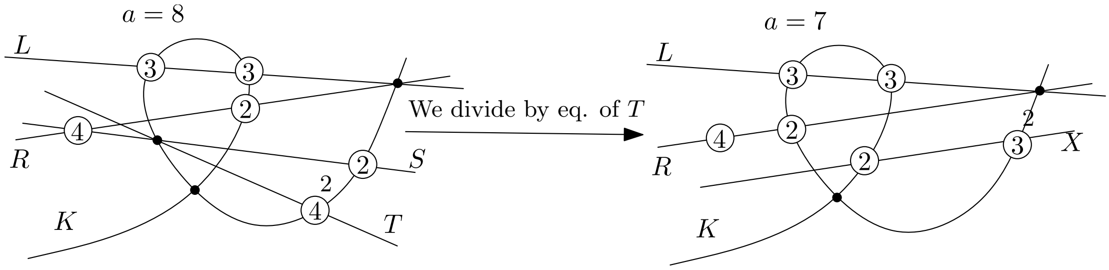


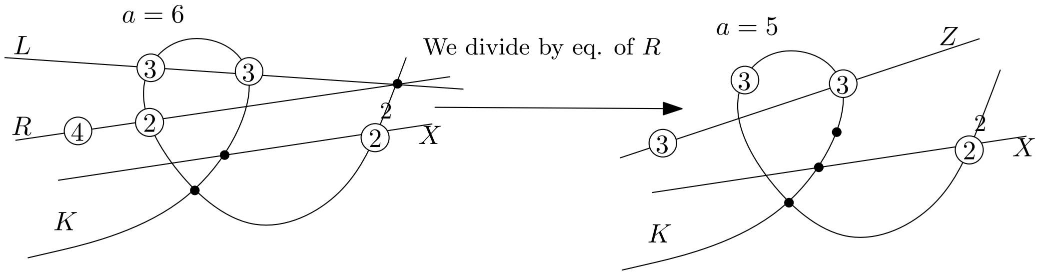
Case $Z_{23} = (7, 7, 7, 7, 7, 7, 7, 6, 6, 6)$ $a = 21$ $r = 6$

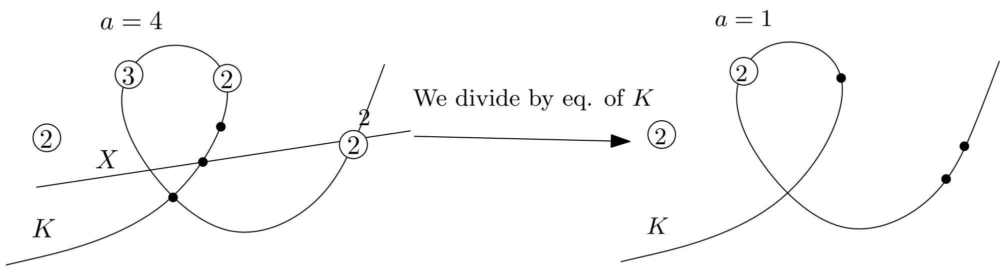




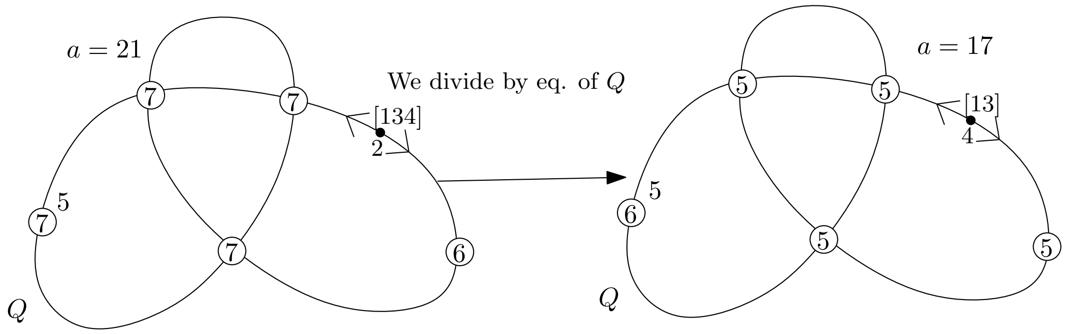

a = 11
Specializing
a = 11
R
L
Q
K
S
2

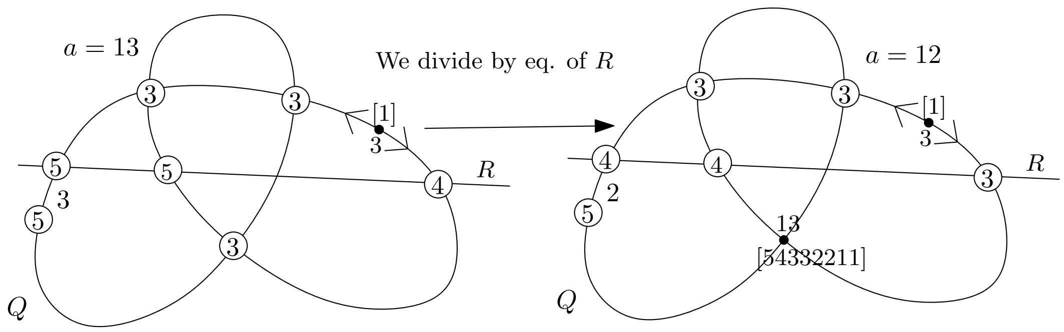

a = 8
We divide by eq. of T
a = 7
L
R
K
S
T
X
2

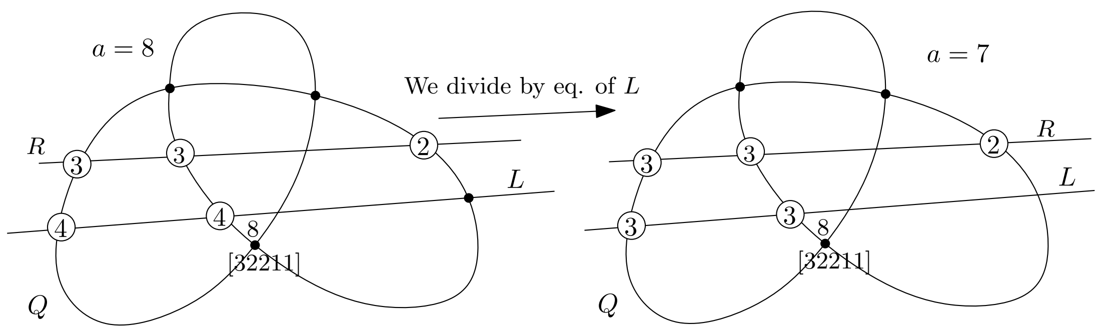

a = 6
We divide by eq. of R
a = 5
L
R
K
X
Z
2

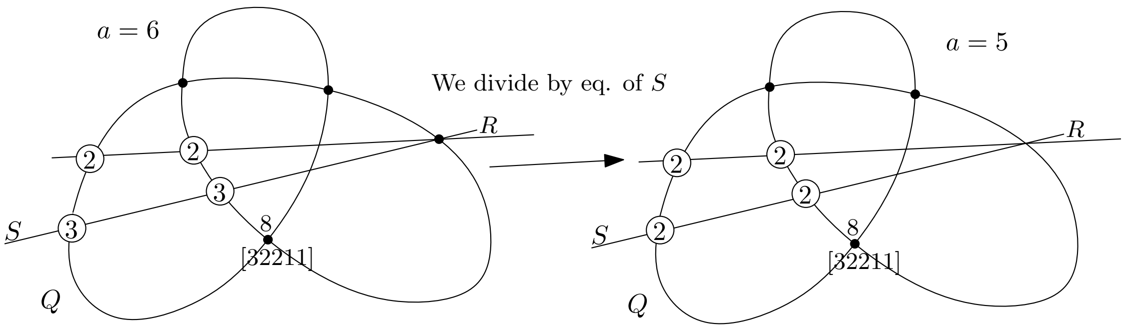

a = 4
We divide by eq. of K
a = 1
X
K
2

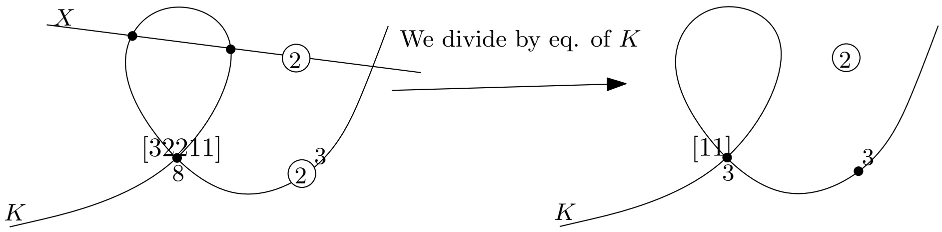
Case $Z_{24} = (7,7,7,7,7,7,7,7,6,4)$ $a = 21$ $r = 2$



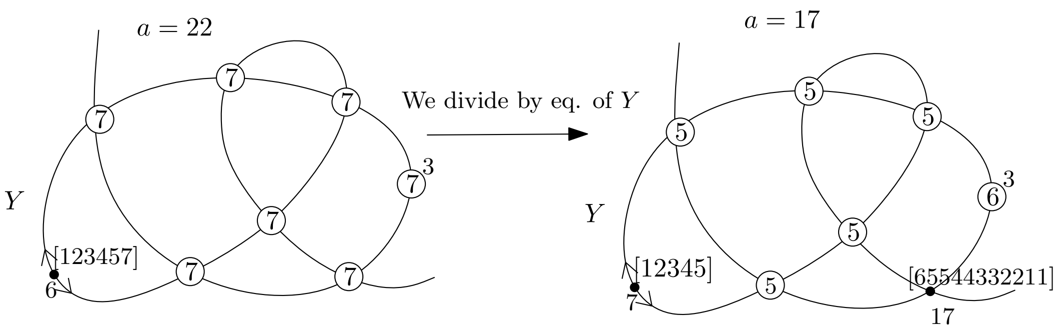


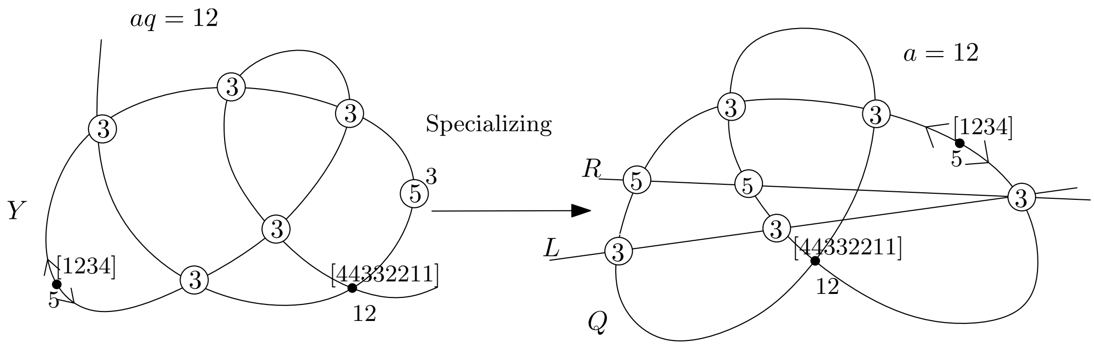




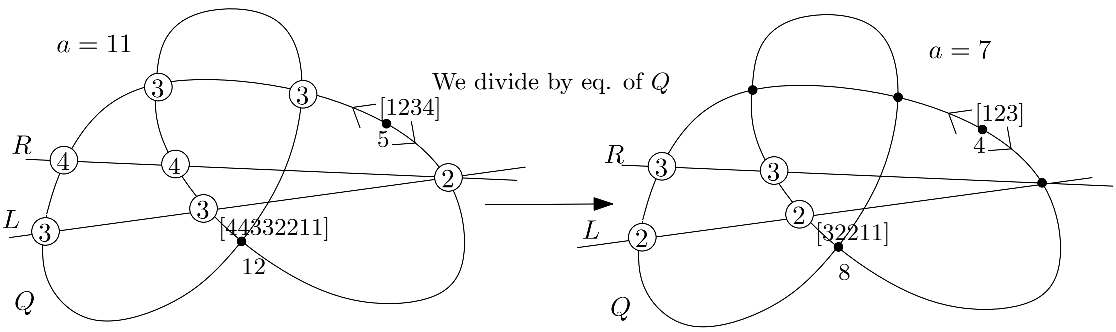


Case $Z_{26} = (7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7)$ $a = 22$ $r = 4$

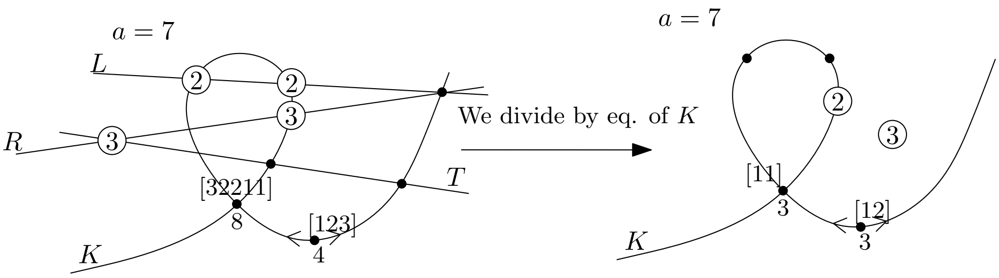




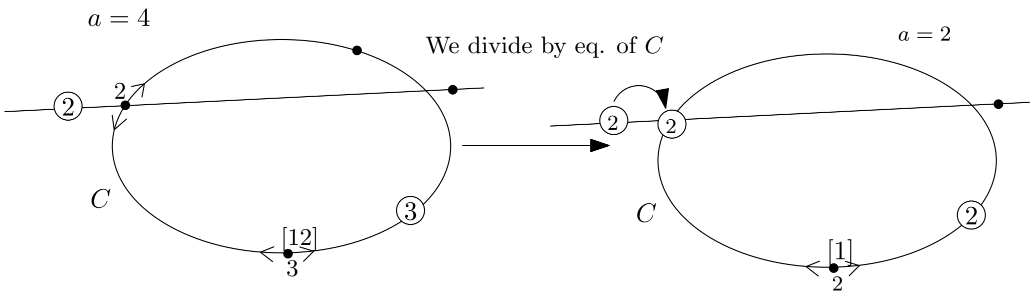
$aq = 12$
Y
[1234]
5
[44332211]
12
Specializing
$a = 12$
R
L
Q
[1234]
5
[44332211]
12



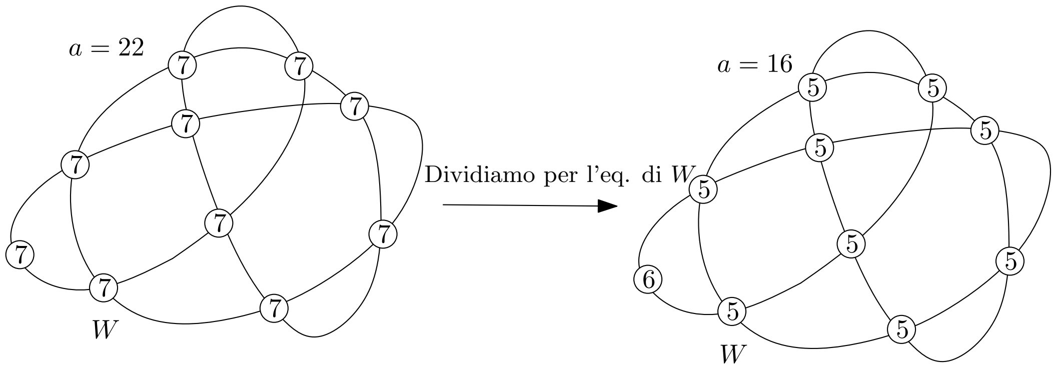
$a = 11$
R
L
Q
[1234]
5
[44332211]
12
We divide by eq. of Q
$a = 7$
R
L
Q
[123]
4
[32211]
8

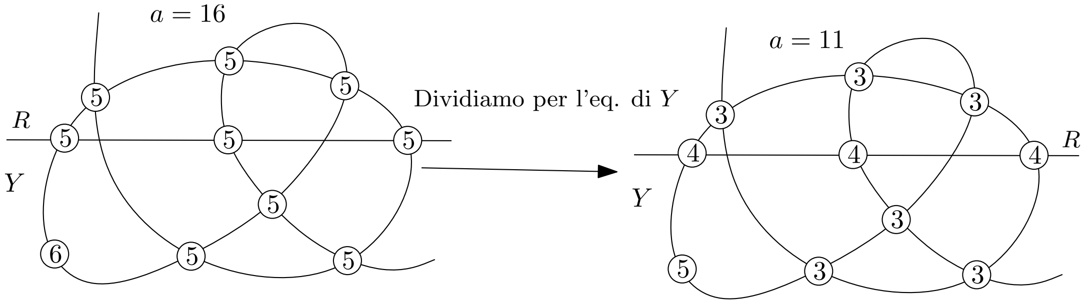


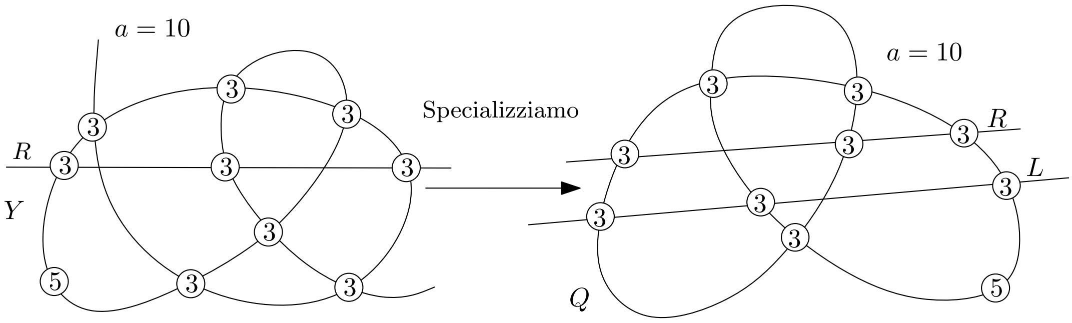
$a = 7$
L
R
T
K
[32211]
8
[123]
4
We divide by eq. of K
$a = 7$
K
[11]
3
[12]
3

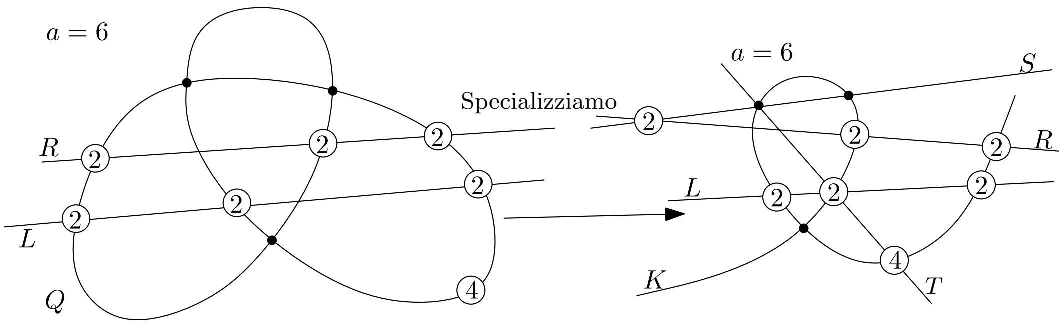


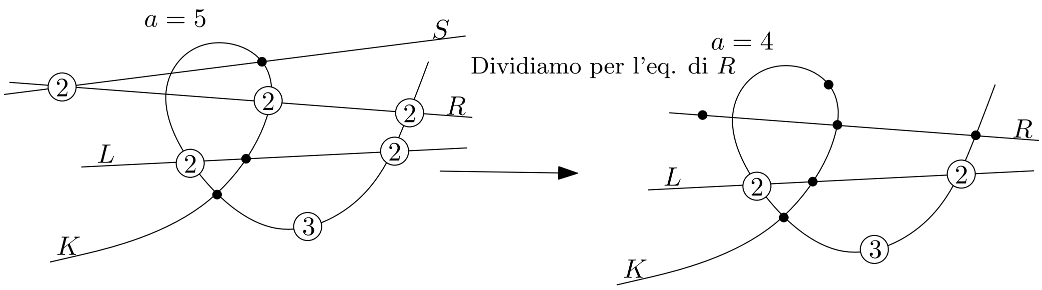
Another way to prove this case

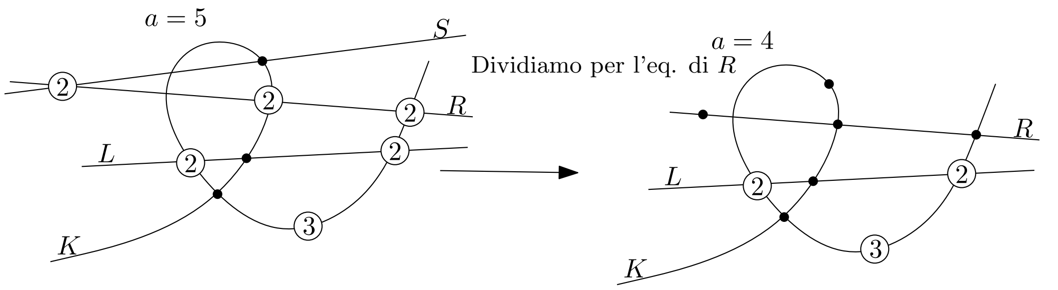




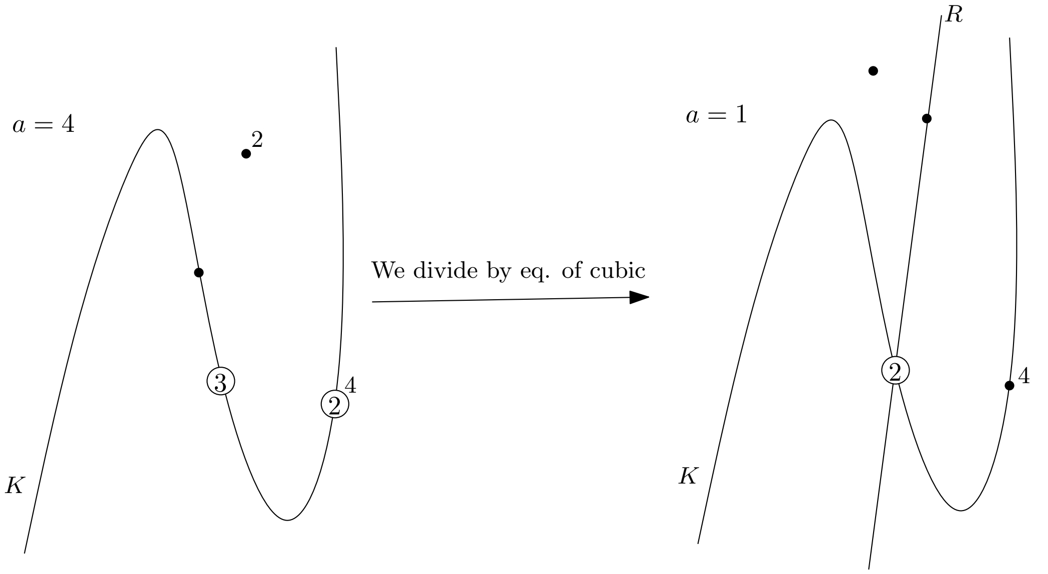

a = 10
Specializziamo
a = 10
R
L
Y
Q


a = 6
Specializziamo
a = 6
S
R
L
K
T
Q

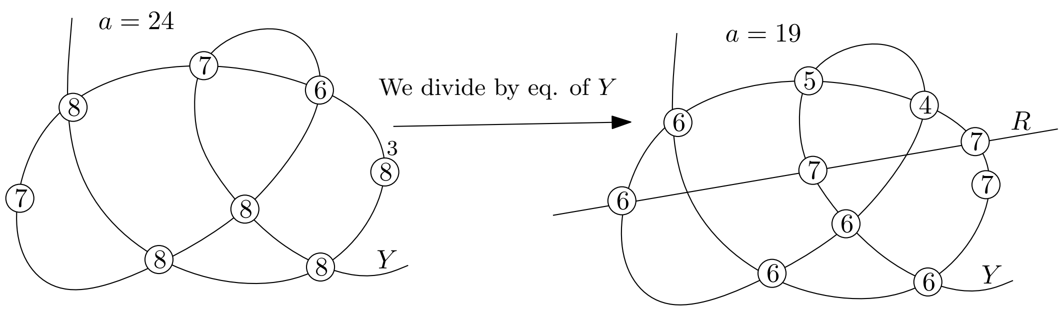

a = 5
S
R
L
K
Dividiamo per l'eq. di R
a = 4
R
L
K

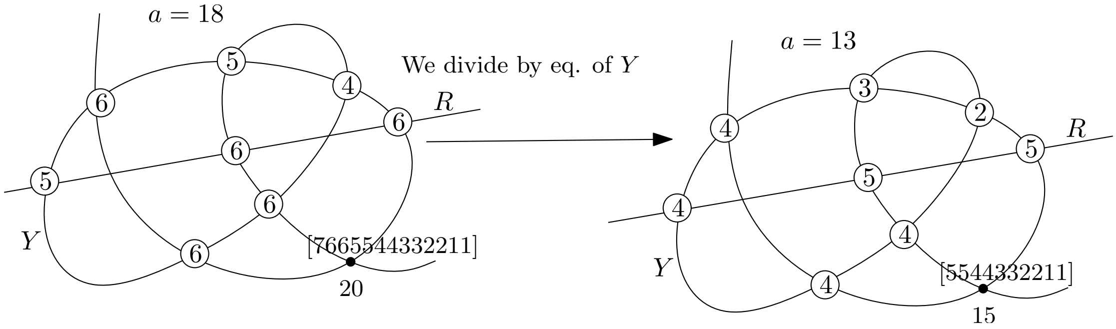


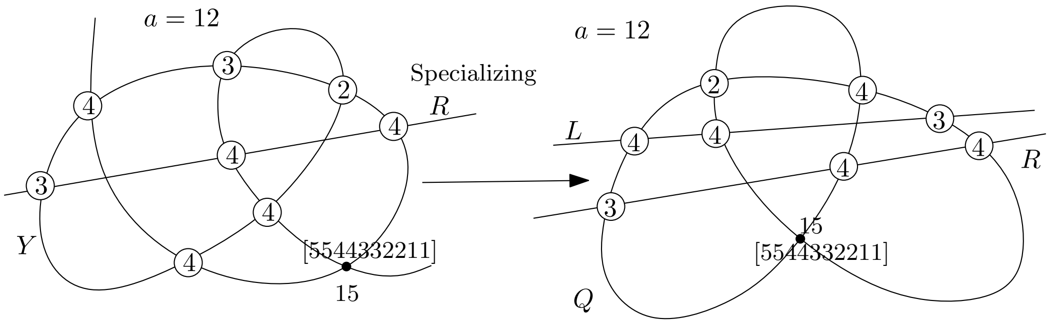
Case $Z_{29} = (3, 2, 2, 2, 2, 1, 1, 1)$ $a = 4$ $r = 6$

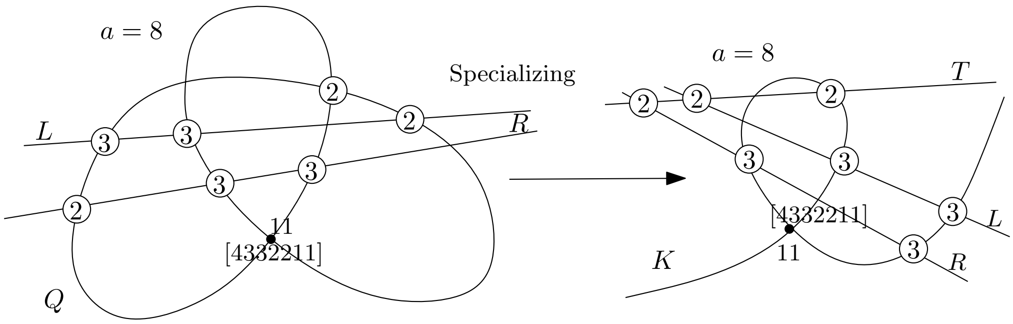


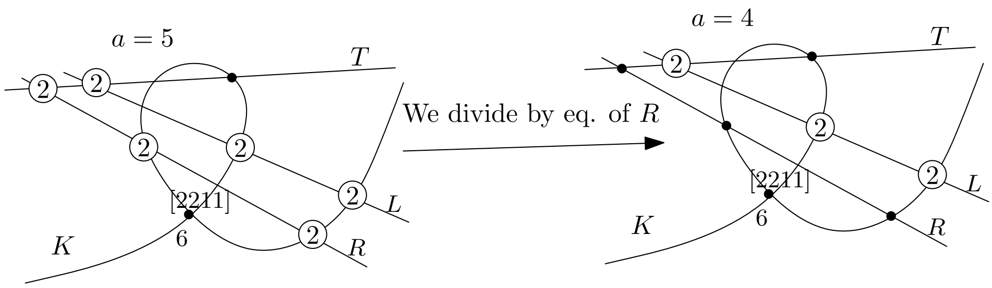
Case $Z_{32} = (8, 8, 8, 8, 8, 8, 8, 7, 7, 6)$ $a = 24$ $r = 4$

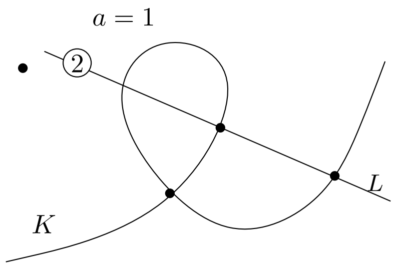



a = 18
Y
R
20
[7665544332211]
We divide by eq. of Y
a = 13
Y
R
15
[5544332211]

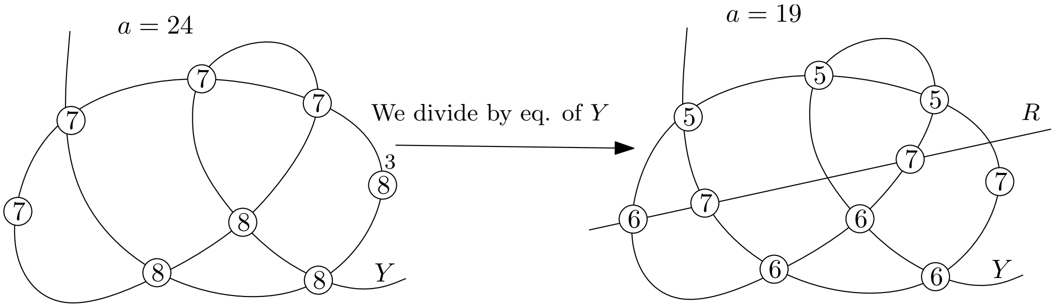

a = 12
Y
R
15
[5544332211]
Specializing
a = 12
L
R
Q
15
[5544332211]

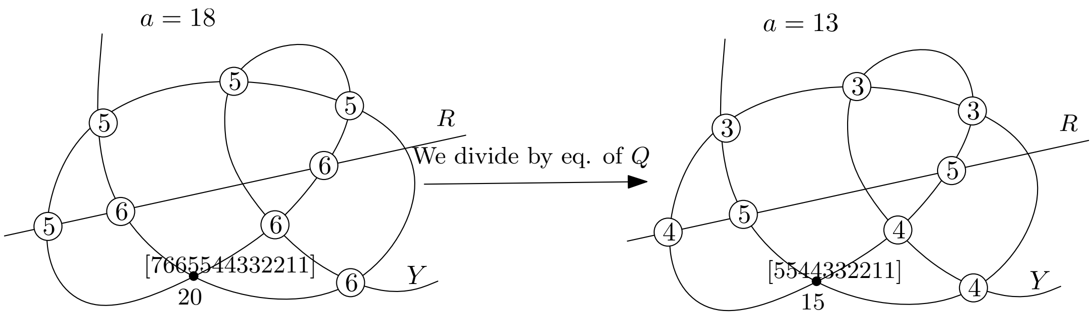

a = 8
L
R
Q
11
[4332211]
Specializing
a = 8
T
L
R
K
[4332211]
11

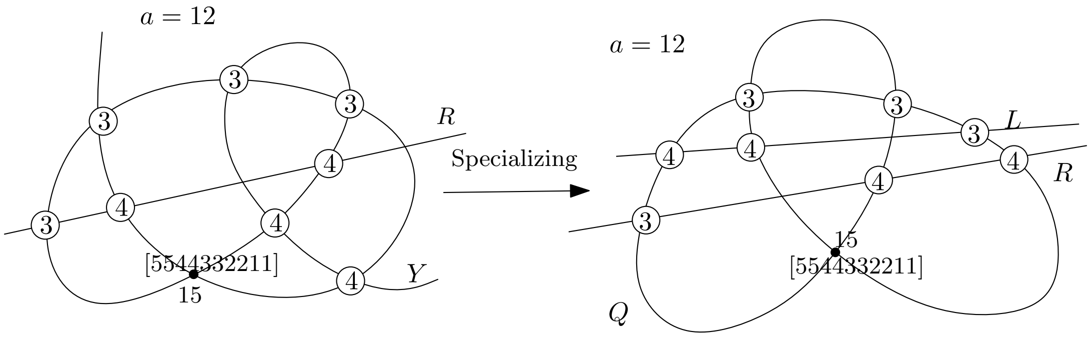


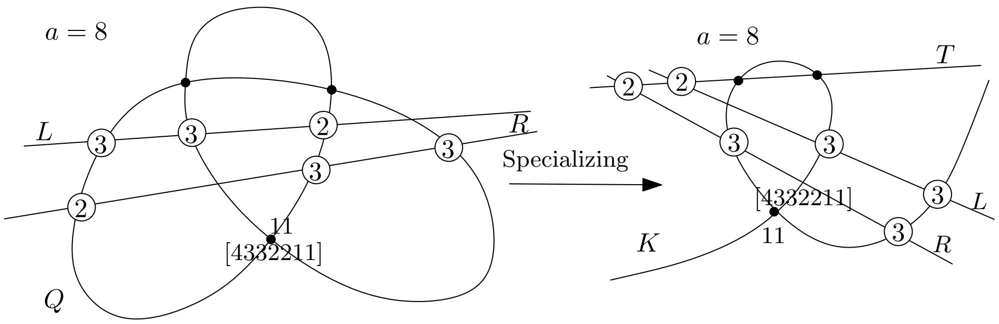


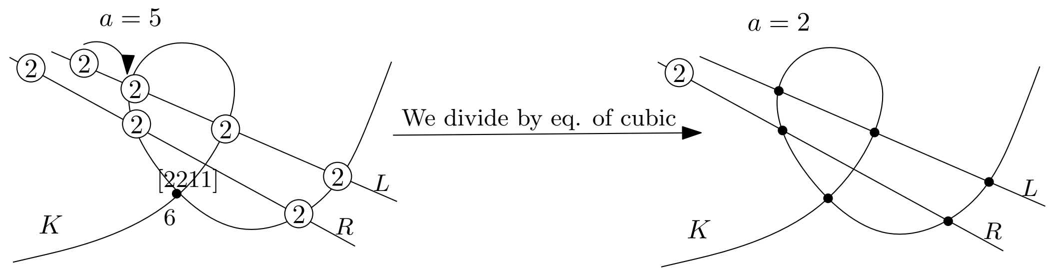
Case $Z_{33} = (8, 8, 8, 8, 8, 8, 7, 7, 7, 7)$ $a = 24$ $r = 3$

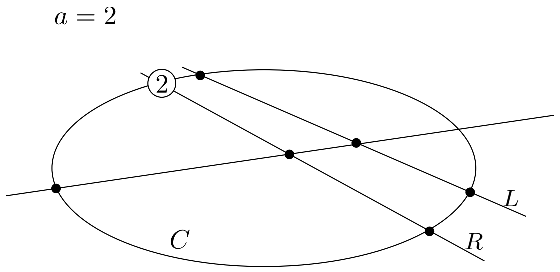



a = 18
5
5
5
5
6
6
6
6
R
Y
[7665544332211]
20
We divide by eq. of Q
a = 13
3
3
3
4
5
5
4
4
R
Y
[5544332211]
15

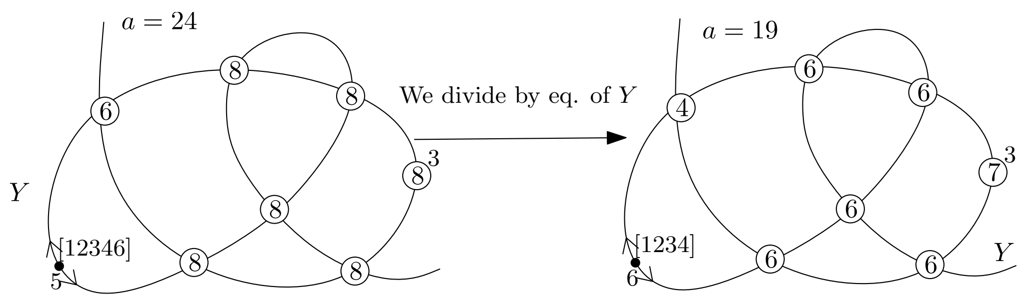

a = 12
3
3
3
3
4
4
4
4
R
Y
[5544332211]
15
Specializing
a = 12
3
3
3
L
4
4
4
4
3
R
15
[5544332211]
Q

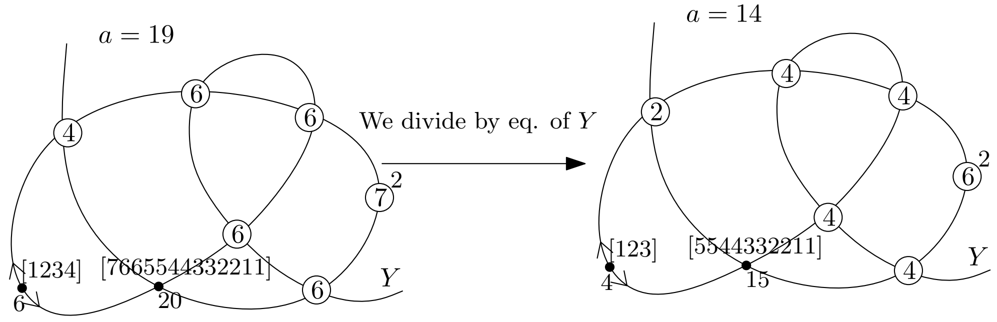

a = 8
L
3
3
2
R
3
2
3
11
[4332211]
Q
Specializing
a = 8
2
2
T
3
3
3
L
[4332211]
11
3
R
K

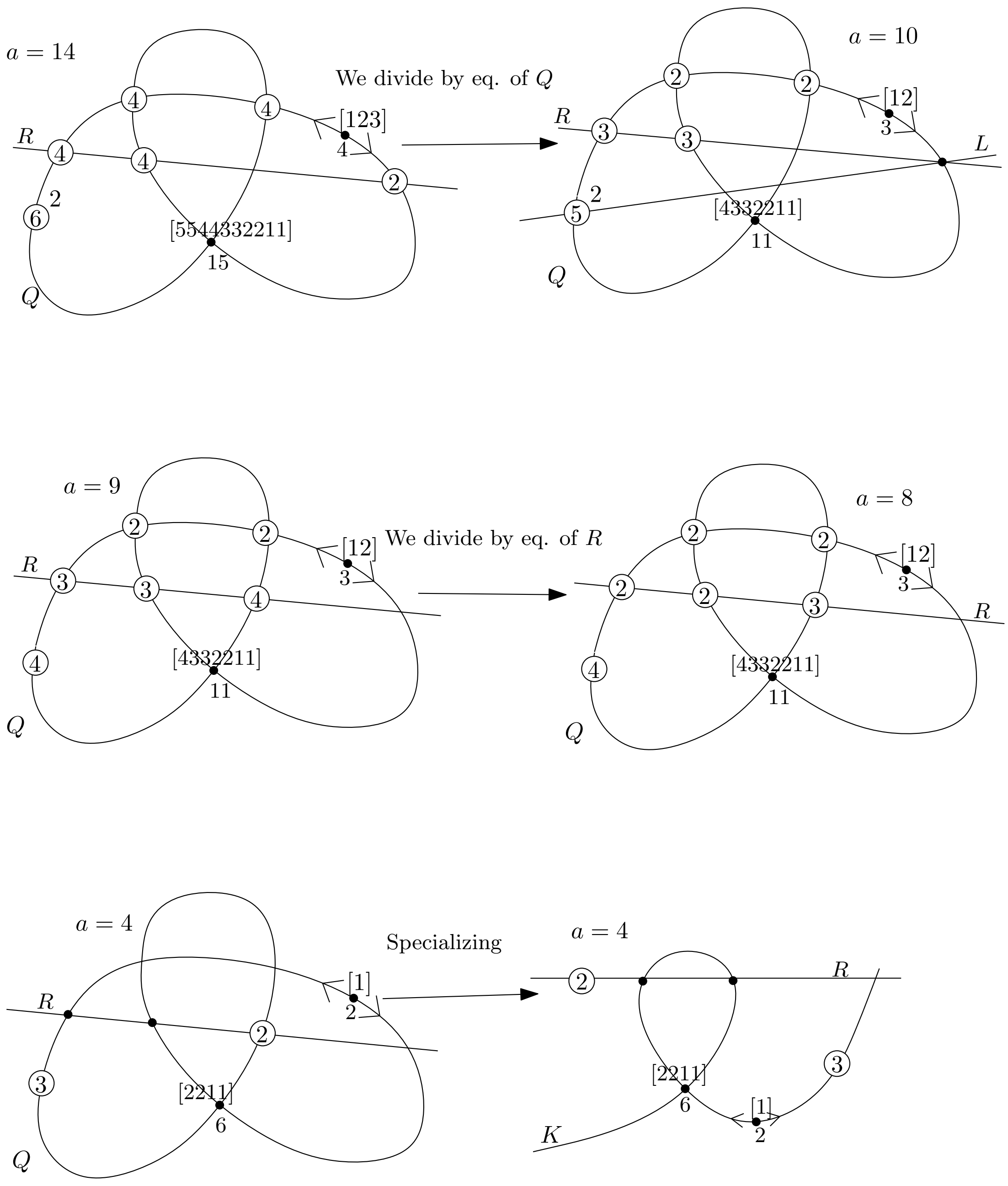


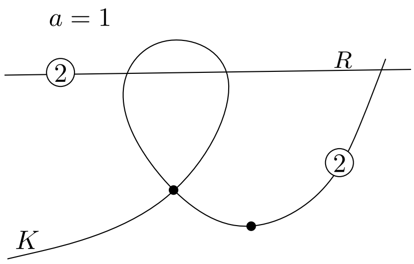


Case $Z_{34} = (8, 8, 8, 8, 8, 8, 8, 8, 6, 6)$ $a = 24$ $r = 5$

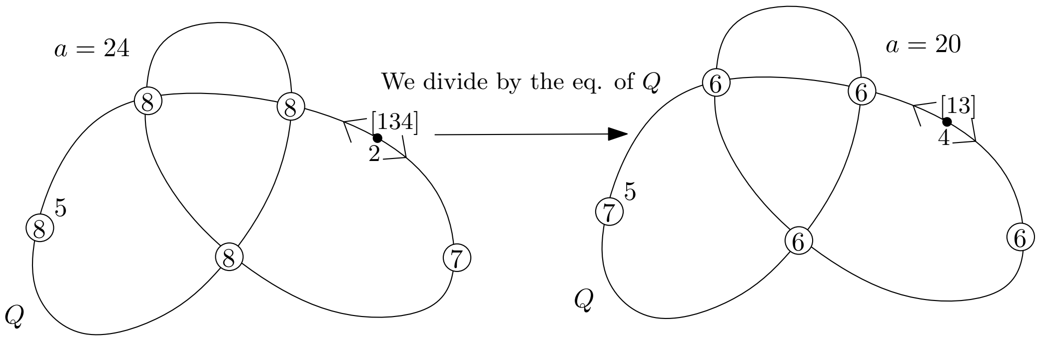


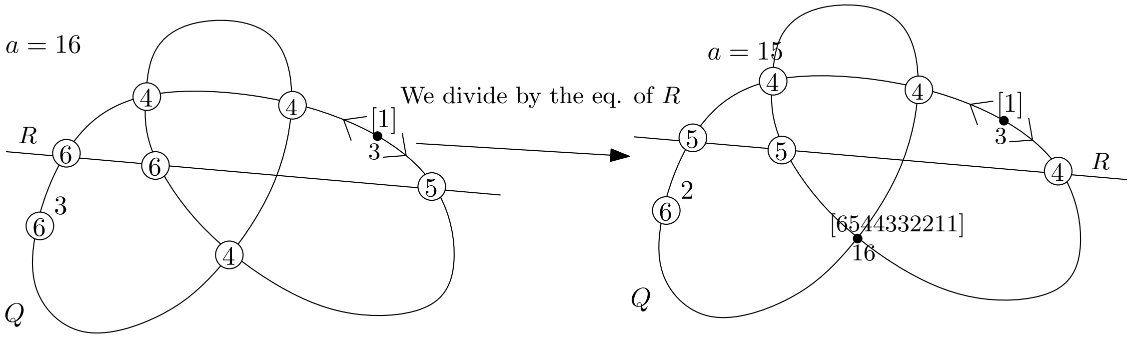


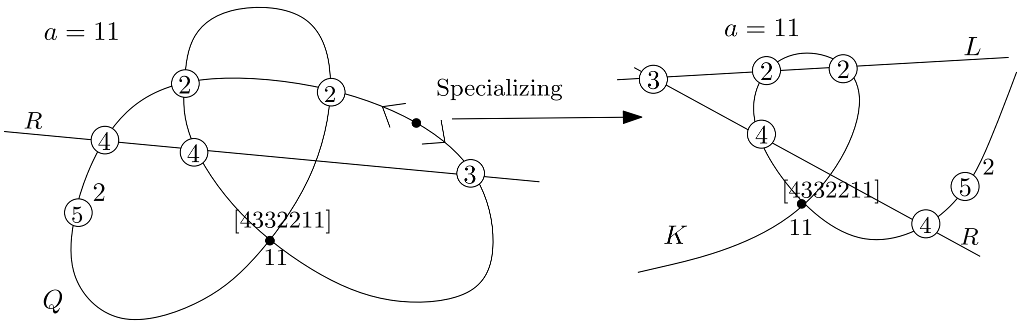

a = 14
We divide by eq. of Q
a = 10
[123]
[5544332211]
15
[12]
[4332211]
11
a = 9
We divide by eq. of R
a = 8
[12]
[4332211]
11
a = 4
Specializing
a = 4
[1]
[2211]
6
R
Q
L
K

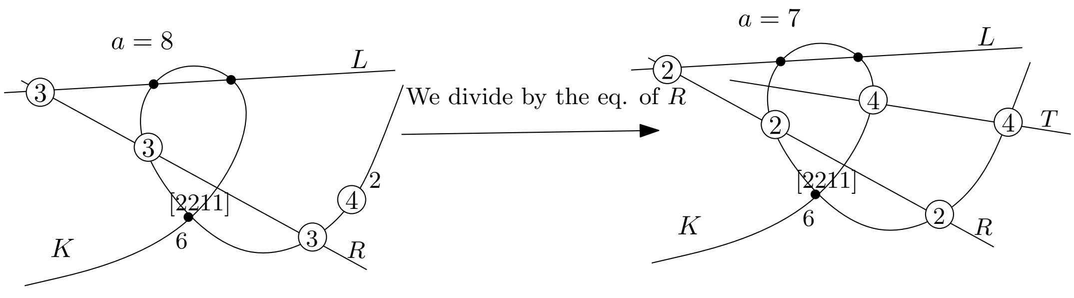


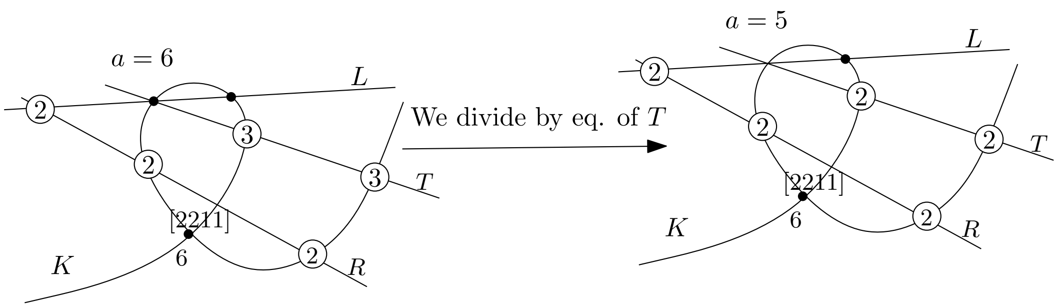
Case $Z_{35} = (8, 8, 8, 8, 8, 8, 8, 8, 8, 7, 4)$ $a = 24$ $r = 1$

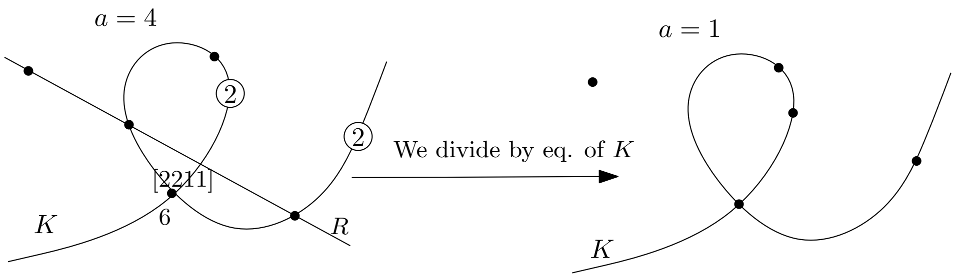




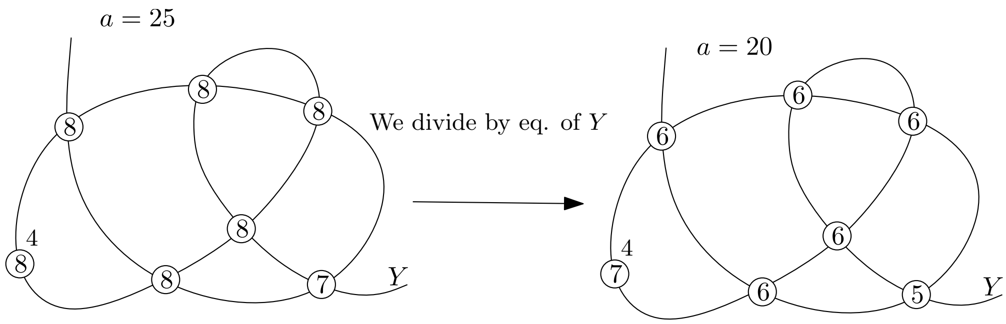

a = 11
R
Q
[4332211]
11
Specializing
a = 11
L
K
R
[4332211]
11

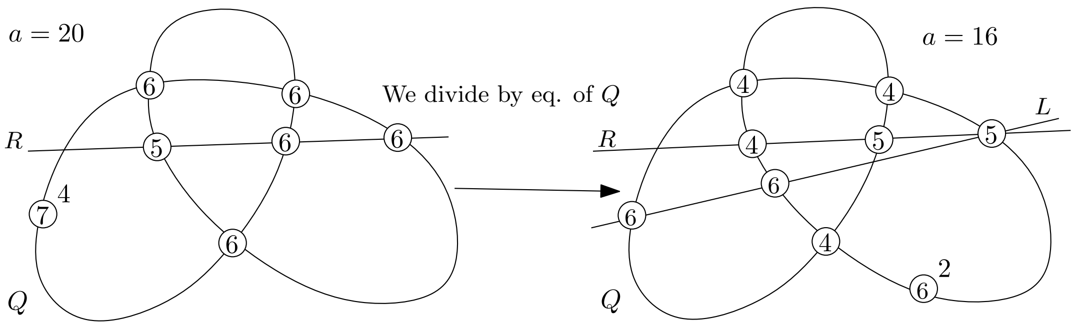

a = 8
L
K
R
[2211]
6
We divide by the eq. of R
a = 7
L
T
K
R
[2211]
6

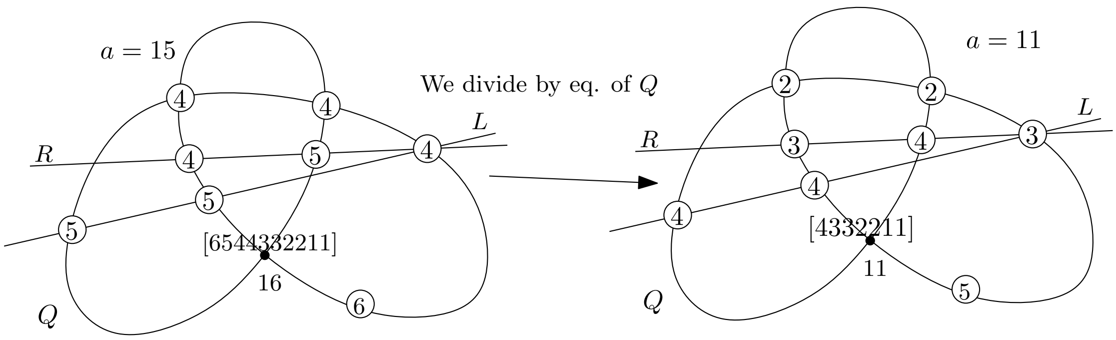

a = 6
L
T
K
R
[2211]
6
We divide by eq. of T
a = 5
L
T
K
R
[2211]
6

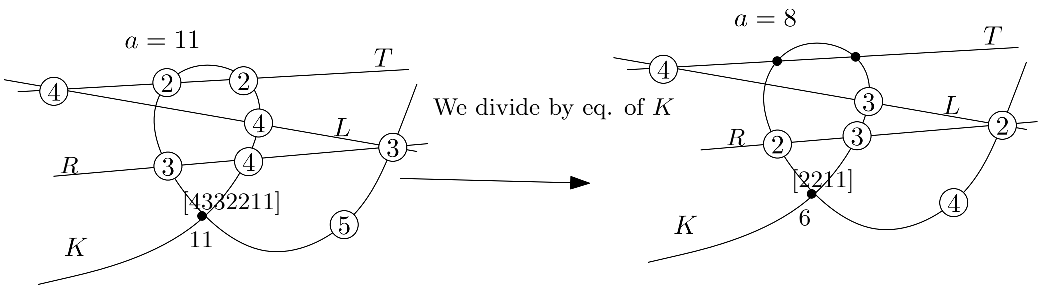


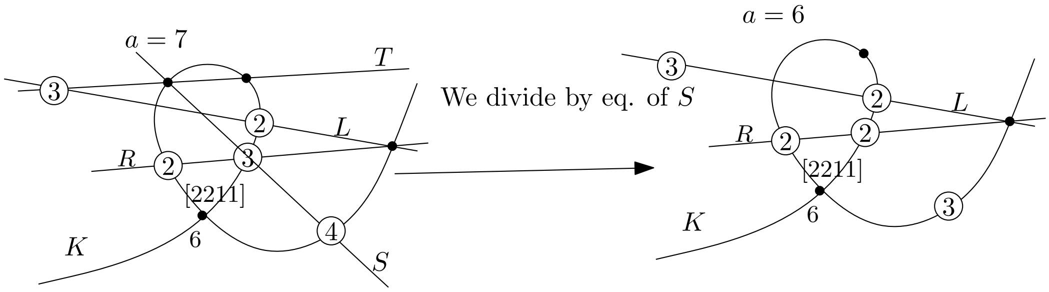
Case $Z_{36} = (8, 8, 8, 8, 8, 8, 8, 8, 8, 7)$ $a = 25$ $r = 1$

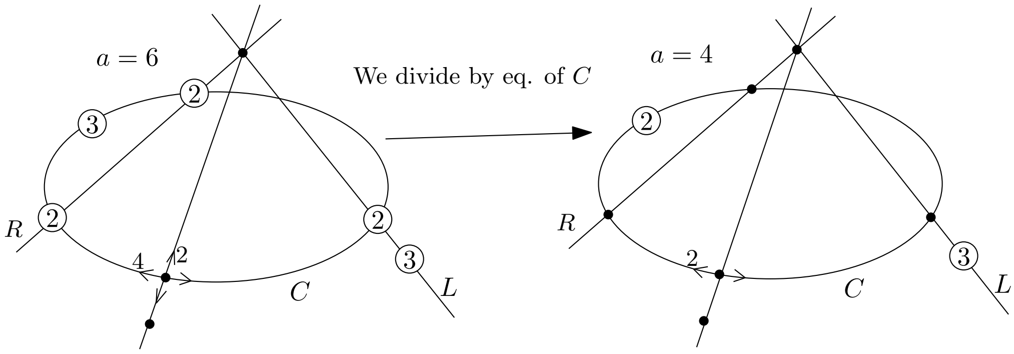


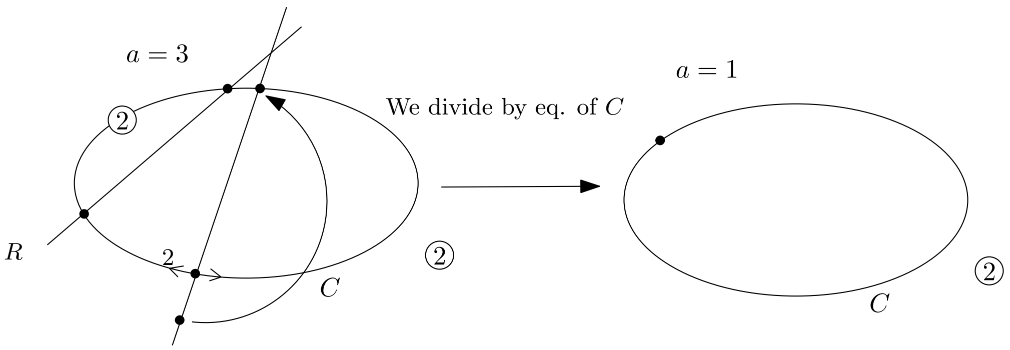



a = 15
We divide by eq. of Q
a = 11
R
L
Q
[6544332211]
16
[4332211]
11


a = 11
We divide by eq. of K
a = 8
T
L
R
K
[4332211]
11
[2211]
6


a = 7
We divide by eq. of S
a = 6
T
L
R
K
S
[2211]
6





For all case that we have just studied the method is true. We conclude summarizing our result in the following theorem

THEOREM 3.6.4 *Every subscheme of $\mathbb{P}^2$ in the form $Z = (b_1P_1 + ... + b_nP_n)$ with $n \leq 10$ and $4 \leq b_i \leq 8$ for all $i = 1, ..., n$, has maximum rank.*

Moreover we have just verified that the conjecture 3.2.7 holds for all linear system with exactly ten assigned point in base locus, with multiplicity minor than 9.

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