Equivalent birational embeddings

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Dedicated to my beloved Fabiano and to my little brother Edoardo

Abstract. Let X be a projective variety of dimension r. We want to understand when two birational embeddings of the same variety are equivalent up to a Cremona transformation of the projective space, in this case we say that they are Cremona equivalent. It is proven that two birational embeddings of X in $\mathbb{P}^n$ with $n \geq r+2$ are Cremona equivalent. To do this, it is produced a chain of Cremona transformations that modify the linear systems giving the two embeddings one into the other. This is done by looking at the two birational embeddings as different projections of a common embedding.

On the other hand, if $n = r+1$, there are birationally divisors that are not Cremona equivalent. The case of plane curves is studied in details. Let C be an irreducible and reduced plane curve of arbitrary genus. It is proven that the curve C is birational to either a line; either a curve $\widetilde{C}$, where the log pair $\left(\mathbb{P}^2, \frac{3}{d}\widetilde{C}\right)$ has canonical singularities, the log canonical divisor nef and Kodaira dimension $\overline{\kappa} = 0$; or a curve $\widetilde{C} \sim \alpha\, C_0 + \beta\, f \subset \mathbb{F}_a$, where the log pair $\left(\mathbb{F}_a, \frac{2}{\alpha}\widetilde{C}\right)$ has canonical singularities and terminal singularities in a neighborhood of the exceptional curve $C_0 \subset \mathbb{F}_a$, the log canonical divisor nef and Kodaira dimension $\overline{\kappa} \leq 1$.

Finally, it is used the theory of &–minimal models to understand whether a rational, irreducible and reduced curve is Cremona equivalent to a line.

Sunto. Sia X una varietá proiettiva di dimensione r. Si vuole capire quando due immersioni birazionali della stessa varietá sono equivalenti a meno di una trasformazione di Cremona dello spazio proiettivo, in tal caso esse si dicono Cremona equivalenti. Si dimostra che due immersioni birazionali di X in $\mathbb{P}^n$ con $n \geq r+2$ sono Cremona equivalenti. Per fare ciò, si produce una catena di trasformazioni di Cremona che modificano l'uno nell'altro i sistemi lineari associati alle due immersioni.

D'altra parte, se $n = r+1$, esistono divisori birazionali che non sono Cremona equivalenti. Il caso delle curve piane è studiato in dettaglio. Sia C una curva piana, irriducibile e ridotta di genere arbitrario. Si dimostra che la curva C è birazionale o ad una retta; o ad una curva $\widetilde{C}$, tale che la coppia logaritmica $\left(\mathbb{P}^2, \frac{3}{d}\widetilde{C}\right)$ ha singolarità canoniche, divisore log canonico nef e dimensione di Kodaira $\overline{\kappa} = 0$; oppure ad una curva $\widetilde{C} \sim \alpha\, C_0 + \beta\, f \subset \mathbb{F}_a$, tale che la coppia logaritmica $\left(\mathbb{F}_a, \frac{2}{\alpha}\widetilde{C}\right)$ ha singolarità canoniche e singolarità terminali in un intorno della curva eccezionale $C_0 \subset \mathbb{F}_a$, divisore log canonico nef e dimensione di Kodaira $\overline{\kappa} \leq 1$.

Infine, è stata utilizzata la teoria dei modelli &–minimali per capire quando una curva razionale, irriducibile e ridotta è Cremona equivalente ad una retta.

Contents

Introduction

Let us consider a rational projective variety $X \subset \mathbb{P}^n$ of dimension r. Then there exists a birational map $\phi : X \dashrightarrow \mathbb{P}^r$. The simplest embedding of $\mathbb{P}^r$, as a projective variety, is the linear one. It is quite natural to ask whether the map ϕ can be extended to a birational map $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ such that $\Phi(X)$ is linear. We shall say that in this case the variety X is *Cremona equivalent* to a linear space.

This extension property reminds us the Abhyankar-Moh property (AMP), [**AM**]. The latter asks about extensions of polynomial embeddings in $\mathbb{C}^n$ to automorphisms of $\mathbb{C}^n$.

The AMP has been studied extensively and seems granted for high codimension smooth varieties, for instance by Jelonek in [**Je, Je1, Je2**], where, in the case of complex field, a positive answer is given for varieties of codimension greater than the dimension.

The AMP can be extended to affine varieties over an infinite field, say k. Then one can ask whether two different embeddings of the same affine variety are equivalent up to an automorphism of k^n. Also in this context, in [**Sr**], Srinivas proved that the answer is positive for varieties with isolated singularities and roughly local dimension half the embedding dimension.

In this thesis, our aim is to solve a similar question in the context of birational geometry of projective varieties. We want to understand when two birational embeddings of the same variety are equivalent up to a Cremona transformation of the projective space, in this case we say that they are Cremona equivalent.

The birational nature of the problem suggests that singularities should not be a main issue. Indeed we are able to treat arbitrarily irreducible and reduced projective varieties. The main novelty comes from the range where the answer is positive. It is not difficult to give examples of rational hypersurfaces that are not Cremona equivalent to a hyperplane. In general birationally equivalent divisors are not Cremona equivalent. What is really surprising is that this is the only case. Namely, we prove that two birational embeddings in $\mathbb{P}^n$ of the same projective variety X of dimension r, over an algebraically closed field, are *Cremona equivalent* as long as $n \geq r+2$.

Here is a brief summary of the contents.

In Chapter 1, we give some preliminary notions, that we will frequently use.

The main features of birational geometry of surfaces in the framework of the Mori program are recalled. The Minimal Model Program (MMP) aims to show that given any n–dimensional complex projective variety X over an algebraically closed field k with $\mathrm{char}(k) = 0$, we have:

- if $\kappa(X, K_X) \geq 0$, then there exists a *minimal model*, that is a variety Y birational to X such that K_Y is nef;
- if $\kappa(X, K_X) = -\infty$, then there is a variety Y birational to X which admits a *Fano fibration*, that is a morphism $Y \to Z$ whose fibres F have ample anticanonical class $-K_F$.

We explicitly present the Log Minimal Model Program in dimension 2, which is the logarithmic generalization of the Mori program for a log pair (S, B), where S is a non–singular projective surface and B is a boundary divisor.

We then interpret the results of Dicks, [**Di**], about the study of &–minimal pairs (S, C) up to birational equivalence, where S is a projective non–singular surface and $C \subset S$ is a reduced curve of arbitrary genus. Dicks gives a classification of &–minimal models for a pair (S, C) and some conditions about the uniqueness of these &–minimal models. In general, a pair (S, C) may have no unique &–minimal model, for example if S is rational. The ambiguity in this case is given by the action of the Cremona group.

Finally, we illustrate some Cremona transformations, that is birational maps $T : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, for $n \geq 2$. In particular, we describe the standard Cremona transformation of $\mathbb{P}^2$, the cubo–cubic Cremona transformation of $\mathbb{P}^3$ and we give the construction of the $(2, n)$ Cremona transformation of $\mathbb{P}^n$.

In Chapter 2, we consider projective varieties $X \subset \mathbb{P}^n$ of dimension $r \leq n-2$ as a first step in the understanding of the problem.

In Section 1, we consider a rational, irreducible and reduced curve $C \subset \mathbb{P}^n$, for $n \geq 3$. We study whether there exists a birational map $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, $n \geq 3$, such that $\Phi(C)$ is a line.

We have already seen that this extension property reminds the Abhyankar-Moh property (AMP). The very first example solved positively was the line in $\mathbb{C}^2$, [**AM**]. This, translated in the dictionary of projective geometry, says that a rational plane curve with a unique singularity is *Cremona equivalent to a line*. Note that an arbitrary rational curve in $\mathbb{P}^2$ is not *Cremona equivalent to a line*. The simplest example is a general projection of $\Gamma \sim (1,a) \subset \mathbb{Q}^2 \subset \mathbb{P}^3$, for $a \geq 5$.

Our results is that this is the unique negative answer.

Theorem 2.1.8. *Let $n \geq 3$ and $C \subset \mathbb{P}^n$ be an irreducible and reduced rational curve. Then C is Cremona equivalent to a line.*

Instead of trying to extend the birational map ϕ we deal with the problem by a series of birational maps that get C closer and closer to a line. First we provide a *birational* resolution of the singularities of C. Namely a birational map $\Omega : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ that resolves the singularities of C. Here we consider sufficently general birational maps all of whose fibers are curves. In this way it is harmless to control the image of a curve and resolve the singularities. Next we single out smooth curves *Cremona equivalent to a line* in any degree. This is accomplished with the same birational maps used in the resolution. Finally we produce a birational map $\Theta : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ between two smooth rational curves of the same degree. The main point here is to recognize a rational curve of degree d as a sublinear system of $\mathcal{O}_{\mathbb{P}^1}(d)$. Then the required map can be interpreted as a way to pass from a subsystem to the other in a finite number of steps.

In Section 2, we study the equivalent birational embeddings of a fixed variety. Let X be a projective irreducible and reduced variety and $\mathcal{L}$ a linear system on X. We say that $\mathcal{L}$ is a birational embedding (in $\mathbb{P}^n$) if $\varphi_{\mathcal{L}} : X \dashrightarrow \mathbb{P}^n$ is birational onto the image.

If $\mathcal{L}$ and $\mathcal{G}$ are two birational embeddings in $\mathbb{P}^n$ of the same variety X, we say that $\mathcal{L}$ is *Cremona equivalent* to $\mathcal{G}$ if there exists a birational map $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ such that $\varphi_{\mathcal{L}} = \Phi \circ \varphi_{\mathcal{G}'}$. We prove the following:

Theorem 2.2.7. *Let X be an irreducible and reduced projective variety of dimension r over an algebraically closed field k. Then two birational embeddings in $\mathbb{P}^n$ are Cremona equivalent as long as $n \geq r+2$.*

To prove the theorem we apply the same strategy used for curves, i.e. we produce a chain of Cremona transformations that modify the linear systems giving the two embeddings one into the other. This is done by looking at the two birational embeddings as different projections of a common embedding.

In Chapter 3, we study the divisorial case. We consider X an irreducible and reduced projective variety of dimension $r \leq 2$ over an algebraically closed field k, with $\mathrm{char}(k) = 0$. We have that two divisorial embeddings are in general not Cremona equivalent. Namely, we prove the following:

Theorem 3.0.2. *Let X be an irreducible and reduced projective variety with $\dim X \leq 2$. Then, there exist infinitely many Cremona inequivalent embeddings of X in $\mathbb{P}^{\dim X+1}$.*

In Section 1, we shall study log pairs $(\mathbb{P}^2, \alpha\, C)$, where C is an irreducible and reduced curve of arbitrary genus. In particular, we shall apply birational transformations to obtain a log pair $(S, \alpha\, \widetilde{C})$, which is a model of $(\mathbb{P}^2, \alpha\, C)$ with canonical singularities, having the log canonical divisor $K_S + \alpha\, \widetilde{C}$ nef and Kodaira dimension $\overline{\kappa}(S, \alpha\, \widetilde{C}) \leq 1$. In this way, we can give a classification of such pairs in terms of a birational equivalence between $(\mathbb{P}^2, \alpha\, C)$ and $(S, \alpha\, \widetilde{C})$. We get the following:

Theorem 3.1.4. *An irreducible and reduced curve $C \subset \mathbb{P}^2$ is birational to one of the following:*

a) *a line;*
b) *a curve $\widetilde{C}$, where the log pair $\big(\mathbb{P}^2, \frac{3}{d}\widetilde{C}\big)$ is a model with canonical singularities, having the log canonical divisor $K_{\mathbb{P}^2} + \frac{3}{d}\widetilde{C} \sim \mathcal{O}$ nef and $\overline{\kappa}\big(\mathbb{P}^2, \frac{3}{d}\widetilde{C}\big) = 0$;*
c) *a curve $\widetilde{C} \subset \mathbb{F}_a$, with $\widetilde{C} \sim \alpha\, C_0 + \beta\, f$, where the log pair $\big(\mathbb{F}_a, \frac{2}{\alpha}\widetilde{C}\big)$ is a model with canonical singularities and terminal singularities in a neighborhood of the exceptional curve $C_0 \subset \mathbb{F}_a$, having the log canonical divisor $K_{\mathbb{F}_a} + \frac{2}{\alpha}\widetilde{C}$ nef and $\overline{\kappa}\big(\mathbb{F}_a, \frac{2}{\alpha}\widetilde{C}\big) \leq 1$.*

In Section 2, we use the results of Dicks to understand whether a rational, irreducible and reduced curve $C \subset \mathbb{P}^2$ is *Cremona equivalent to a line*. We give a new proof of the following result, which is been already shown by Kumar–Murthy in [**KuMu**].

Proposition 3.2.11. *Let $\nu : S \to \mathbb{P}^2$ be a minimal resolution of singularities of the rational curve C and let $C_S = \nu_*^{-1}(C)$ be the strict transform of C. If $C_S^2 \geq -3$, then C is Cremona equivalent to a line.*

The viceversa of the above proposition does not hold. Namely, there exist rational curve C, which is *Cremona equivalent to a line* and having C_S^2 arbitrarily negative.

In Section 3, we give other results about rational irreducible and reduced curve $C \subset \mathbb{P}^2$ in terms of kind and number of singular points and &–minimal model.

Let $C \subset \mathbb{P}^2$ be a rational, irreducible and reduced curve of degree d. Let $m_1 \geq m_2 \geq m_3 \geq \ldots \geq m_k$ be the multiplicity of the singular points of C. We list some particular results:

- Any rational, irreducible and reduced curve $C \subset \mathbb{P}^2$ of degree $d \leq 5$ is *Cremona equivalent to a line*, see Remark 3.2.14.
- If C is a rational curve of degree $d \geq 6$ such that $m_i \leq \frac{d}{3}$, for any $i = 1, \ldots, k$, then C is not Cremona equivalent to a line, see Remarks 3.2.7, 3.3.1.
- If C is a rational curve of degree $d \geq 6$ such that $m_1+m_2+m_3 \leq d$, then C is not Cremona equivalent to a line, see Lemma 3.3.3.
- If C is a rational curve of degree $d \geq 6$ having a singular point of multiplicity $m_1 \geq d-3$, then C is *Cremona equivalent to a line*, see Lemma 3.3.8.
- Let C be a rational curve of degree $d \geq 6$ such that the pair $\left(\mathbb{P}^2, \frac{3}{d}C\right)$ has ordinary canonical singularities.

  Let $\nu : S \to \mathbb{P}^2$ be a minimal resolution of singularities of C and let $C_S = \nu_*^{-1}(C)$ be the strict transform of C. Then, the pair (S, C_S) is &–minimal. Moreover, the pair (S, C_S) is the unique &–minimal model of $(\mathbb{P}^2, C)$, see Lemma 3.3.9.

In Section 4, we give a table containing models of rational plane curves of degree $6 \leq d \leq 10$ with only ordinary singular points, according to Theorem 3.1.4.

Finally, in Section 5, we consider a rational, irreducible and reduced surface $S \subset \mathbb{P}^3$ of low degree and we ask whether whether there exists a birational map $\Psi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ such that $\Psi(S)$ is a plane. We have the following result.

Theorem 3.5.11. *Let $S \subset \mathbb{P}^3$ be a rational, irreducible and reduced cubic surface. Then, there exists a birational map $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ such that $\phi(S) = H$, where H is a plane.*

CHAPTER 1

Preliminaries

1. Minimal Model Program in dimension 2

Let S be a nonsingular projective surface over an algebraically closed field k. Assume that $\mathrm{char}(k) = 0$. We have to introduce some basic objects on S.

Let $\mathrm{Div}(S)$ be the group of Cartier divisors on S and $\mathrm{Pic}(S)$ be the group of line bundles on S. Let $Z^1(S)$ be the group of Weil divisors and let $Z_1(S)$ be the group of 1–cycles on S, i.e. the free abelian group generated, respectively, by prime divisors and reduced irreducible curves. Finally, let $\mathrm{Div}(S) \otimes \mathbb{Q}$ be the group of $\mathbb{Q}$–Cartier divisors on S, that are linear combinations with rational coefficients of Cartier divisors.

There is a pairing

$$\mathrm{Pic}(S) \times Z_1(S) \to \mathbb{Z}$$

defined, for an irreducible reduced curve $C \subset S$, by $(L, C) \to L \cdot C := \deg_C(L_{|C})$, and extended by linearity.

Two invertible sheaves $L_1, L_2 \in \mathrm{Pic}(S)$ are *numerically equivalent*, denoted by $L_1 \equiv L_2$, if $L_1 \cdot C = L_2 \cdot C$, for every curve $C \subset S$. Two 1–cycles C_1, C_2 are *numerically equivalent*, denoted by $C_1 \equiv C_2$, if $C_1 \cdot L = C_2 \cdot L$, for every $L \in \mathrm{Pic}(S)$. Define

$$N^1(S) := \{\mathrm{Pic}(S)/\equiv\} \otimes_{\mathbb{Z}} \mathbb{R}; \qquad N_1(S) := \{Z_1(S)/\equiv\} \otimes_{\mathbb{Z}} \mathbb{R}.$$

By definition, $N^1(S)$ and $N_1(S)$ are dual $\mathbb{R}$–vector spaces and $\equiv$ is the smallest equivalence relation for which this holds.

For any divisor $H \in \mathrm{Pic}(S)$, we can view the class of H in $N^1(S)$ as a linear form on $N_1(S)$. We will use the following notations:

$$H_{\geq 0} := \{x \in N_1(S) \mid H \cdot x \geq 0\}; \qquad H^{\perp} := \{x \in N_1(S) \mid H \cdot x = 0\}.$$

Similarly, we can define $H_{>0}$, $H_{\leq 0}$ and $H_{<0}$.

Since S is a surface, we have that $N^1(S) = N_1(S)$, i.e. the algebraic equivalence of 1–cycles and the numerical equivalence coincide, see [**GH**, p. 163].

Let $\rho(S) = \dim_{\mathbb{R}} N_1(S) = \dim_{\mathbb{R}} N^1(S)$ the Picard number of S. By Néron–Severi theorem [**GH**, p. 461], $\rho(S)$ is a finite number.

We denote by $\mathrm{NE}(S) \subset N_1(S)$ the convex cone generated by effective 1–cycles, i.e.

$$\mathrm{NE}(S) = \{C \in N_1(S) \mid C = \sum r_i C_i,\ \text{with } r_i \in \mathbb{R},\ r_i \geq 0\},$$

where C_i are irreducible curves. Moreover, we denote by $\overline{\mathrm{NE}(S)}$ the closure of $\mathrm{NE}(S)$ in the real topology of $N_1(S)$. $\overline{\mathrm{NE}(S)}$ is called the *Kleiman–Mori cone*. For any $H \in N^1(S)$, we use the notation $\overline{\mathrm{NE}(S)}_{H\geq 0} := \overline{\mathrm{NE}(S)} \cap H_{\geq 0}$. Similarly, we can define $\overline{\mathrm{NE}(S)}_{H>0}$, $\overline{\mathrm{NE}(S)}_{H\leq 0}$ and $\overline{\mathrm{NE}(S)}_{H<0}$.

We have the following:

Definition 1.1.1. A line bundle $L \in N^1(S)$ on S is *numerically eventually free* or *numerically effective*, for short *nef*, if

$$L \cdot C \geq 0, \text{ for any curve } C \subset S,$$

which is equivalent to saying

$$L \geq 0 \text{ on } \mathrm{NE}(S) \quad \text{or} \quad L \geq 0 \text{ on } \overline{\mathrm{NE}(S)}.$$

There is a criterion for ampleness that play an important role in the Mori's theory:

Theorem 1.1.2 (**Kleiman's Criterion for Ampleness**, [**Kl**])**.** *A line bundle L on S is ample if and only if*

$$L > 0 \text{ on } \overline{\mathrm{NE}(S)}.$$

Remark 1.1.3. In the criterion above it is crucial to consider the closure $\overline{\mathrm{NE}(S)}$, since, in general, $L > 0$ on $\mathrm{NE}(S)$ does not imply the ampleness of L.

Denote by K_S the *canonical divisor* of the surface S, that is $K_S \in \mathrm{Div}(X)$ such that $\mathcal{O}_S(K_S) = \Omega^2_S$, where Ω_S is the sheaf of one–forms on S.

Definition 1.1.4. An irreducible and reduced curve E on a nonsingular projective surface S is called a (-1)–curve if $E \cong \mathbb{P}^1$ and $E^2 = -1$. Moreover, an irreducible and reduced curve E on a nonsingular projective surface S is a (-1)–curve if and only if $K_S \cdot E < 0$ and $E^2 < 0$.

Now, we present the main features of birational geometry of surfaces in the framework of the Mori program. We follow the approach of Matsuki [**Ma, BM**].

From the view of the Minimal Model Program (MMP), the so called Mori's Program, our strategy to understand the birational geometry of algebraic surfaces is divided into the following three steps:

MP1. Find a *good representative* in a fixed birational equivalence class.
MP2. Study the properties of the *good representative*.
MP3. Study the birational relation among possibly many choices of the *good representatives*.

We describe the main results from which we obtain the description of the minimal model program in dimension 2:

Theorem 1.1.5 (Castelnuovo's Contractibility Criterion). *Let S be a nonsingular projective surface, $E \subset S$ a (-1)–curve. Then there exists a morphism called the* contraction of E, *$\mu : S \to T$, onto another projective surface T such that*

i) *$\mu(E) = p$, where $p \in T$ is a point, and $\mu : S \setminus E \to T \setminus \{p\}$ is an isomorphism;*
ii) *T is nonsingular.*

In fact, $\mu : S \to T$ is the blow up of T in p.

This theorem leads to a *crude form* of the minimal model program for algebraic surfaces: in dimension 2, after finding a nonsingular projective surface in a given birational equivalence class via resolution of singularities, we can create another in the same birational equivalence class through the operation of blowing up a point, introducing a (-1)–curve. Thus, in search of a good representative suggested in MP1, one has to blow the (-1)–curve down via Castelnuovo's contractibility criterion.

The key criterion for the crude form of MMP is: *is there a (-1)–curve*? This criterion does not provide much global information for an end result of the program, since we know only that it does not have any (-1)–curve. One of Mori's brilliant ideas is to replace this criterion with the one dictated by the canonical divisor K_S of the surface S: *is K_S nef*?

The following theorem states that when K_S is not nef, there is an *extremal contraction*:

Theorem 1.1.6 (**Extremal Contraction**). *If K_S is not nef, then there exists a morphism $\phi : S \to W$ called an* extremal contraction *such that*

i) *ϕ is not an isomorphism;*
ii) *for a curve $C \subset S$*

$$\phi(C) = p \;\Rightarrow\; K_S \cdot C < 0;$$

iii) *all the curves contracted by ϕ are numerically proportional, i.e.*

$$\phi(C) = p \text{ and } \phi(D) = p \;\Rightarrow\; [D] = c[C] \text{ in } H_2(S,\mathbb{R}) \text{ for some } c \in \mathbb{Q}_{>0};$$

iv) *ϕ has connected fibers with W normal and projective.*

The notion of an extremal contraction turns out not only to generalize Castelnuovo's contractibility criterion, but also to provide decisive information on the global structures of the end results of the program.

Moreover, we can give a characterization of an extremal contraction in terms of the geometry of the convex cone generated by effective 1–cycles in the space of the numerical classes of curves: an extremal contraction corresponds to an *extremal ray* of the cone.

What is missing is the characterization of the half–line called an *extremal ray* $R_l = \mathbb{R}_+[C] \subset \overline{\mathrm{NE}(S)} \subset N_1(S)$, which contains the numerical classes of all the curves C contracted by an extremal contraction $\phi : S \to W$. The **Cone Theorem** and the **Contraction Theorem** give the desired characterization. The Cone Theorem describes the extremal rays purely in terms of convex geometry of the cone $\overline{\mathrm{NE}(S)}$ in regard to the intersection with the canonical bundle K_S, while the Contraction Theorem guarantees that each of those extremal rays described in the Cone Theorem is associated to a geometric extremal contraction.

Theorem 1.1.7 (**Cone Theorem in dimension** 2). *Let S be a nonsingular projective surface. Then the closure of the cone of effective curves has the description*

$$\overline{NE(S)} = \overline{NE(S)}_{K_S \geq 0} + \sum R_l,$$

where

$$\overline{NE(S)}_{K_S \geq 0} := \left\{ z \in \overline{NE(S)} \;:\; K_S \cdot z \geq 0 \right\}$$

and the R_l are half–lines such that $R_l \setminus \{0\}$ are in $\overline{NE(S)}_{K_S<0}$ and such that they are of the form

$$R_l = \overline{NE(S)} \cap L^{\perp}$$

for some nef line bundles L.

Moreover, the R_l are discrete in the half–space $N_1(S)_{K_S<0}$.

Theorem 1.1.8 (Contraction Theorem in dimension 2). *For each extremal ray R_l in the half–space $N_1(S)_{K_S<0}$, there exists an extremal contraction $\phi = cont_{R_l} : S \to W$, called the* contraction of an extremal ray *R_l, such that*

$$\phi(C) = p \quad \Leftrightarrow \quad [C] \in R_l \;\; \textit{for any curve } C \subset S.$$

Conversely, any extremal contraction as characterized in Theorem 1.1.6 *is the contraction of an extremal ray.*

The Cone Theorem and the Contraction Theorem finally establish the complete form of the MMP in dimension 2, which provides a good representative required in MP1 in the form of a **minimal model** or a **Mori fiber space** as an end result of the program. The Mori's Program for S produces either a minimal model or a Mori fiber space as an output, depending on whether the canonical divisor K_S is nef or not. This is called the "Easy Dichotomy". We have the following:

Theorem 1.1.9 (Easy Dichotomy Theorem of MMP in dimension 2). *Let S be a nonsingular projective surface. Then an end result of MMP starting from S is a Mori fiber space if and only if there exists a nonempty Zariski open set $U \subset S$ such that for any point $p \in U$ there is an irreducible curve D passing through p with $K_S \cdot D < 0$.*

According to MP2, we now try to study the properties of these end results.

Definition 1.1.10 (Mori Fiber Space in dimension 2). A nonsingular projective surface S with a morphism $\phi : S \to W$ is a **Mori fiber space** if it is an extremal contraction $\phi : S \to W$ with $\dim W < \dim S$, or equivalently if

i) ϕ is a morphism with connected fibers onto a normal projective variety W of $\dim W < \dim S$ (i.e. in dimension 2, W is either a nonsingular projective curve or a point), and
ii) all the curves C in fibers of ϕ are numerically proportional with $K_S \cdot C < 0$.

The following feature of Mori fiber spaces turns out to give an important birational characterization:

Proposition 1.1.11. *Let $\phi : S \to W$ be a Mori fiber space in dimension 2. Then*

$$H^0(S, \mathcal{O}_S(mK_S)) = 0 \quad \forall\, m \in \mathbb{N}.$$

That is to say, $\kappa(S) = -\infty$.

The following is the main structure theorem for Mori fiber spaces in dimension 2:

Theorem 1.1.12 (Characterization of Mori Fiber Spaces in dimension 2). *Let $\phi : S \to W$ be a Mori fiber space in dimension 2. Then one of the following two cases occurs:*

1) $\dim W = 1$. *In this case, every fiber of ϕ is isomorphic to $\mathbb{P}^1$, i.e.*

$$\phi^{-1}(p) \cong \mathbb{P}^1 \quad \forall\, p \in W.$$

More precisely, $\phi : S \to W$ is a $\mathbb{P}^1$–bundle over a nonsingular projective curve W in the algebraic category, i.e. for each point $p \in W$ there exists a Zariski open neighborhood $p \in U \subset W$ such that we have the following commutative diagram:

$$\begin{tikzcd} \phi^{-1}(U) \arrow[r, "\sim"] \arrow[d, "\phi"'] & \mathbb{P}^1 \times U \arrow[d, "p_2"] \\ p \in U \arrow[r, "\sim"] & U \end{tikzcd}$$

2) $\dim W = 0$. *In this case, $S \cong \mathbb{P}^2$ and $\phi : S \cong \mathbb{P}^2 \to W \cong p$, where p is a point, is the obvious morphism.*

We can rephrase the last theorem as the Classification Theorem for extremal rays in dimension 2:

Theorem 1.1.13 (Classification of Extremal Rays in dimension 2). *Let S be a nonsingular projective surface whose canonical bundle K_S is not nef. Then an extremal ray R_l with $R_l \setminus \{0\}$ in the half–space $N_1(S)_{K_S<0}$ is one of the following:*

i) *$R_l = \mathbb{R}_+[l]$, where l is a (-1)–curve with $K_S \cdot l = -1$, and $\mathrm{cont}_{R_l} : S \to W$ is the contraction of the (-1)–curve l;*
ii) *$R_l = \mathbb{R}_+[l]$, where l is a fiber with $K_S \cdot l = -2$ of the algebraic $\mathbb{P}^1$–bundle $\mathrm{cont}_{R_l} : S \to W$ over a nonsingular projective curve W, or*

iii) $R_l = \mathbb{R}_+[l]$, *where* l *is a line in* $S \cong \mathbb{P}^2$ *with* $K_S \cdot l = -3$ *and* $cont_{R_l} : S \to W$ *is the structure morphism from* $\mathbb{P}^2$ *to* $W = Spec\, \mathbb{C}$.

Conversely, any curve as described in i), ii) *or* iii) *spans an extremal ray* $R_l = \mathbb{R}_+[l]$ *with respect to* K_S*, and it has the minimum intersection with* $-K_S$ *among the curves in its numerical class.*

Now, we study minimal models in dimension 2, the other end result of MMP in dimension 2:

Definition 1.1.14 (**Minimal Model in dimension** 2)**.** A nonsingular projective surface S is a **minimal model** if and only if the canonical bundle K_S is nef.

We give the definition of the Kodaira dimension, a fundamental birational invariant, which plays a crucial role in our analysis:

Definition 1.1.15 (**Kodaira Dimension**)**.** Let S be a nonsingular projective variety. The Kodaira dimension $\kappa(S)$ of S is defined to be

$$\kappa(S) = \kappa(S, K_S) := -\infty \qquad \text{if } \ H^0(S, \mathcal{O}_S(mK_S)) = 0 \ \ \forall\ m \in \mathbb{N},$$

$$\kappa(S) = \kappa(S, K_S) := \text{transdeg.}_{\mathbb{C}} \oplus_{m \geq 0} H^0(S, \mathcal{O}_S(mK_S)) - 1$$
$$\text{if } \ H^0(S, \mathcal{O}_S(mK_S)) \neq 0 \ \text{ for some } m \in \mathbb{N}.$$

$\kappa(S, D)$ can be defined similarly for any divisor D on S and for a singular variety T the Kodaira dimension is defined to be that of its desingularization $\kappa(T) = \kappa(S)$, where $\nu : S \to T$ is a birational morphism from a nonsingular projective variety S.

The most important propriety of a minimal model in dimension 2 is the following:

Theorem 1.1.16 (**Existence of an Effective Pluricanonical Divisor for a Minimal Model in dimension** 2)**.** *Let* S *be a minimal model in dimension* 2*. Then* $\kappa(S) \geq 0$*, i.e.* $H^0(S, \mathcal{O}_S(mK_S)) \neq 0$*, for some* $m \in \mathbb{N}$*.*

The next result is the "Hard Dichotomy" Theorem, that gives the characterization of minimal models in terms of the Kodaira dimension, in contrast to the Easy Dichotomy Theorem.

Theorem 1.1.17 (**Hard Dichotomy Theorem of MMP in dimension** 2)**.** *Let* S *be a nonsingular projective surface. Then an end* *result of MMP in dimension 2 starting from S is a minimal model (respectively a Mori fiber space) if and only if $\kappa(S) \geq 0$ (respectively $\kappa(S) = -\infty$).*

Theorem 1.1.18 (**Abundance Theorem in dimension** 2)**.** *Let S be a minimal model in dimension 2. Then the pluricanonical linear system $|mK_S|$ is base point free for sufficiently divisible and large $m \in \mathbb{N}$.*

The Abundance Theorem paves the way to make a fusion between the minimal model program and the Iitaka fibration $\Phi_{|mK_S|} : S \to S_{\text{can}}$, where S_{can} is the *canonical model* of S. Moreover, we have that $S_{\text{can}} = \text{Proj } R$, where $R := \oplus_{m \geq 0} H^0(S, \mathcal{O}_S(mK_S))$.

According to MP3, we now try to study the birational relation between surfaces and we try to discuss the features of the birational relation among the good representatives, namely minimal models when $\kappa \geq 0$ and Mori fiber spaces when $\kappa = -\infty$.

First, we factor any given birational map into a composite of blow ups and blow downs:

Proposition 1.1.19 (**Factorization of Birational Maps in dimension** 2)**.** *Let $\phi : S_1 \dashrightarrow S_2$ be a birational map between two nonsingular projective surfaces. Then there exists a sequence of blow ups $\psi_1 : V \to S_1$ followed by a sequence of blow downs $\psi_2 : V \to S_2$ that factorizes ϕ.*

Uniqueness of the minimal model in a fixed birational equivalence class is the special feature of the birational relation among the good representatives with $\kappa \geq 0$ in dimension 2:

Theorem 1.1.20 (**Uniqueness of the Minimal Model in dimension** 2)**.** *There exists an unique minimal model in a fixed birational equivalence class in dimension 2. More strongly, if $\phi : S_1 \dashrightarrow S_2$ is a birational map between two minimal models in dimension 2, then ϕ is an isomorphism.*

The **Castelnuovo–Noether theorem** decomposing any birational map between two birational Mori fiber spaces into a composite of elementary transformations called "**links**" is the special feature of the birational relation among the good representatives with $\kappa = -\infty$ in dimension 2. We describe the classical Castelnuovo–Noether theorem in the framework of the **Sarkisov program**, which has been developed to give factorization of birational maps among higher–dimensional Mori fiber spaces.

Theorem 1.1.21 (**Castelnuovo–Noether Theorem = Sarkisov Program in dimension** 2)**.** *Let $\Phi : S \dashrightarrow S'$ be a birational map between two Mori fiber spaces $\phi : S \to W$, $\phi' : S' \to W'$ in dimension 2. Then there exists an algorithm, called* ***Sarkisov Program*** *in dimension 2, to decompose Φ into a composite of the following four of "**links**" (elementary transformations):*

1) *Type I:*

$$\begin{tikzcd}
& \mathbb{F}_1 \arrow[dl] \arrow[dd] \\
\mathbb{P}^2 \arrow[d] & \\
pt. & \mathbb{P}^1 \arrow[l]
\end{tikzcd}$$

2) *Type II:*

$$\begin{tikzcd}
& Z \arrow[dl] \arrow[dr] & \\
S \arrow[d] & & S_1 \arrow[d] \\
W & & W_1 \arrow[ll, "\sim"']
\end{tikzcd}$$

where $S \to W$ is a $\mathbb{P}^1$–bundle over a nonsingular projective curve W; $Z \to S$ is a blow up of a point in one ruling; $Z \to S_1$ is the contraction of the strict transform of that ruling to obtain another $\mathbb{P}^1$–bundle $S_1 \to W_1 = W$.

3) *Type III, the inverse of Type I:*

$$\begin{tikzcd}
\mathbb{F}_1 \arrow[dr] \arrow[dd] & \\
& \mathbb{P}^2 \arrow[d] \\
\mathbb{P}^1 \arrow[r] & pt.
\end{tikzcd}$$

4) *Type IV:*

$$\begin{tikzcd} \mathbb{P}^1_1 \times \mathbb{P}^1_2 \arrow[d, "p_1"] & = & \mathbb{P}^1_1 \times \mathbb{P}^1_2 \arrow[d, "p_2"'] \\ \mathbb{P}^1_1 \arrow[dr] & & \mathbb{P}^1_2 \arrow[dl] \\ & pt. & \end{tikzcd}$$

1.1. Mori Program for algebraic surfaces.

We start to describe the first step **MP1** of the Mori's Program in dimension 2.
We take a nonsingular projective surface S by a resolution of singularities with given function field k algebraically closed. We keep contracting (-1)–curves until we get either a surface $S_{\min}$ with the canonical divisor $K_{S_{\min}}$ being nef or a ruled surface S_{mori} over a curve (or $\mathbb{P}^2$ over a point). Hence, the Mori's Program for S produces either a **minimal model** or a **Mori fiber space** as an output, depending on whether the canonical divisor K_S is nef or not.

Indeed, if K_S is nef, the MMP for S produces a **minimal model**, while if K_S is not nef, the Cone Theorem and the Contraction Theorem state that there exists $\phi : S \to W$ the contraction of an extremal ray. At this point, we have two cases: if $\dim W = \dim S$, then the extremal contraction $\phi : S \to W$ is the contraction of a (-1)–curve. We replace S with W and we repeat the Mori's program for W. While if $\dim W < \dim S$, then the extremal contraction $\phi : S \to W$ is by definition a **Mori fiber space**.

Now, we describe the second step **MP2** of the Mori's Program in dimension 2.
The most important property of good representatives is the Hard Dichotomy, which asserts that the Kodaira dimension controls the outcome of MMP: the end result of MMP is either a Mori fiber space or a minimal model depending on whether $\kappa = -\infty$ or $\kappa \geq 0$.

For a surface S with $\kappa(S) \geq 0$, the Abundance theorem claims that a minimal model $S_{\min}$ has a base point free pluri–canonical system, which induces the canonical morphism $\Phi_{|mK_{S_{\min}}|} : S_{\min} \to S_{\text{can}}$, crucial for the understanding of the global structure of S and its moduli. This canonical morphism $\Phi_{|mK_{S_{\min}}|}$ is:

- a birational map to a canonically polarized surface with only rational double points, when $\kappa = 2$;
- an elliptic fibration, when $\kappa = 1$;
- a trivial map to a point, where we know $S_{\min}$ must be either Abelian, bielliptic, K3 or Enriques, when $\kappa = 0$.

For a surface S with $\kappa(S) = -\infty$, the structure of a Mori fiber space is rigid and well understood: either a $\mathbb{P}^1$–bundle over a nonsingular curve or $\mathbb{P}^2$ over a point. They are of course covered by rational curves.

Finally, we describe the third step **MP3** of the Mori's Program in dimension 2.
A minimal model is unique in a fixed birational equivalence class for surfaces with $\kappa \geq 0$, while any birational map among ruled surfaces in a given birational equivalence class is decomposed into a sequence of elementary transformations by Castelnuovo–Noether theorem (i.e. Sarkisov program in dimension 2).

2. Log Minimal Model Program in dimension 2

In this section we discuss the **log birational geometry of surfaces** through the **Log Minimal Model Program** (Log MMP for short) **in dimension** 2.

Definition 1.2.1 (**Logarithmic Pair**). A **logarithmic pair** (also called a **log pair** for short) (S, B) is a pair consisting of a projective surface S and a boundary $\mathbb{Q}$–divisor $B = \sum b_i B_i$ with rational coefficients $0 \leq b_i \leq 1$.

Definition 1.2.2 (**Logarithmic Canonical Bundle**). Let S be a nonsingular projective surface and B a boundary divisor with only normal crossing. The sheaf of logarithmic 2–forms

$$\Omega^2_X(\log B) \cong \mathcal{O}_S(K_S + B)$$

is a line bundle called the **logarithmic canonical bundle** for the log pair (S, B). We call $K_S + B$ the **logarithmic canonical divisor** (or **log canonical divisor** for short) of the log pair (S, B).

Working with the logarithmic category has the following three great advantages:

i) **Iitaka's Philosophy**: it allows us to deal with open surfaces U by considering **logarithmic pairs** (S, B) (called **log pairs** for short), consisting of the compactifications S of U and the boundaries $B = S \setminus U$, which we choose to be of pure codimension one and hence called the **boundary divisor**. Iitaka's philosophy states that a theory or theorem about complete surfaces S dictated by the canonical divisor K_S should find its counterpart in the logarithmic category as a theory or theorem about log pairs (S, B) dictated by the **log canonical divisor** $K_S + B$ and viceversa.

ii) **Generalized Adjunction**: the log canonical divisor $K_S + B$ of a log pair (S, B) naturally restricts to the log canonical divisor on the boundary through the generalized adjunction formula

$$K_S + B_{|B} = K_B + \mathrm{Diff}_S B.$$

It may give an inductional structure to the scheme of arguments when one tries to prove a property of the log canonical divisor $K_S + B$, if the property propagates through the adjunction to the log canonical divisor of the boundary $K_B + \mathrm{Diff}_S B$.

In the presence of singularities on S, the usual adjunction formula $K_S + B_{|B} = K_B$ needs a correction term $\mathrm{Diff}_S B$ called **different** to hold as above, see [**Co**]. Thus the logarithmic category provides a more natural stage for the generalized adjunction to work.

iii) **Logarithmic Ramification Formula**: as we have the ramification formula for a morphism between (smooth) surfaces $f : S' \to S$

$$K_{S'} = f^* K_S + R,$$

we have a similar ramification formula called the *logarithmic ramification formula* for a morphism between logarithmic pairs (smooth with only normal crossings) $f : (S', B_{S'}) \to (S, B_S)$

$$K_{S'} + B_{S'} = f^*(K_S + B_S) + R_{\log}.$$

The logarithmic ramification formula behaves more naturally than the usual one under some circumstances.

Finally, we have the following definitions about singularities of a logarithmic pair, see [**KoMo**]:

Definition 1.2.3. Let (X, B) be a pair, where X is a normal variety and $B = \sum b_i B_i$ is a sum of distinct prime divisors with rational coefficients $0 \leq b_i \leq 1$. Assume that the log canonical divisor $K_X + B$ is $\mathbb{Q}$–Cartier, i.e. there exists $m \in \mathbb{N}$ such that $m(K_X + B)$ is a Cartier divisor. Suppose $f : Y \to X$ is a birational morphism from a normal variety Y. Let $E \subset Y$ denote the exceptional locus of f and $E_i \subset E$ the irreducible exceptional divisors. Let

$$f_*^{-1}B := \sum b_i f_*^{-1} B_i$$

denote the birational transform of B. The two line bundles

$$\mathcal{O}_Y(m(K_Y + f_*^{-1}B))_{|Y-E} \text{ and } f^*\mathcal{O}_X(m(K_X + B))_{|Y-E}$$

are naturally isomorphic. Thus there are rational numbers $a(E_i, X, B)$ such that $m \cdot a(E_i, X, B)$ are integers and

$$\mathcal{O}_Y(m(K_Y + f_*^{-1}B)) \cong f^*\mathcal{O}_X(m(K_X + B))(\sum_i (m \cdot a(E_i, X, B))E_i).$$

By definition $a(B_i, X, B) = -b_i$ and $a(D, X, B) = 0$ for any divisor $D \subset X$ which is different from the B_i. $a(E, X, B)$ is called the *discrepancy* of E with respect to (X, B).

Using numerical equivalence, we can write

$$K_Y + f_*^{-1}B \equiv f^*(K_X + B) + \sum_{E_i \text{ exceptional}} a(E_i, X, B)E_i,$$

or

$$K_Y \equiv f^*(K_X + B) + \sum_{E_i \text{ arbitrary}} a(E_i, X, B)E_i.$$

The **discrepancy of** (X, B) is given by

$$\operatorname{discrep}(X, B) := \inf_E \{a(E, X, B) \ : \ E \text{ is an exceptional divisor over } X\}.$$

That is, E runs through all the irreducible exceptional divisors of all birational morphisms $f : Y \to X$.

Definition 1.2.4 (**Terminal and Canonical Singularities**). Let (X, B) be a pair, where X is a normal variety and $B = \sum b_i B_i$ is a sum of distinct prime divisors with rational coefficients $0 \leq b_i \leq 1$. Assume that the log canonical divisor $K_X + B$ is $\mathbb{Q}$–Cartier. We say that (X, B) has **terminal** (respectively **canonical**) singularities, or we say that the log pair (X, B) is terminal (respectively canonical), by abuse of language, if

$$\operatorname{discrep}(X, B) > 0 \quad (\text{respectively } \operatorname{discrep}(X, B) \geq 0).$$

2.1. Log Birational Geometry of Surfaces. The logarithmic generalization of the Mori program in dimension 2 goes along the main strategies MP1, MP2 and MP3.

MP1. We take a logarithmic pair (S, B) having only **terminal singularities**, where S is a nonsingular projective surface and $B = \sum b_i B_i$ is a boundary $\mathbb{Q}$–divisor with only normal crossing and rational coefficients $0 \leq b_i \leq 1$.

Thanks to the log Cone Theorem and the log Contraction Theorem, if there exists a curve that intersects the log canonical divisor $K_S + B$ negatively, then either this curve generates a log Mori fiber space or it is contracted.

In the end of this process, we obtain either a **log Mori fiber space** $(S_{\mathrm{mori}}, B_{\mathrm{mori}})$ or a **log minimal model** $(S_{\min}, B_{\min})$ such that $K_{S_{\min}} + B_{\min}$ is nef. For details, see [**Ma**, chapter 2, p. 118].

MP2. Again the most important property is the "Hard Dichotomy", which asserts that the logarithmic Kodaira dimension controls the outcome of log MMP: the outcome of log MMP is either a log Mori fiber space or a log minimal model depending on whether $\kappa(K_S + B) = -\infty$ or $\kappa(K_S + B) \geq 0$.

We can prove the Hard Dichotomy theorem of log MMP in dimension 2 thanks to the existence of an effective pluri–log canonical divisor of a log minimal model in dimension 2. Indeed, we have that if $(S_{\min}, B_{\min})$ is a log minimal model in dimension 2, then $\kappa(K_{S_{\min}} + B_{\min}) \geq 0$.

Once $\kappa(K_{S_{\min}} + B_{\min}) \geq 0$ for a log minimal model $(S_{\min}, B_{\min})$ has been established, we obtain the log Abundance theorem for log minimal models in dimension 2, which claims that $(S_{\min}, B_{\min})$ has a base point free pluri–log canonical system.

On the other hand, for a log pair (S, B) with $\kappa(K_S + B) = -\infty$, we have that a log Mori fiber space $(S_{\mathrm{mori}}, B_{\mathrm{mori}})$ is covered by rational curves intersecting $K_{S_{\mathrm{mori}}} + B_{\mathrm{mori}}$ negatively.

MP3. In the genuine birational geometry, we are interested in the relation among good representatives in a given birational equivalence class, where two good representatives are outcomes of one appropriate nonsingular projective variety through MMP if and only if they are birationally equivalent. When we deal with the log pairs and the relation among good representatives obtained through log MMP, the mere notion of birational equivalence does not work well and one is naturally led to the notion of the log MMP relation: we say two or more good representatives are **log MMP–related** if and only if they are outcomes through log MMP of one appropriate log pair consisting of a nonsingular projective surface and a boundary divisor with only normal crossing.

Then, the Uniqueness of the log minimal model in dimension 2 and the log Sarkisov program in dimension 2 hold, replacing the notion of "birational" with that of "log MMP–related".

2.2. Conclusion about log pairs. Let (S, B) be a log pair consisting of a normal surface S and a boundary $\mathbb{Q}$–divisor $B = \sum b_i B_i$ with only normal crossing and rational coefficients $0 \leq b_i \leq 1$.

To (S, B) we can associated:

- a **log resolution** $(\widetilde{S}, \widetilde{B})$, where there exists a proper morphism $f : \widetilde{S} \to S$ from a smooth surface $\widetilde{S}$ such that the union of the support of $f_*^{-1}B$ and of the exceptional locus is a normal crossing divisor.
- a **log minimal model** $(S_{\min}, B_{\min})$, where the pair $(S_{\min}, B_{\min})$ has terminal singularities and $K_{S_{\min}} + B_{\min}$ is nef. We have that the log minimal model is unique in a fixed log MMP–related class.
- a **log canonical model** $\mathrm{Proj}(R(S, B))$, with $R(S, B) = \bigoplus_m H^0(m(K_S + B))$, where the pair (S, B) has canonical singularities and $K_S + B$ is ample. We have that $\mathrm{Proj}(R(S, B))$ is a point if and only if $\kappa(K_S + B) = 0$, $\mathrm{Proj}(R(S, B))$ is a curve if and only if $\kappa(K_S + B) = 1$ and $\mathrm{Proj}(R(S, B))$ is a surface if and only if $\kappa(K_S + B) = 2$.
- a **model with canonical singularities** $(\overline{S}, \overline{B})$, where the pair $(\overline{S}, \overline{B})$ has canonical singularities and $K_{\overline{S}} + \overline{B}$ is nef.

REMARK 1.2.5. Let $(S, \sum b_i B_i)$ be a log pair with canonical singularities. If $K_S + B$ is ample, we have that, for any $\varepsilon > 0$, the pair $(S, \sum(b_i - \varepsilon)B_i)$ is a log minimal model.

3. The &–Minimal Model

In this section we present the results of Dicks in [**Di**] about the study of &–minimal pairs (S, C) or $S\&C$, up to birational equivalence, where S is a projective, nonsingular surface and C is a reduced curve of arbitrary genus on S.

Dicks gives a classification of &–minimal model for a pair (S, C), see [**Di**, Theorem 3.1] and gives some conditions about the uniqueness of &–minimal model, see [**Di**, Proposition 2.2, Theorem 2.3]. In general, a pair (S, C) or $S\&C$ may have no unique &–minimal model, for example if S is rational. The ambiguity in this case is given by the action of the Cremona group.

3.1. Existence and uniqueness of &–minimal model. In the following, we collect the main definitions and theorems of Dicks [**Di**] we are going to use afterwards.

Definition 1.3.1. A projective, non singular surface S with an irreducible curve $C \subset S$ is called a *pair* and denoted (S, C) or $S\&C$.

Definition 1.3.2. Let C be a smooth curve and E a rank 2 vector bundle over C; the projective bundle $\mathbb{P}_C(E)$ is called $\mathbb{P}^1$*–bundle*.

Definition 1.3.3. Let K_S be the canonical bundle on a surface S. In analogy with Definition 1.1.4, a curve l is called $(-n)$*–curve* if $l^2 = -n$ and $K_S \cdot l = -2 + n$.

Definition 1.3.4. A pair (S, C) is &*-minimal* if C is non singular and for every (-1)–curve $l \subset S$ with $l \neq C$, we have $l \cdot C \geq 2$.

Definition 1.3.5. Two pairs (S, C) and (S', C') are birationally equivalent if there is a birational map $f : S \dashrightarrow S'$ such that f (respectively f^{-1}) is an isomorphism at the generic point of C (respectively C').

Definition 1.3.6. An &-minimal pair (S_0, C_0) is an &-minimal model for (S, C) if (S, C) is birationally equivalent to (S_0, C_0).

Given a pair (S, C), it is always possible to find a birationally equivalent pair (S', C') which is &–minimal: we blow up to make C a non-singular curve C'' and then contract (-1)–curves l with $l \neq C''$ and $l \cdot C'' \leq 1$.
We have the following theorem [**Di**, Theorem 3.1], which gives a classification of &-minimal models based on Mori theory:

Theorem 1.3.7 ([**Di**]. Classification of &–minimal model)**.** *Suppose (S, C) is &-minimal. Then one of the following holds:*

1) *$S \cong \mathbb{P}^2$ and C is a line or a conic;*
2) *S is a $\mathbb{P}^1$–bundle (i.e. $S = \mathbb{P}_C(E)$, where E is a rank 2 vector bundle over the smooth curve C) and C is a fibre or a section;*
3) *C is a (-1)–curve;*

4) $C \cong \mathbb{P}^1$, $C^2 \leq -4$ *and* $K_S + (1 + \frac{2}{C^2})C$ *is nef;*
5) $C \cong \mathbb{P}^1$, $C^2 = -2$ *or* -3 *and* K_S *is nef;*
6) $K_S + C$ *is nef.*

Now, we consider the question of when an &–minimal pair can be birational to, but not isomorphic to, an another &–minimal pair. The following proposition [**Di**, Proposition 2.2] gives some conditions under which a pair can be known to have an unique &–minimal model.

Proposition 1.3.8 ([**Di**]). *Suppose that* (S_0, C_0) *is a pair and* $f : (S_0, C_0) \to (S, C)$ *is a morphism such that* $f_{|C_0} : C_0 \to C$ *is an isomorphism; suppose that*

$$K_S + \lambda C \quad \text{is nef for some } \lambda \in \mathbb{Q}, \text{ with } 0 \leq \lambda < 1.$$

Then (S, C) *is the unique &–minimal model of* (S_0, C_0).

This proposition prove that the &–minimal model is unique in a large number of cases. Moreover, this proposition forms a large part of the proof of the following theorem [**Di**, Theorem 2.3] about the uniqueness of the &–minimal model. This theorem is essentially a list of all &–minimal pairs which are not unique in their birational class.

Theorem 1.3.9 ([**Di**]. Uniqueness of &–minimal model). *Let* (S, C) *be an &–minimal pair which is not one of the following:*

1) $S \cong \mathbb{P}^2$ *and* C *is a line, a conic or a cubic;*
2) S *is a* $\mathbb{P}^1$*–bundle and* C *is a fibre or a section;*
3) S *is a* $\mathbb{P}^1$*–bundle and* $C \cdot f = 2$ *for a fiber* f.

Let (S_0, C_0) *be a birationally equivalent pair. Then* (S_0, C_0) *is &–minimal if and only if the two pairs are isomorphic.*

4. Cremona Transformations

A birational map $T : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is called a *Cremona transformation*. It is given by a n–dimensional linear system $L \subset |\mathcal{O}_{\mathbb{P}^n}(d)|$ for some $d \geq 1$. A choice of a basis gives an explicit formula:

$$T : (x_0, \ldots, x_n) \mapsto (P_0(x_0, \ldots, x_n), \ldots, P_n(x_0, \ldots, x_n)),$$

where $P_i(x_0, \ldots, x_n)$ are homogeneous polynomials of degree d. A linear system defining a Cremona transformation is called a *homaloidal linear system*. Let

$$B_L = \bigcap_{D \in L} D$$

be the base locus of L or T, considered as a closed subscheme of $\mathbb{P}^n$. By cancelling the polynomials P_i's by the common divisor, we assume that B_L has no divisorial components. Let $\pi : X \to \mathbb{P}^n$ be the blow-up of the subscheme B_T . There is a morphism $f : X \to \mathbb{P}^n$ such that $f = T \circ \pi$ as rational maps, i.e. the following diagram is commutative

$$\begin{tikzcd} & X \arrow[dl, "\pi"'] \arrow[dr, "f"] & \\ \mathbb{P}^n \arrow[rr, dotted, "T"] & & \mathbb{P}^n \end{tikzcd}$$

In fact, let $L = \mathbb{P}(V)$, where $V \subset H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$, and let $\mathcal{I}$ be the ideal sheaf defining B_L. Then there is a canonical surjection of sheaves

$$\mathcal{O}_{\mathbb{P}^n} \otimes V \to \mathcal{I} \otimes \mathcal{O}_{\mathbb{P}^n}(d).$$

This defines a surjection of the symmetric Algebras

$$\mathrm{Sym}(\mathcal{O}_{\mathbb{P}^n}(-d) \otimes V) \to \mathrm{Sym}(\mathcal{I})$$

and hence a closed embedding

$$X \hookrightarrow \mathrm{Proj}(\mathrm{Sym}(\mathcal{O}_{\mathbb{P}^n}(-d) \otimes V)) \cong \mathbb{P}^n \times \mathbb{P}(V)^*.$$

The composition with the second projection is our map f. It follows from this that

$$f^*\mathcal{O}_{\mathbb{P}^n}(1) = \pi^*\mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{O}_X(-E),$$

where the image of $\pi^*\mathcal{I}$ in $\mathcal{O}_X$ is the ideal sheaf $\mathcal{O}_X(-E)$ of some effective divisor E on X (the exceptional divisor of π). Since f is birational, $f_*\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^n}$, and hence by the projection formula

$$f_*f^*\mathcal{O}_{\mathbb{P}^n}(1) \cong \mathcal{O}_{\mathbb{P}^n}(1) \otimes f_*\mathcal{O}_X \cong \mathcal{O}_{\mathbb{P}^n}(1).$$

This implies that

$$H^0(X, f^*\mathcal{O}_{\mathbb{P}^n}(1)) \cong H^0(X, f_*f^*\mathcal{O}_{\mathbb{P}^n}(1)) \cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)).$$

This shows that the morphism f is given by the complete linear system

$$|f^*\mathcal{O}_{\mathbb{P}^n}(1)| = |\pi^*\mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{O}_X(-E)|.$$

The map T is birational if and only if f is a birational morphism. Assume X is nonsingular. This can be achieved by composing π with resolution of singularities. Let H be a hyperplane in $\mathbb{P}^n$ so that $\mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{O}_{\mathbb{P}^n}(H)$. Then T is birational if and only if

$$f^*(H)^n = (\pi^*(dH) - E)^n = 1,$$

see [**Do**, chapter 7, pp. 153–154].

4.1. Cremona Transformations of the plane.

Quadratic Cremona Transformations. Let $T : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a rational map of $\mathbb{P}^2$ to itself given by polynomials of degree 2, i.e. $T(t_0, t_1, t_2) = (T_0, T_1, T_2) = (P_0(t_0, t_1, t_2), P_1(t_0, t_1, t_2), P_2(t_0, t_1, t_2))$, where $P_i \in \mathbb{C}[t_0, t_1, t_2]_2$, for $i = 1, 2, 3$. The pre-image of a line $V(a_0T_0 + a_1T_1 + a_2T_2)$ is the conic $V(a_0P_0 + a_1P_1 + a_2P_2)$. The pre-image of a general point is equal to the intersection of the pre-images of two general lines, thus the intersection of two conics from the net L of conics spanned by P_0, P_1, P_2. If we want T to define a birational map we need the intersection of two general conics to be equal to 1. This can be achieved if all conics pass through the same set of three points p_1, p_2, p_3 (*base points*). These points must be non-collinear, otherwise all polynomials have a common factor, after dividing, we get a projective transformation. Birational automorphisms of $\mathbb{P}^2$ (*Cremona transformations*) which are obtained by nets of conics through three non-collinear points are called *quadratic transformations*. If we choose a basis in $\mathbb{P}^2$ such that $p_1 = (1, 0, 0)$, $p_2 = (0, 1, 0)$, $p_3 = (0, 0, 1)$ and a basis in L given by the conics $V(T_1T_2)$, $V(T_0T_2)$, $V(T_0T_1)$, then the transformation is given by the formula

$$T : (x_0, x_1, x_2) \mapsto (x_1x_2, x_0x_2, x_0x_1).$$

This is called the *standard Cremona transformation*. For details, see [**Do**, chapter 4, p. 84] and [**SR**, pp. 45–48].

The birational geometry of the plane. We have the following theorem:

Theorem 1.4.1 (**Noether–Castelnuovo**). *The group of birational transformations of the projective plane is generated by linear transformations and the standard Cremona transformation, that is*

$$(x_0 : x_1 : x_2) \mapsto (x_1\, x_2 : x_0\, x_2 : x_0\, x_1),$$

where $(x_0 : x_1 : x_2)$ *are the coordinates of* $\mathbb{P}^2$.

Proof. See [**AnMe**, pp. 60–64]. □

In the study of plane curves will be fundamental another class of transformations between rational surfaces:

Definition 1.4.2 (**Elementary transformations**). Let $\pi : X \to C$ be a smooth ruled surface, see [**Ha**, chapter V, section 2, p. 369]. Let p be a point of X and let f be the fibre of π containing

p. Let $\varepsilon : \widetilde{X} \to X$ be the blow up of the point p. Then the strict transform $\widetilde{f}$ of f on $\widetilde{X}$ is isomorphic to $\mathbb{P}^1$ and $\widetilde{f}^2 = -1$. By Castelnuovo Theorem, see [**Ha**, Theorem 5.7, p. 414], we can blow down $\widetilde{f}$. In other words, there is a morphism $\mu : \widetilde{X} \to X'$ sending $\widetilde{f}$ to a point q and such that μ is the blow up of the point q. If $f' = \mu(E)$, then $f' \cong \mathbb{P}^1$ and ${f'}^2 = 0$. Moreover, the rational map $\pi' : X' \to C$ obtained from π on $X \setminus f \cong X' \setminus f'$ is a morphism, hence $\pi' : X' \to C$ is another smooth ruled surface. Indeed, the fibres of π' are all isomorphic to $\mathbb{P}^1$ and, since π has a section, its strict transform on X' will be a section of π'. This new ruled surface is called the **elementary transformation of X with center p.**

By abuse of language, we call elementary transformation with center p, the birational map $\mathrm{elm}_p : X \dashrightarrow X'$, where $\mathrm{elm}_p = \mu \circ \varepsilon^{-1}$.

Let $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map, which is not an isomorphism. To study the map ω, we start by factorizing it into simpler birational maps, called elementary links, between Mori spaces. We recall that these maps will be either the blow up of a point in $\mathbb{P}^2$ or an elementary transformation of a rational ruled surface.

Consider $\mathcal{H} = \omega_*^{-1}\mathcal{O}(-1)$ the strict transform of lines in $\mathbb{P}^2$; then $\mathcal{H}$ is without fixed components and $\mathcal{H} \subset |\mathcal{O}(d)|$, for some $d > 1$. We have the following:

Theorem 1.4.3. *Let $\mathcal{H} \subset |\mathcal{O}(d)|$ be as above and let $H \in \mathcal{H}$ be its general elements. Then the log pair $(\mathbb{P}^2, \frac{3}{d}H)$ has not canonical singularities. In particular, there is a point $x \in \mathbb{P}^2$ such that*

$$\mathrm{mult}_x H > \frac{d}{3}.$$

Proof. See [**AnMe**, pp. 60–63]. □

We can generalize the above theorem using the following way (see [**Co1**]). Let $\pi : X \dashrightarrow S$ and $\varphi : Y \dashrightarrow W$ be two Mori fiber spaces of dimension ≤ 3. Let $\omega : X \dashrightarrow Y$ be a birational non–biregular map. Let $\mathcal{H}_Y$ be a very ample linear system on Y and $\mathcal{H} = \omega_*^{-1}\mathcal{H}_Y$. Let $H \in \mathcal{H}$ be a general element. By definition of a Mori fiber space, there exists $\mu \in \mathbb{Q}$ such that $K_X + \frac{1}{\mu}H \equiv_\pi 0$. We obtain the following:

Theorem 1.4.4 (Noether–Fano inequalities). *In the above notation, in particular for ω non–biregular and $K_X + \frac{1}{\mu}H \equiv_\pi 0$, either $(X, \frac{1}{\mu}H)$ has not canonical singularities or $K_X + \frac{1}{\mu}H$ is not nef.*

We can interpret a standard Cremona transformation in terms of elementary links:

REMARK 1.4.5. A standard Cremona transformation is given by the following links:

$$\begin{tikzcd} & \mathbb{F}_1 \arrow[r, dashed] \arrow[dl] & \mathbb{F}_0 \arrow[r, dashed] & \mathbb{F}_1 \arrow[dr] & \\ \mathbb{P}^2 & & & & \mathbb{P}^2 \end{tikzcd}$$

Conversely, any map of type

$$\begin{tikzcd} & \mathbb{F}_1 \arrow[r, dashed] \arrow[dl] & \mathbb{F}_a \arrow[r, dashed] & \mathbb{F}_1 \arrow[dr] & \\ \mathbb{P}^2 & & & & \mathbb{P}^2 \end{tikzcd}$$

can be factorized into Cremona transformations.

We note that, since a Cremona transformation is given by conics through 3 not collinear points, the last link above is possible only for $a = 0, 2$. Links of this kind represent birational maps given by conics with either 3 base point or 2 base point plus a tangent direction.

4.2. Cremona transformations of space. Let W_1, W_2 be projective varieties such that $W_i \simeq \mathbb{P}^3$, for $i = 1, 2$. We present the Cremona transformation $T : W_1 \dashrightarrow W_2$. We follow the approach of [**SR**, chapter VIII, section 4]. These transformations classify linear systems of surfaces into classes of birationally equivalent system. An interesting application of these transformations is, for example, the resolution of singularities of space curves and surfaces.

If T is a Cremona transformation of $W_1 \simeq \mathbb{P}^3$ into $W_2 \simeq \mathbb{P}^3$, then T is defined by a 3–dimensional homaloidal linear system of surfaces $\Phi \subset |\mathcal{O}_{W_1}(d)|$, for some $d \geq 1$. Moreover, T^{-1} is defined by a 3–dimensional homaloidal linear system of surfaces $\Psi \subset |\mathcal{O}_{W_2}(d')|$, for some $d' \geq 1$.

Let $D \in \Phi$ be a general surface, we have that D is a rational surface and $D = T^*\Pi'$, for some plane $\Pi' \subset W_2$. Similarly, the general $D' \in \Psi$ is a rational smooth surface and $D' = (T^{-1})^*\Pi$, for some plane $\Pi \subset W_1$. Using the notations of [**SR**], a Φ–surface is a general surface $D \in \Phi$ and a Ψ–surface is a general surface $D' \in \Psi$. Moreover, a Φ–curve is a rational curve C such that $C = T^*l'$, for some line $l' \subset W_2$ and a Ψ–curve is a rational curve C' such that $C = (T^{-1})^*l$, for some line $l \subset W_1$.

Then we have the following:

Theorem 1.4.6 ([**SR**]). *In any space Cremona transformation with generating systems Φ, Ψ, the degree of Φ–surfaces is equal to that of Ψ–curves, and the degree of Ψ–surfaces is equal to that of Φ–curves.*

Hence, when Φ has been chosen, the degrees of the Ψ–surfaces and Ψ–curves are determined; our knowledge of the base elements of Ψ derives from the fundamental surfaces and curves of Φ associated to the base locus of Ψ. In particular, we have that:

- to any fundamental surface A of Φ there corresponds a base point O of Ψ; if Φ_1 is the system of surfaces residual to A in Φ (representing planes through O in W_2), then the multiplicity of O for Ψ is equal to the difference between the degree of the Φ–curves and Φ_1–curves.
- To any simply infinite family of fundamental curves $\mathcal{L}$ of degree λ of Φ there corresponds a λ–fold base curve of Ψ; the degree of this curve is equal to the number of curves $\mathcal{L}$ which lies on a generic Φ–surface.

Moreover, any Ψ–surface is represented on a plane Π of W_1 by means of the system of curves in which Π is met by Φ–surfaces.

At this point, we describe a method (due to Cremona) for the construction of a homaloidal linear system Φ in W_1. We note that the generic (rational) surface D of a homaloidal web is met by the other surfaces of the web in a homaloidal net of curves to which correspond, in the plane representation of D, the curves of a plane homaloidal net.

We choose a rational surface D_0^d that is a generic Φ–surface. In the plane Π in which D_0 is birationally represented, we construct the system of curves $|L|$ which represent the free curves of intersection of D_0 with surfaces of degree d having the same multiple points and curves as D_0. Then we reduce $|L|$ to a homaloidal net $|L^*|$ by imposing fixed components A^* and base points O_i^* on $|L|$. A linear system Φ which meets D_0 in the homaloidal net of curves corresponding to $|L^*|$ is formed by the surfaces of degree d, having the same multiple points and curves as D_0, which have further simple base curves A on D_0, corresponding to A^* on Π, and base points or base contacts at points O_i of D_0 corresponding to O_i^*. Since D_0 is supposedly a generic member of the system, it follows that Φ is a homaloidal web.

Finally, we recall that Cremona transformation of $W_1 \simeq \mathbb{P}^3$ may also be generated by direct geometrical construction or by representing a rational threefold in two different ways, e.g. by two different projection, on space W_1 and W_2.

Now, we give the description and the construction of two Cremona space transformations that are useful in the next arguments.

1) Description of the cubo–cubic Cremona transformation. Let W_1, W_2 be projective varieties such that $W_i \simeq \mathbb{P}^3$, for $i = 1, 2$ and let $C \subset \mathbb{P}^3$ be a general smooth curve of degree 6 and genus 3.

Let $|\mathcal{I}_C(3)| \subset |\mathcal{O}_{W_1}(3)|$ be the linear system of cubic surfaces in $W_1 \simeq \mathbb{P}^3$ containing C, then, by [**Ve**], we have that $|\mathcal{I}_C(3)|$ is homaloidal and defines a Cremona transformation cubo–cubic $\phi : W_1 \dashrightarrow W_2$, with base locus C.

Let $Sec_3(C) \subset W_1$ be the variety of trisecant lines to C, then, by [**Ve**], we have that $Sec_3(C)$ is a surface and that $Sing(Sec_3(C)) = C$, because two distinct trisecant lines to C do not meet in $W_1 \setminus C$.

Let $p \in C$ be a point and consider a Cremona transformation cubo–cubic $\phi : W_1 \dashrightarrow W_2$, with base locus a general curve C. We can describe ϕ in two principal steps:

1) *Blowing ups of the curve C and of the point p.*
Let $\varepsilon : Y = Bl_C W_1 \to W_1$ be the blow up of the curve C in $W_1 \simeq \mathbb{P}^3$ and E_C its exceptional divisor. Let $E_Y = \varepsilon^*_{-1}(Sec_3(C))$ and $F_p = \varepsilon^*_{-1}(p)$ denote, respectively, the strict transforms in Y of the surface $Sec_3(C)$ and of the point p. We have that $E_Y = \mathbb{P}(\mathcal{N}_{C/\mathbb{P}^3})$ is a ruled surface over a curve of genus 3, whose fibers are the strict transforms of trisecant lines to C and, moreover, that $F_p = \mathbb{P}(\mathcal{N}_{C/\mathbb{P}^3,p})$ is a fibre.
Let $\mu_1 : Z_1 = Bl_p W_1 \to W_1$ be the blow up of the point p in $W_1 \simeq \mathbb{P}^3$ and $E_p \simeq \mathbb{P}^2$ its exceptional divisor. Let $C_1 = {\mu_1}^*_{-1}(C)$ denote the strict transform in Z_1 of C. Since C is a smooth curve, then $C_1 \cap E_p = \{q\}$, where q is a point.
Let $\mu_2 : Z_2 = Bl_{C_1} Z_1 \to Z_1$ be the blow up of the curve C_1 in Z_1 and E_{C_1} its exceptional divisor. Let $E'_p = {\mu_2}^*_{-1}(E_p)$ denote the strict transform in Z_2 of E_p, then we have that $E'_p \simeq \mathbb{F}_1$.
Now we consider the contraction of E'_p and we have a birational morphism $\mathrm{cont}_{E'_p} : Z_2 \to Y$ such that $\mathrm{cont}_{E'_p}(E_{C_1}) = E_C$ and $\mathrm{cont}_{E'_p}(E'_p) = F_p$.
2) *Contraction of trisecant lines to the curve C.*
Let $\nu : Y \to W_2$ be the contraction of trisecant lines to C. We have that $\nu(E_Y) = C'$, where C' is a curve of degree 6 and genus 3, and that $\nu(E_C) = Sec_3(C')$, where $Sec_3(C')$ is the surface of trisecant lines to the curve C'.

Note that, if we consider $\varepsilon' : Y' = Bl_{C'}W_2 \to W_2$ the blow up of the curve C' in $W_2 \simeq \mathbb{P}^3$ such that $E_{C'}$ is its exceptional divisor and $E_{Y'} = \varepsilon'^{*}_{-1}(Sec_3(C'))$ is the strict transform of the surface $Sec_3(C')$, and if we consider $\nu' : Y' \to W_1$ the contraction of trisecant lines to C', then we obtain that $\nu'(E_{Y'}) = C$ and $\nu'(E_{C'}) = Sec_3(C)$.

2) Construction of a $T_{(2,n)}$ Cremona transformation. We want to construct a birational map $\psi_n : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, which is a $(2,n)$ Cremona transformation of [**SR**].

We consider $\Lambda_n = |\mathcal{O}_{\mathbb{P}^n}(2) \otimes \mathcal{I}_L \otimes \mathcal{I}_{x_1} \otimes \ldots \otimes \mathcal{I}_{x_n}|$ the linear system of quadric surfaces in $\mathbb{P}^n$ containing a $(n-2)$–dimensional linear space L and n points $x_1, \ldots, x_n$ in general position. It is immediate that $\dim \Lambda_n = n+1$.
By considering a rational normal curve passing through $x_1, \ldots, x_n$ and $(n-1)$–secant to L, we obtain the birationality of ψ_n.

To understand better ψ_n, let us factor it as follows.
Let $\varepsilon : \mathrm{Bl}_L\mathbb{P}^n = X_0 \to \mathbb{P}^n$ be the blow up of the linear space L in $\mathbb{P}^n$. Then X_0 has a natural embedding in $\mathbb{P}^{2n}$ of degree $n+1$. Therefore, X_0 is a minimal degree variety in $\mathbb{P}^{2n}$.

To obtain the map ψ_n, we have now to simply project X_0 from the n points $x_1, \ldots, x_n$.

Consider the projection $\pi_{x_k} : X_{k-1} \subset \mathbb{P}^{2n-k+1} \dashrightarrow X_k \subset \mathbb{P}^{2n-k}$. Clearly, $X_k \subset \mathbb{P}^{2n-k}$ is a minimal degree variety and, in particular, X_{n-1} is quadric in $\mathbb{P}^{n+1}$.

$$\begin{tikzcd}
 & X_0 = \mathrm{Bl}_L\mathbb{P}^n \arrow[dl, "\varepsilon"'] \arrow[r, dotted] & \cdots\cdots \arrow[r, dotted] & X_{n-2} \arrow[r, dotted, "\pi_{x_{n-1}}"] & X_{n-1} = \mathbb{Q}^n \arrow[d, dotted, "\pi_{x_n}"] \\
\mathbb{P}^n \arrow[rrrr, dotted, "\psi_n"] & & & & \mathbb{P}^n
\end{tikzcd}$$

The variety X_k is a cone over $\mathbb{P}^1 \times \mathbb{P}^{n-k}$ and π_{x_k} projects the (strict transform of) hyperplane $H_{x_k} =< L, x_k >$ from the point x_k. Let $L_{x_k} = \pi_{x_k}(H_{x_k})$, then L_{x_k} is a linear space of dimension $n-2$.

Let S^{k-1} be the (strict transform of) linear space spanned by $< x_1, \ldots, x_k >$, then $\psi_n(S^{k-1})$ is a linear space of dimension $k-2$. Note that $\psi_{n|S^{k-1}}$ is given by the linear system $|\Lambda_{k-1} \otimes \mathcal{I}_{x_k}|$. In particular, the fibers of $\psi_{n|S^{k-1}}$ are rational normal curves of degree $k-1$ passing through $x_1, \ldots, x_k$ and $(k-1)$–secant to L.

This description can be interpreted as a chain of elementary transformations of vector bundles. In this way, it gives a factorization of ψ_n in the Sarkisov category.

REMARK 1.4.7. The “fibers” of ψ_n are lines either passing through x_k and meeting L in a point, for $k = 1, \dots, n$, or rational normal curves of degree $n-1$ passing through $x_1, \dots, x_n$ and $(n-1)$–secant to L.

CHAPTER 2

Projective varieties of codimension at least 2

Let us consider a rational projective variety $X \subset \mathbb{P}^n$ of dimension r. Then there exists a birational map $\phi : X \dashrightarrow \mathbb{P}^r$. The simplest embedding of $\mathbb{P}^r$, as a projective variety, is the linear one. It is quite natural to ask whether the map ϕ can be extended to a birational map $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ such that $\Phi(X)$ is linear. We shall say that in this case the variety X is *Cremona equivalent*, see Definition 2.2.3, to a linear space.

This extension property reminds us the Abhyankar-Moh property (AMP), [**AM**]. The latter asks about extensions of polynomial embeddings in $\mathbb{C}^n$ to automorphisms of $\mathbb{C}^n$. More precisely, let A be an algebraic subset of $\mathbb{C}^n$. We say that A has the AMP if for any polynomial embedding $f : A \to \mathbb{C}^n$ there exists a polynomial automorphism F of $\mathbb{C}^n$ such that f is a restriction of this automorphism to the set A. In particular, the affine set $A \subset \mathbb{C}^n$ has the AMP if and only if for any two epimorphisms $f_1, f_2 : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[A]$ there exists an automorphism $F : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n]$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{C}[x_1,\ldots,x_n] \arrow[rr, "F"] \arrow[dr, "f_1"'] & & \mathbb{C}[x_1,\ldots,x_n] \arrow[dl, "f_2"] \\
& \mathbb{C}[A] &
\end{tikzcd}$$

The AMP has been studied extensively and seems granted for high codimension smooth varieties, see for instance [**Je**, Theorems 1.1,1.2]. More precisely, Jelonek proves the followings:

Theorem 2.0.1 ([**Je**], Theorem 2.1)**.** *Let $\varphi : \mathbb{C}^k \times 0 \to \mathbb{C}^n$ be a birational isomorphism onto its image and $n \geq 2k + 1$. There exists a birational isomorphism $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ such that $\mathrm{res}_{\mathbb{C}^k \times 0} \Phi = \varphi$.*

Theorem 2.0.2 ([**Je**], Theorem 2.2)**.** *Let $A \subset \mathbb{C}^n$ be an algebraic variety of dimension k. Let $\varphi : A \to \mathbb{C}^n$ be a rational embedding. If $n \geq 2k+2$, then there exists a birational isomorphism $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ such that $res_A \Phi = \varphi$.*

Hence, in the case of complex field, a positive answer is given for varieties of codimension greater than the dimension.

The AMP can be extended to affine varieties over an infinite field, say k. Then one can ask whether two different embeddings of the same affine variety are equivalent up to an automorphism of k^n. Also in this context the answer is positive for varieties with isolated singularities and roughly local dimension half the embedding dimension, see [**Sr**, Theorems 1,2]. The result (stated in algebraic language) of Srinivas is:

Theorem 2.0.3 ([**Sr**], Theorems 1,2)**.** *Let A be a finitely generated algebra over an infinite field k. Let $m = \dim S_A(\Omega^1_{A|k})$ and $r = \sup\{m, 2\dim A + 1\}$. Then A is generated as k–algebra by r elements. Moreover, suppose that $f : k[x_1, \ldots, x_n] \to A$, $g : k[x_1, \ldots, x_n] \to A$ are two surjections of k–algebras, with $n > r$. Then there is an elementary k–algebra automorphism $\varphi : k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]$ such that $f = g \circ \varphi$.*

Our aim is to solve a similar question in the context of birational geometry of projective varieties. We want to understand when two birational embeddings of the same variety are equivalent up to a Cremona transformation of the projective space, in this case we say that they are Cremona equivalent, see Section 2.

Remark 2.0.4. From Theorem 2.0.3 follows that two different embeddings in k^n of A, where A is an affine variety of dimension 1 with only isolated singularities, are equivalent up to an automorphism of k^n, if $n \geq 4$.

In Section 1, we consider a rational, irreducible and reduced curve $C \subset \mathbb{P}^n$, for $n \geq 3$. Then there exists a birational map $\phi : C \dashrightarrow \mathbb{P}^1$. We study whether the map ϕ can be extended to a birational map $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, $n \geq 3$, such that $\Phi(C)$ is a line. We shall say that in this case the curve C is *Cremona equivalent to a line*. Indeed, we state the following:

Definition 2.0.5. An irreducible reduce rational curve $C \subset \mathbb{P}^n$ is *Cremona equivalent to a line* if there exists a birational map $\Psi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ such that $\Psi(C)$ is a line. More generally we say that C is *Cremona equivalent to a curve* Y if there exists a birational map $\Psi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ such that $\Psi(C) = Y$.

We have already seen above that this extension property reminds us the Abhyankar-Moh property (AMP). The very first example solved positively was the line in $\mathbb{C}^2$, [**AM**]. This, translated in the dictionary of projective geometry, says that a rational plane curve with a unique singularity is *Cremona equivalent to a line*. Note that an arbitrary rational curve in $\mathbb{P}^2$ is not *Cremona equivalent to a line*. The simplest example is a general projection of $\Gamma \sim (1, a) \subset \mathbb{Q}^2 \subset \mathbb{P}^3$, for $a \geq 5$ (see Chapter 3).

The main result of Section 1 is that this is the unique negative answer.

1. Rational curves of $\mathbb{P}^n$

Let consider a rational irreducible reduced curve $C \subset \mathbb{P}^n$, with $n \geq 3$. In this Section, we study whether there exists a birational map $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, $n \geq 3$, such that $\Phi(C)$ is a line. We shall say that in this case the curve C is *Cremona equivalent to a line*, see Definition 2.0.5.

1.1. Birational maps. We work over the complex field. Note that we need an algebraically closed field k with $\mathrm{char}(k) = 0$ only for Definition 2.1.4.

Definition 2.1.1. Let $L \subset P^n$ be a linear space of dimension l. We will indicate with $\pi_L : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-l-1}$ the projection from L.

We start by describing the birational maps that will be used in this section.

The map $\psi_n[L, x_1, \ldots, x_n]$.
The map $\psi_n[L, x_1, \ldots, x_n] : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is the classical Cremona transformation $T_{(2,n)}$ of [**SR**]. For more details, see the description of birational map ψ_n in Section 4 of Chapter 1.

From now on we fix a smooth rational curve C of degree d in $\mathbb{P}^n$, $n \geq 3$, and $r = \max\{d, n\}$.

Definition 2.1.2. The standard rational normal curve N_r of degree r in $\mathbb{P}^r = (x_{r+1} = \ldots = x_{2r-d} = 0) \subseteq \mathbb{P}^{2r-d}$ is the rational curve defined by the minors of order 2 of the following $2 \times r$ matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & \ldots & x_{r-1} \\ x_1 & x_2 & x_3 & \ldots & x_r \end{pmatrix}.$$

Definition 2.1.3. The curve $C \subset \mathbb{P}^n$ is the L-projection of W, if there exists a curve $W \subset \mathbb{P}^N$ and a linear space $L \subset \mathbb{P}^N$ such that

- $\sharp\{L \cap W\} = \deg W - \deg C$
- $\pi_L(W) = C$.

The map χ_C^n.

Let $W \subset \mathbb{P}^{2r-d}$ be a smooth, linearly normal, rational curve of degree r and $L \subset \mathbb{P}^{2r-d}$ a linear space. Assume that C is the L-projection of W.

Let $\omega : \mathbb{P}^{2r-d} \to \mathbb{P}^{2r-d}$ be a projectivity such that $\omega(W) = N_r$. Let $\omega(L) = L'$ and C' the L'-projection of N_r. Note that $\deg C = \deg C'$ and C' is a smooth rational curve.

For $2r - d = 3$ we simply define $\chi_C^n = \omega$. For $2r - d > 3$ the map χ_C^n is given as follows.

$$\begin{tikzcd} W, S_k \subset \mathbb{P}^{2r-d} \arrow[r, "\omega"] \arrow[d, dashed, "\widetilde{\pi_p}"] & \mathbb{P}^{2r-d} \supset S'_k, N_r \arrow[d, dashed, "\widetilde{\pi_{p'}}"'] \\ C \subset \mathbb{P}^n \arrow[r, dashed, "\chi_C^n"] & \mathbb{P}^n \supset C' \end{tikzcd}$$

Let $N = \mathbb{P}^n \subset \mathbb{P}^{2r-d}$ such that $L \cap N = \emptyset$ and $< L, N >= \mathbb{P}^{2r-d}$. Let $M = \mathbb{P}^{n+1}$ satisfying the following conditions: $N \subset M \subset \mathbb{P}^{2r-d}$ and $L \cap M = p$. Let $\widetilde{L} \subset L$ such that $< \widetilde{L}, p >= L$. Then we can factorize the projection $\pi_L : \mathbb{P}^{2r-d} \dashrightarrow \mathbb{P}^n = N$ into the two projection $\pi_{\widetilde{L}} : \mathbb{P}^{2r-d} \dashrightarrow M = \mathbb{P}^{n+1}$ and $\pi_p : \mathbb{P}^{n+1} = M \dashrightarrow N = \mathbb{P}^n$. By applying ω we obtain analogous linear spaces and maps in the target $\mathbb{P}^{2r-d}$. Let $\widetilde{C} = \pi_{\widetilde{L}}(W) \subset M = \mathbb{P}^{n+1}$. Then $\pi_p(\widetilde{C}) = (\pi_p \circ \pi_{\widetilde{L}})(W) = \pi_L(W) = C$. If $\widetilde{C'} = \omega(\widetilde{C})$ and if $p' = \omega(p)$, then $\pi_{\widetilde{L'}}(N_r) = \widetilde{C'}$ and $\pi_{p'}(\widetilde{C'}) = (\pi_{p'} \circ \pi_{\widetilde{L'}})(N_r) = \pi_L(N_r) = C'$.

Consider now a hypersurface $S_k \subset M = \mathbb{P}^{n+1}$ such that $\widetilde{C} \subset S_k$, $\mathrm{mult}_p(S_k) = k - 1$ and p is not a vertex of S_k. If $k >> 0$ such a hypersurface exists. The rational map $\widetilde{\pi_p} = \pi_{p|S_k} : S_k \dashrightarrow \mathbb{P}^n$ is birational as the map $\widetilde{\pi_{p'}} = \pi_{p'|S'_k} : S'_k \dashrightarrow \mathbb{P}^n$, where $S'_k = \omega(S_k) \subset M' = \omega(M)$.

Consider the Cremona transformation $\chi_C^n : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ defined as $\chi_C^n = \widetilde{\pi_{p'}} \circ \omega_{|S_k} \circ \widetilde{\pi}_p^{-1}$. Then

$$\chi_C^n(C) = (\widetilde{\pi_{p'}} \circ \omega_{|S_k} \circ \widetilde{\pi}_p^{-1})(C) = (\widetilde{\pi_{p'}} \circ \omega_{|S_k})(\widetilde{C}) = \widetilde{\pi_{p'}}(\widetilde{C}') = \pi_{p'}(\widetilde{C}') = C'.$$

It is clear that χ_C^n is not defined uniquely by n and C, but we choose this name to simplify the notation. This construction shows that any smooth rational curve is Cremona equivalent to an L-projection of the "standard" rational normal curve N_r, for some r.

The map $\varphi_r^n[L_1, L_2]$.
Let L_1, L_2 be linear spaces in $\mathbb{P}^{2r-d}$ with

- $\dim L_1 = \dim L_2 = 2r - d - n - 1$;
- $\dim L_1 \cap L_2 = 2r - d - n - 2$;
- $\sharp\{L_1 \cap L_2 \cap N_r\} = \max\{r - d - 1, 0\}$.

Let $P = L_1 \cap L_2$, then $\dim P \geq \sharp\{P \cap N_r\} - 1$.
Consider $\pi_P : \mathbb{P}^{2r-d} \dashrightarrow \mathbb{P}^{n+1}$ and let

- $\pi_P(N_r) = Y_{d+1}$, where we assume that Y_{d+1} is smooth;
- $\pi_P(L_1) = p_1$;
- $\pi_P(L_2) = p_2$.

The map $\varphi_r^n[L_1, L_2]$ is given as follows:

$$\begin{tikzcd}
& \mathbb{P}^{2r-d} \arrow[d, dashed, "\pi_P"] & \\
& \mathbb{P}^{n+1} \arrow[dl, dashed, "\pi_{p_1}"'] \arrow[dr, dashed, "\pi_{p_2}"] & \\
\mathbb{P}^n \arrow[rr, dashed, "\varphi_r^n[L_1,L_2]"] & & \mathbb{P}^n
\end{tikzcd}$$

Fix $S \subset \mathbb{P}^{n+1}$ an irreducible and reduced hypersurface of degree k such that $S \supset Y_{d+1}$ and $p_1, p_2 \in Y_{d+1}$ are points of multiplicity $k-1$ for S.

Claim. For $k >> 0$ such a hypersurface exists.

Proof. The curve Y_{d+1} is a smooth rational curve of degree $d+1$, the structure sequence reads

$$0 \longrightarrow \mathcal{I}_{Y_{d+1}}(k) \longrightarrow \mathcal{O}_{\mathbb{P}^{n+1}}(k) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(k(d+1)) \longrightarrow 0.$$

Taking cohomology we get

$$0 \longrightarrow H^0(\mathcal{I}_{Y_{d+1}}(k)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^{n+1}}(k)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^1}(k(d+1))) \ldots.$$

Since $\dim H^0(\mathcal{O}_{\mathbb{P}^1}(k(d+1))) = k(d+1)+1$, then the codimension of $H^0(\mathcal{I}_{Y_{d+1}}(k))$ in $H^0(\mathcal{O}_{\mathbb{P}^{n+1}}(k))$ is at most linear in k.

On the other hand, we can write a hypersurface of degree k in $\mathbb{P}^{n+1}$ with two points of multiplicity $k-1$ as $x_0F + x_1F' = 0$, where $F, F' \in \mathbb{C}[x_2, \dots, x_{n+1}]_{k-1}$ are homogenous polynomials of degree $k-1$ in the n variables $x_2, \dots, x_{n+1}$. Hence, the space of such hypersurfaces has dimension at least $2\binom{(k-1)+(n-1)}{n-1}$, which is a polynomial of degree $n-1$ in the variable k.

For $k >> 0$ this proves the claim. $\square$

Let $S \subset \mathbb{P}^{n+1}$ be the hypersurface of degree k with two points $p_1, p_2 \in S$ of multiplicity $k-1$, then both the projections from p_1 and from p_2 are birational maps $S \dashrightarrow \mathbb{P}^n$. We chose S such that $S \supset Y_{d+1}$. Let $\pi_{p_1} : \mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^n$ and $\pi_{p_2} : \mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^n$ the projections of Y_{d+1} from p_1, p_2, and $\pi_{p_1}(Y_{d+1}) = C$, $\pi_{p_2}(Y_{d+1}) = C'$.

Then, we define $\varphi^n_r[L_1, L_2] = \pi_{p_2|S} \circ (\pi_{p_1|S})^{-1}$, in particular $\varphi^n_r[L_1, L_2](C) = C'$.
In the following, we use the above birational maps to desingularize curves in $\mathbb{P}^n$ and to show that any rational curve is Cremona equivalent to a line. For this purpose we introduce the following definition:

Definition 2.1.4. Let X be an irreducible and reduced curve in $\mathbb{P}^n$. For a point $p \in X$ define

$$d_X(p) := \min\{\text{number of blowing ups of points to resolve } p\}.$$

Note that $d_X(p)$ is a well–defined and finite number.

Let $d(X) = \sum_{x \in X} d_X(x)$. Now, we prove the following:

Proposition 2.1.5. *Let X be a singular, irreducible and reduced curve in $\mathbb{P}^n$. Then there exists a birational map $\mu : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ such that $d(\mu(X)) < d(X)$.*

Proof. Let $x_1 \in X$ be a point such that $d_X(x_1) \geq 1$. Let $x_2, \dots, x_n$ be general points in $\mathbb{P}^n$ and L a general $(n-2)$–dimensional linear space in $\mathbb{P}^n$.

Let $\psi := \psi_n[L, x_1, \dots, x_n]$. From the description given in Definition 2.1.4 and for the generality of our choices, it is immediate that

$d_{\psi(X)}(\psi(x)) = d_X(x)$, for any $x \in X \setminus \{x_1\}$ and

$$\sum_{\psi(y_i)=x_1} d_{\psi(X)}(y_i) = d_X(x_1) - 1.$$

Then $d(\psi(X)) < d(X)$. $\square$

Corollary 2.1.6. *Let $X \subset \mathbb{P}^n$ be a curve and $n \geq 3$. Then X is Cremona equivalent to a smooth curve in $\mathbb{P}^n$.*

We have the following:

Proposition 2.1.7. *There exists at least one smooth rational curve $X \subset \mathbb{P}^n$ of degree d, for all $d \geq 1$, which is Cremona equivalent to a line.*

Proof. We prove the proposition by induction on d.
Initial case: $d \leq 2$. This result is immediate.
Induction step. Assume that the proposition is true for all $d \leq k$.
Let $X_{k+1} \subset \mathbb{P}^n$ be a rational curve of degree $k+1$. Assume that $k+1$ is even. Let $X \subset \mathbb{P}^n$ be a smooth rational curve of degree $\frac{k+1}{2}$ Cremona equivalent to a line. Let $x_1, \ldots, x_n$ be general points in $\mathbb{P}^n$, L a general $(n-2)$–dimensional linear space in $\mathbb{P}^n$ and $\psi := \psi_n[L, x_1, \ldots, x_n]$. Then $\psi(X)$ is the curve we were looking for. To conclude the case $k+1$ is odd, it is enough to consider X of degree $\frac{k+2}{2}$, $x_2, \ldots, x_n \in \mathbb{P}^n$ general and $x_1 \in X$ general. $\square$

Now, we can prove the following important result:

Theorem 2.1.8. *Let $Z \subset \mathbb{P}^n$ be a rational curve, $n \geq 3$. Then there exist a birational map $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ such that $\Phi(Z)$ is a line.*

Proof. Let $Z \subset \mathbb{P}^n$ be a rational curve of degree d. By corollary 2.1.6, we can assume Z is smooth.

By proposition 2.1.7, there exists at least a smooth rational curve $Z' \subset \mathbb{P}^n$ of degree d Cremona equivalent to a line. Let $r = \max\{d, n\}$. The curves $Y = \chi^n_Z(Z)$ and $Y' = \chi^n_{Z'}(Z')$ are projections of $N_r \subset \mathbb{P}^{2r-d}$ from linear spaces L, L' respectively. Consider a chain of linear spaces $L_0 = L, \ldots, L_t = L'$ such that $\dim(L_i \cap L_{i+1}) = 2r - d - n - 2$ and $\sharp\{L_i \cap L_{i+1} \cap N_r\} = \max\{0, r-d-1\}$. The existence of such a chain is immediate when $r = d$ and $L \cap N_r = \emptyset$. In the remaining case by construction $L_i \cap N_r = r - d$ and N_r has not $\mathbb{P}^k$ $(k+2)$-secant. In particular $L_i \cap N_r$ is in general position with respect to linear spaces. It is therefore possible to move the intersection $L_0 \cap N_r$ to $L_t \cap N_r$

one point at a time. Let $Y_i = \varphi_r^n[L_i, L_{i+1}](Y_{i-1})$ and $Y_0 = Y$. Note that $Y_t = Y' = \chi_{Z'}^n(Z')$ is Cremona equivalent to a line, therefore we conclude. □

2. Equivalent birational embeddings

In this Section, leaning on the ideas of the previous one, we want to understand when two birational embeddings of the same variety are equivalent up to a Cremona transformation of the projective space, in this case we say that they are Cremona equivalent. In this generalization, we work over an algebraically closed field k, but we do not need $\mathrm{char}(k) = 0$, that is because we do not use the Hironaka's resolution of singularities.

Let X be a projective irreducible and reduced variety over an algebraically closed field k and $\mathcal{L}$ a linear system on X. Assume that $\mathcal{L}$ is generated by $\{L_0, \dots, L_n\}$. Then the map $\varphi_{\mathcal{L}} : X \dashrightarrow \mathbb{P}(\mathcal{L}^*)$ is given by evaluating the sections of L_i at the points of X.

Definition 2.2.1. We say that $(X, \mathcal{L})$, or simply $\mathcal{L}$, is a birational embedding (in $\mathbb{P}^n$) if $\varphi_{\mathcal{L}} : X \dashrightarrow \mathbb{P}^n$ is birational onto the image. We say that $\varphi_{\mathcal{L}} = \varphi_{\mathcal{M}}$, for two birational embeddings, if there exists a dense open subset $U \subset X$ where $\varphi_{\mathcal{L}}$ and $\varphi_{\mathcal{M}}$ are both defined and have equal restriction.

Remark 2.2.2. Note that given a birational embedding $\mathcal{L}$ in $\mathbb{P}^n$ we can consider it also an embedding into $\mathbb{P}^{n+h}$ by adding h-times the zero section to get

$$\mathcal{L}' = \{L_0, \dots, L_n, 0, \dots, 0\}$$

In all that follows we apply, mainly without mention, this construction to compare birational embeddings into different projective spaces.

We are interested in studying birational embeddings of a fixed variety X. We therefore identify $\mathcal{L}$ with $\mathcal{O}_{\varphi_{\mathcal{L}}(X)}(1)$ via $(\varphi_{\mathcal{L}})_*^{-1}(\mathcal{O}_{\mathbb{P}^n}(1))$.

Let $D \subset X$ be a divisor and consider the linear system $\mathcal{L}_D = \{DL_0, \dots, DL_n\}$. Then we have $\varphi_{\mathcal{L}} = \varphi_{\mathcal{L}_D}$. In what follows we identify $\mathcal{L}$ and $\mathcal{L}_D$.

We extend Definition 2.0.5 to the following:

Definition 2.2.3. Let $\mathcal{L}$ in $\mathbb{P}^n$ and $\mathcal{G} = \{G_0, \dots, G_r\}$ in $\mathbb{P}^r$ be two birational embeddings, assume that $n \geq r$ and let

$$\mathcal{G}' = \{G_0, \dots, G_r, 0, \dots, 0\},$$

obtained by adding $(n-r)$-times the zero section. We say that $\mathcal{L}$ is Cremona equivalent to $\mathcal{G}$, or simply equivalent, if there exists a birational map $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ such that

$$\varphi_{\mathcal{L}} = \Phi \circ \varphi_{\mathcal{G}'}$$

Such a Φ is called a (Cremona) equivalence between $\mathcal{L}$ and $\mathcal{G}$.

Remark 2.2.4. The relation introduced is an equivalence relation on the birational linear systems of a fixed variety. We never ask the linear system neither to be complete nor to be minimally generated by the L_i's.

The equivalence Φ has to be defined on the general point of $\varphi_{\mathcal{G}'}(X)$.

Let $\mathcal{L}$ and $\mathcal{G}$ be two linear systems on X. Then we have the following commutative diagram

$$\begin{tikzcd}
 & \mathbb{P}^N \arrow[dl, dashed, "\pi_L"'] \arrow[dr, dashed, "\pi_G"] & \\
\mathbb{P}^n & X \arrow[l, dashed, "\varphi_{\mathcal{L}}"] \arrow[r, dashed, "\varphi_{\mathcal{G}}"'] \arrow[u, dashed, "\varphi_{\mathcal{L}+\mathcal{G}}"'] & \mathbb{P}^n
\end{tikzcd}$$

where $\mathcal{L}+\mathcal{G} = \{L_iG_j\}$, with $i,j = 0,\dots,n$, and

$$\pi_L \circ \varphi_{\mathcal{L}+\mathcal{G}} = \varphi_{\mathcal{L}_{G_0}} \qquad \pi_G \circ \varphi_{\mathcal{L}+\mathcal{G}} = \varphi_{\mathcal{G}_{L_0}}$$

In other terms L and G are linear spaces spanned, respectively by $\{G_jL_i\}$, with $j = 1,\dots,n$, $i = 0,\dots,n$, and $\{G_jL_i\}$ with $i = 1,\dots,n$, $j = 0,\dots,n$.

This diagram allows us to look at two different embeddings as projections from a common one.

Remark 2.2.5. The projections π_L, π_G in the above diagram play the role of the two projections of rational normal curve $N_r \subset \mathbb{P}^{2r-d}$ from the linear spaces L, L' in the proof of Theorem 2.1.8.

A natural way to construct a birational self-map of $\mathbb{P}^n$ is to consider a hypersurface of degree k in $\mathbb{P}^{n+1}$ with two points of multiplicity exactly $k-1$. Then the projections from the singular points build up the required self-map. The following Lemma allow us to use this trick in a wide contest.

Lemma 2.2.6. *Let $Y \subset \mathbb{P}^{n+1}$ be an irreducible reduced variety and q_1, q_2 two points in $\mathbb{P}^{n+1}$. Let CY_i be the cone over Y with vertex q_i. Assume that* $\dim Y \leq n-2$, $h^0(\mathcal{I}_Y(1)) \neq 0$ *and* $CY_i \not\subset \operatorname{Bs}|\mathcal{I}_Y(1)|$. *Then* *for* $k \gg 0$ *there exists an irreducible reduced hypersurface* $S \in |\mathcal{I}_Y(k)|$ *with:*

- $\mathrm{mult}_{q_i} S = k-1$, *for* $i = 1, 2$,
- $S \not\supset CY_i$.

Proof. Let $l = \langle q_1, q_2 \rangle$ be the line spanned by the q_i's and consider the projections

$$\pi_{q_1} : \mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^n, \ \pi_{q_2} : \mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^n, \ \pi_l : \mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^{n-1}$$

Let $\tilde{Y} = \pi_l(Y)$, $Y_i = \pi_{q_i}(Y)$ be varieties. Then we have the following diagram

$$\begin{tikzcd}
 & Y \subset \mathbb{P}^{n+1} \arrow[dl, dashed, "\pi_{q_1}"'] \arrow[dr, dashed, "\pi_{q_2}"] \arrow[dd, dashed, "\pi_l"'] & \\
Y_1 \subset \mathbb{P}^n \arrow[dr, dashed] & & \mathbb{P}^n \supset Y_2 \arrow[dl, dashed] \\
 & \tilde{Y} \subset \mathbb{P}^{n-1} &
\end{tikzcd}$$

Let us consider $D = (d = 0) \subset \mathbb{P}^{n-1}$ a hypersurface of degree δ with $\tilde{Y} \subset D$. Let $H = (h = 0) \in |\mathcal{I}_Y(1)|$ be a general hyperplane. Define

$$S = (dg_1 + hg_2 = 0) \subset \mathbb{P}^{n+1}$$

where:

g_1 is general of degree $k - \delta$ and multiplicity $k - \delta - 1$ at p_i;
g_2 is general with $\mathrm{mult}_{q_i} hg_2 = k - 1$.

It is easy to check that S satisfies all the requirements. □

We are ready to prove our main result about Cremona equivalent birational embeddings.

Theorem 2.2.7. *Let* X *be an irreducible and reduced projective variety of dimension* r *over an algebraically closed field* k*. Then two birational embeddings in* $\mathbb{P}^n$ *are Cremona equivalent as long as* $n \geq r+2$.

Proof. Let X be an irreducible reduced projective variety. Let $\mathcal{L}$ and $\mathcal{G}$ be two birational embeddings in $\mathbb{P}^n$. Keep in mind that they are both projections of $\mathcal{L} + \mathcal{G}$ and

$$\mathcal{L} = \mathcal{L}_{G_0} = \{L_0G_0, \ldots, L_nG_0\}, \quad \mathcal{G} = \mathcal{G}_{L_0} = \{L_0G_0, \ldots, L_0G_n\}.$$

We want to construct a sequence of Cremona equivalent linear systems $\{\mathcal{A}_i\}$, for $i = 0, \ldots n$, with

- $\mathcal{A}_0 = \mathcal{L}_{G_0} = \{L_0G_0, \ldots, L_nG_0\}$;
- $\mathcal{A}_i = \{L_0G_0, \ldots, L_0G_i, A^i_{i+1}, \ldots, A^i_n\}$, for some $A^i_j \in |\mathcal{L} + \mathcal{G}|$, $j =$ i+1$, \ldots, n$;
- $\mathcal{A}_n = \mathcal{G}_{L_0} = \{L_0G_0, \ldots, L_0G_n\}$.

To prove the theorem we give a recipe that builds $\mathcal{A}_{i+1}$ from $\mathcal{A}_i$. Let $\mathcal{H}_i = \{\mathcal{A}_i, L_0G_{i+1}\}$ be a linear system and $\varphi_{\mathcal{H}_i} : X \to \mathbb{P}(\mathcal{H}^*_i) = \mathbb{P}^{n+1}$ the associated embedding. Then we have $X_i = \varphi_{\mathcal{A}_i}(X) \subset (x_{n+1} = 0) \subset \mathbb{P}^{n+1}$. Let $Z \subset \mathbb{P}^{n+1}$ be the cone over X_i with vertex $q_1 = [0, \ldots, 0, 1]$ and $Y_i := Z \cap H$ a general hyperplane section of Z. Then we have

$$\pi_{q_1}(Y_i) = X_i$$

and Y_i birational to X. Let $q_2 \in (x_0 = \ldots = x_i = x_{n+1} = 0)$ be a general point. In particular $q_2 \notin H$ hence the projection $\pi_{q_2|Y_i}$ is birational onto the image. Let $X_{i+1} := \pi_{q_2}(Y_i)$, then X_{i+1} is birational to X and we define $\mathcal{A}_{i+1} := \mathcal{O}_{X_{i+1}}(1)$, keep in mind Remark 2.2.2. The choice of q_2 gives

$$\mathcal{A}_{i+1} = \{L_0G_0, \ldots, L_0G_{i+1}, A^{i+1}_{i+2}, \ldots, A^{i+1}_n\}$$

for some elements $A^{i+1}_j \in |\mathcal{L} + \mathcal{G}|$.

This reads in the following diagram

$$\begin{tikzcd} & Y_i \subset \mathbb{P}^{n+1} \arrow[dl, dotted, "\pi_{q_1}"'] \arrow[dr, dotted, "\pi_{q_2}"] & \\ \varphi_{\mathcal{A}_i}(X) \subset \mathbb{P}^n & & \mathbb{P}^n \supset \varphi_{\mathcal{A}_{i+1}}(X) \end{tikzcd}$$

By construction $h^0(\mathcal{I}_{Y_i}(1)) \neq 0$ and $\mathrm{Bs}\,|\mathcal{I}_{Y_i}(1)| \not\ni q_i$, for $i = 1, 2$. By hypothesis we have $\dim Y \leq n - 2$, then by Lemma 2.2.6 there exists an irreducible hypersurface $S \in |\mathcal{I}_{Y_i}(k)|$ with $\mathrm{mult}_{q_i} S = k - 1$ and not containing the cones over Y_i with vertex both q_1 and q_2. In particular

$$\pi_{q_1|S} \text{ and } \pi_{q_2|S}$$

are birational maps to $\mathbb{P}^n$ and $\pi^{-1}_{q_1|S}$, $\pi^{-1}_{q_2|S}$ are defined on the general point of $\varphi_{\mathcal{A}_i}(X)$, $\varphi_{\mathcal{A}_{i+1}}(X)$ respectively. Define the map $\Phi_i : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ as follows

$$\Phi_i = \pi_{q_2|S} \circ \pi^{-1}_{q_1|S}$$

By construction Φ_i is an equivalence between $\mathcal{A}_i$ and $\mathcal{A}_{i+1}$. $\square$

We want to stress some by-products of the main Theorem 2.2.7.

Corollary 2.2.8. *Let $X \subset \mathbb{P}^n$ be a subvariety of codimension at least* 2.

If X is rational then it is Cremona equivalent to a linear space.

If X is birational to a smooth subvariety of $\mathbb{P}^n$. Then it is possible to resolve the singularities of X with a Cremona transformation.

The Cremona group of $\mathbb{P}^n$ contains $\mathrm{Bir}(X)$, *the group of birational transformations of X.*

We like to look at the Theorem 2.2.7 as a way to say that the Cremona group of $\mathbb{P}^n$ is really huge.

CHAPTER 3

Divisorial embeddings

In Chapter 2, we have shown that two birational embeddings in $\mathbb{P}^n$ of the same irreducible, reduced, projective variety X of dimension r are *Cremona equivalent* as long as $n \geq r+2$, see Theorem 2.2.7.

In this Chapter, we want to understand whether two divisorial birational embeddings are Cremona equivalent.

In the following, we work over an algebraically closed field k, with char $(k) = 0$.

If we consider a rational variety, it is not difficult to give examples of rational hypersurfaces that are not Cremona equivalent to a hyperplane: let $C \subset \mathbb{P}^2$ be a rational curve with only ordinary double points. If $\deg C \geq 6$ then C is never Cremona equivalent to a line (see Sections 2, 3 for the precise statement). It is easy to produce such curves by projecting a divisor of type $(1, a)$ in a smooth quadric. One can construct similar examples in arbitrary dimension. Hence, in general, birationally equivalent divisors are not Cremona equivalent.

The argument of Noether–Fano inequalities gives us a way to explicitly state that that two divisorial embeddings are in general not Cremona equivalent. Indeed, we have the following:

Lemma 3.0.1. *Let X be an irreducible and reduced projective variety of dimension $n-1$. Let $\mathcal{L}$ and $\mathcal{G}$ be two birational embeddings in $\mathbb{P}^n$ such that $\deg \varphi_{\mathcal{L}}(X) = d$ and $\deg \varphi_{\mathcal{G}}(X) = d'$, where $d' < d$. If $\mathcal{L}$ and $\mathcal{G}$ are Cremona equivalent, then the pair $\left(\mathbb{P}^n, \frac{n+1}{d}\, \varphi_{\mathcal{L}}(X)\right)$ has not canonical singularities, see* Definition 1.2.4.

Proof. Let $\varphi_{\mathcal{L}}(X) = Y$ and $\varphi_{\mathcal{G}}(X) = Y'$. Then $Y, Y' \subset \mathbb{P}^n$ are hypersurfaces of degree d, d' respectively, with $d' < d$. Since $\mathcal{L}$ and $\mathcal{G}$ are two Cremona equivalent birational embeddings in $\mathbb{P}^n$, hence there exists a birational map $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ such that $\Phi(Y) = Y'$.

Consider the following resolution of Φ:

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\ Y \subset \mathbb{P}^n \arrow[rr, dashed, "\Phi"] & & \mathbb{P}^n \supset Y' \end{tikzcd}$$

then we have:

$$\mathcal{O}_Z \sim p^*\Big(\mathcal{O}\Big(K_{\mathbb{P}^n} + \frac{n+1}{d}Y\Big)\Big) = K_Z + \frac{n+1}{d}Y_Z - \sum a_i E_i$$

and

$$q^*\Big(\mathcal{O}\Big(K_{\mathbb{P}^n} + \frac{n+1}{d}Y'\Big)\Big) = K_Z + \frac{n+1}{d}Y_Z - \sum b_i F_i$$

where E_i, respectively F_i, are p, respectively q, exceptional divisors. Let $l \subset \mathbb{P}^n$ be a general line in the right hand side $\mathbb{P}^n$. Since $d' < d$, we get

$$0 > q^{-1}l \cdot \Big(q^*\Big(\mathcal{O}\Big(K_{\mathbb{P}^n} + \frac{n+1}{d}Y'\Big)\Big) + \sum b_i F_i\Big) = (\sum a_i E_i) \cdot q^{-1}l.$$

This proves that at least one $a_i < 0$, then the singularities of the pair $\big(\mathbb{P}^n, \frac{n+1}{d}Y\big)$ are not canonical. □

With Lemma 3.0.1, we can easily shown the following:

Theorem 3.0.2. *Let X be an irreducible and reduced projective variety with $\dim X \leq 2$. Then, there exist infinitely many Cremona inequivalent embeddings of X in $\mathbb{P}^{\dim X+1}$.*

Proof. Assume $\dim X = 2$.
Let A_1, A_2 be two very ample divisors on X of degrees $d_i = \deg A_i \geq 12$, for $i = 1, 2$, with $d_1 \neq d_2$. Then X can be embedded in $\mathbb{P}(|A_i|)$ as a surface X_i of degree d_i, for $i = 1, 2$. We consider the generic projection $\pi_i : \mathbb{P}(|A_i|) \dashrightarrow \mathbb{P}^3$ such that $\pi_i(X_i) = Y_i$, for $i = 1, 2$. We have that Y_i is an irreducible and reduced surface in $\mathbb{P}^3$ of degree d_i and, by a classical result, if $p \in \operatorname{Sing} Y_i$, then $\operatorname{mult}_p Y_i \leq 3$, for $i = 1, 2$.

Therefore, we have obtained two birational embeddings $\mathcal{L}$ and $\mathcal{G}$ of X in $\mathbb{P}^3$ such that $\varphi_{\mathcal{L}}(X) = Y_1$ and $\varphi_{\mathcal{G}}(X) = Y_2$. Since $d_i \geq 12$ and $\operatorname{mult}_p Y_i \leq 3$, for $i = 1, 2$, we have that the pairs $\Big(\mathbb{P}^3, \frac{4}{d_1}Y_1\Big)$ and $\Big(\mathbb{P}^3, \frac{4}{d_2}Y_2\Big)$ have canonical singularities. Finally, since $d_1 \neq d_2$, by Lemma 3.0.1, the birational embeddings $\mathcal{L}$ and $\mathcal{G}$ cannot be Cremona equivalent.

The case of curves can be treated equivalently. □

Remark 3.0.3. Using results of Mather [**M1, M2, M3**], we can generalized Theorem 3.0.2 with $\dim X \leq 14$.

1. Curves of $\mathbb{P}^2$ of arbitrary genus

Let $C \subset \mathbb{P}^2$ be a curve. We shall study log pairs $(\mathbb{P}^2, \alpha\, C)$. In particular, we shall apply birational transformations to obtain a log pair $(S, \alpha\, \widetilde{C})$, which is a model of $(\mathbb{P}^2, \alpha\, C)$ with canonical singularities, having the log canonical divisor $K_S + \alpha\, \widetilde{C}$ nef and Kodaira dimension $\overline{\kappa}(S, \alpha\, \widetilde{C}) \leq 1$. In this way, we can give a classification of such pairs in terms of a birational equivalence between $(\mathbb{P}^2, \alpha\, C)$ and $(S, \alpha\, \widetilde{C})$.

A classical problems of birational geometry of plane curves is to give some birational classifications of them in terms of birational invariants. For example, in [**Ii**], Iitaka study the non–singular, relatively minimal models of the pairs (S, C), where C is a curve on the surface S. Note that, the notion of relatively minimal model of Iitaka is equivalent to the notion of &–minimal model of Dicks in [**Di**], see Section 2. Moreover, Iitaka gives some birational characterization of pairs $(\mathbb{P}^2, C)$ in terms of birational invariants.

Let $C \subset \mathbb{P}^2$ be an irreducible and reduced curve of genus g and degree d. In the following, we denote $m_i = \operatorname{mult}_{p_i} C$ the multiplicities of singular points (including infinitely near singular points) of C, for $i = 1, \ldots, k$, such that $m_1 \geq m_2 \geq m_3 \geq \ldots \geq m_k \geq 2$. Using the notations of Iitaka, we have the following:

Definition 3.1.1 ([**Ii**])**.** A curve $C \subset \mathbb{P}^2$ is said to be a *curve of type* $[d; m_1, \ldots, m_k]$, where d is the degree of C and the multiplicities of all the singular points (including infinitely near singular points) are $m_1, \ldots, m_k$, where $m_1 \geq m_2 \geq m_3 \geq \ldots \geq m_k \geq 2$. Whenever $m_1 = m_2 = \cdots = m_f$, the symbol $[d; m_1^f, m_{f+1}, \ldots, m_k]$. If C is a smooth curve of degree d, C is said to be of type $[d; 1]$.

Notation 3.1.2. It is well known that given a rational surface S, after contracting all exceptional curves on S successively, we have relatively minimal models of S, that are the projective plane $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ or a $\mathbb{P}^1$–bundle over $\mathbb{P}^1$ with a section C_0 of negative self–intersection number. The last surface is denoted by a symbol $\mathbb{F}_a$, where $C_0^2 = -a$. In particular, $\mathbb{F}_0$ denotes the product surface $\mathbb{P}^1 \times \mathbb{P}^1$. The Picard group of $\mathbb{F}_a$, for $a \geq 0$, is generated by the section C_0 and a fiber f of the $\mathbb{P}^1$–bundle.

Let $\widetilde{C}$ be an irreducible curve on $\mathbb{F}_a$. Then there exist integers α and β such that $\widetilde{C} \sim \alpha\, C_0 + \beta\, f$.

We have that $\widetilde{C}\cdot f = \alpha$ and $\widetilde{C}\cdot C_0 = -a\,\alpha+\beta$. Suppose that $\widetilde{C} \neq C_0$, then $\widetilde{C}\cdot C_0 \geq 0$, hence $\beta \geq a\,\alpha$. If $a > 0$, we have that $C_0^2 = -a < 0$ and such a section C_0 is uniquely determined. If $a = 0$, we denote C_0, f by F_1, F_2 respectively, where F_i is the fiber of the projection $\pi_i : \mathbb{F}_0 \to \mathbb{P}^1$, for $i = 1, 2$. We can assume that $\beta \geq \alpha$. Thus α and β are uniquely determined for a given curve $\widetilde{C}$ on $\mathbb{F}_a$.

Let $p \in \widetilde{C} \subset \mathbb{F}_a$ be a point such that $\operatorname{mult}_p \widetilde{C} = m$. Consider $\mu : \mathbb{F}_a \dashrightarrow \mathbb{F}_{a\pm 1}$ an elementary transformation with center p. In the following, we denote by $\widetilde{C}$ again the strict transform of $\widetilde{C}$ with respect to μ. We have two cases.

- If $p \in C_0 \subset \mathbb{F}_a$, then we obtain an elementary transformation $\mu : \mathbb{F}_a \dashrightarrow \mathbb{F}_{a+1}$, where $\widetilde{C} \sim \alpha\,C_0 + (\beta+\alpha-m)f \subset \mathbb{F}_{a+1}$. The map μ introduces a new point $q \in \widetilde{C} \subset \mathbb{F}_{a+1}$ such that $q \notin C_0 \subset \mathbb{F}_{a+1}$ and $\operatorname{mult}_q \widetilde{C} = \alpha - m$.
- If $p \notin C_0 \subset \mathbb{F}_a$, then we obtain an elementary transformation $\mu : \mathbb{F}_a \dashrightarrow \mathbb{F}_{a-1}$, where $\widetilde{C} \sim \alpha\,C_0 + (\beta - m)f \subset \mathbb{F}_{a-1}$. The map μ introduces a new point $q \in \widetilde{C} \subset \mathbb{F}_{a-1}$ such that $q \in C_0 \subset \mathbb{F}_{a-1}$ and $\operatorname{mult}_q \widetilde{C} = \alpha - m$.

Using the notations of Iitaka, we have the following:

Definition 3.1.3 ([**Ii**])**.** A curve $C \sim \alpha\,C_0 + \beta\,f \subset \mathbb{F}_a$, with $a \geq 0$, is said to be a *curve of type* $[\alpha * \beta, a; m_1, \ldots, m_k]$. Whenever $m_1 = m_2 = \cdots = m_f$, the symbol $[\alpha * \beta, a; m_1^f, m_{f+1}, \ldots, m_k]$. If C is a smooth curve, we say that C is of type $[\alpha * \beta, a; 1]$.

Now, we will characterize plane curves. A result similar to the following is given in other terms by Iitaka in [**Ii**].

Theorem 3.1.4. *An irreducible and reduced curve $C \subset \mathbb{P}^2$ is birational to one of the following:*

a) *a line;*
b) *a curve $\widetilde{C}$, where the log pair $\left(\mathbb{P}^2, \frac{3}{d}\widetilde{C}\right)$ is a model with canonical singularities, having the log canonical divisor $K_{\mathbb{P}^2} + \frac{3}{d}\widetilde{C} \sim \mathcal{O}$ nef and $\overline{\kappa}\left(\mathbb{P}^2, \frac{3}{d}\widetilde{C}\right) = 0$;*
c) *a curve $\widetilde{C} \subset \mathbb{F}_a$, with $\widetilde{C} \sim \alpha\,C_0 + \beta\,f$, where the log pair $\left(\mathbb{F}_a, \frac{2}{\alpha}\widetilde{C}\right)$ is a model with canonical singularities and terminal singularities in a neighborhood of the exceptional curve*

$C_0 \subset \mathbb{F}_a$, having the log canonical divisor $K_{\mathbb{F}_a} + \frac{2}{\alpha}\widetilde{C}$ nef and $\overline{\kappa}\big(\mathbb{F}_a, \frac{2}{\alpha}\widetilde{C}\big) \leq 1$.

Proof. We prove the theorem by induction on $\deg C = d$.

FIRST STEP. $d = 1$. We have that $C \subset \mathbb{P}^2$ is a line.

INDUCTION STEP. Suppose that the theorem holds for any irreducible and reduced curve $C \subset \mathbb{P}^2$ of degree $\deg C < d$.

Let $C \subset \mathbb{P}^2$ be an irreducible and reduced curve of type $[d; m_1, \ldots, m_k]$. We have to study the following cases.

First case. $m_1 \leq \frac{d}{3}$. Since $m_i \leq \frac{d}{3}$, for any $i = 1, \ldots, k$, we have that the log pair $\big(\mathbb{P}^2, \frac{3}{d}C\big)$ has canonical singularities. Moreover, since $K_{\mathbb{P}^2} + \frac{3}{d}C \sim \mathcal{O}$, we have that $K_{\mathbb{P}^2} + \frac{3}{d}C$ is nef.

Then, the log pair $\big(\mathbb{P}^2, \frac{3}{d}C\big)$ is a model of $\big(\mathbb{P}^2, \frac{3}{d}C\big)$ with canonical singularities having the log canonical divisor $K_{\mathbb{P}^2} + \frac{3}{d}C$ nef and Kodaira dimension $\overline{\kappa}\big(\mathbb{P}^2, \frac{3}{d}C\big) = 0$.

Second case. $m_1 > \frac{d}{3}$. It immediately follows that $\frac{3}{d} < \frac{2}{d-m_1}$.

If $m_1 = d-1$, then $C \subset \mathbb{P}^2$ is a rational, irreducible and reduced curve having one point p of multiplicity $d-1$. Then there exists a Cremona transformation $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, with center p, p_1, p_2, where $\operatorname{mult}_{p_i} C = 1$, for $i = 1, 2$, such that $\deg \omega(C) = d - 1 < d$, hence, by induction step, we conclude our proof. Moreover, in Section 2, we will show that a such curve is *Cremona equivalent to a line* (see Lemma 3.2.8).

Therefore, in the following, we can assume that $\frac{3}{d} < \frac{2}{d-m_1} \leq 1$.

Let $\nu : \mathbb{F}_1 \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at $p_1 \in C$ such that $\operatorname{mult}_{p_1} C = m_1$. Consider the log pair $\big(\mathbb{F}_1, \frac{2}{d-m_1}\widetilde{C}\big)$, where $\widetilde{C} \sim (d-m_1)C_0 + d\,f \subset \mathbb{F}_1$. The choice of the coefficient $\frac{2}{d-m_1}$ ensures the existence a sequence of elementary transformations $\phi : \mathbb{F}_1 \dashrightarrow \mathbb{F}_a$ such that the log pair $\big(\mathbb{F}_a, \frac{2}{d-m_1}\overline{C}\big)$, where $\overline{C}$ is the strict transform of $\widetilde{C}$ with respect to ϕ, has canonical singularities and terminal singularities in a neighborhood of $C_0 \subset \mathbb{F}_a$.

We have that $\overline{C} \sim (d-m_1)C_0 + \beta\,f$, where $\beta \geq a(d-m_1)$. Since $d - m_1 \geq 2$, we get that $\overline{C}$ is not a section of $\mathbb{F}_a$, i.e. $\overline{C} \neq C_0$, then $\overline{C}$ is nef. We have the following subcases.

i) Let $a \geq 2$. We have that $K_{\mathbb{F}_a} + \frac{2}{d-m_1}\overline{C}$ is nef.

Therefore, the pair $\big(\mathbb{F}_a, \frac{2}{d-m_1}\overline{C}\big)$ is a model of $\big(\mathbb{P}^2, \frac{2}{d-m_1}C\big)$, with canonical singularities and terminal singularities in a neighborhood of C_0, having the log canonical divisor $K_{\mathbb{F}_a}+\frac{2}{d-m_1}\overline{C}$ nef and Kodaira dimension $\overline{\kappa}\big(\mathbb{F}_a, \frac{2}{d-m_1}\overline{C}\big) \leq 1$.

ii) Let $a = 1$. If $K_{\mathbb{F}_1}+\frac{2}{d-m_1}\overline{C}$ is nef, we have that the pair $\big(\mathbb{F}_1, \frac{2}{d-m_1}\overline{C}\big)$ is a model of $\big(\mathbb{P}^2, \frac{2}{d-m_1}C\big)$, with canonical singularities and terminal singularities in a neighborhood of C_0, having the log canonical divisor $K_{\mathbb{F}_1} + \frac{2}{d-m_1}\overline{C}$ nef and Kodaira dimension $\overline{\kappa}\big(\mathbb{F}_1, \frac{2}{d-m_1}\overline{C}\big) \leq 1$.

On the other hand, if $K_{\mathbb{F}_1} + \frac{2}{d-m_1}\overline{C}$ is not nef, we have that $\frac{2}{d-m_1}\overline{C} \cdot C_0 < 1$, then $\frac{2}{d-m_1}(-d + m_1 + \beta) < 1$. From this inequality, we get $\frac{2\beta}{d-m_1} < 3$ and, since $\frac{d-m_1}{2} < \frac{d}{3}$, we have $\beta < d$. Hence, $\overline{C} \sim (d - m_1)C_0 + \beta f \subset \mathbb{F}_1$, where $\beta < d$. Let $\mathrm{cont}_{C_0} : \mathbb{F}_1 \to \mathbb{P}^2$ be the contraction of C_0, then $\mathrm{cont}_{C_0}(\widetilde{C}) = C' \subset \mathbb{P}^2$ is an irreducible and reduced curve of degree $\beta < d$. Therefore, there exists a Cremona transformation $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\omega(C) = C'$, where $\deg C' = \beta < d$ and, by induction step, we can conclude our proof.

- Let $a = 0$. If $K_{\mathbb{F}_0}+\frac{2}{d-m_1}\overline{C}$ is nef, we have that the pair $\big(\mathbb{F}_0, \frac{2}{d-m_1}\overline{C}\big)$ is a model of $\big(\mathbb{P}^2, \frac{2}{d-m_1}C\big)$, with canonical singularities and terminal singularities in a neighborhood of C_0, having the log canonical divisor $K_{\mathbb{F}_0} + \frac{2}{d-m_1}\overline{C}$ nef and Kodaira dimension $\overline{\kappa}\big(\mathbb{F}_0, \frac{2}{d-m_1}\overline{C}\big) \leq 1$.

On the other hand, if $K_{\mathbb{F}_0} + \frac{2}{d-m_1}\overline{C}$ is not nef, we have that $\frac{2}{d-m_1}\overline{C} \cdot F_1 < 2$, i.e. $\beta = \overline{C} \cdot F_1 < d - m_1$. In this case, we consider the log pair $\big(\mathbb{F}_0, \frac{2}{\beta}\overline{C}\big)$ such that $K_{\mathbb{F}_0} + \frac{2}{\beta}\overline{C}$ is nef and we apply the above arguments to this pair to obtain a model with canonical singularities and terminal singularities in a neighborhood of the exceptional curve. Since $\beta < d - m_1$, after finitely many steps, we conclude our proof. □

We have the following:

Remark 3.1.5. Consider the case c) of Theorem 3.1.4, in which the log pair $\big(\mathbb{F}_a, \frac{2}{\alpha}\widetilde{C}\big)$ has terminal singularities and log canonical divisor $K_{\mathbb{F}_a} + \frac{2}{\alpha}\widetilde{C}$ nef. We have that $\big(\mathbb{F}_a, \frac{2}{\alpha}\widetilde{C}\big)$ is a log minimal model, which is unique in a fixed logarithmic class.

Moreover, if $\overline{\kappa} = 1$, then the pair $\big(\mathbb{F}_a, \big(\frac{2}{\alpha} + \varepsilon\big)\widetilde{C}\big)$, for $0 < \varepsilon << 1$, is a log canonical model of $\big(\mathbb{P}^2, \big(\frac{2}{\alpha} + \varepsilon\big)C\big)$ and it is unique.

On the other hand, if $\overline{\kappa} = 0$, we have that $\overline{\kappa}\big(\mathbb{F}_a, \big(\frac{2}{\alpha} + \varepsilon\big)\widetilde{C}\big) = 2$, but the log canonical divisor $K_{\mathbb{F}_a} + \big(\frac{2}{\alpha} + \varepsilon\big)\widetilde{C}$ is not necessarily ample. Hence, the log canonical model of $\big(\mathbb{P}^2, \big(\frac{2}{\alpha} + \varepsilon\big)C\big)$ is either $\big(\mathbb{F}_a, \big(\frac{2}{\alpha} + \varepsilon\big)\widetilde{C}\big)$ or $\big(S(0, 2), \big(\frac{2}{\alpha} + \varepsilon\big)\widetilde{C}\big)$, where $S(0, 2)$ is the quadric cone in $\mathbb{P}^3$.

Consider $C \subset \mathbb{P}^2$ an irreducible and reduced curve as above and such that $m_1 + m_2 + m_3 > d$. If $m_1 + m_2 + m_3 > d$, in general, we cannot state that there exists a Cremona transformation $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\deg \omega(C) < d$. We give the following examples. We remark that, in these examples, it is possible to use also the results of Iitaka in [**Ii**].

Example 3.1.6. *Let $C \subset \mathbb{P}^2$ be an irreducible and reduced curve of degree 7 and genus $g = 7$ having p_1, p_2, p_3 as singularities, where p_1, p_2, p_3 are infinitely near points such that $m_1 = \operatorname{mult}_{p_1} C = 4$, $m_i = \operatorname{mult}_{p_i} C = 2$, for $i = 1, 2$.*

We have that the log pair $\big(\mathbb{F}_3, \frac{2}{3}\widetilde{C}\big)$ is a model of $\big(\mathbb{P}^2, \frac{2}{3}C\big)$ having terminal (and hence canonical) singularities, with $K_{\mathbb{F}_3} + \frac{2}{3}\widetilde{C}$ nef and $\overline{\kappa}\big(\mathbb{F}_3, \frac{2}{3}\widetilde{C}\big) = 1$, like in Theorem 3.1.4.

Moreover, since $\widetilde{C}$ is smooth, we have that $\big(\mathbb{F}_3, \big(\frac{2}{3} + \varepsilon\big)\widetilde{C}\big)$, where $0 < \varepsilon << 1$, is a log canonical model of $\big(\mathbb{P}^2, \big(\frac{2}{3} + \varepsilon\big)C\big)$.

Suppose that there exists a Cremona transformation $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\omega(C) = C'$, where C' is an irreducible and reduced plane curve of degree 6 and genus 7. Then C' has either one 3–ple point or three double points as singularities.

Let $C' \subset \mathbb{P}^2$ be an irreducible and reduced plane curve of degree 6 having only one 3–ple point as singularity.

We have that the log pair $\big(\mathbb{F}_1, \frac{2}{3}\widetilde{C}'\big)$ is a model of $\big(\mathbb{P}^2, \frac{2}{3}C'\big)$ having terminal (and hence canonical) singularities, with $K_{\mathbb{F}_1} + \frac{2}{3}\widetilde{C}'$ nef and $\overline{\kappa}\big(\mathbb{F}_1, \frac{2}{3}\widetilde{C}'\big) = 1$, like in Theorem 3.1.4. *Moreover, since $\widetilde{C}'$ is smooth, we have that $\big(\mathbb{F}_1, \big(\frac{2}{3} + \varepsilon\big)\widetilde{C}'\big)$, where $0 < \varepsilon << 1$, is a log canonical model of $\big(\mathbb{P}^2, \big(\frac{2}{3} + \varepsilon\big)C'\big)$.*

Since C, C' are Cremona equivalent curves and since the log canonical model is unique, we have that the pairs $\big(\mathbb{F}_3, \big(\frac{2}{3}+\varepsilon\big)\widetilde{C}\big)$, $\big(\mathbb{F}_1, \big(\frac{2}{3}+\varepsilon\big)\widetilde{C}'\big)$ must be isomorphic, which is a contradiction. Therefore, there not exists a Cremona transformation $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\omega(C) = C'$.

Let $C' \subset \mathbb{P}^2$ be an irreducible and reduced plane curve of degree 6 having three double points as singularities.

We have that the log pair $\big(\mathbb{P}^2, \frac{1}{2}C'\big)$ is a model of $\big(\mathbb{P}^2, \frac{1}{2}C'\big)$ having canonical singularities, with $K_{\mathbb{P}^2} + \frac{1}{2}C' \sim \mathcal{O}$ nef and $\overline{\kappa}\big(\mathbb{P}^2, \frac{1}{2}C'\big) = 0$, like in Theorem 3.1.4.

Since $\frac{1}{2} < \frac{2}{3}$, then $\big(\mathbb{F}_3, \frac{1}{2}\widetilde{C}\big)$ is a model of $\big(\mathbb{P}^2, \frac{1}{2}C\big)$ with terminal (and hence canonical) singularities with $\overline{\kappa}\big(\mathbb{F}_3, \frac{1}{2}\widetilde{C}\big) < 0$. Since $\big(\mathbb{P}^2, \frac{1}{2}C\big)$ has a model with canonical singularities and log Kodaira dimension $\overline{\kappa} < 0$, *while* $\left(\mathbb{P}^2, \frac{1}{2}C'\right)$ *has a model with canonical singularities and log Kodaira dimension* $\overline{\kappa} = 0$, *we have that there not exists a Cremona transformation* $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ *such that* $\omega(C) = C'$.

Therefore, there not exists a Cremona transformation $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ *such that* $\deg \omega(C) < 7$.

Example 3.1.7. *Let* $D_1, D_2 \subset \mathbb{F}_3$ *be two irreducible and reduced curve such that* $D_i \sim 3C_0 + 11f \subset \mathbb{F}_3$, *for* $i = 1, 2$. *Assume that* D_1 *has a unique singular point* p *with* $\operatorname{mult}_p D_1 = 2$ *such that* $p \in C_0$. *Moreover, we assume that* D_2 *has a unique singular point* q *with* $\operatorname{mult}_q D_2 = 2$ *such that* $q \notin C_0$.

We have that D_1, D_2 *are birational to irreducible and reduced curves of the same type* $[9; 6, 2^3]$ *in* $\mathbb{P}^2$, *see* Definition 3.1.1, *but, by* Theorem 3.1.4, D_1, D_2 *have two different models.*

Via $elm_p : \mathbb{F}_3 \dashrightarrow \mathbb{F}_4$, *we have that the log pair* $\left(\mathbb{F}_4, \frac{2}{3}\overline{D}_1\right)$, *where* $\overline{D}_1 = elm_{p_*}D_1$, *is a terminal model like in* Theorem 3.1.4.

Via $elm_q : \mathbb{F}_3 \dashrightarrow \mathbb{F}_2$, *we have that the log pair* $\left(\mathbb{F}_2, \frac{2}{3}\overline{D}_2\right)$, *where* $\overline{D}_2 = elm_{q_*}D_2$, *is a terminal model like in* Theorem 3.1.4.

In particular, $\left(\mathbb{F}_4, \frac{2}{3}\overline{D}_1\right)$ *is not birational to* $\left(\mathbb{F}_2, \frac{2}{3}\overline{D}_2\right)$. *On the other hand, both* $\left(\mathbb{F}_4, \frac{2}{3}\overline{D}_1\right)$ *and* $\left(\mathbb{F}_2, \frac{2}{3}\overline{D}_2\right)$ *are birational to a curve* $\widetilde{D}_i \subset \mathbb{P}^2$ *of type* $[9; 6, 2^3]$, *for* $i = 1, 2$.

Indeed, we consider the smooth irreducible and reduced curve $\overline{D}_1 \sim 3C_0 + 12f \subset \mathbb{F}_4$. *Then there exists a birational map* $\mu : \mathbb{F}_4 \dashrightarrow \mathbb{F}_1$, *with* $\mu = \mu_1 \circ \mu_2 \circ \mu_3$, *where* $\mu_i : \mathbb{F}_{i+1} \dashrightarrow \mathbb{F}_i$, *for* $i = 1, 2, 3$, *is an elementary transformation with center a general point* $\widetilde{p}_i \in \overline{D}_1 \subset \mathbb{F}_{i+1}$ *such that* $\widetilde{p}_i \notin C_0 \subset \mathbb{F}_{i+1}$. *Each map* μ_i *introduces a new singular point* $\widetilde{q}_i \in C_0 \subset \mathbb{F}_i$ *with* $\operatorname{mult}_{\widetilde{q}_i} \overline{D}_1 = 2$, *for* $i = 1, 2, 3$.

Consider $cont_{C_0} : \mathbb{F}_1 \to \mathbb{P}^2$ *the contraction of* $C_0 \subset \mathbb{F}_1$. *Since* $\overline{D}_1 \sim 3C_0 + 9f \subset \mathbb{F}_1$, *we have that* $cont_{C_0}(\overline{D}_1) = \widetilde{D}_1$, *where* $\widetilde{D}_1$ *is an irreducible and reduced curve of type* $[9; 6, 2^3]$, *with* p_1, p_2, p_3, p_4 *infinitely near singular points.*

Now, we consider the irreducible and reduced curve $D_2 \subset \mathbb{F}_3$ *having a singular point* $q \notin C_0$ *such that* $\operatorname{mult}_q D_2 = 2$. *Then there exists a birational map* $\mu : \mathbb{F}_3 \dashrightarrow \mathbb{F}_1$, *with* $\mu = \mu_1 \circ \mu_2$, *where* $\mu_i : \mathbb{F}_{i+1} \dashrightarrow \mathbb{F}_i$, *for* $i = 1, 2$, *is an elementary transformation with center a general point* $\widetilde{p}_i \in D_2 \subset \mathbb{F}_{i+1}$ *such that* $\widetilde{p}_i \notin C_0 \subset \mathbb{F}_{i+1}$. *Each map* μ_i *introduces a new singular point* $\widetilde{q}_i \in C_0 \subset \mathbb{F}_i$ *with* $\operatorname{mult}_{\widetilde{q}_i} D_2 = 2$, *for* $i = 1, 2$. *Note that* $\mu_2(q)$ *is a point of* $D_2 \subset \mathbb{F}_2$ *such that* $\operatorname{mult}_{\mu_2(q)} D_2 = 2$, $\mu_2(q) \in C_0$ *and it is infinitely near to* $\widetilde{q}_2$.

Consider $cont_{C_0} : \mathbb{F}_1 \to \mathbb{P}^2$ *the contraction of* $C_0 \subset \mathbb{F}_1$*. Since* $D_2 \sim 3C_0 + 9f \subset \mathbb{F}_1$*, we have that* $cont_{C_0}(D_2) = \widetilde{D}_2$*, where* $\widetilde{D}_2$ *is an irreducible and reduced curve of type* $[9; 6, 2^3]$*, with* p_1, p_2, p_3, p_4 *infinitely near singular points.*

Moreover, we have that D_1 *is not birational to any irreducible and reduced curve* $C_1 \subset \mathbb{P}^2$ *of degree* $d \leq 8$*, while* D_2 *is birational to an irreducible and reduced curve* $C_2 \subset \mathbb{P}^2$ *of degree* 8*. Namely, we consider the smooth irreducible and reduced curve* $\overline{D}_2 \sim 3C_0 + 9f \subset \mathbb{F}_2$*.*

We can apply an elementary transformation $elm_{\widetilde{p}} : \mathbb{F}_2 \dashrightarrow \mathbb{F}_1$*, where* $\widetilde{p} \notin C_0$*. Let* $\overline{D}_2$ *denote again the strict transform of* $\overline{D}_2$ *with respect to* $elm_{\widetilde{p}}$*. We have that* $\overline{D}_2 \sim 3C_0 + 8f \subset \mathbb{F}_1$*. The map* $elm_{\widetilde{p}}$ *introduces a new singular points* $\widetilde{q} \in C_0 \subset \mathbb{F}_1$ *with* $\mathrm{mult}_{\widetilde{q}}\,\overline{D}_2 = 2$*.*

Consider $cont_{C_0} : \mathbb{F}_1 \to \mathbb{P}^2$ *the contraction of* $C_0 \subset \mathbb{F}_1$*. Then,* $cont_{C_0}(\overline{D}_2) = C_2$*, where* C_2 *is an irreducible and reduced curve of type* $[8; 5, 2]$*, with* p_1, p_2 *infinitely near singular points.*

Now, we shall study the birational relations among models listed in Theorem 3.1.4.

Proposition 3.1.8. *Let* $(S, \frac{2}{\alpha}C)$ *and* $(S', \frac{2}{\alpha'}C')$ *be two models listed in* Theorem 3.1.4. *Suppose that there exists a birational map* $\varphi : (S, \frac{2}{\alpha}C) \dashrightarrow (S', \frac{2}{\alpha'}C')$*. Then*
- $\alpha = \alpha'$;
- *if* $(S, \frac{2}{\alpha}C)$ *has terminal singularities, then* $(S, \frac{2}{\alpha}C) \cong (S', \frac{2}{\alpha'}C')$*.*

Proof. We prove that $\alpha = \alpha'$. Assume that $\alpha' < \alpha$. Then the log pair $(S', \frac{2}{\alpha}C')$ has terminal singularities, that implies $\overline{\kappa}(S', \frac{2}{\alpha}C') = \overline{\kappa}(S, \frac{2}{\alpha}C) \geq 0$.

Consider the model $(S', \frac{2}{\alpha'}C')$. Then there exists a family of distinct curves $\{Z_\lambda\}$ in S' such that

$$(K_{S'} + \frac{2}{\alpha'}C') \cdot Z_\lambda = 0.$$

Since $\alpha' < \alpha$, i.e. $\frac{2}{\alpha'} > \frac{2}{\alpha}$ we have that $\frac{2}{\alpha'}C' \cdot Z_\lambda > \frac{2}{\alpha}C' \cdot Z_\lambda$, hence

$$(K_{S'} + \frac{2}{\alpha}C') \cdot Z_\lambda < 0,$$

which is a contradiction. Therefore, $\alpha' \geq \alpha$.

Assume that $\alpha' > \alpha$. Likewise, we get a contradiction.
Therefore $\alpha' = \alpha$.

Suppose that the log pair $(S, \frac{2}{\alpha}\, C)$ has terminal singularities, then $(S, \frac{2}{\alpha}\, C)$ is a log minimal model. If the log pair $(S', \frac{2}{\alpha}\, C')$ has canonical singularities, we can blow up the canonical singularities of this pair and we obtain a pair $(\overline{S}', \frac{2}{\alpha}\, \overline{C}')$ with log terminal singularities. Hence $(\overline{S}', \frac{2}{\alpha}\, \overline{C}')$ is a log minimal model. Since there exists a birational map $\varphi : (S, \frac{2}{\alpha}\, C) \dashrightarrow (S', \frac{2}{\alpha}\, C')$, then the two log minimal models $(S, \frac{2}{\alpha}\, C)$ and $(\overline{S}', \frac{2}{\alpha}\, \overline{C}')$ are isomorphic. In particular, we have $S \cong \overline{S}'$, which is a contradiction, that is because $2 = \operatorname{rk} \operatorname{Pic} S \neq \operatorname{rk} \operatorname{Pic} \overline{S}'$.
Therefore, if $(S, \frac{2}{\alpha}\, C)$ is terminal, then $(S', \frac{2}{\alpha}\, C')$ is terminal.

Consider the log pairs $(S, (\frac{2}{\alpha} + \varepsilon)\, C)$ and $(S', (\frac{2}{\alpha} + \varepsilon)\, C')$. If the log canonical divisors $K_S + (\frac{2}{\alpha} + \varepsilon)\, C$ and $K_{S'} + (\frac{2}{\alpha} + \varepsilon)\, C'$ are ample, we have that the log pairs $(S, (\frac{2}{\alpha} + \varepsilon)\, C)$ and $(S', (\frac{2}{\alpha} + \varepsilon)\, C')$ are log canonical models. Since there exists a birational map $\varphi : (S, \frac{2}{\alpha}\, C) \dashrightarrow (S', \frac{2}{\alpha}\, C')$, by uniqueness of Log Canonical model, we have that $(S, (\frac{2}{\alpha} + \varepsilon) C)$ and $(S', (\frac{2}{\alpha} + \varepsilon) C')$ are isomorphic. Therefore, $S \cong S'$ and $C \cong C'$.

If the log canonical divisors $K_S + (\frac{2}{\alpha} + \varepsilon)\, C$ and $K_{S'} + (\frac{2}{\alpha} + \varepsilon)\, C'$ are not ample, by Remark 3.1.5, we have that the log pairs $(S(0,2), (\frac{2}{\alpha} + \varepsilon)\, C)$ and $(S'(0,2), (\frac{2}{\alpha} + \varepsilon)\, C')$ are log canonical models, where $S(0,2)$, $S'(0,2)$ are quadratic cones. Since there exists a birational map $\varphi : (S, \frac{2}{\alpha}\, C) \dashrightarrow (S', \frac{2}{\alpha}\, C')$, by uniqueness of log canonical model, we have that $(S(0,2), (\frac{2}{\alpha} + \varepsilon) C)$ and $(S'(0,2), (\frac{2}{\alpha} + \varepsilon) C')$ are isomorphic. Therefore, $S(0,2) \cong S'(0,2)$ and $C \cong C'$.

This conclude our proof. □

The Proposition 3.1.8 does not hold if the pairs have canonical singularities. We give the following example.

Example 3.1.9. *Let $C \subset \mathbb{P}^2$ be an irreducible and reduced curve of degree 6 having two singular points $p, q \in C$, where p is a node a q is a tacnode. We have that the log pair $\left(\mathbb{P}^2, \frac{1}{2}\, C\right)$ is a model of $\left(\mathbb{P}^2, \frac{1}{2}\, C\right)$ with canonical singularities, having the log canonical divisor $K_{\mathbb{P}^2} + \frac{1}{2} C \sim \mathcal{O}$ nef and Kodaira dimension $\overline{\kappa} = 0$, like in* Theorem 3.1.4.

Let $\nu : \mathbb{F}_1 \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at p. Let $\widetilde{C}$ be the strict transform of C with respect to ν. We have that the log pair $\left(\mathbb{F}_1, \frac{1}{2}\, \widetilde{C}\right)$ is a model of $\left(\mathbb{P}^2, \frac{1}{2}\, C\right)$ with canonical singularities and terminal singularities in a neighborhood of C_0, having the log canonical divisor $K_{\mathbb{F}_1} + \frac{1}{2} \widetilde{C} \sim \mathcal{O}$ nef and Kodaira dimension $\overline{\kappa} = 0$, like in Theorem 3.1.4.

Let $\nu' : \mathbb{F}_1 \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at q. Let $\widetilde{C}$ be the strict transform of C with respect to ν'. Then, there exists a double singular point $\widetilde{q} \in \widetilde{C} \cap C_0 \subset \mathbb{F}_1$. Let $\mu : \mathbb{F}_1 \dashrightarrow \mathbb{F}_2$ be an elementary transformation with center $\widetilde{q}$ and let $\overline{C}$ be the strict transform of $\widetilde{C}$ with respect to μ. We have that the log pair $\left(\mathbb{F}_2, \frac{1}{2}\,\overline{C}\right)$ is a model of $\left(\mathbb{P}^2, \frac{1}{2}\,C\right)$ with canonical singularities and terminal singularities in a neighborhood of C_0, having the log canonical divisor $K_{\mathbb{F}_2} + \frac{1}{2}\overline{C} \sim \mathcal{O}$ nef and Kodaira dimension $\overline{\kappa} = 0$, like in Theorem 3.1.4.

Therefore, the log pairs $\left(\mathbb{P}^2, \frac{1}{2}\,C\right)$, $\left(\mathbb{F}_1, \frac{1}{2}\,\widetilde{C}\right)$ and $\left(\mathbb{F}_2, \frac{1}{2}\,\overline{C}\right)$ are three models of $\left(\mathbb{P}^2, \frac{1}{2}\,C\right)$, which are birational, but not isomorphic.

2. Rational curves of $\mathbb{P}^2$ according to D. Dicks

Let us consider a rational irreducible reduced curve $C \subset \mathbb{P}^2$. In this Section, we ask us whether there exists a birational map $\Phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\Phi(C)$ is a line. In this case, by Definition 2.0.5, with $n = 2$, the curve C is *Cremona equivalent to a line*.

It is easy and boring to show that any rational irreducible reduced curve $C \subset \mathbb{P}^2$ of degree $d \leq 5$ is *Cremona equivalent to a line*, in fact there exists $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ a composition of Cremona transformation such that $\omega(C) = l$, where l is a line.

In analogy with Definition 2.1.4, we have the following:

Definition 3.2.1. For any rational, irreducible and reduced curve C on a smooth surface S and $p \in C$, we put:

$$d_C(p) := \min\{\text{number of blowing ups to resolve the singularity of } C \text{ in } p\}.$$

Let us recall the following theorem ([**Ha**, chapter V, theorem 3.9]):

Theorem 3.2.2 (Embedded resolution of curves in surfaces)**.** *Let C be any curve in the surface S. Then there exists a finite sequence of monoidal transformations $S' = S_n \to S_{n-1} \to \ldots \to S_0 = S$, such that if $f : S' \to S$ is their composition, then the total inverse image $f^{-1}(C)$ with a divisor with normal crossing.*

Remark 3.2.3. a) By Theorem 3.2.2, we have that $d_C(p)$ is well defined. Moreover, $d_C(p)$ is a finite number.

b) If p is a smooth point of $C \subset S$, then $d_C(p) = 0$, while if p is an ordinary singular point of $C \subset S$, then $d_C(p) = 1$.

Remark 3.2.4. Let C be an irreducible and reduced curve of $\mathbb{P}^2$. Let $p \in C$ be a singular point with invariant $d_C(p) > 1$. It is immediate that a general standard Cremona transformation centered in p and in two general points decreases $d_C(p)$.

Let $C \subset \mathbb{P}^2$ be an irreducible and reduced curve of degree d and genus g. Through Theorem 3.2.2, we get the well known:

Proposition 3.2.5. *Let consider a pair $(\mathbb{P}^2, C)$, then there exists a birational map $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\omega(C)$ has only ordinary singular points.*

Proof. Let O_3 be a singular point of C of multiplicity r. We put $d_C(O_3) = i$. We prove the proposition by induction on i.

Initial case: $i = 1$. Then, by Remark 3.2.3 b), the point O_3 is an ordinary singular point of C.

Induction step. Assume that the proposition is true for $d_C(O_3) = i - 1$.

Let $O_1, O_2 \in \mathbb{P}^2$ be two points such that:

1) $O_1, O_2 \notin C$;
2) the lines $l_1 = \langle O_2, O_3\rangle$, $l_2 = \langle O_1, O_3\rangle$ meet C in the point O_3 and in $d - r$ distinct points;
3) the line $l_3 = \langle O_1, O_2\rangle$ meet C in d distinct points.

Let O_1, O_2, O_3 the three fundamental points of the standard Cremona transformation $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. Let $\omega(C) = C_1$ be a curve in $\mathbb{P}^2$. By Remark 3.2.4, we have that:

i) $\deg C_1 = d_1 = 2d - r$;
ii) $\omega(l_3) = L_3$ is an ordinary singular point of C_1 of multiplicity d;
iii) $\omega(l_1) = L_1$ and $\omega(l_2) = L_2$ are ordinary singular points of C_1 of multiplicity $d - r$;
iv) if $q \in C$ is a singular point of multiplicity s, then $\omega(q) \in C_1$ is a singular point of the same kind and multiplicity of q.

Moreover, we have that $C_1 \cap r_1 = \{L_2, L_3\}$, $C_1 \cap r_2 = \{L_1, L_3\}$ and $C_1 \cap r_3 = \{L_1, L_2, p_1, \ldots, p_t\}$, where $\sum_{j=1}^{t} \mathrm{mult}_{p_j} C_1 = r$ and $d_{C_1}(p_J) \leq d_C(O_3) - 1 = i - 1$, for $j = 1, \ldots, t$.

Since $d_{C_1}(p_j) \leq i-1$, we repeat this argument for any p_j, then (the strict transform of) C_1 is a curve in $\mathbb{P}^2$ with only ordinary singular points at (the strict transform of) L_1, L_2, L_3.

To conclude, we apply these arguments for any not ordinary singular point of C different from O_3. □

Let $C \subset \mathbb{P}^2$ be a rational irreducible and reduced curve of degree d. We get an useful proposition, which is a particular case of Lemma 3.0.1.

Proposition 3.2.6. *If C is Cremona equivalent to a line, then the pair $(\mathbb{P}^2, \frac{3}{d}C)$ has not canonical singularities.*

REMARK 3.2.7. We have that if the pair $(\mathbb{P}^2, \frac{3}{d}C)$ has canonical singularities, then C is not *Cremona equivalent to a line*. In particular, if the pair $(\mathbb{P}^2, \frac{3}{d}C)$ has canonical singularities, then C is not Cremona equivalent to any curve C' such that $\deg C' = d' < d$.

Thanks to Proposition 3.2.6 and Remark 3.2.7, we can give the first example of rational, plane curve not Cremona equivalent to a line: the rational, irreducible and reduced curve $C \subset \mathbb{P}^2$ of degree 6 with 10 double points, known as Coble's sextic, see for example [**C, C1**].

Let $C \subset \mathbb{P}^2$ be a rational, irreducible and reduced curve. Let $\nu : S \to \mathbb{P}^2$ be a minimal resolution of singularities of C and let $C_S = \nu_*^{-1}(C)$ be the strict transform of C. Let (S', C'_S) be an &-minimal model for (S, C_S). We have the following preliminary lemmas:

Lemma 3.2.8. *Let $C \subset \mathbb{P}^2$ be a rational curve of degree d having a singular point of multiplicity $d-1$. Then C is Cremona equivalent to a line.*

Proof. We prove the lemma by induction on d.
Initial case. $d = 1$. It is obvious.
induction step. Let assume that the lemma is true for all $d \leq k$.
Let $C \subset \mathbb{P}^2$ be a rational curve of degree $k+1$ and let $p_0 \in C$ be the unique singular point of C of multiplicity k.

Let consider two points $p_1, p_2 \in C$ and three lines $l_0 = \langle p_1, p_2 \rangle$, $l_1 = \langle p_0, p_2 \rangle$, $l_2 = \langle p_0, p_1 \rangle$. Let $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a standard Cremona transformation with center p_0, p_1, p_2 and fundamental lines l_0, l_1, l_2. Then $\omega(C) = C'$ is a plane curve of degree k, having a singular point $q_0 = \omega(l_0)$ of multiplicity $k-1$. By induction step, we conclude our proof. $\square$

REMARK 3.2.9. We can give a direct proof of previous lemma as follows. Let $C \subset \mathbb{P}^2$ be a rational curve of degree d having a singular point of multiplicity $d-1$. Let Λ be the linear system of plane curves of degree d with a singular point p_0 of multiplicity $d-1$ passing simply through $2d-2$ points $p_1, \ldots, p_{2d-2}$. We have that Λ is a 2–dimensional homaloidal linear system, then it defines a Cremona transformation

$T : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $T(C)$ is a line. A such Cremona transformation is called *De Jonquières transformation.*

Lemma 3.2.10. *Let $(\mathbb{F}_n, C')$ be a pair, where C' is a section or a fibre. Then there exists a birational map $\varphi : \mathbb{F}_n \dashrightarrow \mathbb{P}^2$ such that $\varphi(C') = l$, where l is a line of $\mathbb{P}^2$.*

Proof. There exists a birational map $\phi : \mathbb{F}_n \dashrightarrow \mathbb{F}_1$ such that $\phi(C')$ is either a section or a fibre. Let $\varepsilon : \mathbb{F}_1 \dashrightarrow \mathbb{P}^2$ be the natural contraction of the special section $C_0 \subset \mathbb{F}_1$ to the point $p \in \mathbb{P}^2$.

We have three cases.

a) If $\phi(C')$ is a fibre of $\mathbb{F}_1$, we apply the contraction $\varepsilon : \mathbb{F}_1 \dashrightarrow \mathbb{P}^2$, then $\varepsilon(\phi(C'))$ is a line in $\mathbb{P}^2$ through p.
b) If $\phi(C')$ is a section of $\mathbb{F}_1$, with $\phi(C') \neq C_0$, we apply the contraction $\varepsilon : \mathbb{F}_1 \dashrightarrow \mathbb{P}^2$, then $\varepsilon(\phi(C')) = \widetilde{C}$ is a curve of degree d and $\mathrm{mult}_p\, \widetilde{C} = d - 1$. By Lemma 3.2.8, we have that $\widetilde{C}$ is Cremona equivalent to a line.
c) Let $\phi(C') = C_0$. There exists an elementary transformation $\psi : \mathbb{F}_1 \dashrightarrow \mathbb{F}_1$ such that $\psi(C_0) = C''$, where C'' is a section of $\mathbb{F}_1$, with $C'' \neq C_0$. Now, we repeat the arguments of b) for the not special section C'', we obtain that $\varepsilon(C'') = \overline{C} \subset \mathbb{P}^2$ is a curve of degree d and $\mathrm{mult}_p\, \widetilde{C} = d - 1$, which is Cremona equivalent to a line.

This concludes our proof. □

We are ready to prove the following (see [**KuMu**], Corollary 2.5):

Proposition 3.2.11. *Let $\nu : S \to \mathbb{P}^2$ be a minimal resolution of singularities of the rational curve C and let $C_S = \nu_*^{-1}(C)$ be the strict transform of C. If $C_S^2 \geq -3$, then C is Cremona equivalent to a line.*

Proof. Let (S', C'_S) be an &–minimal pair, for which there exists a morphism $f : S \to S'$ such that $f_{|C_S}$ is an isomorphism. Then, by Definitions 1.3.5, 1.3.6, the pair (S', C'_S) is an &–minimal model for (S, C_S).

Let k be the number of (-1)–curves which are contracted by f, then ${C'_S}^2 = {C_S}^2 + k \geq -3 + k \geq -3$. Hence, (S', C'_S) is an &–minimal model for (S, C_S) with ${C'_S}^2 \geq -3$. In particular, the case 4) of the Theorem 1.3.7 doesn't hold.

Let observe that, since S' is rational, the canonical divisor $K_{S'}$ is not nef. In particular, the case 5) of the Theorem 1.3.7 doesn't hold and ${C'_S}^2 \geq -1$.

Let assume that $K_{S'} + C'_S$ is nef. We have that $(K_{S'} + C'_S) \cdot C'_S \geq 0$, then, by adjunction formula, $2g(C'_S) - 2 \geq 0$, i.e. $g(C'_S) \geq 1$, which is a contradiction by rationality of C. Hence $K_{S'} + C'_S$ is not nef and, in particular, the case 6) of the Theorem 1.3.7 doesn't hold.

If the case 1) of the Theorem 1.3.7 holds, there exists a birational map $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\omega(C) = l$, where l is a line of $\mathbb{P}^2$.

If the case 2) of the Theorem 1.3.7 holds, by Lemma 3.2.10, there exists a birational map $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\omega(C) = l$, where l is a line of $\mathbb{P}^2$.

If the case 3) of the Theorem 1.3.7 holds, there exists a birational map $\psi : S' \dashrightarrow S''$ such that $\psi(C'_S) = C''_S$, where for the pair (S'', C''_S) the case 2) of the Theorem 1.3.7 holds. Therefore, there exists a birational map $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\omega(C) = l$, where l is a line of $\mathbb{P}^2$.

This conclude our proof. $\square$

Let show that the result of Proposition 3.2.11 is sharp.

Example 3.2.12. *Let $C = (1, a)$ be a curve in a quadric surface $\mathbb{Q} \subset \mathbb{P}^3$. Let $p \in \mathbb{P}^3 \backslash \mathbb{Q}$ be a point and let $\pi_p : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ be the projection from p. Then $\pi_p(C) = C'$ is a rational plane curve of degree $d = a + 1$ with only nodes as singularities. Let $\nu : S \to \mathbb{P}^2$ be a minimal resolution of singularities of C' and let $C'_S = \nu_*^{-1}(C')$ be the strict transform of C'. If $a \geq 5$, by* Proposition 3.2.6, *the curve C' is not Cremona equivalent to a line and we have that ${C'_S}^2 = (a+1)^2 - 2a(a-1) \leq -4$.*

We have the following:

Remark 3.2.13. Let $\nu : S \to \mathbb{P}^2$ be a minimal resolution of singularities of the rational curve C and let $C_S = \nu_*^{-1}(C)$ be the strict transform of C. If we assume that (S, C_S) is an &–minimal pair, the curve C is *Cremona equivalent to a line* $\Leftrightarrow C_S^2 \geq -3$. We can prove this in the following way:
"$\Rightarrow$". Let $C_S^2 \leq -4$, then the case 4) of Theorem 1.3.7 holds. We have that $K_S + \lambda C$ is nef, where $\lambda = 1 + \dfrac{2}{C_S^2} < 1$. By Proposition 1.3.8 and Theorem 1.3.9, (S, C_S) is the unique &-minimal model of $(\mathbb{P}^2, C)$ and (S, C_S) is not isomorphic to $(\mathbb{P}^2, l)$, where l is a line. Hence, the curve C is not *Cremona equivalent to a line.*
"$\Leftarrow$". It follows immediately by Proposition 3.2.11.

On the other hand, we can observe that, in general, a curve C *Cremona equivalent to a line* is such that the pair (S, C_S) is not &–minimal. For example, consider a rational curve C of degree 6 with four nodes and two points of multiplicity 3. Since $C_S^2 = 2 \geq -3$, by Proposition 3.2.11, the curve C is *Cremona equivalent to a line*. If we consider a conic Γ passing through the four nodes and through one point of multiplicity 3 of C, then $\nu_*^{-1}(\Gamma) = \Gamma_S$ is a (-1)–curve of S meeting C_S in a point. Hence, the pair (S, C_S) is not &–minimal.

Remark 3.2.14. We can prove easily that any rational, irreducible and reduced curve $C \subset \mathbb{P}^2$ of degree $d \leq 5$ is *Cremona equivalent to a line*.

Let $\nu : S \to \mathbb{P}^2$ be a minimal resolution of singularities of the rational curve C and let $C_S = \nu_*^{-1}(C)$ be the strict transform of C. We have that ${C_S}^2$ is minimal if C has $\frac{(d-1)(d-2)}{2}$ double points (not necessarily ordinary) as singularities. In this case ${C_S}^2 = d^2 - 2(d-1)(d-2) = -d^2 + 6d - 4$. Since $d \leq 5$, we obtain that ${C_S}^2 \geq 1 \geq -3$, then, by Proposition 3.2.11, C is *Cremona equivalent to a line*.

Example 3.2.15. *Let $C_0 \subset \mathbb{P}^2$ be a rational, irreducible and reduced curve of degree d_0 having ordinary singular points. Suppose that C_0 is Cremona equivalent to a line. Let $\nu_0 : S_0 \to \mathbb{P}^2$ be a minimal resolution of singularities of C_0 and let $C_{0,S_0} = \nu_*^{-1}(C_0)$ be the strict transform of C_0. We have that $C_{0,S_0}^2 = d_0^2 - \mu$, where μ is the contribute given by singularities of C_0.*

Consider $\omega_0 : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ a Cremona transformation with center a general point of C_0 and two general points in $\mathbb{P}^2$. Let $\omega_0(C_0) = C_1$. We have that C_1 is a rational, irreducible and reduced curve of degree $d_1 = 2d_0 - 1$, having the same singular points of C_0, two points of multiplicity $d_0 - 1$ and one point of multiplicity d_0. Moreover, C_1 is Cremona equivalent to a line. Let $\nu_1 : S_1 \to \mathbb{P}^2$ be a minimal resolution of singularities of C_1 and let $C_{1,S_1} = \nu_*^{-1}(C_1)$ be the strict transform of C_1. Since in C_{1,S_1}^2 the contribute is given by singularities of C_0 and by new singularities introduced by ω_0, we get

$$\begin{aligned} C_{1,S_1}^2 &= d_1^2 - 2(d_0-1)^2 - d_0^2 - \mu = \\ &= (2d_0-1)^2 - 2(d_0-1)^2 - d_0^2 - \mu = d_0^2 - \mu - 1 = C_{0,S_0}^2 - 1. \end{aligned}$$

Recursively, we can repeat the above arguments i times and we obtain a rational, irreducible and reduced curve $C_i \subset \mathbb{P}^2$ of degree $d_i = 2^i(d_0-1)+1$, having the same singular points of C_0, two points of multiplicity $d_j - 1$ and one point of multiplicity d_j, for any $j = 0, \ldots, i-1$.

Moreover, C_i is Cremona equivalent to a line. Let $\nu_i : S_i \to \mathbb{P}^2$ be a minimal resolution of singularities of C_i and let $C_{i,S_i} = \nu_*^{-1}(C_i)$ be the strict transform of C_i. We get

$$C_{i,S_i}^2 = d_i^2 - 2\sum_{j=0}^{i-1}(d_j - 1)^2 - \sum_{j=0}^{i-1} d_j^2 - \mu =$$

$$= (2^i(d_0 - 1) + 1)^2 - 2\sum_{j=0}^{i-1}(2^j(d_0 - 1))^2 - \sum_{j=0}^{i-1}(2^j(d_0 - 1) + 1)^2 - \mu =$$

$$= 1 + (d_0 - 1)^2 - i + 2(d_0 - 1) - \mu = d_0^2 - \mu - i = C_{0,S_0}^2 - i.$$

Note that C_i is an example of curve Cremona equivalent to a line and having C_{i,S_i}^2 arbitrarily negative.

2.1. Rational curves of $\mathbb{P}^2$ according to Kumar–Murthy. The result of Proposition 3.2.11 is been already shown by Kumar–Murthy [**KuMu**]. In [**KuMu**], Kumar–Murthy consider rational curves with self intersection $-n$. They show that for $n \leq 3$ the Kodaira dimension is $-\infty$. This result is related to a theorem of Coolidge, see [**Coo**], which gives necessary and sufficient condition for a plane curve to be transformed into a straight line by a Cremona transformation.

Now, we reproduce the results of Kumar–Murthy, since it will useful in the following.

Let $D \simeq \mathbb{P}^1$ be a smooth, rational, reduced projective curve having negative self intersection and let X a smooth rational projective surface. We have the following result:

Theorem 3.2.16 ([**KuMu**], Theorem 2.1). *Let $D \hookrightarrow X$, $D^2 = -n$, with $n > 0$. Then the following are equivalent:*

a) *(X, D) is equivalent to $(\mathbb{F}_n, D_n)$, where D_n denote the unique section of $\mathbb{F}_n$ with $D_n^2 = -n$, i.e. there exists a rational map $p : X \to \mathbb{F}_n$ which is an isomorphism between a neighborhood of D and that of D_n.*
b) *There exists $\sigma : Y \to X$ a birational morphism such that $\sigma_{|\sigma^{-1}(D)} : \sigma^{-1}(D) \to D$ is an isomorphism and $Y \setminus \sigma^{-1}(D)$ contains an open set of the form $U \times \mathbb{P}^1$, where U is a curve.*
c) *$|mK + nD| = \emptyset, \quad \forall\, m > 0$.*
d) *$\kappa(X, D) = -\infty$.*

Lemma 3.2.17 ([**KuMu**], Lemma 2.2)**.** *Suppose* $(D+K)^2 \leq -2$ *and* $|m(D+K)| \neq \emptyset$*, for some* $m \geq 1$*. Then there exists an exceptional curve of the first kind such that* $E \cap D = \emptyset$*.*

From this Lemma derive some important consequences:

Corollary 3.2.18 ([**KuMu**], Corollary 2.3)**.** *If* $D \hookrightarrow X$ *is a minimal embedding and* $\kappa(X, D) \geq 0$*, then* $(D+K)^2 \geq -1$*.*

Corollary 3.2.19 ([**KuMu**], Corollary 2.4)**.** *The following statements are equivalent:*

a) $\kappa(X, D) = -\infty$;
b) $|2K + D| = \emptyset$;
c) $|2(K + D)| = \emptyset$.

Corollary 3.2.20 ([**KuMu**], Corollary 2.5)**.** *If* $D^2 \geq -3$*, then* $\kappa(X, D) = -\infty$*.*

Proof. Since D is a rational curve, by adjunction formula, we have that $D \cdot (D+K) = -2$. Consider $D \cdot (D+2K)$, we have that:

$$D \cdot (D+2K) = D \cdot (D+K) + D \cdot K = -2 - 2 - D^2 \leq -1 < 0.$$

Since $D \cdot (D+2K) < 0$, we get that $|D+2K| = \emptyset$, hence, by Corollary 3.2.19, $\kappa(X, D) = -\infty$. ☐

Now, we reproduce the proof of Coolidge's Theorem given by Kumar–Murthy, see [**KuMu**, Theorem 2.6].

Let C be any irreducible curve on a smooth rational surface Y. Let $F : X \to Y$ be a birational morphism such that the proper transform D of C is smooth. Let define $\overline{\kappa}(Y, C) = \kappa(X, D)$. We have that $\overline{\kappa}(Y, C)$ is independent of F.

Theorem 3.2.21 (Coolidge)**.** *Let* $C \subset \mathbb{P}^2$ *be an irreducible rational curve. Then there exists a Cremona transformation* σ *of* $\mathbb{P}^2$ *such that* $\sigma(C)$ *is a line if and only if* $\overline{\kappa}(\mathbb{P}^2, C) = -\infty$*.*

Proof. Suppose that there exists $\sigma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\sigma(C) = l$, where l is a line. The Kodaira dimension is a birational invariant in an equivalence class of embeddings, then, since $\overline{\kappa}(\mathbb{P}^2, l) = -\infty$ for a line l, we have that $\overline{\kappa}(\mathbb{P}^2, C) = -\infty$.

Suppose now that $\overline{\kappa}(\mathbb{P}^2, C) = -\infty$. By Theorem 3.2.16, there exists a birational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{F}_n$, for some n, such that $f(C) = D_n$. It is well known that there is a birational map $g : \mathbb{F}_n \dashrightarrow \mathbb{P}^2$ such that $g(D_n)$ is a line.

This conclude the proof. $\square$

REMARK 3.2.22. The plan of the original proof of Coolidge, see [**Coo**], is to reduce the degree of C by quadratic transformation. To prove Theorem 3.2.21, Coolidge show that $\overline{\kappa}(\mathbb{P}^2, C) = -\infty$ if and only if C has no adjoints. We recall that a curve D of degree $n - 3s$ is said to be an s–th adjoint of C if at any point $p \in C$, $\operatorname{mult}_p D \geq (\operatorname{mult}_p C) - s$, where for infinitely near points p of C, C and D are interpreted as the proper transform.

REMARK 3.2.23. By Corollary 3.2.20 and Theorem 3.2.21, we have that if $D^2 \geq -3$, then D is *Cremona equivalent to a line*.

Let $C \subset \mathbb{P}^2$ be a rational, irreducible and reduced curve. We have the following results.

Lemma 3.2.24. *Let* (S, C_S) *be the &–minimal model of* $(\mathbb{P}^2, C)$ *such that the* case 4) of Theorem 1.3.7 *holds. Then* $\kappa(S, C_S) \geq 0$ *and hence* C *is not Cremona equivalent to a line.*

Proof. Consider the pair (S, C_S). If the case 4) of Theorem 1.3.7 holds, we have that $C_S \cong \mathbb{P}^1$, $C_S^2 \leq -4$ and $K_S + \lambda \overline{C}$ is nef, where $\lambda = 1 + \frac{2}{C_S^2}$. Since C_S is smooth, we have that the log pair $(S, \lambda C_S)$ has terminal singularities. Moreover, $K_S + \lambda \overline{C}$ is nef. Hence, the log pair $(S, \lambda C_S)$ is the log minimal model of $(\mathbb{P}^2, \lambda C)$. By log Abundance Theorem, we get $\kappa(S, \lambda C_S) \geq 0$.

Since $C_S^2 \leq -4$, we have that $\lambda = 1 + \frac{2}{C_S^2} < 1$. It follows that $\kappa(S, C_S) \geq 0$. Therefore, the pair (S, C_S) is the &–minimal model of $(\mathbb{P}^2, C)$ with $\kappa(S, C_S) \geq 0$. By Theorem 3.2.21, the curve C is not Cremona equivalent to a line. $\square$

Let $C \subset \mathbb{P}^2$ be a rational, irreducible and reduced curve of degree $d \geq 6$. Let $\nu : S \to \mathbb{P}^2$ be a minimal resolution of singularities of the rational curve C and let $C_S = \nu_*^{-1}(C)$ be the strict transform of C.

Using the notations of Theorem 3.2.21, we have that $\overline{\kappa}(\mathbb{P}^2, \alpha C) = \kappa(S, \alpha C_S)$, for $0 < \alpha \leq 1$. We can prove the following:

Lemma 3.2.25. *We have that* $\overline{\kappa}(\mathbb{P}^2, \frac{1}{2}C) \geq 0$ *if and only if* C *is not Cremona equivalent to a line.*

Proof. "$\Rightarrow$". Suppose that C is *Cremona equivalent to a line*. Since $\overline{\kappa}(\mathbb{P}^2, l) = -\infty$, for a line l, we have that $\overline{\kappa}(\mathbb{P}^2, \lambda\, l) = -\infty$, for any $\lambda \leq 1$. Then $\overline{\kappa}(\mathbb{P}^2, \frac{1}{2}\, l) = -\infty$. Since C is *Cremona equivalent to a line*, it follows that $\overline{\kappa}(\mathbb{P}^2, \frac{1}{2}\, C) = \kappa(S, \frac{1}{2}C_S) = -\infty$. That is because the log Kodaira dimension is an invariant in a fixed birational equivalence class. In this way, we have proved that if $\overline{\kappa}(\mathbb{P}^2, \frac{1}{2}\, C) \geq 0$, then C is not Cremona equivalent to a line.

"$\Leftarrow$". Suppose that $\overline{\kappa}(\mathbb{P}^2, \frac{1}{2}\, C) = \kappa(S, \frac{1}{2}C_S) = -\infty$. We consider the log pair $(S, \frac{1}{2}C_S)$ and we start a Log MMP having as input this pair. The Log MMP for the log pair $(S, \frac{1}{2}C_S)$ gives as output a log pair $(\widetilde{S}, \frac{1}{2}C_{\widetilde{S}})$ such that $\kappa(\widetilde{S}, \frac{1}{2}C_{\widetilde{S}}) = -\infty$, i.e. the log canonical divisor $K_{\widetilde{S}} + \frac{1}{2}C_{\widetilde{S}}$ is not nef.

Since $(S, \frac{1}{2}C_S)$ is a log resolution of the pair $(\mathbb{P}^2, \frac{1}{2}\, C)$, we have that any (-1)–curve $E \subset \widetilde{S}$ contracted by the Log MMP satisfies the following inequality:

$$\Big(K_{\widetilde{S}} + \frac{1}{2}C_{\widetilde{S}}\Big) \cdot E = -1 + \frac{1}{2}C_{\widetilde{S}} \cdot E < 0,$$

i.e. $C_{\widetilde{S}} \cdot E < 2$. Hence, by Definition 1.3.4, the pair $(\widetilde{S}, C_{\widetilde{S}})$ is &–minimal. More precisely, the pair $(\widetilde{S}, C_{\widetilde{S}})$ is an &–minimal model of $(\mathbb{P}^2, C)$.

Now, we study the pair $(\widetilde{S}, C_{\widetilde{S}})$. We have two cases:

First Case: $\widetilde{S} \cong \mathbb{P}^2$. Since $C_{\widetilde{S}}$ is a smooth rational plane curve, we have that $C_{\widetilde{S}} \sim \mathcal{O}(\alpha)$, with $\alpha \leq 2$, i.e. $C_{\widetilde{S}}$ is a line or a conic in $\mathbb{P}^2$. Hence, the curve C is *Cremona equivalent to a line*.

Second Case: $\widetilde{S} \cong \mathbb{F}_a$. We have that $C_{\widetilde{S}} \sim \alpha C_0 + \beta f$, where Pic $\mathbb{F}_a = \langle C_0, f\rangle$. Since $C_{\widetilde{S}}$ is an irreducible curve, then $\beta \geq a\alpha$. Since the log canonical divisor $K_{\widetilde{S}} + \frac{1}{2}C_{\widetilde{S}}$ is not nef, we have that

$$(K_{\widetilde{S}} + \frac{1}{2}C_{\widetilde{S}}) \cdot f = -2 + \frac{1}{2}C_{\widetilde{S}} \cdot f < 0.$$

Hence $\alpha = C_{\widetilde{S}} \cdot f < 4$, i.e. $\alpha \leq 3$.

Consider the adjunction formula $2g(C_{\widetilde{S}}) - 2 = (K_{\widetilde{S}} + C_{\widetilde{S}}) \cdot C_{\widetilde{S}}$, since $C_{\widetilde{S}}$ is a rational curve, we have

$$-2 = (-2C_0 + (-2-a)f) \cdot (\alpha C_0 + \beta f) + (\alpha C_0 + \beta f)^2,$$

that is

$$\alpha(-a\alpha + 2\beta) - \alpha(2+a) + 2(a\alpha - \beta) + 2 = 0.$$

We have the following subcases:

i) $\alpha = 3$. We have:

$$3(-3a + 2\beta) - 3(2 + a) + 2(3a - \beta) + 2 = 0,$$

that is

$$\beta = \frac{3a + 2}{2}.$$

Since $\beta \geq 3a$, we get $a = 0$. Hence, we have $a = 0$, $\beta = 1$, i.e. $\widetilde{S} \cong \mathbb{F}_0$ and $C_{\widetilde{S}} \sim 3F_1 + F_2$. Then $C_{\widetilde{S}}$ is a section of $\mathbb{F}_0$. Hence, for the &–minimal pair $(\widetilde{S}, C_{\widetilde{S}})$ holds the case 2) of Theorem 1.3.7. Therefore, the curve C is *Cremona equivalent to a line*.

ii) $\alpha = 2$. We have:

$$2(-2a + 2\beta) - 2(2 + a) + 2(2a - \beta) + 2 = 0,$$

that is

$$\beta = a + 1.$$

Since $\beta \geq 2a$, we get $a \leq 1$. If $a = 1$, we have $\beta = 2$, i.e. $\widetilde{S} \cong \mathbb{F}_1$ and $C_{\widetilde{S}} \sim 2C_0 + 2f$. Then we consider the contraction of (-1)–curve $C_0 \subset \mathbb{F}_1$ $\mathrm{cont}_{C_0} : \mathbb{F}_1 \to \mathbb{P}^2$ such that $\mathrm{cont}_{C_0}(C_{\widetilde{S}})$ is a smooth conic. Therefore, the curve C is *Cremona equivalent to a line*. While, if $a = 0$, we have $\beta = 1$, i.e. $\widetilde{S} \cong \mathbb{F}_0$ and $C_{\widetilde{S}} \sim 2F_1 + F_2$. Then $C_{\widetilde{S}}$ is a section of $\mathbb{F}_0$. Hence, for the &–minimal pair $(\widetilde{S}, C_{\widetilde{S}})$ holds the case 2) of Theorem 1.3.7. Therefore, the curve C is *Cremona equivalent to a line*.

iii) $\alpha = 1$. We have:

$$-a + 2\beta - 2 - a + 2a - 2\beta + 2 = 0,$$

that holds for any β. We have $\widetilde{S} \cong \mathbb{F}_a$ and $C_{\widetilde{S}} \sim C_0 + \beta f$, with $\beta \geq a$. Then $C_{\widetilde{S}}$ is a section of $\mathbb{F}_a$. Hence, for the &–minimal pair $(\widetilde{S}, C_{\widetilde{S}})$ holds the case 2) of Theorem 1.3.7. Therefore, the curve C is *Cremona equivalent to a line*.

iv) $\alpha = 0$. We have $-2\beta = -2$, that is $\beta = 1$. We get $\widetilde{S} \cong \mathbb{F}_a$ and $C_{\widetilde{S}} \sim f$, i.e. $C_{\widetilde{S}}$ is a fibre of $\mathbb{F}_a$. Hence, for the &–minimal pair $(\widetilde{S}, C_{\widetilde{S}})$ holds the case 2) of Theorem 1.3.7. Therefore, the curve C is *Cremona equivalent to a line*.

We have shown that the &–minimal pair $(\widetilde{S}, C_{\widetilde{S}})$ is isomorphic to $(\mathbb{P}^2, l)$, then the curve C is *Cremona equivalent to a line*. In this way, we have proved that if C is not Cremona equivalent to a line, then $\overline{\kappa}(\mathbb{P}^2, \frac{1}{2}C) \geq 0$.

This conclude our proof. $\square$

Remark 3.2.26. Thanks to Theorem 3.2.21 and Lemma 3.2.25, we have:

$$\kappa\Big(S, \frac{1}{2}C_S\Big) = -\infty \ \Leftrightarrow \ \kappa(S, C_S) = -\infty \ \Leftrightarrow$$
$$C \text{ is } \textit{Cremona equivalent to a line.}$$

This result is the best possible. It is easy to prove that if $\frac{1}{2} \leq \lambda \leq 1$, we have:

$$\kappa(S, \lambda\, C_S) = -\infty \ \Leftrightarrow \ C \text{ is } \textit{Cremona equivalent to a line.}$$

On the other hand, if $\lambda < \frac{1}{2}$, we have:

$$\kappa(S, \lambda\, C_S) = -\infty \ \not\Rightarrow \ C \text{ is } \textit{Cremona equivalent to a line.}$$

Indeed, we give the following example. Let $C_6 \subset \mathbb{P}^2$ be the rational, irreducible, reduced curve of degree 6 with only double points as singularities. Moreover, since $\frac{1}{2} = \frac{3}{d}$, we get that $\overline{\kappa}(\mathbb{P}^2, \frac{1}{2}C_6) = 0$. If $\lambda < \frac{1}{2}$, then $\overline{\kappa}(\mathbb{P}^2, \lambda C_6) = -\infty$, but C_6 is not Cremona equivalent to a line.

3. Other results about rational curves of $\mathbb{P}^2$

In this section, we study some relations between the degree of C and multiplicities of singular points of C and by these we search to understand when C is *Cremona equivalent to a line*.

Let $C \subset \mathbb{P}^2$ be a rational, irreducible and reduced curve of degree d. Let $p_1, \dots, p_k \in C$ be the k (not necessarily ordinary) singular points of multiplicity e_i, for $i = 1, \dots, k$. Let $e_{\max} = \max_{i=1,\dots,k}\{e_i\}$ and $e_{\min} = \min_{i=1,\dots,k}\{e_i\}$. Observe that, by genus formula [**Ha**], we have that

$$\frac{(d-1)(d-2)}{2} = \sum_{i=1}^{k} \frac{e_i(e_i-1)}{2}.$$

Finally, let $\nu : S \to \mathbb{P}^2$ be a minimal resolution of singularities of C and let $C_S = \nu_*^{-1}(C)$ be the strict transform of C.

Remark 3.3.1. We have that $e_i = \operatorname{mult}_{p_i} C \leq \frac{d}{3}$, for any $i = 1, \dots, k \Leftrightarrow (\mathbb{P}^2, \frac{3}{d}C)$ has canonical singularities.

3.1. Results about plane rational curves in terms of kind and number of singular points.

Now, we list some results about Cremona equivalence to a line of a plane rational curve C.

Lemma 3.3.2. *Let $C \subset \mathbb{P}^2$ be a rational, irreducible and reduced curve of degree d. If $e_{\max} = d-2$, then C is Cremona equivalent to a line.*

Proof. Suppose that C has one point of multiplicity $d-2$ and N nodes as singularity. Then $N = \frac{1}{2}[(d-1)(d-2)-(d-2)(d-3)] = d-2$. We have that

$$C_S^2 = d^2 - \sum_{i=1}^{k} e_i^2 = d^2 - (d-2)^2 - 4(d-2) = 4,$$

hence, by Proposition 3.2.11, C is *Cremona equivalent to a line.* □

Lemma 3.3.3. *Let $C \subset \mathbb{P}^2$ be a rational, irreducible and reduced curve of degree $d \geq 6$. Let $m_1 \geq m_2 \geq m_3 \geq \ldots \geq m_k$ be the multiplicity of the singular points of C. If $m_1+m_2+m_3 \leq d$, then C is not Cremona equivalent to a line.*

Proof. If $m_1 \leq \frac{d}{3}$, then C has canonical singularities, hence, by Remark 3.2.7, the curve C is not Cremona equivalent to a line.

Assume $m_1 > \frac{d}{3}$. Suppose that C is *Cremona equivalent to a line*, i.e. there exists a birational map $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\omega(C) = l$, where l is a line.

Let $x \in C$ be a point such that $\operatorname{mult}_x C = m_1$. Since $\operatorname{mult}_x C = m_1 > \frac{d}{3}$, then the pair $(\mathbb{P}^2, \frac{3}{d}C)$ is not canonical at x. Let $\nu : \mathbb{F}_1 \to \mathbb{P}^2$ be the blow up of x with exceptional divisor E. Let C' be the strict transform of C.

We get $K_{\mathbb{F}_1} + \frac{2}{d-m_1}C' \equiv_\pi 0$, where $\pi : \mathbb{F}_1 \to \mathbb{P}^1$ is a Mori space structure. We have that $K_{\mathbb{F}_1} + \frac{2}{d-m_1}C'$ is nef. In fact, let $f \subset \mathbb{F}_1$ be a generic fiber of the ruled structure, then $(K_{\mathbb{F}_1}+\frac{2}{d-m_1}C')\cdot f = -2+2 = 0$ and $(K_{\mathbb{F}_1} + \frac{2}{d-m_1}C') \cdot E = -1 + \frac{2m_1}{d-m_1} = \frac{3m_1-d}{d-m_1} > 0$, since $\frac{d}{3} < m_1 < d$.

Let $x' \in C'$ such that $\operatorname{mult}_{x'} C' = m_2$. If $m_2 \leq \frac{d-m_1}{2}$, then we get that $\frac{d-m_1}{2} \geq m_2 \geq m_i$, for any $i = 3, \ldots, k$. Hence, the pair $(\mathbb{F}_1, \frac{2}{d-m_1}C')$ has canonical singularities. These arguments contradict Noether–Fano inequalities, hence C is not Cremona equivalent to a line.

Assume $\operatorname{mult}_{x'} C' = m_2 > \frac{d-m_1}{2}$, then $(\mathbb{F}_1, \frac{2}{d-m_1}C')$ is not canonical at x'.

We have two cases:

i) if $x' \notin E$, we consider an elementary transformation $\mu : \mathbb{F}_1 \dashrightarrow \mathbb{F}_0$ with center x' such that $\mu(C') = C''$. We get $K_{\mathbb{F}_0} + \frac{2}{d-m_1}C'' \equiv_{\pi_i} 0$, where $\pi_i : \mathbb{F}_0 \to \mathbb{P}^1$ is one of two Mori space structures. Since $m_2+m_3 \leq d-m_1$ and $m_2 > \frac{d-m_1}{2}$, then we have that $m_i \leq m_3 < \frac{d-m_1}{2}$, for any $i = 3, \ldots, k$. Hence, the pair $(\mathbb{F}_0, \frac{2}{d-m_1}C'')$ has canonical singularities.

On the other hand, since $d > m_1 \geq m_2$, we have

$$(K_{\mathbb{F}_0} + \frac{2}{d-m_1}C'') \cdot F_i = -2 + \frac{2(d-m_i)}{d-m_1} = \frac{2(m_1-m_i)}{d-m_1} \geq 0,$$

for $i = 1, 2$, where $\overline{NE(\mathbb{F}_0)} = \langle F_1, F_2 \rangle$ and F_1, F_2 are the strict transform of a line through x, x' respectively. Then $K_{\mathbb{F}_0} + \frac{2}{d-m_1}C''$ is nef. These arguments contradict Noether–Fano inequalities, hence C is not Cremona equivalent to a line.

ii) If $x' \in E$, we consider an elementary transformation $\mu : \mathbb{F}_1 \dashrightarrow \mathbb{F}_2$ with center x' such that $\mu(C') = C''$. We get $K_{\mathbb{F}_2} + \frac{2}{d-m_1}C'' \equiv_{\pi'} 0$, where $\pi' : \mathbb{F}_2 \to \mathbb{P}^1$ is a Mori space structures. Since $m_2 + m_3 \leq d - m_1$ and $m_2 > \frac{d-m_1}{2}$, then we have that $m_i \leq m_3 < \frac{d-m_1}{2}$, for any $i = 3, \ldots, k$. Hence, the pair $(\mathbb{F}_2, \frac{2}{d-m_1}C'')$ has canonical singularities.

On the other hand, since $d > m_1 \geq m_2 > 0$, we have $(K_{\mathbb{F}_2} + \frac{2}{d-m_1}C'') \cdot f' = -2 + \frac{2}{d-m_1}(d-m_1) = -2 + 2 = 0$, and $(K_{\mathbb{F}_2} + \frac{2}{d-m_1}C'') \cdot E' = \frac{2}{d-m_1}(-2(d-m_1) + 2d - m_1 - m_2) = \frac{2(m_1-m_2)}{d-m_1} \geq 0$, where $\overline{NE(\mathbb{F}_2)} = \langle E', f' \rangle$. Then $K_{\mathbb{F}_2} + \frac{2}{d-m_1}C''$ is nef. These arguments contradict Noether–Fano inequalities, hence C is not Cremona equivalent to a line.

These arguments conclude our proof. □

The next two lemmas are particular cases of Lemma 3.3.3.

Lemma 3.3.4. *Let $C \subset \mathbb{P}^2$ be a rational, irreducible and reduced curve of degree d. If C has one singular point of multiplicity $d-h$ and the remaining singular points of multiplicity $\leq \frac{h}{2}$, with $h \geq 4$, then C is not Cremona equivalent to a line.*

Proof. By Lemma 3.3.3, with $m_1 = d-h$, $m_3 \leq m_2 \leq \frac{h}{2}$, it follows that C is not Cremona equivalent to a line. □

Lemma 3.3.5. *Let $C \subset \mathbb{P}^2$ be a rational, irreducible and reduced curve of degree $d \geq 8$. Assume that C has one point of multiplicity r, one point of multiplicity s and the remaining singular points of multiplicity 2. If*

- $r \geq s \geq \lfloor \frac{d}{3} \rfloor + 1 \geq 3$;
- $r + s = d - 2$,

then C is not Cremona equivalent to a line.

Proof. By Lemma 3.3.3, with $m_1 = r$, $m_2 = s$, $m_3 = 2$, it follows that C is not Cremona equivalent to a line. □

Let $C \subset \mathbb{P}^2$ be an irreducible and reduced curve, then, by Proposition 3.2.5, there exists a birational map $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\omega(C)$ has only ordinary singular points.

Definition 3.3.6. Let $C \subset \mathbb{P}^2$ be a rational, irreducible and reduced curve of degree $d \geq 6$. For any point $p \in C$, we consider the invariant $d_C(p)$ (see Definition 2.1.4). For any plane rational curve C, we put:

$$i(C) = \max\{d_C(p) \mid p \in C\}.$$

Let observe that $i(C)$ is well–defined and it is a finite number. Moreover, if C is smooth, then $i(C) = 0$, while if C has only ordinary singularities, then $i(C) = 1$.

Let prove the following results:

Lemma 3.3.7. *Let $C \subset \mathbb{P}^2$ be a rational, irreducible and reduced curve of degree $d \geq 6$. If $e_{\max} = d-3$, then there exists a birational map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\phi(C)$ is a rational, irreducible and reduced curve of degree $\overline{d} \leq d$ and $e_{\max} = \overline{d} - 3$ having only ordinary singular points.*

Proof. Let $C \subset \mathbb{P}^2$ be a rational, irreducible and reduced curve of degree d and invariant $i(C) = i$. Let $p \in C$ be the point such that $\operatorname{mult}_p C = d - 3$.

Let prove the lemma by induction on i.
Initial case. $i = 1$. We have that C has only ordinary singular points.
Induction step. Assume that the lemma holds for any $i \leq j$.

Let $C \subset \mathbb{P}^2$ be a rational, irreducible and reduced curve of degree d and invariant $i(C) = j + 1$. Let $p, p' \in C$ be points such that $\operatorname{mult}_p C = d - 3$ and $\operatorname{mult}_{p'} C = m \leq 3$.

We consider a Cremona transformation with center p, p', q, where $q \in C$ is general, then there exists a birational map $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\omega(C) = C'$, where C' is a rational, irreducible and reduced curve of degree $2d - (d - 3 + m + 1) = d - m + 2 = d' \leq d$, with $i(C') \leq i(C)$. Since $q \in C$ is general, the map ω introduce an ordinary singular point p_1 of multiplicity $d - m - 1 = d' - 3$ and one node.

We have two cases:

i) If $d_{C'}(\widetilde{p}) < i(C) = j+1$, for any point $\widetilde{p} \in C'$, then $i(C') < j+1$. By induction step we can conclude.

ii) If there exists a point $\overline{p} \in C'$ of multiplicity $\overline{m}$ such that $d_{C'}(\overline{p}) = i(C) = j+1$, we consider a a Cremona transformation with center $p_1, \overline{p}, \overline{q}$, where $\overline{q} \in C'$ is general, and we repeat the above arguments. After finitely many steps, we obtain a rational, irreducible and reduced curve $\overline{C} \subset \mathbb{P}^2$ of degree $\overline{d} \leq d'$ having one singular point of multiplicity $\overline{d}-3$ and $i(\overline{C}) < j+1$. By induction step we can conclude.

These arguments conclude our proof. □

Lemma 3.3.8. *Let $C \subset \mathbb{P}^2$ be a rational, irreducible and reduced curve of degree $d \geq 6$. If $e_{\max} \geq d-3$, then C is Cremona equivalent to a line.*

Proof. If $e_{\max} = d-1$, then there exists a birational map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\phi(C)$ is a rational, irreducible and reduced curve of degree d having only ordinary singular points, with $e_{\max} = d-1$. By Lemma 3.2.8, the curve $\phi(C)$ is *Cremona equivalent to a line*. Hence C is *Cremona equivalent to a line*.

If $e_{\max} = d-2$, then there exists a birational map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\phi(C)$ is a rational, irreducible and reduced curve of degree d having only ordinary singular points, with $e_{\max} = d-2$. By Lemma 3.3.2, the curve $\phi(C)$ is *Cremona equivalent to a line*. Hence C is *Cremona equivalent to a line*.

If $e_{\max} = d-3$, then there exists a birational map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\phi(C) = \overline{C}$ is a rational, irreducible and reduced curve of degree $\overline{d} \leq d$ having only ordinary singular points, with $e_{\max} = \overline{d}-3$. Hence, in the following, we consider a rational, irreducible and reduced curve $\overline{C} \subset \mathbb{P}^2$ of degree $\overline{d}$ having only ordinary singular points, with $e_{\max} = \overline{d}-3$.

We prove the lemma by induction on $\overline{d}$.

Initial case. Assume $\overline{d} = 6$ and $e_{\max} = 3$. Let $\nu : S \to \mathbb{P}^2$ be a minimal resolution of singularities of $\overline{C}$ and let $\overline{C}_S = \nu_*^{-1}(\overline{C})$ be the strict transform of $\overline{C}$. If $e_{\max} = 3$, we have that $\overline{C}_S^2$ is minimum when $\overline{C}$ has only one point of multiplicity 3 and seven nodes as singularities. In this case we get

$$\overline{C}_S^2 = \overline{d}^2 - \sum_{i=1}^{k} e_i^2 = 36 - 9 - 28 = -1,$$

then, by Proposition 3.2.11, the curve $\overline{C}$ is *Cremona equivalent to a line*. By minimality of $\overline{C}_S^2$, we can conclude that if $e_{\max} = 3$, then $\overline{C}$ is

Cremona equivalent to a line.
INDUCTION STEP. Assume that the lemma holds for $\overline{d} \leq h$.
Let $\overline{C}$ be a rational, irreducible and reduced curve of degree $h+1$ having only ordinary singular points. Suppose that $e_{\max} = (h+1) - 3 = h - 2$. The remaining singular points of $\overline{C}$ is such that $e_i \leq 3$. We have three cases:

1) If $\overline{C}$ has at least two nodes as singularities, we consider a Cremona transformation $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with center p_1, p_2, p_3, where $\operatorname{mult}_{p_1} \overline{C} = h - 2$, $\operatorname{mult}_{p_2} \overline{C} = 2$ and $\operatorname{mult}_{p_3} \overline{C} = 2$. We have that $\omega(\overline{C})$ is a rational curve of degree $2(h+1) - (h-2) - 4 = h$ having a singular point of multiplicity $(h+1) - 4 = h - 3$. By Lemma 3.3.7, $\omega(\overline{C})$ is Cremona equivalent to a rational, irreducible and reduced curve of degree $h' \leq h$ and $e_{\max} = h' - 3$ having only ordinary singular points. By induction step, we can conclude that $\overline{C}$ is *Cremona equivalent to a line.*
2) If $\overline{C}$ has only one node as singularities, we consider a Cremona transformation $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with center p_1, p_2, p_3, where $\operatorname{mult}_{p_1} \overline{C} = h-2$, $\operatorname{mult}_{p_2} \overline{C} = 3$ and $\operatorname{mult}_{p_3} \overline{C} = 2$. We have that $\omega(\overline{C})$ is a rational curve of degree $2(h+1) - (h-2) - 5 = h - 1$ having a singular point of multiplicity $(h+1) - 5 = h - 4 = (h-1) - 3$. By Lemma 3.3.7, $\omega(\overline{C})$ is Cremona equivalent to a rational, irreducible and reduced curve of degree $h' \leq h - 1$ and $e_{\max} = h' - 3$ having only ordinary singular points. By induction step, we can conclude that $\overline{C}$ is *Cremona equivalent to a line.*
3) If $\overline{C}$ has only singular points of multiplicity 3, we consider a Cremona transformation $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with center p_1, p_2, p_3, where $\operatorname{mult}_{p_1} \overline{C} = h - 2$, $\operatorname{mult}_{p_2} \overline{C} = 3$ and $\operatorname{mult}_{p_3} \overline{C} = 3$. We have that $\omega(\overline{C})$ is a rational curve of degree $2(h+1) - (h-2) - 6 = h - 2$ having a singular point of multiplicity $(h+1) - 6 = h - 5 = (h-2) - 3$. By Lemma 3.3.7, $\omega(\overline{C})$ is Cremona equivalent to a rational, irreducible and reduced curve of degree $h' \leq h - 2$ and $e_{\max} = h' - 3$ having only ordinary singular points. By induction step, we can conclude that $\overline{C}$ is *Cremona equivalent to a line.*

This concludes our proof. □

3.2. Results about plane rational curves in terms of MMP.

Lemma 3.3.9. *Let $C \subset \mathbb{P}^2$ be a rational, irreducible and reduced curve of degree $d \geq 6$ such that the pair $\left(\mathbb{P}^2, \frac{3}{d}C\right)$ has ordinary canonical* *singularities. Let $\nu : S \to \mathbb{P}^2$ be a minimal resolution of singularities of C and let $C_S = \nu_*^{-1}(C)$ be the strict transform of C. Then, the pair (S, C_S) is &–minimal. Moreover, the pair (S, C_S) is the unique &–minimal model of $(\mathbb{P}^2, C)$.*

Proof. Let $p_1, \dots, p_N \in C$ be the ordinary singular points of C such that $\operatorname{mult}_{p_i} C = e_i$, with $e_i \leq \frac{d}{3}$, for any $i = 1, \dots, N$. The resolution $\nu : S \to \mathbb{P}^2$ is the blow up of $\mathbb{P}^2$ at $p_1, \dots, p_N$ with exceptional divisors $E_1, \dots, E_N$. Then $\nu^* C = C_S + e_i E_1 + \cdots + e_N E_N$ and C_S is smooth.

We have to prove that for any (-1)–curve $l \subset S$, we have $l \cdot C_S \geq 2$. The (-1)–curves in S are the exceptional divisors $E_1, \dots, E_N$ and the rational, irreducible and reduced curves $R \subset \mathbb{P}^2$ of degree m, having $\operatorname{mult}_{p_i} R = k_i \geq 0$, for $i = 1, \dots, N$, with $\sum_{i=1}^{N} k_i^2 = m^2 + 1$.

We have that

$$C_S \cdot E_j = (\nu^* C - e_1 E_1 - \ldots - e_N E_N) \cdot E_j = -e_j E_j^2 = e_j \geq 2,$$

for any $j = 1, \dots, N$.

Moreover, since $\nu^* R = R_S + k_1 E_1 + \ldots + k_N E_N$, we have that

$$C_S \cdot R_S = (\nu^* C - e_1 E_1 - \ldots - e_N E_N) \cdot (\nu^* R - k_1 E_1 - \ldots - k_N E_N) =$$

$$= \nu^* C \cdot \nu^* R + e_1 k_1 E_1^2 + \ldots + e_N k_N E_N^2 = dm - \sum_{i=1}^{N} e_i k_i.$$

Since R is a rational curve of degree m, by genus formula, we have

$$\sum_{i=1}^{N} \frac{k_i(k_i - 1)}{2} = \frac{(m-1)(m-2)}{2} = \frac{m^2 - 3m + 2}{2}.$$

Then, we get $\sum_{i=1}^{N} k_i = \sum_{i=1}^{N} {k_i}^2 - m^2 + 3m - 2$. By hypothesis, $\sum_{i=1}^{N} k_i^2 = m^2 + 1$, hence $\sum_{i=1}^{N} k_i = 3m - 1$. Since $e_i \leq \frac{d}{3}$. for any $i = 1, \dots, N$, we obtain that

$$C_S \cdot R_S = dm - \sum_{i=1}^{N} e_i k_i \geq dm - \frac{d}{3} \sum_{i=1}^{N} k_i = dm - \frac{d}{3}(3m - 1) = \frac{d}{3} \geq 2,$$

since $d \geq 6$.

Therefore, (S, C_S) is an &–minimal pair, i.e. (S, C_S) is the &–minimal model of $(\mathbb{P}^2, C)$. Now, we have to prove that (S, C_S) is the unique &–minimal model of $(\mathbb{P}^2, C)$.
Since the pair $\left(\mathbb{P}^2, \frac{3}{d}C\right)$ has canonical singularities, by Remark 3.2.7, C is not Cremona equivalent to a line. Hence the cases 1), 2), 3) of Theorem 1.3.7 don't hold.

Since C is rational and C_S is smooth, by adjunction formula, we have that $(K_S + C_S) \cdot C_S = -2 < 0$, then $K_S + C_S$ is not nef and, in particular, the case 6) of Theorem 1.3.7 doesn't hold.

Since S is rational, the canonical divisor K_S is not nef. In particular, the case 5) of Theorem 1.3.7 doesn't hold.

Therefore, the case 4) of Theorem 1.3.7 holds, i.e. $C_S \cong \mathbb{P}^1$, $C_S^2 \leq -4$ and $K_S + \lambda C_S$ is nef, where $\lambda = 1 + \frac{2}{C_S^2} < 1$. By Proposition 1.3.8 and Theorem 1.3.9, (S, C_S) is the unique &–minimal model of $(\mathbb{P}^2, C)$. $\square$

Remark 3.3.10. Under the hypotheses of Lemma 3.3.9, we consider (S, C_S) be the unique &–minimal model of the pair $(\mathbb{P}^2, C)$. Since the case 4) of Theorem 1.3.7 holds for the pair (S, C_S), then, by Lemma 3.2.24, we have that $\kappa(S, C_S) \geq 0$.

We have the following:

Corollary 3.3.11. *Let $C, C' \subset \mathbb{P}^2$ be rational, irreducible and reduced curves of degrees $d, d' \geq 6$ such that the pairs $\left(\mathbb{P}^2, \frac{3}{d}C\right)$, $\left(\mathbb{P}^2, \frac{3}{d'}C'\right)$ have ordinary canonical singularities. If $\sharp(Sing\ C) \neq \sharp(Sing\ C')$, then there not exists a birational map $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\varphi(C) = C'$.*

Proof. Let $\nu : S \to \mathbb{P}^2$ be a minimal resolution of singularities of C and let $C_S = \nu_*^{-1}(C)$ be the strict transform of C. Let $\nu' : S' \to \mathbb{P}^2$ be a minimal resolution of singularities of C' and let $C'_S = \nu_*^{-1}(C')$ be the strict transform of C'. Since the pairs $\left(\mathbb{P}^2, \frac{3}{d}C\right)$, $\left(\mathbb{P}^2, \frac{3}{d'}C'\right)$ have canonical singularities, by Lemma 3.3.9, (S, C_S) and (S', C'_S) are the unique &–minimal models of $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C')$ respectively. Moreover, by hypothesis, $\sharp(\text{Sing } C) \neq \sharp(\text{Sing } C')$, i.e. rk (Pic (S)) $\neq$ rk (Pic (S')). Then we have that (S, C_S) and (S', C'_S) are not isomorphic, hence there not exists a birational map $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\varphi(C) = C'$. $\square$

Remark 3.3.12. Let C' be a rational, irreducible and reduced curve of degree 7 having 15 nodes as singularities. Since the pair $\left(\mathbb{P}^2, \frac{3}{d'}C'\right)$ has canonical singularities, by Remark 3.2.7, C' is not Cremona equivalent to a line. Moreover, by Corollary 3.3.11, we have that there not exists a birational map $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\varphi(C) = C'$, where C is a rational, irreducible and reduced curve of degree 6 having 10 nodes as singularities, which is not Cremona equivalent to a line. Hence C' is a rational curve not Cremona equivalent to a line and not Cremona equivalent to a rational curve not Cremona equivalent to a line of degree $d < 7$.

The condition $\sharp(\text{Sing } C) \neq \sharp(\text{Sing } C')$ is not necessary to the not Cremona equivalence between C and C'. Namely, there exist rational curves C, C' having ordinary canonical singular points, with $\sharp(\text{Sing } C) = \sharp(\text{Sing } C')$, such that there not exists a birational map $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\varphi(C) = C'$.

Example 3.3.13. *Let C be the rational, irreducible and reduced curve of degree 6 with ten nodes as singularities. Let C' be the rational, irreducible and reduced curve of degree 15 with nine points of multiplicity 5 and one node as singularities. We have that $\sharp(\textit{Sing } C) = \sharp(\textit{Sing } C')$.*

Let $p : S \to \mathbb{P}^2$ be a minimal resolution of singularities of C and let $C_S = p_*^{-1}(C)$ be the strict transform of C. By Lemma 3.3.9, *(S, C_S) is the (unique) &–minimal model of $(\mathbb{P}^2, C)$, then for any (-1)–curve $R_S \subset S$, we have $R_S \cdot C_S = 2$.*

Suppose that there exists a birational map $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\varphi(C) = C'$, i.e. C, C' are Cremona equivalent curves.
Let $q : S' \to \mathbb{P}^2$ be a minimal resolution of singularities of C' and let $C'_{S'} = q_*^{-1}(C')$ be the strict transform of C'. Since C, C' are Cremona equivalent curves and since, by Lemma 3.3.9, *the pairs $(\mathbb{P}^2, C)$, $(\mathbb{P}^2, C')$ has unique &–minimal models, then there exists an isomorphism $f : (S, C_S) \to (S', C'_{S'})$. Hence we can consider the pair (S, C_S) as the &–minimal model of $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C')$, where $p_*^{-1}(C) = C_S = q_*^{-1}(C')$.*

$$\begin{tikzcd} & C_S \subset S \arrow[dl, "p"'] \arrow[dr, "q"] & \\ C \subset \mathbb{P}^2 \arrow[rr, dotted, "\varphi"] & & \mathbb{P}^2 \supset C' \end{tikzcd}$$

Since C' has singular points of multiplicity 5, then there exists a (-1)–curve $l_S \subset S$ such that $C_S \cdot l_S = 5$, which is a contradiction.

Therefore, there not exists a birational map $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\varphi(C) = C'$.

Remark 3.3.14. Let $C, C' \subset \mathbb{P}^2$ be rational, irreducible and reduced curves of degrees $d, d' \geq 6$. Let $\nu : S \to \mathbb{P}^2$ be a minimal resolution of singularities of C and let $C_S = \nu_*^{-1}(C)$ be the strict transform of C. Let $\nu' : S' \to \mathbb{P}^2$ be a minimal resolution of singularities of C' and let $C'_S = \nu_*^{-1}(C')$ be the strict transform of C'. If there exists a birational map $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\varphi(C) = C'$, then ${C_S}^2 = {C'_S}^2$.

In general, we have that $\sharp(\text{Sing } C) = \sharp(\text{Sing } C')$ not implies that ${C_S}^2 = {C'_S}^2$. Namely, there exist rational curves C, C' having $\sharp(\text{Sing } C) = \sharp(\text{Sing } C')$, but ${C_S}^2 \neq {C'_S}^2$. Hence, by Remark 3.3.14, there not exists a birational map $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\varphi(C) = C'$.

Example 3.3.15. *Let C be the rational, irreducible and reduced curve of degree 7 with only nodes as singularities. Let C' be the rational, irreducible and reduced curve of degree 15 with eight points of multiplicity 5, two points of multiplicity 3 and five node as singularities. Note that $\sharp(\mathit{Sing}\ C) = \sharp(\mathit{Sing}\ C') = 15$. On the other hand, we get*

$$ {C_S}^2 = d^2 - \sum_{i=1}^{15} (\text{mult}_{p_i} C)^2 = 49 - 60 = -11 $$

and

$$ {C'_S}^2 = d'^2 - \sum_{i=1}^{15} (\text{mult}_{p'_i} C')^2 = 225 - 200 - 18 - 20 = -13. $$

Then ${C_S}^2 \neq {C'_S}^2$. Hence, by Remark 3.3.14, *there not exists a birational map $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\varphi(C) = C'$.*

Moreover, we have that ${C_S}^2 = {C'_S}^2$ not implies that C, C' are Cremona equivalent curves. Namely, there exist rational curves C, C' having ${C_S}^2 = {C'_S}^2$, such that there not exists a birational map $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\varphi(C) = C'$.

Example 3.3.16. *Let C be the rational, irreducible and reduced curve of degree 6 with ten nodes as singularities. Let C' be the rational, irreducible and reduced curve of degree 15 with nine points of multiplicity 5 and one node as singularities (see* Example 3.3.13*). Note that $\sharp(\mathit{Sing}\ C) = \sharp(\mathit{Sing}\ C') = 10$. We have that*

$$ {C_S}^2 = d^2 - \sum_{i=1}^{10} (\text{mult}_{p_i} C)^2 = 36 - 40 = -4 $$

and

$$C'^{\,2}_S = d'^2 - \sum_{i=1}^{10} (\operatorname{mult}_{p'_i} C')^2 = 225 - 225 - 4 = -4.$$

Then ${C_S}^2 = {C'_S}^2$. *On the other hand, in* Example 3.3.13 *we have proved that there not exists a birational map* $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ *such that* $\varphi(C) = C'$.

Finally, we have the following:

Proposition 3.3.17. *Let* $C, C' \subset \mathbb{P}^2$ *be rational, irreducible and reduced curves of degrees* $d, d' \geq 6$, *having ordinary canonical singularities. Let* $\nu : S \to \mathbb{P}^2$ *be a minimal resolution of singularities of* C *and let* $C_S = \nu_*^{-1}(C)$ *be the strict transform of* C. *Let* $\nu' : S' \to \mathbb{P}^2$ *be a minimal resolution of singularities of* C' *and let* $C'_S = \nu_*^{-1}(C')$ *be the strict transform of* C'. *There exists a birational map* $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ *such that* $\varphi(C) = C'$, *i.e.* C, C' *are Cremona equivalent curves if and only if* $S \cong S'$ *and* $C_S \cong C'_{S'}$.

Proof. "$\Leftarrow$". It is obvious.
"$\Rightarrow$". Since the pairs $\left(\mathbb{P}^2, \frac{3}{d}C\right)$, $\left(\mathbb{P}^2, \frac{3}{d'}C'\right)$ have canonical singularities, by Lemma 3.3.9, (S, C_S) and (S', C'_S) are the unique &–minimal models of $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C')$ respectively.

Since C, C' are Cremona equivalent curves and since the pairs $(\mathbb{P}^2, C)$, $(\mathbb{P}^2, C')$ have unique &–minimal models, then there exists an isomorphism $f : (S, C_S) \to (S', C'_{S'})$. Hence, $S \cong S'$ and $C_S \cong C'_{S'}$ and we conclude our proof. $\square$

4. Models of rational plane curves of low degree with ordinary singularities

In this Section, we give a table containing models of rational plane curves of degree $6 \leq d \leq 10$ with only ordinary singular points, according to Theorem 3.1.4.

Let $C \subset \mathbb{P}^2$ be a rational, irreducible and reduced curve of degree d. In the previous Sections, we have proved the following useful results:

- If C has an unique singular point of multiplicity $d-1$, we have that the curve C is *Cremona equivalent to a line*, see Lemma 3.2.8.
- If C has a singular point of multiplicity $d-2$, by Lemma 3.3.2, we have that C is *Cremona equivalent to a line*.

- If C is a rational curve of degree $d \geq 6$ having a singular point of multiplicity $d-3$, then, by Lemma 3.3.8, C is *Cremona equivalent to a line*.
- In Example 3.2.12 is proved that any rational curve C of degree $d \geq 6$ having only ordinary nodes as singularities is not Cremona equivalent to a line.
- By Remarks 3.2.7, 3.3.1, if $e_i \leq \frac{d}{3}$, for any $i = 1, \ldots, k$, then C is not Cremona equivalent to a line.
- By Remark 3.2.14, any rational, irreducible and reduced curve $C \subset \mathbb{P}^2$ of degree $d \leq 5$ is *Cremona equivalent to a line*.

Let $C \subset \mathbb{P}^2$ be a rational, irreducible and reduced curve of degree $6 \leq d \leq 10$, having only ordinary singular points of multiplicity m_i, for $i = 1, \ldots, k$, with $m_1 \geq m_2 \geq m_3 \geq \cdots \geq m_k \geq 2$. By Definition 3.1.1, the curve C is said to be of type $[d; m_1, \ldots, m_k]$. We can summarize the possible models of such curves according to Theorem 3.1.4 in the following table.

Remark 3.4.1. The assumption allow to drastically reduced the length of the table.

Ordinary singularities avoid situations like in Example 3.1.7, while the low degree ensures that the general fiber $f \subset \mathbb{F}_a$ is the strict transform of the line in $\mathbb{P}^2$ through the point of multiplicity m_1.

Type of the curve C	Model	Type of the curve $\widetilde{C}$
$[6; m_1, \ldots, m_k]$, with $m_1 \geq 3$	$(\mathbb{P}^2, \widetilde{C})$, where $\widetilde{C} = l$	$[1; 1]$
$[6; 2^{10}]$	$(\mathbb{P}^2, \frac{1}{2}\widetilde{C})$, with $\overline{\kappa} = 0$	$[6; 2^{10}]$
$[7; m_1, \ldots, m_k]$, with $m_1 \geq 4$	$(\mathbb{P}^2, \widetilde{C})$, where $\widetilde{C} = l$	$[1; 1]$
$[7; 3^3, m_4, \ldots, m_k]$	$(\mathbb{P}^2, \widetilde{C})$, where $\widetilde{C} = l$	$[1; 1]$
$[7; 3^2, 2^9]$	$(\mathbb{P}^2, \frac{1}{2}\widetilde{C})$, with $\overline{\kappa} = 0$	$[6; 2^{10}]$

Type of the curve C	Model	Type of the curve $\widetilde{C}$
$[7;3,2^{12}]$	$(\mathbb{F}_1,\frac{1}{2}\widetilde{C})$, with $\overline{\kappa}=1$	$[4*7,1;2^{12}]$
$[7;2^{15}]$	$(\mathbb{P}^2,\frac{3}{7}\widetilde{C})$, with $\overline{\kappa}=0$	$[7;2^{15}]$
$[8;m_1,\ldots,m_k]$, with $m_1\geq 5$	$(\mathbb{P}^2,\widetilde{C})$, where $\widetilde{C}=l$	$[1;1]$
$[8;4^3,m_4,\ldots,m_k]$, $[8;4^2,3,m_4,\ldots,m_k]$, $[8;4,3^3,m_5,\ldots,m_k]$	$(\mathbb{P}^2,\widetilde{C})$, where $\widetilde{C}=l$	$[1;1]$
$[8;4^2,2^9]$, $[8;4,3^2,2^9]$	$(\mathbb{P}^2,\frac{1}{2}\widetilde{C})$, with $\overline{\kappa}=0$	$[6;2^{10}]$
$[8;4,3,2^{12}]$	$(\mathbb{F}_1,\frac{1}{2}\widetilde{C})$, with $\overline{\kappa}=1$ $(\mathbb{F}_2,\frac{1}{2}\widetilde{C})$, with $\overline{\kappa}=1$	$[4*7,1;2^{12}]$ $[4*9,2;2^{12}]$
$[8;4,2^{15}]$	$(\mathbb{F}_1,\frac{1}{2}\widetilde{C})$, with $\overline{\kappa}=1$	$[4*8,1;2^{15}]$
$[8;3^6,m_7,\ldots,m_k]$	$(\mathbb{P}^2,\widetilde{C})$, where $\widetilde{C}=l$	$[1;1]$
$[8;3^5,2^6]$	$(\mathbb{P}^2,\frac{1}{2}\widetilde{C})$, with $\overline{\kappa}=0$	$[6;2^{10}]$
$[8;3^4,2^9]$	$(\mathbb{F}_1,\frac{1}{2}\widetilde{C})$, with $\overline{\kappa}=1$	$[4*7,1;2^{12}]$
$[8;3^3,2^{12}]$	$(\mathbb{P}^2,\frac{3}{7}\widetilde{C})$, with $\overline{\kappa}=0$	$[7;2^{15}]$
$[8;3^2,2^{15}]$	$(\mathbb{F}_0,\frac{2}{5}\widetilde{C})$, with $\overline{\kappa}=0$	$[5*5,0;2^{16}]$
$[8;3,2^{18}]$	$(\mathbb{F}_1,\frac{2}{5}\widetilde{C})$, with $\overline{\kappa}=1$	$[5*8,1;2^{18}]$
$[8;2^{21}]$	$(\mathbb{P}^2,\frac{3}{8}\widetilde{C})$, with $\overline{\kappa}=0$	$[8;2^{21}]$

Type of the curve C	Model	Type of the curve $\widetilde{C}$
$[9; m_1, \ldots, m_k]$, with $m_1 \geq 6$	$(\mathbb{P}^2, \widetilde{C})$, where $\widetilde{C} = l$	$[1; 1]$
$[9; 5, 4^2, m_4, \ldots, m_k]$, $[9; 5, 4, 3^2, m_5, \ldots, m_k]$, $[9; 5, 3^4, m_6, \ldots, m_k]$	$(\mathbb{P}^2, \widetilde{C})$, where $\widetilde{C} = l$	$[1; 1]$
$[9; 5, 4, 3, 2^9]$, $[9; 5, 3^3, 2^9]$	$(\mathbb{P}^2, \frac{1}{2}\widetilde{C})$, with $\overline{\kappa} = 0$	$[6; 2^{10}]$
$[9; 5, 4, 2^{12}]$, $[9; 5, 3^2, 2^{12}]$	$(\mathbb{F}_1, \frac{1}{2}\widetilde{C})$, with $\overline{\kappa} = 1$ $(\mathbb{F}_2, \frac{1}{2}\widetilde{C})$, with $\overline{\kappa} = 1$	$[4 * 7, 1; 2^{12}]$ $[4 * 9, 2; 2^{12}]$
$[9; 5, 3, 2^{15}]$	$(\mathbb{F}_1, \frac{1}{2}\widetilde{C})$, with $\overline{\kappa} = 1$ $(\mathbb{F}_2, \frac{1}{2}\widetilde{C})$, with $\overline{\kappa} = 1$ $(\mathbb{F}_3, \frac{1}{2}\widetilde{C})$, with $\overline{\kappa} = 1$	$[4 * 8, 1; 2^{15}]$ $[4 * 10, 2; 2^{15}]$ $[4 * 12, 3; 2^{15}]$
$[9; 5, 2^{18}]$	$(\mathbb{F}_1, \frac{1}{2}\widetilde{C})$, with $\overline{\kappa} = 1$	$[4 * 9, 1; 2^{18}]$
$[9; 4^4, m_5, \ldots, m_k]$, $[9; 4^3, 3, m_5, \ldots, m_k]$, $[9; 4^2, 3^4, m_7, \ldots, m_k]$, $[9; 4, 3^7, 2]$	$(\mathbb{P}^2, \widetilde{C})$, where $\widetilde{C} = l$	$[1; 1]$
$[9; 4^3, 2^{10}]$, $[9; 4^2, 3^3, 2^7]$	$(\mathbb{P}^2, \frac{1}{2}\widetilde{C})$, with $\overline{\kappa} = 0$	$[6; 2^{10}]$
$[9; 4, 3^6, 2^4]$	$(\mathbb{P}^2, \frac{1}{2}\widetilde{C})$, with $\overline{\kappa} = 0$ $(\mathbb{F}_2, \frac{1}{2}\widetilde{C})$, with $\overline{\kappa} = 0$	$[6; 2^{10}]$ $[4 * 8, 2; 2^9]$
$[9; 4^2, 3^2, 2^{10}]$	$(\mathbb{F}_1, \frac{1}{2}\widetilde{C})$, with $\overline{\kappa} = 1$	$[4 * 7, 1; 2^{12}]$
$[9; 4, 3^5, 2^7]$	$(\mathbb{F}_1, \frac{1}{2}\widetilde{C})$, with $\overline{\kappa} = 1$ $(\mathbb{F}_2, \frac{1}{2}\widetilde{C})$, with $\overline{\kappa} = 1$ $(\mathbb{F}_3, \frac{1}{2}\widetilde{C})$, with $\overline{\kappa} = 1$	$[4 * 7, 1; 2^{12}]$ $[4 * 9, 2; 2^{12}]$ $[4 * 11, 3; 2^{12}]$

Type of the curve C	Model	Type of the curve $\widetilde{C}$
$[9;4^2,3,2^{13}]$, $[9;4,3^4,2^{10}]$	$(\mathbb{P}^2,\frac{3}{7}\widetilde{C})$, with $\overline{\kappa}=0$	$[7;2^{15}]$
$[9;4^2,2^{16}]$	$(\mathbb{F}_0,\frac{2}{5}\widetilde{C})$, with $\overline{\kappa}=0$	$[5*5,0;2^{16}]$
$[9;4,3^3,2^{13}]$	$(\mathbb{F}_0,\frac{2}{5}\widetilde{C})$, with $\overline{\kappa}=0$ $(\mathbb{F}_2,\frac{2}{5}\widetilde{C})$, with $\overline{\kappa}=0$	$[5*5,0;2^{16}]$ $[5*10,2;2^{16}]$
$[9;4,3^2,2^{16}]$	$(\mathbb{F}_1,\frac{2}{5}\widetilde{C})$, with $\overline{\kappa}=1$	$[5*8,1;2^{18}]$
$[9;4,3,2^{19}]$	$(\mathbb{F}_0,\frac{2}{5}\widetilde{C})$, with $\overline{\kappa}=1$	$[5*6,0;2^{20}]$
$[9;4,2^{22}]$	$(\mathbb{F}_1,\frac{2}{5}\widetilde{C})$, with $\overline{\kappa}=1$	$[5*9,1;2^{22}]$
$[9;3^f,2^h]$	$(\mathbb{P}^2,\frac{1}{3}\widetilde{C})$, with $\overline{\kappa}=0$	$[9;3^f,2^h]$
$[10;m_1,\dots,m_k]$, with $m_1\geq 7$	$(\mathbb{P}^2,\widetilde{C})$, where $\widetilde{C}=l$	$[1;1]$
$[10;6,4^3,m_5,\dots,m_k]$, $[10;6,4^2,3,m_5,\dots,m_k]$, $[10;6,4,3^3,m_6,\dots,m_k]$, $[10;6,3^5,m_7,\dots,m_k]$	$(\mathbb{P}^2,\widetilde{C})$, where $\widetilde{C}=l$	$[1;1]$
$[10;6,4^2,2^9]$, $[10;6,3^4,2^9]$	$(\mathbb{P}^2,\frac{1}{2}\widetilde{C})$, with $\overline{\kappa}=0$	$[6;2^{10}]$
$[10;6,4,3^2,2^9]$	$(\mathbb{P}^2,\frac{1}{2}\widetilde{C})$, with $\overline{\kappa}=0$ $(\mathbb{F}_2,\frac{1}{2}\widetilde{C})$, with $\overline{\kappa}=0$	$[6;2^{10}]$ $[4*8,2;2^9]$
$[10;6,4,3,2^{12}]$, $[10;6,3^3,2^{12}]$	$(\mathbb{F}_1,\frac{1}{2}\widetilde{C})$, with $\overline{\kappa}=1$, $(\mathbb{F}_2,\frac{1}{2}\widetilde{C})$, with $\overline{\kappa}=1$	$[4*7,1;2^{12}]$, $[4*9,2;2^{12}]$

Type of the curve C	Model	Type of the curve $\widetilde{C}$
$[10;6,4,2^{15}]$, $[10;6,3^2,2^{15}]$	$(\mathbb{F}_1,\frac{1}{2}\widetilde{C})$, with $\overline{\kappa}=1$, $(\mathbb{F}_2,\frac{1}{2}\widetilde{C})$, with $\overline{\kappa}=1$, $(\mathbb{F}_3,\frac{1}{2}\widetilde{C})$, with $\overline{\kappa}=1$	$[4*8,1;2^{15}]$, $[4*10,2;2^{15}]$, $[4*12,3;2^{15}]$
$[10;6,3,2^{18}]$	$(\mathbb{F}_1,\frac{1}{2}\widetilde{C})$, with $\overline{\kappa}=1$, $(\mathbb{F}_2,\frac{1}{2}\widetilde{C})$, with $\overline{\kappa}=1$, $(\mathbb{F}_3,\frac{1}{2}\widetilde{C})$, with $\overline{\kappa}=1$,	$[4*9,1;2^{18}]$, $[4*11,2;2^{18}]$, $[4*13,3;2^{18}]$
$[10;6,2^{21}]$	$(\mathbb{F}_1,\frac{1}{2}\widetilde{C})$, with $\overline{\kappa}=1$	$[4*10,1;2^{21}]$
$[10;5^3,m_4,\dots,m_k]$, $[10;5^2,4^2,m_5,\dots,m_k]$, $[10;5^2,4,3,m_5,\dots,m_k]$, $[10;5^2,3^4,m_7,\dots,m_k]$, $[10;5,4^3,m_5,\dots,m_k]$, $[10;5,4^2,3^3,m_7,\dots,m_k]$, $[10;5,4,3^6,2^2]$	$(\mathbb{P}^2,\widetilde{C})$, where $\widetilde{C}=l$	$[1;1]$
$[10;5^2,4,2^{10}]$, $[10;5^2,3^3,2^7]$, $[10;5,4^2,3^2,2^8]$	$(\mathbb{P}^2,\frac{1}{2}\widetilde{C})$, with $\overline{\kappa}=0$	$[6;2^{10}]$
$[10;5,4,3^5,2^5]$, $[10;5,3^8,2^2]$	$(\mathbb{P}^2,\frac{1}{2}\widetilde{C})$, with $\overline{\kappa}=0$, $(\mathbb{F}_2,\frac{1}{2}\widetilde{C})$, with $\overline{\kappa}=0$	$[6;2^{10}]$, $[4*8,2;2^9]$
$[10;5^2,3^2,2^{10}]$, $[10;5,4^2,3,2^{11}]$	$(\mathbb{F}_1,\frac{1}{2}\widetilde{C})$, with $\overline{\kappa}=1$	$[4*7,1;2^{12}]$
$[10;5,4,3^4,2^8]$, $[10;5,3^7,2^5]$	$(\mathbb{F}_1,\frac{1}{2}\widetilde{C})$, with $\overline{\kappa}=1$, $(\mathbb{F}_2,\frac{1}{2}\widetilde{C})$, with $\overline{\kappa}=1$, $(\mathbb{F}_3,\frac{1}{2}\widetilde{C})$, with $\overline{\kappa}=1$	$[4*7,1;2^{12}]$, $[4*9,2;2^{12}]$, $[4*11,3;2^{12}]$

Type of the curve C	Model	Type of the curve $\widetilde{C}$
$[10;5^2,3,2^{13}]$, $[10;5,4^2,2^{14}]$, $[10;5,4,3^3,2^{11}]$, $[10;5,3^6,2^8]$	$(\mathbb{P}^2,\frac{3}{7}\widetilde{C})$, with $\overline{\kappa}=0$	$[7;2^{15}]$
$[10;5^2,2^{16}]$	$(\mathbb{F}_0,\frac{2}{5}\widetilde{C})$, with $\overline{\kappa}=0$	$[5*5,0;2^{16}]$
$[10;5,4,3^2,2^{14}]$, $[10;5,3^5,2^{11}]$	$(\mathbb{F}_0,\frac{2}{5}\widetilde{C})$, with $\overline{\kappa}=0$, $(\mathbb{F}_2,\frac{2}{5}\widetilde{C})$, with $\overline{\kappa}=0$	$[5*5,0;2^{16}]$, $[5*10,2;2^{16}]$
$[10;5,4,3,2^{17}]$, $[10;5,3^4,2^{14}]$	$(\mathbb{F}_1,\frac{2}{5}\widetilde{C})$, with $\overline{\kappa}=1$	$[5*8,1;2^{18}]$
$[10;5,4,2^{20}]$	$(\mathbb{F}_0,\frac{2}{5}\widetilde{C})$, with $\overline{\kappa}=1$	$[5*6,0;2^{20}]$
$[10;5,3^3,2^{17}]$	$(\mathbb{F}_0,\frac{2}{5}\widetilde{C})$, with $\overline{\kappa}=1$, $(\mathbb{F}_2,\frac{2}{5}\widetilde{C})$, with $\overline{\kappa}=1$	$[5*6,0;2^{20}]$, $[5*11,2;2^{20}]$
$[10;5,3^2,2^{20}]$	$(\mathbb{F}_1,\frac{2}{5}\widetilde{C})$, with $\overline{\kappa}=1$	$[5*9,1;2^{22}]$
$[10;5,3,2^{23}]$	$(\mathbb{F}_0,\frac{2}{5}\widetilde{C})$, with $\overline{\kappa}=1$	$[5*7,0;2^{24}]$
$[10;5,2^{26}]$	$(\mathbb{F}_1,\frac{2}{5}\widetilde{C})$, with $\overline{\kappa}=1$	$[5*10,1;2^{26}]$
$[10;4^6]$, $[10;4^5,3,m_7,\ldots,m_k]$, $[10;4^4,3^3,m_8,\ldots,m_k]$, $[10;4^3,3^6]$	$(\mathbb{P}^2,\widetilde{C})$, where $\widetilde{C}=l$	$[1;1]$
$[10;4^5,2^6]$, $[10;4^4,3^2,2^6]$, $[10;4^3,3^5,2^3]$	$(\mathbb{P}^2,\frac{1}{2}\widetilde{C})$, with $\overline{\kappa}=0$	$[6;2^{10}]$

Type of the curve C	Model	Type of the curve $\widetilde{C}$
$[10; 4^4, 3, 2^9]$	$(\mathbb{F}_1, \frac{1}{2}\widetilde{C})$, with $\overline{\kappa} = 1$, $(\mathbb{F}_2, \frac{1}{2}\widetilde{C})$, with $\overline{\kappa} = 1$, $(\mathbb{F}_3, \frac{1}{2}\widetilde{C})$, with $\overline{\kappa} = 1$	$[4 * 7, 1; 2^{12}]$, $[4 * 9, 2; 2^{12}]$, $[4 * 11, 3; 2^{12}]$
$[10; 4^3, 3^4, 2^6]$	$(\mathbb{F}_1, \frac{1}{2}\widetilde{C})$, with $\overline{\kappa} = 1$	$[4 * 7, 1; 2^{12}]$
$[10; 4^4, 2^{12}]$	$(\mathbb{F}_1, \frac{1}{2}\widetilde{C})$, with $\overline{\kappa} = 1$	$[4 * 8, 1; 2^{15}]$
$[10; 4^3, 3^3, 2^9]$	$(\mathbb{P}^2, \frac{3}{7}\widetilde{C})$, with $\overline{\kappa} = 0$	$[7; 2^{15}]$
$[10; 4^3, 3^2, 2^{12}]$	$(\mathbb{F}_0, \frac{2}{5}\widetilde{C})$, with $\overline{\kappa} = 0$	$[5 * 5, 0; 2^{16}]$
$[10; 4^3, 3, 2^{15}]$	$(\mathbb{F}_1, \frac{2}{5}\widetilde{C})$, with $\overline{\kappa} = 1$	$[5 * 8, 1; 2^{18}]$
$[10; 4^3, 2^{18}]$	$(\mathbb{P}^2, \frac{3}{8}\widetilde{C})$, with $\overline{\kappa} = 0$	$[8; 2^{21}]$
$[10; 4^2, 3^f, 2^h]$	$(\mathbb{P}^2, \frac{1}{3}\widetilde{C})$, with $\overline{\kappa} = 0$	$[9; 3^{f+1}, 2^{h+1}]$
$[10; 4^2, 2^{24}]$	$(\mathbb{F}_0, \frac{1}{3}\widetilde{C})$, with $\overline{\kappa} = 0$	$[6 * 6, 0; 2^{25}]$
$[10; 4, 3^f, 2^h]$	$(\mathbb{F}_1, \frac{1}{3}\widetilde{C})$, with $\overline{\kappa} = 1$	$[6 * 10, 1; 3^f, 2^h]$
$[10; 3^f, 2^h]$	$(\mathbb{P}^2, \frac{3}{10}\widetilde{C})$, with $\overline{\kappa} = 0$	$[10; 3^f, 2^h]$

We give some examples of computations of certain cases listed in the above table. Similarly, we can prove the remaining ones.

- Let $C \subset \mathbb{P}^2$ be a rational, irreducible curve of type $[8; 4, 3, 2^{12}]$. Since C has ordinary singular points, there exists a Cremona transformation $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\omega(C) = C'$, where $C' \subset \mathbb{P}^2$ is a rational, irreducible and reduced curve of type $[7; 3, 2^{12}]$. Let $p_1 \in C'$ be the point of multiplicity 3. If p_1 is not infinitely near to any double points of C', then the log pair

$(\mathbb{F}_1, \frac{1}{2}\widetilde{C})$, where $\widetilde{C}$ is curve of type $[4*7, 1; 2^{12}]$, is a model of $(\mathbb{P}^2, \frac{1}{2}C)$ with canonical singularities and terminal singularities in a neighborhood of the (-1)–curve $C_0 \subset \mathbb{F}_1$, having $K_{\mathbb{F}_1} + \frac{1}{2}\widetilde{C}$ nef and $\overline{\kappa}(\mathbb{F}_1, \frac{1}{2}\widetilde{C}) = 1$, like in Theorem 3.1.4. If there exists a double point $q \in C'$ such that $p_1 \equiv q$, then the log pair $(\mathbb{F}_2, \frac{1}{2}\widetilde{C})$, where $\widetilde{C}$ is curve of type $[4*9, 2; 2^{12}]$, is a model of $(\mathbb{P}^2, \frac{1}{2}C)$ with canonical singularities and terminal singularities in a neighborhood of $C_0 \subset \mathbb{F}_2$, having $K_{\mathbb{F}_2} + \frac{1}{2}\widetilde{C}$ nef and $\overline{\kappa}(\mathbb{F}_2, \frac{1}{2}\widetilde{C}) = 1$, like in Theorem 3.1.4.

- Let $C \subset \mathbb{P}^2$ be a rational, irreducible curve of type $[8; 3^2, 2^{15}]$. To obtain a model listed in Theorem 3.1.4, since C has ordinary singular points, we have to blow up the points of multiplicity $m > \frac{d-m_1}{2} = \frac{5}{2}$. Then, we consider the blow up of p_1 and we apply an elementary transformation with center p_2, hence we have that the log pair $(\mathbb{F}_0, \frac{2}{5}\widetilde{C})$ is a model of $(\mathbb{P}^2, \frac{2}{5}C)$ with terminal singularities. Since with the elementary transformation with center p_2 we have introduced a singular point of multiplicity 2, then $\widetilde{C}$ is an irreducible and reduced curve of type $[5*5, 0; 2^{16}]$. Therefore, $(\mathbb{F}_0, \frac{2}{5}\widetilde{C})$ has $K_{\mathbb{F}_0} + \frac{2}{5}\widetilde{C} \sim \mathcal{O}$ nef and $\overline{\kappa}(\mathbb{F}_0, \frac{2}{5}\widetilde{C}) = 0$.
- Let $C \subset \mathbb{P}^2$ be a rational, irreducible curve of type $[8; 2^{21}]$. By Theorem 3.1.4, we have that the log pair $(\mathbb{P}^2, \frac{3}{8}C)$ is a model of $(\mathbb{P}^2, \frac{3}{8}C)$ with canonical singularities, having $K_{\mathbb{P}^2} + \frac{3}{8}C \sim \mathcal{O}$ nef $\overline{\kappa}(\mathbb{P}^2, \frac{3}{8}C) = 0$.
- Let $C \subset \mathbb{P}^2$ be a rational, irreducible curve of type $[9; 5, 3, 2^{15}]$. Since C has ordinary singular points, there exists a Cremona transformation $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\omega(C) = C'$, where $C' \subset \mathbb{P}^2$ is a rational, irreducible and reduced curve of type $[8; 4, 2^{15}]$. Let $p_1 \in C'$ be the point of multiplicity 4. If p_1 is not infinitely near to any double points of C', then the log pair $(\mathbb{F}_1, \frac{1}{2}\widetilde{C})$, where $\widetilde{C}$ is curve of type $[4*8, 1; 2^{15}]$, is a model of $(\mathbb{P}^2, \frac{1}{2}C)$ with canonical singularities and terminal singularities in a neighborhood of the (-1)–curve $C_0 \subset \mathbb{F}_1$, having $K_{\mathbb{F}_1} + \frac{1}{2}\widetilde{C}$ nef and $\overline{\kappa}(\mathbb{F}_1, \frac{1}{2}\widetilde{C}) = 1$, like in Theorem 3.1.4. If there exists an unique double point $q_1 \in C'$ such that $p_1 \equiv q_1$, then the log pair $(\mathbb{F}_2, \frac{1}{2}\widetilde{C})$, where $\widetilde{C}$ is curve of type $[4*10, 2; 2^{15}]$, is a model of $(\mathbb{P}^2, \frac{1}{2}C)$ with canonical singularities and terminal singularities in a neighborhood of $C_0 \subset \mathbb{F}_2$, having $K_{\mathbb{F}_2} + \frac{1}{2}\widetilde{C}$

nef and $\overline{\kappa}(\mathbb{F}_2, \frac{1}{2}\widetilde{C}) = 1$, like in Theorem 3.1.4. Finally, if there exist two double points $q_1, q_2 \in C'$ such that $p_1 \equiv q_1 \equiv q_2$, then the log pair $(\mathbb{F}_3, \frac{1}{2}\widetilde{C})$, where $\widetilde{C}$ is curve of type $[4*12, 3; 2^{15}]$, is a model of $(\mathbb{P}^2, \frac{1}{2}C)$ with canonical singularities and terminal singularities in a neighborhood of $C_0 \subset \mathbb{F}_3$, having $K_{\mathbb{F}_3} + \frac{1}{2}\widetilde{C}$ nef and $\overline{\kappa}(\mathbb{F}_3, \frac{1}{2}\widetilde{C}) = 1$, like in Theorem 3.1.4.

- Let $C \subset \mathbb{P}^2$ be a rational, irreducible curve of type $[9; 5, 2^{18}]$. To obtain a model listed in Theorem 3.1.4, since C has ordinary singular points, we have to blow up the points of multiplicity $m > \frac{d-m_1}{2} = 2$. Then, we consider the blow up of p_1, hence we have that the log pair $(\mathbb{F}_1, \frac{1}{2}\widetilde{C})$ is a model of $(\mathbb{P}^2, \frac{1}{2}C)$ with canonical singularities and terminal singularities in a neighborhood of the (-1)–curve $C_0 \subset \mathbb{F}_1$, where $\widetilde{C}$ is an irreducible and reduced curve of type $[4*9, 1; 2^{18}]$. Therefore, $(\mathbb{F}_1, \frac{1}{2}\widetilde{C})$ has $K_{\mathbb{F}_1} + \frac{1}{2}\widetilde{C}$ nef and $\overline{\kappa}(\mathbb{F}_1, \frac{1}{2}\widetilde{C}) = 1$.
- Let $C \subset \mathbb{P}^2$ be a rational, irreducible curve of type $[9; 4, 3, 2^{19}]$. To obtain a model listed in Theorem 3.1.4, since C has ordinary singular points, we have to blow up the points of multiplicity $m > \frac{d-m_1}{2} = \frac{5}{2}$. Then, we consider the blow up of p_1 and we apply an elementary transformation with center p_2, hence we have that the log pair $(\mathbb{F}_0, \frac{2}{5}\widetilde{C})$ is a model of $(\mathbb{P}^2, \frac{2}{5}C)$ with terminal singularities. Since with the elementary transformation with center p_2 we have introduced a singular point of multiplicity 2, then $\widetilde{C}$ is an irreducible and reduced curve of type $[5*6, 0; 2^{20}]$. Therefore, $(\mathbb{F}_0, \frac{2}{5}\widetilde{C})$ has $K_{\mathbb{F}_0} + \frac{2}{5}\widetilde{C}$ nef and $\overline{\kappa}(\mathbb{F}_0, \frac{2}{5}\widetilde{C}) = 1$.

5. Rational surfaces of low degree in $\mathbb{P}^3$

Let us consider a rational irreducible reduced surface $S \subset \mathbb{P}^3$. In analogy with Definition 2.0.5, we have:

Definition 3.5.1. An irreducible reduce rational surface $S \subset \mathbb{P}^n$, $n \geq 3$, is *Cremona equivalent to a plane* if there exists a birational map $\Psi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ such that $\Psi(C)$ is a plane.

In the following, we ask us whether a rational irreducible and reduced surface $S \subset \mathbb{P}^3$ of degree $d \leq 3$ is *Cremona equivalent to a plane*.

5.1. Quadric surfaces.

Proposition 3.5.2. *Let $Q \subset \mathbb{P}^3$ be an irreducible and reduced quadric surface. Then there exists a birational map $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ such that $\phi(Q) = \mathbb{P}^2$, i.e. Q is Cremona equivalent to a plane.*

Proof. Let Q be a quadric surface of $\mathbb{P}^3$ with $C \subset Q$ a smooth conic and let $P = \langle C \rangle$.

Consider $\Lambda = |\mathcal{O}_{\mathbb{P}^3}(2) \otimes \mathcal{I}_C|$ the linear system of quadric surfaces of $\mathbb{P}^3$ containing C.

Let $\varepsilon : X = \mathrm{Bl}_C\mathbb{P}^3 \longrightarrow \mathbb{P}^3$ be the blow up of the conic C in $\mathbb{P}^3$ with exceptional divisor E. Denote $\Lambda' = \varepsilon^*_{-1}(\Lambda)$, $P' = \varepsilon^*_{-1}(P)$, $Q' = \varepsilon^*_{-1}(Q) \in \Lambda'$ the strict transforms of Λ, P, Q respectively. We have that $\Lambda' = \varepsilon^*\Lambda - E = \varepsilon^*\mathcal{O}_{\mathbb{P}^3}(2) - E$, $P' = \varepsilon^*P - E$ and $K_X = \varepsilon^*K_{\mathbb{P}^3} + E = \varepsilon^*\mathcal{O}_{\mathbb{P}^3}(-4) + E$.

Consider a line $l \subset P$, then l meets C in two points. Let l' be the strict transform of l with respect to the blow up of C. We have that $P' \cdot l' = (\varepsilon^*P - E) \cdot l' = 1 - 2 = -1$. Then $P' = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-1))$ and we can contract P'. Since C is a complete intersection of Q and P, then $P' \cap Q' = \emptyset$ and Λ' is a base points free linear system. Hence, Λ' defines a morphism $\varphi : X \to \mathbb{P}^{k-1}$, which is the contraction of P', where $k = \dim|\Lambda'| = \dim|\Lambda| = 5$. Then the contraction of P' is a morphism $\varphi : X \to \mathbb{P}^4$ such that $\varphi(X) = \overline{Q}$, where $\overline{Q}$ is a quadric hypersurface of $\mathbb{P}^4$. Hence, there exists a birational map $\psi : \mathbb{P}^3 \dashrightarrow \overline{Q}$, where $\psi = \varphi^{|\overline{Q}} \circ \varepsilon^{-1}$, such that $\psi(Q) = \mathcal{O}_{\overline{Q}}(1)$, i.e. $\psi(Q)$ is a section of $\overline{Q}$. We denote $Q'' = \psi(Q)$, then Q'' is a quadric surface in $\overline{Q}$.

Let p be a general point of Q'' and let $\pi_p : \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ be the projection from the point p. Since $p \in Q'' \subset \overline{Q}$, then we get a birational map $\pi_{p|\overline{Q}} : \overline{Q} \dashrightarrow \mathbb{P}^3$ such that $\pi_{p|\overline{Q}}(Q'') = \mathbb{P}^2$.

Then there exists a birational map $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$, where $\phi = \pi_{p|\overline{Q}} \circ \psi$, such that $\phi(Q) = \mathbb{P}^2$. □

Remark 3.5.3. We can also prove that an irreducible and reduced quadric surface Q is Cremona equivalent to a plane thanks to the existence of the Cremona transformation $T_{(2,3)}$.

5.2. Cubic surfaces. Let $S \subset \mathbb{P}^3$ be a rational, irreducible and reduced cubic surface having only rational double points as singularities. First of all, we prove two preliminary lemmas.

Lemma 3.5.4. *Let $S \subset \mathbb{P}^3$ be a rational, irreducible and reduced cubic surface having only rational double points as singularities. Then there exists $\Gamma \subset S$, where Γ is a twisted cubic.*

Proof. It is well known that any rational, irreducible and reduced cubic surface S having only rational double points is the image of $\widehat{\mathbb{P}}$ in $\mathbb{P}^3$ via an immersion $\varphi : \widehat{\mathbb{P}} \to \mathbb{P}^3$ determined by δ, where $\widehat{\mathbb{P}} = \mathrm{Bl}_{p_1,\ldots,p_6}\mathbb{P}^2$ is the surface of the blow up of $\mathbb{P}^2$ in six points $p_1, \ldots, p_6$ and δ is the linear system of plane cubic curves with assigned base points $p_1, \ldots, p_6$, having not fixed components.

Since δ is without fixed components, i.e. there not exist four collinear points, then we can always assume that there are three not collinear points. Suppose that p_1, p_2, p_3 are not collinear points. We have two cases:

- if p_4, p_5, p_6 are not collinear points, then we consider $C_1 \subset \mathbb{P}^2$ a smooth conic passing through p_1, p_2, p_3. Let $\pi : S \to \mathbb{P}^2$ be the projection and let $E_1, \ldots, E_6 \subseteq S$ be the exceptional curves, then we have that $\pi^*C_1 = \Gamma_1 + E_1 + E_2 + E_3$, where $\Gamma_1 \subset S$ is a rational curve of degree 3. Since p_4, p_5, p_6 are not collinear points, then there not exists a line L_1 such that $C_1 \cup L_1 \in \delta$, hence Γ_1 is not plane, i.e. $\Gamma_1 \subset S$ is a twisted cubic.
- If p_4, p_5, p_6 are collinear points, then we consider the two triples of points $\{p_1, p_2, p_4\}$ and $\{p_3, p_5, p_6\}$. We have that p_1, p_2, p_4 are not collinear points, otherwise p_1, p_2, p_4, p_5, p_6 are collinear points, which is a contradiction. On the other hand, we have that p_3, p_5, p_6 are not collinear points, otherwise p_3, p_4, p_5, p_6 are collinear points, which is a contradiction. Consider $C_2 \subset \mathbb{P}^2$ a smooth conic passing through p_1, p_2, p_4, then we have that $\pi^*C_2 = \Gamma_2 + E_1 + E_2 + E_4$, where $\Gamma_2 \subset S$ is a rational curve of degree 3. Since p_3, p_5, p_6 are not collinear points, then there not exists a line L_2 such that $C_2 \cup L_2 \in \delta$, hence Γ_2 is not plane, i.e. $\Gamma_2 \subset S$ is a twisted cubic.

This conclude our proof. □

Lemma 3.5.5. *Let $p \in S$ be an A_n point and D_1 a smooth curve through p. Let $\mathcal{L}$ be a linear system of Cartier divisors with $Bsl\mathcal{L} = D_1$ and $D \in \mathcal{L}$ a general element. Let $D = D_1 + D_2$, then D_2 is smooth and $Diff_p D_1 = Diff_p D_2$, where $Diff_p D_i$ denote the different of D_i at $p \in S$, for $i = 1, 2$, see* [**Co**].

Proof. We prove the lemma by induction on n.
INITIAL CASES. If $n = 0$, i.e. $p \in S$ is a smooth point, then $\mathrm{Diff}_p D_1 = \mathrm{Diff}_p D_2 = 0$ and D_2 is smooth.

Let $\varepsilon : \overline{S} \to S$ be the blow up of p with exceptional divisor E and $\overline{D}_1$, $\overline{D}_2$ the strict transforms of D_1, D_2 respectively.

If $n = 1$, i.e. if E is smooth, then it is immediately that $\mathrm{Diff}_p D_1 = \mathrm{Diff}_p D_2 = \frac{1}{2}$ and D_2 is smooth.

INDUCTION STEP. Assume that the lemma holds for any $i \leq n-1$. Let $p \in S$ be an A_n point and let $E = E_1 + E_2$ be a pair of (-2)–curves. We have two cases:

i) if $\overline{D}_1 \not\ni E_1 \cap E_2$, then we can assume that $\overline{D}_1 \cap E_1 \neq \emptyset$ and $\overline{D}_2 \cap E_2 \neq \emptyset$. We have that $\varepsilon^* D_j = \overline{D}_j + \frac{1}{2} E_j + \delta_j E_{3-j}$, for some δ_j and for $j = 1, 2$. Since
$$\varepsilon^*(K_S + D_j) \cdot \overline{D}_j = 2g(\overline{D}_j) - 2 + \mathrm{Diff}_p D_j,$$
for $j = 1, 2$, we obtain that
$$\varepsilon^*(K_S + D_j) \cdot \overline{D}_j = (K_{\overline{S}} + \overline{D}_j + \frac{1}{2} E_j + \delta_j E_{3-j}) \cdot \overline{D}_j =$$
$$(K_{\overline{S}} + \overline{D}_j) \cdot \overline{D}_j + \frac{1}{2} = 2g(\overline{D}_j) - 2 + \frac{1}{2},$$
for $j = 1, 2$. Hence $\mathrm{Diff}_p D_1 = \mathrm{Diff}_p D_2 = \frac{1}{2}$ and D_2 is smooth.

ii) If $\overline{D}_1 \ni E_1 \cap E_2$, then $\overline{D}_2 \ni E_1 \cap E_2$. Since $E_1 \cap E_2$ is an A_{n-2} point, by induction step, we have that $\mathrm{Diff}_p D_1 = \mathrm{Diff}_p D_2$ and D_2 is smooth.

These arguments conclude our proof. □

We are ready to prove the following:

Proposition 3.5.6. *Let $S \subset \mathbb{P}^3$ be a rational, irreducible and reduced cubic surface having only rational double points as singularities. Let $\Gamma \subset S \subset \mathbb{P}^3$ be a twisted cubic and let $S' \in |\mathcal{I}_\Gamma(3)|$ be general. Let $S \cap S' = \Gamma \cup R$, then $R \subset \mathbb{P}^3$ is a smooth curve of degree 6 and genus 3. Moreover, there exists a birational map $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ such that $\phi(S) = H$, where H is a plane.*

Proof. By hypothesis $\Gamma \subset S \cap S'$, then $S \cdot S' = S_{|_{S'}} = \Gamma + R$. Since $\deg S_{|_{S'}} = 9$ and $\deg \Gamma = 3$, we have that $\deg R = 6$.

The surface $S' \in |\mathcal{I}_\Gamma(3)|$ is general, then S' is a smooth cubic surface containing Γ. By [**BW**], we have that S has only A_n singular points with $n \leq 5$, moreover $\mathrm{Bsl}|\mathcal{I}_\Gamma(3)| = \Gamma$, therefore the general intersection $S \cap S'$ is singular only along Γ. By Lemma 3.5.5, with $D_1 = \Gamma$, $D_2 = R$ and $\mathcal{L} = |\mathcal{I}_\Gamma(3)|$, we have that $\mathrm{Diff}\, \Gamma = \mathrm{Diff}\, R$ at any singular points of S and R is smooth.

We have to compute the genus g of R. By adjunction formula, we get

$$2g-2=(K_S+R)\cdot R-\text{Diff } R=K_S\cdot R+R^2-\text{Diff } R \tag{3.5.1}$$

and

$$-2=(K_S+\Gamma)\cdot\Gamma-\text{Diff }\Gamma=K_S\cdot\Gamma+\Gamma^2-\text{Diff }\Gamma. \tag{3.5.2}$$

Consider $K_S\cdot R=\mathcal{O}_S(-1)\cdot R=-6$ and $K_S\cdot\Gamma=\mathcal{O}_S(-1)\cdot\Gamma=-3$, hence, by (3.5.1) and by (3.5.2) respectively, we obtain that $2g-2=-6+R^2-\text{Diff } R$ and $\Gamma^2=1+\text{Diff }\Gamma$.

We have

$$27=(S\cdot S)_{|_{S'}}=(\Gamma+R)^2=\Gamma^2+R^2+2\Gamma\cdot R, \tag{3.5.3}$$

then we have $27=R^2+1+\text{Diff }\Gamma+2\Gamma\cdot R$, i.e.

$$R^2=26-2\Gamma\cdot R-\text{Diff }\Gamma. \tag{3.5.4}$$

Hence, we get

$$2g-2=-6+26-2\Gamma\cdot R-\text{Diff }\Gamma-\text{Diff } R=20-2\Gamma\cdot R-2\text{Diff }\Gamma,$$

i.e. $g=11-\Gamma\cdot R-\text{Diff }\Gamma$.

On the other hand, we can write

$$R^2+2\Gamma\cdot R=R\cdot(R+2\Gamma)=R\cdot(R+\Gamma)+R\cdot\Gamma=R\cdot S_{|_{S'}}+R\cdot\Gamma,$$

where, by (3.5.4), $R\cdot S_{|_{S'}}+R\cdot\Gamma=26-\text{Diff }\Gamma$ and, since $R\subset S_{|_{S'}}$, $R\cdot S_{|_{S'}}=18$. Therefore, we have that $R\cdot\Gamma=26-\text{Diff }\Gamma-18=8-\text{Diff }\Gamma$, hence $g=11-8+\text{Diff }\Gamma-\text{Diff }\Gamma=3$.

Since R is a smooth curve of degree 6 contained in a cubic surface S, then R is not a plane curve.

Let prove that R is not contained in a quadric surface $Q\subset\mathbb{P}^3$. Assume the curve R contained in a nonsingular quadric surface $Q\subset\mathbb{P}^3$, then R is of type $(2,4)$, see [**Ha**, chapter IV, Remark 6.4.1]. On the other hand, we have that R is a complete intersection of Q and the cubic surface S, then R is of type $(3,3)$ on Q, which is a contradiction. If R is contained in a quadric cone $Q\subset\mathbb{P}^3$, then R is a complete intersection of Q with the cubic surface S and, in this case, the genus of R is 4, see [**Ha**, chapter IV, Remark 6.4.1, p. 352], which is a contradiction. Hence R is not contained in a quadric surface of $\mathbb{P}^3$, therefore we have that $H^0(\mathbb{P}^3,\mathcal{I}_R(2))=0$.

Consider the exact sequence

$$0\to\mathcal{I}_R\to\mathcal{O}_{\mathbb{P}^3}\to\mathcal{O}_R\to 0,$$

twisting by $\mathcal{O}_{\mathbb{P}^3}(2)$, we obtain

$$0 \to \mathcal{I}_R(2) \to \mathcal{O}_{\mathbb{P}^3}(2) \to \mathcal{O}_R(2) \to 0.$$

Taking cohomology we get

$$0 \to H^0(\mathcal{I}_R(2)) \to H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_R(2)) \to H^1(\mathcal{I}_R(2)) \to 0.$$

Since $H^0(\mathcal{I}_R(2)) = 0$, we have that $H^0(\mathcal{O}_{\mathbb{P}^3}(2)) = H^0(\mathcal{O}_R(2))$, then $H^1(\mathcal{I}_R(2)) = 0$. Hence $R \subset \mathbb{P}^3$ is a smooth irreducible curve such that $H^1(\mathbb{P}^3, \mathcal{I}_R(2)) = 0$, i.e. R is 2–normal, see [**ACGH**, chapter III, ex. D-1, p. 140]. Moreover, we have that $\mathcal{O}_R(1)$ is non special. In fact, consider D a divisor of degree 6 on R, then $\mathcal{L}(D) = \mathcal{O}_R(1)$. Since $\deg D = 6 > 2g - 2 = 4$, we have that $\deg(K_R - D) < 0$, where K_R is the canonical divisor on R, and then $l(K_R - D) = \dim H^1(\mathcal{O}_R(1)) = 0$, i.e. $H^1(\mathcal{O}_R(1)) = 0$, hence $\mathcal{O}_R(1)$ is non special.

Since $R \subset \mathbb{P}^3$ is a smooth, irreducible and 2–normal curve, by [**ACGH**, chapter III, ex. D-5, p. 140], we have that R is 3–normal, i.e. $H^1(\mathbb{P}^3, \mathcal{I}_R(3)) = 0$. In particular, R is k–normal, for $k \geq 3$.

Consider $\Lambda = |\mathcal{I}_R(3)|$ the linear system of cubic surface in $\mathbb{P}^3$ containing the smooth curve R of degree 6 and genus 3. We have to prove that $\dim \Lambda = 3$.

Consider the ideal $\mathcal{I}_R$ of R, we have that $H^1(\mathcal{I}_R(2)) = 0$, $H^2(\mathcal{I}_R(1)) = H^1(\mathcal{O}_R(1)) = 0$ and $H^3(\mathcal{I}_R) = H^2(\mathcal{O}_R) = 0$, then $H^j(\mathbb{P}^3, \mathcal{I}_R(3 - j)) = 0$, for any $j > 0$, hence $\mathcal{I}_R$ is 3–regular. It implies that we can consider a resolution of $\mathcal{I}_R$ given by Hilbert–Burch Theorem, see [**Do**, Theorem 9.3.6, p. 247] and [**Ei**, Theorem 20.15, p. 502]:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4)^3 \to \mathcal{O}_{\mathbb{P}^3}(-3)^4 \to \mathcal{I}_R \to 0.$$

Twisting by $\mathcal{O}_{\mathbb{P}^3}(3)$, we obtain

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1)^3 \to \mathcal{O}_{\mathbb{P}^3}^4 \to \mathcal{I}_R(3) \to 0$$

and taking cohomology we get

$$0 \to H^0(\mathcal{O}_{\mathbb{P}^3}(-1))^3 \to H^0(\mathcal{O}_{\mathbb{P}^3})^4 \to H^0(\mathcal{I}_R(3)) \to H^1(\mathcal{O}_{\mathbb{P}^3}(-1))^3$$
$$\to H^1(\mathcal{O}_{\mathbb{P}^3})^4 \to H^1(\mathcal{I}_R(3)) \to H^2(\mathcal{O}_{\mathbb{P}^3}(-1))^3 \to \dots.$$

Since $H^1(\mathcal{I}_R(3)) = 0$, then

$$\dim H^0(\mathcal{I}_R(3)) = 4 \dim H^0(\mathcal{O}_{\mathbb{P}^3}) - 3 \dim H^0(\mathcal{O}_{\mathbb{P}^3}(-1)) = 4.$$

Moreover, we can prove that R is projectively normal and, by Castelnuovo–Mumford Theorem, see [**Ei**, section 20.5, p. 504], we have that $\mathcal{I}_R$ is generated by cubic forms, hence $\mathrm{Bsl}(\Lambda) = R$.

Let $D \in \Lambda$ be general. Since there exists $S' \in \Lambda$, where S' is a smooth cubic surface, then the general cubic surface $D \in \Lambda$ is smooth.

We have to prove that $|\mathcal{O}_D(3H - R)|$ ia a linear system of rational normal cubic curves of self–intersection 1. More precisely $|3H-R|$ defines the contraction $\sigma : D \to \mathbb{P}^2$ of six lines of D to $\mathbb{P}^2$. It is enough to prove that two normal rational cubic curves intersect in an unique point in $S \setminus \mathrm{Sing}(S)$.

Let D_1, $D_2 \in \Lambda$ be general cubic surfaces. We have that ${D_1}_{|_{S'}} = \Gamma_1 + R$ and ${D_2}_{|_{S'}} = \Gamma_2 + R$, then

$$\Gamma_1 \cdot \Gamma_2 = ({D_1}_{|_{S'}} - R) \cdot ({D_2}_{|_{S'}} - R) =$$

$$= {D_1}_{|_{S'}} \cdot {D_2}_{|_{S'}} - 2R \cdot {D_1}_{|_{S'}} + R^2 = 27 - 36 + 10 = 1.$$

Therefore, we obtain a birational map $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$, which is a cubo–cubic Cremona transformation with base locus R, see [**Ve**, section 0.3.2, pp. 30–34], such that $\phi(S) = H$, where H is a plane in $\mathbb{P}^3$, i.e. S is *Cremona equivalent to a plane.* $\square$

Let $S \subset \mathbb{P}^3$ be a rational, irreducible and reduced cubic surface with a double line L and which is not a cone. We have the following:

Proposition 3.5.7. *Let $S \subset \mathbb{P}^3$ be a rational, irreducible and reduced cubic surface with a double line L and which is not a cone. Then there exists a birational map $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ such that $\phi(S) = H$, where H is a plane.*

Proof. Let $S' \subset \mathbb{P}^4$ be a cubic scroll and let $p \in \mathbb{P}^4 \setminus S'$ be a general point. Since S' is a minimal degree surface, then the ideal $\mathcal{I}_{S'}$ of S' is generated by quadric forms, hence there exists a quadric hypersurface $Q \subset \mathbb{P}^4$ such that $Q \in |\mathcal{I}_{S'}(2) \otimes \mathcal{I}_p|$. Let $\pi_p : \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ be the projection from the point p, then we have a birational map ${\pi_p}_{|Q} : Q \dashrightarrow \mathbb{P}^3$ such that ${\pi_p}_{|Q}(S') = S$, where S is a cubic surface in $\mathbb{P}^3$. Consider the conic $C \subset S'$ such that $p \in \langle C \rangle$, then ${\pi_p}_{|Q}(C) = L \subset S$ is a double line. Therefore, $S \subset \mathbb{P}^3$ is a rational, irreducible and reduced cubic surface such that $\mathrm{Sing}(S) = L$, where L is a double line.

Let $p' \in S'$ be a general point and let $\pi_{p'} : \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ be the projection from p'. Hence we have a birational map ${\pi_{p'}}_{|Q} : Q \dashrightarrow \mathbb{P}^3$ such that ${\pi_{p'}}_{|Q}(S') = \widetilde{Q}$, where $\widetilde{Q}$ is a quadric surface in $\mathbb{P}^3$.

By Proposition 3.5.2, we have that there exists a birational map $\phi' : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ such that $\phi'(\widetilde{Q}) = H$, where H is a plane.

Hence, there exists a birational map $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$, where

$$\phi = \phi' \circ \pi_{p'|Q} \circ (\pi_{p|Q})^{-1},$$

such that $\phi(S) = H$, where H is a plane. □

REMARK 3.5.8. Thanks to Propositions 3.5.2, 3.5.6 and 3.5.7, we have that if $S \subset \mathbb{P}^3$ is a rational, irreducible and reduced surface of degree $d \leq 3$ which is not a cone, then S is *Cremona equivalent to a plane*.

5.3. Cones. Let T be a 3–dimensional scroll on $\mathbb{P}^2$, i.e. there exists $\mathcal{E}$ a locally free sheaf of rank 2 on $\mathbb{P}^2$ such that $T = \mathbb{P}_{\mathbb{P}^2}(\mathcal{E})$, and let $\varphi : T \to \mathbb{P}^2$ be the projection.

Consider $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ a standard Cremona transformation, we have that ω induces a birational transformation on the scroll T in the following way:

$$\begin{tikzcd} T = \mathbb{P}(\mathcal{E}) \arrow[r, dashed, "\Phi"] \arrow[d, "\varphi"'] & \widetilde{T} = \mathbb{P}(\mathcal{E}') \arrow[d, "\widetilde{\varphi}"] \\ \mathbb{P}^2 \arrow[r, dashed, "\omega"] & \mathbb{P}^2 \end{tikzcd}$$

We obtain a birational map $\Phi : T \dashrightarrow \widetilde{T}$, where $\widetilde{T}$ is a scroll on $\mathbb{P}^2$, i.e. there exists $\mathcal{E}'$ a locally free sheaf of rank 2 on $\mathbb{P}^2$ such that $\widetilde{T} = \mathbb{P}_{\mathbb{P}^2}(\mathcal{E}')$, and where $\widetilde{\varphi} : \widetilde{T} \to \mathbb{P}^2$ is the projection.

We have to describe the birational transformation $\Phi : T \dashrightarrow \widetilde{T}$ induced by ω. Since a Cremona transformation is the composition of blow ups of points with contractions of (-1)–curves, we can factor Φ by elementary links in the Sarkisov category. The links are described in the following:

CLAIM. i) Let $p \in \mathbb{P}^2$ be a point and let $f = \varphi^{-1}(p)$ be the fibre of the scroll T at p. Let $\varepsilon_p : \widehat{\mathbb{P}} \to \mathbb{P}^2$ be the blow up of p in $\mathbb{P}^2$. Then, the blow up of a point $p \in \mathbb{P}^2$ corresponds to the blow up of the fibre $f \subset T$. We have $\varepsilon_f : \widehat{T} \to T$ the blow up of $f \subset T$, where $\widehat{T}$ is a scroll on $\widehat{\mathbb{P}}$ and $\widehat{\varphi} : \widehat{T} \to \widehat{\mathbb{P}}$ is the projection.
ii) Let $B \subset \widehat{\mathbb{P}}$ be a (-1)–curve, then $D = \widehat{\varphi}^*(B) \simeq \mathbb{F}_a$, for some $a \geq 0$. We have two cases:

- if $D \simeq \mathbb{F}_0$, then the contraction of the (-1)–curve B corresponds to the blow down of D;
- if $D \simeq \mathbb{F}_1$, then the contraction of the (-1)–curve B corresponds to the flop of D.

We have the following:

Lemma 3.5.9. *Let $T, \widetilde{T}$ be two rational scrolls on $\mathbb{P}^2$, i.e. $T = \mathbb{P}_{\mathbb{P}^2}(\mathcal{E})$, $\widetilde{T} = \mathbb{P}_{\mathbb{P}^2}(\mathcal{E}')$, where $\mathcal{E}, \mathcal{E}'$ are locally free sheaves of rank 2 on $\mathbb{P}^2$. Consider $\Phi : T \dashrightarrow \widetilde{T}$ the birational map induced by a standard Cremona transformation on $\mathbb{P}^2$. We have the following statements:*

i) *if $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(-1)$, then $\mathcal{E}' = \mathcal{O} \oplus \mathcal{O}$;*
ii) *if $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}$, then $\mathcal{E}' = \mathcal{O} \oplus \mathcal{O}$.*

Proof. We prove both statements at the same time.

Consider the scroll $T = \mathbb{P}_{\mathbb{P}^2}(\mathcal{E})$. Let $\varphi : T \to \mathbb{P}^2$ be the projection and let f be a general fibre of the scroll T.

Let $E \subset T$ be a section, then we have two different cases:

i) if $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(-1)$, we have that $(E, E_{|E}) = (\mathbb{P}^2, \mathcal{O}(-1))$ and $E \cdot f = 1$;
ii) if $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}$, we have that $(E, E_{|E}) = (\mathbb{P}^2, \mathcal{O})$ and $E \cdot f = 1$.

Consider a standard Cremona transformation $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with center $p_1, p_2, p_3 \in \mathbb{P}^2$. Let $f_i = \varphi^{-1}(p_i)$ be the fibre of the scroll T at p_i, for $i = 1, 2, 3$.

We have that ω induces a Cremona transformation $\widetilde{\omega} : E \dashrightarrow \widetilde{E}$ with center x_1, x_2, x_3, where $x_i = f_i \cap E$, for $i = 1, 2, 3$.

Therefore, we obtain a birational map $\Phi : T \dashrightarrow \widetilde{T}$, where $\widetilde{T} = \mathbb{P}_{\mathbb{P}^2}(\mathcal{E}')$. Let $\widetilde{\varphi} : \widetilde{T} \to \mathbb{P}^2$ be the projection.

$$\begin{tikzcd}
E \subset T = \mathbb{P}(\mathcal{E}) \arrow[r, dashed, "\Phi"] \arrow[d, "\varphi"'] & \widetilde{T} = \mathbb{P}(\mathcal{E}') \supset \widetilde{E} \arrow[d, "\widetilde{\varphi}"] \\
\mathbb{P}^2 \arrow[r, dashed, "\omega"] & \mathbb{P}^2
\end{tikzcd}$$

To know $\widetilde{T}$, we have to factor the birational map Φ by elementary links in the Sarkisov category. The links are described in the following way:

$$\begin{tikzcd}
E \subset T \arrow[d, "\varphi"] & \widehat{T} \supset \widehat{E} \arrow[l, "\widehat{\varepsilon}"'] \arrow[r, dashed, "\xi"] \arrow[d, "\widehat{\varphi}"'] & \widetilde{T} \supset \widetilde{E} \arrow[d, "\widetilde{\varphi}"] \\
\mathbb{P}^2 & \widehat{\mathbb{P}} \arrow[l, "\varepsilon"'] \arrow[r, "\psi"] & \mathbb{P}^2
\end{tikzcd}$$

Let $\varepsilon : \widehat{\mathbb{P}} \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at p_1, p_2, p_3. Note that in $\widehat{\mathbb{P}}$ there are six (-1)–curves: A_1, A_2, A_3 the exceptional divisors of the blow up ε and B_1, B_2, B_3 the strict transforms of the lines $l_1 = \langle p_2, p_3 \rangle$, $l_2 = \langle p_1, p_3 \rangle$, $l_3 = \langle p_1, p_2 \rangle$ respectively.

By the claim, the blow up $\varepsilon : \widehat{\mathbb{P}} \to \mathbb{P}^2$ corresponds to $\widehat{\varepsilon} : \widehat{T} \to T$, which is the blow up of the fibers $f_1, f_2, f_3 \subset T$. We have that $\widehat{T}$ is a scroll on $\widehat{\mathbb{P}}$ and let $\widehat{\varphi} : \widehat{T} \to \widehat{\mathbb{P}}$ be the projection. In particular, $\widehat{\varepsilon}_{|\widehat{E}} : \widehat{E} \to E$ is the blow up of E at x_1, x_2, x_3.

Consider the (-1)–curves $A_1, A_2, A_3 \subset \widehat{\mathbb{P}}$. Since $\mathcal{N}_{f_{i|T}} \sim \mathcal{O} \oplus \mathcal{O}$, we have that $\widehat{\varphi}^*(A_i) \simeq \mathbb{F}_0$, for $i = 1, 2, 3$.

Consider the (-1)–curves $B_1, B_2, B_3 \subset \widehat{\mathbb{P}}$. Let $\widehat{\varphi}^*(B_i) = D_i$ and let $G_i = D_i \cap \widehat{E}$ be the exceptional section of the rational scroll D_i, for $i = 1, 2, 3$.

At this point, we have two different cases:

i) if $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(-1)$, we have that $\mathcal{E}_{|L} \sim \mathcal{O} \oplus \mathcal{O}(-1)$, where L is a general line in $\mathbb{P}^2$. Since $(G_i \cdot G_i)_{|D_i} = -1$ and $(G_i \cdot G_i)_{|\widehat{E}} = 2$, then $\mathcal{N}_{G_{i|\widehat{T}}} \sim \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and hence $D_i \simeq \mathbb{F}_1$, for $i = 1, 2, 3$.

Let $r \subset E$ be a general line passing through x_1 such that $x_2, x_3 \notin r$. Since $(E, E_{|E}) = (\mathbb{P}^2, \mathcal{O}(-1))$, then $E \cdot r = -1$. Let $\widehat{r} \subset \widehat{E}$ be the strict transform of r with respect to the blow up $\widehat{\varepsilon}$. We have that

$$\widehat{E} \cdot \widehat{r} = \widehat{\varepsilon}^* E \cdot \widehat{\varepsilon}^* r = E \cdot r = -1.$$

Let $\psi : \widehat{\mathbb{P}} \to \mathbb{P}^2$ be the contraction of (-1)–curves B_1, B_2, B_3. By the claim, the contraction ψ corresponds to $\xi : \widehat{T} \dashrightarrow \widetilde{T}$ the flops of the rational scrolls D_1, D_2, D_3.

Let $\widetilde{E}, \widetilde{r}$ be the strict transforms of $\widehat{E}, \widehat{r}$, respectively, with respect to ψ. Since the flops of the rational scroll D_i is the composition of the blow up of G_i with the contraction of the residual $\widehat{D_i} \simeq \mathbb{P}^2$, where $\widehat{D_i}$ is the strict transform of D_i after the blow up of G_i, for $i = 1, 2, 3$, then we have $\widetilde{E} = \psi^*_{-1}\widehat{E} + F_1 + F_2 + F_3$ and $\widetilde{r} = \psi^*_{-1}\widehat{r}$, where F_i is the exceptional divisor of the blow up of G_i, for $i = 1, 2, 3$. Hence

$$\widetilde{E} \cdot \widetilde{r} = (\psi^*_{-1}\widehat{E} + F_1 + F_2 + F_3) \cdot (\psi^*_{-1}\widehat{r}) =$$

$$\psi^*_{-1}\widehat{E} \cdot \psi^*_{-1}\widehat{r} + F_1 \cdot \widetilde{r} + F_2 \cdot \widetilde{r} + F_3 \cdot \widetilde{r}.$$

Since $\widehat{r} \cap G_2 = \emptyset = \widehat{r} \cap G_3$ and $\widehat{r} \cap G_1 \neq \emptyset$, we get $F_2 \cdot \widetilde{r} = 0 = F_3 \cdot \widetilde{r}$ and $F_1 \cdot \widetilde{r} = 1$. Therefore,

$$\widetilde{E} \cdot \widetilde{r} = \psi^*_{-1}\widehat{E} \cdot \psi^*_{-1}\widehat{r} + F_1 \cdot \widetilde{r} = \widehat{E} \cdot \widehat{r} + 1 = -1 + 1 = 0.$$

ii) if $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}$, we have that $\mathcal{E}_{|L} \sim \mathcal{O} \oplus \mathcal{O}$, where L is a general line in $\mathbb{P}^2$. Then $\mathcal{N}_{G_{i|\widehat{T}}} \sim \mathcal{O}(-1) \oplus \mathcal{O}$ and hence $D_i \simeq \mathbb{F}_0$, for $i = 1, 2, 3$.

Let $r \subset E$ be a general line passing through x_1 such that $x_2, x_3 \notin r$. Since $(E, E_{|E}) = (\mathbb{P}^2, \mathcal{O})$, then $E \cdot r = 0$. Let $\widehat{r} \subset \widehat{E}$ be the strict transform of r with respect to the blow up $\widehat{\varepsilon}$. We have that

$$\widehat{E} \cdot \widehat{r} = \widehat{\varepsilon}^* E \cdot \widehat{\varepsilon}^* r = E \cdot r = 0.$$

Let $\psi : \widehat{\mathbb{P}} \to \mathbb{P}^2$ be the contraction of (-1)–curves B_1, B_2, B_3. By the claim, the contraction ψ corresponds to $\xi : \widehat{T} \dashrightarrow \widetilde{T}$ the blows down of the rational scrolls D_1, D_2, D_3.

Let $\widetilde{E}, \widetilde{r}$ be the strict transforms of $\widehat{E}, \widehat{r}$, respectively, with respect to ψ. Therefore,

$$\widetilde{E} \cdot \widetilde{r} = \psi_{-1}^* \widehat{E} \cdot \psi_{-1}^* \widehat{r} = \widehat{E} \cdot \widehat{r} = 0.$$

Consider the scroll $\widetilde{T} = \mathbb{P}_{\mathbb{P}^2}(\mathcal{E}')$. Since $r \subset E$ is a general line passing through x_1, then $\widetilde{\omega}(r) = \widetilde{r}$ is a general line in $\widetilde{E} \subset \widetilde{T}$. We have that $\widetilde{E} \simeq \mathbb{P}^2$ and $\widetilde{E} \cdot \widetilde{r} = 0$. Hence $\widetilde{E}_{|\widetilde{E}} \sim \mathcal{O}$ and, by the theory of deformations, $h^0(\widetilde{E}) > 1$. Let $\widetilde{E}_1, \widetilde{E}_2 \in H^0(\widetilde{E})$, then $\widetilde{E}_1 \cap \widetilde{E}_2 = \emptyset$, i.e. $\widetilde{E}_1 \cdot \widetilde{E}_2 \sim \mathcal{O}$.

Therefore, since $\widetilde{T}$ is a scroll on $\mathbb{P}^2$, there exists a morphism $\widetilde{T} \to \mathbb{P}^2$, where the fibers are $\mathbb{P}^1$ and there exists a morphism $\widetilde{T} \to \mathbb{P}^1$, where the fibers are $\mathbb{P}^2$, given by a pencil in the base point free linear system $H^0(\widetilde{E})$. We can conclude that $\widetilde{T} = \mathbb{P}^1 \times \mathbb{P}^2$, i.e. in both cases $\mathcal{E}' = \mathcal{O} \oplus \mathcal{O}$.

These arguments conclude our proof. □

Proposition 3.5.10. *Let $S \subset \mathbb{P}^3$ be a cone over C, where $C \subset \mathbb{P}^2$ is a rational, irreducible and reduced curve of degree $d \geq 3$, which is Cremona equivalent to a line. Then, there exists a birational map $\Psi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ such that $\Psi(S) = \Pi$, where Π is a plane, i.e. S is Cremona equivalent to a plane.*

Proof. Let $\nu : T \to \mathbb{P}^3$ be the blow up of the vertex $v \in S \subset \mathbb{P}^3$ and let E be its exceptional divisor. We have that T is a scroll on $\mathbb{P}^2$, i.e. $T = \mathbb{P}_{\mathbb{P}^2}(\mathcal{E})$, where $\mathcal{E}$ is a locally free sheaf of rank 2 on $\mathbb{P}^2$. Let $\varphi : T \to \mathbb{P}^2$ be the projection and let f be a general fibre of the scroll T.

Consider the section $E \subset T$, we have that $(E, E_{|E}) = (\mathbb{P}^2, \mathcal{O}(-1))$ and $E \cdot f = 1$, then there exists an invertible sheaf $\mathcal{L}$ on $\mathbb{P}^2$ such that $\mathcal{L} \otimes \mathcal{E} = \mathcal{O} \oplus \mathcal{O}(-1)$. Hence $T = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(-1))$.

Since the curve $C \subset \mathbb{P}^2$ is Cremona equivalent to a line, there exists a birational map $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\omega(C) = l$, where l is a line.

By Noether–Castelnuovo theorem, we have that the birational map ω on the projective plane is a composition of linear transformations and of standard Cremona transformations.

Moreover, if we consider a scroll $T' = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(-1))$ or $T' = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O})$, by Lemma 3.5.9, we get that a standard Cremona transformation induces a birational map $T' \dashrightarrow T''$, where $T'' = \mathbb{P}^1 \times \mathbb{P}^2$.

Therefore, the birational map $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ induces a birational map $\Phi : \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(-1)) \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^2$ described in the following diagram:

$$\begin{tikzcd}
E \subset \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-1)) \arrow[r, dotted, "\Phi"] \arrow[d, "\varphi"'] & \mathbb{P}^1 \times \mathbb{P}^2 \supset \widetilde{E} \arrow[d, "\widetilde{\varphi}"] \\
C \subset \mathbb{P}^2 \arrow[r, dotted, "\omega"] & \mathbb{P}^2 \supset l
\end{tikzcd}$$

In the diagram above, $\widetilde{\varphi} : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$ is the projection and $\widetilde{E} = \Phi(E)$, where $(\widetilde{E}, \widetilde{E}_{|\widetilde{E}}) = (\mathbb{P}^2, \mathcal{O})$.

Let $\widehat{S} \subset \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(-1))$ be the strict transform of S with respect to the blow up ν, in particular $\widehat{S} = \varphi^*(C)$. Then, $\Phi(\widehat{S}) = \widetilde{S} \subset \mathbb{P}^1 \times \mathbb{P}^2$, where $\widetilde{S} = \widetilde{\varphi}^*(l)$.

Consider $\widetilde{T} = \mathbb{P}^1 \times \mathbb{P}^2 = \mathbb{P}_{\mathbb{P}^2}(\mathcal{E}')$. Since $\mathcal{E}'_{|l} \sim \mathcal{O} \oplus \mathcal{O}$, we have that $\mathcal{N}_{L_{|\widetilde{T}}} \sim \mathcal{O}(-1) \oplus \mathcal{O}$, where $L = \widetilde{S} \cap \widetilde{E}$ is the exceptional section of $\widetilde{S}$. Hence $\widetilde{S} \simeq \mathbb{F}_0$.

Consider the Segre immersion $\mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$, then $\mathbb{P}^1 \times \mathbb{P}^2$ is a smooth 3–fold of degree 3 in $\mathbb{P}^5$. Moreover, since there exists an embedding of the rational ruled surface $\mathbb{F}_0 \simeq \widetilde{S}$ as a rational scroll of degree 2 in $\mathbb{P}^3$, we have that $\widetilde{S} \subset \mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ is the non–singular quadric surface in $\mathbb{P}^3 \hookrightarrow \mathbb{P}^5$, see [**Ha**, chapter V, Corollary 2.19, p. 381].

Let $p \in \widetilde{S} \subset \mathbb{P}^1 \times \mathbb{P}^2$ be a point and let $\pi_p : \mathbb{P}^5 \to \mathbb{P}^4$ be the projection from p. We have that $\pi_p(\mathbb{P}^1 \times \mathbb{P}^2) = Q^3$, where Q^3 is a quadric cone in $\mathbb{P}^4$ of vertex w. Moreover, $\pi_p(\widetilde{S}) = \widetilde{\Pi}$, with $\widetilde{\Pi}$ a plane in Q^3.

Let $q \in Q^3$ be a point such that $q \neq w$. Let $\pi_q : \mathbb{P}^4 \to \mathbb{P}^3$ be the projection in $\mathbb{P}^3$ from q. Since q is a smooth point of Q^3, we have that $\pi_{q_{|Q^3}} : Q^3 \dashrightarrow \mathbb{P}^3$ is a birational map. Hence $\pi_q(\widetilde{\Pi}) = \Pi \subset \mathbb{P}^3$ is a plane.

Therefore, we obtain a birational map $\Psi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$, where $\Psi = \pi_{q_{|Q^3}} \circ \pi_{p_{|\mathbb{P}^1 \times \mathbb{P}^2}} \circ \Phi \circ \nu^*_{-1}$, such that $\Psi(S) = \Pi$, where Π is a plane.

These arguments conclude our proof. □

Summarizing, we have the following:

Theorem 3.5.11. *Let $S \subset \mathbb{P}^3$ be a rational, irreducible and reduced cubic surface. Then, there exists a birational map $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ such that $\phi(S) = H$, where H is a plane, i.e. S is Cremona equivalent to a plane.*

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