Boscovich's geometrical principle of Continuity, and the "mysteries of the infinity"

Abstract. In this paper we give a detailed account of Boscovich's geometrical principle of continuity. We also compare his ideas with those of his forerunners and successors, in order to cast some light on his possible sources of inspiration and to underline the elements of novelty in his approach to the subject.

Sunto. In questo lavoro si presenta in modo dettagliato il principio di continuità geometrica di Boscovich. Si confrontano le sue idee con quelle dei suoi precursori e successori, con l'obiettivo di far luce sulle sue possibili fonti di ispirazione e sottolineare gli elementi di novità del suo approccio all'argomento.

Keywords. R. Boscovich. Principle of Geometrical Continuity. Conic Sections.
MSC Code 01 A 50; 01 A 45; 01 A 55.

1. Introduction

The law, or principle, of (geometrical) continuity can be synthetically enounced as follows: let a figure be conceived to undergo a certain *continuous* variation, and let some *general* property concerning it be granted as true, so long as the variation is confined *within* certain limits; then the same property will belong to *all* the successive states of the figure (that is, all the states which admit the property being expressed), the enunciation being *modified* (occasionally) according to known rules.[1] This principle has, to some extent, been accepted and employed in all ages of the history of mathematics, but it is only in modern times that its validity as a means of discovery or proof has been achieved. In fact, although many results of this principle are to be found in the writings of Gaspard Monge, and the members of his "school", only after Jean-Victor Poncelet's *Traité des propriétés projective des figures* did it come to be universally acknowledged [Poncelet 1822].

As will be seen in section 2, the first germ of the principle of continuity is to be found in the *principle of analogy* introduced by Johannes Kepler, without any explicit enunciation, in his study *De coni sectionibus*, included in chapter IV of *Ad Vitellionem paralipomena quibus astronomiae pars optica traditur* [Kepler 1604, 92-96]. As introduced by Kepler, the principle of continuity was largely independent from algebraic considerations, although the consideration of the behaviour of certain functions in limiting conditions was fundamental to his study. Only later were developments suggested by the occurrence of negative and imaginary roots in second degree equations applied to geometry.

Traces of the principle also appear in Girard Desargues' *Brouillon project* [Desargues 1639], in some works of Wilhelm Gottfried Leibniz, Isaac Newton and other authors of the eighteenth century.

The earliest thorough treatment of this subject is found in two of Roger Joseph Boscovich's works, the dissertation *De transformatione locorum geometricorum ubi de continuitatis lege, ac de quibusdam infiniti mysteriis*, included in the third volume of his *Elementa universae matheseos* [Boscovich 1754a], and the dissertation *De continuitatis lege* [Boscovich 1754b]. With these essay, Boscovich wanted to cast light on the role of the principle of continuity in natural sciences and in geometry, and to raise it to the rank of a general method, aiming to make it the starting point for his major work *Theoria philosophiae naturalis* [Boscovich 1758]. In the first essay, he proceeded by

[1] See [Poncelet 1822, xiii].

geometrical reflections, and in the second one, also through the analysis of certain natural phenomena, he stressed the philosophical aspects of the principle.

The mechanical applications of the principle of continuity and its philosophical features, which also involve the concept of *continuous*, as developed in the dissertation *De continuitatis lege* have been widely investigated.[2] On the contrary, although there are some contributions concerning Boscovich's principle of geometrical continuity, as expounded in the dissertation *De transformatione locorum geometricorum*,[3] an in-depth study of Boscovich's ideas, and achievements, on this topic still seems to be lacking.

The goal of this paper is to give an account of Boscovich's approach to the principle of geometrical continuity through a detailed analysis and discussion of the eleven rules ("canones"), by which he operated in the transformation of geometrical loci. Then, by illustrating and commenting a series of relevant examples, among the many offered by Boscovich in order to validate the rules, we penetrate the "mysteries of infinity", as he called certain phenomena which arise in geometry when some quantity grows beyond any limit.[4] Moreover, we aim to cast some light on his possible sources of inspiration and, in order to underline the novel elements of some of Boscovich's geometrical ideas, to investigate to what extent his achievements anticipated certain concepts later introduced by Lazar Carnot and Poncelet.

Our paper expands and completes the pioneering works of Taylor, and Manara and Spoglianti.

We end this introduction with a biographical note on Boscovich.[5]

Roger Joseph Boscovich (Ruggiero Giuseppe Boscovich in Italian, and Ruđer Josip Bošković in Croatian) was born in the Republic of Ragusa, in Dalmatia (today Dubrovnik in Croatia), in 1711. His mother, Paola Bettera, was the daughter of a rich merchant of Italian origin. Roger was educated at the Jesuit Collegium Ragusinum, and, in 1725, he crossed the Adriatic Sea to reach Rome, where he studied at the Collegio Romano, where his mathematical teacher was Orazio Borgondio (1675-1741). Borgondio was preceded on the chair of mathematics by Gilles François de Gottignies (1630-1689) who was a pupil of Andreas Tacquet (1612-1660) and Gregorius Saint-Vincent (1584-1667). In 1732, on completion of his course he was required to teach there for five years. He continued to study Newton's *Opticks* and *Principia* from 1735 onwards. During those years he also made several astronomical observations, in particular on the transit of Mercury across the solar disc in 1736, publishing his results in *De Mercurio novissimo infra solem transitu* (1737). The same year, after completing his preliminary training, he followed a course of theology which would lead him into priesthood. In 1740, he was nominated professor of mathematics at the Collegio Romano, and, in 1744, he was ordained a priest. In 1754, he published the *Elementa universae matheseos*, in three volumes, which was reprinted three years later [Boscovich 1757]. He was author of about a hundred books and papers on optics, astronomy, meteorology and trigonometry, with special interest in mathematical physics. In 1758, he published his "magnum opus" the *Theoria philosophiae naturalis reducta ad unicam legem virium in natura existentium*
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[2] See [Boscovich 1961] [Boscovich 1991] [Boscovich 1993] and, in particular, [Stipanić 1975], [Stipanić 1993], [Homann 1993], [Schubring 2005, 179—182], [Guzzardi 2014], [Martinović 2015], and the reference quoted there.

[3] See [Taylor 1881, lxxiii—lxxviii], [Manara, Spoglianti 1979], [Martinović 1991].

[4] In a letter to G.S. Conti (written in Italian), on February 26th, 1762, quoted by Guzzardi [2014, 24-25], Boscovich explained what he meant by "mysteries of infinity": "Everywhere… the infinity enters, our limited and finite mind is lost, because our ideas are too weak to clearly conceive it. Therefore, these I call *misteries of infinity*, and I distinguish them from the absurdies, as I found in the actual extension of an absolutely infinite line. The absurdies make me believe a thing is impossible; the mysteries, the difficulties of conceiving, the clouds that veil our imagination, only make me think of the weakness of our mind". This locution will later be cleared through the discussion of some examples.

[5] For more detailes see for instance [Hill 1961], [Casini 1971], for an account of his entire work see [Proverbio 2007], for his mathematical genealogy and work see [Pepe 2010a], [Pepe 2010b], and for his conception of matter [Guzzardi 2014].

existentium [Boscovich 1758], which endeavoured to create a system of Natural Philosophy by reducing to a single law all the forces of nature. The work, which was the first mathematical theory of atomism, made its author internationally famous. Being given permission to travel through Europe, in 1759, Boscovich went to Paris, and after six months moved to London, where, in January 1761, he was elected fellow of the Royal Society. In 1764, he was appointed professor of mathematics at the University of Pavia, and then took part in the foundation of the Brera Observatory. When, in 1773, Pope Clement XIV dissolved the Jesuit order, Boscovich returned to Paris, where he had the position of Director of Optics for the French Navy, and became a French citizen. In the following years he was involved in several scientific and priority disputes, and, no longer feeling at ease in France, he decided to return to Italy in 1783. He remained for two years in Bassano, and then, in 1786, he moved to Brera. In the last years of his life, Boscovich suffered from mental disorders and died in Milan in 1787.

2. The principle of continuity from Kepler to Boscovich

In this section we give a brief historical account of the principle of continuity, presenting the fundamental ideas of the main authors who developed and applied this principle in its various forms.[6] At the basis of this principle there is the concept of "continuity", a concept that has always attracted the attention of philosophers and scientists. We owe to Aristotle the first known definition of continuity, reported by D.E. Smith as follows: "A thing is continuous when of any two successive parts the limits at which they touch, are one and the same, and are, as the word implies, held together" [Smith 1923, 93]. This definition was adopted even during the Renaissance, and long after.

Kepler was the first to use the principle of continuity, which he called *analogy*, by considering what happens to the properties of a geometrical figure under a continuous transformation which preserves certain ratios among the various parts of the figure itself [Kepler 1604, 92]. Looking at the conic sections cut by a plane rotating around a straight line, he recognised that among the conic sections there exists an order due to their properties, and ,"speaking analogically rather than geometrically" (analogicè magis quam Geometricè loquendo), the conic sections are connected in "continuous" manner: from the (double) straight line one passes through an infinity of hyperbolas to the parabola (which stands in the middle), and thence through an infinity of ellipses to the circle, and vice-versa. He constructed a very interesting continuous plane system of conics (fig. 1) in which all the elements share a vertex, axis, and a focus. By observing the behavior of those conics, when one of the foci remains fixed and the other goes to infinity, he was led to new concepts which had a fundamental role in the development of geometry. First of all, that of "point at infinity" of a straight line, which he introduced by defining the second focus of the parabola, that he called "*caeco foco*" (blind focus), as the point of the axis infinitely far from the vertex of the parabola, and which has to be considered on both sides of the axis. Any straight line parallel to the axis is the reflection of a ray issuing from the first focus, which is directed to the blind focus. Thus he implicitly recognized the "points at infinity" as the butt of a set of parallel lines. He also conceived the straight line as a "closed line", like a circle, by means of its point at infinity, which, in fact, is thought to be "located" at infinite distance on both sides of the straight line.[7]

[6] For the history of the principle of continuity in mathematical thinking we refer to [Knobloch 1991].

[7] See [Taylor 1900], [Field 1987], and for a detailed analysis of Kepler's system of conics [Del Centina 2016a].

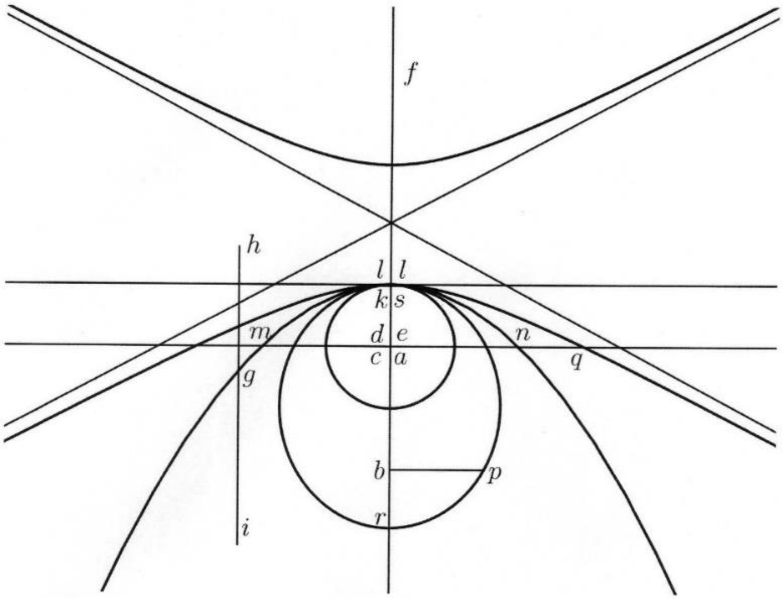


Figure 1. Kepler's continuous plane system of conics. All conics in the system share the fixed focus $a = c = d = e$, the vertex l, and the axis fr. By moving the second focus b along the axis (first in a), one passes from the circle through infinitely many ellipses to the parabola, when point b is at infinity, and then through infinitely many hyperbolas, whose second focus re-appears upon the axis as f, to the double-line (coinciding in the figure with the tangent at the vertex l to all conics, counted twice), as the limiting figure of the collapsing hyperbolas.

In the same work Kepler gave the mechanical constructions for the ellipse and the hyperbola, and he used analogy in order to get the mechanical construction of the parabola [Kepler 1604, 96]. He started from that of the ellipse with foci A, B and vertex C (fig. 2). The ends of a string, of length $AC + BC = c$, are pinned at the foci A and B, and a pen P is placed at C; if the pen P moves around the foci, keeping the string stretched, it draws an ellipse. Kepler observed that when focus B is pushed along the axis to infinity, keeping the other focus A fixed, the initial ellipse changes continuously into more elongated ellipses until, at the limit, it becomes a parabola, and all focal rays PB have become parallel to the axis. So, in order to construct the parabola mechanically, he fixed a point E on the axis AC, took a string pinned in A and of length $l = EC + CA$; then by moving the end E of the string along the perpendicular EK to the axis, keeping the string stretched by the pen P, initially placed at C, and PE' parallel to the axis (so that the distance of P from AC is always equal to EE'), the pen P draws the parabola (fig. 2). To get this result, Kepler likely used analogy by reasoning as follows. If H be the intersection point between the perpendicular to the axis AC issuing from E and the straight line BP (fig. 2), then one has

$$BE + EC + AC = c = AP + PH + HB.$$

It is readily seen that by moving B away from A, along the straight line CE, both HB and EB grow beyond any limit and the difference $HB - EB$ decreases, until it becomes 0 when B is at infinity. Hence, in the limiting case of the parabola, H has become E', and one has $EC + AC = AP + PE'$.

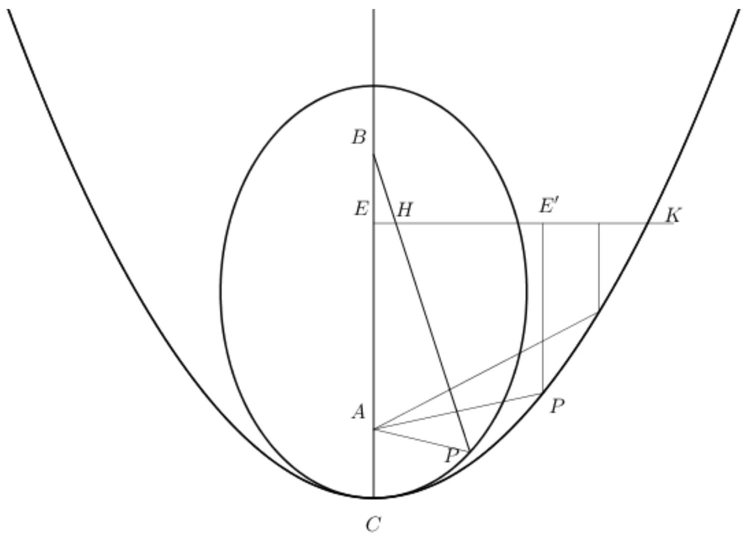


Figure 2. Construction of the ellipse, and Kepler's construction of the parabola by use of analogy.

According to Kepler one must admire analogies, especially in geometry "since they restrict the infinity of cases between their respective extremes and the mean with however many absurd phrases, and place the whole essence of any subject vividly before the eyes".[8]

Kepler's new ideas set out in the *De coni sectionibus* did not escape the attention of the mathematicians of his time. Henry Briggs comprehended and accepted Kepler's way of looking at parallels as straight lines to, or from, a point at infinity in one direction or its opposite. In a letter to Kepler dated "10 Calendae Martiis 1625",[9] Briggs stated a theorem to which he came by using analogy (fig. 3):

Let A be a vertex of a conic section (ellipse or hyperbola), B and C its foci, and make $AB = AD$ draw the circle of centre C and radius CD. Then all points of the conic are at the same distance from the focus B and from the circle.[10] In the ellipse is $NB = NM$, in the hyperbola is $OX = OB$, in the parabola, (whose second focus is missing, or rather is infinitely distant, the straight line DF takes the place of the circle), is $PB = FP$.[11]

When C becomes infinitely far from B, the ellipse is transformed into a parabola, the circle into a horizontal straight line (which is the common tangent at D to both circles), and the equality $NB = NM$, or $OX = OB$, by the principle of continuity, becomes $PB = PF$, which expresses the focus-directrix property of the parabola.

So, by the principle of continuity a property which holds for the ellipse and the hyperbola is extended to the parabola, as the limiting figure when the second focus of the conic has reached infinity. Moreover, with his reasoning Briggs clearly assimilates a straight line to a circle whose centre is located infinitely far away.

[8] "In Geometria praecipuè suspiciendos, dum infinitos casus interiectos intra sua extrema, mediumque, quantumvis absurdis locutionibus concludunt, totamque rei alicuius essentiam luculenter ponunt ob oculos" [Kepler 1604, 95]. English translation by W.H. Donahue, in [Kepler 2000, 109].

[9] The letter is published in [Kepler 1859, 405-407].

[10] This circle is today known as *directrix circle*.

[11] "Si A sit vertex sectionis, et B, C foci, et AB , AD aequales, et centro C, radio CD describatur peripheria: quodlibet punctum sectionis eandem servabit distantiam a foco B et dicta peripheria. Eruntque in Ellipsi NB, NM in Hyperbole OX, OB, in Parabola (cui focus alter deest, vel distat infinite, et idcirco recta DF vicem obtinet peripheriae) PB, FP aequales" [Kepler 1859, 406].

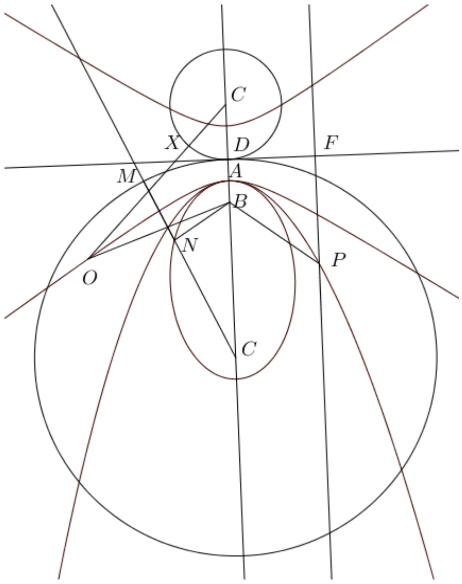


Figure 3. Brigg's theorem. All points of the ellipse are at equal distance from the focus B and from the (lower) circle with centre at C and radius CD, i.e. $NB = NM$. The same is for the other (upper) circle with respect to the hyperbola.

In his *Brouillon project*,[12] Desargues seems to adopt the principle of continuity several times, for instance, when he defines the four points involution, or when, similarly to Kepler, he writes that one can pass from one conic to another though infinitely many conics, and the parabola stands in the middle between ellipsis and hyperbolas [Desargues 1639, 8 and 15]. Desargues also alludes to the "continuous" and to the "mysteries of the infinity" [Desargues 1639, 1] and in the *Advertissement* (the errata corrige attached to his work), when he, referring to the occurrence of infinitely small and infinitely great quantities, seems to say that the mind is sometimes unable to understand the results of reasoning (see note to "*Page* 30" in the *Advertissement*).

Blaise Pascal, in the *Generatio conisectionum*, where he adopts and extends Desargues' ideas, says that "the parabola is in the medium between ellipse and hyperbola".[13]

It seems that the ideas of Desargues and Pascal, led Leibniz to a "dynamical" vision of the geometry. In the years 1672-1676, Leibniz was in Paris, where he greatly increased his mathematical knowledge. In 1673, he was informed on Desargues' perspective through La Hire's *Nouvelle méthode en géométrie* [La Hire 1673], and Bosse's *Manière universelle* [Bosse 1648].[14] In 1675, as is confirmed by a letter he addressed to Gallois [Debuiche 2013, 366], Leibniz recognised, as a fundamental aspect of the Desarguesian theory, the assimilation of converging straight lines to parallels, "whose intersection point is located at infinity". The following year Leibniz had the opportunity to read the manuscript of Pascal's treatise on conic sections [Taton 1962, 198]. After his stay in Paris, Leibniz came back to London, and, in January 1677, he initiated his work on *Characteristica Geometrica* [Leibniz 1679], by which he intended to develop a geometry of situation, that was to have extended and perfected the Cartesian geometry, including the Euclidean one, equipped with its own language and formalism. Here, a line is not seen as a set of points, but as "produced" by the motion of a point, as the motion of a line produces a surface, and that of a surface produces a solid. Moreover, in this work, all properties of conic sections can be deduced

[12] Reprinted and commented in [Taton 1951]. For an English translation [Field and Gray 1987].
[13] "Denique patet, Parabolam tenere medium inter Antobolam et hyperbola" [Gerhardt 1892, 202].
[14] See [Echeverria 1994].

from that of a “universal” one, without referring to their kind, ellipse, parabola, hyperbola, circle or (double) straight line [Echeverria, 1994, 284-285; Debuiche, 2013].

Leibniz believed that the *principe de l’Ordre Général* (principle of the universal order), which he later called principle of continuity,[15] is very useful in reasoning, even if it is neither used enough, nor known in all its range. He formulated it in a brief text entitled *Extrait d’une Lettre de M. L. sur un Principe Général…*:

> When the difference between two instances in a given series or that which is proposed can be decreased until it becomes smaller than any given quantity, the corresponding difference in what is sought or in their results must of necessity also be decreased or become less than any given quantity whatever. Or to put it more commonly, when two instances or given objects approach each other continuously, so that one at last passes over into the other, it is necessary for their consequences or results (or the unknown) to do so also. This depends on a more general principle: that, as the given objects are ordered, so the unknown are ordered also.[16]

According to Leibniz “this principle has its origin in the infinity, and it is absolutely necessary in geometry, although it operates to advantage in physics as well”.[17]

He gave several examples of applications of the principle, and as a geometrical one he adduced the continuous transformation of one conic section into another, and explained how the parabola can be seen as the limiting case of the ellipse when one of the two foci goes infinitely far from the other. In particular, he observed that in the limit the rays coming from the remote focus differ from straight lines parallel to the axis less than any given quantity. He claimed that all theorems valid for the ellipse remain, in general, valid for the parabola [Leibniz 1687, 746-747].

The similarity with the ideas expressed by Pascal, Desargues and Kepler, is evident. Nevertheless, as Kline [1972, 843] remarks, it was only after Leibniz that the principle of continuity was recognised and used constantly.

Certainly with the development of differential calculus in the late seventeenth century, the consideration of infinitely small and infinitely great quantities connected with the concept of limit, somehow influenced the use, and even the same definition, of the principle of continuity. Moreover, the idea of imaginary points also became a necessary supplement to the law of continuity. This may have originated from the geometry of Descartes.

Isaac Newton [1704a, 151] in the *Opticks* admitted that an algebraic equation in x and y, which represent coordinates, may have imaginary roots. According to Taylor [1900, 218], to say that “is to say in effect that there are what may be called imaginary points”. In fact, Newton extended the conclusions at which he arrived under the hypothesis of reality of some points, to the case in which these become imaginary. In the third edition of the *Principia*, increased and amended [Newton 1726, book I, section 5, proposition XXIV],[18] after the proof in which an auxiliary straight line which is supposed to meet the sought conic in two points is used, he observed “But once the constructions have been demonstrated for the case in which the straight line does cut the trajectory, the constructions for the case in which it does not cut the trajectory also can be found; and for the

[15] Leibniz called it “principle of continuity” only starting in the early 1690s [Jorgensen 2009, 224].

[16] “ … lorsque la différence des deux cas peut être diminuée au-dessous de toute grandeur donnée, in datis, ou dans ce qui est posé, il faut qu’elle se puisse trouver aussi diminuée au-dessous de toute grandeur donnée in quaesitis, ou dans ce qui en résulte ; Ou pour parler familièrement : lorsque les cas (ou ce qui est donné) s’approchent continuellement, et se perdent enfin l’un dans l’autre, il faut que les suites ou évènements (ou ce qui est demandé) le fassent aussi. Ce qui dépend encore d’un principe plus général, sçavoir : datis ordinatis etiam quaesita sunt ordinata” [Leibniz 1687, 746 ; Leibniz 1989].

[17] “ Il [ce principe] a son origine de *l’infini*, il est absolument nécessaire dans la Géométrie, mais il réussit encore dans la Physique …” [Leibniz 1687, 745].

[18] To describe a trajectory that will pass through three given points and touch two straight lines given in position.

sake of brevity I do not take the time to demonstrate them further”.[19] What Newton seems to be doing is to extend the validity of a certain functional relation which holds when it produces real solutions, also to the case of imaginary solutions. In a certain sense this is what Boscovich, and later Poncelet, would do by introducing the “ideal chord” (see later). In the subsequent lemma XXII, Newton considered certain transformations of a curve $\mathcal{F}$ into another of the same order, defined as follows: if a and b are given constants, and X, Y represent, respectively, the abscissa and the ordinate of a point P on $\mathcal{F}$, then the abscissa x and the ordinate y of the corresponding point on the transformed curve $\mathcal{F}'$ satisfy the conditions $Xx = ab$, $Yx = ay$. When Newton observed that under the transformation, tangents to $\mathcal{F}$ are transformed into tangents to $\mathcal{F}'$, he was using the principle of continuity. In fact, he considered the tangent in A as the limiting position of a straight line, intersecting the curve in A, B, when B approaches A.

Traces of the principle of geometrical continuity can be found in Euler [1748, 73-74], Simson [1750, 210] (first published in 1735), and MacLaurin [1748]. In this last work a (real) geometrical interpretation of certain imaginary roots was given, similarly to what Boscovich was to do six years later. But, very likely, these contributions had no influence on him.

It goes without saying that Boscovich read, and studied, Newton’s works, expecially the *Principia*, and, at least in part accepting his approach to natural philosophy, Boscovich promoted the introduction of the Newtonian theory of gravitation in the Jesuit Colleges around the 1740s, see [Casini 1983, 143-171, Pepe 1998, 410-411].

In 1745, Boscovich had to deal with the themes of natural philosophy at the Collegio Romano for the first time, and the very same year he published the essay *De Viribus vivis dissertatio* [Boscovich 1745], which may be seen as the first step toward the *Theoria* [Boscovich 1758]. This work was followed by the *De materiae divisibilitate et principiis corporum*, written in 1748 [Boscovich 1757], where he tackled the question of the divisibility of matter, already present in *De Natura, et usu Infinitorum, et Infinite parvorum* [Boscovich 1741]. Boscovich considered the concepts of indivisible physical points and infinite divisibility of matter, to be closely linked. He qualified the first as mathematical points, without extension and dimension, but having real properties, such as the *inertia*, that is the quality of maintaining the status of motion. The points in their movement, as simple mathematical points do, produce curves (trajectories) which are not composed by points but just bound by points [Guzzardi 2014, 23]. We stress that the idea of straight lines and curves generated by the motion of a mathematical point, seems to come directly from Leibniz.

In 1754, Boscovich published his book on conic sections [Boscovich 1754a], to which he added the essay on the transformation of figures, where he discussed the law of geometrical continuity more extensively, and explicitly, than all other mathematicians of his century, attempting to formalize it by means of a series eleven rules. The same year, Boscovich composed the *De continuitatis lege* [Boscovich 1754b], in which he provided an in-depth analysis of the concept of continuity, aiming to ground the theory of forces on the “celebrated principle” introduced by Leibniz in 1687 [Guzzardi 2014, 34]. He wrote:

> In every continuous quantity one needs to distinguish between what is the end or the limit, from what is limited. The first must be indivisible, by the reason that it is an end, the second has to be infinitely divisible… From the nature itself of the limit, it follows that a limit cannot be contiguous to another limit. In fact, we need to consider the continuous as what lies between its limits. Neither one can be the end of the preceding, nor can the other be the beginning of the subsequent, as, according to the nature of the continuous, their limit must be in common.[20]

[19] “Sed demonstratis constructionibus ubi recta illa trajectoriam secat, innotescunt constructiones, ubi non secat; iisque ultra demonstrandis brevitatis gratia non immoror.” [Newton 1726, 87 ; Newton 1999, 493].

[20] “In quavis continua quantitate distingui debet id, quod est terminus, seu limes, ab eo, cujus est terminus. Illud primum in ea ratione, in qua est terminus, debet esse indivisibile, hoc secundum debet esse divisibile in infinitum. …Ex ipsa

According to Schubring [2005, 179] the principle of continuity was, for Boscovich, the fundamental concept not only for physics, but also for geometry, "the latter being understood as the theory of the curves of motion". In the *De continuitatis lege* he adopted, in the Jesuit tradition, the definition of continuity proper of Aristotle, which he explained by the example of "a broken line" [Boscovich 1754b, v], (fig. 4).[21]

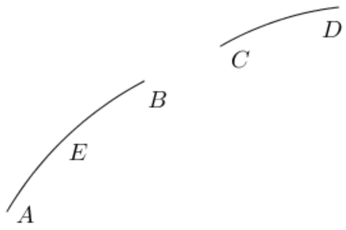


Figure 4. Boscovich's figure 1, which illustrates his example of the "broken line" to explain the definition of continuity: a line $AEBCD$ is broken at B and C, with an interruption of the continuity by the jump between B and C; whilst the line AEB is continuous because E is the common endpoint of the parts AE and EB.

Boscovich adopted Leibniz's definition of the principle of continuity by translating into Latin, word by word, what Leibniz had written in his reply to Malebranche [Boscovich 1754b, xliv; Guzzardi 2014, 34-35], and he also reported the definition of Bernoulli [Boscovich 1754b, xlv].

The principle of geometrical continuity resurfaced in Monge [1798], but without being explicitly stated, and in [Carnot 1801], in the form of "principle of correlation", but it was with the aforementioned treatise on the projective properties of figures by Poncelet [1822] that the principle of continuity took a fundamental place in the development of geometry.

3. The *Elementorum Universae Matheseos tomus III*

Boscovich began to be interested in the principle of continuity early in the 1740s, when he was also working on a textbook on planimetry and stereometry, or, in his words, the "vestibule of geometry",[22] which was published in two volumes only in 1752 under the title *Elementa Universae Matheseos*. Two years later, the work was reprinted with the addition of a third volume which included the *Sectionum Conicarum Elementa*, and the dissertation *De transformatione locorum geometricorum*. Most of the first part, where Boscovich entered the realm of "geometry which never operates by leaps",[23] had already been worked out in 1747 [Martinović 2015, 62]. The second part, completed by the end of 1753 or at the latest by the beginning of the 1754, likely originated from Boscovich's study on the transformation of one conic section into another which he had started to tackle when he was developing the first part. The three volumes met with wide approval, and the work was reissued in Venice in 1757.[24]

As regards the *Sectionum Conicarum Elementa*, probably Boscovich learned the first elements by Borgondio and certainly knew the well-known treatise on conic sections of Gregorius Saint-Vincent, *Opus geometricum quadraturae circuli et sectionum coni* (1647), but this work did not

termini natura consequitur etiam illud, terminum termino continguum esse non posse. Nam semper haberi debet, illud continuum ipsis interjacens, cujus ii ipsi termini sunt. Neque alter potest esse finis paecedentis, et alter principium sequentis, cum ex natura continui (num. 6) communis esse debeat eorum terminus". [Boscovich 1754b, v-vi]. Stipanić recognised the similarity between the ideas of Boscovich and Dedekind's axiom of continuity [Stipanić 1993, 480-481].

[21] In this regard see also [Guzzardi 2014, 23-25].

[22] … in ipso nimirum Geometriae vestibulo usi olim sumus [Boscovich 1754b, vii].

[23] Geometria, quae nihil usquam operatur per saltum [Boscovich 1754a, xviii].

[24] For the history of the various editions see [Pepe 2010b].

inspire him, and his approach to the conic sections and to the transformation of the curves is completely innovative.

A characteristic feature of the third volume is that there are over 270 figures distributed in seven tables, included at the end. Boscovich inserted them for an understanding of the text, and the several examples by which he illustrated his ideas and concepts on the transformation of the geometrical loci, which othervise would have remained obscure.

3.1 *The Sectionum Conicarum Elementa*

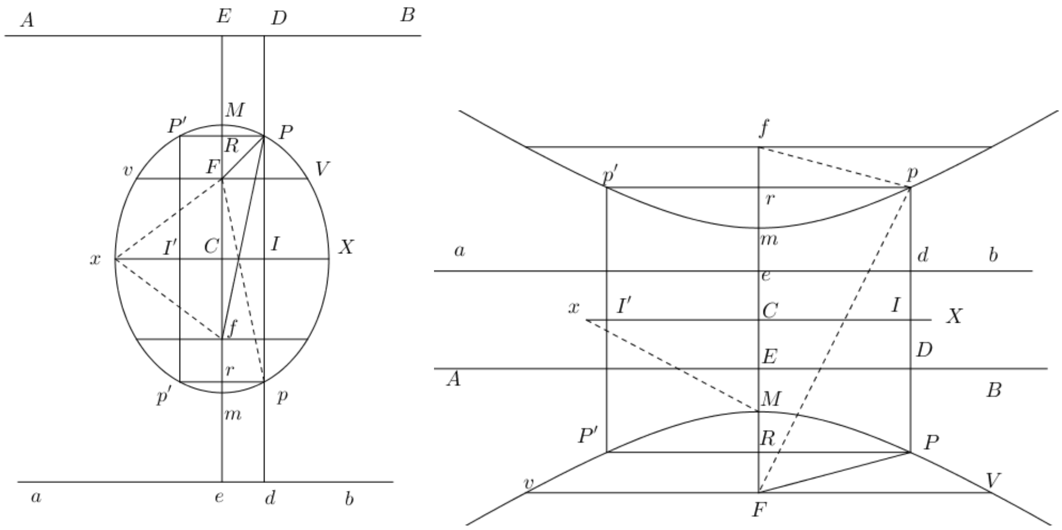
At the very beginning of his treatise Boscovich defines a conic section by assigning directrix, focus and "given ratio" (i.e. determining ratio, or eccentricity):

> If from any point *P* of a curve, a straight line *PD* is drawn perpendicular to an unbounded straight line *AB*, given in position, and another straight line *PF* is issued to a given point *F*, which lies outside *AB*, and if *PF* and *PD* are constantly in the same ratio, then I call the line [described by *P*] conic section, ellipse, parabola, or hyperbola, according to the ratio, if it is less than equality [i.e. < 1], or equal [i.e. = 1], or bigger than equality [i.e. > 1].[25]

Although this property was known to Pappus,[26] it was Newton who brought it fully to light in the *Principia*,[27] and on it Boscovich founded his whole theory of conics.

Then, in No. 90 he pointed out that in the ellipse and in the hyperbola (see fig. 5), the distance between the foci, the transverse axis, and the distance between the two directrices, are in continuous proportion, as *FM* to *ME*, that is:

$$Ff : Mm = Mm : Ee = FM : ME$$



(a) (b)

Figure 5. (a) Boscovich's figure 19; (b) Boscovich's figure 20.

The adopted definition of conic section gave Boscovich the opportunity "to move", or "to deform", one conic into another, even degenerate. Similarly to Kepler, he carried this out by costructing certain (continuous) plane systems of conics.

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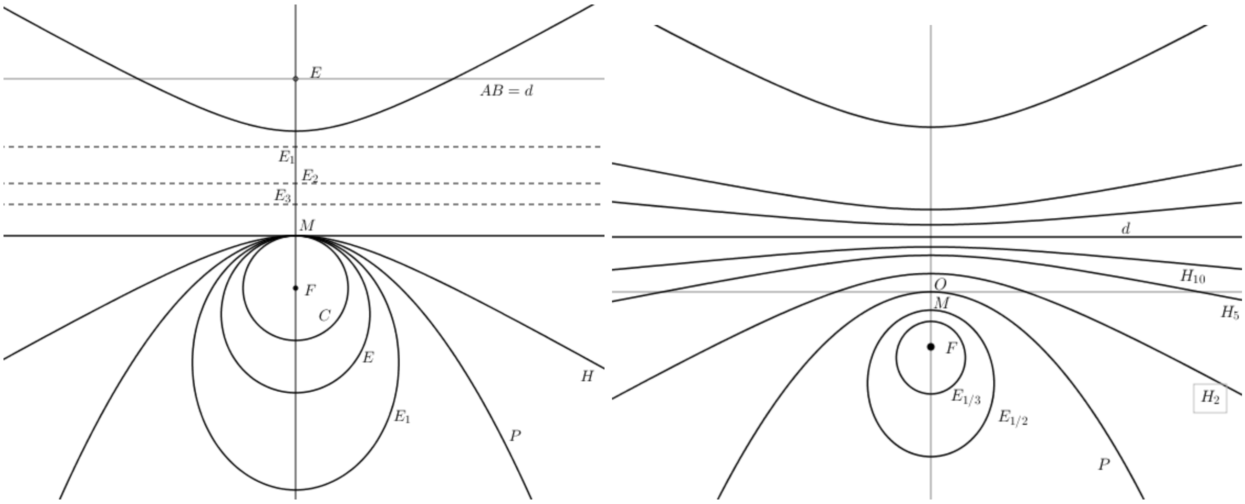
[25] "Si ex omnibus punctis P cujusdam lineae ducta PD perpendiculari ad rectam AB indefinitam positione datam, et alia recta PF ad punctum F datum extra ipsam AB, fuerit semper FP ad PD in ratione data; lineam illam dico Sectionem Conicam, Ellipsim, Parabolam, vel Hyperbolam, prout illa ratio fuerit minoris inaequalitatis, aequalitatis, vel majoris inaequalitatis" [Boscovich 1754a, 1].

[26] See his *Mathematical Collections*, lib. VII, prop. 238 in [Pappus 1588, 303 versus].

[27] See Lib. I sect. 4 and prop. 20 in [Newton 1687, 63]. See also [Newton 1707, 149].

In No. 107, Boscovich fixed the focus *F*, the vertex *M*, and left the directrix *AB* = *d* free to move parallel to itself, from the initial position through point *E*, corresponding to the ellipse even denoted by *E* (see fig. 6a). If *d* moves in the direction of *M*, the initial ellipse is transformed into more elongated ellipses, as for example E_1 when *d* has reached the position E_1, until it becomes the parabola *P* when *d* has reached the position E_2; then the parabola is transformed into hyperbolas as for example *H* when *d* is in position E_3, and so on until the hyperbola collapses onto the tangent line at *M* to all the conics of the system, when *d* has reached the position *M*. When *d* moves in the opposite direction, the initial ellipse *E* is transformed into less and less elongated ellipses until coinciding with circle *C* when point *E* (and the directrix *d*) has gone to infinity.



(a) (b)

Figure 6. (a) Transformation of the conics when the focus *F* and the vertex *M* are fixed and the directrix *AB=d* is free to move parallel to itself; (b) Transformation of the conics when the focus *F* and the directrix *d* are maintained fixed and the eccentricity *e* varies from 0 up to infinity.

We observe that this system of conics is equivalent to that introduced by Kepler, in fact, to move the directrix is equivalent to move the second focus, as Kepler did.

In No. 110, Boscovich considered the system of conics in which the focus *F* and the directrix *d* are maintained fixed and the eccentricity *e* may vary from 0 up to infinity (see fig. 6b). Starting (for instance) from $e = ½$, corresponding to the ellipse $E_{1/2}$, by decreasing *e* we have infinitely many ellipses, among which $E_{1/3}$ corresponding to $e = 1/3$, until the ellipse crashes on *F* when $e = 0$. In the opposite direction, when *e* increases, the ellipse $E_{1/2}$ is transformed, through infinitely many more elongated ellipses, into the parabola *P*, for $e = 1$, then the parabola is transformed into hyperbolas, as for instance H_2, H_5 and H_{10} corresponding to $e = 2, 5, 10$, until the hyperbola collapses onto the directrix *d* when $e = \infty$.

In the same paragraph, Boscovich also constructed the system of conics defined by making the directrix and the eccentricity fixed, and allowing the focus to move along the axis towards the directrix. If the eccentricity is < 1, the ellipses of the system (all similar to each other) degenerate in point *E* (i.e. the intersection point of the directrix with the transverse axis of the ellipses); if the eccentricity is 1, the parabolas degenerate into a straight line perpendicular to the axis (i.e. they degenerate into a double-line); if the eccentricity is > 1, the two vertices become closer and closer, and the hyperbolas degenerate into a pair of straight lines obliques to the directrix.

In No. 140, Boscovich introduced what Taylor [1881, VI] later defined “the characteristic feature of a masterly though neglected work”, this is a circle to which Boscovich gave no name, but that is known today as "eccentric circle".[28] To be precise, the *eccentric circle* of a conic, with respect to any given point, is the circle centered on that point, whose radius is equal to the distance of the centre from the directrix times the eccentricity.

The occasion for introducing the eccentric circle arose from the problem of finding the intersections of a straight line with a conic having the focus *F*, the directrix AB and the eccentricity (*Propositio* III). Clearly this is the same as drawing the conic, when one focus, the directrix and the eccentricity are given. Boscovich proceeded as follows.

To find the intersections of a "general" straight line HK with the conic, Boscovich considered intersection H of the straight line with the directrix AB, and the eccentric circle with respect to a point L (see fig. 7), then he drew the straight line though L parallel to the given straight line, which will intersect AB in O. Next he issued from O the line parallel to the line HF, which meets the eccentric circle in points t, T, and draws from F the focal radii parallel to tL, TL. These two straight lines through F will meet the given line HK in P, p. Then, if LG, PD are perpendicular to the directrix, it results that

$$PF : PD = LT : LG = e,$$

that is point P belongs to the conic. The same holds true for point p.

We may observe that under this transformation secants and tangents to the eccentric circle are transformed into secants and tangents to the conic. This shows a powerful but quite elementary method by means of which the properties of a conic may be inferred from those of a circle, avoiding the cone (that is "without projection and section").

We shall be returning to this proposition in section 5, when we comment on Boscovich's rule 9.

Boscovich stressed (No. 110) the usefulness of considering continuous transformations of one conic section into another because "they may disclose their true geometrical inner nature". Most likely, in dealing with the transformation of the conic sections, Boscovich was led to tackle the problem of the transformation of geometric loci in its generality, and this resulted in the aforementioned dissertation.

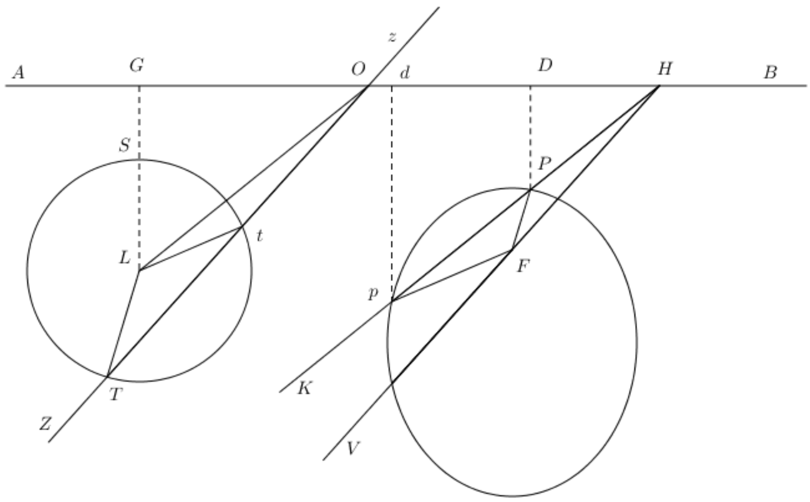


Figure 7. Boscovich's figure 41, which illustrates the construction of a conic by means of an eccentric circle.

3.2 *De transformatione locorum geometricorum*

It would seem that Boscovich was aware of the originality of his conceptions, but cautiously abstained from overstressing the novelty of his speculations, as he wrote:

[28] Following Taylor, we use this terminology, but other authors call it "generating circle" or "auxiliary circle".

It (the *De transformatione locorum geometricorum*) contains many things worthy of knowing which I have not found elsewhere, and many things that can also be found in other places but nowhere have I found them reduced to sound rules, and studied by the geometrical method. However, I cannot claim it as completely new because I cannot exclude it may be found in some unknown very old publication.[29]

Apart from Leibniz and Bernoulli, Boscovich [1754b, iv, xliv-xlv], did not mention any of his predecessors, but it goes without saying that he was directly, or indirectly, greatly inspired by the works of Newton and Kepler.

In his dissertation Boscovich introduced the question of the transformation of geometrical loci by discussing several examples, carrying out ample motivation in favour of the principle of continuity, and then proceeding to enunciate eleven rules, to apply in the transformation which constitutes the formal core of his essay. In fact, he wrote:

On the other hand, in the transformations of conic sections, we have variety of points and the passage through zero or through infinity, and the return from it. By approaching the infinity, the points often remain real, either visible in some place, or hidden in the infinity, or may even have become imaginary. The same happens for the segments which end in those points, at times they maintain the same direction, at times they change it, often they vanish, often they are produced to infinity, they can also have an extension greater than infinity. So the transformation of conic sections is suitable to state and confirm certain rules, which are widely observed in geometrical situations, and the examples of the rules are deduced from the elements of these curves. From these rules, and their application to the same Conicarum Sectionum Elementa, it appears what these curves have in common, and then what admits a common proof, as well as what cannot be transferred from one to another, and the very reason for the anomaly is revealed, and our aim to enhance the cited elements is achieved. All rules depend on what we have so far remarked, and they are like its fruits. We will discuss each of them individually, in order to reveal their meaning, and we will give examples and applications to conic sections.[30]

With these words Boscovich explained the twofold relationship between the transformation of conic sections, and his theory of transformations of geometrical loci. The theory is at the same time the *fruit* of the proved transformation of conics, and a *means of verification and discovery*, when it is applied to the study of conic sections when the problems cannot be immediately interpreted. For example, in No. 302, after having proved in No. 300 the chords theorem for nonsingular conics (*Propositio* VI),[31] he observes that the same property can be transferred to the limiting form when the conic splits into two straight lines.

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[29] "Multa autem continet, quae licet scitu sane dignissima, ego quidem nusquam alibi offendi, multa, quae licet alibi etiam occurrant saepe, nusquam ego quidem ad certos reperi redacta canones, et geometrica methodo pertractata. Ea tamen pro novis venditare non audeo; cum mihi quidem inscitiae meae culpa, nova esse possint, licet fortasse sint apud Litterariam Remp[ublicam] vetustissima". [Boscovich 1754a, xviii].

[30] "Porro in hujusmodi transformationibus Sectionum Conicarum aliarum in alias habentur punctorum multiplices et transit per nihilum, ac per infinitum, et regressus inde. Ipsi autem appulsus ad infinitum, vel nihilum saepe puncta retinent in statu reali, vel alicubi conspicua, vel infinito obruta, ibique velut delitescentia, quandoque etiam ad imaginarietatem deturbant, adeoque linearum, quae ipsis terminatur, habetur jam perseverantia in eadem directione, jam directionis mutatione, jam impossibilitas, et saepe annihilatio, ac evanescentia, saepe productio in infinitum, saepe etiam circuitus quidam per infinitum, et quaedam veluti plusquam infinita extensio. Hinc haec ipsa Conicarum Sectionum transformatio aptissima est, ad declarandos, confirmandosque quosdam canones, qui per universam late Geometriam observantur, et eorum exempla ex demonstrates harum curvarum elementis depromenda. Ex ipsis autem canonibus, eorumque applicatione ad haec ipsa Conicarum Sectionum Elementa patebit etiam, quae hisce curvis communia sint, et communem demonstrationem siscipiant, quae ab altera ad alteram transferri non possint, et ipsa ejus anomaliae ratio se prodet, ac nostrum in hisce elementis adornandis consilium palam fiet. Ejusmodi vero canones ex iis, quae huc usque vidimus pendent omnes, et sunt eorum quidam veluti fructus. Proponemus autem singulos, ac eorum rationem proseremus, exempla dabimus, et applicationem ad Conicas Sectiones" [Boscovich 1754a, 367-368].

[31] Let two chords of a conic be given which meet the curve respectively at *A*, *B* and *C*, *D* and intersect each other at *P*, then the ratio $AP \times BP : CP \times DP$ does not depend on point *P* but only on the directions of the two chords. This theorem is commonly known as the "chords theorem".

The different behaviour of conics at infinity, or when some points pertinent to them transit through it, induced Boscovich to elucidate some mysteries of the infinity, and to provide a more in-depth understanding of the law of geometric continuity.

Boscovich first observed that every member of a geometrically defined locus must have the same nature and properties, descending from its definition, which is not given by an analytical expression, but rather by means of a geometrical construction as is the case for conics. He claimed that since the character of an element of a geometrical locus is determined by its definition, it retains the properties resulting from the definition (No. 674). Therefore, even if the proof is achieved for a determined geometrical configuration, the reasoning remains valid in infinitely many cases even when "a different disposition modifies the scheme so that a sort of artifice is necessary to save the analogy and to keep the validity of the solution and the demonstration".[32]

Afterwards Boscovich enlarged the family of the loci, to which his analysis applied, by including some curves defined analytically, namely, those in which the ratio between the abscissae, "even however multiplied", is equal to the ratio, direct or inverse, "even however multiplied", of the ordinates; that is, he considered integer powers of the coordinates in direct or inverse ratio. Hence, among his geometrical loci he also included the families of the parabolas and of the hyperbolas of higher order [Boscovich 1754a, No. 693]. We return to these questions in section 5, when we present and discuss rule 11. However, we remark here that the extension to curves of higher degree interested Boscovich in connection with his search for a general law of forces. In fact, many of the considerations on the behaviour of curves at infinity are more extensively treated by Boscovich in *De continuitate lege*, and finally in *Philosophiae naturalis theoria*. We limit ourselves to discuss only what is in *De transformatione locorum geometricorum*, referring to [Guzzardi 2014, 54-60] for all aspects connected with natural philosophy.

As a consequence of his analysis of the behaviour of conics under transformations Boscovich pointed out:

> let a problem be solved [by some geometrical construction] in a generic situation; if the disposition of the given objects changes [even] a little bit, then the construction also changes and sometimes considerably; certain additions transform into subtractions, straight lines [segments] and angles change direction, certain terms become impossible, others grow to the infinity, and an intersection, necessary to solve the problem, disappears when two straight lines that were intersecting become parallel, circles whose centre is moved to the infinity and become straight lines, and others similar accidents. In these cases, Geometry follows the immutable law that nothing happens by leaps. Thus, to preserve continuity, it is often useful to reach the infinity and sometimes to go further, so causing occurrences which cannot be called with better name than mysteries of the infinity, and which increase till the absurdities become truths.[33]

Boscovich then set forth a typology of changes that can arise when geometrical constructions are subject to a transformation, and he established how to operate in order to preserve the principle of continuity. He distinguished the following cases: 1) passage from sums to differences and vice-versa; 2) change of direction in straight lines and angles; 3) appearance of impossible and imaginary quantities; 4) points which go to infinity, whether they are intersections, centres of circles, or generally solutions to problems. So the first part of the *De transformationem locorum*

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[32] "Multo tamen saepius in ipsis casibus positio diversa ita schema perturbat, ut artificio quodam sit opus, ad servandam analogiam, et retinendam solutionis, ac demonstrationis vim" [Boscovich 1754a, 399].

[33] "Ubi nimirum problema quodpiam generaliter solveris; mutata nonnihil datorum dispositione; plerumque ipsa constructio mutari plurimum debet; quaedam summa in differentias abeunt, quaedam ractarum et angulorum directiones mutantur, quidam termini evadunt impossibiles, quidam in infinitum excrescunt ita, ut intersectio, quae ad problematis solutionem necessaria erat, nusquam sit, ut ubi binae rectae convergentes abeunt in parallelas, quidam circuli, abeunte centro in infinitum, mutantur in rectas lineas; ac alia eiusmodi accidunt sane multa. In iis autem constantissimas quasdam leges observat Geometria, quae nihil usquam operatur per saltum. Sed in eiusmodi continuitate servanda occurrunt saepe quidam progressus in infinitum, et quidam transitus per infinitum, qui secum trahunt quaedam, quae haud suo, an alio melius nomine appellari possint, quam mysteriorum quorundam infiniti, quae tamen eo excrescunt, ut in vera demum absurda videantur recidere" [Boscovich 1754a, xviii-xix].

geometricorum, namely No. 676-759, is devoted to illustrating through examples all of these possible changes.

4. Some of Boscovich's examples

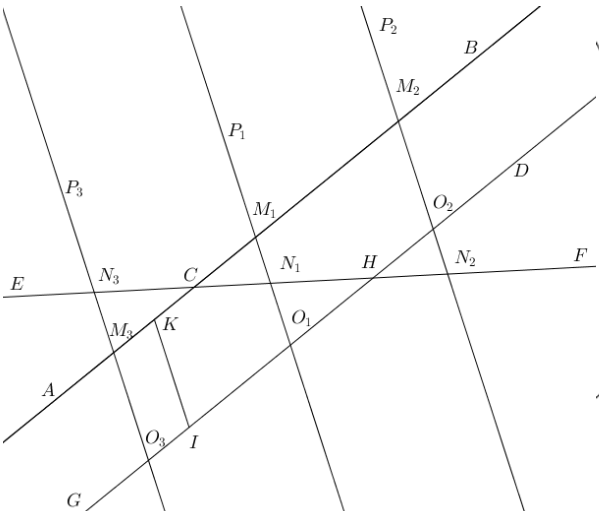
As an introduction to the rules and as an aid to understand, not only the rules, but also the "mysteries of the infinity", we present here a selection of the many examples given by Boscovich in support of the principle of (geometrical) continuity.

4.1 *From positive to negative by transiting through zero or infinity*

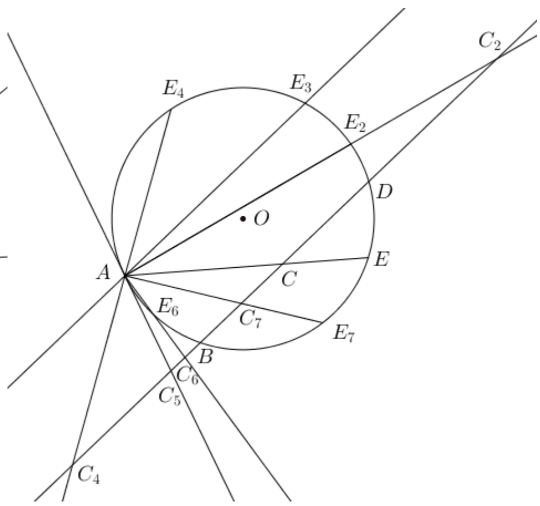
The need to change signs to segments according to their orientation, is performed by Boscovich in No. 676, see fig. 8a.

Two parallel straight lines *AB*, *DG* intersect a transversal *EF* at *C* and *H*, respectively. From a point *P*, a straight line *l* is drawn which intersects the three lines *AB*, *DG*, *EF* respectively at *M*, *O*, *N*, so that *MN* is the segment on *l* between the first and the third straight line, and *ON* is the segment on *l* between the second and the third straight line. Then from any point *K* on *AB*, the parallel is issued to *l* which intersects *DG* at *I*. If point *N* lies between *C* and *H*, as it is for point N_1 in fig. 8a, then the segment *KI* equals the sum of the segments *MN* and *ON*, but if point *N* does not lie between *C* and *H* (as point N_2 or N_3), the segment *KI* is no longer the sum of the two segments *MN* and *ON* but their difference.[34] The transition from the sum to the difference occurs when or *M* = *C* or *N* = *H*, i.e. when the segment *MN* or the segment *ON* reduces to a point, and its length is zero.

Boscovich observed that the orientation of any variable segment can be chosen arbitrarily, but if the segment changes its orientation in the transformation, in order to preserve the analogy (in this case to maintain the previous functional relation), its sign has to be changed.



(a)



(b)

Figure 8. (a) Boscovich's figure 239, which illustrates that, under a transformation, segments may change their orientation; (b) Boscovich's figure 240, which illustrates that, under a transformation, angles may change their orientation.

[34] "Ubicumque punctum P erit collocatum ita, ut N_1 cadat inter C et H, solutio problematis rite procedet. At si P jaceat in P_2, vel P_3 ita, ut N cadat extra CH, vel in N_2, ad partes H, vel in N_3 ad partes C, eadem constructio prima fronte videbitur fallere. Nam in utroque casu earundem rectarum MN, NO non erit summa, sed differentia MO, quae aequatur CI [KI]". [Boscovich 1754a, 399].

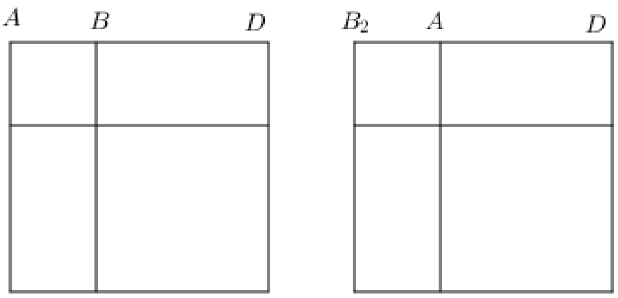
The need to change signs to arches/angles according to their orientation, is performed by Boscovich in No. 681, see fig. 8b.

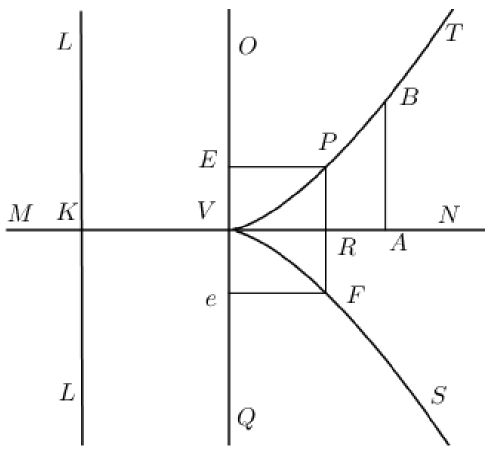
Let *AE* and *BD* be two chords of a circle which intersect in point *C* which can be either inside or outside the circle. Boscovich observed that in the first case the measure of the angle $\widehat{ACB}$ is half the sum of the arches *AB* and *DE*, that is $\left(\widehat{AOB} + \widehat{DOE}\right)/2$; while if point *C* , as point C_2, is outside the circle the measure of the angle $\widehat{ACB}$ is equal to the difference of the two arches *AB* and *DE*. Boscovich remarked that in the second case, if the arch *DE* is taken negatively, the measure of the angle $\widehat{ACB}$ is still half of the sum of the two arches *AB* and *DE*.[35] He considered the chord *AE* as rotating (counterclockwise) around point *A*, and when $E = D$ the arch *DE* is zero, and then, when *E* overtakes *D* the arch *DE* changes orientation, and must be considered as negative. Boscovich also observes that, when $E = E_3$ the arch DE_3 is equal to the arch *AB* and the straight line AE_3 is parallel to the straight line *BD* so that the angle between the two chords is equal to zero (point C_3 is at infinity), when $E = A$, the chord *AE* is the tangent to the circle at point *A*, and when *E* overtakes *B*, the arch *DE* changes its sign again, as there is another passage through zero when $E = B$.

Boscovich seems to be saying that by rotating the chord *AE* around *A*, the geometry of the figure changes continuously, so that the angle between the two chords *AE* and *BD* also changes continuously and the angles involved pass from positive to negative angles by the transit through the zero, and that the algebraic relation $\widehat{ACB} = \left(\widehat{AOB} + \widehat{DOE}\right)/2$, which holds when the intersection point *C* is inside the circle, holds for any possible mutual position of the two chords.

The functional relation is maintained.

Boscovich remarked (No. 686) that several propositions of the second book of Euclid's *Elements*, such as propositions four and seven, geometric representations for $(a + b)^2$ and $(a - b)^2$, can be led to one another, by taking the sign into account when the direction is changed. In proposition four, point *B* is taken between *A* and *D*, and the sum of the squares on *AB* and on *BD* plus twice the rectangle with sides *AB*, *BD* is equal to the square on *AD*; in proposition seven, point B_2 is taken out of the segment *AD*, and the sum of the squares on AB_2 and on B_2D is equal to the square on *AD* plus twice the rectangle with sides AB_2, B_2D (fig. 9 a,b). These are two statements of the same theorem, and the seventh proposition results from the fourth if it is noted that AB_2 has the opposite orientation with respect to *AB*, while DB_2 has the same orientation as *DB* (or vice versa).





[35] "… mensura anguli ACB est semisumma arcuum AB, DE a rectis ipsum continentibus interceptorum (Cor. 4 Pr. 9, Geom.). At si punctum C_2 jaceat extra circulum; ea ipsa mensura anguli AC2B evadit differentia arcuum AB, DE_2 quod nimirum directio arcus DE_2 est contraria directioni DE, quae si negative modo sumatur: adhuc pro mensura habebitur semisumma." [The measure of the angle ACB is the half-sum of the arcs AB, DE (Cor. 4 Pr. 9, Geom.). But if point C_2 is outside the circle, the measure of the angle AC_2B become the difference of the arcs AB, DE_2, as the direction of the arc DE_2 is opposite to the direction of the arc DE, if it is computed negatively, for the measure you will have the half-sum.] (Boscovich 1754a, 302-303).

(a) (b) (c)

Figure 9. (a) Proposition IV of Euclid's *Elements*: the square on *AD* equals the sum of the squares on *AB* and on *BD*, plus twice the rectangle $AB \times BD$; (b) Proposition VII of Euclid's *Elements*: the sum of the squares on AB_2 and B_2D equals twice the rectangle $AB_2 \times B_2D$ plus the square on *AD*; (c) Boschovich's fig. 250, the branch *VS* of the curve is the reflection of the branch *VT*.

It is only in No. 714, after having discussed many examples in its support, that Boscovich introduces the principle of geometrical continuity:

> It is wonderful that in geometry the continuity law is ever preserved, according to which nothing changes by leaps; no thing arises or vanishes at once, but always passes from one magnitude to another through all the intermediate stages. No arc of a geometric loci breaks somewhere, but it turns, or reflects, on itself as in figure 250 [fig. 9c] at point *V*, and either it returns on itself as in the ellipse, or it goes to the infinity as the arcs of the hyperbola and of the parabola, or it rounds on itself in an endless spiral shape, that is, when it moves away from a point, on the one hand, it goes to the infinity and, on the other, it approaches [another point] without reaching it and without being truncated somewhere, as happens with that curve called logarithmic spiral, of whose nature we will deal elsewhere, or it goes to the infinity with two spiral shapes, which occurs in a lot of other spirals. Moreover, the coordinates, the normal lines, the straight lines under the tangent lines, the tangent lines, the angles between the tangent lines and the axe, or with any other given straight line, or with a straight line defined by the same geometric locus, the curvature, the direction of the curve, and any other quantity, all these quantities change without any leaps, by passing through all the intermediate magnitudes of the same sort.[36]

So, according to Boscovich not only the geometrical curves are continuous in themselves, but also objects, as tangent, or quantities, as curvature, connected with them change with continuity.

In the subsequent No. 715, Boscovich stressed that the change of sign never takes place "per saltum" but "per gradus continuos" and that this can occur in two ways: either by transit through zero or by transit through infinity, as he showed by the following example (No. 716). From a point *C* not lying on a straight line *AB* (fig. 10) draw the perpendicular *CH* to *AB*, and a straight line *CP*. If *CP* rotates by continuous motion around *C*, "immotum ordine *NKOQ*" (i.e. counterclockwise), the sign of the segment *HP* changes, from positive to negative or viceversa, when *P* passes though *H*, and it changes again when the line *CP* becomes parallel to *AB*, i.e. when point *P* is at infinity on the one side of *AB*, (positive or negative), and in the next instant it is at infinity on the opposite site, and *HP* has changed its sign. He wrote:

> Let *FG* rotate continuously, so that point P' continues to go away from *H*, so that HP' increases, and by passing through any degree of finites magnitudes, it becomes infinite when L' is in *O*. In this case, the intersection P' will be nowhere, as it is swallowed in that huge Ocean of the infinity.[37]

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[36] "Mirum sane, quam sibi ubique costans sit Geometria potissimum in lege continuationis servanda, cuius vi nihil uspiam mutatur per saltum; aut totum simul exoritur, aut evanescit, sed a quacumque magnitudine ad aliam quamcumque semper itur per intermedias omnes. Nullus Loci Geometrici arcus uspiam abrumpitur, sed vel in gyrum torquetur, vel in se ipsum reflectitur, ut in fig. 250 in V, ac vel in se ipsum redit, ut in Ellipsi, vel in infinitum protenditur, ut crura hyperbolica, et parabolica, vel spiris infinitis circumagitur, aut recedendo a puncto quodam ex altera tantum parte in infinitum, et ex altera accedendo semper, quin ad ipsium pertingat unquam, et quin tamen uspiam abrumpatur, quod et illi accedit, quam spiralem logarithmicam apellant, et cuius naturam alibi persequemur, vel demum binis saltem spirarum ordinibus recedendo in infinitum, quod aliae multae spirales praestant. Ac ordinatae normales, subnormales, tangentes, anguli tangentium cum axe, vel cum recta data quavis, vel cum recta utcumque per eundem Locum Geometricum definita, curvatura ipsa, directio curvae, ac quidvis aliud fine ullo saltu mutatur semper transeundo per omnes intermedias quantitates eiusdem generis" [Boscovich 1754a, 326-327].

[37] "Pergat *FG* converti, et Punctum P' perpetuo recedet ab *H*, aucta perpetuo HP' per omnes finitarum magnitudinum gradus in infinitum, donec L'abeat in *O*, quo casu intersectio P'in illo infiniti quodam velut immenso pelago quodammodo absorpta nusquam jam erit" [Boscovich 1754a, 328].

Boscovich underlined that when the passage occurs through the zero, point *P* passes through *H*, and coincides with *H* itself for one moment, but when the passage occurs at infinity, in that moment point *P* disappears (No. 718).

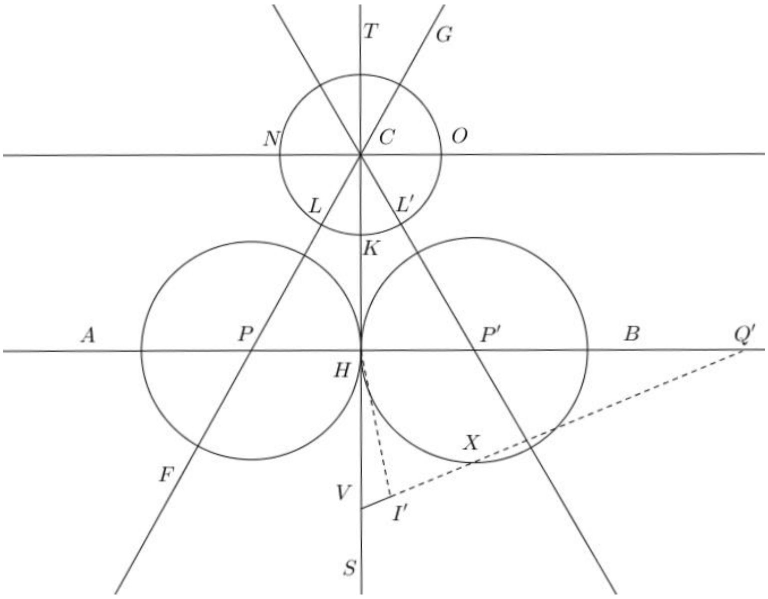

Figure 10. Boscovich's figure 264, and illustrates the change of sign which occurs by transiting through zero or through infinity.

4.2 *Occurrence of imaginary quantities and their real representation*

In No. 682 Boscovich observed that, as for segments, also rectangles and squares may become negative, according to the orientation of their sides, and following the sign-rule of algebra. He also remarked (No. 684) that it is impossible to find a real side of a negative square, in fact, as in arithmetic and algebra, he underlined, it becomes "quantitas imaginaria", that is imaginary, and he later stressed that, as for the change from positive to negative, "also the change from real to imaginary never takes place per saltum, but gradually and this change can happen only by transition through zero or infinity".[38]

An example of the occurrence of imaginary quantities is offered by the search of the mean proportional between two segments differently oriented, which is the same as finding the side of a square whose area is equal to that of a given negative rectangle (No. 685).

The following example is particularly interesting because in it Boscovich also shows how to obtain a real representation of an imaginary magnitude, see fig. 11.

Let a straight line *AD* be given, and let *C* be the midpoint of the segment *AD*, and *B* any other point on *AD*. Draw the circle of centre *C* and diameter *AD*, and the straight line through *B* orthogonal to the diameter *AD*, which intersects the circle in two points both denoted by *G*. By a well known property of the chords of a circle, the square of the side *BG* has an area equal to the area of the rectangle of sides *AB* and *BD*, and this is true for any position of point *B* between *A* and *D*. Boscovich notices that, if point *B* is taken outside the segment *AD* (as B_2 from the part of *A* or as B_3 from the part of *D*), one side of the rectangle changes orientation, so that the area of the rectangle of sides *AB* and *BD* becomes negative, and the same is true for the square so that its side becomes impossible, that is, not real. In fact, the previous construction is not possible since, in this case, the perpendicular to *AD* issuing from *B* does not intersect the circle. Then, to preserve the analogy (that is the law of continuity), Boscovich suggested another construction: let the tangent lines to the circle be drawn from point (for instance) B_3, and let H_2 be the points of tangency. The segments

[38] "Eodem pacto realis quantitas nunquam in imaginariam abibit per saltum, sed semper gradatim, nec unquam is transitus fiet, nisi ubi ea devenerit vel ad nihilum, vel ad infinitum" [Boscovich 1754a, 362].

B_3H_2 are equal and by the power theorem we have $(B_3H_2)^2 = AB_3 \times DB_3$, so B_3H_2 is the required mean proportional. Boscovich remarked that if on the perpendicular to *AD* issuing from B_3, points L_2 are taken so that $B_3 L_2 = B_3 H_2$, these points belong to a (branch of a) hyperbola which is tangent to the circle at point *D*. The same construction for points B_2 gives rise to the (other branch of the) same hyperbola.

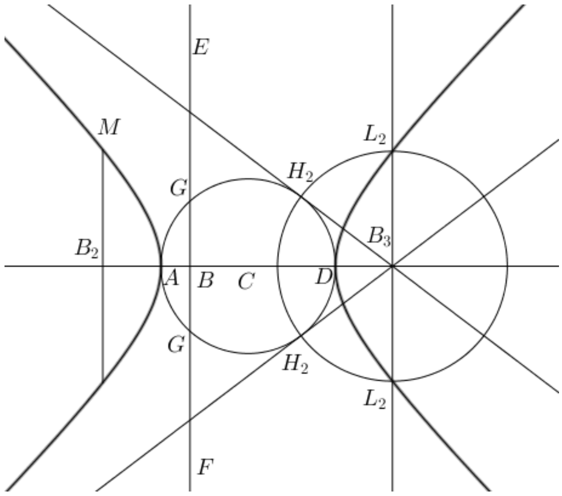

Figure 11. Boscovich's figure 242, which illustrates the construction of the "supplementary" conic by which Boscovich got the real representation of the imaginary chord.

In particular, Boscovich stressed that, in the construction with the circle, one has *AB*: *BG* = *BG*: *BD*, and that, in the construction with the hyperbola, one has AB_3: B_3L_2 = B_3L_2: B_3D, so that the functional relation is maintained. This is a very interesting remark to which we return in section 6, when introducing Poncelet's concepts of "ideal chord" and "supplementary conic", both anticipated by Boscovich.

4.3 *The straight line as a circle with infinite radius and centre at infinity*.

To Boscovich, the transit through infinity meant that the two extremities of a straight line had to be considered connected, as if the straight line was a circle of infinite radius with its centre placed at an infinite distance away. He wrote:

> One such example will be shown here, in which the passage and the connection of the infinity will be shown, and it will be clear that the straight line must be considered as a circle whose ray is infinity and the centre placed at an infinite distance, as if it hides and then comes back from the opposite side.[39]

Boscovich illustrated this idea with an example, still referring to his figure 264, here completed in fig. 10. He reasoned as follows.

Let *AB* be a straight line, *C* a point outside it, and *TS* the perpendicular to *AB* from *C*, which intersects *AB* in *H*. Consider a straight line through *C* intersecting *AB* in point *P*, and a small circle with its centre at *C* in point *L*. Moreover, let the circle of centre *P* and radius *PH* be drawn, whose curvature, according to Boscovich, is inversely proportional to *PH*. Then, if the straight line *CLP* rotates counterclockwise around *C*, when point *P* approaches point *H*, the radius *PH* decreases continuously passing through all finite degrees of magnitude, until it becomes zero when $P = H$, at the same time the curvature of the circle increases beyond any limit, until it becomes infinite and the circle is collapsed on *H*. Continuing to rotate the straight line, point *P* overtakes *H* and the circle reappears on the other side of the perpendicular *ST*, with centre at a point P' and radius HP' which,

[39] "Unum ex huiusmodi exemplis hic proferemus, in quo quidem omnino videbitur demonstrari immediatus ille transitus, et infiniti nexus, ac patebit, rectam lineam haberi debere pro circulo, cuius radius sit infinitus, et cuius centrum in infinita illa distantia, quodammodo velut obrutum delitescat, ac deinde ex parte opposita regrediatur" [Boscovich 1754a, 332-333].

continuing the rotation, increases up to infinity, passing through all finite degrees of magnitude, in the meantime, its curvature decreases continuously, until it becomes zero, and the circle becomes the straight line *CH* (infinitely produced). He wrote:

> Let L' continue to moving toward *O*: let the ray continue to increase, and it increases to infinity by passing through all degrees of finite magnitude. Meanwhile, the curvature of the circle decreases passing through all finite degrees of magnitude, and the circumference becomes closer and closer to the straight line *CH* produced infinitely in both the directions in *S* and *T*, beyond any limits. So there is any point *V* on the straight line, at whatever distance from *H*, that cannot be approached by a circumference beyond any fixed distance VI'.[40]

To show this Boscovich proceeded as follows (see again fig. 10). On the straight line *ST* he fixed a point V at any distance from *H*, and then he took a point I' not on *ST* so that VI' was as small as desired. *H* and I' having been joined, let the angle $\widehat{HI'X}$ be drawn equal to the angle $\widehat{I'HB}$. Then the straight line $I'X$ necessarily meets the line *AB* somewhere, let's say at point Q', being the sum of the previous angles less than two right angles. Clearly $Q'H = Q'I'$ and the circle with its centre in Q' and radius $Q'H$ passes through I'.

Boscovich then asked: "What happens to the circle, when *L* approches *O*, and point P' has gone to infinity, so that it will be nowhere?" His answer is: "It must coincide with the same straight line *ST* produced infinitely".[41]

Ensued by a clarification:

> That nexus of infinity is reached in two ways: first the straight line *ST* must be considered as an endless circle whose centre is at infinite distance on the side of *B*, or on the side of *A*, whose parts at infinity are connected, so that the same circumference goes away from *H* to *T*, comes back to *H* passing from *S*, the centre of the circle is at infinite distance from the line *ST* on both sides of *AB*, which in turn are joined at infinity, and the circle goes from *H* to *T* coming back to *H* passing from *S*, with continuous motion without interruption.[42]

4.4 *Hyperbola and parabola have to be considered as closed curves*

Boscovich clearly showed with an abundance of geometrical illustrations that certain properties of the hyperbola can derive from properties of the ellipse by change of sign. This "quasi elliptic" nature of the hyperbola, as Taylor [1881, 311] says, also appears in Boscovich's demonstration that the hyperbola and the parabola can be considered as closed curves (No. 737-739).

To show this for the hyperbola, referring to his fig. 269 (our fig. 12), he argued as follows.

Let *D* be one vertex of the hyperbola and let a straight DA_1 be drawn through *D*, which intersects the hyperbola in point P_1. If the line DA_1 rotates with continuity and counterclockwise around *D*, the intersection point P_1 starts to describe part of the branch to which *D* belongs, until the straight line has become A_2B_2 which is parallel to an asymptote, and point P_1 has gone to infinity. Continuing to rotate, the point reappears on the opposite branch of the hyperbola and describes it, passing through points P_3 and P_4 etc., until the straight line in the position A_5B_5 has become parallel

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[40] "Pergat jam moveri L' versus O: perget augeri circulis radius, et ipse circulus, ac per omnes magnitudinum finitarum gradus excrescent in infinitum. Interea vero curvatura circuli decrescet partier ultra quoscumque limites et peripheria ad rectam CH utrinque in infinitum productam in S, et T accedet partier ultra quoscumque limites ita, ut nullum sit punctum V ejusdem rectae in quacumque distantia ab H assumptum, ad quod ea peripheria aliquando non accedat ultra quoscumque limites distantiae quoscumque Vl' utcumquae parvae" [Boscovich 1754a, 335].

[41] "Jam vero quinam futurus est peripheriae status in ipso appulsu *L* ad *O*, in quo punctum P' ita in infinitum recessit, ut nusquam jam esset? "Debuit sane congruere cum ipsa recta *ST* in infinitum producta" [Boscovich 1754a, 336].

[42] "Inde autem duplici via nexus ille infiniti videtur erui: primo quidem, quia recta ipsa infinita ST debet considerari tanquam circulus quidam infinitus, cuius centrum sit in infinita quadam distantia, sive ex parte B, sive ex parte A, quae partes in ipso infinito copulentur quodammodo, et conjungantur, ut ipsa circuli peripheria ab H versus T digressa ad ipsum H ex parte S redeat quodammodo ductu continuo, nec usquam abrupto" [Boscovich 1754a, 337].

to the other asymptote, and continuing to rotate, the point newly reappears on the first branch of the hyperbola, and describes it passing through points P_6, D etc.

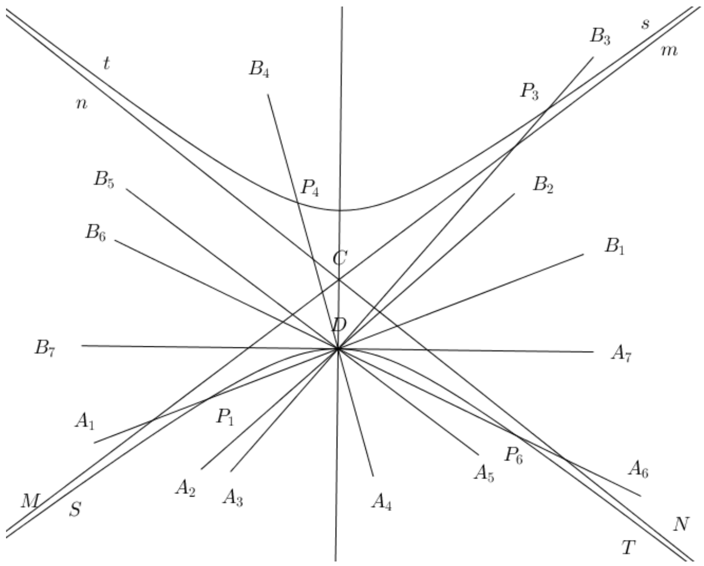
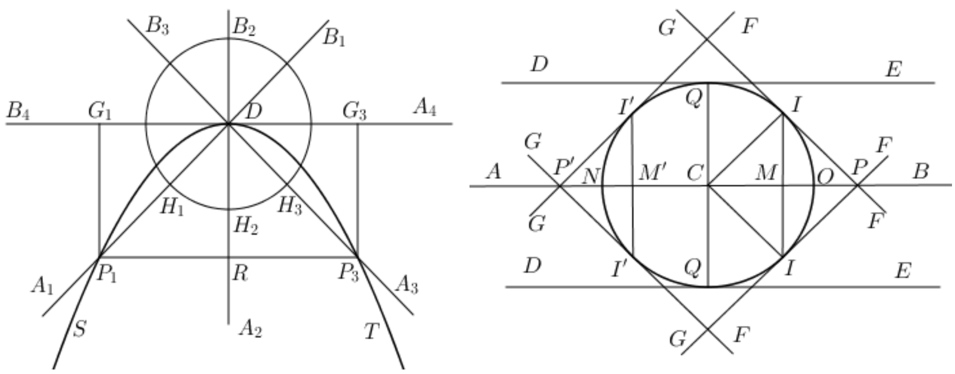


Figure 12. Boscovich's figure 269, which illustrates the hyperbola as a curve closed.

Then Boscovich wrote:

In this way, when the straight line *AB* performs by a continuous motion half turn, at the same time point *P* draws with continuous motion the branch of the hyperbola, and the hyperbola has to be considered, in some sense, a continuous [closed] curve which comes back on itself at the opposite infinite distances, the branch *t* joined with *T* and the branch *s* with *S*. The continuous path is: DP_1S (*Infinity*) sP_3P_4 *t (Infinity)* TP_6D.[43]

In No. 740, Boscovich claimed that also the parabola has to be considered as a curve closed on itself. He showed this by applying the same reasoning as before in reference to his figure 270, here fig. 13a: "Also the parabola is in a sense a continuous curve, which comes back on itself in this order DP_1S *(infinity)*TP_3D".[44]



(a) (b)

[43] "Atque hoc quidem pacto, ubi recta *AB* dimidiam conversionem absolverit motu continuo, Punctum itidem *P* motu continuo percurret utcumque Hyperbolae ramum, et Hyperbola ipsa habenda erit pro curva quadam continua, quae quodammodo in orbem redeat etiam ipsa, et in infinitis illis oppositis distantiis quodammodo veluti conjugatur, connectaturque, crure *t* conjuncto cum *T*, et *s* cum *S*. Ductus autem eius continuus est DP_1S (*Infinitum*) sP_3P_4 *t (Infinitum)* TP_6D" [Boscovich 1754a, 345].

[44] "Erit autem Parabola etiam ipsa curva quaedam continua in se quodammodo rediens hoc ordine DP_1S *(infinitum)* TP_3D" [Boscovich 1754a, 347].

Figure 13. (a) Boscovich's figure 270, which illustrates the parabola as a curve closed; (b) Boscovich's figure 265. Point *P* is the intersection of the tangents to the circle at points *I*.

Next Boscovich claimed that this property of closedness belongs to other curves:

> Generally, in all the geometric figures, […] if a curve has a branch which estends to infinity, it has necessarily one more branch which comes from infinity and at infinity the two branches are connected, on the same side or on the opposite side. This occurrence is due to the continuity law, which in geometry is always preserved. Proof of this can be obtained by means of the algebraic calculation and where the applications of algebra to geometry will be treated, everything will be proved .[45]

We recall Boscovich's intention to publish a book on the application of algebra to geometry.

In No. 753-754, Boscovich commented on "negative quantities", and "quantities greater than infinity". In 753, by considering the straight line as closed line at infinity, he interpreted a quantity which "re-emerges after it has disappeared being gone at infinity", not as a negative one but rather as positive and "greater than infinity". On the base of fig. 265 (see our fig. 13b), Boscovich observed that going *M* toward to *C*, while *CM* decreases *CP* increases, until *M* coincides with *C*, and point *P* is gone to infinity. Then *M*, continuing its motion, passes to the opposite side at *M'*, and also *P* goes to the opposite side at *P'*. If the direction *CB* is considered positive, the direction *CA* have to be considered negative, and vice versa. So, two segments *CP'* have to be considered, one in the direction of *CB*, i.e. $COB\infty AP'$, and the other in the direction of *CA*, i.e. the segment *CNP'*. Then he remarked that, when point *M* becomes *M'* by passing through zero, point *P* becomes *P'* by passing through infinity, so in the proportion $CM : CO = CO : CP$ to *CM'*, which is less than zero and negative, will correspond $COB\infty AP'$, which is positive and greater than infinity: the first term of in the proportion is negative, the fourth is positive and greater than infinity, and their product is positive. "In this example", Boscovich stressed, "the mysteries of infinity are evident".[46]

In No. 754, claiming that these considerations are very useful in the conic sections, Boscovich also wrote:

> The consideration of the two *AC* in fig. 271, that is *ARC* and $AM\infty NC$, is of great utility in the conic sections, where to the finite axis of the ellipse *MCm*, in fig. 9 [our fig. 14a], we counterpose as analogous, not the finite axis of the hyperbola Mn in fig. 11 [our fig.14b], but the axis $MH\infty hm$ prolonged to infinity. Similarly, in fig. 269 [our fig. 12] where the straight lines A_1B_1, A_2B_2, transform in A_3B_3, the straight line DP_1 in passing through infinity will transform in DP_3 negative. This [DP_1], will be transformed, if the direct analogy is respected as required by the mysteries of the infinity, in $DA_3\infty BP_3$, positive until now, and prolonged to infinity, […] but it is commonly considered for this the other *DP* finite and negative, which is the counter-analogous to that, if this term may be used, and its complement is to a certain infinite circle, because with negative quantities subtraction has to be substituted by addition, where it occurs by transit

[45] "Generaliter autem in figuris omnibus geometricis, […] si quod crus in infinitum abeat, semper habebitur crus alterum ex infinito regrediens vel ex eadem parte, vel ex contraria cum ipso in illa infinita distantia connexum quodammodo, quod omnino ad continuitatis legem ubique in Geometria servatam religiosissime est necessarium, ac ope calculi algebraici generaliter demonstrari potest, et ubi de applicatione Algebrae ad Geometriam agendum erit, omnino demonstrabitur" [Boscovich 1754a, 349-350].

[46] Some years later, in a letter to the mathematician G. Calandrelli, Boscovich, clarifying his thinking, wrote as follows: "I think that the multiplication of two negative quantities gives a positive one only when the negative quantities are considered less than zero [i.e. after having overstepped the zero], but a negative quantity to a positive one, is not as a positive one to a negative one: the fourth proportional quantity is not negative but more than infinity, according to the idea of the passage through the infinity given by me in the third volume of my elements", quoted in Italian in Guzzardi [2014, 51].

through infinity, it is necessary that the first quantity be considered analogous not to the negative one but the positive transited through infinity, which according to the previous idea is said greater than infinity.[47]

5. Boscovich's theory of transformations

"These things exposed…", writes Boscovich, "I continue by introducing the theory of transformations, which I complete by a double definition of analogy and 11 rules".[48]

5.1 *The two kinds of analogy*

Boscovich's theory of transformations begins with an explanation of the basic concepts, and first of all, that of "analogous figures" (No. 760), as he wrote:

> First we call *analogous* those points which are determined in the same way in both situations of their geometrical construction, before and after the transformation, that is, those points which are determined by intersection of geometrical loci, of straight lines with other straight lines, with the circle, with the contour of conic sections, with curves, defined by means of the same law.[49]

His definition is illustrated with some examples. Looking at fig. 8a, points M_1, M_2, M_3 are analogous to one another being defined by the same intersection law under the transformation induced by the parallel translation of the straight line *l*. The same is for points O_1, O_2, O_3 and points N_1, N_2, N_3.

Boscovich continued by saying:

> Analogous are the vertices *M* among them, as well the vertices *m*, of the transver axes of ellipse, parabola and hyperbola, being defined by the same law, that is by the constant ratio of the distances from the focus *F* and the directrix *AB*.[50]

He also called analogous the lines ending in two analogous points, the surfaces terminated by two analogous lines, the solids terminated by analogous surfaces. Examples are the straight lines M_1O_1, M_2O_2, M_3O_3 in fig. 8a; the focal rays *FM* of the ellipse, parabola and hyperbola, among them; the transversal chords *VFv* through the foci of the ellipse, parabola and hyperbola (see fig. 14).

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[47] "Consideratio tamen binarum *AC* in figura 271 nimirum *ARC*, & $AM\infty NC$, usum etiam in Sectionibus Conicis contemplandis paullo inferius habebit praestantissimum, ubi axi Ellipseos *MCm* finito in fig. 9 ostendemus prorsus, & directe analogum, non axem finitum Hyperbolae *Mm* in fig. 11, sed axem $MH\infty hm$ traductum per infinitum. Pariter in fig. 269, ubi recta A_1B_1 per A_2B_2 abit in A_3B_3, concipitur DP_1 per infinitum abire in DP_3 negativam. Abit illa, si analogia spectetur directa, & ab infiniti mysteriis petita, in $DA_3\infty BP_3$ adhunc positivam, & per infinitum traductam, & proprietates prioris quecumque a directione pendent, cum huius directione conspirant. Sed considerari solet pro ipsa illa altera *DP* finita, ac negativa, quae huic contranaloga est, si hac voce uti licet, & est eius complementum ad quendam veluti infinitum circulum, qua idea nobis infra opus erit ad ostendendum illud etiam, posse rationem reddi, cur in negativis quantitatibus subtractio additioni substituenda sit etiam, ubi obvenerint ex transitu puncti per infinitum, licet quantitati, quae habebatur ante discessum in infinitum, sit prorsus, & directe analoga non haec quantitas negativa, sed positiva illa per infinitum traducta, quae juxta illam superiorem ideam plusquam infinita diceretur" [Boscovich 1754a, 361-362].

[48] "His expositis, et tanquam materia quadam novi cujusdam aedificii praeparata, ad ordinandam transformationum theoriam progredior num. 760, quam duplicis analogiae definitione, et 11 Canonibus complector" [Boscovich 1754a, p. xxi].

[49] "In primis *Analoga* dicemus puncta, quae eodem modo determinantur in utroque ejusdem geometricae constructionis statu, ante nimirum transformationem, et post, quae nempe determinantur per concursum eorundem Locorum Geometricorum, rectarum cum aliis rectis, cum circulo, cum Sectionis Conicae perimetro, cum lineis per ejusmodi concursus definitis eadem lege" [Boscovich 1754a, 368].

[50] "Analogi sunt tam vertices *M*, quam *m* in fig. 9, 10, 11 axium transversorum Ellipseos, Parabolae, Hyperbolae, qui ubique eadem lege determinantur per rationem constantem ex foco *F* assumpto; et recta directrice *AB*" [Boscovich 1754a, 369].

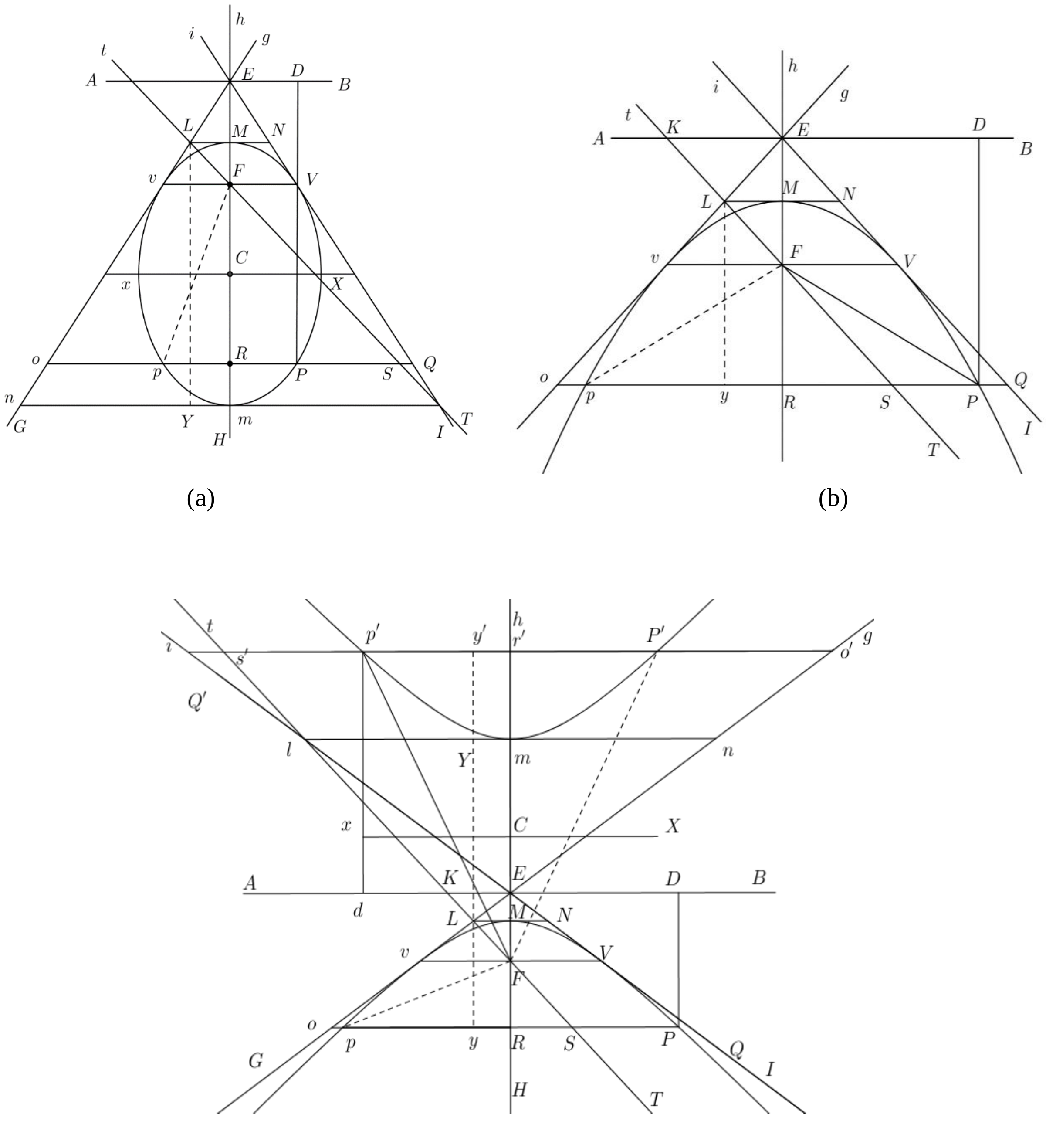

Figure 14. Boscovich’s figures 9, 10, 11. Examples of analogous points, of analogous lines in the conic sections: the vertices *M* of the transverse axes, the focal rays *FM*, the transversal chords *VFv*.

Then, in No. 761, he defined two kind of analogy:

There are two kinds of analogy, the *primary*, the main one, is when after the transformation the direction of the quantity remains unchanged, or it changes an even number of times. The *secondary*, also called *anti-analogy*, is when the direction of the quantity changes only one time, or an odd number of times.[51]

[51] “Deinde bina huius analogiae genera distinguimus: alterum *Primarium*, et summum; cum post transformationem manet directio quantitatis definitae; vel mutatur numero mutationum pari; alterum *Secundarium*, cum directio quantitatis mutatur semel; vel numero mutationum impair, quae posset etiam *Antianologia* dici” [Boscovich 1754a, 369].

Examples of primary analogy are those among all the straight lines *MO*'s; between the straight lines M_1N_1, M_2N_2; between the straight lines N_1O_1, N_3O_3 (see fig. 8a). As regards the conics (see fig. 14), examples of primary analogy are the focal rays *FM*'s; the transversal chords through a focus, when they maintain the orientation. One more example is offered by the finite transverse axes *Mm* of ellipse and the axes $M\infty m$ of hyperbola (which has to be considered close to infinity), when one of the transformations described in No. 107 and No. 110 is applied, so that point *m* moves toward *H*, reaches infinity and then goes to the opposite side as point *h*.

An example of secondary analogy is that between the straight lines M_1N_1, M_3N_3, as well as that between the straight lines N_1O_1, N_2O_2 (see fig. 8a). As for the conics, examples of secondary analogy are to be found between the focal rays *Fm*, and between the finite axes *Mm*, which have opposite orientation after the transformation (see fig. 14a, c).

Before introducing the rules, Boscovich stressed their universal validity:

The transformation of the conic sections [one into the other] is a suitable way to state and confirm the rules which are followed by the whole geometry, and the examples of the rules are taken from the Elements of these curves [Boscovich here refers to his *Sectionum Conicarum Elementa*]. The common characters of these curves, which deserve the same demonstration, but also what is not common to these curves, and the reason for the anomaly, arise from the same rules and from their application to conic sections. So our project to adorn these Elements appears clear. In effect, all the rules result from what has been performed so far and they are its fruit. We will present the rules individually providing justification and examples as well as giving their applications to the conic sections. Some mysteries of the infinity will appear and increase until they make us believe in the impossibility of an infinite extension, and we will give the theory of the indefinite quantities, either indefinitely small or indefinitely large, that in another work will be performed. [52]

Stressing the universal validity of these rules Boscovich had claimed in the *Praefatio:*

Moreover, all the Rules have been proved: the examples which have been placed first are recalled: each of them is applied to the Conic Sections to point out the analogy and their use in these Elements of mine is manifested [53]

We should like to underline, however, that they have to be intended as rules with which to operate in the transformations, justified by the examples, rather than proved theorems.

5.2 *The eleven Rules.*

In each geometrical problem Boscovich distinguished three elements: the *statement* or "enunciatio", the *proof* or "demonstratio", and the *solution* or "solutio". The rules show what happens under transformation to these three elements of the problem. We stress that in Boscovich's thinking, all transformations of geometrical loci involve basic objects, that is, points, segments, angles, as well as proportions etc.

Rule 1: *If the quantities upon which the solution of the problem or the statement of the theorem depends, after the transformation, remain unchanged in the form of primary analogy, and if there is*

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[52] "Hinc haec ipsa Conicarum Sectionum transformatio aptissima est, ad declarandos, confirmandosque quosdam canones, qui per universam late Geometriam observantur, et eorum exempla ex demonstratis harum curvarum elementis depromenda. Ex ipsis autem canonibus, eorumque applicatione ad haec ipsa Conicarum Sectionum Elementa patebit etiam, quae hisce curvis communia sint, et communem demonstrationem suspiciant, quae ab altera ad alteram tranferri non possint, et ipsa eius anomaliae ratio se prodet, ac nostrum in hisce elementis adornandis consilium palam fiet. Eiusmodi vero canones ex iis, quae huc usque vidimus pendent omnes, et sunt eorum quidam veluti fructus. Proponemus autem singulos, ac eorum rationem proferemus, exempla dabimus, et applicationem ad Conicas Sectiones. Occurrent autem identidem quaedam etiam infiniti mysteria, quae eo usque excrescent, ut infiniti extensi impossibilitatem demum suadeant, ac ad indefinitorum, sive indefinite parva sint, sive indefinite magna, theoriam, quam alio opere pertractabimus, nos deducent" [Boscovich 1754a, 367-368].

[53] "Porro singuli Canones demonstrantur accurate: singulorum exempla ex iis, quae praemissa fuerant proferuntur: singula ad Conicarum Sectionum naturam, et analogiam contemplandam applicantur, ac eorum usus in hisce meis earundem elementis concinnandis ostenditur" [Boscovich 1754a, xxiii].

no transit through infinity, then the solution, the statement and the proof remain verbatim the same. If, instead, some of the quantities transit through infinity and at infinity they are conceived as joined and connected, by maintaining one end, in those [quantities] which depend only on the direction, everything remains unchanged, in those [quantities] which pertain to the magnitude, ratio should be regarded equal to that [ratio] which derives from their definition, without considering any transit through infinity.[54]

The example of No. 676 is again adduced (see subsection 4.1).

As regards the second part of the rule, Boscovich refered to the property of a light ray coming from one focus which is reflected by the conic section; in the case of the ellipse, after reflection, the ray converges to the other focus, while in the hyperbola the reflected ray passes through infinity and then converges to the second focus (No. 770).

Moreover, to justify the part of the rule where the magnitude is considered, Boscovich recalled that in the ellipse, and in the hyperbola, the distance between the foci, the transverse axis, and the distance between the two directrices are in continuous proportion in the ratio $FM{:}ME$ (see subsection 3.1 and fig. 5). So, according to Boscovich, in the case of the hyperbola, the arc $F\infty f$ of the infinite circle should be larger than the arc $M\infty m$, and the latter larger than the arc $E\infty e$, because they are in the constant ratio of $FM{:}ME$, which is > 1, even if it seems to be the converse, that is, the first arc less than the second one and the second less than the third. A great mystery of the infinity is highlighted here, claimed Boscovich, who tried to solve it by explaining that "One must consider the first circle [corresponding to the arc $F\infty f$] extended to infinity much more than the second one [corresponding to the arc $M\infty m$] and the second much more than the third [corresponding to the arc $E\infty e$], so that ∞ has different distances in different circles, and this is due to the condition and the nature of the segments which are considered infinitely extended."[55]

Therefore, Boscovich's conclusion as to the mystery of the infinity, is as follows:

> In order, therefore, to maintain the analogy, an infinity may have to be greater than another infinity, even infinitely greater, and also that the ratio between an infinite quantity and a finite one, is equal to the ratio between a finite quantity and zero.[56]

Boscovich returned to these mysteries in rule 11.

Rule 2: *If certain quantities remain analogous under the second kind of analogy, in the statements, and in the proofs, those which have changed direction an odd number of times have to be considered as negative, to be precise, if out of two quantities only one is changed in this way, the sum will become a difference, which, in its turn, will result positive or negative according to the quantity which has changed: if it is smaller or bigger than the other respectively; if both quantities change their direction, the sums remain sums and the subtractions remain subtractions, but, in this case, they have to change from positive to negative if they enter other problems or theorems. In the proofs, in which proportions are used, if only one term of the ratio, no matter whether the first or*

---

[54]"Si quantitates, a quibus solutio problematis pendet, vel enunciatio theorematis, maneant omnes post transformationem analogae primo analogia genere, nec ullus habeatur transitus per infinitum; manebit eadem solutio, enunciatio, demonstratio, nulla re, nulla verbo mutato. Quod si aliquae ex iis per infinitum traductae et in ipso infinito copulatae, ac connexae inter se concipiantur, extante utroque extremo; in iis, quae a sola directione pendent, manebunt itidem omnia; in iis, quae ad magnitudinem pertinent, censeri debet earum ratio eadem, quae oritur ex ea lege, qua determinantur prorsus analoga illi, quam haberent si per infinitum non transissent" [Boscovich 1754b, 373].

[55] Videtur hoc ingens quoddam infiniti mysterium… Debet igitur concipi ille circulus primus in infinito ipso extensus longe ultra secundum, secundus longe ultra tertium ita, ut illud ∞ in aliis ejusmodi circulis in alia distantia infinita sit, pro conditione, & natura rectarum, quae per infinitum traductae concipiantur [Boscovich 1754b, 378-379].

[56]"Quin etiam fieri posset, ut ad analogiam servandam infinitum infinito etiam infinities majus, sive in ratione, quam habet infinita quantitas ad finitam, finita ad nihilum, haberi debeat" [Boscovich 1754a, 380].

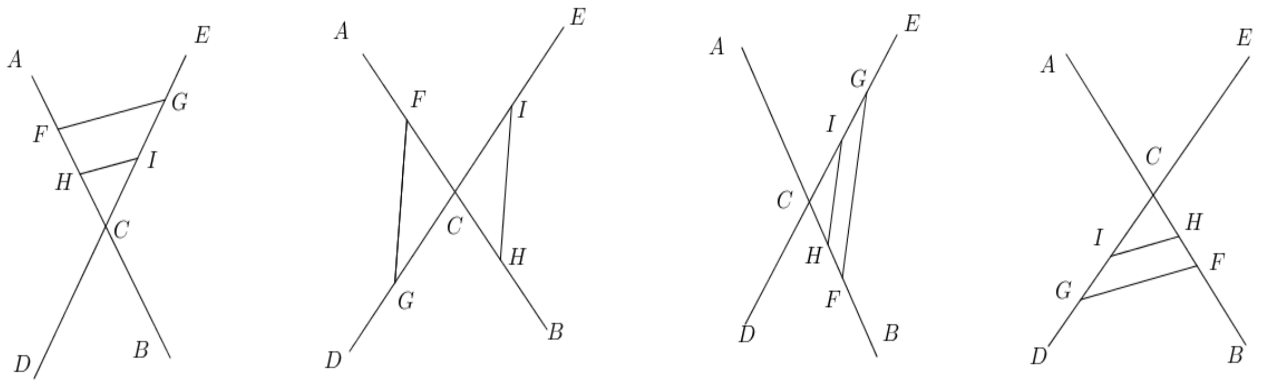
the second, becomes negative, componendo has to be substituted by dividendo, and vice-versa; otherwise if both the terms of the ratio become negatives, the reasoning must be maintained.[57]

An example of the application of this rule is given in No. 691. The figure concerns the problem Boscovich dealt with in No. 689, i.e. to find the segment *CG* fourth proportional after the given segments *CH*, *CF*, *CI* (fig. 16). We can observe that it is possible to pass from the diagram (a) to the next (b) by moving point *H*, which crosses point *C* where *CH* = 0 and then becomes negative. At the same time, point *G* has reached the new position on the opposite side with respect to *C*, by passing through infinity, when *FG* became parallel to *ED*. In the diagram (c), besides *H*, even *F* has changed its position, and in diagram (d) besides *H* and *F*, also *I* has changed its position.

In fig. 15a from the proportion $CF: CH = CG: CI$, by dividendo, it results that $FH: CH = GI: CI$. In fig. 15b where CH, CG have changed orientation, the same is found i.e. $H: CH = GI: CI$, by componendo. In fig. 15c, where the first two terms CF, CH have changed direction, and in fig. 15d, where all the terms have changed direction, the same is found by dividendo.

So Boscovich concluded:

The reason is clear, as when one of the terms changes position, the sum of the first and the second, or the sum of the third and the fourth, becomes a difference, or the difference becomes a sum, instead, when none of the terms changes or all the terms change, the sum and the difference remain.[58]



(a) (b) (c) (d)

Figure 15. Boscovich's figures 243-246 which illustrates how to find the segment fourth proportional after *CH*, *CF*, *CI*, as example of application of rule 2.

The following rule clarifies how to operate with proportions when a transformation is applied.

Rule 3: *If after a transformation, some terms of a proportion maintain the analogy of the second kind, the proportion is also maintained, but [only if], in whatever way proportions arise, or the changes never occur, or the number of the changes is even; in equal rectangles and in equal*

---

[57] "Si aliquae quantitates maneant analogae solo secundario analogiae genere, computanda erunt in enunciationibus, et demonstrationibus negativo modo ea, quae directionem mutarunt numero impare mutationum, ut nimirum si e binis altera tantum mutetur eo pacto, summa abeat in differentiam, qua pro positiva habeatur, vel pro negativa, prout ea, qua mutavit, erat minor, vel major, et viceversa: si mutetur utraque, summa, et differentia remaneant partier summae, et differentiae, sed e positivis in negativas abiisse censeantur, ubi ad ulteriora vel theoremata, vel problemata adhibenda sint. In demonstrationibus vero per proportiones institutis argumentationi per compositionem substitui debet argumentatio per divisionem, et viceversa, ubi e binis terminis rationis tam prima, quam secunda abierit in negativum alter tantummodo; retinendum argumentationis genus, si utraque mutet rationis utriuslibet" [Boscovich 1754a, 380].

[58] "Ratio est manifesta, quia summa primi, et secundi, vel tertii, et quarti mutatur in differentiam, vel differentia in summam, ubi alter ex iis positionem mutat, manet vero summa, vel differentia, si vel neuter mutet, vel uterque" [Boscovich 1754a, 312].

solids, the number of the mutations is either in all even [terms] or in all odd [terms], and a term outcoming from a proportion, or a term coming from anywhere, has to be considered negative or positive according to the number of the changes of those quantities from which it results, odd or even.[59]

The following rule gives some rules to operate with angles.

Rule 4: *The angle, a side of which has changed its direction, has to be replaced with its supplementary, i.e. the angle shaped by the unchanged side and the prolongation of the side which changed direction; the angle in which both the sides have changed their direction has to be replaced with its opposite angle and in order to maintain the statement, as regards the side which has changed its direction, the same letter must be utilised, in both cases, so that one is on the side of the analogous point in the anti-analogy, and the other is on the opposite side; in the statements and in the demonstrations it is to be avoided, as far as possible, that congruent angles become vertical angles, that an angle which is outside the parallel lines becomes inside and corresponding or alternate; the whole will depend on the number of changes, however in individual cases, it is easy to conceive the replacement to perform, if the two rules are followed. In general, when the vertex of the angle changes from inside to outside the strip formed by two parallel lines, the angles between the sides which were vertical angles, will become the same angle; the angles between the sides and the parallel lines, from interior angles become corresponding angles, and viceversa if the point [the vertex] changes from outside to inside the strip. Moreover, if the vertex of the angle changes again from outside the strip passing through it, once more it is outside the strip, but on the other side of the parallel lines, the angle does not change, while, as regards the angles formed with the parallel lines, the ones which are exterior become interior, and viceversa*.[60]

Boscovich's fourth rule provided some rules to operate with the symbols so that the statements were still valid after the transformation. According to Manara and Spoglianti [1979, 168] this attitude foreshadows some aspects of modern formal logic.

Rule 5: *When an angle changes passing from one part to another [i.e. changes of sign], this may happen only by transiting through zero or through two right-angles; if we consider the angle whose change takes place by transiting through zero, this has to be considered negative, and in the operations of addition it has to be computed negatively so that, a sum becomes a difference, [but] if only one [of the angles] is changed; if we consider the angle whose change takes place by transiting through two right-angles, the resulting angle has to be substituted by its complement to*

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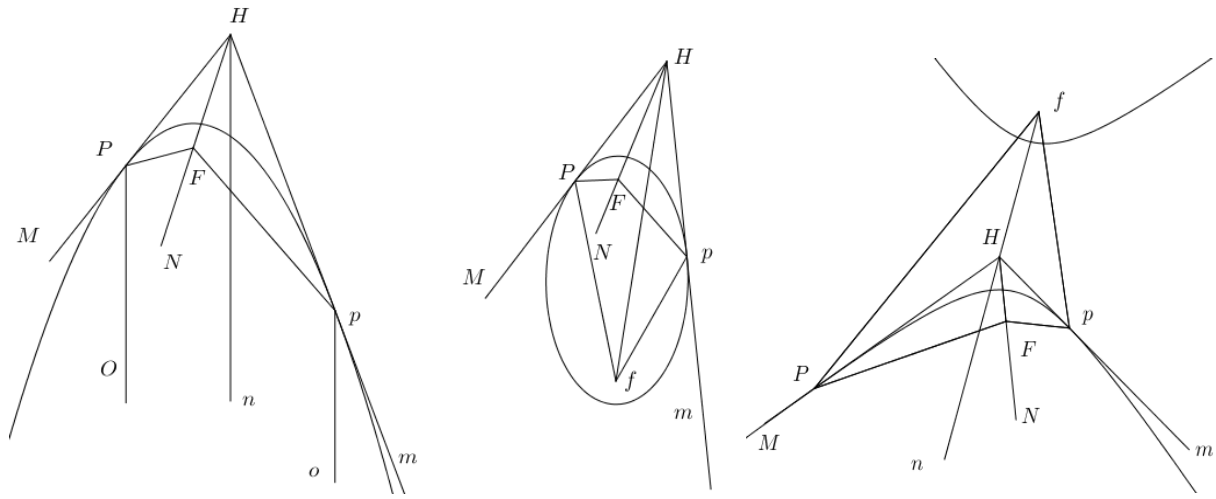
[59] "Si in aliqua proportione termini aliqui post transformationem maneant analogi secundario analogiae genere, manebit proportio: sed in proportionibus utcumque compositis nunquam mutatio habebitur, nisi numero pari, in rectangulis, vel solidis aequalibus debebit, vel in omnibus haberi mutationum numerus par vel impar in omnibus, et terminus, qui invenitur proportionibus quibuscumque, vel quovis ductu, censendus erit negativus, vel positivus, prout mutationum numerus fuerit in iis, a quibus pendet, impar, vel par" [Boscovich 1754a, 383-384].

[60] "Angulo, cuius alterum crus tantummodo directionem mutavit, succedit is, qui ejus est complementum ad duos rectos, sive quem continet crus non mutatum cum crure mutato producto: angulo, cuius utrumque mutavit directionem, succedit is, qui ipsi ad verticem opponitur, et ut enunciatio maneat, in crure quod directionem mutavit, communis aliqua littera opponenda est in binis casibus sita ad partes oppositus ita, ut altera jaceat ad partem puncti analogi secundario analogiae genere, altera ad partem oppositam; in demonstrationibus vero, ut et in enunciationibus cavendum semper fieri posse, ut anguli, qui congruebant, fiant ad verticem oppositi, qui erat externus in parallelis, evadat internus, et oppositus, vel alternus; atque ea a numero mutationum pendebunt, ita tamen, ut in singulis casibus admodum facile deprehendatur sustitutio facienda in demonstratione, notatis illis binis successionum regulis. Generaliter autem ubi vertex anguli, qui erat intra binas parallelas, abeat extra; angulus ipse enunciatus concursu crurum cum iis parallelis hinc, et inde ad verticem oppositus, fiet communis, anguli vero crurum cum parallelis mutabuntur ex alternis in externos, ac internos, et oppositos, et viceversa si punctum abeat inter parallelas. Quod si extra fuerit, et abeat extra, sed ad partes alterius parallela, manebit ipse angulus, et anguli ad parallelas, qui erant externi, fient interni, et viceversa" [Boscovich 1754a, 392].

four right-angles, which will be called, as is usual to say in Geometry, convex, or, as some say, skew, and the analogy will be better respected.[61]

Boscovich considered the case of an angle which was the sum of two others, and which, after the transformation, becomes the difference of the other two. Referring to his figure 240, here fig. 8b, he observed that this may happen when point E becomes point E_2 by passing through zero, and the angle $\widehat{ACB}$, which was the sum of the angles $\widehat{AEB}$ and $\widehat{DBE}$, becomes $\widehat{AC_2B}$, which is the difference between $\widehat{AE_2B}$ and $\widehat{DBE_2}$.

The same rule applies in the conic sections, as Boscovich showed in No. 803. In No. 184 he had considered a conic section and a point H not on it, and the two tangents to the conic issued from H which touch the conic at P and p. Referring to his figures 57, 58, 59, here fig. 16a,b,c he had established the following: in the ellipse and in the hyperbola, if F and f are their foci, one has respectively, $2\widehat{PHp} = \widehat{PFp} - \widehat{Pfp}$ and $2\widehat{PHp} = \widehat{PFp} + \widehat{Pfp}$.[62] In the parabola one has $2\widehat{PHp} = \widehat{PFp}$,[63] since, Boscovich stated in No. 803, the second focus f has gone to infinity, and the angle $\widehat{Pfp}$ has become zero. In the transformation which allows passage from the ellipse to the hyperbola, transiting through the parabola, the focus reappears on the opposite side of the axis, and the angle $\widehat{Pfp}$ by passing through zero becomes negative, and the difference (the case of the ellipse) becomes a sum (the case of the hyperbola).



(a) (b) (c)

Figure 16. Boscovich's figures 57, 58, 59. The sum of angles becomes the difference by passing from the ellipse to the hyperbola, through the parabola.

Rule 6: *The square of a segment, positive or negative, is positive and every positive square has two sides [roots], a positive one and a negative one. Moreover, if a certain square is equal to a*

[61] "Ubi anguli hiatus ab altera plaga ad alteram transit, quod fieri potest vel transeundo per nihilum, vel transeundo per binos rectos, si accipiatur is, qui eiusmodi mutatione oritur transeundo per nihilum, habendus est pro negativo, et in summis negativo modo computandus ita, ut summa in differentias abeant, altero tantum e binis mutato; at si eo abeat transeundo per binos rectos, angulo orto juxta communem Geometriae nomenclaturam debet substitui eis complementum ad 4 rectos, qui si appellatur angulus convexus, vel ut aliqui solent gibbus, saepe analogia multo melius servabitur" [Boscovich 1754a, 400].

[62] The angles $\widehat{PFp}, \widehat{Pfp}$ are, in the ellipse, those whose openings face each other, in the hyperbola, those whose openings are oriented in the same direction.

[63] In No. 184 Boscovich wrote that, in the parabola, the angle $\widehat{PFp}$ is that whose apex faces point H ("cuspis anguli spectet concursum tamgentium").

rectangle in which one of its sides changes direction, the square has to be considered real, but negative, and analogous, by the analogy of second kind, to the initial square, but its side will be imaginary and impossible, lacking the analogous of the initial side. If both the sides of the rectangle change their direction, both the sides of the square will be real, the positive and the negative one, and each of them will be analogous to each side of the initial square, according to the analogy of the first kind.[64]

Here follows his corollary: *The mean proportional between two segments, both positive or both negative, is double, one positive and one negative, of the same length but of opposite direction. The mean proportional between two segments, one negative and one positive, is imaginary. However, it is possible to have both the means of the same length but opposite direction, one positive and one negative*.[65]

Boscovich considered the rule to be very useful in the elements of the conic sections and here his overall vision appears clearly. In fact, after observing that the corollary had been made manifest in the problem to determine the mean proportional segment between two given ones (see section 4.2), i.e. a square equal to a given rectangle (No. 685), he pointed out an elegant relation between the hyperbola and the ellipse, by means of which some problems can be reduced to only one, in a single statement (No. 808). He had already observed (No. 761) the *primary* analogy between the transverse axis of the ellipse, which is finite, and the axis of the hyperbola $M\infty m$ which is extended to infinity and passes through infinity. Now he recalled the proportion which holds for ellipse and hyperbola (fig. 14a, c):

$$Mm : VFu = MR \times Rm : RP^2$$

where *Mm* is the principal axis, *VFu* the chord which passes through the focus *F*, *R* a point on the axis. He observes that in the ellipse, until *R* is between *M* and *m*, the corresponding point on the ellipse *P* can be found and the segment *RP* is real. If *R* is outside the segment *Mm* (on the side of *M* or on the side of *m*) *MR* or *Rm* becomes negative and also their rectangle and the square RP^2. So *RP* is immaginary. In the case of the hyperbola, if *R* is outside the segment *Mm*, or on the side of *H* or on the side of *h*, in the proportion *MR* and *Rm* have an opposite direction, so their rectangle is negative. However, in this case also *Mm* is negative, while *VFu* is positive, so the fourth proportional, RP^2, is positive and the segment *RP* is real. Vice versa, if *R* is between *M* and *m*, *MR* and *Rm* will have the same direction, so the rectangle *MRm* is positive, while *Mm* has changed. In the proportion the fourth proportional is real and negative. So on the hyperbola there is no point, and this is the imaginary case.

In No. 811, Boscovich noticed that while in the ellipse any diameter intersects the ellipse, it is not so in the hyperbola; i.e. only the diameters included in the angles between the asymptotes (i.e. those containing the two branches of the hyperbola) intersect the hyperbola, while the others have an imaginary intersection. In particular, in the ellipse any diameter and its conjugate properly intersect the ellipse, while in the hyperbola the conjugate diameter of a given properly intersecting diameter does not intersect the hyperbola. So, Boscovich affirmed that to maintain the analogy (that

[64] "Quadratum lineae tam positivae, quam negativae est positivum, et quodvis quadratum positivum bina habet latera alterum positivum alterum negativum. Si autem quoddam quadratum aequale fuerit rectangulo, cuius latus alterum directionem mutet; ipsum quidem quadratum censendum erit reale, sed negativum, et quadrato primi analogum secundo genere analogiae; at ejus latus fiet imaginarium, et impossibile, deficiente ibi termino analogo lateri quadrati prioris; si directionem mutet utrumque rectanguli latus, erit reale utrumque latus quadrati positivum, et negativum, et singula ex his erunt analoga primo analogiae genere singulis lateribus prioris quadrati" [Boscovich 1754b, 403].

[65] "Inter binas rectas tam simul positivas, quam simul negativas media proportionalis est duplex, altera positiva, altera negativa, quae longitudine sunt aequales, directione contraria. Inter binas alteram positivam, negativam alteram media proportionalis realis non habetur, sed in impossibilem et imaginariam utraque transit: haberi autem possunt binae mediae longitudine aequales, sed positione contraria altera positiva, altera negativa" [Boscovich 1754b, 404-405].

is the principle of continuity) one also has to consider the conjugate hyperbola of the given one,[66] so that any diameter properly intersects the given hyperbola or its conjugate (see fig. 17).

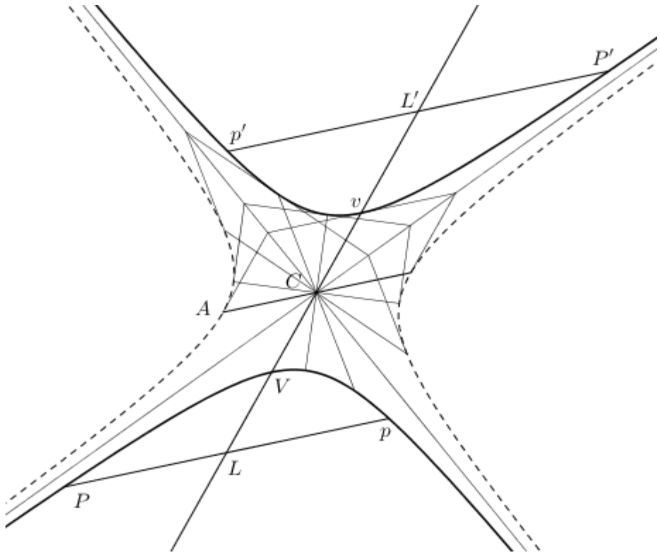


Figure 17. The construction of the conjugate diameters of the given hyperbola. It draws inspiration from Boscovich's figures 52, 83, 84.

Boscovich seems to say that in order to maintain the functional relation $LP^2 : VL \times Lv = CA^2 : CV^2$, which holds between the abscissae and the ordinates relative to a diameter and its conjugate (see fig. 18), the conjugate diameter has to be considered as a diameter of the conjugate hyperbola. Therefore, Boscovich gave another example of real representation of the "imaginary", similar to Poncelet's "ideal chord" (see section 6), in this case we have the pencil of straight lines through the centre of the hyperbola, instead of the "pencil" of the parallel straight lines, that are the chords considered in subsection 4.2 (fig. 11).

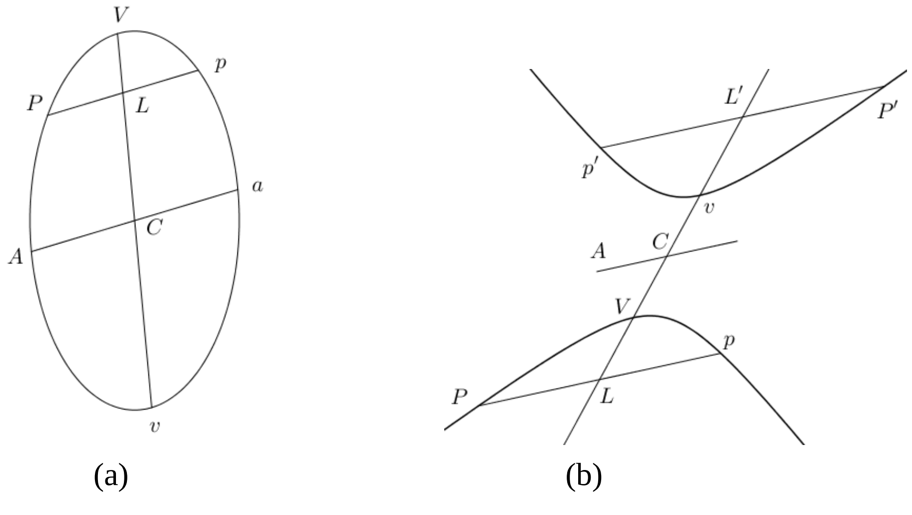


Figure 18. Boscovich's figures 134 and 135 which illustrates the functional relation $LP^2 : VL \times Lv = CA^2 : CV^2$, which holds between the abscissae and the ordinates relative to a diameter and its conjugate: (a) the case of the ellipse; (b) the case of the hyperbola.

[66] Boscovich gave the definition of conjugate hyperbolas in No. 170: two hyperbolas are called conjugated if the transverse axis of one of them is the conjugate axis of the other, and viceversa. Boscovich observed that two conjugate hyperbolas have the same asymptotes and equal focal distance from the common centre.

In particular, we have the real representation of the conjugate axis of the hyperbola, whose square is negative (fig. 5b). Boscovich's reasoning is the following:

The square of the conjugate semi-axis, CX^2 is equal to the rectangle *MFm* whose sides are the distances of one focus from the two vertices (i.e. $MF \times Fm$). The transverse semi-axis of the hyperbola *Mm* is analogous "secundario analogiae genere" to the transverse semi-axis of the ellipse, and also the rectangle *MFm* in the hyperbola is analogous "secundario analogiae genere" to the rectangle *MFm* in the ellipse, so the square of the conjugate semi-axis in the hyperbola is negative and its side is imaginary (No. 820). The identity which holds for the ellipse $CF^2 = CM^2 - CX^2$, where CM and CX are the semi-axes, also holds for the hyperbola, by taking into accont the sign of CX^2.

Rule 7: *In a proportion, if both terms of one ratio remain finite, and one term of the other ratio becomes zero, or infinite, then also the other term will become zero, or infinite, in the same way. Moreover, if the two extreme [medium] terms of the proportion remain constant and one of the two extreme terms becomes zero, then the other extreme term becomes infinite, and vice-versa. The same happens for the medium terms, if the extreme terms remain constant: thus, equivalently, if any rectangle remains equal to itself [in area] and one of its sides goes to zero or to infinity, the other side, on the contrary, goes to infinity or to zero.*[67]

Rule 8: *If [in a transformation] two straight lines, which meet in a point, become parallel, the point [of intersection] goes to infinity, so that it is nowhere, while the angle between them vanishes and its supplementary angle has to be considered equal to two right-angles. Vice-versa, if [in a transformation] the intersection point goes to infinity, so that it is nowhere, or one of the two angles goes to zero and the other becomes two right-angles, then the two straight lines become parallel.*[68]

Rule 9: *If two straight lines, which pass through the same point, move so that they become superimposed, and the angle between them vanishes, [then] other two straight lines [which remain] respectively parallel to them, become parallel to each other or superimposed. If in two similar triangles the corresponding vertices [in a transformation] go to lie on the corresponding opposite bases, then the distances of the points which replace the intersection of the sides from both ends of their bases, are in the same ratio, reciprocally, and with their bases.*[69]

The second part of rule 9 can be rendered as follows: after a transformation which mantains the similarity of the triangles, the vertices C and C_1 of two similar triangles ABC and $A_1B_1C_1$ go to lie on the corresponding opposite bases, AB and A_1B_1, respectively in points H and H_1 (see fig. 19)[70].

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[67] "Si in quatuor proportionis cujuspiam terminis binis utriuslibet rationis maneant finiti, reliquorum autem alter, abeat in nihilum, vel ita in infinitum, ut alterum saltem ejus extremum nusquam jam sit; alter abibit pariter in nihilum vel in infinitum eodem pacto. Quod si binis extremis [sic] manentibus alter ex extremis abeat in nihilum, vel in infinitum, alter contra abibit in infinitum, vel in nihilum, et idem in mediis continget, si bini extremi maneant: ac si, quod eodem redit, quoddam rectangulum finito rectangulo aequale maneat, ac alterum ejus latus abeat in nihilum, vel in infinitum, alterum contra abibit in infinitum, vel in nihilum" [Boscovich 1754a, 429-430].

[68] "Si binae rectae, quae ad quoddam punctum convergebant, parallelae fiant, illud punctum ita in infinitum recedit, ut nusquam jam sit, angulus vero, quem ad partes in ipso finito remanentes continebant, evanescit; ac is, quem altera continebat cum altera producta, censeri debet, ut in duas rectas definens. Si vero e contrario concursus ita in infinitum recedat, ut nusquam jam sit, vel angulus ex altera parte evanescat, ex altera abeat in duos rectos, illae ipsae rectae evadunt parallelae" [Boscovich 1754a, 432-433].

[69] "Si binae rectae ex eodem puncto digressae superponantur, earum angulo evanescent; binae aliae; quae iis parallelae erant singulae singulis, evadent inter se parallelae, vel pariter superponentur. Quod si in binis triangulis similibus vertex utriusque abeat in basim, lateribus basi supperpositis; binae distantiae puncti in quod abit vertex, quod punctum succedit intersectioni laterum, a binis extremis ipsius basis tam ad se invicem, quam ad ipsam basim, erunt utrobique in eadem ratione" [Boscovich 1754a, 440].

[70] When under the transformation point C moves and goes to lie on the opposite base, it does not necessarily follow a rectilinear "trajectory", it may even follow a curvilinear path provided the similarity of the triangles is maintained (see the next figure).

Then, the distances of H and H_1 from the extremes of their bases are in the same ratio, reciprocally, and with their bases, i.e. $AH: A_1H_1 = HB: H_1B_1 = AB: A_1B_1$. So, the relation of similarity which holds for any couple of triangles, however small the distances of the vertices from the extremities of their bases are, also holds at the limit, when the vertices go to lie on the opposite base AB, or its prolongation. The reason is to maintain the continuity, because any other ratio would occur "per saltum".

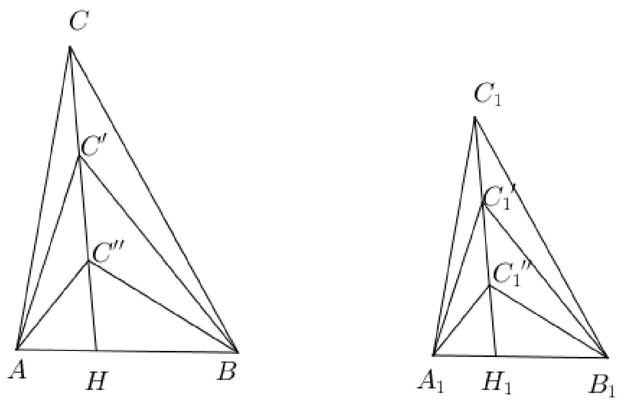

Figure 19. It illustrates the second part of rule 9, where the sides AC' and BC' are respectively parallel to the sides A_1C_1' and $B_1\,C'_1$, and so on, and point C moves along a given line.

As an application of this rule, Boscovich referred to the construction of the points intersection of a conic section, defined by giving a focus, directrix and eccentricity, with a straight line passing through the focus (No. 856). We have seen in section 3 how Boscovich constructed these points when the straight line does not pass through the focus, by means of the eccentric circle. Here he applied rule 9 in order to find the intersections in this limiting case. He argued as follows: when the straight line HK (see fig. 7) rotates around H, until it coincides with the line HFV, the vertices t, T, p, P of the triangles LTt, FPp go to lie on the corresponding opposite bases. The previous proportions are maintained during the transformation and this is true also in the limiting configuration, so that the sought points P, p on the straight line HV are defined by $FP: PH = LT: LO$ and $Fp: pH = Lt: LO$, respectively (No. 856).

Rule 10: *If the radius of the circle becomes infinite, so that, one of its points remaining fixed, the centre is nowhere [is gone to infinity], the circle is transformed into a straight line, and vice-versa the straight line will be considered as an infinite circle*.[71]

Boscovich missed the straight line at infinity. This straight line was defined by Poncelet by passing into a three dimensional space, to be the intersection of all planes parallel to the given one. The circle of infinite radius, as thought by Boscovich, in Poncelet splits into a straight line and the line at infinity.

The following rule concerns the behaviour at the limit of the ratio of two magnitudes which grow indefinitely, in the cases in which their difference remains finite as well as growing indefinitely, and the ratio (at the limit) can be finite, infinite or zero. With these considerations Boscovich introduced the concepts of "rate of growth", or "order of infinity", of a quantity which grows beyond any limit.

Boscovich had been interested in the behaviour of the ratio of two quantities which may change their magnitudes, becoming even infinitely small or infinitely great, already many years before [Boscovich 1741]. The aim of this short treatise was to ground Newton's method of fluxions on justifiable rules providing a coherent definition of infinitesimal [Homann 1993, 410]. In it Boscovich argued with the same geometrical examples that he discussed in the *De transformatione*.

[71] "Si circuli radius in infinitum abeat ita, ut altero extremo manente, centrum nusquam jam sit peripheria circuli abibit in rectam lineam, et recta linea viceversa habenda erit pro peripheria circuli infiniti" [Boscovich 1754a, 442].

Rule 11: *If one end of two segments goes to infinity, so that the segments become infinite, their ratio at infinity, has to be estimated as the same to which they tend, while they grow infinitely, beyond any limits: and if other two segments, which remain finite, have the same ratio as the previous two which are growing beyond any limit, their ratio is the same as that of the two segments at infinity. The ratio to which two quantities tend when they grow indefinitely, beyond any limit, which one considers as their ratio in the moment they become infinite, can be a ratio of equality or of finite inequality [i.e. 1 or a finite number], or increasing or decreasing beyond any limit; it will always be a ratio of equality, when their difference persists finite or zero; and the difference will remain finite or zero when two segments, having a common end, become infinite when the common end goes to infinity, while the other two ends of the segments remain fixed.* [72]

Boscovich claimed that the first part was a consequence of the principle of continuity, then he illustrated the rule by means of some examples. Here below are those offered in No. 864.

He considered two parallel straight lines HI and FG, through the fixed points H and F, which met the straight line DE respectively at I and G (see fig. 15a). Suppose the straight lines HI and FG rotate counterclockwise respectively around H and F but remain parallel to each other, then points I and G move along DE, the segments CI and CG grow indefinitely, until they become infinite when FG and HI are parallel to DE. Since, whatever the position of points I and G is, the triangles CHI and CFG are similar, the ratio CI/CG is always equal to CH/CF which is a constant. Therefore the value of the ratio CI/CG, when CI and CG become infinite, i.e. at the limit, is finite and equal to CH/CF.

In the next example Boscovich considered the hyperbola of equation $xy = 1$ and the cuspidal cubic $yx^2 = 1$ (fig. 20). These curves meet at point $B = (1,1)$, and both have as their asymptotes the x-axis and the y-axis. Let V denote the origin, A the projection of B on the x-axis, and let R be a point on the positive side of the x-axis. Moreover, let I and E be the intersection of the vertical straight line through R respectively with the hyperbola and the cuspidal cubic. When R approaches V, the segments RE and RI grow indefinitely. If H denotes the intersections between the straight line VI with the vertical straight line through A, by the similarity of the triangles VRI and VAH, we have $RE{:}RI = AH{:}RI = AV{:}RV$. Then, since AV is constant while RV goes to zero, the ratio $RE{:}RI$ goes to infinity.

[72] "Si binae rectae altero saltem utriusque limite ita in infinitum abeunte, ut nusquam jam sit, infinitae evadant; debent in ipso infinito censeri, ut assecutae illam rationem, ad quam ultra quoscumque limites accesserunt, dum in infinitum excrescerent: et si binae aliae rectae fuerint semper in earum ratione, ac illis excrescentibus in infinitum, remaneant finitae; habebunt accurate eam rationem ipsam, ad quam illae ultra quoscumque limites accesserant. Ratio autem, ad quam accedent binae quantitates, dum in infinitum excrescent, et quam assecutae censeri debent, ubi jam infinitae sint, potest esse ratio aequalitatis, vel inaequalitatis finite, vel excrescens, aut decrescens ultra quoscumque limites; erit tamen semper ratio aequalitatis, ubi differentia ipsarum finita maneat, vel nulla, et differentia semper manebit finite, vel nulla; si binae rectae terminatae ad idem punctum ab aliis binis punctis abierint in infinitum, manentibus his binis punctis, et abeunte in infinitum illo communi ita, ut nusquam jam sit" [Boscovich 1754b, 443-444].

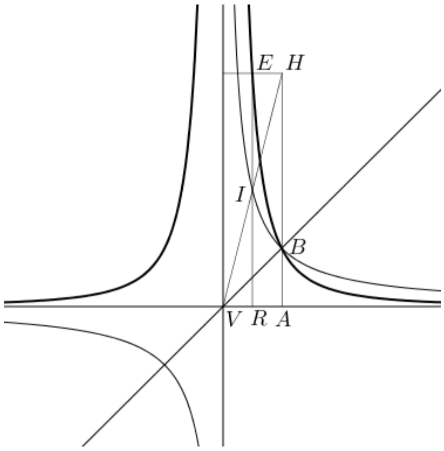

Figure 20. Boscovich's fig. 260. The ratio $RE\colon RI$ goes to infinity when RV goes to zero.

In No. 871, Boscovich explained how Rule 11 may be useful to understand what happens when one passes from ellipsis (or hyperbolas) to the parabola. He wrote:

In the application of the theorems concerning the ellipse and the hyperbola to the parabola [this rule] is very useful. As the segments Fm and mE in fig. 9 [here fig. 14a] must assume a ratio of equality [i.e. their ratio becomes 1] when [the ellipse] becomes a parabola, the vertex m of the transverse axis in the parabola in fig. 10 [here fig. 14b] is nowhere [goes to infinity], and both segments become infinite. [73]

Then he continued by saying:

The theorems regarding the ellipse and the hyperbola of number 74 can be transferred to the parabola: that is, the squares of the semiordinates RP in fig 9 and 11 [here fig. 14a,c] of the transverse axis are as the rectangles MRp [really MRm] of the abscissae from both vertices.[74]

We underline that this means the ratio $RP^2\colon MR \times Rm$ is constant, that is it does not depend on the position of point R on the transverse axis: $RP^2\colon MR \times Rm = R'P'^2\colon MR' \times R'm$.

Afterwards, he added:

While the conic section becomes a parabola, figure 10 [here fig. 14b], m goes to infinity so that it is nowhere. Hence if one assumes two semi-ordinates, that is two *Rm*'s, which have become infinite, their ratio has become of equality, since their difference is equal to the distance between the two points R's [they are infinities of the same order]. Therefore those rectangles are as the [corresponding] abscissa MR, from the vertex M which remains to the parabola. This does, in fact, happen and we have expounded it in the above cited section.[75]

Clearly this is to say $RP^2\colon MR = R'P'^2\colon MR'$, that is the ratio $RP^2\colon MR$ does not depend on the position of point R on the transverse axis.

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[73] "In applicatione theorematum Ellipseos, vel Hyperbolae ad Parabolam summus ejus usus haberi potest. Quoniam rectae Fm, mE in fig. 9 [here fig. 14a] debent acquirere rationem aequalitatis, ubi ea in Parabolam migrat, vertex m axis transversi in ipsa Parabola in fig. 10 [here fig. 14b] nusquam jam est, et ipse evadunt absolutè infinitae" [Boscovich 1754a, 450].

[74] "Hinc vero ad Parabolam transfertur theorema pertinens ad Ellipsim, et Hyperbolam propositum num. 74, quod nimirum quadrata semirdinatarum RP in fig. 9 et 11 [here fig. 14 a,c] ad axem transversum sint, ut rectangula MRp [sic] sub abscissis a binis verticibus" [Boscovich 1754a, 450-451].

[75] "Dum eae mutantur in Parabolam figurae 10 [here fig. 14b], abit m in infinitum ita, ut usquam jam sit. Quare si binae assumantur semiordinatae, binarum mR, quae ad eas pertinent, et evadunt absolute infinitae, ratio evadit ratio aequalitatis, cum differentia ipsarum sit illa finite distantia binorum punctorum R. Quare illa rectangula erunt, ut solae abscissae MR a vertice M, qui in Parabola manet. Id autem ita se habere constat; id enim ipsum eodem numero pariter proposuimus" [Boscovich 1754a, 451].

Hence Boscovich remarked that the property of the ellipse and hyperbola described in No. 74,[76] can be transferred to the parabola. The same happens regarding the other properties described in No. 357:

In the same way one can transfer from the ellipse and hyperbola to the parabola the property of all the diameters of number 357, when the centre of the parabola has gone to infinity; at the same time with it [the centre], even the other vertex of any diameter has gone to infinity so that it is nowhere.[77]

Therefore, Boscovich extended the properties of the diameters from ellipses and hyperbolas to the parabola, by means of the principle of continuity, highlighting what is common to all conics.

6. Boscovich's innovative ideas

After the publication of the *Géométrie* by Descartes in 1637, with the developments of analysis and algebra and the full acquisition of complex numbers, there slowly emerged, in the late seventeenth century, the desire to give pure geometry the same generality as these disciplines. The aim was to unify apparently different theorems, often depending on the displacement of parts of the figure connected with the problem under study, in order to discover the "common", or "intrinsic", properties of geometrical loci. Let us recall, for example, Leibniz's "universal conic", or Newton's projective classification of plane curves of third order, mentioned in section 2. Descartes himself had already noticed that several algebraic formulas, whose terms only differ in some + or − sign, often express an analogous property of various figures, which have the same construction, but which differ in the mutual position of some of their parts. Moreover, according to Bompiani [1925], several authors, including Newton and d'Alembert, had some difficulties in interpreting properly the roots of certain equations which translate geometrical problems into algebra.

In this frame, at the beginning of the ninetheenth century, Carnot and Poncelet proposed methods of proof which could be applied not only to one specific figure, but to a range of figures, independently from the position, or reality, of certain their elements. According to [Nabonnand 2011], their general approach developed along three lines: to adopt a principle of transformation of figures; to find methods of proof to be applicable to certain classes of figures, to find the properties which are independent from the contingent situation. As we have noticed in sections 4 and 5, Boscovich had already tried to do that, and to fully understand his innovative thinking we recall the basic facts on which the two French geometers founded their methods.

6.1 *Carnot's principle of correlation.*

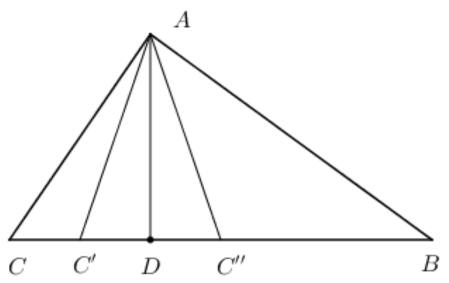
In [Carnot 1801], Carnot proposed his "principe de corrélation" (principle of correlation), which he later perfected in [Carnot 1803].

In one figure, Carnot made a distinction between the relations of "position" of its elements, which express the mutual position of points, straight lines, circles and so on, and the relations of "magnitude", such as length of segments, measure of angles, ratios, etc. He recognised that changes in position of certain elements in the initial figure, may induce changes in orientation, and then in sign, of other elements of the transformed figure. So, Carnot conceived a set of rules by which to associate to each part of the initial figure, which he called *primitive*, the corresponding part in the transformed figure, which he called *correlative*, so that it may be possible to detect how the changes in position reflect on the formulas expressing a property of the primitive figure. He distinguished

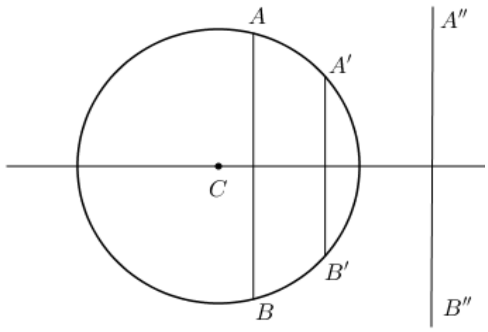
[76] "Quadrata semiordinatarum axis transversi sunt in Parabola, ut abscissae a vertice, in Ellipsi, et Hyperbola, ut rectangula sub binis abscissis a verticibus" [Boscovich 1754a, 24].

[77] "atque eo modo ex Ellipsi, et Hyperbola ad Parabolam transfertur eadem proprietas generalius pro diametris omnibus proposita numer. 357 cum abeunte in infinitum centro in Parabola; simul cum ipso cujusvis diametri alter vertex in infinitum abeat, nec usquam jam sit" [Boscovich 1754a, 451].

three kinds of correlation, *direct*, *indirect*, and *complex*, according to whether, in the correlation no changes in sign occur, some changes in sign occur, but not under square or higher roots, or changes in sign occur also under square or higher roots (see fig. 21a,b).



(a)



(b)

Figure 21. (a) Let the triangle *ABC* be given, and suppose *C* moves continuously toward *B* along *CB*, reaching the positions *C'*, *D* and *C''*, then the traingles *ABC'* and *ABD* are in direct correlation with the first, while the triangle *ABC''* is in inverse correlation, because the segment *CD*, has become *C''D*, which has different orientation; (b) Let the straight line *AB* intersects the circle of centre *C*, and suppose *AB* moves continuously parallel to itself, reaching the positions *A'B'* and *A''B''*, then the second configuration is directly correlative to the first, but no longer the third, which is instead in complex correlation with the first.

In the first case, the formulae which hold for the primitive figure also hold for the correlative, while, in the other cases, suitable changes are needed (change in signs, or the introduction of complex factors) in order to render the formulae applicable to the correlative figure [Carnot 1803, 73]. Carnot called *explicit* formulas those which are immediately applicable to the studied configuration, and *implicit* those which could not be applied without any change of signes. For instance, referring to fig.21a, *BC* = *BD* + *DC* becomes *BC'* = *BD* + *DC'*, which is the same as before, but this does no hold when *C* as reached the position *C''*. Then, *BC* = *BD* + *DC* and *BD* = *BC* – *CD* are explicit formulas, for the first and second triangles, but for the third they are implicit, while in this case *BD* = *BC''* + *C''D* is explicit.

Next he built a system in which it was possible to operate according these rules. This consisted of the formation of a *table of comparison*, “tableaux de corrélation”, representing the properties of the two figures, the primitive and the correlative, so that, once the table was formed, the formulas applicable to the correlative figure were automatically deduced from those established for the primitive figure, see [Nabonnand 2011, 24-25].

There are many similarities between Boscovich’s “analogy of figures”, intended as correspondence among parts of two figures as recalled in our subsection 5.1, and the definition of “correlation” given by Carnot. Moreover, Boscovich spoke of “primary” and “secondary” analogy, with the same meaning as “direct” and “indirect” correlation, and the rules 1-5 may be seen as corresponding to Carnot’s “correlation table”. Boscovich recognized that the secondary analogy may produce imaginary elements, and rule 6, which deals with “imaginaries”, anticipated Carnot’s “complex correlation”.

Carnot [1801, 177-188] treated the complex correlation only at the very end of his book, and briefly. To illustrate it, he considered the following example connected with the problem of finding the mean proportional between two magnitudes. Suppose *a*, *x* and *y* are three magnitudes, that is positive quantities, such that *a* > *x*, and *y* is the mean proportional between a + *x* and a – *x*, that is *a* + *x* : *y* = *y* : *a* – *x*. Then, *x*, *y* satisfy the equation $x^2 + y^2 = a^2$.

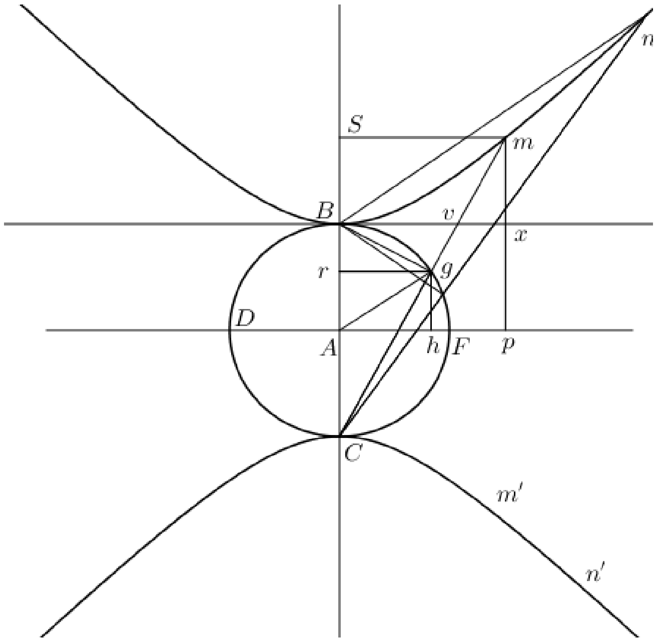


Figure 22. Figure 28 in [Carnot 1801] which illustrates the complex correlation. This is similar to Boscovich's figure 242 (see our fig.12).

Carnot observed that if, by varying the data for insensible degrees, x becomes greater than a, the above proportion is no longer valid, being $a – x < 0$, and it has to be substituted by $(a + x) : y = y : (x – a)$, which yelds to the curve $x^2 - y^2 = a^2$. The two configurations are neither in direct nor indirect correlation, but in complex correlation, since it is possible to pass from one to the other only multiplying y by $\sqrt{-1}$. In particular, the two curves, that is the circle and the equilateral hyperbola, are in complex correlation. Then, referring to his fig. 28 (our fig. 22), Carnot [1801, 184-186] argued as follows. For any point *g* on the circle, by a well known property one has *Br* : *gr* = *gr* : *Cr*, and *gr* (= *y*) is the half chord intercepted by the straight line *gr* on the circle. This proportion is maintained by *SB* : *Sm* = *Sm* : *SC*, where *m* is the point intercepted on the hyperbola $x^2 - y^2 = a^2$ by the straight line *Cg*. In an implicit and cumbersome way, Carnot seems to saying that when the straight line *gr*, moves horizontally and overpasses point *B*, to maintain the first proportion the half chord *gr* of the circle has to be substituted by the *Sm*, which results the half chord of the hyperbola correlative to the circle.

This is an argument which echoes that used by Boscovich in No. 685, and here described in subsection 4.2.

Carnot ended his book by hastily observing that it can be similarly established the correlation between the ellipse and the (nonequilateral) hyperbola, and that there are analogies between the two curves. These analogies were, instead, clearly highlighted by Boscovich in No. 811 (see section 5, comments to rule 6).

6.2 *Poncelet's principle of continuity and the ideal chord*.

In his *Sur la loi des signes de position en géométrie, la loi et le principle de continuité*, Poncelet [1815] refuted the work of Carnot. He realized that it was not by a set of complicated rules that the passage from one figure to another can be reached, and one can give geometry the same extension as algebra. According to him, to achieve this goal, it was necessary to give geometrical figures an indeterminate form, by leaving indeterminate certain relations of position, or ratios, among the magnitudes of some of their parts. Preserving the nomenclature established by Carnot, Poncelet expressed his idea as follows: when a figure varies by infinitesimal degrees, without altering the initial conditions, the properties of the primitive figure continue to be applicable to the successive states of the system, as long as one takes into account the changes which occurred. Contrary to Carnot, in Poncelet's thinking, two correlated figures are not seen as separate, but as two stages of a continuous system whose elements all maintain the property under consideration.

Poncelet found that the reason for the greater generality of algebra, and analysis, resided in the adoption of the principle of continuity in these disciplines. So, according to Poncelet, to give geometry the same extension as algebra this principle also had to be adopted in geometry in a proper form. In the *Considerations philosophiques et techniques sur le principe de continuité dans les lois géométriques*, starting from a primitive figure, in which —as required by Carnot— everything is real and well defined, Poncelet [1818] considered the displacement, by continuous movement, (that is by "infinitesimal degrees"), of the parts of the given figure, and in this regard he wrote:

> In any case, there is no difficulty in admitting the principle for every real and absolute stage of the figure; from this it follows that there is no contradiction in admitting it, at least ideally, for all those stages of the figure where certain objects have become impossible, … by pronouncing on the permanence of relations, one does not pronounce on the nature and the absolute existence of the objects or of the magnitudes concerning these. … Now, I would even say that they have become neither absurd nor insignificant, and they still serve to characterize and determine the true nature of the system to which they apply.[78]

Next he pointed out:

> We see that they not only apply to the stages of a figure in which the correlation with the primitive is simply indirect, and consequently real, but even to those where certain parts of the figure have vanished or have become *imaginary*, or *infinite*, thus losing their geometrical individuality, that is, to all stages which have only preserved an ideal correlation with the primitive stage of the system. Considered from a certain point of view, these general consequences constitute what one can properly call the principle of continuity or permanence of the mathematical relations of the figure.[79]

So, for Poncelet, the principle of continuity is a sort of what was to become the "principle of permanence of functional relations", which he considered as an evident unquestionable truth.

Poncelet based his *Traité* on the *principle of continuity*, and on the *principle of projection*, the latter concerning the transformation of one figure into another under the operations of "projection and section", whose proof he based on the former. Therefore, the principle of continuity in itself is not a principle of "transformation" of figures, but rather the basis on which to build such a theory.[80]

Like Boscovich, Poncelet too justified his assumption with several examples, often similar to those given by Boscovich. For instance, Poncelet considered two straight lines intersecting in a point P, then he remarked that by rotating, with continuous motion, one of the two around one of its points $\neq P$, "the intersection point moves away from its original position beyond any given

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[78] "Quoi qu'il en soit, il n'y a évidemment aucune difficulté à admettre le principe pour tous les états réels et absolus d'une figure ; or, il résulte de là même qu'il ne saurait y avoir de contradiction à l'y admettre, au moins idéalement, pour tous ceux des états de cette figure où certains objets seraient devenus impossibles, … en prononçant sur la permanence des relations, il ne prononce nullement sur la nature et l'existence absolue des objets ou des grandeurs que ces relations concernent. … Or, je dis qu'alors même elles ne sont devenues ni absurdes ni insignifiantes, et qu'elles servent encore à caractériser et à déterminer la véritable nature du système auquel elles s'appliquent" [Poncelet 1864, 338].

[79] "On voit qu'elles s'appliquent, non-seulement aux états d'une figure dont la corrélation avec la primitive est simplement indirecte et par conséquent réelles, mais encore à tous ceux où certaines parties de la figure sont devenues nulles, *imaginaires*, *infinies*, en perdant ainsi leur existence géométrique individuelle, c'est-à-dire à tous les états qui n'auraient plus conservé qu'une corrélation *idéale* avec l'état primitif du système. Considérées sous un certain point de vue, ces conséquences générales constituent ce qu'on peut appeler proprement le principe de la continuité ou permanence des relations mathématiques de la grandeur figurée" [Poncelet 1864, 349].

[80] See [Del Centina 2016b, section 3], and [Nabonnand 2015] for details.

distance, until it disappears (at infinity) when the two straight lines have become parallel".[81] Then he remarked:

> since, in the case under examination… the real point of intersection [the point P] is always unique, in order to preserve the continuity, it is necessary that this be unique also when the two straight lines have become parallel, and from this, of necessity, the following paradoxical consequence is derived: the two extremities of an infinite straight line meet and merge at infinity.[82]

An argument we find also in Boscovich.

Poncelet correctly looked at a circle with its centre at infinity as a pair of straight lines, one at finite distance and one at infinity, that is, the straight line at infinity of the plane of the circle. So when Boscovich deformed the circle, maintaining one of its points fixed and pushing its centre at infinity, he missed considering the straight line at infinity. In particular, Boscovich did not have any perception of the "projective plane", that is, the plane completed by its "line at infinity".

There is another more important point in which Boscovich seems to anticipate Poncelet: the construction of the "ideal chord". To introduce this concept, referring to his figure 6 (our fig. 23), Poncelet argued as follows.

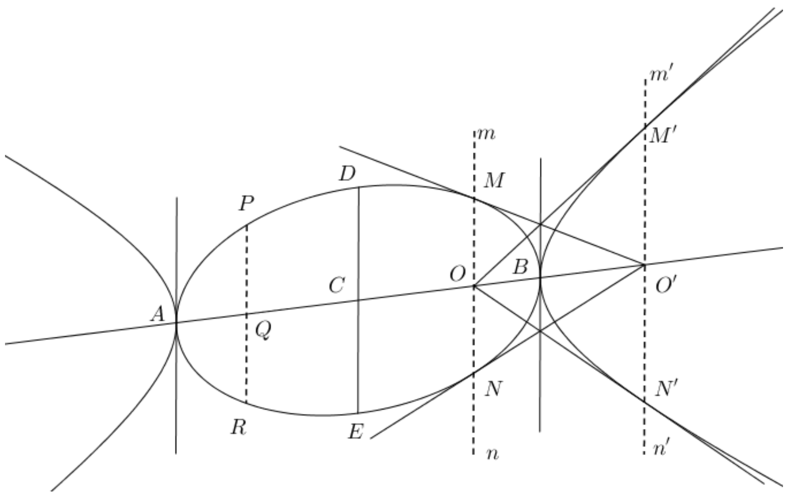
Let γ be a conic, and let mn be a straight line intersecting γ in points M and N. Let d be the conjugate diameter to mn, and let A, B be the points where d intersects γ, and say O the intersection between d and mn. Then

$$(OM)^2 = p(OA)(OB),$$

where p is a real number which does not change for any other straight line parallel to mn and intersecting γ. Let $m'n'$ be a straight line parallel to mn but not intersecting γ, to which Poncelet associated a real segment $M'N'$ on the line $m'n'$, such that M' satisfies the condition

$$(O'M')^2 = p(O'A)(O'B)$$

where $O' = d \cap m'n'$, and $O'M' = O'N'$. Then Poncelet called $M'N'$ the *ideal chord* intercepted by $m'n'$.[83] Moving $m'n'$ maintaining it parallel to mn, points M' and N' describe a new conic that Poncelet called *supplementary* (to the given conic with respect to the direction *mn*).



[81] "… le point de leur intersection commune, après s'être écarté de son origine à une distance plus grand que toute distance donnée, finira par cesser tout à fait d'être, circonstance qui aura lieu quand ces deux droites seront devenues parallèles" [Poncelet, 1864, 346].

[82] "… puisque, dans le cas examinée de deux droites situées dans une même plan, le point d'intersection réelle est toujours unique, il faut aussi admettre, pour la continuité, qu'il est encore unique quand ces deux droites deviennent parallèles, et de la dérive cette conséquence paradoxale, mois pour tout nécessaire : «Les deux extrémités d'une droite indéfinie se rejoignent et se confondent à l'infini»" [Poncelet 1864, 347].

[83] Let us observe that from points M' and N' one may recover the pair of conjugate complex points in which the line $m'n'$ intersects γ, and vice-versa. For details, see Del Centina [2016b, section 3] and Nabonnand [2015].

Figure 23. Figure 6 in [Poncelet 1822] which illustrates Poncelet's construction of the ideal chord $M'N'$, and of the supplementary conic: the straight line *mn* is the polar of O' with respect to the given conic, then points M' and N' are the intersection of $m'n'$ with the straight lines through O and the points where the tangent to the given conic at *B* meets the tangents to the given conic issuing from O'.

Poncelet remarked that the new conic, determined in this way, is a hyperbola, or an ellipse, according to whether the first is an ellipse, or a hyperbola, and is a parabola if the first is also a parabola. He called the new conic *supplementary* to the first, and vice-versa. If one of the two conics is a circle, the other will be an equilateral hyperbola.

As we have seen in section 4.2, Boscovich had the same idea, though only in the case of a circle, and used the "conservation of functional relations", to extend a property of conic sections to the case in which points become imaginary. Moreover, Boscovich recognized that the ends of the ideal chord, when it moves parallel to itself, describe the "supplementary" hyperbola (in the sense of Poncelet) of the given circle (see comment to rule 6). Then, for the hyperbola, Boscovich considered the "ideal" conjugate diameter to a "real" one, and claimed that its ends describe the conjugate hyperbola of the given one.

All this shows how forward-looking Boscovich's ideas were.

7. Final remarks and conclusions

As we have pointed out through his own words in subsection 3.1 and 3.2, Boscovich's *Sectionum Conicarum Elementa* also aimed at expounding the theory of conic sections from a unified point of view, and he must be credited with trying, for the first time in history, to place the principle of continuity on secure grounds by elaborating a series of rules with which can be soundly applied to the transformation of geometrical loci.

Boscovich looked at this principle more from the point of view of a philosopher, and scientist of nature, than from the point of view of a mathematician and geometer. This clearly emerges from the *De Continuitatis Lege* [Boscovich 1754b], in which he turned his attention mainly to the system of nature. Nevertheless, geometry was an instrument by which to confirm the validity of the principle of continuity, and, in order to avoid criticism, Boscovich frequently made use of geometrical examples and constructions. The transformation of conic sections from one to another, which was the object of profound reflexion and keen consideration, led him to develop the theory of transformations of geometrical loci, which was both "the fruit" of the proved transformation of conics, and a "means of verification and discovery", in fact, the *De Transformatione Locorum Geometricorum* and the *Sectionum Conicarum Elementa* can be considered complementary to each other, given the frequent references between the two parts, and many shared figures.

As may be inferred from the second section of this paper, it appears clear that for certain concepts and ideas Boscovich was indebted, either directly or indirectly, to Kepler; suffice it to think of the concepts of "point at infinity", and of "parallel straight lines as converging to the same point at infinity", to consider the straight line "as closed on itself at infinity", and the same for the parabola and the hyperbola, to see the parabola as "the separating element between a continuous system of ellipsis and a continuous system of hyperbolas".

Boscovich was certainly indebted to Newton for the definition of conic sections through directrix, focus and given ratio, which allowed him to give substance to Kepler's ideas, and to look at the transformation of one conic into another as a means to discover their "inner properties", as discussed here in section 3.1.

If Kepler used the principle of continuity without stating it, Boscovich adopted the definition given by Leibniz, and, on the themes of "infinity" and "continuous", his ideas were essentially those of Aristotle. In fact, according to him, only the "potential infinity" exists, because the "infinity" is something that takes place only in the process of formation. For Boscovich [1741, No. 12] the infinitely large and the infinitely small do not exist in themselves – that is as fixed quantities ("constants") –, they may exist only in their possibility of always becoming something else, that is, they can be increased or decreased beyond any limit [Guzzardi 2014, 24]. Nevertheless, Boscovich was able to recognize, as we have seen by analysing rules 10 and 11, the existence of different "order of infinity" and "infinitesimal", and to give examples through the rate by which a quantity increases or decreases beyond any limit. But, according to him, and as we have remarked in sections 4 and 5, there still remained "mysteries which exceed the human comprehension". He would have liked to add a fourth volume to his work, devoted mostly to analysis, in which, together with a twelfth rule, he would also have treated the infinitesimals geometrically and algebraically. Unfortunately, this volume never appeared.

As we have observed, Boscovich did not transform one conic into another "by projection and section" passing through three dimensional space, but he "deformed" one conic into another inside a continuous plane system. Likely for this reason, although the principle of continuity was available to him, and he had somehow anticipated the idea of "conservation of functional relation", he was unable to give to the theory of conics the direction later impressed by Poncelet.

The *Sectionum Conicarum Elementa*, together with the *De Transformatione Locorum Geometricorum*, are Boscovich's most important mathematical works. Referring to the latter Lalande wrote:

> Where the author treats the law of continuity, and several mysteries of the infinity, with a Geometry always enlightened by the flame of the most subtle Metaphysics, whose help skilled geometers often lacked, and without which one often strays in the labyrinth of the infinity.[84]

Despite Boschovich's extensive presentation on the usefulness of the principle of continuity in geometry, no further reflection on this principle, or application in geometry, was ever carried out through the eighteenth century. This may be explained by the fact that Boscovich attached his essay to an (apparently) elementary treatise on conic sections, which in turn was soon forgotten. It was so that, when in the very early nineteenth century the renewal of synthetic geometry occurred with the School of Monge, Boscovich's work had no role on the future development of geometry, and, although he anticipated important concepts, that would come to complete maturity only in Poncelet's hands, his geometrical work remained confined to his century, and we should not emphasize Boscovich's position as a forerunner of modern theories, as some have tried to do.

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[84] "où l'auteur traite de la Loi de continuité, & de plusieurs mystères de l'infini, avec une Géométrie toujours éclairée du flambeau de la Métaphysique la plus subtile, dont le secours n'a que trop souvent manqué à des Géomètres habiles & sans laquelle on s'égare souvent dans le dédale de l'infini", *Journal des Sçavans*, Septembre 1766, p. 604. Lalande stated that Boscovich was "one of the greatest mathematicians of his century".

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