Rend. Sem. Mat. Univ. Pol. Torino - Vol. 63, 1 (2005)
Polynom. Interp. Proceed.

A. Calabri*

ON THE REPRESENTATION OF ENRIQUES SURFACES AS DOUBLE PLANES

Abstract. In this paper we give a short proof of the well-known representation of Enriques surfaces as double planes, by using the properties of the adjoint linear system to the branch curve.

Enriques surfaces play a fundamental role in the classification of complex algebraic surfaces: historically they have been the first examples of irrational surfaces with geometric genus $p_g = 0$ and irregularity $q = 0$. Indeed, in 1894, Enriques suggested in a letter to Castelnuovo that these properties were fulfilled by (the normalization of) a sextic surface in $\mathbb{P}^3(\mathbb{C})$ having the six edges of a tetrahedron as double lines. Soon later, in 1896, Castelnuovo proved his celebrated rationality criterion, which states that an algebraic surface is rational if and only if it is regular and has bi-genus $P_2 = 0$.

In 1906, Enriques proved in [10] that every surface with $P_2 = 1$ and $P_3 = q = 0$ is isomorphic to his original example and he gave a rather complete treatment of these surfaces, which have justly been named after him. In particular Enriques showed that they can be represented as *double planes*, i.e. as double covers of $\mathbb{P}^2$, branched along a reduced curve of degree 8 as in the statement of Theorem 1 below.

A modern approach to Enriques surfaces has been carried out by Averbukh in [2, 15] and by Artin in [1]. The former one, in particular, showed again how to represent them as double planes. Equivalently, Enriques surfaces can be realized as double coverings of a quadric surface in $\mathbb{P}^3$, and these models have turned out to be very useful to study them, e.g. they allowed Horikawa to determine the periods of Enriques surfaces, see [14].

Nowadays, one usually says that Y is an Enriques surface if $q(Y) = 0$ and K_Y is a non-trivial element of 2-torsion in $\mathrm{Pic}(Y)$. In particular Y is supposed to be minimal. It is very well-known that Enriques surfaces form an irreducible family of dimension 10 and they are a distinguished class among surfaces with Kodaira dimension zero, which include also abelian, hyperelliptic and K3 surfaces. For a detailed account of the properties of Enriques surfaces, we refer the readers to the very interesting book [9] by Cossec and Dolgachev, where they considered Enriques surfaces in any characteristic; in particular see Chapter IV therein for a comprehensive report on their projective models (cf. also pp. 270–288 in [3]).

In this paper we present a short proof of the well-known representation of Enriques surfaces as double planes. Namely we will prove the following:

*The author is a member of G.N.S.A.G.A.–I.N.d.A.M. "Francesco Severi" and is partially supported by E.C. project EAGER, contract n. HPRN-CT-2000-00099.

THEOREM 1. *A smooth model of a double plane $\pi : X \to \mathbb{P}^2$ is a surface of Kodaira dimension zero with $q = p_g = 0$ if and only if there is a Cremona transformation $\gamma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that the induced normal double plane, birationally equivalent to $\pi : X \to \mathbb{P}^2$, is branched along a reduced curve of degree 8 which has two lines L_1, L_2 as irreducible components and the residual sextic has the following singularities:*

1. *a double point at $p_0 = L_1 \cap L_2$;*
2. *a tacnode at a point $p_i \in L_i$, $i = 1, 2$, where L_i is the tacnodal tangent.*

Either p_1 or p_2 may possibly be infinitely near of the first order to p_0.

Let $\pi : X \to \mathbb{P}^2$ be a double plane and let $\rho : Y \to S$ be its canonical resolution, branched over the smooth curve B. One sees that, if Y has Kodaira dimension $-\infty$, then $|B + mK_S| = \emptyset$, for every $m \geq 2$, and in [5] we saw how to use these conditions in order to classify rational and ruled double planes.

If Y has Kodaira dimension zero and $p_g(Y) = q(Y) = 0$, i.e. Y is birationally equivalent to an Enriques surface, one sees that $p_a(B/2) = 0$, $|B/2 + K_S| = \emptyset$, $|B + mK_S| = \emptyset$ for $m > 2$ and $|B + 2K_S| = \{D\}$, where D is a curve which does not move (see Lemma 1 below). We will show that these conditions are enough to find a Cremona transformation $\gamma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ as in the statement of Theorem 1.

In other words, our proof is based only on the properties of double covers and on the numerical characters (plurigenera and irregularity) of Enriques surfaces, with no need to use the geometry of curves on them.

In Section 1, we will fix notation and recall some well-known facts about double coverings. Then, in Section 2, we will prove Theorem 1.

Let us finally remark that a representation of Enriques surfaces as *fourfold* covers of $\mathbb{P}^2$ has been described by Verra in [16] and by Casnati and Ekedahl in [8].

1. Notation and preliminaries.

We consider algebraic varieties defined over the field of complex numbers $\mathbb{C}$. Let $\kappa(X)$ denote the Kodaira dimension of an algebraic variety X. A double plane $\pi : X \to \mathbb{P}^2$ is a double covering of the projective plane $\mathbb{P}^2$, i.e. π is a finite flat morphism of degree 2. Two double planes π and $\rho : Z \to \mathbb{P}^2$ are said to be *birationally equivalent* if there exists two birational maps $\gamma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ and $\varphi : Z \dashrightarrow X$ such that $\pi \circ \varphi = \gamma \circ \rho$.

In particular, if X is normal, a Cremona transformation $\gamma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ uniquely determines the birational map $\varphi : Z \dashrightarrow X$, where Z is normal, and we will say that $\rho : Z \to \mathbb{P}^2$ is the double plane induced by π and γ.

Let us recall some well-known facts about double coverings (see, e.g., [3]). A double covering $\rho : Y \to S$ of any smooth rational surface S is uniquely determined by its branch curve C in S. Moreover C is smooth if and only if Y is smooth, and C is reduced if and only if Y is normal. If C is not reduced, say $C = \sum_i m_i C_i$, where the C_i's are the irreducible components of C and $m_i \geq 1$, then the normalization Y^ν of Y is a double covering of S branched over $\sum_i \varepsilon_i C_i$, where $\varepsilon_i = m_i \mod 2 \in \{0, 1\}$.

Let $\pi : X \to \mathbb{P}^2$ be a normal double plane, branched along a reduced curve C. If C is not smooth, there exists a birational morphism $\sigma : S \to \mathbb{P}^2$ such that the normalization Y of $X \times_{\mathbb{P}^2} S$ is smooth. The induced double covering $\rho : Y \to S$ is usually called *the canonical resolution* of π (see [3, p. 87] or [6]).

Let B be the branch curve of ρ and $\tilde{C}$ be the strict transform in S of C. Then $B = \tilde{C} + \sum_i \varepsilon_i E_i$, where $\varepsilon_i \in \{0, 1\}$ and the E_i's are the irreducible exceptional curves in S. Let us say that E_i is *branched* if $\varepsilon_i = 1$, and *unbranched* otherwise. Recall that B is an *even* divisor in S, i.e. $B/2$ is well-defined in the Picard group $\mathrm{Pic}(S)$ of S, and $\rho_*(\mathcal{O}_Y) \cong \mathcal{O}_S \oplus \mathcal{O}_S(-B/2)$, thus the *plurigenera* of Y are

$$P_m(Y) = h^0(S, mB/2 + mK_S) + h^0(S, (m-1)B/2 + mK_S),$$

for all $m \geq 1$, whereas its *irregularity* is $q(Y) = p_g(Y) - p_a(B/2)$.

In order to describe the singularities of C, it is convenient to recall and to use the classical notions of infinitely near points (cf. [13, p. 392], [12, v. 2, pp. 336–386], [7], or [5] in this setting). Let us write the birational morphism $\sigma : S \to \mathbb{P}^2$ as $\sigma = \sigma_n \circ \cdots \circ \sigma_1 \circ \sigma_0$, where $\sigma_i : S_i \to S_{i-1}$ is the blow-up at a point $x_i \in S_{i-1}$ and $\mathbb{P}^2 = S_{-1}$, $S = S_n$. One says that x_k is *infinitely near* to x_j, and we write $x_k > x_j$, if $x_k \in (\sigma_{k-1} \circ \cdots \circ \sigma_j)^{-1}(x_j)$. By $x_k >^s x_j$ we mean that x_k is infinitely near *of order s* to x_j. We say that x_k is *proper* if it is not infinitely near to x_j, for any $j \neq k$. In other words, a proper point really belongs to $\mathbb{P}^2$.

Let us denote by E_i (E_i^*, resp.) the strict (total, resp.) transform in S of the exceptional curve $\sigma_i^{-1}(x_i) \subset S_i$ of σ_i. Recall that $E_i = E_i^* - \sum_j q_{ij} E_j^*$ in $\mathrm{Pic}(S)$, where $q_{ij} \in \{0, 1\}$. One says that x_j is *proximate* to x_i if and only if $q_{ij} = 1$.

In $\mathrm{Pic}(S)$, write $\tilde{C} = 2dL - \sum_i c_i E_i^*$, where L is a total transform of a line, $2d = \tilde{C} \cdot L = \deg(C)$ and $c_i = \tilde{C} \cdot E_i^*$ is usually called the *multiplicity* of C at x_i. Then $B = \tilde{C} + \sum_i \varepsilon_i E_i = 2dL - \sum_i b_i E_i^*$, where $b_i = B \cdot E_i^* = c_i - \varepsilon_i + \sum_{j \neq i} \varepsilon_j q_{ji}$. Let us say that b_i is the *virtual* multiplicity of the branch curve of π at x_i.

Notice that if $x_k > x_j$, then $c_k \leq c_j$, because $\tilde{C} \cdot E_j \geq 0$. But the same is not true for the b_i's: it may happen that $x_k >^1 x_j$ and $b_k > b_j$. This occurs if and only if $b_k = b_j + 2$, $c_k = c_j$ and $\varepsilon_j = 1$. In that case, let us say that x_j (x_k, resp.) is a *defective* (*excessive*, resp.) point. One can check that x_j is defective if and only if E_i is a branched and $E_i^2 = -2$, or, equivalently, if and only if $\rho^{-1}(E_i)$ is a (-1)-curve in Y (see, e.g., [6] for more details).

For example, if C has a triple point $x_j \in \mathbb{P}^2$ with a triple point x_k infinitely near to it, i.e. in our notation $x_k >^1 x_j$ and $c_k = c_j = 3$, then $b_j = 2$, $\varepsilon_j = 1$ and $b_k = 4$, thus x_j is defective and x_k is excessive.

Regarding Cremona transformations $\gamma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, recall that Noether-Castelnuovo Theorem states that γ is the composition of finitely many *quadratic* Cremona transformations, i.e. such that the pull-back of the net of lines is a net of conics passing through three simple points, which can be proper or infinitely near. In particular, if these three points are x_0, x_1, x_2, with virtual multiplicity b_0, b_1, b_2, one checks that the branch curve of the induced normal double plane has degree $4d - b_0 - b_1 - b_2$ and virtual multiplicities $2d - b_1 - b_2$, $2d - b_0 - b_2$, $2d - b_0 - b_1$ at the points corresponding to x_0, x_1, x_2, respectively (cf., e.g., Lemma 5.1 in [5]).

2. Proof of Theorem 1.

First we determine some properties of the branch curve, and its adjoint linear systems, of a double plane whose canonical resolution is a surface Y of Kodaira dimension zero with $p_g = q = 0$. This clearly forces $P_{2n} = 1$ and $P_{2n+1} = 0$, for every $n \geq 1$, and the minimal model W of Y is such that $K_W^2 = 0$ (see, e.g., Lemma VIII.1 in [4]).

LEMMA 1. *Let $\pi : X \to \mathbb{P}^2$ be a normal double plane and $\rho : Y \to S$ its canonical resolution, branched over the smooth curve B in the smooth rational surface S. If Y is such that $\kappa(Y) = p_g(Y) = q(Y) = 0$, then*

(i) $|B/2 + K_S| = \emptyset$*;*

(ii) $p_a(B/2) = 0$*;*

(iii) $h^0(S, B + 2K_S) = 1$*, i.e.* $|B + 2K_S| = \{D\}$*;*

(iv) $|B + mK_S| = \emptyset$ *for* $m > 2$*.*

Proof. The double cover formulas for $p_g(Y)$ and $q(Y)$ recalled in §1 imply trivially (i) and (ii). If $m \geq 3$ is odd, say $m = 2n + 1$ with $n > 0$, then $P_m(Y) = 0$ forces $|nB + mK_S| = \emptyset$, therefore $|B + mK_S| = \emptyset$, because B is effective.

Since $P_2(Y) = 1$, one has either $|B + 2K_S| = \emptyset$ or $|B/2 + 2K_S| = \emptyset$, where the former (the latter, resp.) linear system corresponds to the invariant (anti-invariant, resp.) part of $|2K_Y|$. Note that the Riemann-Roch Theorem and $K_W^2 = 0$ imply that $h^0(-2K_W) > 0$, hence $\mathcal{O}_W(2K_W) \cong \mathcal{O}_W$. This means that the invariant part of $|2K_Y|$ is not empty, i.e. $|B+2K_S| = \{D\}$ is a curve which does not move. Since $P_{2n}(Y) = 1$, $n > 1$, it follows that $|B + 2nK_S| \subset |nB + 2nK_S| = \{nD\}$, and the inclusion is strict because B is effective and B cannot be part of D. Therefore $|B + 2nK_S| = \emptyset$, for $n > 1$, which concludes the proof. $\square$

Now we want to show how to use the above properties (i)-(iv) in order to find a Cremona transformation $\gamma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ as in the statement of Theorem 1. This can be easily shown by applying the techniques we used to classify rational double planes. Indeed, the key results in [5] are Propositions 9.4 and 9.12, which can be stated together as follows:

PROPOSITION 1. *Let $\pi : X \to \mathbb{P}^2$ be a normal double plane, branched along a reduced curve C of degree $2d$, and let $\rho : Y \to S$ be its canonical resolution, branched along the smooth curve B (cf. notation in §1). Suppose that $p_a(B/2) \geq -1$. If $|B+mK_S| = \emptyset$ for every $m \geq m_0$, where m_0 is a fixed integer with $m_0 \leq 2d/3$, then there exists a Cremona transformation $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that the induced double plane is branched along a curve of degree $2d'$ with a point x_0 of maximal virtual multiplicity $> 2(d' - m_0)$.* $\square$

The main idea of the proof of the previous proposition is that the conditions $|B + mK_S| = \emptyset$, $m \geq m_0$, imply that the branch curve has singularities of "large" multiplicity at some points. This should imply that one can apply a quadratic Cremona transformation, centered at these points, which makes the branch curve somewhat "simpler", and then go on inductively. Proposition 9.4 in [5] shows that the following configuration of the singular points $x_0, \dots, x_n$ of the branch curve is such that one does not easily see which quadratic Cremona transformation "simplifies" the branch curve:

($\star$) there is a point x_0 with $b_0 \geq 2(d - m_0)$ and each point x_i such that $b_i > d - b_0/2$, say for $i = 1, \dots, h$, is excessive, say $x_i >^1 x_{h+i}$, with $b_i = 2 + d - b_0/2$ and such that there is a line L_i passing through x_0, x_i, x_{h+i}.

In this case, moreover, L_i is an irreducible component of the branch curve.

Proposition 9.12 in [5] shows that, if $p_a(B/2) \geq -1$, then configuration ($\star$) may occur only if $h = 3$ and $b_0 = b_1$, in which case one can apply two quadratic transformations centered at $x_1, \dots, x_6$ and again one can "simplify" the branch curve.

In our situation Proposition 1 clearly implies the following:

COROLLARY 1. *Let $\pi : X \to \mathbb{P}^2$ be a normal double plane, branched along a reduced curve C of degree $2d \geq 10$, and let $\rho : Y \to S$ be its canonical resolution. If Y is birationally equivalent to an Enriques surface, then there exists a Cremona transformation $\delta : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that the induced double plane is branched along a curve of degree $2d'$ with a point x_0 of maximal virtual multiplicity $b'_0 = 2d' - 4$.*

Proof. By Lemma 1, we can apply Proposition 1 with $m_0 = 3$. This implies the assertion with $b'_0 \geq 2d' - 4$. On the other hand, if $b'_0 \geq 2d' - 2$, then $\kappa(Y) = -\infty$ (cf., e.g., Lemma 8.6 in [5]) and we get a contradiction. □

Now we are ready to conclude the proof of Theorem 1.

Let $\pi : X \to \mathbb{P}^2$ be a normal double plane, branched along a reduced curve C of degree $2d$, with usual notation introduced in §1.

If $2d \leq 4$, then Y has Kodaira dimension $-\infty$.

Suppose that $2d = 6$. If the maximal virtual multiplicity is $b_0 \geq 4$, then again $\kappa(Y) = -\infty$. Otherwise, $b_0 \leq 2$ and $p_g(Y) = h^0(S, B/2 + K_S) = h^0(S, \mathcal{O}(S)) = 1$.

This forces $2d \geq 8$. Suppose that $2d = 8$. Again, if the maximal virtual multiplicity is $b_0 \geq 6$, then $\kappa(Y) = -\infty$. Let h be the number of points x_i with virtual multiplicity $b_i = 4$. Lemma 1, (ii), says that $0 = p_a(B/2) = 3 - h$, therefore $h = 3$. After re-ordering the indexes, we may assume that x_0, x_1, x_2 are the points with $b_0 = b_1 = b_2 = 4$.

Suppose that all of them are excessive, say $x_i >^1 x_{i+3}$, with $b_{i+3} = 2$, $i = 0, 1, 2$. Then we may assume that $x_3 \in \mathbb{P}^2$ and either $x_4 \in \mathbb{P}^2$ or $x_4 >^1 x_0$. In both cases the quadratic Cremona transformation centered at x_0, x_3, x_4 induces a normal double plane branched along a curve of degree 8 with a point, corresponding to x_1, which is not excessive and of virtual multiplicity 4.

So we may assume that $x_0 \in \mathbb{P}^2$. Note that, if we could find two points x_i and x_j with $b_i = 4$ and $b_j \geq 2$ such that there exists a quadratic Cremona transformation centered at x_0, x_i, x_j, then the induced normal double plane would be branched along a curve of degree ≤ 6, which contradicts our assumptions, according to the previous discussion.

This implies that both x_1, x_2 must be excessive, say $x_1 >^1 x_3$ and $x_2 >^1 x_4$, and moreover that there are two lines L_1, L_2 passing through x_0, x_3, x_1 and x_0, x_4, x_2, respectively. Note that this is configuration ($\star$) with $m_0 = h = 2$ and that

$$|B + 2K_S| = E_3 + E_4 + |2L - 2E_0^* - E_1^* - \cdots - E_4^*| = \{E_3 + E_4 + L_1 + L_2\},$$

which agrees with Lemma 1, (iii), where, abusing a little of notation, we denote by L_i also the strict transform in S of the line L_i, $i = 1, 2$. Note also that L_i is clearly also an irreducible component of the branch curve C, because it meets C at x_0, x_i, x_{i+2}, where C has multiplicity $c_0 = 4$, $c_i = 3$, $c_{i+2} = 3$, respectively. Setting $p_0 = x_0$ and $p_i = x_{i+2}$, $i = 1, 2$, this proves Theorem 1, in case $2d = 8$.

In order to conclude the proof of Theorem 1, it suffices to show that, if $2d \geq 10$, then there exists a Cremona transformation $\delta : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that the induced normal double plane has degree $< 2d$.

By Corollary 1, we know that $b_0 = 2d - 4$. Note that either

(i) $x_0 \in \mathbb{P}^2$, thus $c_0 \geq b_0 = 2d - 4$; or

(ii) there is no proper point of virtual multiplicity $2d - 4$ and x_0 is excessive, with $x_0 >^1 x_i \in \mathbb{P}^2$, for some i, thus $c_0 = c_i = 2d - 5$.

Consider first the latter case. Then $2d = 10$, otherwise the line $\overline{x_i x_0}$ would be a double component of the branch curve C, contradicting the assumption that C is reduced. Thus $b_0 = 6$, $b_i = 4$ and $c_0 = c_i = 5$. By Lemma 1, (i), we have that

$$\emptyset = |B/2 + K_S| = \overline{x_0 x_i} + E_i + |L - E_0^* - \cdots|$$

hence there is a point x_j with $b_j = 4$ such that either the quadratic Cremona transformation δ centered at x_0, x_i, x_j is well-defined, or $x_j >^1 x_k$, with $b_k \geq 2$, and the quadratic Cremona transformation δ' centered at x_0, x_i, x_k is well-defined. In both two situations, the branch curve of the induced normal double plane has degree < 10, which concludes the proof in case (ii).

Consider finally case (i). If there is a point x_i with $b_i \geq 6$, then apply a quadratic transformation centered at x_0, x_i and a general point x in the plane, thus the branch curve of the induced normal double plane has degree $\leq 2d - 2$ and the proof is done. So we may assume that, apart x_0, all other x_i's have $b_i \leq 4$. By Lemma 1, (ii), we have that $0 = p_a(B/2) = (d - 1)(d - 2)/2 - h$, where h is the number of points x_i, say $x_1, \ldots, x_h$, with $b_i = 4$.

We claim that there are two points x_i and x_j, with $b_i = 4$ and $b_j \geq 2$, such that the quadratic Cremona transformation centered at x_0, x_i, x_j is well-defined, therefore the branch curve of the induced normal double plane will have degree $\leq 2d-2$ and the proof of Theorem 1 will be concluded.

Indeed, either all the points $x_1,\dots,x_h$ are excessive, or there is a point x_i, with $b_i=4$, and such that either $x_i\in\mathbb{P}^2$ or $x_i>^1 x_0$. If all the x_i's are excessive, then $x_i>^1 x_{j(i)}$ with $b_{j(i)}=2$, and moreover there is one of them, say x_k, such that either $x_{j(k)}\in\mathbb{P}^2$ or $x_{j(k)}>^1 x_0$.

Note that $x_1,\dots,x_h$ cannot be all proximate to x_0, because C has multiplicity $2d-4$, or $2d-5$, at x_0, with $d>4$, and $h=(d-1)(d-2)/2$. Thus we cannot find a quadratic transformation as above only if the points x_i are as in configuration $(\star)$, with $m_0=2$. In that case, let L_i, $i=1,\dots,h$, be the strict transform in S of the line passing through x_0, x_{h+i}, x_i. For every $i=1,\dots,h$, the curve L_i should be a component of B and also of $|B+2K_S|$, which is

$$|B+2K_S|=\sum_{i=h+1}^{h}E_{2h}+|(2d-6)(L-E_0^*)-\sum_{i=1}^{2h}E_i^*|$$

and we get a contradiction with Lemma 1, (iii), which says that $h^0(S,B+2K_S)=1$, because we should have $h=(d-1)(d-2)/2$ such lines. $\square$

References

[1] ARTIN M., *On Enriques surfaces*, Harvard thesis, 1960.

[2] AVERBUKH B.G., *On the special types of Kummer and Enriques surfaces*, Izv. Akad. Nauk SSSR Ser. Mat. **29** (1965), 1095–1118, also as Appendix in [15].

[3] BARTH W., PETERS C. AND VAN DE VEN A., *Compact complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 4, Springer-Verlag 1984.

[4] BEAUVILLE A., *Complex algebraic surfaces*, London Math. Soc. Lecture Notes **68**, Cambridge Univ. Press 1983.

[5] CALABRI A., *On rational and ruled double planes*, Annali Mat. Pura Appl. **181** (2002), 365–387.

[6] CALABRI A., FERRARO R., *Explicit resolutions of double point singularities of surfaces*, Collect. Math. **53** (2002), 99–131.

[7] CASAS-ALVERO E., *Singularities of plane curves*, London Math. Soc. Lect. Notes **276**, Cambridge University Press 2000.

[8] CASNATI G., EKEDAHL T., *Covers of algebraic varieties. I. A general structure theorem, covers of degree 3, 4 and Enriques surfaces*, J. Algebraic Geom. **5** (3) (1996), 439–460.

[9] COSSEC F., DOLGACHEV I., *Enriques surfaces I*, Progress in Mathematics **76**, Birkhäuser 1989.

[10] ENRIQUES F., *Sopra le superficie algebriche di bigenere uno*, Mem. Soc. Italiana delle Scienze detta dei XL, ser. 3, **14** (1906), 327–352, also in *Memorie scelte di Geometria. 2*, Zanichelli 1959, 241–272.

[11] ENRIQUES F., *Le superficie algebriche*, Zanichelli 1949.

[12] ENRIQUES F., CHISINI O., *Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche*, Zanichelli 1915–1934.

[13] HARTSHORNE R., *Algebraic geometry*, Graduate Texts in Mathematics **53**, Springer 1978.

[14] HORIKAWA E., *On the periods of Enriques surfaces*, I, Math. Ann. **234** (1978), 73–108; II, Math. Ann. **235** (1978), 217–246.

[15] SHAFAREVICH I. (ED.), *Algebraic surfaces*, Proc. Steklov Math. Inst. **75** (1967).

[16] VERRA A., *On Enriques surfaces as a fourfold cover of* $\mathbb{P}^2$, Math. Ann. **266** (1983), 241–250.

AMS Subject Classification (2000): 14J28.

Alberto CALABRI, Dipartimento di Matematica, Università degli Studi di Bologna, Piazza di Porta San Donato 5, 40126 Bologna, ITALY
e-mail: calabri@dm.unibo.it