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Chiara Boiti · Elisabetta Gallucci

The overdetermined Cauchy problem for ω-ultradifferentiable functions

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Abstract. In this paper we study the Cauchy problem for overdetermined systems of linear partial differential operators with constant coefficients in some spaces of ω-ultradifferentiable functions in the sense of Braun et al. (Results Math 17(3–4):207–237, 1990), for non-quasianalytic weight functions ω. We show that existence of solutions of the Cauchy problem is equivalent to the validity of a Phragmén–Lindelöf principle for entire and plurisubharmonic functions on some irreducible affine algebraic varieties.

1. Introduction

In this paper we consider the Cauchy problem for overdetermined systems of linear partial differential operators with constant coefficients in some classes $\mathcal{E}_{(\omega)}$ of ω-ultradifferentiable functions, for a non-quasianalytic weight.

We consider weight functions ω in the sense of [16], but we relax their condition $\log(1+t) = o(\omega(t))$, for $t \to +\infty$, by a weaker condition in the spirit of [4] (see condition (γ) in Definition 2.1), since we work only in the Beurling setting. This allows to consider the space of C^∞ functions as a particular case of the space $\mathcal{E}_{(\omega)}$ for $\omega(t) = \log(1+t)$ and will be particularly useful in the description of the space $\mathcal{S}_\omega$ of ω-ultradifferentiable Schwartz functions in the forthcoming paper [6]. For this reason we collect, in Sect. 2, those results of [16], in the Beurling setting, which are still valid under the weaker condition (γ) of Definition 2.1.

Note, on the contrary, that in the Roumieu case the stronger condition $\log(1+t) = o(\omega(t))$ is required (see, for instance, Remark 2.15, or [18] for more details).

In Sect. 3 we investigate the overdetermined Cauchy problem in the frame of Whitney ω-ultradifferentiable functions, in the spirit of [10,12,26], in order to bypass the question of formal coherence of the data, which naturally arises in the overdetermined case.

Indeed, in the classical Cauchy problem for a linear partial differential equation with initial data on a hypersurface, smooth initial data together with the equation

C. Boiti (✉)· E. Gallucci: Dipartimento di Matematica e Informatica, Università di Ferrara, Via Machiavelli n. 30, 44121 Ferrara, Italy e-mail: chiara.boiti@unife.it

E. Gallucci: e-mail: elisabetta.gallucci@student.unife.it

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allow to compute the Taylor series of a smooth solution at any given point of the hypersurface.

This leads, in the case of systems of linear partial differential equations, to the notion of formally non-characteristic hypersurface that was considered in [2,3,26].

In the case of overdetermined systems, the question of the formal coherence of the data would be particularly intricate, so that the above remarks suggest further generalizations of the Cauchy problem, where the assumption that the initial data are given on a formally non-characteristic hypersurface is dropped, and we allow formal solutions (in the sense of Whitney) of the given system on any closed subset as initial data.

Using Whitney functions, we can thus consider a more general framework in which two quite arbitrary sets are involved. We take K_1 and K_2 closed convex subsets of $\mathbb{R}^N$ with $K_1 \subsetneq K_2$ for $j = 1, 2$, thinking at K_1 as the set where the initial data are given, and at K_2 as the set where we want to find a solution of the following Cauchy problem:

$$\begin{cases} A_0(D)u = f \\ u|_{K_1} \equiv \varphi, \end{cases} \tag{1.1}$$

where $A_0(D)$ is an $a_1 \times a_0$ matrix of linear partial differential operators with constant coefficients, $\varphi \in \left(W_{K_1}^{(\omega)}\right)^{a_0}$, $f \in \left(W_{K_2}^{(\omega)}\right)^{a_1}$ are the given Cauchy data in the Whitney classes of ω-ultradifferentiable functions of Beurling type on K_1 and K_2 respectively, and $u|_{K_1} \equiv \varphi$ means that they are equal in the Whitney sense, i.e. with all their derivatives.

It comes out (see Sect. 3) that, in order to find a solution $u \in \left(W_{K_2}^{(\omega)}\right)^{a_0}$ of the Cauchy problem (1.1), the function f must satisfy some integrability conditions. These may be written as

$$\begin{cases} A_1(D)f = 0 \\ f|_{K_1} \equiv 0, \end{cases} \tag{1.2}$$

for a matrix $A_1(D)$ of linear partial differential operators with constant coefficients obtained by a Hilbert resolution of $\mathcal{M} = \operatorname{coker}\left({}^t A_0(\zeta) : \mathcal{P}^{a_1} \to \mathcal{P}^{a_0}\right)$, where $\mathcal{P} = \mathbb{C}[\zeta_1, \ldots, \zeta_N]$ (see (3.6)).

The rows of the matrix $A_1(D)$ give a system of generators for the module of all integrability conditions for f that can be expressed in terms of partial differential operators, and if $A_1(\zeta) \not\equiv 0$ we say that the Cauchy problem is *overdetermined*.

We prove in Theorem 3.19 that the Cauchy problem (1.1), for f satisfying (1.2), admits at least a solution if and only if a Phragmén–Lindelöf principle holds on the complex characteristic varieties V associated to $\mathcal{M}$. Moreover, splitting $\mathbb{R}^N \simeq \mathbb{R}^k_t \times \mathbb{R}^n_x$, this could be done also allowing different scales of regularity in the t-variables and in the x-variables (see Remark 3.25).

Relating the existence of solutions of the Cauchy problem to the validity of a Phragmén–Lindelöf principle may be very useful. For instance, in the case of associated characteristic varieties V of dimension 1, it was found in [9] a complete characterization of algebraic curves that satisfy the Phragmén–Lindelöf principle, by means of Puiseux series expansions on the branches of V at infinity: it comes out that the exponents and coefficients of the Puiseux series expansions are strictly related to the classes of functions where the Cauchy problem admits at least a solution. Since Puiseux series expansions can be computed by several computer programs, such as MAPLE for instance, this characterization may be very useful (see Example 3.26).

2. Ultradifferentiable functions

In the present section we follow [16] to define and enlighten properties of the space $\mathcal{E}_{(\omega)}$ of ω-ultradifferentiable functions of Beurling type for the following class of weights:

Definition 2.1. Let $\omega : [0, \infty) \to [0, \infty)$ be a continuous increasing function. It will be called a *non-quasianalytic weight function* $\omega \in \mathcal{W}'$ if it has the following properties:

(α) $\exists\, K \geq 1 : \omega(2t) \leq K(1 + \omega(t)) \qquad \forall t \geq 0,$
(β) $\displaystyle\int_1^\infty \frac{\omega(t)}{t^2} dt < \infty,$
(γ) $\exists\, a \in \mathbb{R},\ b > 0 : \omega(t) \geq a + b\log(1 + t), \qquad \forall t \geq 0,$
(δ) $\varphi : [0, \infty) \to [0, \infty), \qquad \varphi(t) := \omega(e^t)$ is convex.

For $z \in \mathbb{C}^N$ we write $\omega(z)$ for $\omega(|z|)$, where $|z| = \sum_{j=1}^N |z_j|$.

Remark 2.2. Condition (β) is the condition of non-quasianalyticity and it will ensure, in the following, the existence of functions with compact support (cf. Remark 2.13).

Remark 2.3. In the original paper [16], instead of condition (γ), the following stronger condition was considered:

$$\lim_{t\to\infty} \frac{\log(1 + t)}{\omega(t)} = 0. \tag{2.1}$$

In this case we say that $\omega \in \mathcal{W}$.

Then we can define the *Young conjugate* φ^* of φ by

$$\begin{aligned} \varphi^* : [0,\infty) &\longrightarrow \mathbb{R} \\ y &\longmapsto \sup_{x\geq 0}(xy - \varphi(x)). \end{aligned}$$

There is no loss of generality to assume that ω vanishes on [0, 1] (cf. also [1]). Then φ^* has only non-negative values, it is convex and increasing, satisfies $\varphi^*(0) = 0$ and $(\varphi^*)^* = \varphi$ (cf. [15,16]). Moreover if $\lim_{x\to\infty} \frac{x}{\varphi(x)} = 0$ then $\lim_{y\to\infty} \frac{y}{\varphi^*(y)} = 0$. Note that (2.1) implies $\lim_{x\to\infty} \frac{x}{\varphi(x)} = 0$.

Example 2.4. The following functions $\omega \in \mathcal{W}'$ are examples of non-quasianalytic weight functions (eventually after a change in the interval $[0, \delta]$ for suitable $\delta > 0$):

$$\omega(t) = t^{\alpha}, \qquad 0 < \alpha < 1 \tag{2.2}$$
$$\omega(t) = (\log(1+t))^{\beta}, \qquad \beta \geq 1 \tag{2.3}$$
$$\omega(t) = t(\log(e+t))^{-\beta}, \qquad \beta > 1.$$

They are all weight functions also in $\mathcal{W}$, except (2.3) for $\beta = 1$.

For the sake of completeness we collect here some results of [16] which are clearly valid also for $\omega \in \mathcal{W}'$ (see [17,18] for more details):

Lemma 2.5. *Let $\omega \in \mathcal{W}'$. Then*

$$\omega(x+y) \leq K(1+\omega(x)+\omega(y)), \qquad \forall x, y \in \mathbb{C}^N.$$

Lemma 2.6. *For $\omega \in \mathcal{W}'$ and $\varphi(t) = \omega(e^t)$, there exists $L > 0$ such that*

$$\varphi^*(y) - y \geq L\varphi^*\left(\frac{y}{L}\right) - L, \qquad \forall y \geq 0.$$

Lemma 2.7. *For $\omega \in \mathcal{W}'$ and $\varphi(t) = \omega(e^t)$ we have that $\dfrac{\varphi(x)}{x}$ and $\dfrac{\varphi^*(s)}{s}$ are increasing.*

Lemma 2.8. *Let $\omega \in \mathcal{W}'$. Then there exists a weight function $\sigma \in \mathcal{W}$ with $\omega(t) = o(\sigma(t))$.*

Proposition 2.9. *Let $\omega \in \mathcal{W}'$. Then for each $N \in \mathbb{N}$ there exists $\delta_N > 0$ such that for every $\varepsilon > 0$ there exists $H \in \mathcal{C}^{\infty}(\mathbb{R}^N)$, $H \neq 0$, with*

$$\mathrm{supp}(H) \subset [-\varepsilon, \varepsilon]^N$$
$$\int_{\mathbb{R}^N} |\hat{H}(t)| e^{\delta_N \omega(t)} dt < \infty.$$

Proof. See [16], Corollary 2.5 and Remark after Corollary 2.6. □

Proposition 2.10. *Let $\omega \in \mathcal{W}'$. Then for each $N \in \mathbb{N}$ and $\varepsilon > 0$ there exists $H \in \mathcal{C}^{\infty}(\mathbb{R}^N)$, $H \neq 0$, with*

$$\mathrm{supp}(H) \subset [-\varepsilon, \varepsilon]^N$$
$$\int_{\mathbb{R}^N} |\hat{H}(t)| e^{m\omega(t)} dt < \infty, \qquad \forall m > 0.$$

Proof. See [16], Corollary 2.6 and the related Remark. □

Let us now prove the following result in the Beurling case, for $\omega \in \mathcal{W}'$, referring to [16] for the analogous result in the Roumieu case for $\omega \in \mathcal{W}$.

Lemma 2.11. *Let $\omega \in \mathcal{W}'$ and $f \in \mathcal{D}(\mathbb{R}^N)$. If there is $B > 0$ such that*

$$\int_{\mathbb{R}^N} |\hat{f}(t)| e^{B\omega(t)} dt := C < \infty,$$

then

$$\sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in \mathbb{R}^N} |f^{(\alpha)}(x)| e^{-B\varphi^*\left(\frac{|\alpha|}{B}\right)} \leq \frac{C}{(2\pi)^N}. \tag{2.4}$$

If (2.4) holds for $f \in \mathcal{D}(\mathbb{R}^N)$ and $B > 0$ then there is $D > 0$, depending only on ω, N and B, and there is $L > 0$ depending only on ω and N, such that for $K = \operatorname{supp} f$ and $m_N(K)$ its Lebesgue measure, we have that

$$|\hat{f}(z)| \leq m_N(K) \frac{CD}{(2\pi)^N} e^{\left(H_K(\operatorname{Im} z) + \left(\frac{1}{b} - \frac{B}{L}\right)\omega(z)\right)} \qquad \forall z \in \mathbb{C}^N, \tag{2.5}$$

where $b > 0$ is the constant of condition (γ) in Definition 2.1.

Proof. The proof of (2.4) is the same as for $\omega \in \mathcal{W}$, so that we refer to [16, Lemma 3.3] for it. Let us prove (2.5).

By condition (α) there is $L > 0$ such that

$$\omega(Nr) \leq L\omega(r) + L \qquad \forall r > 0. \tag{2.6}$$

Let now $z \in \mathbb{C}^N$ be given, let l be the index with

$$|z_l| = \max_{1 \leq j \leq N} |z_j|$$

and assume $|z_l| > 1$. Write then

$$\hat{f}(z) = \int_K f(t) e^{-i<t,z>} dt = \int_K \left(\frac{\partial^j}{\partial t_l^j} f(t)\right) \cdot \frac{1}{(iz_l)^j} \cdot e^{-i<t,z>} dt$$

by partial integration, for all $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

In view of (2.4), this implies that, for all $j \in \mathbb{N}_0$:

$$|\hat{f}(z)| \leq m_N(K) \frac{C}{(2\pi)^N} e^{\left(B\varphi^*\left(\frac{j}{B}\right) - j\log|z_l| + H_K(\operatorname{Im} z)\right)}. \tag{2.7}$$

Now, note that for every $x > 0$ there exists $j \in \mathbb{N}_0$ such that $j \leq Bx < j + 1$, and hence from (2.6) and (γ)

$$\begin{aligned}
\sup_{j \in \mathbb{N}_0} \left(j \log|z_l| - B\varphi^*\left(\frac{j}{B}\right)\right) &= B \sup_{j \in \mathbb{N}_0} \left(\frac{j+1}{B} \log|z_l| - \varphi^*\left(\frac{j}{B}\right)\right) - \log|z_l| \\
&\geq B \sup_{x > 0} \left(x \log|z_l| - \varphi^*(x)\right) - \log|z_l| \\
&= B\varphi^{**}(\log|z_l|) - \log|z_l| = B\omega(z_l) - \log|z_l| \\
&\geq B\omega\left(\frac{z}{N}\right) - \log|z| \geq \frac{B}{L}\omega(z) - 1 - \log|z|
\end{aligned}$$

$$\geq \frac{B}{L}\omega(z) - 1 - \frac{\omega(z)}{b} + \frac{a}{b}$$
$$= \left(\frac{B}{L} - \frac{1}{b}\right)\omega(z) + \left(\frac{a}{b} - 1\right). \tag{2.8}$$

By passing to the infimum over all $j \in \mathbb{N}_0$ in (2.7) and by using (2.8) we obtain:

$$|\hat{f}(z)| \leq m_N(K)\frac{C}{(2\pi)^N}e^{\left\{\left(\frac{1}{b}-\frac{B}{L}\right)\omega(z)+(1-\frac{a}{b})+H_K(\mathrm{Im}\, z)\right\}}$$
$$= m_N(K)\frac{CD}{(2\pi)^N}e^{\left\{\left(\frac{1}{b}-\frac{B}{L}\right)\omega(z)+H_K(\mathrm{Im}\, z)\right\}},$$

where $D = e^{(1-\frac{a}{b})}$. □

Definition 2.12. Let $\omega \in \mathcal{W}'$ and let $K \subset \mathbb{R}^N$ be a compact set. For $\lambda > 0$ we define the Banach space

$$\mathcal{D}_\lambda(K) = \left\{ f \in C^\infty(\mathbb{R}^N) |\ \mathrm{supp}\, f \subset K \ \text{ and } \ \|f\|_\lambda := \int_{\mathbb{R}^N} |\hat{f}(t)|e^{\lambda\omega(t)}dt < \infty \right\}. \tag{2.9}$$

We set

$$\mathcal{D}_{(\omega)}(K) = \operatorname*{proj\,lim}_{\lambda\to\infty} \mathcal{D}_\lambda(K)$$

endowed with the topology of the projective limit.

For an open set $\Omega \subset \mathbb{R}^N$ we define then

$$\mathcal{D}_{(\omega)}(\Omega) = \operatorname*{ind\,lim}_{K\subset\subset\Omega} \mathcal{D}_{(\omega)}(K)$$

where the inductive limit is taken over all compact subsets of Ω. We endow $\mathcal{D}_{(\omega)}(\Omega)$ with the inductive limit topology.

The elements of $\mathcal{D}_{(\omega)}(\Omega)$ are called *ω-ultradifferentiable functions of Beurling type with compact support.*

For $\omega \in \mathcal{W}$ we also recall from [16] the analogous definition in the Roumieu case:

$$\mathcal{D}_{\{\omega\}}(K) = \operatorname*{ind\,lim}_{\lambda\to 0} \mathcal{D}_\lambda(K)$$

endowed with the topology of the inductive limit, and for an open set $\Omega \subset \mathbb{R}^N$,

$$\mathcal{D}_{\{\omega\}}(\Omega) = \operatorname*{ind\,lim}_{K\subset\subset\Omega} \mathcal{D}_{\{\omega\}}(K),$$

where the inductive limit is taken over all compact subsets K of Ω, endowed with the inductive limit topology.

The elements of $\mathcal{D}_{\{\omega\}}(\Omega)$ are called *ω-ultradifferentiable functions of Roumieu type with compact support.*

Remark 2.13. As in [16], we have the following:

(1) Let $K \subset \mathbb{R}^N$ with non-empty interior. If $\omega \in \mathcal{W}'$ then $\mathcal{D}_{(\omega)}(K) \neq \{0\}$; if $\omega \in \mathcal{W}$ then $\mathcal{D}_{\{\omega\}}(K) \neq \{0\}$ and moreover $\mathcal{D}_{(\omega)}(K) \subset \mathcal{D}_{\{\omega\}}(K)$.
(2) For $\omega, \sigma \in \mathcal{W}'$ we have that $\mathcal{D}_{(\omega)}(\mathbb{R}) \subset \mathcal{D}_{(\sigma)}(\mathbb{R})$ iff $\sigma = O(\omega)$.
(3) We say that two functions ω and σ are equivalent if $\omega = O(\sigma)$ and $\sigma = O(\omega)$. Note that if $\omega \leq \sigma \leq C\omega$ for some $C > 0$ and if $\psi(x) = \sigma(e^x)$, then

$$C\varphi^*\left(\frac{y}{C}\right) \leq \psi^*(y) \leq \varphi^*(y) \qquad \forall y > 0.$$

With this formula, it's easy to see that definitions and most theorems in the sequel don't change if ω is only equivalent to a weight function.

Lemma 2.11 and the classical Paley–Wiener Theorem for $\mathcal{D}(K)$ imply the following Paley–Wiener theorem for ω- ultradifferentiable functions in the Beurling setting (we refer to [16] for the Roumieu case):

Theorem 2.14. *(Paley–Wiener Theorem for ω-ultradifferentiable functions of Beurling type) Let $\omega \in \mathcal{W}'$, $K \subset \mathbb{R}^N$ a convex compact set and $f \in L^1(\mathbb{R}^N)$. The following are equivalent:*

(1) $f \in \mathcal{D}_{(\omega)}(K)$,
(2) $f \in \mathcal{D}(K)$ *and for all* $k \in \mathbb{N}$

$$\sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in \mathbb{R}^N} |f^{(\alpha)}(x)| e^{\left(-k\varphi^*(\frac{|\alpha|}{k})\right)} < \infty,$$

(3) *for all* $k \in \mathbb{N}$ *there is* $C_k > 0$ *such that*

$$|\hat{f}(z)| \leq C_k e^{(H_K(\operatorname{Im} z) - k\omega(z))} \qquad \forall z \in \mathbb{C}^N.$$

Proof. **(1)** $\Rightarrow$ **(2)**:
If $f \in \mathcal{D}_{(\omega)}(K)$ then, by definition, $f \in \mathcal{D}(K)$ and for every $\varepsilon > 0$

$$\int_{\mathbb{R}^N} |\hat{f}(t)| e^{\varepsilon\omega(t)} dt =: C_\varepsilon < \infty.$$

So, by Lemma 2.11, for all $\varepsilon > 0$

$$\sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in \mathbb{R}^N} |f^{(\alpha)}(x)| e^{\left(-\varepsilon\varphi^*\left(\frac{|\alpha|}{\varepsilon}\right)\right)} < \infty, \tag{2.10}$$

and hence (2), since $\dfrac{\varphi^*(x)}{x}$ is increasing.
(2) $\Rightarrow$ **(3)**:
If $f \in \mathcal{D}(K)$ satisfies (2) then it also satisfies (2.10) for every $\varepsilon > 0$ since $\varphi^*(s)/s$ is increasing, and hence, by Lemma 2.11, there exists $D_\varepsilon > 0$ such that

$$|\hat{f}(z)| \leq D_\varepsilon e^{\left(H_K(\operatorname{Im} z) + \left(\frac{1}{b} - \frac{\varepsilon}{L}\right)\omega(z)\right)}. \tag{2.11}$$

Therefore for every $\tilde{\varepsilon} > 0$ we can choose

$$\varepsilon = L\left(\tilde{\varepsilon} + \frac{1}{b}\right) > 0$$

in (2.10), so that (2.11) becomes

$$|\hat{f}(z)| \le D_\varepsilon e^{(H_K(\operatorname{Im} z) - \tilde{\varepsilon}\omega(z))},$$

and hence (3).

(3) $\Rightarrow$ **(1)** :

By (3) and (γ) we have that for all $\lambda > 0$, taking $k \in \mathbb{N}$ with $k > \lambda$, there exist $C_\lambda, C'_\lambda > 0$ such that

$$\begin{aligned} \int_{\mathbb{R}^N} |\hat{f}(t)| e^{\lambda\omega(t)} dt &\le C_\lambda \int_{\mathbb{R}^N} e^{(-k+\lambda)\omega(t)} dt \\ &\le C_\lambda \int_{\mathbb{R}^N} e^{(-k+\lambda)(a+b\log(1+t))} dt \\ &= C'_\lambda \int_{\mathbb{R}^N} (1+t)^{b(\lambda-k)} dt. \end{aligned} \tag{2.12}$$

For $k > \dfrac{N+1}{b} + \lambda$ the above integral is finite and hence there exists $C''_\lambda > 0$ such that

$$\int_{\mathbb{R}^N} |\hat{f}(t)| e^{\lambda\omega(t)} dt \le C''_\lambda.$$

To prove that $f \in \mathcal{D}(K)$ note that (3) and (γ) imply that for every $k \in \mathbb{N}$ there exists $C_k > 0$ such that

$$\begin{aligned} \left|\hat{f}(z)\right| &\le C_k e^{H_K(\operatorname{Im} z) - k\omega(z)} \\ &\le C_k e^{H_K(\operatorname{Im} z) - k(a+b\log(1+|z|))} \\ &= C_k e^{-ak} e^{H_K(\operatorname{Im} z)} (1+|z|)^{-bk} \qquad \forall z \in \mathbb{C}^N. \end{aligned}$$

Therefore for every $n \in \mathbb{N}$ there exists $C_n > 0$ such that

$$\left|\hat{f}(z)\right| \le C_n e^{H_K(\operatorname{Im} z)} (1+|z|)^{-n} \qquad \forall z \in \mathbb{C}^N.$$

By the classical Paley–Wiener Theorem we finally have that $f \in \mathcal{D}(K)$ and hence the theorem is proved. □

Remark 2.15. The inequality (2.12) highlights the sufficiency of condition (γ) on the weight ω: by the arbitrariety of k we can allow a fixed $b > 0$ to make the integral convergent. On the contrary, this is not possible in the Roumieu case, where k is fixed and hence $\log(1+t) = o(\omega(t))$ is required.

For a sequence $\mathbb{P} = (p_n)_{n\in\mathbb{N}}$ of continuous functions $p_n : \mathbb{C}^N \to \mathbb{R}$, we define

$$A_{\mathbb{P}}(\mathbb{C}^N) := \left\{ f \in \mathcal{O}(\mathbb{C}^N) \mid \text{for all } n : \sup_{z\in\mathbb{C}^N} |f(z)|e^{-p_n(z)} < \infty \right\},$$

where $\mathcal{O}(\mathbb{C}^N)$ is the set of all entire functions on $\mathbb{C}^N$.

Let ω be a weight function and $K \subset \mathbb{R}^N$ a convex compact set. Define

$$\mathbb{P}_K := \{p_n : z \mapsto H_K(\operatorname{Im} z) - n\omega(z),\ n \in \mathbb{N}\}.$$

From the Paley–Wiener Theorem 2.14 we get:

Proposition 2.16. *Let $\omega \in \mathcal{W}'$ and K a compact convex set of $\mathbb{R}^N$. Then, if*

$$\mathcal{D}_{(\omega)}(K) \cong A_{\mathbb{P}_K}(\mathbb{C}^N).$$

The isomorphism is given by the Fourier–Laplace transform.

We can collect here below some more properties on these spaces of ω-ultradifferentiable functions with compact support, for $\omega \in \mathcal{W}'$, similarly as in [16] for $\omega \in \mathcal{W}$.

Corollary 2.17. *Let $K \subset \mathbb{R}^N$ be compact and $\Omega \subset \mathbb{R}^N$ be open. Let $\omega \in \mathcal{W}'$. Then $\mathcal{D}_{(\omega)}(K)$ is a (FN)-space, i.e. a nuclear Fréchet space.*

Lemma 2.18. *Let $\omega \in \mathcal{W}'$, $f \in \mathcal{D}(\mathbb{R}^N)$, $g \in \mathcal{D}_{(\omega)}(\mathbb{R}^N)$. Then we have:*

(1) $f * g \in \mathcal{D}_{(\omega)}(\mathbb{R}^N)$,
(2) $\operatorname{supp}(f * g) \subset \operatorname{supp} f + \operatorname{supp} g$,
(3) $\widehat{f * g}(z) = \hat{f}(z)\hat{g}(z)$.

Lemma 2.19. *Let $K_1, K_2 \subset \mathbb{R}^N$ be compact sets with $K_1 \subset \mathring{K}_2$.*

(a) *Let $\omega, \sigma \in \mathcal{W}'$ with $\sigma \le \omega$. Then for all $f \in \mathcal{D}_{(\sigma)}(K_1)$ there is a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{D}_{(\omega)}(K_2)$ with $\lim_{n\to\infty} f_n = f$ in $\mathcal{D}_{(\sigma)}(K_2)$.*
(b) *Let $\omega \in \mathcal{W}'$. Then for all $f \in \mathcal{D}(K_1)$ there is a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{D}_{(\omega)}(K_2)$ with $\lim_{n\to\infty} f_n = f$ in $\mathcal{D}(K_2)$.*

Proposition 2.20. *Let $\omega, \sigma \in \mathcal{W}$ with $\sigma = o(\omega)$. Then the inclusions*

$$\mathcal{D}_{(\omega)}(\Omega) \hookrightarrow \mathcal{D}_{\{\omega\}}(\Omega) \hookrightarrow \mathcal{D}_{(\sigma)}(\Omega) \hookrightarrow \mathcal{D}(\Omega)$$

are continuous and sequentially dense for each open set $\Omega \subset \mathbb{R}^N$.

Let us now introduce the algebra of ω-ultradifferentiable functions of Beurling type with arbitrary support:

Definition 2.21. For $\omega \in \mathcal{W}'$ and an open set $\Omega \subset \mathbb{R}^N$ we define

$$\mathcal{E}_{(\omega)}(\Omega) := \Bigg\{ f \in \mathcal{C}^\infty(\Omega) \mid \text{for all compact } K \subset \Omega \text{ and all } m \in \mathbb{N}$$

$$p_{K,m}(f) := \sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in K} |f^{(\alpha)}(x)| e^{\left(-m\varphi^*\left(\frac{|\alpha|}{m}\right)\right)} < \infty \Bigg\}.$$

The space $\mathcal{E}_{(\omega)}(\Omega)$ carries the metric locally convex topology given by the seminorms $p_{K,m}$ where K is a compact subset of Ω and $m \in \mathbb{N}$.

The elements of $\mathcal{E}_{(\omega)}(\Omega)$ are called *ω-ultradifferentiable functions of Beurling type*.

For the Roumieu case we consider $\omega \in \mathcal{W}$ and recall, from [16]:

$$\mathcal{E}_{\{\omega\}}(\Omega) := \Bigg\{ f \in \mathcal{C}^\infty(\Omega) \mid \text{for all compact } K \subset \Omega \text{ there is } m \in \mathbb{N} \text{ with}$$

$$\sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in K} |f^{(\alpha)}(x)| e^{\left(-\frac{1}{m}\varphi^*(m|\alpha|)\right)} < \infty \Bigg\}.$$

The elements of $\mathcal{E}_{\{\omega\}}(\Omega)$ are called *ω-ultradifferentiable functions of Roumieu type*.

Example 2.22. For ω as in (2.2) the space $\mathcal{E}_{\{\omega\}}(\Omega)$ is the classical Gevrey class of order $\frac{1}{\alpha}$. For ω as in (2.3) with $\beta = 1$, the space $\mathcal{E}_{(\omega)}(\Omega)$ is the space $\mathcal{E}(\Omega)$ of $\mathcal{C}^\infty$ functions in Ω.

Remark 2.23. In general the spaces of ω-ultradifferentiable functions defined as in Definition 2.21 are different from the Denjoy–Carleman classes of ultradifferentiable functions as defined in [21] (cf. [14]).

Take, for instance, the weight function $\omega(t) = \max\{0, (\log|t|)^s\}$, for $s > 1$. It does not satisfy the condition

$$2\omega(t) \leq \omega(Ht) + H \qquad \forall t \geq 0$$

for any $H \geq 1$ (cf. [14, Example 20]). Therefore, by [14, Cor. 16], the space $\mathcal{E}_{(\omega)}(\Omega)$ cannot be considered as a Denjoy–Carleman class $\mathcal{E}_{(M_p)}(\Omega)$ for any weight sequence $\{M_p\}_{p \in \mathbb{N}_0}$ and any open subset Ω of $\mathbb{R}^N$.

As in [16], we have the following properties of the space $\mathcal{E}_{(\omega)}(\Omega)$, for $\omega \in \mathcal{W}'$, referring to [16] for the analogous properties of $\mathcal{E}_{\{\omega\}}(\Omega)$, with $\omega \in \mathcal{W}$.

Proposition 2.24. *Let $\omega \in \mathcal{W}'$. The space $\mathcal{E}_{(\omega)}(\Omega)$ is a locally convex algebra with continuous multiplication.*

Lemma 2.25. *Let $\omega \in \mathcal{W}'$, $\Omega \subset \mathbb{R}^N$ be open, $K_1 \subset \mathring{K}_2 \subset K_2 \subset \cdots \subset \Omega$ be an exhaustion of Ω by compact sets. Choose $\chi_j \in \mathcal{D}_{(\omega)}(K_j)$ with $0 \leq \chi_j \leq 1$ and $\chi|_{K_{j-1}} \equiv 1$. We thus have maps*

$$\mathcal{D}_{(\omega)}(K_{j+1}) \to \mathcal{D}_{(\omega)}(K_j)$$
$$f \mapsto \chi_j f$$

by which

$$\mathcal{E}_{(\omega)}(\Omega) = \operatorname{proj}\lim_{j\to\infty} \mathcal{D}_{(\omega)}(K_j).$$

Lemma 2.26. *Let $\omega \in \mathcal{W}'$ and K a compact subset of an open set $\Omega \subset \mathbb{R}^N$. Then $\mathcal{D}_{(\omega)}(K)$ carries the topology which is induced by $\mathcal{E}_{(\omega)}(\Omega)$.*

Proposition 2.27. *The following properties hold:*

(1) *For $\omega \in \mathcal{W}'$ the inclusion $\mathcal{D}_{(\omega)}(\Omega) \hookrightarrow \mathcal{E}_{(\omega)}(\Omega)$ is continuous and has dense image.*
(2) *Let $\omega, \sigma \in \mathcal{W}$ with $\sigma = o(\omega)$, then the inclusion $\mathcal{E}_{\{\omega\}}(\Omega) \hookrightarrow \mathcal{E}_{(\sigma)}(\Omega)$ is continuous and has dense image.*

Proposition 2.28. *Let $\omega \in \mathcal{W}'$, $\Omega \subset \mathbb{R}^N$ be open and let $\{\Omega_j\}_{j\in\mathbb{N}}$ be an open covering of Ω. Then there are $f_j \in \mathcal{D}_{(\omega)}(\Omega_j)$ with $0 \le f_j \le 1$ such that $\sum_{j=1}^{\infty} f_j = 1$ and $\{\operatorname{supp} f_j\}_{j\in\mathbb{N}}$ is locally finite.*

Proposition 2.29. *Let $\omega \in \mathcal{W}'$, Ω_1, Ω_2 be given open subsets of $\mathbb{R}^N$, let $g : \Omega_1 \to \Omega_2$ be real-analytic, and let $f \in \mathcal{E}_{(\omega)}(\Omega_2)$. Then $f \circ g \in \mathcal{E}_{(\omega)}(\Omega_1)$. In particular, $\mathcal{E}_{(\omega)}(\Omega)$ contains all real-analytic functions on Ω.*

Let us now introduce the ω-ultradistributions of Beurling type with compact and arbitrary support:

Definition 2.30. Let $\omega \in \mathcal{W}'$ and $\Omega \subset \mathbb{R}^N$ an open set.

(1) The elements of $\mathcal{D}'_{(\omega)}(\Omega)$ are called *ω-ultradistributions of Beurling type.*
(2) For an ultradistribution $T \in \mathcal{D}'_{(\omega)}(\Omega)$ its support supp T is the set of all points such that for every neighbourhood U there is $\varphi \in \mathcal{D}_{(\omega)}(U)$ with $< T, \varphi > \neq 0$.

Analogously, we have ω-ultradistributions of Roumieu type $\mathcal{D}'_{\{\omega\}}(\Omega)$, for $\omega \in \mathcal{W}$, as in [16].

Remark 2.31. By Proposition 2.20, the definition of support of an ultradistribution T doesn't depend on the choice of the class $\mathcal{D}_{(\omega)}(\Omega)$, for $\omega \in \mathcal{W}'$, as long as it contains T. In particular, if T is a distribution $T \in \mathcal{D}'(\Omega)$, then the support defined above is the usual one.

As in [16, Prop. 5.3], the elements of $\mathcal{E}'_{(\omega)}(\Omega)$ can be identified with distributions in $\mathcal{D}'_{(\omega)}(\Omega)$ with compact support:

Proposition 2.32. *For $\omega \in \mathcal{W}'$, an ultradistribution $T \in \mathcal{D}'_{(\omega)}(\Omega)$ can be extended continuously to $\mathcal{E}_{(\omega)}(\Omega)$ iff* supp T *is a compact subset of Ω.*

Definition 2.33. Let $\omega \in \mathcal{W}'$ and $\Omega \subset \mathbb{R}^N$ be open. For $f \in \mathcal{E}_{(\omega)}(\Omega)$ and $T \in \mathcal{D}'_{(\omega)}(\Omega)$ we define $fT \in \mathcal{D}'_{(\omega)}(\Omega)$ by

$$\langle fT, \varphi\rangle = \langle T, f\varphi\rangle \qquad \forall\varphi \in \mathcal{D}_{(\omega)}(\Omega).$$

This makes $\mathcal{D}'_{(\omega)}(\Omega)$ an $\mathcal{E}_{(\omega)}(\Omega)-$module.

Definition 2.34. Let $\omega \in \mathcal{W}'$. For an ultradistribution $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^N)$, and for $f \in \mathcal{E}_{(\omega)}(\mathbb{R}^N)$ we define the convolution

$$T_\mu(f) := \mu * f : \mathbb{R}^N \to \mathbb{C},$$

by

$$\mu * f(x) = \langle \mu_y, f(x-y) \rangle .$$

As in [16, Prop. 6.3]:

Proposition 2.35. *For $\omega \in \mathcal{W}'$ the convolution map*

$$T_\mu : \mathcal{E}_{(\omega)}(\mathbb{R}^N) \to \mathcal{E}_{(\omega)}(\mathbb{R}^N)$$

is continuous.

Notation. For $z \in \mathbb{C}^N$ we set

$$f_z(x) = e^{-i\langle x,z\rangle}, \qquad x \in \mathbb{R}^N.$$

For each $\lambda > 0$ we have

$$\begin{aligned}
&\sup_{\alpha\in\mathbb{N}_0^N} \sup_{x\in\mathbb{R}^N} |f_z^{(\alpha)}(x)| e^{-\lambda\varphi^*\left(\frac{|\alpha|}{\lambda}\right)} \\
&\quad = \sup_{\alpha\in\mathbb{N}_0^N} \sup_{x\in\mathbb{R}^N} |z^\alpha||e^{-i\langle x,z\rangle}| e^{-\lambda\varphi^*\left(\frac{|\alpha|}{\lambda}\right)} \\
&\quad \le \sup_{x\in\mathbb{R}^N} e^{\langle x,\operatorname{Im} z\rangle} \cdot \exp\left\{ \sup_{\alpha\in\mathbb{N}_0^N} \left(|\alpha| \log|z| - \lambda\varphi^*\left(\frac{|\alpha|}{\lambda}\right)\right)\right\} \\
&\quad = e^{H_K(\operatorname{Im} z)} \exp\left\{ \sup_{\alpha\in\mathbb{N}_0^N} \lambda \left(\frac{|\alpha|}{\lambda} \log|z| - \varphi^*\left(\frac{|\alpha|}{\lambda}\right)\right)\right\} \\
&\quad = e^{H_K(\operatorname{Im} z)} \exp\left\{\lambda\varphi^{**}(\log|z|)\right\} \\
&\quad = e^{H_K(\operatorname{Im} z)+\lambda\omega(z)}.
\end{aligned} \tag{2.13}$$

Thus $f_z \in \mathcal{E}_{(\omega)}(\Omega)$ for all ω and Ω.

Definition 2.36. Let $\omega \in \mathcal{W}'$. The *Fourier–Laplace transform* $\hat{\mu}$ of $\mu \in \mathcal{E}'_{(\omega)}(\Omega)$ is defined by

$$\hat{\mu} : z \mapsto \langle \mu, f_z \rangle .$$

Note that for $\varphi \in \mathcal{D}_{(\omega)}(\Omega)$:

$$\begin{aligned}
\widehat{\mu * \varphi}(z) &= \int_{\mathbb{R}^N} \mu * \varphi(t) f_z(t) dt = \int_{\mathbb{R}^N} \langle \mu_y, f_z(t)\varphi(t-y) \rangle dt \\
&= \int_{\mathbb{R}^N} \langle \mu_y, f_z(s+y)\varphi(s) \rangle ds = \int_{\mathbb{R}^N} \langle \mu, f_z \rangle \varphi(s) f_z(s) ds \\
&= \langle \mu, f_z \rangle \int_{\mathbb{R}^N} \varphi(s) f_z(s) ds = \hat{\mu}(z)\hat{\varphi}(z).
\end{aligned}$$

Theorem 2.37. *(Paley–Wiener theorem for ω-ultradistributions of Beurling type) Let $\omega \in \mathcal{W}'$ and $K \subset \mathbb{R}^N$ compact and convex. If $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^N)$ with* supp $\mu \subset K$ *then $\hat{\mu}$ is entire and there exist $C, \lambda > 0$ such that*

$$\left|\hat{\mu}(z)\right| \leq Ce^{H_K(\operatorname{Im} z)+\lambda\omega(z)} \qquad \forall z \in \mathbb{C}^N. \tag{2.14}$$

This holds, in particular, for K equal to the convex hull of supp μ. *Moreover,*

$$\langle \mu, \varphi \rangle = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \hat{\mu}(-t)\hat{\varphi}(t)dt \qquad \forall \varphi \in \mathcal{D}_{(\omega)}(\mathbb{R}^N). \tag{2.15}$$

Conversely, if g is an entire function on $\mathbb{C}^N$ that satisfies (2.14), i.e.

$$|g(z)| \leq Ce^{H_K(\operatorname{Im} z)+\lambda\omega(z)} \qquad \forall z \in \mathbb{C}^N,$$

for some $C, \lambda > 0$, then there exists $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^N)$ such that $\hat{\mu} = g$ and supp $\mu \subset$ *K.*

Proof. Let us first prove that if $f \in \mathcal{E}_{(\omega)}(\mathbb{R}^N)$ with $f|_K \equiv 0$ then $\langle \mu, f \rangle = 0$.

To this aim we assume, without loss of generality, that $0 \in \mathring{K}$ and define

$$f_t(x) := f(tx), \qquad 0 < t < 1.$$

Then $\langle \mu, f_t \rangle = 0$. Let us to prove that

$$\lim_{t \to 1^-} \langle \mu, f_t \rangle = \langle \mu, f \rangle. \tag{2.16}$$

We have that $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^N)$, so μ is a linear and continuous function on $\mathcal{E}_{(\omega)}(\mathbb{R}^N)$ and to prove (2.16) it's sufficient to prove that $f_t \to f$ in $\mathcal{E}_{(\omega)}(\mathbb{R}^N)$. Therefore, fix $\tilde{K} \subset \mathbb{R}^N$ compact, $m \in \mathbb{N}$ and prove that

$$\sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in \tilde{K}} |D^\alpha f_t(x) - D^\alpha f(x)| e^{-m\varphi^*\left(\frac{|\alpha|}{m}\right)} \to 0. \tag{2.17}$$

Indeed,

$$\begin{aligned} |D^\alpha f_t(x) - D^\alpha f(x)| = |D^\alpha f(tx) - D^\alpha f(x)| &= \left|t^\alpha (D^\alpha f)(tx) - D^\alpha f(x)\right| \\ &\leq \left|t^\alpha (D^\alpha f)(tx) - t^\alpha D^\alpha f(x)\right| + \left|t^\alpha D^\alpha f(x) - D^\alpha f(x)\right| \\ &= t^\alpha \left|(D^\alpha f)(tx) - D^\alpha f(x)\right| + (1 - t^\alpha) \left|D^\alpha f(x)\right|, \end{aligned}$$

so, for $0 < t < 1$, we have that

$$\begin{aligned} &\sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in \tilde{K}} |D^\alpha f_t(x) - D^\alpha f(x)| e^{-m\varphi^*\left(\frac{|\alpha|}{m}\right)} \\ &\quad \leq \sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in \tilde{K}} |(D^\alpha f)(tx) - D^\alpha f(x)| e^{-m\varphi^*\left(\frac{|\alpha|}{m}\right)} \\ &\qquad + (1 - t^\alpha) \sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in \tilde{K}} |D^\alpha f(x)| e^{-m\varphi^*\left(\frac{|\alpha|}{m}\right)}. \end{aligned} \tag{2.18}$$

We observe that $(1-t^\alpha)\to 0$ for $t\to 1^-$ and $\sup_{\alpha\in\mathbb{N}_0^N}\sup_{x\in\tilde{K}}|D^\alpha f(x)| e^{-m\varphi^*\left(\frac{|\alpha|}{m}\right)} = C_{\tilde{K}} < \infty$ because $f\in\mathcal{E}_{(\omega)}(\mathbb{R}^N)$.

To estimate also the first addend of (2.18) let us remark that it's not restrictive to assume $0\in\tilde{K}$, since we can enlarge $\tilde{K}$. Therefore, denoting by $\mathrm{ch}(\tilde{K})$ the convex hull of $\tilde{K}$, by the Lagrange Theorem we have that there exists $\xi\in\mathrm{ch}(\tilde{K})$ on the segment of extremes x and tx, such that

$$\begin{aligned}
&\sup_{\alpha\in\mathbb{N}_0^N}\sup_{x\in\tilde{K}}|(D^\alpha f)(tx)-D^\alpha f(x)|e^{-m\varphi^*\left(\frac{|\alpha|}{m}\right)}\\
&\quad=\sup_{\alpha\in\mathbb{N}_0^N}\sup_{x\in\tilde{K}}\left|\left\langle\nabla(D^\alpha f)(\xi),tx-x\right\rangle\right|e^{-m\varphi^*\left(\frac{|\alpha|}{m}\right)}\\
&\quad\leq\sup_{\alpha\in\mathbb{N}_0^N}\left\{\sup_{\xi\in\mathrm{ch}(\tilde{K})}\|\nabla D^\alpha f(\xi)\|\cdot(1-t)\cdot\sup_{x\in\tilde{K}}\|x\|\cdot e^{-m\varphi^*\left(\frac{|\alpha|}{m}\right)}\right\}\\
&\quad\leq C(1-t)\sup_{\alpha\in\mathbb{N}_0^N}\sup_{\xi\in\mathrm{ch}(\tilde{K})}\|\nabla D^\alpha f(\xi)\|e^{-m\varphi^*\left(\frac{|\alpha|}{m}\right)},
\end{aligned}$$

for some $C>0$.

However,

$$\|\nabla D^\alpha f(\xi)\|\leq\sum_{j=1}^N\left|D_jD^\alpha f(\xi)\right|\leq\sum_{j=1}^N\sup_{|\beta|=1}\left|D^\beta D^\alpha f(\xi)\right|=N\sup_{|\beta|=1}\left|D^{\alpha+\beta}f(\xi)\right|,$$

so

$$\begin{aligned}
&\sup_{\alpha\in\mathbb{N}_0^N}\sup_{x\in\tilde{K}}|(D^\alpha f)(tx)-D^\alpha f(x)|e^{-m\varphi^*\left(\frac{|\alpha|}{m}\right)}\\
&\quad\leq CN(1-t)\sup_{\tilde{\alpha}\in\mathbb{N}^N}\sup_{\xi\in\mathrm{ch}(\tilde{K})}\left|D^{\tilde{\alpha}}f(\xi)\right|e^{-m\varphi^*\left(\frac{|\tilde{\alpha}|}{m}\right)},
\end{aligned}$$

where $\sup_{\tilde{\alpha}\in\mathbb{N}^N}\sup_{\xi\in\mathrm{ch}(\tilde{K})}\left|D^{\tilde{\alpha}}f(\xi)\right|e^{-m\varphi^*\left(\frac{|\tilde{\alpha}|}{m}\right)}<\infty$ by definition of $f\in\mathcal{E}_{(\omega)}(\mathbb{R}^N)$. Then, from (2.18), we have obtained (2.17), i.e.

$$f_t\to f\qquad\text{in }\mathcal{E}_{(\omega)}(\mathbb{R}^N).$$

Therefore (2.16) holds true.

Since $\langle\mu,f_t\rangle=0$ for all $t\in(0,1)$, then $\langle\mu,f\rangle=0$. This can be done for all $f\in\mathcal{E}_{(\omega)}(\mathbb{R}^N)$ with $f|_K=0$, and hence there exists $C,\lambda>0$ such that

$$|\langle\mu,f\rangle|\leq Cp_{K,\lambda}(f)\qquad\forall f\in\mathcal{E}_{(\omega)}(\mathbb{R}^N).\tag{2.19}$$

For $f_z(x)=e^{-i<x,z>}$, we observe that

$$p_{K,\lambda}(f_z)=\sup_{\alpha\in\mathbb{N}_0^N}\sup_{x\in\mathbb{R}^N}|f_z^{(\alpha)}(x)|e^{-\lambda\varphi^*\left(\frac{|\alpha|}{\lambda}\right)}\leq e^{H_K(\mathrm{Im}\,z)+\lambda\omega(z)}$$

by (2.13).

Substituting in (2.19) with $f = f_z$ and remembering that $\hat{\mu}(z) = \langle \mu, f_z \rangle$, we obtain (2.14); moreover $\hat{\mu}$ is entire because f_z is entire (cf. also [16, Prop. 7.2]).

To prove (2.15) we observe that if $\varphi \in \mathcal{D}_{(\omega)}(\mathbb{R}^N)$, then

$$\begin{aligned}\langle \mu, \varphi \rangle = \mu * \check{\varphi}(0) &= \mathcal{F}^{-1}\left(\widehat{\mu * \check{\varphi}}\right)(0) \\ &= \mathcal{F}^{-1}\left(\hat{\mu} \cdot \hat{\check{\varphi}}\right)(0) \\ &= \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \hat{\mu}(t)\hat{\varphi}(-t)dt \\ &= \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \hat{\mu}(-t)\hat{\varphi}(t)dt.\end{aligned}$$

Conversely, let g be entire on $\mathbb{C}^N$ statisfying (2.14) and define μ by

$$\langle \mu, f \rangle = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} g(-t)\hat{f}(t)dt, \qquad f \in \mathcal{D}_{(\omega)}(\mathbb{R}^N).$$

Then $\mu \in \mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ with $\operatorname{supp} \mu \subset K$, hence $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^N)$ by Proposition 2.32, and $\hat{\mu} = g$ by (2.15) (see also [16, Prop. 7.3]). □

Proposition 2.38. *Let $\Omega \subset \mathbb{R}^N$ be open and $\omega \in \mathcal{W}$. For every $\mu \in \mathcal{D}'_{\{\omega\}}(\Omega)$ there is a weight function $\sigma \in \mathcal{W}$ with $\sigma = o(\omega)$ such that $\mu \in \mathcal{D}'_{(\sigma)}(\Omega) \subset \mathcal{D}'_{\{\omega\}}(\Omega)$. The analogous statement holds for $\mathcal{E}'_{\{\omega\}}(\Omega)$.*

Proof. See [16], Proposition 7.6. □

Let us close this section with the following result on tensor product spaces in the Beurling setting, which is the analogous to [16, Thm. 8.1], for $\omega \in \mathcal{W}'$:

Theorem 2.39. *Let $\omega \in \mathcal{W}'$. Let $K_j \subset \mathbb{R}^{N_j}$ be compact and $\Omega_j \subseteq \mathbb{R}^{N_j}$ be open, for $j = 1, 2$. Then we have the following isomorphisms:*

$$\begin{aligned}\mathcal{D}_{(\omega)}(K_1)\widehat{\otimes}\mathcal{D}_{(\omega)}(K_2) &\simeq \mathcal{D}_{(\omega)}(K_1 \times K_2) \\ \mathcal{E}_{(\omega)}(\Omega_1)\widehat{\otimes}\mathcal{E}_{(\omega)}(\Omega_2) &\simeq \mathcal{E}_{(\omega)}(\Omega_1 \times \Omega_2).\end{aligned}$$

3. The Cauchy problem for overdetermined systems

In this section we consider the Cauchy problem for overdetermined systems of linear partial differential operators with constant coefficients in the class of ω-ultradifferentiable functions of Beurling type defined in the previous section.

To bypass the question of formal coherence of the initial data, that could be especially intricate in the overdetermined case (cf. [2,3,26]), we consider initial data in the Whitney sense, in the spirit of [10,12,26].

Let F be a locally closed subsets of $\mathbb{R}^N$, so that there exists an open subset Ω of $\mathbb{R}^N$ with $F \subset \Omega$ and $\bar{F} \cap \Omega = F$.

For $\omega \in \mathcal{W}'$ we denote by $\mathcal{I}^{(\omega)}(F, \Omega)$ the subspace of functions in $\mathcal{E}_{(\omega)}(\Omega)$ which vanish of infinite order on F:

$$\mathcal{I}^{(\omega)}(F, \Omega) = \left\{ f \in \mathcal{E}_{(\omega)}(\Omega) : D^\alpha f \equiv 0 \text{ on } F, \ \forall \alpha \in \mathbb{N}_0^N \right\}.$$

Definition 3.1. Let $\omega \in \mathcal{W}'$. We define the space $W_F^{(\omega)}$ of *Whitney ω-ultradifferentiable functions* on F by the exact sequence

$$0 \longrightarrow \mathcal{I}^{(\omega)}(F, \Omega) \longrightarrow \mathcal{E}_{(\omega)}(\Omega) \longrightarrow W_F^{(\omega)} \longrightarrow 0; \tag{3.1}$$

i.e.

$$W_F^{(\omega)} \simeq \mathcal{E}_{(\omega)}(\Omega) / \mathcal{I}^{(\omega)}(F, \Omega).$$

We endow $W_F^{(\omega)}$ with the quotient topology.

Denoting by $\mathcal{P} = \mathbb{C}[\zeta_1, \ldots, \zeta_N]$ the ring of complex polynomials in N indeterminates, we consider $W_F^{(\omega)}$ as a unitary left and right $\mathcal{P}-$module by the action of $p(\zeta)$ on $u \in W_F^{(\omega)}$ described by

$$p(\zeta)u = up(\zeta) = p(D)u, \tag{3.2}$$

by the formal substitution $\zeta_j \leftrightarrow D_j = \frac{1}{i}\partial_j$.

Since $\mathcal{I}^{(\omega)}(F, \Omega)$ is a closed ideal and a differential $\mathcal{P}$-submodule of $\mathcal{E}_{(\omega)}(\Omega)$, the space $W_F^{(\omega)}$ is Fréchet–Schwartz, a Fréchet algebra and a differential $\mathcal{P}$-submodule. These topological and algebraic structures are independent from the choice of the open neighbourhood Ω of F with $\bar{F} \cap \Omega = F$ (cf. also [12] and [13]).

Let us denote by $\mathrm{Aff}(F)$ the affine span of $F \subset \mathbb{R}^N$ and let $\mathrm{Int}_{\mathrm{Aff}}(F)$ be the interior of F as a subspace of $\mathrm{Aff}(F)$.

Lemma 3.2. *Let $\omega \in \mathcal{W}'$ and F a closed convex subset of $\mathbb{R}^N$. Then the family of seminorms*

$$p_{K,\lambda}(f) = \sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in K} |D^\alpha f(x)| e^{-\lambda \varphi^* \left(\frac{|\alpha|}{\lambda} \right)}$$

for $\lambda > 0$ and $K \subset\subset F$, defines the topology of $W_F^{(\omega)}$.

Proof. Since F is closed and convex, then $\overline{\mathrm{Int}_{\mathrm{Aff}}(F)} = F$ and F satisfies Whitney's property (P): for all $K \subset\subset F$ there exists $C_K > 0$ such that every pair x, y of points of K can be joined by a rectifiable curve in F of length not exceeding $C_K |x - y|$.

The thesis then follows from [13]. □

As a consequence, we have the following:

Lemma 3.3. *Let $\omega \in \mathcal{W}'$ and F a convex and closed subset of $\mathbb{R}^N$. Then*

$$\left(W_F^{(\omega)} \right)' \simeq \mathcal{E}'_{(\omega)}(F).$$

Proof. The thesis can be proved by dualizing the exact sequence (3.1), as in [25] (cf. also [22]), or it can be deduced from [23, Prop. 3.6] and the Paley–Wiener Theorem 2.37 (cf. also [12]). □

Let us now consider a pair K_1, K_2 of closed convex subsets of $\mathbb{R}^N$ with $K_1 \subsetneq K_2$.

Given an $a_1 \times a_0$ matrix $A_0(\zeta)$ with polynomial entries, by (3.2) we consider the corresponding operator $A_0(D)$. We want to solve, in the Whitney's sense, the Cauchy problem

$$\begin{cases} A_0(D)u = f \\ u|_{K_1} \equiv 0, \end{cases} \tag{3.3}$$

where $u|_{K_1} \equiv 0$ means that u vanishes with all its derivatives on K_1.

Let us remark that if ${}^tQ(\zeta) : \mathcal{P}^{a_1} \to \mathcal{P}$ is such that

$${}^tA_0(\zeta){}^tQ(\zeta) \equiv 0, \tag{3.4}$$

then, in order to solve the Cauchy problem (3.3), f must satisfy the integrability condition

$$Q(D)f = 0, \tag{3.5}$$

because of $Q(D)f = Q(D)A_0(D)u = 0$.

Since $\mathcal{P}$ is a Noetherian ring, the collection of all vectors ${}^tQ(\zeta)$ satisfying (3.4) form a finitely generated $\mathcal{P}-$module. So we can insert the map ${}^tA_0(\zeta) : \mathcal{P}^{a_1} \to \mathcal{P}^{a_0}$ into a Hilbert resolution:

$$\begin{aligned} 0 \longrightarrow \mathcal{P}^{a_d} \xrightarrow{{}^tA_{d-1}(\zeta)} \mathcal{P}^{a_{d-1}} \longrightarrow \cdots \longrightarrow \\ \mathcal{P}^{a_2} \xrightarrow{{}^tA_1(\zeta)} \mathcal{P}^{a_1} \xrightarrow{{}^tA_0(\zeta)} \mathcal{P}^{a_0} \longrightarrow \mathcal{M} \longrightarrow 0, \end{aligned} \tag{3.6}$$

where $\mathcal{M} = \operatorname{coker}{}^tA_0(\zeta) = \mathcal{P}^{a_0} \big/ {}^tA_0(\zeta)\mathcal{P}^{a_1}$ and the matrix ${}^tA_1(\zeta)$ is obtained from a basis of the integrability conditions (3.4). The sequence is exact, i.e. $\operatorname{Im}{}^tA_j = \operatorname{Ker}{}^tA_{j+1}$.

Therefore a necessary condition to solve (3.3), is that f satisfies the following integrability condition:

$$A_1(D)f = 0. \tag{3.7}$$

Moreover, every necessary condition for the solvability of (3.3), which can be expressed in terms of linear partial differential equations, is a consequence of (3.7) (cf. [26]).

Definition 3.4. If $A_1(\zeta) \not\equiv 0$ then the Cauchy problem (3.3) is called *overdetermined* and, to solve it, the condition (3.7) has to be satisfied.

Let us remark that if u solves (3.3), then also f must vanish with all its derivatives on K_1, so that we look for solutions $u \in \left(W^{(\omega)}_{K_2}\right)^{a_0}$ of (3.3) when f satisfies

$$\begin{cases} f \in \left(W^{(\omega)}_{K_2}\right)^{a_1} \\ A_1(D)f = 0 \\ f|_{K_1} \equiv 0. \end{cases} \tag{3.8}$$

Remark 3.5. By Whitney's extension theorem it's not restrictive to consider zero Cauchy data (see [5,13,22,23]).

Let us denote by $\mathcal{I}^{(\omega)}(K_{1,}K_2)$ the space of Whitney ω-ultradifferentiable fuctions on K_2 which vanish of infinite order on K_1 :

$$\mathcal{I}^{(\omega)}(K_{1,}K_2) = \left\{ f \in W^{(\omega)}_{K_2} : D^\alpha \ f|_{K_1} \equiv 0 \ \forall \alpha \in \mathbb{N}_0^N \right\}.$$

The Cauchy problem (3.3)–(3.8) is then equivalent to:

$$\begin{cases} \text{given } f \in \left(\mathcal{I}^{(\omega)}(K_1, K_2)\right)^{a_1} \text{ such that } A_1(D)f = 0 \\ \text{find } u \in \left(\mathcal{I}^{(\omega)}(K_1, K_2)\right)^{a_0} \text{ such that } A_0(D)u = f. \end{cases} \tag{3.9}$$

Remark 3.6. By the isomorphisms

$$\operatorname{Ext}^0_{\mathcal{P}}\left(\mathcal{M}, \mathcal{I}^{(\omega)}(K_1, K_2)\right) \simeq \operatorname{Ker} A_0(D) = \{u \in \mathcal{I}^{(\omega)}(K_1, K_2)^{a_0} : \ A_0(D)u = 0\}$$
$$\operatorname{Ext}^1_{\mathcal{P}}\left(\mathcal{M}, \mathcal{I}^{(\omega)}(K_1, K_2)\right) \simeq \frac{\operatorname{Ker} A_1(D)}{\operatorname{Im} A_0(D)},$$

we have:

(1) *uniqueness* of solutions of the Cauchy problem (3.9) is equivalent to the condition

$$\operatorname{Ext}^0_{\mathcal{P}}(\mathcal{M}, \mathcal{I}^{(\omega)}(K_1, K_2)) = 0;$$

(2) *existence* of solutions of (3.9), is equivalent to the condition

$$\operatorname{Ext}^1_{\mathcal{P}}(\mathcal{M}, \mathcal{I}^{(\omega)}(K_1, K_2)) = 0;$$

(3) *existence and uniqueness* of a solution of (3.9), is equivalent to the condition

$$\operatorname{Ext}^0_{\mathcal{P}}(\mathcal{M}, \mathcal{I}^{(\omega)}(K_1, K_2)) = \operatorname{Ext}^1_{\mathcal{P}}(\mathcal{M}, \mathcal{I}^{(\omega)}(K_1, K_2)) = 0.$$

Remark 3.7. The above Remark 3.6 highlights the algebraic invariance of the problem: uniqueness and/or existence of solutions of the Cauchy problem (3.9) depend only on the module $\mathcal{M}$ and not on it's presentation by a particular matrix ${}^tA_0(D)$.

Note also that we have the short exact sequence

$$0 \longrightarrow \mathcal{I}^{(\omega)}(K_1, K_2) \longrightarrow W^{(\omega)}_{K_2} \longrightarrow W^{(\omega)}_{K_1} \longrightarrow 0,$$

that implies the long exact sequence

$$\begin{aligned} 0 &\longrightarrow \operatorname{Ext}^0_{\mathcal{P}}\left(\mathcal{M}, \mathcal{I}^{(\omega)}(K_1, K_2)\right) \longrightarrow \operatorname{Ext}^0_{\mathcal{P}}\left(\mathcal{M}, W^{(\omega)}_{K_2}\right) \longrightarrow \operatorname{Ext}^0_{\mathcal{P}}\left(\mathcal{M}, W^{(\omega)}_{K_1}\right) \longrightarrow \\ &\longrightarrow \operatorname{Ext}^1_{\mathcal{P}}\left(\mathcal{M}, \mathcal{I}^{(\omega)}(K_1, K_2)\right) \longrightarrow \operatorname{Ext}^1_{\mathcal{P}}\left(\mathcal{M}, W^{(\omega)}_{K_2}\right) \longrightarrow \operatorname{Ext}^1_{\mathcal{P}}\left(\mathcal{M}, W^{(\omega)}_{K_1}\right) \longrightarrow \\ &\longrightarrow \operatorname{Ext}^2_{\mathcal{P}}\left(\mathcal{M}, \mathcal{I}^{(\omega)}(K_1, K_2)\right) \longrightarrow \cdots . \end{aligned} \tag{3.10}$$

As in [25] (cf. also [5]) we have that $W^{(\omega)}_{K_i}$ are injective $\mathcal{P}-$modules, i.e. $\mathrm{Ext}^j_{\mathcal{P}}(\mathcal{M}, W^{(\omega)}_{K_i}) = 0$ for $i = 1, 2$ and for all $j \geq 1$.

Therefore, the complex (3.10) reduces to:

$$0 \longrightarrow \mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{M}, \mathcal{I}^{(\omega)}(K_1, K_2)\right) \longrightarrow \mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{M}, W^{(\omega)}_{K_2}\right) \longrightarrow$$
$$\mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{M}, W^{(\omega)}_{K_1}\right) \longrightarrow \mathrm{Ext}^1_{\mathcal{P}}\left(\mathcal{M}, \mathcal{I}^{(\omega)}(K_1, K_2)\right) \longrightarrow 0.$$

In particular

$$\mathrm{Ext}^j_{\mathcal{P}}(\mathcal{M}, \mathcal{I}^{(\omega)}(K_1, K_2)) = 0 \qquad \forall j > 1. \tag{3.11}$$

Remark 3.8. From Remark 3.6 and the above considerations, it follows that uniqueness and/or existence of solutions of the Cauchy problem (3.9) is related to injectivity and/or surjectivity of the homomorphism

$$\mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{M}, W^{(\omega)}_{K_2}\right) \longrightarrow \mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{M}, W^{(\omega)}_{K_1}\right). \tag{3.12}$$

The injectivity of (3.12) is equivalent to the fact that the dual homomorphism

$$\mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{M}, W^{(\omega)}_{K_1}\right)' \longrightarrow \mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{M}, W^{(\omega)}_{K_2}\right)' \tag{3.13}$$

has a dense image.

Moreover, surjectivity is equivalent to have a dense and closed image. But (3.12) has a closed image if and only if (3.13) has a closed image (cf. [19], Ch. IV, § 2, n. 4, Thm. 3), so that the surjectivity of (3.12) is equivalent to the fact that the dual homorphism (3.13) is injective and has a closed image.

By Remarks 3.6 and 3.8, and [26, Prop. 1.1–1.2], we have that:

Proposition 3.9. *Let $\omega \in \mathcal{W}'$ and $K_1,\ K_2$ closed convex subsets of $\mathbb{R}^N$ with $K_1 \subsetneq K_2$, for $j = 1, 2$. Let $\mathcal{M}$ be a unitary $\mathcal{P}-$module of finite type and denote by* $\mathrm{Ass}(\mathcal{M})$ *the set of all prime ideals associated to $\mathcal{M}$.*

Then the following statements are equivalent:

(1) *The Cauchy problem* (3.9) *admits* ***at most one*** *solution;*
(2) $\mathrm{Ext}^0_{\mathcal{P}}(\mathcal{M}, \mathcal{I}^{(\omega)}(K_1, K_2)) = 0$;
(3) $\mathrm{Ext}^0_{\mathcal{P}}(\mathcal{P}/\wp, \mathcal{I}^{(\omega)}(K_1, K_2)) = 0$ *for all* $\wp \in \mathrm{Ass}(\mathcal{M})$*;*
(4) *The homomorphisms*

$$\mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W^{(\omega)}_{K_2}\right) \to \mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W^{(\omega)}_{K_1}\right)$$

are injective for all $\wp \in \mathrm{Ass}(\mathcal{M})$*;*

(5) *The homomorphisms*

$$\mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W^{(\omega)}_{K_1}\right)' \to \mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W^{(\omega)}_{K_2}\right)'$$

have a dense image for all $\wp \in \mathrm{Ass}(\mathcal{M})$.

Proposition 3.10. *Let $\omega \in \mathcal{W}'$, $\mathcal{M}$ a unitary $\mathcal{P}-$module of finite type and K_1, K_2 closed convex subsets of $\mathbb{R}^N$ with $K_1 \subsetneq K_2$, for $j = 1, 2$. Then the following statements are equivalent:*

(1) *The Cauchy problem* (3.9) *admits* ***at least*** *a solution;*
(2) $\mathrm{Ext}^1_{\mathcal{P}}(\mathcal{M}, \mathcal{I}^{(\omega)}(K_1, K_2)) = 0$;
(3) $\mathrm{Ext}^1_{\mathcal{P}}(\mathcal{P}/\wp, \mathcal{I}^{(\omega)}(K_1, K_2)) = 0$ *for all* $\wp \in \mathrm{Ass}(\mathcal{M})$;
(4) *The homomorphisms*

$$\mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W^{(\omega)}_{K_2}\right) \to \mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W^{(\omega)}_{K_1}\right) \tag{3.14}$$

are surjective for all $\wp \in \mathrm{Ass}(\mathcal{M})$;
(5) *The homomorphisms*

$$\mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W^{(\omega)}_{K_1}\right)' \to \mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W^{(\omega)}_{K_2}\right)' \tag{3.15}$$

are injective and have a closed image, for all $\wp \in \mathrm{Ass}(\mathcal{M})$.

Proposition 3.11. *Let $\omega \in \mathcal{W}'$, $\mathcal{M}$ a unitary $\mathcal{P}-$module of finite type and K_1, K_2 closed convex subsets of $\mathbb{R}^N$ with $K_1 \subsetneq K_2$, for $j = 1, 2$. Then the following statements are equivalent:*

(1) *The Cauchy problem* (3.9) *admits* ***one and only one*** *solution;*
(2) $\mathrm{Ext}^0_{\mathcal{P}}(\mathcal{M}, \mathcal{I}^{(\omega)}(K_1, K_2)) = \mathrm{Ext}^1_{\mathcal{P}}(\mathcal{M}, \mathcal{I}^{(\omega)}(K_1, K_2)) = 0$;
(3) $\mathrm{Ext}^0_{\mathcal{P}}(\mathcal{P}/\wp, \mathcal{I}^{(\omega)}(K_1, K_2)) = \mathrm{Ext}^1_{\mathcal{P}}(\mathcal{P}/\wp, \mathcal{I}^{(\omega)}(K_1, K_2)) = 0$ *for all* $\wp \in \mathrm{Ass}(\mathcal{M})$;
(4) *The homomorphisms*

$$\mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W^{(\omega)}_{K_2}\right) \to \mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W^{(\omega)}_{K_1}\right)$$

are isomorphisms for all $\wp \in \mathrm{Ass}(\mathcal{M})$;
(5) *The homomorphisms*

$$\mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W^{(\omega)}_{K_1}\right)' \to \mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W^{(\omega)}_{K_2}\right)'$$

are isomorphisms for all $\wp \in \mathrm{Ass}(\mathcal{M})$.

The overdetermined Cauchy problem (3.9) is thus reduced to the study of the dual homomorphism

$$\mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W^{(\omega)}_{K_1}\right)' \to \mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W^{(\omega)}_{K_2}\right)', \qquad \wp \in \mathrm{Ass}(\mathcal{M}).$$

Let us start with some preliminary results. We first need the following:

Definition 3.12. For a prime ideal $\wp$ of $\mathcal{P}$ we define the *complex characteristic variety of* $\mathcal{P}/\wp$ by:

$$V = V(\wp) := \{\zeta \in \mathbb{C}^N : p(-\zeta) = 0 \ \forall p \in \wp\}. \tag{3.16}$$

This is an affine algebraic variety.

Lemma 3.13. *Let $\omega \in \mathcal{W}'$, $\wp$ a prime ideal of $\mathcal{P}$ and $K \subset \mathbb{R}^N$ a convex and closed subset of $\mathbb{R}^N$. Then we have the following isomorphism:*

$$\operatorname{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W_K^{(\omega)}\right)' \simeq \mathcal{E}'_{(\omega)}(K)\big/\wp(D)\otimes\mathcal{E}'_{(\omega)}(K) \tag{3.17}$$

with

$$\wp(D)\otimes\mathcal{E}'_{(\omega)}(K) := \left\{\sum_{h=1}^{r} p_h(D)T_h \,:\; T_h \in \mathcal{E}'_{(\omega)}(K)\right\},$$

where $p_1(\zeta), \dots, p_r(\zeta)$ are generators of $\wp$.

Proof. For any closed subspace F of a Fréchet space E, the dual F' of F is isomorphic (cf. [24, Prop. 6.14]) to:

$$F' \simeq E'/F^0,$$

where F^0 is the annihilator of F, defined by

$$F^0 := \left\{T \in E' \,:\, T(f) = 0, \;\; \forall f \in F\right\}.$$

Then, since $\operatorname{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W_K^{(\omega)}\right)$ is a closed subspace of the Fréchet space $W_K^{(\omega)}$, we have

$$\operatorname{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W_K^{(\omega)}\right)' \simeq \left(W_K^{(\omega)}\right)'\Big/\left(\operatorname{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W_K^{(\omega)}\right)\right)^0,$$

and, by Lemma 3.3,

$$\operatorname{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W_K^{(\omega)}\right)' \simeq \mathcal{E}'_{(\omega)}(K)\Big/\left(\operatorname{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W_K^{(\omega)}\right)\right)^0, \tag{3.18}$$

with

$$\left(\operatorname{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W_K^{(\omega)}\right)\right)^0 = \left\{T \in \mathcal{E}'_{(\omega)}(K) : T(u) = 0, \;\forall u \in \operatorname{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W_K^{(\omega)}\right)\right\}.$$

Let us now remark that the affine algebraic variety $V(\wp)$ associated to $\wp$, as defined in (3.16), can be written as:

$$V = V(\wp) = \{\zeta \in \mathbb{C}^N \,:\; p_h(-\zeta) = 0, \; \forall h = 1, \dots, r\}.$$

Then

$$p_h(D_x)e^{-i<x,\zeta>} = p_h(-\zeta)e^{-i<x,\zeta>} = 0 \qquad \forall \zeta \in V(\wp).$$

But

$$\begin{aligned}\operatorname{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W_K^{(\omega)}\right) &= \operatorname{Ker} A_0(D) \\ &= \left\{u \in W_K^{(\omega)} \,:\; p_h(D)u = 0 \;\; \forall h = 1, \dots, r\right\},\end{aligned}$$

so that

$$e^{-i<\cdot,\zeta>} \in \mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W_K^{(\omega)}\right) \Leftrightarrow \zeta \in V(\wp). \tag{3.19}$$

Therefore the Fourier–Laplace transform $\hat{T}(\zeta)$ of an element $T \in \left(\mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W_K^{(\omega)}\right)\right)^0$ is an entire function which satisfies:

$$\hat{T}(\zeta) = \left\langle T, e^{-i<\cdot,\zeta>}\right\rangle = 0 \qquad \forall \zeta \in V(\wp).$$

By the Nullstellensatz (see [27]), there exist entire functions $F_1(\zeta), \ldots, F_r(\zeta)$ such that

$$\hat{T}(\zeta) = \sum_{h=1}^{r} p_h(-\zeta)F_h(\zeta) \qquad \forall \zeta \in \mathbb{C}^N.$$

By the Paley–Wiener Theorem 2.37, the Fourier–Laplace transform of a distribution $T \in \mathcal{E}'_{(\omega)}(K)$ is characterized by an estimate of the form

$$\left|\hat{T}(\zeta)\right| \le Ce^{H_{\sigma_T}(\mathrm{Im}\,\zeta)+\alpha\omega(\zeta)}, \tag{3.20}$$

for some $C > 0$, $\alpha \in \mathbb{N}$, where $\sigma_T \subset K$ is the convex hull of supp T.

If K is not compact, we choose $K_\alpha \subset \mathring{K}_{\alpha+1}$ compact and such that $K = \underset{\alpha}{\cup} K_\alpha$, while if K is compact, we choose $K_\alpha = K$ for all α.

Since $\sigma_T \subset K_\alpha$ for some α, then (3.20) implies that there exist $C > 0$, $\alpha \in \mathbb{N}$ such that

$$\left|\hat{T}(\zeta)\right| \le Ce^{H_{K_\alpha}(\mathrm{Im}\,\zeta)+\alpha\omega(\zeta)}. \tag{3.21}$$

Define

$$\psi_\alpha(\zeta) := H_{K_\alpha}(\mathrm{Im}\,\zeta) + \alpha\omega(\zeta);$$

since ω is plurisubharmonic by condition (δ) of Definition 2.1 (cf. [20, Thm. 1.6.7]), then $\psi_\alpha(\zeta)$ is plurisubharmonic in $\mathbb{C}^N$.

Moreover we have that for every $k_0 > 0$ there exists $k_1 > 0$ such that

$$|\psi_\alpha(\mathrm{Im}\,z + \mathrm{Im}\,\zeta) - \psi_\alpha(\mathrm{Im}\,\zeta)| \le k_1 \qquad \text{for } |z| \le k_0. \tag{3.22}$$

Indeed,

$$\begin{aligned} |\psi_\alpha(z+\zeta) - \psi_\alpha(\zeta)| &= \left|H_{K_\alpha}(\mathrm{Im}\,z + \mathrm{Im}\,\zeta) + \alpha\omega(z+\zeta) - H_{K_\alpha}(\mathrm{Im}\,\zeta) - \alpha\omega(\zeta)\right| \\ &\le \left|H_{K_\alpha}(\mathrm{Im}\,z + \mathrm{Im}\,\zeta) - H_{K_\alpha}(\mathrm{Im}\,\zeta)\right| + \alpha\left|\omega(z+\zeta) - \omega(\zeta)\right|. \end{aligned}$$

Now observe that

$$\begin{aligned} &H_{K_\alpha}(\mathrm{Im}\,z + \mathrm{Im}\,\zeta) - H_{K_\alpha}(\mathrm{Im}\,\zeta) \\ &\quad \le H_{K_\alpha}(\mathrm{Im}\,z) + H_{K_\alpha}(\mathrm{Im}\,\zeta) - H_{K_\alpha}(\mathrm{Im}\,\zeta) \le c \qquad |z| \le k_0, \end{aligned}$$

for some $c > 0$ and

$$H_{K_\alpha}(\operatorname{Im}\zeta) - H_{K_\alpha}(\operatorname{Im}z + \operatorname{Im}\zeta) \le H_{K_\alpha}(\operatorname{Im}\zeta) - \langle x, \operatorname{Im}z\rangle - \langle x, \operatorname{Im}\zeta\rangle \qquad \forall x \in K_\alpha. \tag{3.23}$$

Moreover, by definition of supremum, for all $\varepsilon > 0$ there exists $\bar{x} \in K_\alpha$ such that

$$\langle \bar{x}, \operatorname{Im}\zeta\rangle > H_{K_\alpha}(\operatorname{Im}\zeta) - \varepsilon.$$

So, choosing such $\bar{x}$ in (3.23) we have

$$H_{K_\alpha}(\operatorname{Im}\zeta) - H_{K_\alpha}(\operatorname{Im}z + \operatorname{Im}\zeta) \le \varepsilon + c', \qquad \text{if } |z| \le k_0,$$

for some $c' > 0$, hence there exists $k_1 > 0$ such that

$$\left| H_{K_\alpha}(\operatorname{Im}z + \operatorname{Im}\zeta) - H_{K_\alpha}(\operatorname{Im}\zeta)\right| \le k_1, \qquad |z| \le k_0.$$

Furthermore, by Lemma 2.5 we have that

$$\omega(z+\zeta) \le C(1 + \omega(z) + \omega(\zeta)),$$

for some $C > 0$ and hence for every $k_0 > 0$ there exists $k_1' > 0$ such that

$$|\omega(z+\zeta) - \omega(\zeta)| \le k_1', \qquad |z| \le k_0.$$

Therefore (3.22) is proved.

We can therefore apply the Ehrenpreis Fundamental Theorem (see [20, Thm. 7.7.13], and [5,18] for more details) and obtain that we can choose the entire functions F_h satisfying

$$|F_h(\zeta)| \le C' e^{H_{K_\alpha}(\operatorname{Im}\zeta) + \alpha\omega(\zeta) + m'\log(1+|\zeta|)},$$

for some $C' > 0$, $m' \in \mathbb{N}$.

By condition (γ)

$$m'\log(1+|\zeta|) \le \frac{m'}{b}\omega(\zeta) - \frac{m'a}{b},$$

so there exist C'', $C''' > 0$ and $\alpha' \in \mathbb{N}$ such that

$$|F_h(\zeta)| \le C'' e^{H_{K_\alpha}(\operatorname{Im}\zeta) + \alpha'\omega(\zeta)} \le C''' e^{H_{K_{\alpha''}}(\operatorname{Im}\zeta) + \alpha''\omega(\zeta)}$$

with $\alpha'' = \max\{\alpha, \alpha'\}$.

Hence, by the Paley–Wiener Theorem 2.37:

$$F_h = \widehat{T_h}$$

for some $T_h \in \mathcal{E}'_{(\omega)}(K)$.

We have thus proved that if $T \in \left(\operatorname{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W_K^{(\omega)}\right)\right)^0$, then

$$\hat{T}(\zeta) = \sum_{h=1}^{r} p_h(-\zeta)\widehat{T_h}(\zeta) = \sum_{h=1}^{r} \widehat{p_h(D)T_h}(\zeta), \qquad \text{with } T_h \in \mathcal{E}'_{(\omega)}(K).$$

This result implies that

$$T \in \wp(D) \otimes \mathcal{E}'_{(\omega)}(K),$$

and so, by (3.18),

$$\left(\mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W_K^{(\omega)}\right)\right)' \simeq \mathcal{E}'_{(\omega)}(K)/\wp(D) \otimes \mathcal{E}'_{(\omega)}(K).$$

□

Let us define $\mathcal{O}_{\psi_\alpha}(\mathbb{C}^N)$ as the space of holomorphic functions u on $\mathbb{C}^N$ which satisfy for some $C > 0$ and for all $\zeta \in \mathbb{C}^N$:

$$|u(\zeta)| \leq Ce^{\psi_\alpha(\zeta)} = Ce^{H_{K_\alpha}(\mathrm{Im}\,\zeta)+\alpha\omega(\zeta)}. \tag{3.24}$$

We can then consider the inductive limit

$$\mathcal{O}_\psi(\mathbb{C}^N) := \operatorname*{ind\,lim}_{\alpha\to\infty} \mathcal{O}_{\psi_\alpha}(\mathbb{C}^N).$$

From the Paley–Wiener Theorem 2.37, by Fourier–Laplace transform we have the following isomorphism:

$$\mathcal{E}'_{(\omega)}(K) \simeq \mathcal{O}_\psi(\mathbb{C}^N).$$

Therefore, from Lemma 3.13:

$$\mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W_K^{(\omega)}\right)' \simeq \mathcal{O}_\psi(\mathbb{C}^N)/\wp(D) \otimes \mathcal{O}_\psi(\mathbb{C}^N). \tag{3.25}$$

Let V be a reduced affine algebraic variety. Denote by $\mathcal{O}_{\psi_\alpha}(V)$ the space of holomorphic functions on V (i.e. complex valued continuous functions on V which are restrictions of entire functions on $\mathbb{C}^N$) that satisfy (3.24) for some $\alpha \in \mathbb{N}$, $C > 0$ and for all $\zeta \in V$. Consider then the inductive limit

$$\mathcal{O}_\psi(V) = \operatorname*{ind\,lim}_{\alpha\to\infty} \mathcal{O}_{\psi_\alpha}(V).$$

We have the following:

Proposition 3.14. *Let $\omega \in \mathcal{W}'$, $\wp$ a prime ideal of $\mathcal{P}$ with associated algebraic variety $V = V(\wp)$, and K a closed convex subset of $\mathbb{R}^N$. Then we have a natural isomorphism:*

$$\mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W_K^{(\omega)}\right)' \simeq \mathcal{O}_\psi(V).$$

Proof. By (3.25) we have to prove the following isomorphism:

$$\mathcal{O}_\psi(\mathbb{C}^N)/\wp(D) \otimes \mathcal{O}_\psi(\mathbb{C}^N) \simeq \mathcal{O}_\psi(V).$$

First of all we prove that the homomorphism

$$\mathcal{O}_\psi(\mathbb{C}^N)/\wp(D) \otimes \mathcal{O}_\psi(\mathbb{C}^N) \to \mathcal{O}_\psi(V) \tag{3.26}$$

is injective: if $f \in \mathcal{O}_\psi(\mathbb{C}^N)$ is zero on V, then by the Nullstellensatz there exist entire functions f_h on $\mathbb{C}^N$ such that

$$f(\zeta) = \sum_{h=1}^{r} p_h(-\zeta) f_h(\zeta) \qquad \forall \zeta \in \mathbb{C}^N.$$

Since f satisfies (3.24) by assumption, from the Ehrenpreis Fundamental Theorem [20, Thm. 7.7.13] (see also [5, 18] for more details), we can choose f_h satisfying (3.24) too, hence $f_h \in \mathcal{O}_\psi(\mathbb{C}^N)$ and this implies that $f \in \wp \otimes \mathcal{O}_\psi(\mathbb{C}^N)$. So we have obtained that f is the zero element of $\mathcal{O}_\psi(\mathbb{C}^N)\big/\wp(D) \otimes \mathcal{O}_\psi(\mathbb{C}^N)$, proving the injectivity of the homomorphism (3.26).

On the other hand, the homomorphism (3.26) is surjective: if $f \in \mathcal{O}_\psi(V)$, then $f \in \mathcal{O}(\mathbb{C}^N)$ and satisfies (3.24) for some $\alpha \in \mathbb{N}$, $C > 0$ and for all $\zeta \in V$. By the Ehrenpreis Fundamental Theorem [20, Thm. 7.7.13], there exist $g \in \mathcal{O}(\mathbb{C}^N)$, with $f = g$ on V, and two constants $C' > 0$ and $n \in \mathbb{N}$ such that

$$\sup_{\mathbb{C}^N} |g| e^{-\psi_\alpha - n\log(1+|\zeta|)} \leq C' \sup_V |f| e^{-\psi_\alpha}.$$

Since the right-hand side is finite because f satisfies (3.24) on V, we have that

$$|g(\zeta)| \leq C'' e^{\psi_\alpha(\zeta) + n\log(1+|\zeta|)} \leq C''' e^{\psi_{\alpha'}(\zeta)} \qquad \forall \zeta \in \mathbb{C}^N$$

for some C'', $C''' > 0$ and $\alpha' \in \mathbb{N}$. So $g \in \mathcal{O}_\psi(\mathbb{C}^N)$. □

Proposition 3.14 will be crucial in the study of the homomorphism (3.15) related to the study of existence and/or uniqueness of solutions of the Cauchy problem (3.9).

To this aim we take K_1 and K_2 closed and convex sets, with $K_1 \subsetneq K_2$.

Then we define, for $j = 1, 2$:

$$\psi_\alpha^j(\zeta) := H_{K_\alpha^j}(\operatorname{Im}\zeta) + \alpha\omega(\zeta)$$

for K_α^j compact convex set with $K_\alpha^j \subset \mathring{K}_{\alpha+1}^j$ and $\cup_\alpha K_\alpha^j = K_j$, for each $j = 1, 2$.

We consider the inductive limits

$$\mathcal{O}_{\psi^j}(\mathbb{C}^N) := \operatorname*{ind\,lim}_{\alpha\to+\infty} \mathcal{O}_{\psi_\alpha^j}(\mathbb{C}^N), \qquad j = 1, 2. \tag{3.27}$$

From the above considerations we have the following:

Remark 3.15. The study of the homomorphism (3.15) is reduced to the study of the homomorphism

$$\mathcal{O}_{\psi^1}(V) \to \mathcal{O}_{\psi^2}(V). \tag{3.28}$$

By Proposition 3.10 the existence of solutions of the Cauchy problem (3.9) is equivalent to the surjectivity of the homomorphism (3.14). But (3.14) has always a dense image, by the following:

Lemma 3.16. *Let $\omega \in \mathcal{W}'$, $\wp$ a prime ideal and K_1, K_2 closed convex subsets of $\mathbb{R}^N$ with $K_1 \subsetneq K_2$, for $j = 1, 2$. Then the homomorphism*

$$\mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W^{(\omega)}_{K_2}\right) \to \mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W^{(\omega)}_{K_1}\right) \tag{3.29}$$

has always a dense image.

Proof. By Lemma 3.13

$$\mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W^{(\omega)}_{K_1}\right)' \simeq \mathcal{E}'_{(\omega)}(K_1)\big/\wp(D) \otimes \mathcal{E}'_{(\omega)}(K_1).$$

Let $T \in \mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W^{(\omega)}_{K_1}\right)'$ which vanish on

$$\mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W^{(\omega)}_{K_2}\right) = \left\{u \in W^{(\omega)}_{K_2} : \; p_h(D)u = 0 \;\; \forall h = 1, 2, \ldots, r\right\},$$

where $p_1(\zeta), \ldots, p_r(\zeta)$ are generators of $\wp$. We must prove that $T \equiv 0$.

By (3.19) we have that

$$\hat{T}(\zeta) = T(e^{-i\langle\cdot,\zeta\rangle}) = 0 \qquad \forall \zeta \in V(\wp),$$

moreover by the Nullstellensatz (cf. [27]) and the Ehrenpreis Fundamental Theorem (cf. [20]),

$$T = \sum_{h=1}^{r} p_h(D) T_h$$

for some $T_h \in \mathcal{E}'_{(\omega)}(K_1)$, i.e. $T \in \wp(D) \otimes \mathcal{E}'_{(\omega)}(K_1)$. This shows that T is identically zero as an element of the space $\mathcal{E}'_{(\omega)}(K_1)\big/\wp(D) \otimes \mathcal{E}'_{(\omega)}(K_1) \simeq \mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\wp, W^{(\omega)}_{K_1}\right)'$, and hence the homomorphism (3.29) has a dense image.

□

Remark 3.17. By Proposition 3.10 and Lemma 3.16, the Cauchy problem (3.9) admits at least a solution if and only if the homomorphism (3.14) has a closed image, i.e. if and only if the dual homomorphism (3.15) has a closed image, by [19, Ch. IV, § 2, n. 4, Thm. 3] (see also Remark 3.8).

By Proposition 3.14 we thus have that the Cauchy problem (3.9) admits at least a solution if and only if the homomorphism (3.28) has a closed image.

By Theorem 5.1 of [12] this condition is equivalent to the validity of the following Phragmén–Lindelöf principle:

Theorem 3.18. *(Phragmén–Lindelöf principle for holomorphic functions) Let V be a reduced affine algebraic variety and $\mathcal{O}_{\psi^1}(V)$ and $\mathcal{O}_{\psi^2}(V)$ be defined as in (3.27). Then the following are equivalent:*

(i) $\mathcal{O}_{\psi^1}(V) \hookrightarrow \mathcal{O}_{\psi^2}(V)$ *has closed image;*

(ii) $\forall\alpha\in\mathbb{N}$, $\exists\beta\in\mathbb{N}$ *such that*

$$\mathcal{O}_{\psi^1}(V)\cap\mathcal{O}_{\psi^2_\alpha}(V)\subset\mathcal{O}_{\psi^1_\beta}(V);$$

(iii) *the following Phragmén–Lindelöf principle holds:*

$$(PL)\begin{cases}\forall\alpha\in\mathbb{N},\ \exists\beta\in\mathbb{N},\ C>0 \text{ such that}\\ \text{if } f\in\mathcal{O}(V) \text{ satisfies for some costants } \alpha_f\in\mathbb{N},\ c_f>0\\ \begin{cases}|f(\zeta)|\le e^{\psi^2_\alpha(\zeta)} & \forall\zeta\in V\\ |f(\zeta)|\le c_f e^{\psi^1_{\alpha_f}(\zeta)} & \forall\zeta\in V\end{cases}\\ \text{then it also satisfies:}\\ |f(\zeta)|\le Ce^{\psi^1_\beta(\zeta)} \qquad \forall\zeta\in V.\end{cases}$$

Summarizing, by Remark 3.17 and Theorem 3.18, we have the following:

Theorem 3.19. *(Phragmén–Lindelöf principle for the existence of solutions) Let $\omega\in\mathcal{W}'$. The Cauchy problem (3.9) admits **at least** a solution if and only if the following Phragmén–Lindelöf principle holds for all $\wp\in\mathrm{Ass}(\mathcal{M})$ and $V=V(\wp)$:*

$$(PL)'\begin{cases}\forall\alpha\in\mathbb{N},\ \exists\beta\in\mathbb{N},\ C>0 \text{ such that}\\ \text{if } f\in\mathcal{O}(V) \text{ satisfies for some costants } \alpha_f\in\mathbb{N},\ c_f>0\\ \begin{cases}|f(\zeta)|\le \exp\left\{H_{K^2_\alpha}(\mathrm{Im}\,\zeta)+\alpha\omega(\zeta)\right\} & \forall\zeta\in V\\ |f(\zeta)|\le c_f\exp\left\{H_{K^1_{\alpha_f}}(\mathrm{Im}\,\zeta)+\alpha_f\omega(\zeta)\right\} & \forall\zeta\in V\end{cases}\\ \text{then it also satisfies:}\\ |f(\zeta)|\le C\exp\left\{H_{K^1_\beta}(\mathrm{Im}\,\zeta)+\beta\omega(\zeta)\right\} \qquad \forall\zeta\in V.\end{cases}$$

Let us now recall the definition of plurisubharmonic functions on an affine algebraic variety $V\subset\mathbb{C}^N$:

Definition 3.20. A function $u:V\to[-\infty,+\infty)$ is called *plurisubharmonic* on V if it is locally bounded from above, plurisubharmonic in the usual sense on V_{reg}, the set of all regular points of V, and satisfies

$$u(\zeta)=\limsup_{\substack{z\in V_{\mathrm{reg}}\\ z\to\zeta}}u(z)$$

at the singular points of V.

By $\mathrm{psh}(V)$ we denote the set of all functions that are plurisubharmonic on V.

By Theorem 1.2 of [11], Theorem 3.19 is equivalent to the following:

Theorem 3.21. *(Phragmén–Lindelöf principle for plurisubharmonic functions) Let $\omega\in\mathcal{W}'$. The Cauchy problem (3.9) admits **at least** a solution if and only if the following Phragmén–Lindelöf principle holds for all $\wp\in\mathrm{Ass}(\mathcal{M})$ and $V=V(\wp)$:*

$$(PL)_\omega \begin{cases} \forall\alpha\in\mathbb{N},\ \exists\beta\in\mathbb{N},\ C>0 \textit{ such that} \\ \textit{if } u\in\mathrm{psh}(V) \textit{ satisfies for some costants } \alpha_u\in\mathbb{N},\ c_u>0 \\ \begin{cases} u(\zeta)\le H_{K^2_\alpha}(\mathrm{Im}\,\zeta)+\alpha\omega(\zeta) & \forall\zeta\in V \\ u(\zeta)\le H_{K^1_{\alpha_u}}(\mathrm{Im}\,\zeta)+\alpha_u\omega(\zeta)+c_u & \forall\zeta\in V \end{cases} \\ \textit{then it also satisfies:} \\ u(\zeta)\le H_{K^1_\beta}(\mathrm{Im}\,\zeta)+\beta\omega(\zeta)+C \qquad \forall\zeta\in V. \end{cases}$$

Also the problem of existence of a unique solution of the Cauchy problem (3.9) can be easily treated by the study of the dual homomorphism (3.15). In particular, by Propositions 3.11 and 3.14, we have:

Theorem 3.22. *Let $\omega\in\mathcal{W}'$. The Cauchy problem* (3.9) *admits* ***one and only one*** *solution if and only if, for all $\wp\in\mathrm{Ass}(\mathcal{M})$ and $V=V(\wp)$, the homomorphism*

$$\mathcal{O}_{\psi^1}(V)\hookrightarrow\mathcal{O}_{\psi^2}(V)$$

is an isomorphism.

By Theorem 5.2 of [12] we can finally state the following:

Theorem 3.23. *Let $\omega\in\mathcal{W}'$. The Cauchy problem* (3.9) *admits* ***one and only one*** *solution if and only if, for all $\wp\in\mathrm{Ass}(\mathcal{M})$ and $V=V(\wp)$, one of the following equivalent conditions holds:*

(i) $\mathcal{O}_{\psi^1}(V)\hookrightarrow\mathcal{O}_{\psi^2}(V)$ *is an isomorphism;*

(ii) $\forall\alpha\in\mathbb{N}$, $\exists\beta\in\mathbb{N}$ *such that*

$$\mathcal{O}_{\psi^2_\alpha}(V)\subset\mathcal{O}_{\psi^1_\beta}(V);$$

(iii) $\forall\alpha\in\mathbb{N}$, $\exists\beta\in\mathbb{N}$, $C>0$ *such that*

$$\sup_{\zeta\in V}\left|f(\zeta)e^{-H_{K^1_\beta}(\mathrm{Im}\,\zeta)-\beta\omega(\zeta)}\right|\le C\sup_{\zeta\in V}\left|f(\zeta)e^{-H_{K^2_\alpha}(\mathrm{Im}\,\zeta)-\alpha\omega(\zeta)}\right|$$

for all $f\in\mathcal{O}(V)$.

Remark 3.24. Clearly condition *(i)* (resp. *(ii)*, *(iii)*) of Theorem 3.23 implies condition *(i)* (resp. *(ii)*, *(iii)*) of Theorem 3.18.

Remark 3.25. If we split $\mathbb{R}^N\simeq\mathbb{R}^k_t\times\mathbb{R}^n_x$, by Theorem 2.39 we could also allow different scales of regularity in the t-variables and in the x-variables, as in [12] or [9].

We can allow, for instance, weight functions of the form

$$\omega(\tau,\zeta)=\sigma_{\alpha_1}(|\tau|)+\sigma_{\alpha_2}(|\zeta|), \qquad \text{for } (\tau,\zeta)\in\mathbb{C}^k\times\mathbb{C}^n, \tag{3.30}$$

for $0\le\alpha_1,\alpha_2<1$, where each $\sigma_\alpha:\ [0,+\infty)\to[0,+\infty)$ is defined by

$$\sigma_\alpha(t):=\begin{cases} t^\alpha & \text{if } 0<\alpha<1 \\ \log(1+t) & \text{if } \alpha=0. \end{cases}$$

This is a weight function under our condition (γ), weaker than the corresponding one $\log(1+t)=o(\omega(t))$ of [16].

Example 3.26. Let us consider the operator

$$P(D_t, D_x) := \Big[(2D_x - D_t)^3 - D_t - D_x\Big]\Big[(D_x^3 - D_t)^2 + D_x^5\Big](D_x - D_t^4) + 1$$

and the associated algebraic variety

$$V = \Big\{(\tau, \zeta) \in \mathbb{C}^2 : \ P(-\tau, -\zeta) = 0\Big\}.$$

Since this is an algebraic curve, one can use for instance MAPLE to compute the Puiseux series expansions on its branches at infinity (cf. [7, Example 5.1]) and prove, by [9, Thm. 5.16], that V satisfies the Phragmén–Lindelöf principle $(PL)_\omega$, for ω as in (3.30), if and only if

$$\max\{\alpha_1, \alpha_2\} \geq \frac{1}{3}, \quad \alpha_2 \geq \frac{1}{4} \ \text{ and } \ \max\{3\alpha_1, \alpha_2\} \geq \frac{1}{2}.$$

This means that there is a very specific region of the plane $\mathbb{R}^2_{(\alpha_1,\alpha_2)}$ that exactly determines in which classes of (small) Gevrey functions (or C^∞ functions if $\alpha_j = 0$) the associated Cauchy problem admits at least a solution and in which classes it does not.

We refer to [7,8] and [9] for more examples in this direction.

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