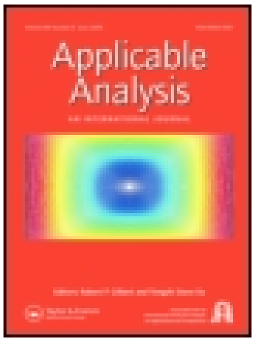

Taylor & Francis
Taylor & Francis Group

Applicable Analysis
An International Journal

ISSN: 0003-6811 (Print) 1563-504X (Online) Journal homepage: http://www.tandfonline.com/loi/gapa20

An existence and uniqueness theorem for the Navier–Stokes equations in dimension four

Vincenzo Coscia

To cite this article: Vincenzo Coscia (2016): An existence and uniqueness theorem for the Navier–Stokes equations in dimension four, Applicable Analysis, DOI: 10.1080/00036811.2016.1263837

To link to this article: http://dx.doi.org/10.1080/00036811.2016.1263837

Published online: 02 Dec 2016.

APPLICABLE ANALYSIS, 2016
http://dx.doi.org/10.1080/00036811.2016.1263837

Taylor & Francis
Taylor & Francis Group

An existence and uniqueness theorem for the Navier–Stokes equations in dimension four

Vincenzo Coscia

Dipartimento di Matematica e Informatica, Università di Ferrara, Ferrara, Italy

ABSTRACT
We prove that the steady state Navier–Stokes equations have a solution in an exterior Lipschitz domain of $\mathbb{R}^4$, vanishing at infinity, provided the boundary datum belongs to $L^3(\partial\Omega)$.

ARTICLE HISTORY
Received 22 October 2016
Accepted 18 November 2016

COMMUNICATED BY
G. Panasenko

KEYWORDS
Stationary Navier Stokes equations; 4*D* exterior Lipschitz domains; boundary–value problem

AMS SUBJECT CLASSIFICATIONS
76D05; 35Q30; 31B10; 76D03

1. Introduction

In this paper, we shall consider the steady–state boundary–value problem for the Navier–Stokes equations[1]

$$\begin{aligned} \nu\Delta \boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nabla p = \boldsymbol{0} &\quad \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 &\quad \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{a} &\quad \text{on } \partial\Omega \\ \lim_{r\to+\infty} \boldsymbol{u}(x) = \boldsymbol{0} &\quad \text{on } \partial\Omega \end{aligned} \tag{1}$$

in the four-dimensional exterior domain

$$\Omega = \mathbb{R}^4 \setminus \bigcup_{i=1}^{m} \overline{\Omega}_i$$

where Ω_i are bounded Lipschitz domains with connected boundaries such that $\Omega_i \cup \Omega_j = \varnothing$, $i \neq j$. In (1), $\boldsymbol{u}$ and p are the kinetic and pressure fields respectively, $\nu > 0$ is the kinematical viscosity coefficient and $\boldsymbol{a}$ is an assigned field on $\partial\Omega$.

Strictly connected to (1) is its linearized version, the Stokes equations

$$\begin{aligned} \nu\Delta \boldsymbol{u} - \nabla p = \boldsymbol{0} &\quad \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 &\quad \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{a} &\quad \text{on } \partial\Omega \\ \lim_{r\to+\infty} \boldsymbol{u}(x) = \boldsymbol{0} &\quad \text{on } \partial\Omega. \end{aligned} \tag{2}$$

CONTACT Vincenzo Coscia vincenzo.coscia@unife.it

It is well known that if $\boldsymbol{a} \in L^3(\partial\Omega)$, then (2) has a solution[2] $(\boldsymbol{u}_s, p_s)$ analytical in Ω, such that $\boldsymbol{u}_s = O(r^{-2})$, $p_s = O(r^{-3})$ and which satisfies $(2)_3$ in the sense of the non tangential convergence, *i.e*, there is a finite cone Γ such that

$$\lim_{x(\in\Gamma_\xi)\to\xi} \boldsymbol{u}(x) = \boldsymbol{a}(\xi) \tag{3}$$

for almost all $\xi \in \partial\Omega$, where $\Gamma_\xi \subset \Omega$ is a cone with vertex at ξ, congruent to Γ.

In Section 3, we prove the following

Theorem 1: *If $\boldsymbol{a} \in L^3(\partial\Omega)$ and*

$$\mathcal{F} = \frac{1}{4\omega}\sum_{i=1}^{m}\left|\int_{\partial\Omega_i} \boldsymbol{a}\cdot\boldsymbol{n}\right| \max_{\partial\Omega}\frac{1}{|x-x_i|^2} < \nu, \tag{4}$$

where ω is the measure of the surface of the unit ball of $\mathbb{R}^4$, x_i fixed point of Ω_i, and $\boldsymbol{n}$ is the outward (with respect to Ω) unit normal to $\partial\Omega$, then (1) has a solution $(\boldsymbol{u}, p) \in [L^4(\Omega) \cap C^\infty(\Omega)] \times C^\infty(\Omega)$. Moreover, $\boldsymbol{u}$ is unique in $L^4(\Omega)$, provided $3\|\boldsymbol{u}_s\|_{L^4(\Omega)} < 4\nu$ and

$$\|\boldsymbol{u}_s\|_{L^4(\Omega)} + \frac{\|\boldsymbol{u}_s\|^2_{L^4(\Omega)}}{\frac{4}{3}\left(\nu - \frac{3}{4}\|\boldsymbol{u}_s\|_{L^4(\Omega)}\right)} < \frac{4\nu}{3}, \tag{5}$$

where $\boldsymbol{u}_s$ is the solution to (2) in Ω with boundary datum $\boldsymbol{a}$.

In the next section, we collect the main preliminary tools we shall need to get our results. For domains of class $C^{1,1}$ and boundary data in $W^{1/4,4}(\partial\Omega)$ problem (2) in bounded domains has been considered by several authors (see p.297 of [1] and the references therein).

2. Preliminary results

The fundamental solution to (2) writes

$$\begin{aligned} \mathcal{U}_{ij}(x-y) &= -\frac{1}{4\omega\nu|x-y|^2}\left\{\delta_{ij} + \frac{2(x_i-y_i)(x_j-y_j)}{|x-y|^2}\right\},\\ \varpi_i(x-y) &= \frac{x_i-y_i}{\omega|x-y|^4}. \end{aligned} \tag{6}$$

The Stokes simple layer potential with density $\boldsymbol{\psi} \in L^q(\partial\Omega)$ is defined by

$$\begin{aligned} \boldsymbol{v}[\boldsymbol{\psi}](x) &= \int_{\partial\Omega} \boldsymbol{\mathcal{U}}(x-\xi)\cdot\boldsymbol{\psi}(\xi)\mathrm{d}a_\xi,\\ P[\boldsymbol{\psi}](x) &= \int_{\partial\Omega} \boldsymbol{\varpi}(x-\xi)\cdot\boldsymbol{\psi}(\xi)\mathrm{d}a_\xi, \end{aligned} \tag{7}$$

and is a solution to Stokes' Equations $(2)_{1,2}$ in $\mathbb{R}^4 \setminus \partial\Omega$. The trace of $(7)_1$ on $\partial\Omega$ is a continuous operator

$$\mathcal{S} : L^q(\partial\Omega) \to W^{1,q}(\partial\Omega). \tag{8}$$

For $q = 2$, $\mathcal{S}$ is Fredholm with index zero and $\mathrm{Kern}\,\mathcal{S} = \mathrm{Kern}\,\mathcal{S}' = \{\boldsymbol{n}\}$,[2,3] where[3]

$$\mathcal{S}' : W^{-1,2}(\partial\Omega) \to L^2(\partial\Omega) \tag{9}$$

is the adjoint of $\mathcal{S}$. Hence, if $\boldsymbol{a} \in L^2(\partial\Omega)$, then there is $\boldsymbol{\psi} \in W^{-1,2}(\partial\Omega)$ such that the pair

$$\begin{aligned} \boldsymbol{u}_s(x) &= \boldsymbol{v}[\boldsymbol{\psi}] + \boldsymbol{\sigma}(x), \\ p_s(x) &= P[\boldsymbol{\psi}](x), \end{aligned} \tag{10}$$

is the unique solution to (2), which vanishes at infinity and takes the value $\boldsymbol{a}$ on $\partial\Omega$ according to (3),[2,3] where

$$\boldsymbol{\sigma}(x) = \frac{1}{\omega}\sum_{i=1}^{m} \frac{(x_i - x)}{|x - x_i|^4} \int_{\partial\Omega_i} \boldsymbol{a}\cdot\boldsymbol{n}\,.$$

By classical stability results, the Fredholm property of (8) can be extended in a neighborhood of $(2-\epsilon, 2+\epsilon)$, with ϵ depending on $\partial\Omega$. If $\partial\Omega$ is of class C^1, then (8) is Fredholm for all $q \in (1, +\infty)$. By results of [4], if $\boldsymbol{a} \in L^3(\partial\Omega)$, then the above solution belongs to $L^4(\Omega)$ and

$$\|\boldsymbol{u}_s\|_{L^4(\Omega)} \le \gamma \|\boldsymbol{a}\|_{L^3(\partial\Omega)}. \tag{11}$$

3. Proof of Theorem 1

Following [5], let $\boldsymbol{a}_\epsilon \in W^{1,2}(\partial\Omega)$ be such that

$$\int_{\partial\Omega} |\boldsymbol{a} - \boldsymbol{a}_\epsilon|^3 < \epsilon, \tag{12}$$

for small positive ϵ. Let $\boldsymbol{u}_{s\epsilon}$ be the solution to the Stokes problem with boundary value $\boldsymbol{a} - \boldsymbol{a}_\epsilon$. For $\boldsymbol{u} \in L^4(\Omega)$, denote by $\mathcal{K}[\boldsymbol{u}]$ the solution to the Stokes problem with boundary value $-\mathrm{tr}_{\,|\partial\Omega}\mathcal{V}[\boldsymbol{u}]$, where

$$\mathcal{V}[\boldsymbol{u}](x) = \int_\Omega \mathcal{U}(x-y)(\boldsymbol{u}\cdot\nabla\boldsymbol{u})(y)\mathrm{dv}_y.$$

Consider the functional equation

$$\boldsymbol{u}'(x) = \boldsymbol{u}_{s\epsilon} + (\mathcal{K} + \mathcal{V})[\boldsymbol{u}]. \tag{13}$$

By virtue of (11), (12) we see that $\boldsymbol{u}'$ is a contraction in $L^4(\Omega)$. Hence, it follows that (13) has a fixed point $\boldsymbol{u}_\epsilon$ which is a solution to $(1)_{1,2}$ taking the value $\boldsymbol{a} - \boldsymbol{a}_\epsilon$ on $\partial\Omega$ and such that

$$\|\boldsymbol{u}_\epsilon\|_{L^4(\Omega)} \le c\epsilon. \tag{14}$$

Let $\boldsymbol{v}_s$ be the solution to the Stokes problem with boundary value $\boldsymbol{a}_\epsilon$:

$$\boldsymbol{v}_\epsilon = \boldsymbol{v}[\boldsymbol{\psi}] + \boldsymbol{\sigma}_\epsilon, \quad \boldsymbol{\sigma}_\epsilon(x) = -\frac{\boldsymbol{x}}{\omega|\boldsymbol{x}|^4}\int_{\partial\Omega} \boldsymbol{a}_\epsilon\cdot\boldsymbol{n}. \tag{15}$$

Let us look for a solution $\boldsymbol{w} \in W_0^{1,2}(\Omega_R)$ to

$$\begin{aligned} \nu\Delta\boldsymbol{w} - (\boldsymbol{u}_\epsilon + \boldsymbol{v}_s + \boldsymbol{w})\cdot\nabla(\boldsymbol{v}_s + \boldsymbol{w}) - (\boldsymbol{w} + \boldsymbol{v}_s)\cdot\nabla\boldsymbol{u}_\epsilon - \nabla Q = \boldsymbol{0}, \\ \mathrm{div}\,\boldsymbol{w} = 0, \end{aligned} \tag{16}$$

in $\Omega_R = \Omega \cap S_R$, for large R. To this end, it is sufficient to show that the set of all solutions to (16) are bounded in $W^{1,2}(\Omega)$ (see, e.g. [1]). Following a classical *reductio ad absurdum* argument of J. Leray [6], let us suppose that a sequence of solutions $\boldsymbol{w}_k \in W_0^{1,2}(\Omega)$ to (16) exists such that

$$\lim_{k\to+\infty} J_k = +\infty, \quad J_k = \|\nabla\boldsymbol{w}_k\|_{L^2(\Omega)}. \tag{17}$$

Setting $\boldsymbol{w}'_k = \boldsymbol{w}_k/J_k$, a straightforward argument shows that $\boldsymbol{w}'_k$ tends strongly in $L^q(\Omega_R)$, $q < 4$, and weakly in $W^{1,2}(\Omega_R)$, to a field $\boldsymbol{w}' \in W^{1,2}_0(\Omega)$, with $\|\nabla \boldsymbol{w}'\|_{L^2\Omega)} \le 1$, which satisfies the Euler equations

$$\begin{aligned} \boldsymbol{w}' \cdot \nabla \boldsymbol{w}' + \nabla Q' = \mathbf{0} \quad & \text{in } \Omega_R \\ \operatorname{div} \boldsymbol{w}' = 0 \quad & \text{in } \Omega_R, \end{aligned} \tag{18}$$

for some field $Q' \in W^{1,4/3}(\Omega_R)$ constant on $\partial\Omega$ and ∂S_R (say, Q_0 on $\partial\Omega$ and Q_R on ∂S_R) and

$$\nu = \int_\Omega \boldsymbol{w}' \cdot \nabla \boldsymbol{w}' \cdot (\boldsymbol{u}_\epsilon + \boldsymbol{v}_s). \tag{19}$$

Let us extend $\boldsymbol{w}'$ to Ω by setting $\boldsymbol{w}' = \mathbf{0}$ in $\complement S_R$. Since

$$\begin{aligned} \left| \int_\Omega \boldsymbol{w}' \cdot \nabla \boldsymbol{w}' \cdot (\boldsymbol{u}_\epsilon + \boldsymbol{v}_s) \right| &\le \|\boldsymbol{u}_\epsilon\|_{L^4(\Omega)} \|\boldsymbol{w}'\|_{L^4(\Omega)} \|\nabla \boldsymbol{w}'\|_{L^2(\Omega)} \le c\epsilon, \\ \int_\Omega \boldsymbol{w}' \cdot \nabla \boldsymbol{w}' \cdot \boldsymbol{v}[\boldsymbol{\psi}] &= -\int_{\partial\Omega_R} \nabla Q' \cdot \boldsymbol{v}[\boldsymbol{\psi}] \\ &= -Q_R \int_{\partial S_R} \boldsymbol{v}[\boldsymbol{\psi}] \cdot \boldsymbol{n} - Q_0 \int_{\partial\Omega} \boldsymbol{v}[\boldsymbol{\psi}] \cdot \boldsymbol{n} = 0, \end{aligned}$$

and, taking into account that $\nabla \boldsymbol{w}' \cdot \nabla \boldsymbol{w}'^\top| = \hat{\nabla} \boldsymbol{w}'|^2 - \tilde{\nabla} \boldsymbol{w}'|^2$, with $\hat{\nabla} \boldsymbol{w}'$, $\tilde{\nabla} \boldsymbol{w}'$ symmetric and skew parts of $\nabla \boldsymbol{w}'$ respectively and $\|\nabla \boldsymbol{w}'\|^2_{L^2(\Omega)} = 2\|\tilde{\nabla} \boldsymbol{w}'\|^2_{L^2(\Omega)} = 2\|\hat{\nabla} \boldsymbol{w}'\|^2_{L^2(\Omega)}$,

$$\begin{aligned} \left| \int_\Omega \boldsymbol{w}' \cdot \nabla \boldsymbol{w}' \cdot \boldsymbol{\sigma}_\epsilon \right| &= \left| \int_{\partial\Omega} \boldsymbol{a}_\epsilon \cdot \boldsymbol{n} \right| \left| \int_\Omega \frac{\nabla \boldsymbol{w}' \cdot \nabla \boldsymbol{w}'^\top}{2\omega|\boldsymbol{x}|^2} \right|, \\ &\le \left| \int_{\partial\Omega} (\boldsymbol{a}_\epsilon - \boldsymbol{a}) \cdot \boldsymbol{n} \right| \left| \int_\Omega \frac{\nabla \boldsymbol{w}' \cdot \nabla \boldsymbol{w}'^\top}{2\omega|\boldsymbol{x}|^2} \right| \\ &\quad + \left| \int_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{n} \right| \left| \int_\Omega \frac{|\hat{\nabla} \boldsymbol{w}'|^2 - |\tilde{\nabla} \boldsymbol{w}'|^2}{2\omega|\boldsymbol{x}|^2} \right| \le c\epsilon + \mathcal{F} \end{aligned}$$

from (19) it follows

$$\nu - c\epsilon - \mathcal{F} \le 0. \tag{20}$$

Therefore, since ϵ can be chosen small as we want, we see that the *hypothesis ad absurdum* (17) implies that (4) is not true and this gives the desired uniform estimate. Set $R = k \in \mathbb{N}$ for $k \ge k_0$ and denote by $(\boldsymbol{w}_k, Q_k)$ a solution to (16). By repeating *ad litteram*, the above by contradiction argument (with obvious modification), we see that the sequence $\{\boldsymbol{w}_k\}$ is uniformly bounded in $D^{1,2}(\Omega)$[4] and this is sufficient to conclude that (16) has a solution $\boldsymbol{w} \in D^{1,2}_0(\Omega)$ in Ω.[1,7] Clearly, the field $\boldsymbol{u} = \boldsymbol{u}_\epsilon + \boldsymbol{v}_s + \boldsymbol{w}$ gives the desired solution to (1). If $(\boldsymbol{u} + \boldsymbol{w}, p + Q)$ is another solution to (1), with $\boldsymbol{u} + \boldsymbol{w} \in L^4(\Omega)$ and $\boldsymbol{w} \in D^{1,2}_0(\Omega)$, then a simple computation and Schwarz' inequality and Sobolev's inequality yield

$$\begin{aligned} \nu \int_\Omega |\nabla \boldsymbol{w}|^2 &= \int_\Omega \boldsymbol{w} \cdot \nabla \boldsymbol{w} \cdot \boldsymbol{u} \le \|\boldsymbol{u}\|_{L^4(\Omega)} \|\boldsymbol{w}\|_{L^4(\Omega)} \|\nabla \boldsymbol{w}\|_{L^2(\Omega)} \\ &\le \tfrac{3}{4} \|\boldsymbol{u}\|_{L^4(\Omega)} \|\nabla \boldsymbol{w}\|^2_{L^2(\Omega)}. \end{aligned}$$

Hence if $3\|\boldsymbol{u}\|_{L^4(\Omega)} < 4\nu$, then $\boldsymbol{w} = \mathbf{0}$. Write $\boldsymbol{u} = \boldsymbol{u}_s + \boldsymbol{w}$, where $\boldsymbol{u}_s$ is the solution (10) to the Stokes problem with boundary datum $\boldsymbol{a}$, and assume $3\|\boldsymbol{u}_s\|_{L^4(\Omega)} < 4\nu$. Therefore, $\boldsymbol{w} \in D^{1,2}_0(\Omega)$ is a solution to the equations

$$\begin{aligned} \nu\Delta\boldsymbol{w} - (\boldsymbol{u}_s + \boldsymbol{w})\cdot\nabla(\boldsymbol{u}_s + \boldsymbol{w}) - \nabla p = \mathbf{0} &\quad \text{in } \Omega \\ \operatorname{div}\boldsymbol{w} = 0 &\quad \text{in } \Omega, \\ \boldsymbol{w} = \mathbf{0} &\quad \text{on } \partial\Omega, \\ \lim_{x\to\infty}\boldsymbol{w}(x) = \mathbf{0}. & \end{aligned} \tag{21}$$

By a standard argument

$$\begin{aligned} \nu\int_\Omega |\nabla\boldsymbol{w}|^2 &= \int_\Omega \boldsymbol{w}\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{u}_s + \int_\Omega \boldsymbol{u}_s\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{u}_s \\ &\le \tfrac{3}{4}\|\boldsymbol{u}\|_{L^4(\Omega)}\|\nabla\boldsymbol{w}\|^2_{L^2(\Omega)} + \|\boldsymbol{u}_s\|^2_{L^4(\Omega)}\|\nabla\boldsymbol{w}\|_{L^2(\Omega)}. \end{aligned}$$

Hence

$$\tfrac{4}{3}\big(\nu - \tfrac{3}{4}\|\boldsymbol{u}_s\|_{L^4(\Omega)}\big)\|\boldsymbol{w}\|_{L^4(\Omega)} \le \big(\nu - \tfrac{3}{4}\|\boldsymbol{u}_s\|_{L^4(\Omega)}\big)\|\nabla\boldsymbol{w}\|_{L^2(\Omega)} \le \|\boldsymbol{u}_s\|^2_{L^4(\Omega)}. \tag{22}$$

By (22) and Minkowski inequality

$$\|\boldsymbol{u}\|_{L^4(\Omega)} \le \|\boldsymbol{u}_s\|_{L^4(\Omega)} + \|\boldsymbol{w}\|_{L^4(\Omega)} \le \|\boldsymbol{u}_s\|_{L^4(\Omega)} + \frac{\|\boldsymbol{u}_s\|^2_{L^4(\Omega)}}{\tfrac{4}{3}\big(\nu - \tfrac{3}{4}\|\boldsymbol{u}_s\|_{L^4(\Omega)}\big)}.$$

Hence it follows that (5) yields uniqueness in $L^4(\Omega)$. Note that by (11), (5) is implied by $\gamma\|\boldsymbol{a}\|_{L^3(\partial\Omega)} < 4\nu/3$ and

$$\gamma\|\boldsymbol{a}\|_{L^3(\partial\Omega)} + \frac{\gamma^2\|\boldsymbol{a}\|^2_{L^3(\Omega)}}{\tfrac{4}{3}\big(\nu - \tfrac{3\gamma}{4}\|\boldsymbol{a}\|_{L^3(\partial\Omega)}\big)} < \frac{4\nu}{3}.$$

□

Remark 1: It is clear that the proof of existence of a solution to (1) in $L^4(\Omega)$ requires only that the corresponding Stokes problem has a solution $\boldsymbol{u}_s \in L^4(\Omega)$. Hence, it follows that if $\partial\Omega$ is of class $C^{1,1}$, then Theorem 1 can be extended to boundary data $\boldsymbol{a} \in W^{-1/4,4}(\partial\Omega)$.[3]

4. Some remarks in higher dimensions

If Ω is an exterior domain of $\mathbb{R}^m$ $(m > 3)$ of class C^1, then for $\boldsymbol{a} \in L^{m-1}(\partial\Omega)$ the Stokes problem

$$\begin{aligned} \nu\Delta\boldsymbol{u} - \nabla p = \mathbf{0} &\quad \text{in } \Omega \\ \operatorname{div}\boldsymbol{u} = 0 &\quad \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{a} &\quad \text{on } \partial\Omega, \\ \lim_{x\to\infty}\boldsymbol{u}(x) = \mathbf{0}, & \end{aligned} \tag{23}$$

has a solution $\boldsymbol{u}_s \in L^m(\Omega)$ and $\|\boldsymbol{u}_s\|_{L^m(\Omega)} \le \gamma\|\boldsymbol{a}\|_{L^{m-1}(\partial\Omega)}$ (see [3]). For $\boldsymbol{u} \in L^m(\Omega)$ consider the functional equation

$$\boldsymbol{u}'(x) = \boldsymbol{u}_s + (\mathcal{K} + \mathcal{V})[\boldsymbol{u}] \tag{24}$$

in $L^m(\Omega)$, where the operator $\mathcal{K} + \mathcal{V}$ is defined in the proof of Theorem 1. Taking into account that

$$\|(\mathcal{K} + \mathcal{V})[\boldsymbol{u}]\|_{L^m(\Omega)} \le c\|\boldsymbol{u}\|^2_{L^m(\Omega)},$$

If $\|\boldsymbol{a}\|_{L^{m-1}(\partial\Omega)}$ is sufficiently small, then $\boldsymbol{u}'$ is a contraction in a ball of $L^m(\Omega)$ and the fixed point of (24) is a C^∞ solution to (1). Moreover, if $\boldsymbol{u} + \boldsymbol{w} \in L^m(\Omega)$ is another solution to (1), then by Schwarz's inequality and Sobolev's inequality

$$\nu\int_\Omega |\nabla\boldsymbol{w}|^2 = \int_\Omega \boldsymbol{w}\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{u} \le \tfrac{(m-1)}{(m-2)\sqrt{m}}\|\boldsymbol{u}\|_{L^m(\Omega)}\|\nabla\boldsymbol{w}\|^2_{L^m(\Omega)}.$$

Hence, it follows that the above solution is unique in the ball

$$\|\boldsymbol{u}\|_{L^m(\Omega)} < \frac{\nu\sqrt{m}(m-2)}{m-1}.$$

It is clear that for domains of class C^1 we can repeat the argument in the proof of Theorem 1 to see that for $\boldsymbol{a} \in L^{m-1}(\Omega)$ and fluxes obeying a condition of the type (4), then the Equations (1) have a solution $\boldsymbol{u} = \boldsymbol{u}_\epsilon + \boldsymbol{v}_s + \boldsymbol{w}$, with $\boldsymbol{u}_\epsilon$, $\boldsymbol{v}_s$ regular in Ω and $\boldsymbol{w} \in D_0^{1,2}(\Omega)$. Up to date, we have not general results assuring that $\boldsymbol{w}$ is regular.

Notes

1. For the main notation, we follow the monograph.[1]
2. See [3] and Section 2.
3. In such a case, (7) has to be understood as the value of the functional $\boldsymbol{\psi}$ at $\boldsymbol{\mathcal{U}}$.
4. $D^{1,2}(\Omega) = \{\varphi \in L^1_{loc}(\Omega)\colon \|\nabla\varphi\|_{L^2(\Omega)} < +\infty\}$ and $D_0^{1,2}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to $\|\nabla\varphi\|_{L^2(\Omega)}$.

Disclosure statement

No potential conflict of interest was reported by the author.

References

[1] Galdi GP. An introduction to the mathematical theory of the Navier–Stokes equations. Steady-state problems. New York: Springer; 2011.
[2] Mitrea M, Taylor M. Navier–Stokes equations on Lipschitz domains in Riemannian manifolds. Math. Ann. 2001;321:955–987.
[3] Russo R. On Stokes' problem. In: Rannacher R, Sequeira A, editors. Advances in mathematica fluid mechanics. Berlin: Springer-Verlag; 2010. p. 473–511.
[4] Shen Z. A note on the Dirichlet problem for the Stokes system in Lipschitz domains. Proc Amer Math Soc. 1995;123:801–811.
[5] Marušić-Paloka E. Solvability of the Navier–Stokes system with L^2 boundary data. Appl Math Optim. 2000;41:365–375.
[6] Leray J. Étude de diverses équations intégrales non linéaire et de quelques problèmes que pose l'hydrodynamique [Study of different nonlinear integral equations and of some problems related to hydrodynamics]. J Math Pures Appl. 1933;12:1–82.
[7] Temam R. Navier–Stokes equations. Amsterdam: North-Holland; 1977.