

**QUADRATIC FAMILIES OF ELLIPTIC CURVES AND
UNIRATIONALITY OF DEGREE 1 CONIC BUNDLES**

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Let K be a number field and $a_i(t) \in K[t]$ polynomials of degree 2. We consider the family of elliptic curves

$$E_t := (y^2 = a_3(t)x^3 + a_2(t)x^2 + a_1(t)x + a_0(t)) \subset \mathbb{A}_{xy}^2 \quad (*)$$

parametrized by $t \in K$. Our aim is to show that there are many values $t \in K$ for which the corresponding elliptic curve E_t has rank ≥ 1 . Conjecturally this should hold for a positive proportion of them; see the survey [RS02]. We prove rank ≥ 1 for about the square root of all $t \in K$, listed by height.

We are only interested in *nontrivial* families, when at least two of the curves E_t are smooth, elliptic and not isomorphic to each other over K . Thus $a_3(t)$ is not identically 0 and not all the $a_i(t)$ are constant multiples of the same square $(t-c)^2$.

We view the whole family as a single algebraic surface in \mathbb{A}_{xyt}^3 and look at the distribution of K -points. The resulting surface has degree 5 but its closure in \mathbb{P}^3 is very singular. We prove the following.

Theorem 1. *Let k be any field of characteristic $\neq 2$ and $a_0(t), \dots, a_3(t) \in k[t]$ polynomials of degree 2 giving a nontrivial family of elliptic curves. Then the surface*

$$S := (y^2 = a_3(t)x^3 + a_2(t)x^2 + a_1(t)x + a_0(t)) \subset \mathbb{A}_{xyt}^3 \quad (**)$$

is unirational over k .

The proof has two parts. First assume that we know a k -point $p \in S$ that is not a 6-torsion point on the corresponding elliptic curve. Then we have geometrically clear and quite explicit formulas to prove unirationality. The second, harder part is to show that there are such k -points. We have not been able to turn this part of the proof into explicit formulas; see Remark 40.

From any sufficiently general k -point of S we obtain families of elliptic curves of rank ≥ 1 .

Corollary 2. *Let k be an infinite field of characteristic $\neq 2$. Then there are infinitely many different rational functions $q(u) \in k(u)$ that are quotients of degree 2 polynomials such that the rank of the elliptic curve E_t as in (*) is ≥ 1 for all but finitely many values $t = q(u)$ where $u \in k$.*

Unirationality can also be used to exhibit points of small height. For a Zariski open subset $U \subset S$, let $N(U, B)$ be the number of K -points of height $\leq B$ in U . Manin's conjecture [FMT89] suggests that $N(U, B)$ should grow at least like $B^{3/2}$, using the naive height function $\text{ht}(x, y, t) := \text{ht}(x) + \text{ht}(t)$. Theorem 1 implies that $N(U, B)$ grows at least like a power of B .

Corollary 3. *Let K be a number field. There is an $\epsilon > 0$ such that for every S as above and for every Zariski open subset $U \subset S$,*

$$N(U, B) \geq c(S) \cdot B^\epsilon \quad \text{for } B \gg 1,$$

where $c(S) > 0$ depends on S .

The proof gives an explicit value for ϵ but it seems to be small.

4 (Connection with conic bundles). We can rewrite

$$a_3(t)x^3 + \cdots + a_0(t) = A(x)t^2 + B(x)t + C(x)$$

where A, B, C are cubics. Projection to the x -axis exhibits S as the family of conics

$$F_x := (y^2 = A(x)t^2 + B(x)t + C(x)) \subset \mathbb{A}_{y,t}^2.$$

The conic F_x is singular iff x is a root of the discriminant $B(x)^2 - 4A(x)C(x)$; in general this happens for 6 different values of x . We get a possible 7th singular fiber at infinity. After suitable birational transformations the fiber at infinity is isomorphic to

$$F_\infty := (\tilde{a}_3(s, t) = 0) \subset \mathbb{P}_{stw}^2,$$

where $\tilde{a}_3(s, t) := s^2 a_3(t/s)$ is the homogenization of $a_3(t)$. Thus S is birational to a conic bundle with ≤ 7 singular fibers; see Definition 6.

A very degenerate case is when $B(x)^2 - 4A(x)C(x) \equiv 0$. These are easy to enumerate by hand. The only non-rational surface occurs when $a_i(t) = c_i(t - \alpha)^2$ for every i . Then setting $z := y/(t - \alpha)$ gives the new equation

$$z^2 = c_3x^3 + c_2x^2 + c_1x + c_0.$$

Thus S is birational to the product of an elliptic curve with \mathbb{P}^1 .

Our main technical result, Theorem 7, says that every conic bundle with 7 singular fibers is unirational over its field of definition. Not every conic bundle with 7 singular fibers can be written as (**), but unirationality is easier for the other ones; see Section 4.

Various coincidences among the $a_i(t)$ lead to simpler conic bundles. If $a_3(t)$ is reducible in $k[t]$ then F_∞ is reducible over k and we can contract either of its irreducible components to get a conic bundle with 6 singular fibers.

If $a_i(t) = c_i q(t)$ for every i then the substitution $y = zq(t)$ transforms the equation into

$$q(t)z^2 = c_3x^3 + c_2x^2 + c_1x + c_0.$$

The corresponding surface is a conic bundle with 4 singular fibers. In this special case Manin's conjecture [FMT89] suggests that $N(S, B)$ should grow like B^2 . In the complex multiplication case these were studied in [Mun11] where the bound $N(S, B) \geq \epsilon \cdot B^{2-\eta}$ is proved for every $\eta > 0$.

5 (Connection with higher dimensional conic bundles). Let k be an algebraically closed field. A (birational) conic bundle over k is a morphism $\pi : X \rightarrow Y$ whose generic fiber is isomorphic to a conic. It is a long standing open problem to understand which conic bundles are rational or unirational. These questions are interesting only when Y itself is unirational.

In many cases, for instance when $\dim X = 3$, one can realize Y as a conic bundle $\tau : Y \dashrightarrow B$, hence we have $\tau \circ \pi : X \dashrightarrow B$ whose generic fiber X_B is a 2-dimensional conic bundle over the function field $k(B)$. Thus if X_B is (uni)rational over $k(B)$ then X is (uni)rational over k .

The conjecture of [Isk87] says that rationality is almost equivalent to $(K_{S_B}^2) \geq 5$. Our Corollary 8 implies that X is unirational if $(K_{S_B}^2) \geq 1$.

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1. MINIMAL CONIC BUNDLES

Definition 6 (Conic bundles). Let k be a field. A surface with a *pencil of rational curves* over k is a projective, geometrically irreducible surface T together with a morphism $\pi : T \rightarrow B$ to a smooth, projective curve B such that the geometric generic fiber of π is isomorphic to \mathbb{P}^1 . M. Noether proved in 1870 that if $k = \mathbb{C}$ then T is birational (over B) to the trivial family $\mathbb{P}^1 \times B \rightarrow B$ [Noe70].

This is no longer true if k is not algebraically closed and the birational properties of such surfaces can be quite subtle. The generic fiber $F_{k(B)}$ is isomorphic to a conic and T is birational (over B) to $\mathbb{P}^1 \times B$ iff $F_{k(B)} \cong \mathbb{P}_{k(B)}^1$. The latter holds iff $F_{k(B)}$ has a $k(B)$ -point. This in turn is equivalent to $\pi : T \rightarrow B$ having a section.

A *conic bundle* over k is a smooth, projective, geometrically irreducible surface S together with a morphism $\pi : S \rightarrow B$ to a smooth, projective curve B such that geometric fibers of π are plane conics (either smooth or a pair of lines).

A conic bundle is called *minimal* if it can not be obtained from another conic bundle $\pi_1 : S_1 \rightarrow B$ by blowing up points. If every fiber of π is smooth then $\pi : S \rightarrow B$ is minimal; these are the trivial examples.

As a generalization of Noether's result, Iskovskikh [Isk79] and Mori [Mor82] proved that every surface with a pencil of rational curves is birational (over B) to a minimal conic bundle. Thus, up-to birational equivalence, it is sufficient to study minimal conic bundles.

It is a long standing interesting question to understand which conic bundles are rational or unirational. If $\pi : S \rightarrow B$ is unirational then there is a dominant morphism $\mathbb{P}^2 \dashrightarrow B$, hence $B \cong \mathbb{P}^1$ by Lüroth's theorem; see also Lemma 25. Thus from now on we restrict our attention to minimal conic bundles $\pi : S \rightarrow \mathbb{P}^1$.

A basic numerical invariant of a minimal conic bundle is the *number of singular fibers*, denoted by $\delta(S)$. People coming from the theory of del Pezzo surfaces prefer to use instead the self intersection of the canonical class (K_S^2) , also called the *degree*. These are related by the formula

$$(K_S^2) = 8 - \delta(S).$$

As with del Pezzo surfaces, the birational complexity of S increases as the degree decreases.

Conic bundles with $\delta(S) \in \{0, 1, 2, 3\}$ are rational if they have a k -point; see [Man72, Sec.IV.8] or [Kol96, III.3.13].

Conic bundles with $\delta(S) \in \{4, 5\}$ are unirational if they have a k -point. This has been essentially known to Segre [Seg51], Iskovskikh [Isk70] and Manin [Man72], though most of their work focused on smooth del Pezzo surfaces over infinite fields. Extra difficulties over small fields were resolved in [Kol02]. For the conic bundle case see also [Mes96a].

Unirationality of degree 2 del Pezzo surfaces with a k -point is still not fully proved; see [STVA14, FvL14] for the known results.

We treat conic bundles with $\delta(S) = 6$ in Section 8.

In this note we settle the next case and prove the following

Theorem 7. *Let k be a field of characteristic $\neq 2$ and $\pi : S \rightarrow \mathbb{P}^1$ a conic bundle with 7 singular fibers. Then S is unirational over k .*

Moreover, if $|k| \geq 53$ then π has a degree 8 rational multi-section. That is, there is a map $\sigma : \mathbb{P}^1 \rightarrow S$ such that $\pi \circ \sigma : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ has degree 8.

Combining with the earlier results gives the following more uniform statement.

Corollary 8. *Let k be a field of characteristic $\neq 2$ and $\pi : S \rightarrow \mathbb{P}^1$ a conic bundle with ≤ 7 singular fibers. The following are equivalent.*

- (1) S has a k -point.
- (2) S is unirational over k .

Conic bundles with ≥ 9 singular fibers seem to behave quite differently and various heuristics suggest contradictory possibilities. Working with forms similar to (**) shows that such conic bundles appear as quadratic families of genus ≥ 2 hyperelliptic curves. This suggests that they should contain few k -points and they should not be unirational. On the other hand, they become rational after a finite field extension, so they are not far from being unirational.

This leaves conic bundles with 8 singular fibers as a quite interesting case. Some of these are unirational by [Mes94]. We hope to return to them in the future.

Unirationality of conic bundles—with any number of singular fibers—over local fields and sufficiently large finite fields is treated in [Yan85, Mes96b, Kol99, KS03]. These, however, seem to be special properties of these fields rather than of conic bundles.

9 (Proof of Corollary 2). Pick any rational point $p \in S(k)$ that lies on a smooth fiber F . Then $F \cong \mathbb{P}_k^1$ and projection of F to the t -axis is described by a degree 2 rational function $q(u)$. We are done unless we get torsion points for infinitely many $u \in K$. By [Sil94, Thm.III.11.4] this can happen only if the whole fiber F consists of m -torsion points for some fixed m , in which case the modular curve $X_1(m)$ is rational. Thus $m \leq 12$ and there are only finitely many such fibers. \square

10 (Proof of Corollary 3). A map $\mathbb{P}^2 \dashrightarrow S \dashrightarrow \mathbb{P}_x^1 \times \mathbb{P}_t^1$ is described by 4 homogeneous polynomials of some degree d :

$$(u:v:w) \mapsto (H_1(u:v:w) : H_2(u:v:w), H_3(u:v:w) : H_4(u:v:w)).$$

Thus if $\text{ht}(u:v:w) \leq B^{1/d}$ then the image point has height $\leq B$ (up to constant factors). This gives $\epsilon \geq \frac{2}{d}$. \square

A difficulty in computing the best ϵ is that the map $\phi : \mathbb{P}^2 \dashrightarrow S$ does not determine d since composing ϕ with a birational self-map of \mathbb{P}^2 can increase or decrease the value of d . The geometric description suggests that it might be possible to find H_i of degree ≤ 8 , but so far we have managed to write down only much higher degree examples.

2. THE GEOMETRY OF CONIC BUNDLES

We give a quick summary of the geometry of rational surfaces and conic bundles. This is a classical topic. For some modern treatments see [Kol96, Secs.III.2–3], [KSC04, Chap.3] or [Dol12, Chap.8].

11 (Rational surfaces). Let \bar{k} be an algebraically closed field and S a smooth rational surface over \bar{k} . Set $r := (K_S^2)$.

Basic examples are \mathbb{P}^2 and $\mathbb{F}_e := \text{Proj}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(e))$ for $e \geq 0$.

Every other S is obtained either from \mathbb{P}^2 by blowing up $9 - r$ points or from some \mathbb{F}_e by blowing up $8 - r$ points. (There can be many such representations.)

Note that $h^0(\mathbb{P}^2, -K) = 10$, $h^0(\mathbb{F}_e, -K) = 9$ for $0 \leq e \leq 3$ and $h^0(\mathbb{F}_e, -K) = e + 6$ for $e \geq 3$. Thus we see that $\dim | -K_S| \geq r$.

If $C \in | -K_S|$ is irreducible and reduced then, by the adjunction formula,

$$2p_a(C) - 2 = (C \cdot (C + K_S)) = 0, \quad \text{hence } p_a(C) = 1.$$

Otherwise every irreducible component of C is a smooth rational curve.

Definition 12 (Weak del Pezzo surfaces). Classically, a smooth, projective, geometrically irreducible surface S is called a *del Pezzo surface* if $-K_S$ is ample, but modern usage allows S to have Du Val singularities.

A smooth, projective, geometrically irreducible surface S is called a *weak del Pezzo surface* if $(K_S^2) > 0$ and the following equivalent conditions hold.

- (1) S is the minimal resolution of a del Pezzo surface with Du Val singularities.
- (2) $-K_S$ is nef, that is, $(C \cdot K_S) \leq 0$ for every irreducible curve $C \subset S$.
- (3) $| -K_S|$ has no fixed components.
- (4) There is an irreducible curve $C \in | -K_S|$.

In these cases $\dim | -K_S| = (K_S^2)$ is called the *degree* of S . (If $-K_S$ is ample and $d := (K_S^2) \geq 3$ then $| -K_S|$ gives an embedding $\phi_S : S \hookrightarrow \mathbb{P}^d$ as a surface of degree d .) For a curve $C \subset S$ the intersection number $(C \cdot K_S)$ is called the *degree* of C . (If $d := (K_S^2) \geq 3$ then this is the same as the degree of $\phi_S(C) \subset \mathbb{P}^d$.)

Definition 13 (Minimal conic bundles). For any conic bundle $\pi : S \rightarrow \mathbb{P}^1$ we use F to denote a fiber of π . The adjunction formula gives that $(F \cdot K_S) = -2$.

Assume next that $\pi : S \rightarrow \mathbb{P}^1$ is minimal. Let $p \in \mathbb{P}^1$ be any point and F_p the fiber of π over p . Then a fiber F is isomorphic, over $k(p)$, either to a smooth conic or to a conjugate pair of lines. In particular, F_p is irreducible over $k(p)$.

If $\pi : S \rightarrow \mathbb{P}^1$ is a degree 1, weak del Pezzo conic bundle then the linear system $| -K_S|$ is a pencil of elliptic curves with a unique base point; henceforth denoted by p^* . We use F^* to denote the fiber containing p^* .

Definition 14 (Bertini involution). (See [Dol12, Sec.8.8] or [KSC04, 3.42].) Let S be a del Pezzo surface of degree 1, possibly with Du Val singularities. Then $| -2K_S|$ gives a morphism of degree two $\pi_0 : S \rightarrow Q$ (where Q is a quadric cone in \mathbb{P}^3). The base point p^* is the preimage of the vertex of Q . The corresponding Galois involution $\tau : S \rightarrow S$ is called the *Bertini involution*.

Let $C \in | -K_S|$ be a geometrically irreducible curve; it has arithmetic genus 1. Using the base point $p^* \in C$ as the identity of the group structure, the Bertini involution sends $q \in C$ to $-q \in C$.

15 (Curves of low degree). Let T be a degree 1 del Pezzo surface, possibly with Du Val singularities. The following is the list of degree 1 curves on T ; with degree as in Definition 12.

- (1) Double covers of a ruling of Q . These are members of $| -K_T|$.
- (2) In some cases $\pi_0^{-1}(Q \cap P)$ is reducible for a plane $P \subset \mathbb{P}^3$ not passing through the vertex of Q . (If T is smooth, this happens for 120 planes.)

Both components are then smooth, rational. These are the (-1) -curves on T ; they do not pass through p^* .

The following is the list of degree 2 curves on T .

- (3) Usually $\pi_0^{-1}(Q \cap P)$ is irreducible for a plane $P \subset \mathbb{P}^3$. Such curves do not pass through p^* .
- (4) In some cases $\pi_0^{-1}(Q \cap Q')$ is reducible for a quadric $Q' \subset \mathbb{P}^3$.

From the above list we can read off the connection between the Bertini involution and the conic bundle structure.

16 (Bertini involution and the conic bundle structure). Let T be a degree 1 del Pezzo surface, possibly with Du Val singularities, and $|F|$ a pencil of degree 2 rational curves. It has at least 1 member F^* passing through p^* .

If F^* is reducible then at least one of the irreducible components must be as in case (15.1). Furthermore, either both irreducible components are as in case (15.1) or one of them is as in case (15.1) and the other as in case (15.2). In the first case p^* is the singular point of F^* and F^* is invariant under the Bertini involution. In the second case p^* is a smooth point of F^* and F^* is not invariant under the Bertini involution.

If F^* is irreducible then F^* must be as in case (15.4). Thus F^* is not invariant under the Bertini involution.

Note further that $\pi_0^*|\mathcal{O}_Q(1)|$ is a complete linear system, thus if one member of $|F|$ is invariant under the Bertini involution then every member is invariant.

Furthermore, if a pencil $|F|$ is invariant under an involution then at least 2 members are invariant, hence, in our case, every member is invariant.

We have proved the following.

Proposition 17. *Let $\pi : S \rightarrow \mathbb{P}^1$ be a weak del Pezzo conic bundle of degree 1. Then the Bertini involution preserves the conic bundle structure iff p^* is a singular point of a singular fiber of π . \square*

3. CLASSIFICATION UP-TO ISOMORPHISM

Lemma 18. *Let $\pi : S \rightarrow B$ be a minimal conic bundle with at least 1 singular fiber. Then every line bundle on S is of the form $\mathcal{O}_S(-aK_S) \otimes \pi^*L_B$ where L_B is a line bundle on B . In particular, every curve $C \subset S$ is either a fiber or the projection $\pi|_C : C \rightarrow B$ has even degree.*

Proof. Assume for simplicity that k is perfect. Let L be a line bundle on S and $F \subset S$ a reducible geometric fiber with irreducible components $F' + F''$. If $(L \cdot F') \neq (L \cdot F'')$ then the conjugates of F' form a conjugation invariant set of pairwise disjoint (-1) -curves. This contradicts the minimality assumption. Thus $(L \cdot F') = (L \cdot F'')$ hence L has even degree on the generic fiber. Therefore $L(aK_S)$ is trivial on every smooth fiber of π for some $a \in \mathbb{Z}$. By the above considerations, $L(aK_S)$ is also trivial on the singular fibers. By cohomology and base change, this implies that

$$L(aK_S) \cong \pi^*\pi_*(L(aK_S)).$$

Essentially the same argument works if k is not perfect; see [Mor82, Chap.2]. \square

Lemma 19. *Let k be a field, B a smooth conic and $\pi : S \rightarrow B$ a minimal conic bundle of degree $r := (K_S^2) \geq 1$. Then one of the following holds.*

- (1) S is a weak del Pezzo surface,
- (2) $r = 2$ and $|-K_S| = C + |2F|$ where C is a conjugate pair of disjoint smooth rational curves with self intersection -3 .
- (3) $r = 1$ and $|-K_S| = C + |F|$ where C is a geometrically irreducible smooth rational curve with self intersection -3 .
- (4) $\pi : S \rightarrow \mathbb{P}^1$ is a \mathbb{P}^1 -bundle.

Proof. If π has a section then $\pi : S \rightarrow \mathbb{P}^1$ is a \mathbb{P}^1 -bundle. Thus assume from now on that there are no sections.

Write $|-K_S| = C + |M|$ where C is the fixed part and $|M|$ the mobile part. If $C = 0$ then S is a weak del Pezzo surface; see Definition 12.3.

If $C \neq 0$ then C can not contain a fiber, so C is a union of multi-sections. Furthermore,

$$2 = (-K_S \cdot F) = (C \cdot F) + (M \cdot F) \geq (C \cdot F) \geq 2,$$

the last inequality by Lemma 18. Thus $(C \cdot F) = 2$ and so we have 3 possibilities for C : geometrically irreducible, conjugate pair of sections or geometrically nonreduced; the latter can happen only in characteristic 2. In all cases $(M \cdot F) = 0$, thus $M \sim sF$ for some s and $s = \dim |sF| = \dim |M| = \dim |-K_S| \geq r$. Note that $\deg \omega_C = (C \cdot (C + K_S)) = -s(C \cdot F) = -2s$ and $(C^2) = ((-K_S - sF)^2) = r - 4s$. We distinguish several cases.

If C is geometrically irreducible and reduced then $\deg \omega_C \geq -2$. Thus $s = 1$ hence $r = 1$ and C is a smooth, geometrically rational curve with $(C^2) = -3$; this is case (3).

If C is disconnected then $C_{\bar{k}}$ is the disjoint union of 2 components that are conjugate over k . Thus either $\deg \omega_C \geq 0$ (this is not possible for us) or $\deg \omega_C = -4$ and $C_{\bar{k}}$ is the disjoint union of 2 smooth, rational curves. If $r = s = 2$ then $(C^2) = -6$; this is case (2). Otherwise $r = 1$ and (C^2) is odd, but this is not possible.

If C is connected, geometrically reduced and reducible then $\deg \omega_C \geq -2$, hence $s = r = 1$ and $(C^2) = -3$. In this case $C_{\bar{k}} = C_1 + C_2$ and so $(C^2) = 2(C_1^2) + 2(C_1 \cdot C_2)$ is even, a contradiction. This case can not happen.

Finally, if C is irreducible but geometrically nonreduced then C intersects only 1 of the geometric irreducible components of every singular fiber. This is impossible by Lemma 18. \square

Corollary 20. *A degree 1 conic bundle over k has a k -point.*

Proof. If S is weak del Pezzo then the unique base point of $|-K_S|$ is a k -point. If S is as in (19.3) then C is a smooth, rational curve and $(C^2) = -3$ gives a degree 3 point on C . Thus $C \cong \mathbb{P}^1$.

We still need to deal with the cases when S is not minimal, thus S is obtained from a minimal conic bundle T of degree $m + 1$ by blowing up m points. In particular, T has a 0-cycle of degree 1. For $m \geq 3$ this easily implies that T has a k -point [Man72, Sec.IV.8].

If $m = 2$ then T is birational to a cubic surface with a degree 2 point, hence it has a k -point as well.

If $m = 1$ then T has a k -point (the one we blow up) and so does S . \square

Remark 21. It is not known if a cubic surface with a 0-cycle of degree 1 has a k -point or not, see [Cor76]. There are degree 2 del Pezzo surfaces with a degree 3 point—and hence with a 0-cycle of degree 1—but without k -points.

Definition 22. We divide degree 1, smooth, minimal conic bundles $\pi : S \rightarrow \mathbb{P}^1$ into three types up-to isomorphism. By Lemma 19, either we are in case (19.3) or $|-K_S|$ is a pencil with a unique base point $p^* \in S(k)$. In the latter case let F^* denote the fiber of π containing p^* . We have three possibilities for S .

Type G: (General case) S is a weak del Pezzo surface and p^* lies on a smooth fiber F^* . Therefore $F^* \cong \mathbb{P}_k^1$.

Type S: (Special case) S is a weak del Pezzo surface and p^* lies on a singular fiber F^* . In this case p^* is the unique singular point of F^* .

Type E: (Exceptional case) S is not a weak del Pezzo surface.

(The birational classification is different. As a consequence of Theorem 7 and Proposition 33 we see that every minimal conic bundle of degree 1 is birational to a minimal conic bundle of Type G.)

Next we give more detailed descriptions of the two main Types.

23 (Type G). After blowing up p^* , the pencil $|-K_S|$ becomes base point free and gives a morphism $\pi_2 : B^*S \rightarrow \mathbb{P}_{uv}^1$ whose general fiber is a genus 1 curve (smooth if $\text{char } k \neq 2, 3$). We thus have a morphism

$$(\pi, \pi_2) : B^*S \rightarrow \mathbb{P}_{xy}^1 \times \mathbb{P}_{uv}^1.$$

Since $(-K_S \cdot F) = 2$, the morphism (π, π_2) has degree 2. Assume that there is an irreducible exceptional curve D . Then D is contained in a fiber of π , but every fiber of π is irreducible by Lemma 18. Thus (π, π_2) is finite. If $\text{char } k \neq 2$, let $C_S \subset \mathbb{P}_{xy}^1 \times \mathbb{P}_{uv}^1$ be its branch curve. A general fiber F of π has genus 0 and π_2 restricts to a double cover $F \rightarrow \mathbb{P}_{uv}^1$. Thus the latter has 2 branch points. A general fiber E of π_2 has genus 1 and π restricts to a double cover $E \rightarrow \mathbb{P}_{xy}^1$. Thus the latter has 4 branch points. Therefore C_S has bidegree $(2, 4)$. We can thus write the equation of C_S as

$$a_4(u, v)x^4 + a_3(u, v)x^3y + a_2(u, v)x^2y^2 + a_1(u, v)xy^3 + a_0(u, v)y^4 = 0, \quad (23.1)$$

where the a_i are homogeneous of degree 2.

If $\text{char } k \neq 2$ then a double cover of a smooth surface is smooth iff the branch curve is smooth, hence C_S is smooth.

We will repeatedly use 2 observations connecting the a_i to the properties of B^*S .

- (2) The curve defined by (23.1) is singular along $y = 0$ iff there is a (u_0, v_0) that is a double root of a_4 and root of a_3 . In particular, if $a_4 \equiv 0$ then C_S is singular at the roots of a_3 .
- (3) The preimage of $(y = 0)$ in B^*S is reducible (resp. geometrically reducible) iff a_4 is a square in $k[u, v]$ (resp. in $\bar{k}[u, v]$).

We can choose the x, y -coordinates such that F^* lies over $(y = 0)$. Note that the corresponding fiber of $\pi_B^*S \rightarrow \mathbb{P}_{xy}^1$ is the union of the exceptional curve $E^* \subset B^*S$ and of the birational transform of F^* . Thus $a_4(u, v)$ is a square. We can choose the u, v -coordinates such that $a_4(u, v) = v^2$. Since $(y = v = 0)$ is a smooth point of C_S , we see that u^2 appears in $a_3(u, v)$ with non-zero coefficient. With these choices, in affine coordinates x, u, z , the equation of B^*S is

$$z^2 = x^4 + a_3(u)x^3 + a_2(u)x^2 + a_1(u)x + a_0(u) \quad (23.4)$$

$\deg a_i \leq 2$ and $\deg a_3 = 2$ where we use the shorthand $a_i(u) := a_i(u, 1)$.

We can thus think of a conic bundle of Type G as a quadratic family of genus 1 curves with two k -points.

Warning. This double cover gives a biregular involution on B^*S , hence a birational involution on S but, by Proposition 17, it is *not* the Bertini involution; see also Paragraph 28.

24 (Type S). Here p^* is the singular point of F^* and, after blowing up p^* , the pencil $| - K_S |$ becomes base point free. We again have a morphism

$$(\pi, \pi_2) : B^*S \rightarrow \mathbb{P}_{xy}^1 \times \mathbb{P}_{uv}^1.$$

The birational transform of F^* (consisting of a conjugate pair of (-2) -curves) is contracted. As above, if $\text{char } k \neq 2$ then, (π, π_2) branches along a curve B_S of bidegree $(2, 4)$, however B_S has 2 singular points at the 2 images of the birational transform of F^* . These 2 points lie on the same fiber, namely F^* , of the 1st projection. Thus $B_S = E_0 \cup C_S$ where $C_S \subset \mathbb{P}_{xy}^1 \times \mathbb{P}_{uv}^1$ is a smooth curve of bidegree $(2, 3)$. We can thus write the equation of C_S as

$$a_3(u, v)x^3 + a_2(u, v)x^2y + a_1(u, v)xy^2 + a_0(u, v)y^3 = 0, \quad (24.1)$$

where the a_i are homogeneous of degree 2. We can choose the x, y -coordinates such that $E_0 = (y = 0)$. Then we have the following.

- (2) a_3 is irreducible in $k[u, v]$.
- (3) The curve defined by (24.1) is singular along $x = 0$ iff there is a (u_0, v_0) that is a double root of a_0 and root of a_1 .

Indeed, the conjugate pair of irreducible components of F^* map to the points $(y = a_3 = 0)$, proving (2) while (3) is the same argument as (23.2).

In suitable coordinates we can write the affine equation as

$$z^2 = a_3(u)x^3 + a_2(u)x^2 + a_1(u)x + a_0(u), \quad (24.4)$$

where $\deg a_i \leq 2$ and $\deg a_3 = 2$.

As before, this double cover gives a birational involution on S . By Proposition 17 it is the Bertini involution.

4. UNIRATIONALITY FOR TYPE G AND E

Lemma 25 (Enriques criterion). *A surface with a pencil of rational curves $\pi : T \rightarrow B$ is unirational iff it has a rational multi-section. That is, a morphism $\sigma : \mathbb{P}^1 \rightarrow T$ such that $\pi \circ \sigma : \mathbb{P}^1 \rightarrow B$ is nonconstant.*

Proof. Assume that there is a dominant map $p : \mathbb{P}^n \dashrightarrow T$. Over an infinite base field, a general line $L \subset \mathbb{P}^n$ gives a morphism $\sigma : \mathbb{P}^1 \rightarrow T$ such that $\pi \circ \sigma : \mathbb{P}^1 \rightarrow B$ is nonconstant.

Over finite fields one needs to be a little more careful; see [Kol02, Lem.12].

Conversely, assume that there is a morphism $\sigma : \mathbb{P}^1 \rightarrow T$ such that $\pi \circ \sigma : \mathbb{P}^1 \rightarrow B$ is nonconstant. Then $\mathbb{P}^1 \times_B T \rightarrow \mathbb{P}^1$ is a surface with a pencil of rational curves that has a section. Thus, as we noted in Definition 6, $\mathbb{P}^1 \times_B T$ is rational hence T is unirational. \square

Proposition 26. *A degree 1 conic bundle $\pi : S \rightarrow \mathbb{P}^1$ of Type E is unirational.*

Proof. We are in case (19.3), thus $|-K_S| = C + |F|$ where C is a geometrically irreducible smooth rational curve with self intersection -3 . Thus $\mathcal{O}_S(C)|_C$ is a line bundle of odd degree, hence $C \cong \mathbb{P}^1$. Thus S is unirational by (25). \square

Proposition 27. *A degree 1 conic bundle $\pi : S \rightarrow \mathbb{P}^1$ of Type G is unirational.*

More precisely, the Bertini involution (as in Definition 14) does not preserve the conic bundle structure and $\tau(F^) \subset S$ is a rational multi-section of degree 8.*

Proof. As in Paragraph 14, $|-2K_S|$ gives a morphism $\pi_0 : S \rightarrow Q$ and, as we discussed in Paragraph 16, the conic bundle structure is given by a pencil of rational curves $|F|$ coming from quadric sections $Q' \cap Q$ whose preimages in S are reducible. These preimages are thus members of $|-4K_S|$ and the Bertini involution interchanges $|F|$ with $|-4K_S - F|$. Since $(F \cdot (-4K_S - F)) = 8$, the curves in $|-4K_S - F|$ give degree 8 multi-sections.

The fiber F^* is rational over k , thus $\tau(F^*)$ is a rational multi-section. So S is unirational by the Enriques criterion 25. \square

28 (Explicit formulas). Fixing a value of $u = u_0$ set $a_i := a_i(u_0)$. We get an elliptic curve

$$z^2 = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0.$$

The base point p^* is the point at infinity on the $(z \sim x^2)$ -branch. Then the linear system $|2p^*|$ is given by parabolas of the form

$$z = x^2 + \frac{1}{2}a_3x + \lambda.$$

Note that

$$(x^2 + \frac{1}{2}a_3x + \lambda)^2 = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

is a degree ≤ 2 equation in x and there is a unique choice, namely

$$\lambda = \frac{1}{8}(4a_2 - a_3^2),$$

that gives a linear equation. The root of this equation gives the x -coordinate of $-p'$ where p' is the point at infinity on the $(z \sim -x^2)$ -branch. Thus $-p'$ is given by

$$\left(x := \frac{(4a_2 - a_3^2)^2 - 64a_0}{64a_1 - 8a_3(4a_2 - a_3^2)}, z := x^2 + \frac{1}{2}a_3x + \frac{1}{8}(4a_2 - a_3^2) \right).$$

Thus, in the conic bundle case, the restriction of τ to F^* is given by

$$\begin{aligned} x(u) &= \frac{(4a_2(u) - a_3(u)^2)^2 - 64a_0(u)}{64a_1(u) - 8a_3(u)(4a_2(u) - a_3(u)^2)} \quad \text{and} \\ z(u) &= x(u)^2 + \frac{1}{2}a_3(u)x(u) + \frac{1}{8}(4a_2(u) - a_3(u)^2). \end{aligned}$$

In the type G case $\deg a_3 = 2$, so the numerator of $x(u)$ has degree 8 and the denominator degree 6. In particular, $x(u)$ is never constant, in agreement with our earlier computations in Paragraph 16.

5. UNIRATIONALITY FOR TYPE S; FIRST CONSTRUCTION

In the special case there are no obvious rational curves on S but the smooth fibers of π are conics and usually some of them are rational. However, the Bertini involution maps every fiber to itself, so it does not yield new rational curves.

We look instead at a degree 4 endomorphism $m_2 : S \dashrightarrow S$ obtained as follows. (More generally, we have a degree r^2 endomorphism $m_r : S \dashrightarrow S$ for every $r \in \mathbb{Z}$.)

29 (Doubling map). Let $\pi : S \rightarrow \mathbb{P}^1$ be a degree 1, minimal, weak del Pezzo conic bundle of type S. Let $C \in |-K_S|$ be a geometrically irreducible curve. Using the base point $p^* \in C$ as the identity of the group structure, the doubling map m_2 sends $q \in C$ to $2q \in C$. Note that $q, -q$ lie on the same fiber and so $2q, -2q$ also lie on the same fiber. Thus the x -coordinate of $m_2(q)$ depends only on $(x(q), u(q))$.

So pick a smooth fiber F , say over $x = 0$. Let $q_0 \in F(\bar{k})$ have coordinates $(0, y_0, u_0)$. By explicit computation we get that

$$x(m_2(q_0)) = \frac{a_1(u_0)^2 - 4a_2(u_0)a_0(u_0)}{4a_3(u_0)a_0(u_0)}. \quad (29.1)$$

We know that a_3, a_0 are not identically zero since the vanishing of either one would give a reducible, hence singular, branch curve C_S . If this expression is non-constant then $m_2(F) \subset S$ is a multi-section of π . Next we describe the cases when this construction does not give a rational multi-section.

Proposition 30. *Let $\pi : S \rightarrow \mathbb{P}^1$ be a degree 1, minimal conic bundle of Type S and $F_0 \subset S$ a smooth fiber of π . Then one of the following holds.*

- (1) $m_2(F_0) \subset S$ is a multi-section of π ,
- (2) $m_4(F_0) \subset S$ is a multi-section of π ,
- (3) F_0 consists of 2-torsion points and $m_r(F_0) \subset F_0 \cup \{p^*\}$ for every $r \in \mathbb{Z}$.
- (4) F_0 consists of 3-torsion points and $m_r(F_0) \subset F_0 \cup \{p^*\}$ for every $r \in \mathbb{Z}$.

Proof. We are in case (1) unless (29.1) is constant. Then, for some $\alpha \in k$,

$$a_1^2 - 4a_2a_0 = 4\alpha a_3a_0 \quad \text{and hence} \quad a_1^2 = 4a_0(a_2 + \alpha a_3).$$

We claim that a_1 is a constant multiple of a_0 . Since a_0 divides a_1 this holds once a_0 has only simple roots. (We need to work projectively, that is, if $\deg a_0 < 2$ then there is a root at infinity of multiplicity $2 - \deg a_0$.) If u_0 is a double root of a_0 then it is also a root of a_1 but then C_S is singular at $x = 0, u = u_0$ by (24.3), a contradiction. Thus there is a $\beta \in k$ such that

$$a_1 = 2\beta a_0 \quad \text{and so} \quad \beta^2 a_0 = (a_2 + \alpha a_3).$$

Expressing a_2 and substituting we get the equation

$$a_3x^2(x - \alpha) + a_0(\beta x + 1)^2. \quad (30.5)$$

Note further that the fiber F_α is isomorphic to the fiber F_0 . (It is the image of F_0 under m_2 .) Thus, $m_2(F_\alpha) \subset S$ is a multi-section of π , except when the polynomial given by the substitution $x = x' + \alpha$, namely

$$a_3(x' + \alpha)^2x' + a_0(\beta x' + \alpha\beta + 1)^2, \quad (30.6)$$

has the same form as (30.5). Rearranging by powers of x' we get that

$$a'_0 = a_0(\alpha\beta + 1)^2 \quad \text{and} \quad a'_1 = a_3\alpha^2 + 2a_0\beta(\alpha\beta + 1).$$

Thus $a'_0 \mid a'_1$ holds iff $a_0 \mid a_3$, unless $\alpha = 0$. If $a_0 \mid a_3$ then the polynomial (30.5) is reducible and C_S is singular, so this can not happen.

Finally, if $\alpha = 0$, we are down to just 2 exceptional types corresponding to $\beta \neq 0$ and $\beta = 0$:

- (7) ($z^2 = a_3(u)x^3 + a_0(u)(x-1)^2$). In this case F_1 is another smooth fiber and $m_2(F_1) \subset S$ is a multi-section of π_1 .
- (8) ($z^2 = a_3(u)x^3 + a_0(u)$). There are no other obvious smooth fibers.

In both cases F_0 consists of 3-torsion points and every multiplication map m_r sends F_0 to itself. \square

Corollary 31. *Let S be a degree 1, minimal conic bundle of Type S over k .*

- (1) *If k is infinite then there is a degree 2 field extension K/k such that S_K is unirational.*
- (2) *If k is finite and $|k| \geq 19$ then S is unirational.*

Proof. Since an elliptic curve has at most 11 non-trivial 2- or 3-torsion points, $\pi : S \rightarrow \mathbb{P}^1$ has at most 11 fibers that consist of 2- or 3-torsion points. Pick any point $v \in \mathbb{P}^1(k)$ such that the fiber F_v over v is smooth and does not consist of 2- or 3-torsion points. Since F_v is a conic, it has a point in some degree 2 field extension K/k . Then Proposition 30 produces a rational multi-section $\sigma : \mathbb{P}_K^1 \rightarrow S_K$, hence S_K is unirational.

If k is finite then F_v has a k -point, so we get unirationality over k . \square

6. BIRATIONAL MAPS OF CONIC BUNDLES

32 (Elementary transformations). Let $\pi : S \rightarrow B$ be a conic bundle over an algebraically closed field, $F \subset S$ a smooth fiber and $q \in F$ a point. The elementary transformation of S centered at q is obtained by first blowing up q and then contracting the birational transform of F . We get another conic bundle of the same degree and a birational map $\rho_q : S \dashrightarrow S_q$.

More generally, let $Q \subset S$ be a finite collection of points, each contained in a smooth fiber such that every fiber contains at most 1 point. Let F_Q be the union of all fibers that contain a point of Q . We can then blow up Q to get $\alpha_Q : T_Q \rightarrow S$ with exceptional curve E_Q . Let $F'_Q \subset T_Q$ denote the birational transform of F_Q . We can next contract F'_Q to get $\beta_Q : T_Q \rightarrow S_Q$ and the composite $\rho_Q : S \dashrightarrow S_Q$. These maps and surfaces fit into a diagram

$$\begin{array}{ccc} & T_Q & \\ \alpha_Q \swarrow & & \searrow \beta_Q \\ S & \xrightarrow{\rho_Q} & S_Q \end{array} \quad (32.1)$$

If S and Q are defined over k then so is $\rho_Q : S \dashrightarrow S_Q$.

In order to compute the birational transform of $-K_{S_Q}$ note that

$$K_{T_Q} \sim \alpha_Q^* K_S + E_Q \sim \beta_Q^* K_{S_Q} + F'_Q \quad \text{and} \quad F'_Q \sim \alpha_Q^* F_Q - E_Q.$$

These together imply that

$$\beta_Q^* (-K_{S_Q}) \sim \alpha_Q^* (-K_S + F_Q) - 2E_Q. \quad (32.2)$$

Equivalently, if the linear system $| -K_{S_Q} |$ is not empty then

$$\rho_Q^* | -K_{S_Q} | = | -K_S + F_Q | (-2Q). \quad (32.3)$$

That is, the pull-back of $| -K_{S_Q} |$ by ρ_Q consists of those curves in $| -K_S + F_Q |$ that have multiplicity ≥ 2 at each point of Q .

Consider now the very special case when there is an irreducible curve $C \in | -K_S |$ and Q consists of smooth points of C . Then

$$C + F_Q \in | -K_S + F_Q | (-2Q) \quad (32.4)$$

and $C_Q := \rho_Q(C) \in | -K_{S_Q} |$. In particular, S_Q is also weak del Pezzo by (12.4). Note that the line bundle $\mathcal{O}_S(C + F_Q)$ does not depend on the choice of C and Q

since it is isomorphic to $\mathcal{O}_S(-K_S + nF)$ where $n := |Q|$. Thus, using (32.3) and the natural isomorphism $C \cong C_Q$ we see that

$$\mathcal{O}_{S_Q}(C_Q)|_{C_Q} \cong \mathcal{O}_S(C + F_Q)|_C(-2Q) \cong \mathcal{O}_S(-K_S + nF)|_C \otimes \mathcal{O}_C(-2Q). \quad (32.5)$$

As a first application we describe how to transform conic bundles of Types S or E into a conic bundle of Type G.

Proposition 33. *Let S be a degree 1, minimal conic bundle of Type S or E over a field k with $\text{char } k \neq 2$ and $|k| \geq 53$. Then S is birational to a conic bundle of Type G iff it has at least one k -point that lies on a smooth fiber. Thus every such S has a degree 8 rational multi-section.*

Proof. We start with Type E. Let $C \subset S$ be the rational double section and F a smooth fiber with a k -point. Pick $q \in F(k)$ that is not on C . We get an elementary transformation $\rho_q : S \dashrightarrow S_q$. Note that $C \in |-K_S - 2F|$ by (19.2), hence $C + 2F \in |-K_S|(-2q)$ and so $C_q \in |-K_{S_q}|$. Thus S_q is weak del Pezzo by (12.4).

Assume next that S is of type S. Pick a k -point q lying on a smooth fiber F . Note that $F \cong \mathbb{P}_k^1$ since F is a conic with a k point.

Let $C \in |-K_S|$ be the unique curve passing through q . It is better to think of C as determined by the point pair $F \cap C = \{q, q'\}$.

The restriction of $|-K_S|$ to F is a degree 2 pencil on F , hence it has at most 2 ramification points since $\text{char } k \neq 2$. Thus $q \neq q'$ holds for all but 2 pairs $\{q, q'\}$. Since the elliptic pencil has at most 12 singular fibers, C is geometrically irreducible and $q \neq q'$ for all but 26 points $q \in F(k)$.

We are in the case considered in (32.4), thus the map ρ_q transforms C into a curve $C \cong C_q \in |-K_{S_q}|$ and, by (32.5),

$$\mathcal{O}_{S_q}(C_q)|_{C_q} \cong \mathcal{O}_C(3p^* - 2q).$$

(Here we used that S is of Type S, thus $F|_C \sim 2p^*$.) Therefore the base point p_q^* of S_q is the unique point on $C \cong C_q$ such that $p_q^* \sim 3p^* - 2q$. Hence S_q is of Type G iff p_q^* lies on a smooth fiber.

If $p_q^* = p^*$ then $2p^* \sim 2q$ and so C is either singular at q or is tangent to F at q . This was excluded.

If p_q^* lies on a singular fiber, then it is the unique singular point of the fiber. Since any two curves in $|-K_S|$ meet only at p^* , in this case p_q^* uniquely determines $\{q, q'\}$. This gives another 24 possible exceptions. Thus, with at most 50 exceptions, p_q^* lies on a smooth fiber of π and S_q is of Type G. The last assertion then follows from Proposition 27. \square

7. UNIRATIONALITY FOR TYPE S; SECOND CONSTRUCTION

Let $T \subset \mathbb{P}^3$ be a cubic surface. Given 2 points q, q' on T the line connecting them usually intersects T in a unique 3rd point $p(q, q')$. This gives a rational map $\text{Sym}^2(T) \dashrightarrow T$ that is very useful in studying rational points on T ; see for instance [Kol08]. Note that one can also think of $p(q, q')$ as the base point of the linear system $|-K_T|(-q - q')$.

Generalizing the latter point of view gives similar maps for degree 1 conic bundles. This will complete the proof of Theorem 7.

Construction 34. Let S be a degree 1, weak del Pezzo conic bundle. The linear system $| -K_S + nF |$ has dimension $3n + 1$ and self-intersection $4n + 1$.

Therefore, given any set of n points $Q = \{q_1, \dots, q_n\}$, the linear system

$$| -K_S + nF |(-2Q)$$

has dimension at least 1 and there is a—possibly empty—open subset $U \subset S^n$ such that $| -K_S + nF |(-2Q)$ has the following properties for every $(q_1, \dots, q_n) \in U$.

- (1) The dimension is 1.
- (2) The general member is geometrically irreducible and, after blowing up Q , it has arithmetic genus 1.
- (3) The base points consist of Q (with multiplicity 2) and one more point; we denote it by p_Q^* .

We will check that $U \neq \emptyset$. If this holds then $Q \mapsto p_Q^*$ defines a morphism $U \rightarrow S$ which descends to a rational map $\Phi_n : \text{Sym}^n(S) \dashrightarrow S$.

Proposition 35. *Let k be a field and S a degree 1, weak del Pezzo conic bundle. Then the above $\Phi_n : \text{Sym}^n(S) \dashrightarrow S$ is defined and dominant.*

Proof. It is sufficient to check these claims over the algebraic closure of k . Thus we may assume to start with that k is algebraically closed. We will exhibit a rather special set of points $Q = \{q_1, \dots, q_n\}$ such that $(q_1, \dots, q_n) \in U$.

Pick an irreducible curve $C \in | -K_S |$ and smooth points $q_1, \dots, q_n \in C$ such that each q_i is contained in a smooth fiber F_i of π and the F_i are all distinct. Then S_Q is another degree 1, weak del Pezzo conic bundle hence $| -K_{S_Q} |$ has dimension 1 and its general member is geometrically irreducible and has arithmetic genus 1. Thus, by (32.3),

$$\rho_Q^* | -K_{S_Q} | = | -K_S + F_Q |(-2Q)$$

also has dimension 1, its general member is geometrically irreducible and, after blowing up the q_i , has arithmetic genus 1. Thus the extra base point p_Q^* is the preimage (under ρ_Q) of the base point of $| -K_{S_Q} |$ (which we also usually denote by p_Q^*). Furthermore, (32.5) says that

$$\mathcal{O}_{C_Q}(p_Q^*) \cong \mathcal{O}_{S_Q}(C_Q)|_{C_Q} \cong \mathcal{O}_S(-K_S + nF)|_C \otimes \mathcal{O}_C(-2Q).$$

Thus $\Phi_n(q_1, \dots, q_n)$ is the unique point on C such that

$$\Phi_n(q_1, \dots, q_n) \sim p^* + n(F|_C) - 2\sum_i q_i.$$

By varying the points q_i inside C we see that $\Phi(U) \cap C$ is dense in C and by varying C we conclude that $\Phi(U)$ is dense in S . \square

Remark 36. Note that while taking the points q_i on the same member of $| -K_S |$ makes the geometric computation easy, it does not give new k -points unless C already has nontrivial k -points.

Proposition 37. *Let X be a geometrically irreducible k -variety and K/k a field extension of degree n . If X is unirational over K then $\text{Sym}^n(X)$ is unirational over k .*

Proof. Since X_K is unirational, there is a dominant rational map $\phi_K : \mathbb{A}^r \dashrightarrow X$ defined over K . Taking the Weil restriction (see, for instance [BLR90, Sec.7.6]) gives a dominant rational map

$$\mathfrak{R}_{K/k}(\phi_K) : \mathbb{A}_k^{rn} \cong \mathfrak{R}_{K/k}(\mathbb{A}^r) \dashrightarrow \mathfrak{R}_{K/k}(X),$$

defined over k . Composing with the natural map $\mathfrak{R}_{K/k}(X) \rightarrow \text{Sym}^n(X)$ shows that $\text{Sym}^n(X)$ is unirational over k . \square

Corollary 38. *Let k be a field and S a degree 1, weak del Pezzo conic bundle over k . Then S is unirational over k .*

Proof. Since S is geometrically rational, it is unirational over a finite field extension K/k . By Proposition 37 we conclude that $\text{Sym}^n(X)$ is unirational over k where $n = \deg(K/k)$. Thus X is unirational by Proposition 35. \square

Remark 39. Note that the above argument proves the unirationality part of Theorem 7 without using any of the earlier constructions. However, a direct application of this argument gives a rather large value for n . The Galois group of \bar{k}/k acts on the 14 (-1) -curves in the 7 singular fibers. Thus all of these (-1) -curves are defined over a Galois extension K/k whose degree divides $7!2^7$. Over K we can contract 7 of these curves to see that S_K is birational to a \mathbb{P}^1 -bundle over \mathbb{P}^1 , hence rational. Thus Corollary 38 gives a dominant rational map

$$\Phi_n : \mathbb{P}^{2n} \dashrightarrow S \quad \text{where} \quad n = 7!2^7 = 645120.$$

Remark 40. In order to get something that is more computable, one should use Corollary 31 and apply Proposition 37 with $n = 2$. We have not been able to write an explicit formula for Φ_2 but the proof gives the following description of the 8-fold section for the equation (**).

First some notation. Let t_0, t'_0 be the roots of $a_3(t) = 0$ and set $p := ((1:0), (t_0:1))$ and $p' := ((1:0), (t'_0:1))$. Given polynomials $f(u), g(u), F(u), G(u) \in K[u]$ let $C(f, g, F, G) \subset \mathbb{P}_x^1 \times \mathbb{P}_t^1$ denote the (closure of the) image of the map

$$u \mapsto (f(u)/g(u), F(u)/G(u)).$$

Step 1. Pick $x, y \in K$ at random. Solving (**) for t we get a conjugate pair of points $r, r' \in S$.

Step 2. Using the elliptic curves (*), compute $r_2 \sim -2r$ and $r'_2 \sim -2r'$. Let $q, q' \in \mathbb{P}_x^1 \times \mathbb{P}_t^1$ denote the projections of r_2, r'_2 .

Step 3. We are now looking for the unique member C^* of the rational pencil $|-4K_S + 7F|(-8q - 8q')$ that passes through p^* . Choosing a birational map $\mathbb{P}_u^1 \rightarrow C^*$ such that $\infty \mapsto p^*$, in the representation of Paragraph 24 the map $\mathbb{P}_u^1 \rightarrow C^* \hookrightarrow S$ is given as follows.

- (i) There are polynomials f, g, F, G, H of degrees 8, 6, 17, 17, 26 such that
- (ii) $C(f, g, F, G)$ has multiplicity 8 at q, q' ,
- (iii) $C(f, g, F, G)$ has multiplicity 3 at p, p' and
- (iv) $a_3(F/G)(f/g)^3 + \dots + a_0(F/G) = (H/(gG))^2$.

(Explanation: We know that C^* is a degree 8 multi-section, thus $f(u)/g(u)$ is a rational function of degree 8. We chose $\mathbb{P}_u^1 \rightarrow C^*$ such that $\infty \mapsto p^*$. Since p^* is a singular point of the fiber at infinity, this corresponds to a double pole at infinity, thus $\deg g = 6$. Furthermore, $C^* \in |-4K_S + 7F|$ and so $(-K_S \cdot C^*) = 18$. Blowing up p^* reduces the intersection number by 1, hence $\deg F, \deg G \leq 17$. The expression on the left side of (iv) has denominator g^3G^2 but this can be a square only if a g factor gets canceled. This gives $\deg H = 26$.)

Step 4. The 8-fold section is the image of the map

$$u \mapsto \left(\frac{f(u)}{g(u)}, \frac{H(u)}{g(u)G(u)}, \frac{F(u)}{G(u)} \right).$$

We can thus obtain many rational points using the above map.

Step 5. Once we have a rational point $q = (x_0, y_0, t_0) \in S$, we also have a simpler degree 8 rational multi-section. In this case we need to find polynomials f, g, F, G, H of degrees 8, 6, 9, 9, 18 such that $C(f, g, F, G)$ has multiplicity 8 at q and 3 at p, p' . The rest is the same as above.

Step 6. A subtlety with this method is that these formulas work only for points in general position and it is difficult to describe these conditions explicitly. Another problem is that $\text{Aut}(\mathbb{P}^1, \infty)$ acts on all solutions and it is not clear how to rigidify the set-up.

8. UNIRATIONALITY OF HIGHER DEGREE CONIC BUNDLES

We complete the proof of Corollary 8 by proving that minimal conic bundles $\pi : S \rightarrow \mathbb{P}^1$ of degree ≥ 2 are unirational over k provided they have a k -point.

As we noted after Corollary 8, the only possibly new case is the unirationality of degree 2, minimal conic bundles. We reduce it to the already established degree 1 case. One could use the methods of the previous sections to get a more direct argument but some case analysis would still remain.

As before, we consider only fields whose characteristic is different from 2. The main reason is that many proofs involve conics or involutions and in this way we avoid inseparability problems. There would be further difficulties with very small fields, especially with \mathbb{F}_2 .

Proposition 41. *Let k be a field and $\pi : S \rightarrow \mathbb{P}^1$ a degree 2, minimal conic bundle over k with a k -point $p \in S(k)$. Then S is unirational over k .*

Proof. By Lemma 42 it is enough to consider the case when S is weak del Pezzo.

If p lies on a smooth fiber of π then the blow-up $B_p S$ is a degree 1 conic bundle over k hence unirational by Corollary 38.

If p lies on a singular fiber F_p , then $B_p S$ is not a conic bundle. We can however contract the birational transform of F_p to get a degree 1 conic bundle over k , albeit with a conjugate pair of A_1 -singularities. It is easy to see that Corollary 38 works for degree 1 conic bundles with Du Val singularities.

It is, however, cleaner to follow Construction 34 directly. Given $Q = \{q_1, \dots, q_n\}$ we consider the linear system $|-K_S + nF|(-2Q - p)$. As in Proposition 35 we show that, for general Q , $|-K_S + nF|(-2Q - p)$ has a unique unassigned base point p_Q^* and it defines a dominant rational map $\Phi_n : \text{Sym}^n(S) \dashrightarrow S$. Thus S is unirational by Proposition 37. \square

Lemma 42. *Let k be a field and $\pi : S \rightarrow \mathbb{P}^1$ a degree 2, minimal conic bundle over k . Then S is birational to a degree 2, minimal, weak del Pezzo conic bundle.*

Proof. There is nothing to do if S is weak del Pezzo. Otherwise, we are in the exceptional case (19.2), hence S contains a conjugate pair of disjoint (-3) -curves A, A' and $|-K_S| = A + A' + |2F|$.

Pick a fiber F_1 defined over k . Then $A \cap F_1$ determines a degree 2 field extension k'/k . Assume now that there is a smooth fiber F_2 that is defined over k' but not over k . Then $A \cap F_2$ is a k' -point, thus F_2 is rational over k' . Thus it has a k' -point r not on $A + A'$; let \bar{r} denote its conjugate. Since F_2 is not defined over k , r, \bar{r} lie on different smooth fibers.

The elementary transformation centered at $r + \bar{r}$ gives $S \dashrightarrow S_r$. The birational transform of $A + A'$ is $A_q + A'_q \in |-K_{S_r}|$, a conjugate pair of (-1) -curves meeting at 2 points. Thus S_r is a weak del Pezzo surface by (12.4).

One can always find the required F_2 , save when $k = \mathbb{F}_3$ and the 6 singular fibers lie over the 6 points of $\mathbb{F}_9 \setminus \mathbb{F}_3$. In this case the fibers over the \mathbb{F}_3 points are smooth and rational. We can thus pick 2 \mathbb{F}_3 -points in 2 different smooth fibers to play the role of r, \bar{r} .

(Actually, there is no minimal conic bundle where the 6 singular fibers lie over the 6 points of $\mathbb{F}_9 \setminus \mathbb{F}_3$. Indeed, these points form 3 conjugacy classes, but class field theory (see for instance [Wei67, Sec.XIII.6]) implies that, over a finite field, the singular fibers form an even number of conjugacy classes.)

□

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