Meccanica manuscript No.
(will be inserted by the editor)

A model of imperfect interface with damage

Elena Bonetti · Giovanna Bonfanti ·
Frédéric Lebon · Raffaella Rizzoni

Received: date / Accepted: date

Abstract In this paper two models of damaged materials are presented. The first model is a model of glue including micro-cracks evolution and which has two different regimes, one in traction and one in compression. The second model is model of interface derived from the first one by an asymptotic analysis. Simple numerical examples are presented.

Keywords Thin film · Bonding · Asymptotic analysis · Damage · Imperfect interface

1 Introduction

These last years, the study of imperfect interface between solids became a subject of a very large interest for scientists and the industries, in particular because of the development of composite materials [1–3,5,7,8,10,11,21–23,25, 26,28,29]. It is extremely important in particular to control the damage between the fiber and the matrix. Precise models of damaged interface are thus necessity to design structures.

E. Bonetti
Department of Mathematics, University of Pavia, Italy
E-mail: elena.bonetti@unipv.it

G. Bonfanti
Department of Mathematics, University of Brescia, Italy
E-mail: giovanna.bonfanti@unibs.it

F. Lebon
Laboratory of Mechanics and Acoustics, Aix-Marseille University, UPR7051, CNRS, Centrale Marseille, France
E-mail: lebon@lma.cnrs-mrs.fr

R. Rizzoni
Department of Engineering, University of Ferrara, Italy
E-mail: raffaella.rizzoni@unife.it

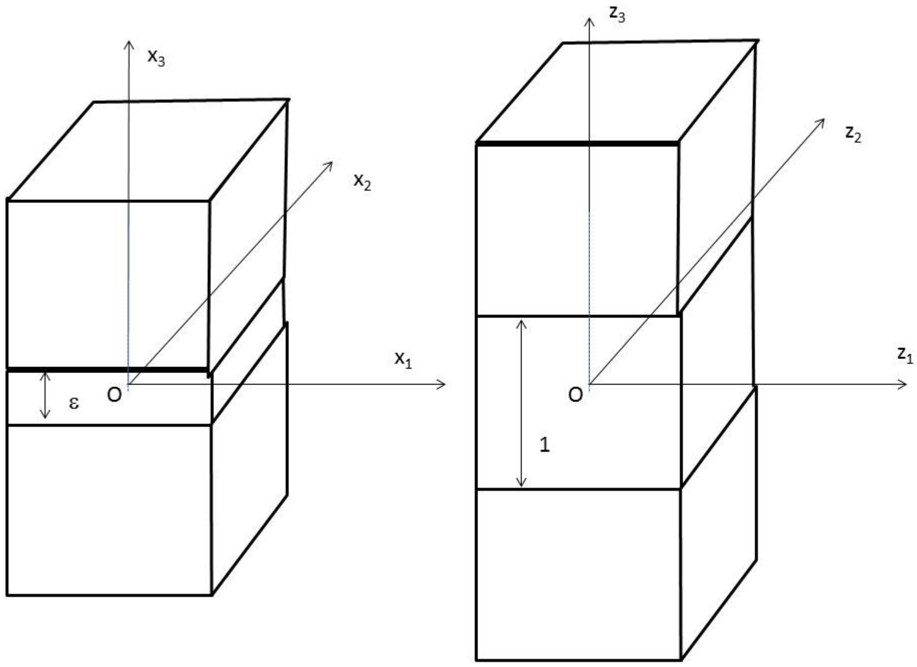

Fig. 1 Composite body: initial structure and rescaled structure

In this paper, a model of imperfect interface including damage is proposed. This model is based on the asymptotic analysis [4, 9, 12, 13, 29, 31, 32, 34] of a composite made by two elastic solids bonded together by a third thin one, which has a non linear behavior. In order to obtain impenetrability conditions, the glue is micro-cracked and has two different regimes, one in traction and one in compression.

The paper is divided in three parts. In the first section, the problem of composite body made by three deformable solids bonded together, two adherents and an adhesive is presented. The adhesive is a non linear material. The constitutive equation of this material takes into account damage and the non symmetry between traction and compression. In the second part of the paper, an asymptotic expansion method is introduced and applied at the problem introduced in the first section. A model of imperfect interface is derived. In the third section, the model is applied to a particular cracked material. A simple example in one dimension is studied.

2 The three-dimensional equations of the composite body

In the following a composite body made by three deformable solids (see figure 1), two adherents and an adhesive (also called a glue), and occupying the domain Ω^ε is considered. The dependance of the domain Ω^ε on the parameter ε will be precized in the following. An orthonormal Cartesian basis $(O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is introduced and let (x_1, x_2, x_3) be taken to denote the three coordinates of a particle. The origin lies at the center of the adhesive midplane and the x_3-axis runs perpendicular to the open bounded set S, $S = \{(x_1, x_2, x_3) \in \Omega^\varepsilon : \; x_3 = 0\}$ which will be called in the following the interface. The adhesive, also called interphase is occupying the domain B^ε, defined by $B^\varepsilon = \{(x_1, x_2, x_3) \in \Omega^\varepsilon : \; |x_3| < \frac{\varepsilon}{2}\}$. Note that ε is the thickness of the glue. The adherents are occupying respectively the domains $\Omega^\varepsilon_\pm$ defined by $\Omega^\varepsilon_\pm = \{(x_1, x_2, x_3) \in \Omega : \; \pm x_3 > \frac{\varepsilon}{2}\}$. The two-dimensional domains $S^\varepsilon_\pm$ are taken to denote the interfaces between the adhesive and the adherents, $S^\varepsilon_\pm = \{(x_1, x_2, x_3) \in \Omega : \; x_3 = \pm\frac{\varepsilon}{2}\}$. On a part S_g of the boundary $\partial\Omega^\varepsilon$, an external load g is applied, and on a part S_u of $\partial\Omega^\varepsilon$ such that $S_g \cap S_u = \emptyset$, the displacement is imposed to be equal to 0. Moreover, it is assumed that $S_u \cap B^\varepsilon = \emptyset$ and $S_g \cap B^\varepsilon = \emptyset$. A body force f is applied in $\Omega^\varepsilon_\pm$. In the following, u^ε is taken to denote the displacement field, σ^ε the Cauchy stress tensor and $e(u^\varepsilon)$ the strain tensor. Under the small strain hypothesis we have $e_{ij}(u^\varepsilon) = \frac{1}{2}(u^\varepsilon_{i,j} + u^\varepsilon_{j,i})$, where the comma is the partial derivative.

The two adherents are supposed to be elastic, thus

$$\sigma^\varepsilon_{ij} = a^\pm_{ijhk} e_{hk}(u^\varepsilon) \tag{1}$$

or

$$\sigma^\varepsilon = \partial\psi^\pm_{,e}(e(u^\varepsilon)), \tag{2}$$

where the free energy $\psi^\pm = \frac{1}{2} a^\pm e(u^\varepsilon) : e(u^\varepsilon)$, $a^\pm$ is the fourth order elasticity tensor verifying the usual conditions of positivity and symmetry.

The adhesive is a generalized Kachanov-type material. In the Kachanov theory [24,35], the constitutive equations are obtained after the homogenization of a micro-cracks material, with k families of cracks. The elastic coefficients depend on the lenghts l_k of these cracks. In the following, only a family of cracks is considered and l is taken the denote the lenght of these cracks. This parameter can be considered as a damage parameter. It is observed also, that in this theory the stiffness of the material $b^\varepsilon(l)$ takes the form $b^\varepsilon \approx \varepsilon b(l)$ [8].

As an example, for a crack orthogonal to $\mathbf{e}_3$, the Young modulus in the third direction E_3 is equal to

$$E_3 = \frac{E_0}{1 + 2\rho C E_0}, \tag{3}$$

where

$$C = \frac{\Pi}{2} \frac{1}{\sqrt{E_0}} \left(\frac{1}{\mu_0} - 2\frac{\nu_0}{E_0} + \frac{2}{E_0} \right)^{1/2} \tag{4}$$

and E_0 (resp. μ_0, resp. ν_0) is the Young modulus (resp. the shear modulus, resp. the Poisson ratio) of the undamaged material and ρ is the density of cracks i.e. $\rho = \frac{l^3}{V}$ in 3 dimensions and $\rho = \frac{l^2}{S}$ in 2 dimensions, where V (resp. S) is the volume (resp. the surface) of the representative elementary domain. Note that V and S are proportional to the thickness of the interphase ε.

In order to avoid the penetration along the cracks, it is supposed that the elasticity coefficients do not depend on the lenght of the cracks in compression, only in traction. In conclusion, in the interphase, two regimes are considered,

$$\sigma^\varepsilon_{ij} = \begin{cases} \varepsilon b_{ijhk}(l) e_{hk}(u^\varepsilon) & \text{if } e^s(u^\varepsilon) \geq 0 \\ e^s(u^\varepsilon) B_{ijhk}\delta_{hk} + \varepsilon b_{ijhk}(l) e^d_{hk}(u^\varepsilon) & \text{if } e^s(u^\varepsilon) \leq 0 \end{cases} \tag{5}$$

where b and B are two fourth order elasticity tensors verifying the usual conditions of positivity and symmetry, $e^s = \frac{1}{3} tr(e(u^\varepsilon))$ (resp. $e^d = e(u^\varepsilon) - \frac{1}{3} tr(e(u^\varepsilon) Id)$ is the spheric (resp. deviatoric) part of e and δ is the Kroenecker symbol.

Now, the possible evolution of the lenght l is introduced. Following the general theory proposed in [6], a pseudo-potential of dissipation ϕ is considered. For the sake of simplicity, this potential which is rate dependent is chosen as quadratic i.e.

$$\phi(\dot{l}) = \frac{1}{2} \eta^\varepsilon \dot{l}^2 + I_{\{\dot{l} \geq 0\}}(\dot{l}), \tag{6}$$

where I_A is the indicator function of the set A, $I_A(x) = 0$ if $x \in A$, $I_A(x) = +\infty$, otherwise and η^ε a viscosity parameter. This indicator function imposes to $\dot{l}$ to be positive i.e. the crack length increases. The free energy associated to the constitutive equation of the interphase (eqs. 5) is given by

$$\psi^i(e(u^\varepsilon), l)) = \psi^{i-s}(e(u^\varepsilon)) - \omega^\varepsilon l + I_{[l_0, +\infty[}(l) \tag{7}$$

where ω^ε is a (negative) parameter similar to the Dupr's energy [6,7], l_0 is a given initial crack lenght and

$$\psi^{i-s}(e(u^\varepsilon)) = \begin{cases} \frac{1}{2} \varepsilon b(l) e(u^\varepsilon) : e(u^\varepsilon) & \text{if } e^s(u^\varepsilon) \geq 0 \\ \frac{1}{2} (e^s(u^\varepsilon))^2 B_{ijhk} \delta_{ij} \delta_{hk} + \frac{1}{2} \varepsilon b(l) e^d(u^\varepsilon) : e^d(u^\varepsilon) & \text{if } e^s(u^\varepsilon) \leq 0 \end{cases} \tag{8}$$

Note that η^ε and ω^ε are volumic densities and thus these two coefficients are inversely proportional to ε. In next sections, we will denote $\eta^\varepsilon = \eta/\varepsilon$ and $\omega^\varepsilon = \omega/\varepsilon$.

Thus, the evolution equation is written

$$\eta^\varepsilon \dot{l} = \begin{cases} \left(\omega^\varepsilon - \frac{1}{2}\varepsilon b_{,l}(l) e(u^\varepsilon) : e(u^\varepsilon)\right)_+ & \text{if } e^s(u^\varepsilon) \geq 0 \\ \left(\omega^\varepsilon - \frac{1}{2}\varepsilon b_{,l}(l) e^d(u^\varepsilon) : e^d(u^\varepsilon)\right)_+ & \text{if } e^s(u^\varepsilon) \leq 0 \end{cases} \tag{9}$$

where $()_+$ is the positive part of a function. In the following l is supposed to be independant of x_3 i.e. the interphase and the representative volume thicknesses are equal. A more general hypothesis is presented in appendix.

The equations governing the equilibrium problem of the composite structure are written as follows:

$$\begin{cases} \sigma^\varepsilon_{ij,j} + f_i = 0 & \text{in } \Omega^\varepsilon_\pm \\ \sigma^\varepsilon_{ij} n_j = g_i & \text{on } S_g \\ \sigma^\varepsilon_{ij,j} = 0 & \text{in } B^\varepsilon \\ [[\sigma^\varepsilon_{i3}]] = 0 & \text{on } S^\varepsilon_\pm \\ [[u^\varepsilon_i]] = 0 & \text{on } S^\varepsilon_\pm \\ u^\varepsilon_i = 0 & \text{on } S_u \\ \sigma^\varepsilon_{ij} = a^\pm_{ijhk} e_{hk}(u^\varepsilon) & \text{in } \Omega^\varepsilon_\pm \\ \sigma^\varepsilon_{ij} = \varepsilon b_{ijhk}(l) e_{hk}(u^\varepsilon) \text{ if } e^s(u^\varepsilon) \geq 0 & \text{in } B^\varepsilon \\ \sigma^\varepsilon_{ij} = e^s(u^\varepsilon) B_{ijhk}\delta_{hk} + \varepsilon b_{ijhk}(l) e^d_{hk}(u^\varepsilon) \text{ if } e^s(u^\varepsilon) \leq 0 & \text{in } B^\varepsilon \\ \eta^\varepsilon \dot{l} = \left(\omega^\varepsilon - \frac{1}{2}\varepsilon b_{,l}(l) e(u^\varepsilon) : e(u^\varepsilon)\right)_+ \text{ if } e^s(u^\varepsilon) \geq 0 & \text{in } B^\varepsilon \\ \eta^\varepsilon \dot{l} = \left(\omega^\varepsilon - \frac{1}{2}\varepsilon b_{,l}(l) e^d(u^\varepsilon) : e^d(u^\varepsilon)\right)_+ \text{ if } e^s(u^\varepsilon) \geq 0 & \text{in } B^\varepsilon \end{cases} \tag{10}$$

where $[[f]]$ denotes the jump of f along $S^\varepsilon_\pm$ i.e. $f((\varepsilon/2)^\pm) - f((\varepsilon/2)^\mp)$. We recall that $f(a^+) = \lim_{x \longrightarrow a, x>a} f(x)$ and $f(a^-) = \lim_{x \longrightarrow a, x<a} f(x)$.

3 The asymptotic expansion method

Since the thickness of the interphase is very small, it is natural to seek the solution of problem (10) using asymptotic expansions with respect to the parameter ε [14–18, 20, 33]. In particular, the following asymptotic series with integer powers are assumed:

$$\begin{cases} \mathbf{u}^\varepsilon = \mathbf{u}^0 + \varepsilon\, \mathbf{u}^1 + o(\varepsilon) \\ \sigma^\varepsilon = \sigma^0 + \varepsilon\, \sigma^1 + o(\varepsilon). \end{cases} \tag{11}$$

The domain is then rescaled (see figure 1) using a classical procedure:

– In the adhesive, the following change of variable is introduced

$$(x_1, x_2, x_3) \in B^\varepsilon \to (z_1, z_2, z_3) \in B, \text{ with } (z_1, z_2, z_3) = (x_1, x_2, \frac{x_3}{\varepsilon})$$

and it is set $\hat{\mathbf{u}}^\varepsilon(z_1, z_2, z_3) = \mathbf{u}^\varepsilon(x_1, x_2, x_3)$ and $\hat{\sigma}^\varepsilon(z_1, z_2, z_3) = \sigma^\varepsilon(x_1, x_2, x_3)$, where $B = \{(x_1, x_2, x_3) \in \Omega : \ |x_3| < \frac{1}{2}\}$.

– In the adherent, the following change of variable is introduced

$$(x_1, x_2, x_3) \in \Omega^\varepsilon_\pm \to (z_1, z_2, z_3) \in \Omega_\pm, \text{ with } (z_1, z_2, z_3) = (x_1, x_2, x_3 \pm 1/2 \mp \varepsilon/2)$$

and it is set $\bar{\mathbf{u}}^\varepsilon(z_1, z_2, z_3) = \mathbf{u}^\varepsilon(x_1, x_2, x_3)$ and $\bar{\sigma}^\varepsilon(z_1, z_2, z_3) = \sigma^\varepsilon(x_1, x_2, x_3)$, where $\Omega_\pm = \{(x_1, x_2, x_3) \in \Omega : \ \pm x_3 > \frac{1}{2}\}$. The external forces is assumed to be independent of ε. As a consequence, it is set $\bar{f}(z_1, z_2, z_3) = f(x_1, x_2, x_3)$ and $\bar{g}(z_1, z_2, z_3) = g(x_1, x_2, x_3)$.

The governing equations of the rescaled problem are as follows:

$$\begin{cases} \bar{\sigma}^\varepsilon_{ij,j} + \bar{f}_i = 0 & \text{in } \Omega_\pm \\ \bar{\sigma}^\varepsilon_{ij} n_j = \bar{g}_i & \text{on } \bar{S}_g \\ \hat{\sigma}^\varepsilon_{ij,j} = 0 & \text{in } B \\ \bar{\sigma}^\varepsilon_{i3} = \hat{\sigma}^\varepsilon_{i3} & \text{on } S_\pm \\ \bar{u}^\varepsilon_i = \hat{u}^\varepsilon_i & \text{on } S_\pm \\ \bar{u}^\varepsilon_i = 0 & \text{on } \bar{S}_u \\ \bar{\sigma}^\varepsilon_{ij} = \bar{a}^\pm_{ijhk} \bar{e}_{hk}(\bar{u}^\varepsilon) & \text{in } \Omega_\pm \\ \hat{\sigma}^\varepsilon_{ij} = \varepsilon \hat{b}_{ijhk}(l) \hat{e}_{hk}(\hat{u}^\varepsilon) \text{ if } \hat{e}^s(\hat{u}^\varepsilon) \geq 0 & \text{in } B \\ \hat{\sigma}^\varepsilon_{ij} = \hat{e}^s(\hat{u}^\varepsilon) \hat{B}_{ijhk} \delta_{hk} + \varepsilon \hat{b}_{ijhk}(l) \hat{e}^d_{hk}(\hat{u}^\varepsilon) \text{ if } \hat{e}^s(\hat{u}^\varepsilon) \leq 0 & \text{in } B \\ \hat{\eta}^\varepsilon \dot{\hat{l}} = \left(\hat{\omega}^\varepsilon - \frac{1}{2} \varepsilon \hat{b}_{,l}(l) \hat{e}(\hat{u}^\varepsilon) : \hat{e}(\hat{u}^\varepsilon) \right)_+ \text{ if } \hat{e}^s(\hat{u}^\varepsilon) \geq 0 & \text{in } B \\ \hat{\eta}^\varepsilon \dot{\hat{l}} = \left(\hat{\omega}^\varepsilon - \frac{1}{2} \varepsilon \hat{b}_{,l}(l) \hat{e}^d(\hat{u}^\varepsilon) : \hat{e}^d(\hat{u}^\varepsilon) \right)_+ \text{ if } \hat{e}^s(\hat{u}^\varepsilon) \leq 0 & \text{in } B \end{cases} \tag{12}$$

where $S_\pm = \{(x_1, x_2, x_3) \in \Omega : \ x_3 = \pm\frac{1}{2}\}$ and $\bar{\cdot}, \hat{\cdot}$ denote the rescaled operators in the adherents and in the adhesive, respectively.
In view of 11 the displacement and stress fields are written as asymptotic expansions

$$\begin{cases} \hat{\sigma}^\varepsilon = \hat{\sigma}^0 + \varepsilon\, \hat{\sigma}^1 + o(\varepsilon) \\ \hat{u}^\varepsilon = \hat{u}^0 + \varepsilon\, \hat{u}^1 + o(\varepsilon) \\ \bar{\sigma}^\varepsilon = \bar{\sigma}^0 + \varepsilon\, \bar{\sigma}^1 + o(\varepsilon) \\ \bar{u}^\varepsilon = \bar{u}^0 + \varepsilon\, \bar{u}^1 + o(\varepsilon), \end{cases} \tag{13}$$

in the rescaled adhesive and adherents, respectively.

3.1 Expansions of the equilibrium equations in the adherents

Substituting 13 into the first, second, sixth and seventh equations of 12, it is obtained at the first order of expansion (power 0)

$$\begin{cases} \bar{\sigma}^0_{ij,j} + \bar{f}_i = 0 & \text{in } \Omega_\pm \\ \bar{\sigma}^0_{ij} n_j = \bar{g}_i & \text{on } \bar{S}_g \\ \bar{u}^0_i = 0 & \text{on } \bar{S}_u \\ \bar{\sigma}^0_{ij} = \bar{a}^\pm_{ijhk} \bar{e}_{hk}(\bar{u}^0) & \text{in } \Omega_\pm \end{cases} \tag{14}$$

3.2 Expansions of the equilibrium equations in the adhesive

Substituting 13 into the third equation of 12 it is deduced that the following conditions hold in B (power -1):

$$\hat{\sigma}^0_{i3,3} = 0, \tag{15}$$

i.e. $\hat{\sigma}^0_{i3}$ does not depend on z_3, that it can be expressed as

$$\left[\hat{\sigma}^0_{i3}\right] = 0, \tag{16}$$

where $[f] = f(x_1, x_2, \frac{1}{2}) - f(x_1, x_2, -\frac{1}{2})$. In the adhesive the strain field becomes:

$$\hat{e}(\hat{u}^\varepsilon) = \varepsilon^{-1}\hat{e}^{-1} + \hat{e}^0 + \varepsilon \hat{e}^1 + o(\varepsilon) \tag{17}$$

where

$$\begin{aligned} \hat{e}^{-1}_{33} &= \hat{u}^0_{3,3} \\ \hat{e}^{-1}_{\alpha 3} &= \frac{1}{2}\hat{u}^0_{\alpha,3}, \ \alpha = 1,2 \end{aligned} \tag{18}$$

It is obvious to remark that

$$\hat{e}^s(\hat{u}) = \frac{1}{3}(\varepsilon^{-1}\hat{u}^0_{3,3} + \hat{u}^0_{1,1} + \hat{u}^0_{2,2} + \hat{u}^1_{3,3} + o(1)) \tag{19}$$

and that at the first order of expansion the sign of the spheric part of the strain tensor is the sign of $\hat{u}^0_{3,3}$. The eighth equation in 12 is considered i.e. $\hat{u}^0_{3,3} \geq 0$. At the first order in the expansions (power 0), it is obtained

$$\hat{\sigma}^0_{i3} = \hat{b}_{i3j3}(l)\hat{u}^0_{j,3} \tag{20}$$

It is observed that $\hat{\sigma}^0_{i3}$ (and thus $\hat{u}^0_{i,3}$) does not depend on z_3, thus

$$\hat{\sigma}^0_{i3} = \hat{b}_{i3j3}(l)\left[\hat{u}^0_j\right] \tag{21}$$

which is the classical equation of soft interface. Denoting $\hat{b}_{i3j3}(l) = K^{33}_{ij}(l)$, it is obtained

$$\hat{\sigma}^0 \mathbf{e}_3 = K^{33}(l)\left[\hat{u}^0\right] \ \text{if} \ \left[\hat{u}^0_3\right] \geq 0 \text{ on } S^\pm \tag{22}$$

Now the nineth equation in 12 is considered. The expansion gives at the first order (power -1)

$$0 = \hat{B}_{ii33}\hat{u}^0_{3,3} \text{ if } \hat{u}^0_{3,3} \leq 0 \tag{23}$$

or due the property of positivity of the tensor B

$$\hat{u}^0_{3,3} = 0 \text{ if } \hat{u}^0_{3,3} \leq 0 \tag{24}$$

Function $\hat{u}^0_{3,3}$ being constant in the third direction, it can be deduced that

$$\left[\hat{u}^0_3\right] = 0 \text{ if } \left[\hat{u}^0_3\right] \leq 0 \tag{25}$$

Note that at the second order in the expansions (power 0), it is obtained

$$\hat{\sigma}^0_{i3} = \hat{b}_{i3j3}(l)\hat{u}^0_{j,3} \tag{26}$$

In conclusion, it is obtained on $S^{\pm}$

$$\begin{array}{c} \hat{\sigma}^0\mathbf{e}_3 = K^{33}(l)\left[\hat{u}^0\right] \text{ if } \left[\hat{u}^0_3\right] \geq 0 \\ \hat{\sigma}^0\mathbf{e}_3 = K^{33}(l)\left[\hat{u}^0\right], \left[\hat{u}^0_3\right] = 0 \text{ if } \left[\hat{u}^0_3\right] \leq 0 \end{array} \tag{27}$$

Now the two last equations in 12 are considered. If we consider that the lenght l increases, the first term in the expansion gives (power -1)

$$\hat{\eta}\dot{\hat{l}} = \hat{\omega} - \frac{1}{2}(\hat{b}_{,l}(l))_{i3j3}\hat{u}^0_{j,3} : \hat{u}^0_{i,3} = \hat{\omega} - \frac{1}{2}(K^{33}_{,l}(l))_{ij}\hat{u}^0_{j,3}.\hat{u}^0_{i,3}$$

Now, this equation can be integrated along the third direction in two steps, considering that $\hat{\sigma}^0_{i3} = K^{33}_{ij}(l)\hat{u}^0_{j,3}$ or $\hat{u}^0_{,3} = (K^{33}(l))^{-1}\hat{\sigma}^0\mathbf{e}_3$. Thus, it is obtained

$$\hat{\eta}\dot{\hat{l}} = \hat{\omega} - \frac{1}{2}K^{33}_{,l}(l)\left[\hat{u}^0\right].\left[\hat{u}^0\right]$$

This equation can be decomposed, as classical, into normal and tangential parts

$$\hat{\eta}\dot{\hat{l}} = \hat{\omega} - \frac{1}{2}K^{33}_{N,l}(l)\left[\hat{u}^0_N\right]^2 - \frac{1}{2}K^{33}_{T,l}(l)\left[\hat{u}^0_T\right].\left[\hat{u}^0_T\right]$$

3.3 Matching between the adhesive and the adherents

Substituting 15 into the fourth and fifth equations of 12, it is deduced that the following conditions hold on $S_{\pm}$:

$$\begin{array}{c} \hat{\sigma}^0_{i3}(z_1, z_2, \pm\frac{1}{2}) = \bar{\sigma}^0_{i3}(z_1, z_2, \pm\frac{1}{2}) = \sigma^0_{i3}(x_1, x_2, \pm\frac{\varepsilon}{2}) \approx \sigma^0_{i3}(x_1, x_2, 0) \\ \hat{u}^0_i(z_1, z_2, \pm\frac{1}{2}) = \bar{u}^0_i(z_1, z_2, \pm\frac{1}{2}) = u^0_i(x_1, x_2, \pm\frac{\varepsilon}{2}) \approx u^0_i(x_1, x_2, 0^{\pm}) \end{array} \tag{28}$$

In conclusion, it s obtained

$$\begin{cases} \sigma^0_{ij,j} + f_i = 0 & \text{in } \Omega_\pm \\ \sigma^0_{ij} n_j = g_i & \text{on } S_g \\ u^0_i = 0 & \text{on } S_u \\ \sigma^0_{ij} = a^\pm_{ijhk} e_{hk}(u^0) & \text{in } \Omega_\pm \\ \sigma^0_{i3} = K^{33}_{ij}(l) \left[u^0_j\right]_+ & \text{on } S \\ \eta \dot{l} = \left(\omega - \frac{1}{2} K^{33}_{,l}(l) \left[u^0\right]_+ . \left[u^0\right]_+ \right)_+ & \text{on } S \end{cases} \tag{29}$$

where $\left[u^0\right]_+ = \left[u^0\right]$ if $\left[u^0_3\right] \geq 0$, $\left[u^0\right]_+ = \left(\left[u^0_1\right], \left[u^0_2\right], 0\right)^T$ if $\left[u^0_3\right] \leq 0$.

It is obtained a model of imperfect soft interface with unilateral contact and damage evolution. Note that the variable l (lenght variable) is similar to the density of adhesion (no dimension variable) introduced by M. Frmond in [5]. This intensity of adhesion can be interpreted mechanically as the ratio l/l_0.

4 An example in 2D: Kachanov material

4.1 Constitutive equations

In two dimensions (in the plan $(O, \mathbf{e}_1, \mathbf{e}_2)$), the problem can be rewritten as

$$\begin{cases} \sigma^0_{ij,j} + f_i = 0 & \text{in } \Omega_\pm \\ \sigma^0_{ij} n_j = g_i & \text{on } S_g \\ u^0_i = 0 & \text{on } S_u \\ \sigma^0_{ij} = a^\pm_{ijhk} e_{hk}(u^0) & \text{in } \Omega_\pm \\ \sigma^0_{i2} = K^{22}_{ij}(l) \left[u^0_j\right]_+ & \text{on } S \\ \eta \dot{l} = \left(\omega - \frac{1}{2} K^{22}_{,l}(l) \left[u^0\right]_+ . \left[u^0\right]_+ \right)_+ & \text{on } S \end{cases} \tag{30}$$

For the Kachanov model of homogenized cracked material [24,35], the stiffness matrix is written under engineer notations as

$$K^\varepsilon = \begin{bmatrix} E_0 & \dfrac{\nu_0 L \varepsilon}{2 l^2 C} & 0 \\ \dfrac{\nu_0 L \varepsilon}{2 l^2 C} & \dfrac{L \varepsilon}{2 l^2 C} & 0 \\ 0 & 0 & \dfrac{L \varepsilon}{l^2 C} \end{bmatrix} \tag{31}$$

where L is the lenght of the interphase, C is given in (4). It is supposed that the crack is along x_1 axis. Matrix K^{33} (in fact K^{22} in this configuration), which is diagonal, reads

$$K^{22} = \begin{bmatrix} \dfrac{L}{2 l^2 C} & 0 \\ 0 & \dfrac{L}{l^2 C} \end{bmatrix} \tag{32}$$

and its derivative reads

$$K_{,l}^{22} = \begin{bmatrix} \dfrac{-L}{l^3C} & 0 \\ 0 & \dfrac{-2L}{l^3C} \end{bmatrix} \tag{33}$$

The last equation in (30) (evolution of l) is written

$$\eta\dot{l} = \left(\omega + \frac{L}{2l^3C}\left[u_1^0\right]^2 + \frac{L}{l^3C}\left[u_2^0\right]_+^2\right)_+$$

4.2 A focus on the crack lenght evolution

As a first approach of the evolution of l, we can consider the case $\eta = 0$ (the viscosity is equal to zero) and in order to simplify $\left[u_1^0\right] = 0$. As we can see on figure 2, in compression $\left[u_2^0\right] = 0$ (unilateral contact). In traction, until $\dfrac{L}{l^3C}\left[u_2^0\right]_+^2 \leq -\omega$ i.e. $\left[u_2^0\right] \leq \sqrt{-\dfrac{\omega l^3 C}{L}} = \delta_2$ the stiffness is constant and the relation between σ_{22} and $\left[u_2^0\right]$ is linear, with the slope equal to $\dfrac{L}{2l^2C}$. We take σ_{22}^{max} to denote $\dfrac{1}{2}\sqrt{-\dfrac{\omega L}{Cl}}$, the maximum value of σ_{22}. When the threshold is reached, we have

$$l = \left(-\frac{L\left[u_2^0\right]^2}{\omega C}\right)^{1/3}$$

and then

$$\sigma_{22} = \frac{1}{2}\left(\frac{L\omega^2}{C\left[u_2^0\right]}\right)^{1/3}$$

This relation is represented on figure 2. On figure 2, the relation between σ_{22} and $\left[u^0\right]$ is represented approximatively (using implicit Euler integration schema) for various values of the viscosity.

4.3 A simple academic example

In this section, the example of a simple bar of length L_0 is considered (see figure 3). The bar is bonded on its left part and loaded on its right part. The volumic force is neglected. The force on the right part is given by

$$F(t) = \begin{cases} F_0 t & t \leq t_1 \\ F_0 t_1 & t_1 \leq t \leq t_2 \\ \dfrac{F_0 t_2}{t_2 - t_3} t - \dfrac{F_0 t_2}{t_3(t_2 - t_3)} & t_2 \leq t \leq t_3 \end{cases} \tag{34}$$

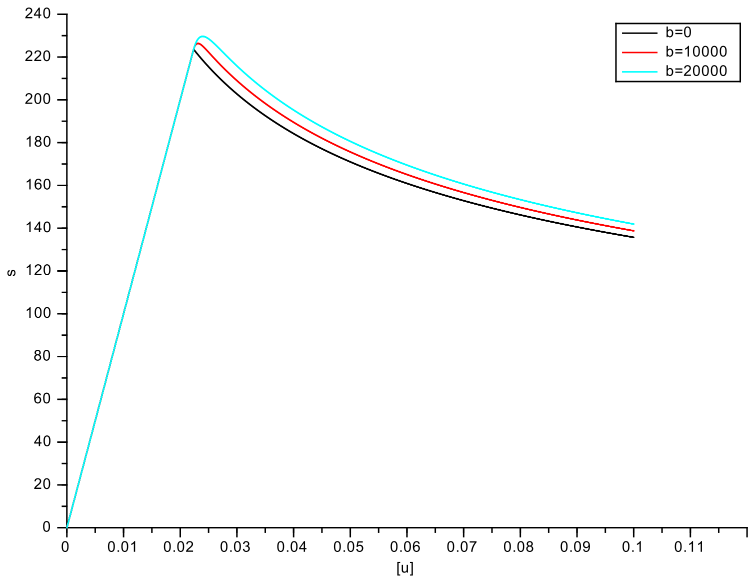

Fig. 2 Evolution of normal stress vs jump of displacement for various valors of viscosity (normalized values, $L = 1,\ C = 100,\ \omega = -50$)

Fig. 3 A simple example

where F_0 and $t_i,\ i = 1, 2, 3$ are given. It is obvious to show that the displacement field u takes the form

$$u(x) = \frac{F(t)}{E}x + u_0,$$

where E is the Young's modulus of the bar and u_0 is given by the interface law $\sigma = F(t) = \frac{C}{l^2}u_0$. The lenght l is given by the equation

$$b\dot{l} = \left(w + \frac{C}{l^3}u_0^2\right)_+.$$

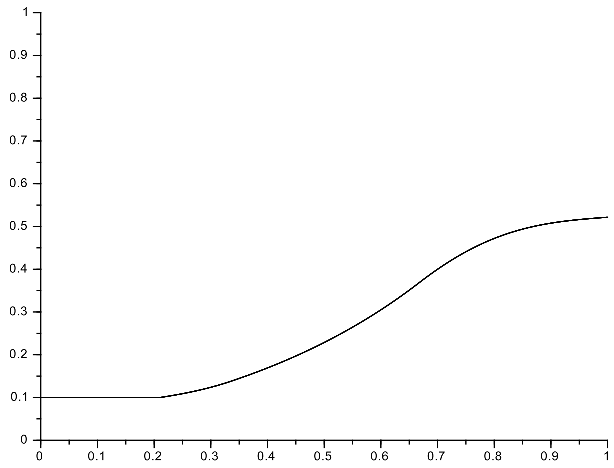

Fig. 4 Evolution of the crack lenght along time ($C = 1,\ \omega = 1,\ b = 1,\ F_0 = 15,\ l_0 = 0.1,\ T_3 = 1,\ T_1 = T_3/3,\ T_2 = 2T_3/3$)

We take l_0 to denote the initial lenght of the crack. If $t \leq \sqrt{\dfrac{\omega C}{l_0}}$, then $l = l_0$. If $t \geq \sqrt{\dfrac{\omega C}{l_0}}$, then the crack lenght can be computed by a simple implicit Euler schema, i.e.

$$l^{k+1} = \frac{l^k - \omega \dfrac{\Delta t}{b}}{1 - \dfrac{\Delta t (F(t))^2}{bC}}$$

where Δt is taken to denote the time step, $l^k = l(t^k)$ and $t^k = k\Delta t$.

We can observe on figure 4 the increase of the crack lenght along time and on figure 5 the decrease of the stiffness of the glue in relation with the increase of the crack lenght for academic values of the coefficients. The model proposed here seems qualitatively coherent. During the first part of the loading (linear increase) and if $t < 0.21$, the crack lenght remains constant ($l = l_0$). When $t > 0.21$ the crack lenght increases. We can observe the change of curvature corresponding to the variation of the loading. On figure 5, we can see that the stiffness is divided by more than 25 during the process.

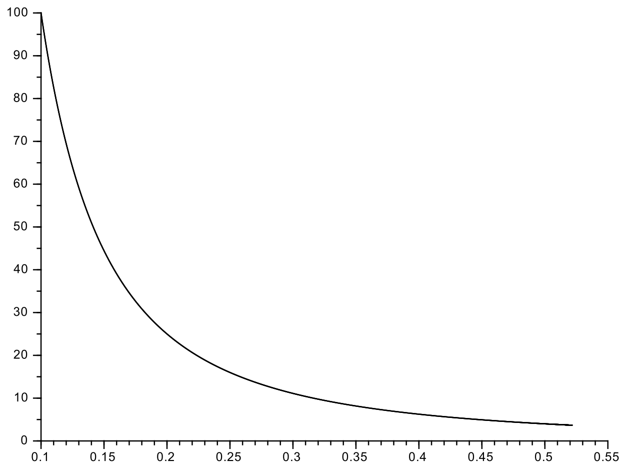

Fig. 5 Evolution of the stiffness vs crack lenght $C = 1,\ \omega = 1,\ b = 1,\ F_0 = 15,\ l_0 = 0.1,\ T_3 = 1,\ T_1 = T_3/3,\ T_2 = 2T_3/3)$

5 Conclusion

In the first part of this paper, a model of damaged material was proposed. This model is based on homogenization techniques and thermodynamics principles. The damage was governed by the evolution of the crack lenght at the micro-scale. In a second part of the paper, a model of imperfect interface has been derived from the asymptotic study of a three phase composite with perfectly bonding conditions between two adherents and an adhesive having the behavior studied in the first part of the paper. The expansion at the first level yields a model of imperfect interface taking into account damage. This last model was studied on a simple example in one dimension. It was shown that this model is qualitatively efficient.
In the future, the model of imperfect interface proposed in this paper will be mathematically analyzed [] and numerically implemented to test its reliability and efficiency. Other kind of cracked materials [27] and damage evolutions models will also be studied. Stochastic processes can be also introduced to take into account the variability of crack lengthes.

Acknowledgements This research was partially supported by "Partenariats Hubert Curien (PHC)", Galile Program 2015, Project number 32288WH.

References

1. Benveniste, Y. and Miloh, T., *Imperfect soft and stiff interfaces in two-dimensional elasticity*, Mechanics of Materials, 33 (6), 309-323 (2001).
2. Bornert P.G., Bretheau, T., Gilormini P., *Homogénéisation en mécanique des matériaux, Tome 1 : Matériaux aléatoires élastiques et milieux périodiques* . Hermes Sciences, Paris (2001).
3. Challamel, N., Girhammar, U.A., *Boundary-layer effect in composite beams with interlayer slip*, Journal of Aerospace Engineering, 24 (2), 199-209 (2011).
4. Dumont, S.,Lebon, F., Rizzoni, R., *An asymptotic approach to the adhesion of thin stiff films*, Mech. Res. Com., 58, 24-35 (2014).
5. Frmond M., *Adhrence des solides*, Journal de Mcanique Thorique et Applique, 6, 383-407 (1987).
6. Frmond M., *Non-Smooth Thermo-mechanics*, Springer (2001).
7. Fouchal F., Lebon F., Titeux, I., *Contribution to the modelling of interfaces in masonry construction*, Const. Buld. Mat., 23, 24282441 (2009).
8. Fouchal F., Lebon F., Raffa M.L., Vairo G., *An Interface Model Including Cracks and Roughness Applied to Masonry*, The Open Civ. Eng. J., 8, 263-271 (2014).
9. Geymonat, G., Krasucki, F., Lenci, S., *Mathematical analysis of a bonded joint with a soft thin adhesive*, Math. Mech. Sol, 4 (2), 201-225 (1999).
10. Hashin, Z., *Thin interphase/imperfect interface in elasticity with application to coated fiber composites*, Journal of the Mechanics and Physics of Solids, 50 (12), 2509-2537 (2002).
11. Ivanov, I.V., Velchev, D.S., Kneć, M., Sadowski, T., *Computational models of laminated glass plate under transverse static loading*, in “Shell-line structures”, Non-classical Theories and Applications, Adv. Struct. Mat., Altenbach H., Eremeyev V.A. Editors, Springer, Berlin, 469-490 (2011).
12. Klarbring A., *Derivation of the adhesively bonded joints by the asymptotic expansion method*, International Journal of Engineering Science, 29, 493-512 (1991).
13. Lebon, F., Ould Khaoua, A., Licht, C., *Numerical study of soft adhesively bonded joints in finite elasticity*, Comp. Mech.., 21, 134-140 (1998).
14. Lebon, F., Rizzoni, R., *Asymptotic study on a soft thin layer: The non-convex case*, Mech. Adv. Mat. Struct., 15, 12-20 (2008).
15. Lebon, F., Rizzoni, R., *Asymptotic analysis of a thin interface: The case involving similar rigidity*, Int. J. Engng. Sci., 48, 473-486 (2010).
16. Lebon, F., Rizzoni, R., *Asymptotic behavior of a hard thin linear elastic interphase: An energy approach*, Int. J. Sol. Struct., 48, 441-449 (2011).
17. Lebon, F., Rizzoni, R., *Modeling a hard, thin curvilinear interface*, Discr. Cont. Dyn. Syst. Series S, 6 (6), 1569-1586 (2013).
18. Lebon, F., Rizzoni, R., Ronel-Idrissi, S., *Asymptotic analysis of some non-linear soft thin layers*, Comp. and Struct., 82, 1929-1938-486 (2004).
19. Lebon, F., Ronel-Idrissi, S., *Asymptotic analysis of Drucker-Prager and Mohr-Coulomb soft thin interfaces*, Steel Comp. Struct., 4, 133-147 (2004).
20. Lebon, F., Zaittouni, S., *Asymptotic modelling of interfaces taking contact conditions into account: Asymptotic expansions and numerical implementation*, Int. J. Engn. Sci., 48, 111-127 (2010).
21. Lenci, S. , Clementi, F., Warminski, J., *Nonlinear free dynamics of a two-layer composite beam with different boundary conditions*, Meccanica, 50 (3), 675-688 (2015).
22. Li Y.D, Xiong T., Dong L., *A new interfacial imperfection coupling model (IICM) and its effect on the facture behavior of a layered multiferroic composite: Anti-plane case*, European Journal of Mechanics - A/Solids, 52, 26-36 (2015).
23. Lopez-Realpozo, J.C..Rodriguez-Ramos, R. . Guinovart-Diaz, R. . Bravo-Castillero, J., Otero, J.A. . Sabina, F.J., Lebon, F., Dumont, S., Sevostianov, I., Effective elastic shear stiffness of a periodic fibrous composite with non-uniform imperfect contact between the matrix and the fibers, International Journal of Solids and Structures, 51 (6), 1253-1262 (2014).
24. Mauge C, Kachanov M., *Effective elastic properties of an anisotropic material with arbitrarily oriented interacting cracks*, J. Mech. Phys. Solids, 42, 561-584 (1994).

25. J.A. Nairn, *Numerical implementation of imperfect interfaces*, Comp. Mat. Sci. 40, 525-536 (2007).
26. Pelissou, C., Lebon, F., *Asymptotic modeling of quasi-brittle interfaces*, Comp. Struct., 87, 1216-1223 (2009).
27. Ponte Castaneda, M., Willis, J., *The effect of spatial distribution on the effective behavior of composite materials and cracked media*, J. Mech. Phys. Sol., 43, 1919-1951 (1995)
28. Rekik A., Lebon, F., *Identification of the representative crack length evolution in a multi-level interface model for quasi-brittle masonry*, Int. J. Solids Struct., 47 (22-23), 3011-3021 (2010)
29. Rekik A., Lebon, F., *Homogenization methods for interface modeling in damaged masonry*, Adv. Eng. Software, 46 (1), 35-42 (2012)
30. Rizzoni, R., Lebon, F., *Asymptotic analysis of an adhesive joint with mismatch strain*, Eur. J. Mech. A/Sol., 36, 1-8 (2012).
31. Rizzoni, R., Lebon, F., *Imperfect interfaces as asymptotic models of thin curved elastic adhesive interphases*, Mech. Res. Comm., 51, 39-50 (2013).
32. Rizzoni, R., Dumont, S., Lebon, F., *On Saint Venant - Kirchhoff Imperfect Interfaces* preprint (2015).
33. Rizzoni, R., Dumont, S., Lebon, F., Sacco, S., *Homogenization methods for interface modeling in damaged masonry*, Int. J. Solids Struct., 51, 4137-4148 (2014).
34. Serpilli, M., *Asymptotic analysis of a multimaterial with a thin piezoelectric interphase*, Meccanica, 49 (7), 1641-1652 (2014).
35. Tsukrov, I., Kachanov, M., *Effective moduli of an anisotropic material with elliptical holes of arbitrary orientational distribution*, Int. J. Solids Struct., 37 (41), 5919-5941 (2000).

Appendix

In this appendix, it is supposed that the crack lenght l depends on x_3. A stratified interphase with n layers is considered. We take $l_k,\ k = 1, ..., n$ to denote the crack lenght in each layer. Introducing the rescaling and let

$$h_0 = -\frac{1}{2} < h_1 < ... < h_{n-1} < h_n = \frac{1}{2} \tag{35}$$

Note that $h_k - h_{k-1}$ is the thickness of the $k-th$ layer. The eighth and ninth equations in 12 are considered. At the first order in the expansions (power 0), it is obtained

$$\hat{\sigma}^0_{i3} = \hat{b}_{i3j3}(l_k)\hat{u}^0_{j,3},\ k = 1, ..., n \tag{36}$$

By integration, it is obtained

$$(h_k - h_{k-1})\,\hat{\sigma}^0_{i3} = \hat{b}_{i3j3}(l_k)\left[\hat{u}^0_j\right]^k_{k-1},\ k = 1, ..., n \tag{37}$$

where $[f]^k_{k-1} = f(z_1, z_2, h_k) - f(z_1, z_2, h_{k-1})$. Thus,

$$\left[\hat{u}^0_j\right]^k_{k-1} = (h_k - h_{k-1})\,\hat{K}^{33}(l_k)\hat{\sigma}^0_{i3},\ k = 1, ..., n \tag{38}$$

and

$$\left[\hat{u}^0_j\right] = \sum_{k=1}^{n}(h_k - h_{k-1})\left(\hat{K}^{33}\right)^{-1}(l_k)\hat{\sigma}^0_{i3},\ k = 1, ..., n \tag{39}$$

We take $\left(\hat{K}^{33}_n\right)^{-1}(l)$ to denote $\sum_{k=1}^{n}(h_k - h_{k-1})\left(\hat{K}^{33}\right)^{-1}(l_k)$, and thus

$$\sigma^0_{i3} = \hat{K}^{33}_n(l)\left[\hat{u}^0_j\right]. \tag{40}$$