SELF-LINKED CURVES AND NORMAL BUNDLE.

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Abstract. We give necessary conditions on the degree and the genus of a smooth, integral curve $C \subset \mathbb{P}^3$ to be selk-linked (i.e. locus of simple contact of two surfaces). We also give similar results for set theoretically complete intersection curves with a structure of multiplicity three (i.e. locus of 2-contact of two surfaces).

Introduction.

The motivation of this note is the following question, raised in [5]: does there exist a smooth, integral curve $C \subset \mathbb{P}^3$, of degree 8, genus 3, which is self-linked? We recall that a curve is self-linked if it is the locus of (simple) contact of two surfaces (see Section 1). This question in turn is motivated by the following fact (proved in [5], Proposition 7.5): let $S \subset \mathbb{P}^3$ be a surface with ordinary singularities. Let $C \subset S$ be a smooth, irreducible curve which is the set theoretic complete intersection (s.t.c.i.) of S with another surface. If $C \not\subset Sing(S)$, then C is self-linked (on S) (see Remark 7 for a precise statement). We recall that the problem to know whether or not every smooth irreducible curve $C \subset \mathbb{P}^3$ is a s.t.c.i. is still open. The study of self-linked curves is a first step in this long standing open problem. Self-linked curves have been studied by many authors (see [5] and the bibliography therein).

In this note we show that, as expected, no curve of degree 8, genus 3 is self-linked. This follows from our main result (Theorem 4) which gives necessary conditions on the invariants of a curve to be self-linked. As a consequence we obtain that if $d \geq 13$ and $d > g - 3$, then no curve of degree d, genus g can be self-linked (Corollary 6).

In the last section we obtain similar results for curves which are set theoretic complete intersections with a triple structure (i.e. curves admitting a triple structure which is a complete intersection).

Throughout this note we work over an algebraically closed field of characteristic zero.

1. Generalities.

We denote by $C \subset \mathbb{P}^3$ a smooth, irreducible curve of degree d, genus g. The curve C is *self-linked* if it is (algebraically) linked to itself by a complete intersection $F_a \cap F_b$ of two surfaces of degrees a, b. In particular $2d = ab$. This is equivalent to say that there exists a double structure, C_2, on C which is a complete intersection of type (a, b).

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Let's observe that if C is not a complete intersection, then $C \cap Sing(F_a) \neq \emptyset$ and $C \cap Sing(F_b) \neq \emptyset$. This follows from the fact (see [5], Lemma 7.6) that for a surface $S \subset \mathbb{P}^3$, $Pic(S)/Pic(\mathbb{P}^3)$ is a torsion free abelian group.

The two surfaces F_a, F_b are tangents almost everywhere along C. Moreover at every point $x \in C$ one of the two is smooth (otherwise the embedding dimension of the intersection would be three). So F_a, F_b define a sub-line bundle $L \subset N_C$. Abusing notations $L = N_{C,F_a} = N_{C,F_b}$. The quotient $N_C^* \to L^* \to 0$ defines the double structure C_2, hence:

$$0 \to L^* \to \mathcal{O}_{C_2} \to \mathcal{O}_C \to 0 \tag{1}$$

By the exact sequence of liaison:

$$0 \to \mathcal{I}_{C_2} \to \mathcal{I}_C \to \omega_C(4-a-b) \to 0$$

we see that $\mathcal{I}_{C,C_2} \simeq \omega_C(4-a-b)$. This means that $L^* = \omega_C(4-a-b)$. In particular:

$$\deg(L) =: l = d(a+b-4) - 2g + 2 \tag{2}$$

Remark 1. *If C is a complete intersection, then C is self-linked. If C is a curve on a quadric cone, then C is self-linked. In all these cases N_C splits.*

On the other hand it is easy to give examples of curves which are not self-linked. Let $C \subset \mathbb{P}^3$ be a smooth, irreducible curve whose degree, d, is an odd prime number. Assume $h^0(\mathcal{I}_C(2)) = 0$. If C is self-linked by $F_a \cap F_b$, then $2d = ab, a \leq b$. Since d is prime, $a = 2$, in contradiction with the assumption $h^0(\mathcal{I}_C(2)) = 0$.

A less evident fact: if $C \subset \mathbb{P}^3$ is a smooth subcanonical curve (i.e. $\omega_C \simeq \mathcal{O}_C(a)$ for some $a \in \mathbb{Z}$) which is not a complete intersection, then C is not self-linked (see [1]).

We can add a further class of examples:

Lemma 2. *Let C be a smooth, irreducible curve lying on a smooth quadric $Q \subset \mathbb{P}^3$. If C is not a complete intersection and $\deg(C) > 4$, then C is never self-linked.*

Proof. Assume C self-linked by $F_a \cap F_b$, $a \leq b$. Let (α, β), $\alpha < \beta$, denote the bi-degree of C on Q. If $F_a = Q$, then $F_b \cap Q$ is a curve of bi-degree $(b,b) = (2\alpha, 2\beta)$. It follows that $\alpha = \beta$ and C is a complete intersection. So we may assume that F_a is not a multiple of Q. The intersection $F_a \cap Q$ consists of C and of curve A of bi-degree $(a-\alpha, a-\beta)$. Since A is not empty (C is not a complete intersection) we have $a > \alpha$ and $a \geq \beta$. It follows that: $2a > \alpha + \beta = d$. So $a > d/2$. Since $ab = 2d$, we get $b = 2d/a \geq a > d/2$, so $a \leq 3$ hence $d \leq 5$. If $d = 5$, then $(a,b) = (2,5)$ in contradiction with $a > d/2$. Hence $d \leq 4$. □

If $d < 5$, then C is rational or elliptic, see Theorem 4. This lemma is in contrast with the fact that every curve on a quadric cone is self-linked.

2. The Gauss map associated to $L \subset N_C$.

We first recall some constructions associated to a sub-bundle of N_C. In what follow we don't assume C self-linked, C is just any smooth, irreducible curve not contained in a plane. If L is a sub-bundle of N_C, then $L(-1) \subset N_C(-1)$ comes from a rank two vector bundle: $\mathcal{T}_L \subset T_{\mathbb{P}^3}(-1)|C$. At each point $x \in C$, $\mathcal{T}_L(x) \subset T_{\mathbb{P}^3}(-1)(x) = V/d_x$, defines a plane of $\mathbb{P}^3$

containing the tangent line T_xC (here we see $\mathbb{P}^3$ as the projective space of lines of the four dimensional vector space V and $d_x \subset V$ is the line corresponding to the point $x \in \mathbb{P}^3$).

Local computations show that the plane $\mathcal{T}_L(x)$ is the Zariski tangent plane to the double structure C_2 defined by $N_C^* \to L^* \to 0$.

Now the bundle $\mathcal{T}_L$ defines the Gauss map $\varphi_L : C \to D \subset \mathbb{P}_3^*$ ($\varphi_L(x) = \mathcal{T}_L(x)$). It is known that φ_L can't be constant and that D can't be a line ([2], [6] Theorem 1.6). By the Nakano's exact sequence $\varphi_L^*(\mathcal{O}_{\mathbb{P}_3^*}(1)) = T_{\mathbb{P}^3}(-1)|C/\mathcal{T}_L$, which has degree $d - \deg(\mathcal{T}_L)$. Since $L(-1) = \mathcal{T}_L/T(-1)_C$, we get:

$$\deg(\varphi_L^*(\mathcal{O}_{\mathbb{P}_3^*}(1))) = \deg(\varphi_L).\deg(D) = 3d + 2g - 2 - l \tag{3}$$

Now consider the dual curve of D, $D^* \subset \mathbb{P}^3$ (defined by the osculating planes of D). The tangent surface $Tan(D^*)$ is called the *characteristic surface of* L and is denoted by $S_L^\vee$. This surface is the envelope surface of the family of planes $\{\mathcal{T}_L(x)\}_{x\in C}$. Since the $\mathcal{T}_L(x)$ are the tangent spaces to the double structure C_2, we have $C_2 \subset S_L^\vee$ (see also [8] Lemma 2.1.2).
If D is a plane curve, then $S_L^\vee$ is the cone over the (plane) dual curve D^*.

We will need the following result, which is contained in [7]:

Lemma 3. *A smooth, integral curve $C \subset \mathbb{P}^3$, of degree 9, genus 7 is never self-linked.*

Proof. If C is self linked it is by a complete intersection of type $(3, 6)$. If the cubic surface, F_3, is normal, then by (the proof of) Theorem 3.1 in [7], we should have $9.6 \le 6.7$, which is not the case. If the cubic is ruled we conclude with Propositions 3.4, 3.5 of [7]. Finally if F_3 is a cone, it has to be the cone over a smooth cubic curve (see the proof of Theorem 5.1 of [7]). But a degree 9 curve on such a cone is a complete intersection $(3, 3)$, hence has genus 10. □

Now we can state and prove our main result:

Theorem 4. *Let $C \subset \mathbb{P}^3$ be a smooth, irreducible curve of degree d, genus g. Assume $d \ge 5$ and $h^0(\mathcal{I}_C(2)) = 0$. If C is self-linked by a complete intersection of type (a, b), then one of the following occurs:*
$g = 3, d = 6$ and $(a, b) = (3, 4)$, or:

$$g \ge 4 \text{ and } 4g \ge d(a + b - 7) + 12 \tag{4}$$

Proof. From (2) and 3 we get

$$r := \deg(\varphi_L^*(\mathcal{O}_{\mathbb{P}_3^*}(1)) = \deg(\varphi_L).\deg(D) = 4g - 4 - d(a + b - 7) \tag{5}$$

Hence we have:

$$4g - 4 - r = d(a + b - 7) \text{ and } 2d = ab. \tag{6}$$

The assumption $h^0(\mathcal{I}_C(2)) = 0$ implies $b \ge a \ge 3$ and $\deg(D) \ge 3$. Indeed we already know that $\deg(D) \ge 2$. If we have equality, then $C \subset S_L^\vee$ which is a cone over the dual conic D^*. So we have: $r \ge 3$.

If $g \le 1$, $4g - 4 - d(a + b - 7) \ge 3$ implies $a + b \le 6$, hence $(a, b) = (3, 3)$, which is impossible. So $g \ge 2$. If $2 \le g \le 3$, we get $(a, b) = (3, 4)$, hence $d = 6$. Moreover $r = 4$ if $g = 2$ and $r = 8$ if $g = 3$.

Assume first that φ_L is bi-rational. Then $D \subset \mathbb{P}_3^*$ is an integral curve of degree r and geometrical genus g. If D is not contained in a plane, then $g \leq p_a(D) \leq G(r,2)$, where $G(r,2)$ is given by the Halphen-Castelnuovo's bound: $G(r,2) = (r-2)^2/4$ if r is even, $G(r,2) = (r-1)(r-3)/4$, if r is odd. It follows that $g \leq G(7,2) = 6$. Since $g \geq 2$ we immediately get $r \geq 5$. From what we said above, this implies $g \geq 3$, hence $d \geq 6$. We have $4g-4-r \leq 15$ and from (6), since $d \geq 6$, $a+b-7 \leq 2$. It follows that $(a,b;d) = (3,4;6),(4,4;8),(3,6;9),(4,5;10)$. From (6) we get: $4(g-1) = r, r+8, r+18, r+20$ and we see that there is no solution with $5 \leq r \leq 7$, $3 \leq g \leq 6$.

In conclusion if $r \leq 7$ and if φ_L is bi-rational, then D is a plane curve of degree r and geometric genus $g \geq 2$. We have $2 \leq g \leq (r-1)(r-2)/2 = p_a(D)$. Moreover C_2 lies on the cone, K, over the (plane) dual curve D^*. Finally since φ_L is bi-rational, C is a unisecant on the cone K. This implies that $\deg(D^*) + \varepsilon = d$ (+), where $\varepsilon = 1, 0$, according to whether C passes through the vertex of the cone or not.

Since $g \geq 2$, we get $r \geq 4$.

If $r = 4$ then $2 \leq g \leq 3$ and we already know that $d = 6$. If $g = 3$, D is smooth and $\deg(D^*) = 12$, in contradiction with (+). If $g = 2$, D has one double point which can be a node, a cusp or a tacnode. It follows that $\deg(D^*) = 10, 9$ or 8. In any case we get a contradiction with (+).

If $r = 5$, then $2 \leq g \leq 6$ and from (6) we get $4g - 9 = d(a+b-7)$. Since $d \geq 5$, the cases $2 \leq g \leq 3$ are impossible. If $g = 4$, the only possibility is $d = 7, a+b = 8$. Hence $a = b = 8$, but then again $d = ab/2 = 8$: contradiction. In the same way we see that the cases $g = 5, 6$ are impossible.

If $r = 6$ then $2 \leq g \leq 10$ and $4g - 10 = d(a+b-7)$, with $d = ab/2$. Observe that if $a+b-7 = 1$, then $a = b = 4$ and $d = 8$, if $a+b-7 = 2$, then $(a,b,d) = (3,6,9)$ or $(4,5,10)$. We get that for $g < 10$ the only possibility is $g = 7, d = 9, (a,b) = (3,6)$, which is excluded by Lemma 3. Finally if $g = 10$, then D is smooth. It follows that $d = \deg(D^*) + \varepsilon = 30 + \varepsilon$. Since (6) yields $30 = d(a+b-7)$, we get $d = 30$ and $a = b = 4$, which is impossible.

If $r = 7$ then $2 \leq g \leq 15$ and $4g - 11 = d(a+b-7)$. For most values of $g \leq 15$, $4g-11$ is a prime number and anyway it always has a simple factorization into prime numbers. Bearing in mind that if $a+b-7 = 1$, then $a = b = 4$ and $d = 8$; if $a+b-7 = 2$ the $(a,b,d) = (3,6,9)$ or $(4,5,10)$ and if $a+b-7 = 3$, then $(a,b,d) = (4,6,12)$, we easily see that there are no solutions.

In conclusion if $r \leq 7$ and φ_L is bi-rational, then the only possibility is for $r = 6$, $d = 9$, $g = 7$ and $(a,b) = (3,6)$ (in this case D is a plane curve with a triple point).

Now for $3 \leq r \leq 7$, $r = \deg(\varphi_L).\deg(D)$ and $\deg(D) \geq 3$, we see that if φ_L is not bi-rational, then $r = 6$, $\deg(\varphi_L) = 2$ and $\deg(D) = 3$.

If D is not contained in a plane it is a twisted cubic. The dual curve D^* is again a twisted cubic and $S^\vee = Tan(D^*)$ is a quartic surface. Since $C_2 \subset S^\vee$, $S^\vee = AF_a + BF_b$. If $b > 4$, it follows that $F_a = S^\vee$, i.e. $a = 4$. From (6) we get: $4g = d(d-6)/2 + 10$. Since $b = d/2$, d is even, hence $d \equiv 0, 2 \pmod 4$ and we see that the previous equation never gives an integral value for g. This shows $b \leq 4$, hence $(a,b,d) = (3,4,6),(4,4,8)$. Plugging these values into (6) we get a contradiction.

It follows that D must be a cubic plane curve. If D is smooth (has a node, a cusp), then $\deg(D^*) = 6$ (4 or 3). Since φ_L has degree two, C is a bi-secant on the cone $S^\vee$ over D^*. It follows that $d = 2\deg(D^*) + \varepsilon$. Since $C_2 \subset S^\vee$, $S^\vee = AF_a + BF_b$. If $b > \deg(D^*)$, then

$F_a = S^\vee$ and $a = \deg(D^*)$. It follows that $b = 2d/\deg(D^*)$. This implies $b = 4$. It follows that $(a, b, d) = (3, 4, 6), (4, 4, 8)$. Plugging these values into (6) we get a contradiction.

In conclusion we must have $r \geq 8$. □

Remark 5. *Because of Lemma 2 the assumption $h^0(\mathcal{I}_C(2)) = 0$ is harmless.*

There exist smooth curves of degree 6, genus 3 which are self-linked ([4], [3]).

This improves Theorem 7.8 of [5]. It follows from (4) that no curve of degree 8, genus 3 can be self-linked. This answers to a question raised in [5] (Introduction and Remark 7.19).

Corollary 6. *let $C \subset \mathbb{P}^3$ be a smooth, irreducible curve of degree $d > 4$, genus g, with $h^0(\mathcal{I}_C(2)) = 0$. If C is self-linked, then:*

$$g \geq \frac{d(\sqrt{8d} - 7)}{4} + 3 \tag{7}$$

Moreover if $d \geq 13$ and $d > g - 3$ no curve of degree d, genus g can be self-linked.

Proof. If $2d = ab, a \geq 2$, then $a + b$ varies from $d + 2$ ($a = 2, b = d$) to $2\sqrt{2d}$ ($a = b = \sqrt{2d}$). The inequality then follows from (4).

A curve with $d > g-3$ and $d \geq 13$ cannot lie on a quadric cone. Moreover if $d \geq 13$, then $2d = ab \geq 26$. It follows that $a + b \geq 11$ and inequality (4 is never satisfied if $d > g - 3$. □

Remark 7. *A reduced surface $S \subset \mathbb{P}^3$ is said to have* ordinary singularities *if its singular locus consists of a double curve, R, the surface having transversal tangent planes at most points of R, plus a finite number of pinch points and non-planar triple points. As proved in [5], Proposition 7.5, if a smooth curve is a set theoretic complete intersection on S with ordinary singularities and if $C \not\subset Sing(S)$, then C is self-linked (on S).*

3. Triple structures.

To conclude let's see how this approach applies also to set theoretic complete intersections (s.t.c.i.) with a triple structure. Assume $F_a \cap F_b = C_3$, where C_3 is a triple structure on a smooth, irreducible curve of degree d, genus g (i.e. C_3 is a locally Cohen-Macaulay (in our case l.c.i.) scheme with $Supp(C_3) = C$ and $ab = 3d$). The complete intersection $F_a \cap F_b$ links C to a double structure, C_2, on C. By liaison we have: $p_a(C_2) - g = d(a+b-4)/2$. Now C_2 (which as any double structure on C is a locally complete intersection curve) corresponds to a sub-line bundle $L \subset N_C$. From the exact sequence (1), we get:

$$l := \deg(L) = \frac{d}{2}(a + b - 4) - g + 1 \tag{8}$$

Theorem 8. *Let $C \subset \mathbb{P}^3$ be a smooth, connected curve of degree d, genus g. Assume C does not lie on a plane nor on a quadric cone. If there exists on C a triple structure which is the complete intersection of two surfaces of degrees a, b, then:*

$$3g \geq \frac{d}{2}(a + b - 10) + 6 \tag{9}$$

In particular: $g \geq \frac{d}{6}(\sqrt{12d} - 10) + 1$.

Proof. As before we consider the Gauss map φ_L. By (3) and (8), we have:

$$r := \deg(\varphi_L).\deg(D) = 3g - 3 - \frac{d}{2}(a + b - 10).$$

We know that $r \geq 2$ and if equality C lies on a quadric cone. So we may assume $r \geq 3$ and (9) follows. For the second inequality, if $ab = 3d$, then $a + b \geq 2\sqrt{3d}$. □

Combining with Corollary 6 we get:

Corollary 9. *Let $C \subset \mathbb{P}^3$ be a smooth, connected curve of degree d, genus g. If C is not contained in a plane nor in a quadric cone and if $g < \dfrac{d(\sqrt{12d} - 10) + 6}{6}$, then C cannot be a s.t.c.i. with a structure of multiplicity $m \leq 3$.*

By the way let us observe the following elementary fact:

Lemma 10. *Let $C \subset \mathbb{P}^3$ be a smooth, connected curve of degree d, genus g. Let s denote the minimal degree of a surface containing C. Assume C is the set theoretic complete intersection of two surfaces of degrees $a, b; a \leq b$ and that a is minimal with respect to this property. Let $md = ab$. If $a > s$ or if $h^0(\mathcal{I}_C(s)) > 1$, then $m \geq d/s^2$.*

Proof. Assume $C = F_a \cap F_b$ as sets with $a \leq b$ and $ab = md$. If $S \in H^0(\mathcal{I}_C(s))$, then $S^m \in H^0_*(\mathcal{I}_X)$, where X denotes the $m-1$-th infinitesimal neighbourhood of C ($\mathcal{I}_X = \mathcal{I}_C^m$). It follows that $S^m \in (F_a, F_b)$. So $S^m = AF_a + BF_b$. If $b > sm$, then $S^m = AF_a$ and since S is integral, we get $S^t = F_a$. It follows that $S \cap F_b = C$ as sets. By minimality of a, it follows that $F_a = S$. This is excluded by our assumptions ($a > s$ or $h^0(\mathcal{I}_C(s)) > 1$). So $b \leq sm$. Thus $m \geq b/s$, hence $m^2 \geq ab/s^2 = md/s^2$ and the result follows. □

Let $C \subset Q$, Q a smooth quadric surface. Assume C is the s.t.c.i. of two surfaces of degrees a, b. Then if $d > 3$ and C is not a complete intersection, it is easy to see that $b \geq a > 2$. Hence $m \geq d/4$, where $dm = ab$.

References

[1] Beorchia, V.-Ellia, Ph.: *Normal bundle and complete intersections*, Rend. Sem. Mat. Univers. Politecn. Torino, vol. 48, 4, 553-562 (1990)

[2] Eisenbud, D.-Van de Ven, A.: *On the normal bundles of smooth rational space curves*, Math. Ann., **256**, 453-463 (1981)

[3] Ellia, Ph.: *Exemples de courbes de $\mathbb{P}^3$ à fibré normal semi-stable, stable*, Math. Ann., **264**, 389-396, (1983)

[4] Gallarati, D.: *Ricerche sul contatto di superficie algebriche lungo curve*, Mémoire Acad. Roy. Belge **32**, 1-78 (1960)

[5] Hartshorne, R.-Polini, C.: *Divisors class groups of singular surfaces*, Preprint arXiv: 1301. 3222v1 [math.AC], 15 Jan 2013 (2013)

[6] Hulek, K.-Sacchiero, G.: *On the normal bundle of elliptic space curves*, Arch. Math., **40**, 61-68 (1983)

[7] Jaffe, D.: *On set theoretic complete intersections in $\mathbb{P}^3$*, Math. Ann., **285**, 165-176 (1989)

[8] Ramella L.: *Sur les schémas définissant les courbes rationnelles lisses de $\mathbb{P}^3$ ayant fibré normal et fibré tangent restreint fixés*, Mémoire (nouvelle sèrie) de la Soc. Math. de France, **54**, (1993)

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