Wave-Front Tracking for the Equations of Non-Isentropic Gas Dynamics

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Abstract

We study the model equations of polytropic gas dynamics, which constitute a system of three hyperbolic conservation laws. Global in time BV-solutions were obtained by T.-P. Liu (Indiana Univ. Math. J., 1978) provided that $(\gamma-1)$ times the total variation of the initial data is sufficiently small; here γ is the adiabatic coefficient. The aim of this paper is to give an alternative proof by exploiting the Dafermos-Bressan-Risebro wave-front tracking scheme. An original feature is the use of the *path decomposition method* to obtain pathwise estimates of the approximate solutions; these estimates show the decay properties of the solutions and play a crucial role in proving the stability of the wave-front tracking scheme.

1 Introduction

The equations of gas dynamics in one-space dimension are given in Lagrangian coordinates by

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ \left(\mathcal{E} + \frac{1}{2}u^2\right)_t + (pu)_x = 0, \end{cases} \tag{1.1}$$

for $(x,t) \in \mathbb{R} \times \mathbb{R}_+$. Here above u is the velocity, p the pressure, v the specific volume and $\mathcal{E}$ the internal energy; p and v are positive quantities. Temperature and entropy are denoted by Θ, S, respectively, and satisfy the first and second law of thermodynamics: $d\mathcal{E} = \Theta dS - p dv$, [10]. The gas is assumed to be *ideal*, i.e., $pv = R\Theta$, and *polytropic*: $\mathcal{E} = C_v\Theta + \mathcal{E}_0$; here, R, C_v and $\mathcal{E}_0$ are constants. As a consequence, the entropy S is expressed as

$$S = C_v \left(\log p + \gamma \log v\right) + \text{const},$$

where $\gamma = 1 + R/C_v > 1$. By setting $\mathcal{E}_0 = -\frac{a^2}{\gamma-1}$, so that $\mathcal{E}$ makes sense for $\gamma \to 1$, we have

$$\mathcal{E} = \frac{pv - a^2}{\gamma - 1}, \quad p = a^2 v^{-\gamma} e^{\frac{\gamma-1}{R}S}.$$

In the limit case $\gamma = 1$ one has $p = a^2 v^{-1}$ and $\mathcal{E} = a^2(-\log v + S/R)$; hence the system (1.1) coincides with the equation of *isothermal gas dynamics*.

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We shall discuss the existence of solutions, defined globally in time, to the initial value problem for the equations (1.1). The initial data are given by

$$(v, u, S)|_{t=0} = \big(\overline{v}(x), \overline{u}(x), \overline{S}(x)\big), \tag{1.2}$$

with

$$\overline{v}(x) \geq v_* > 0 \tag{1.3}$$

and $\overline{v}$, $\overline{u}$, $\overline{S} \in \mathrm{BV}(\mathbb{R})$, the space of functions having bounded total variation. We denote by $\mathrm{TV}(f)$ the total variation of a function $f \in \mathrm{BV}(\mathbb{R})$.

If the total variation of the initial data is sufficiently small, then Glimm's existence theorem [11] applies for any $\gamma \geq 1$ and guarantees the existence of global solutions. Hence, the issue is to study the case of initial data whose total variation is not necessarily small. The following results are by now classic.

Theorem 1.1 (Liu [15])**.** *Assume* $1 < \gamma \leq \frac{5}{3}$ *and* (1.3)*. If* $(\gamma - 1)\mathrm{TV}(\overline{v}, \overline{u}, \overline{S})$ *is sufficiently small, then there exists a global in time* BV*-solution to* (1.1)-(1.2)*.*

Theorem 1.2 (Nishida [17])**.** *Assume* $\gamma = 1$ *and* (1.3)*. If the total variation* $\mathrm{TV}(\overline{v}, \overline{u}, \overline{S})$ *of the initial data is finite, then there exists a global in time* BV*-solution to* (1.1)-(1.2)*.*

Of course, if $\gamma = 1$ then the two first equations in (1.1) decouple from the third one and the entropy S is computed by means of v and the initial data $\overline{S}$. Another celebrated paper, due to Nishida and Smoller [18], dealt previously with the system of *isentropic* gas dynamics and a result analogous to Theorem 1.1 was proved. Later on, Temple [22] provided a different proof of Theorem 1.1, basing on the observation that the polytropic gas model $1 < \gamma \leq \frac{5}{3}$ can be treated as a perturbation of the isothermal model $\gamma = 1$. All of these results were proved by exploiting the *random choice method* due to Glimm [11].

In recent years the *wave-front tracking scheme*, which was initiated by Dafermos [8] for a single conservation laws and then developed by Bressan [6, 7], and Risebro [19] for hyperbolic systems of conservation laws, has been proved a better alternative to Glimm's scheme. This approach has the following advantages:

- Free from random sampling;
- Based on simple interaction estimates;
- Useful in studying asymptotic properties;
- Appropriate for establishing the continuous dependence on the initial data.

The wave-front tracking scheme was successfully applied to the case of isentropic gas dynamics by Asakura [3]. About continuous dependence both in the isentropic case and for the full system (1.1) we also refer to [13].

This paper is the natural continuation of the analysis begun in Asakura [3]. Our main goal is to provide a proof to both theorems above by the front tracking scheme. In doing this, a difficult issue is the control of the total variation of the approximate solutions; to this end we exploit a rather new technique, the *path decomposition method*, which was first fully exploited by Asakura in [3]. Notice that, although the notion of path was introduced by Temple and Young [21], the idea of the decomposition indeed goes back to Asakura [2]. It consists, roughly speaking, in the following.

Paths are broken lines, in the xt-plane, which follow shock and entropy fronts. Each front is decomposed into finitely many segments, whose positions coincide with that of the front but belong to different paths. A notion of *strength* is then introduced for each segment, in such a way that the strength of a wave is the sum of the strengths of the segments that decompose it. A *generation order* is also defined for each segment of the path. The goal of this apparently complicated construction is to deal separately with primary paths and secondary paths. In the former case, the strength is decreasing along the path and then the control of the variation is trivial. The latter case takes into account, for instance, both the "new" shocks generated through an interaction (for example, shocks that belong to families different from the interacting ones) and a part of those shocks whose strength is amplified in the interaction (this is the case, for instance, of a shock emerging from the interaction of two shocks, all of them belonging to the same family).

Nishida's lemma [16] states that, in the case of isothermal gas dynamics, the total amount of shock waves in Glimm approximate solutions does not increase. We shall obtain estimates of the approximate solutions along paths, showing that the strength of each path decreases at the rate of c^{n-1} (for $0 < c < 1$) as n, the generation order, increases. This is the "pathwise" version for system (1.1) of Nishida's lemma and plays a crucial role in proving both the stability of the wave-front tracking scheme and the decay property of the weak entropy solutions (see Glimm and Lax [12]). If $1 < \gamma \leq \frac{5}{3}$, we also introduce *secondary waves* and show that the total amount of these waves is $(\gamma - 1)$ times the *interaction potential*; this leads to a further understanding both of the assumption made in Theorem 1.1 and of the aforementioned perturbation method of Temple [22].

The main result of this paper is the following.

Theorem 1.3. *Under the same assumptions of Theorem 1.1, the wave-front tracking scheme is stable and provides a global* BV*-solution to* (1.1)–(1.2).

As we emphasized above, the interest of this result lies more in the techniques of proof than in its bare statement.

The system (1.1) is close to the following one, which arises in the modelling of phase transitions in fluids:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v, \lambda)_x = 0, \\ \lambda_t = 0. \end{cases} \tag{1.4}$$

Global BV-solutions are obtained by Amadori and Corli [1], Asakura and Corli [5], Holden, Risebro and Sande [14] for $p = a^2(\lambda)v^{-1}$ in [1, 5] and $p = a^2 v^{-\lambda}$ in [14]. System (1.4) is simpler than (1.1) in the sense that, if the initial data $\overline{\lambda}$ is constant in an interval $[x_0, x_1]$, then in that region it reduces to the system of isothermal or isentropic gas dynamics. However, the study of (1.4) is somewhat more difficult than that of (1.1), because the former lacks of any appropriate parameter as $\epsilon = \frac{1}{2}(\gamma - 1)$ in the latter, where ϵ times the total variation of the initial data is assumed arbitrarily small. However, some basic ideas and methods developed in [5] are adopted in the present paper. We emphasize that in [5] only shock fronts (and not entropy fronts, as it is also the case here) gave rise to paths.

The paper is organized as follows. In Section 2 we quickly introduce the notation concerning the wave curves and the Riemann problem [7, 9, 20]. Then, the front-tracking scheme is described in Section 3 following Bressan [7]. Section 4 deals with the (very) technical subject of the local interaction estimates. The main result of this section, namely Lemma 4.3, was first given by Liu [15]; however, the interested reader can find fully detailed proofs in [4]. In Section 5 we finally enter the core of the paper by proving the global interaction estimates, which parallel those of Liu [15] but are suitably adapted to our different scheme. We define paths and their strengths in Section 6; a precise description of the decomposition by paths is given there. Estimates along paths are obtained in Section 7 and the stability of the front-tracking scheme is finally proved in Section 8. Our estimates also imply the asymptotic stability of the BV-solutions obtained in this paper (see Asakura [3, Section 8]).

2 The Riemann Problem

The quantity $\eta = S/R$ is called the *dimensionless entropy*. As we shall see in (2.8), system (1.1) admits stationary waves connecting states with $p_+ = p_-$, $u_+ = u_-$, $\eta_+ \neq \eta_-$; then, it is useful to choose $U = (p, u, \eta)$ as independent variables. Note that v can be written by using p and η; we have

$$v = a^{\frac{2}{\gamma}} e^{\frac{\gamma-1}{\gamma}\eta} p^{-\frac{1}{\gamma}} \quad \text{and} \quad \sqrt{-v_p(p,\eta)} = \gamma^{-\frac{1}{2}} a^{\frac{1}{\gamma}} e^{\frac{\gamma-1}{2\gamma}\eta} p^{-\frac{\gamma+1}{2\gamma}}.$$

The quasi-linear equations associated to (1.1) are

$$p_t - \frac{u_x}{v_p} = 0, \quad u_t + p_x = 0, \quad \eta_t = 0.$$

From these equations we find that the characteristic speeds are

$$\lambda_1(U) = -\frac{1}{\sqrt{-v_p(p,\eta)}}, \quad \lambda_2(U) = \frac{1}{\sqrt{-v_p(p,\eta)}}, \quad \lambda_0(U) = 0$$

and the corresponding characteristic vector fields may be written as

$$R_1(U) = {}^t(1, -\sqrt{-v_p(p,\eta)}, 0), \quad R_2(U) = {}^t(1, \sqrt{-v_p(p,\eta)}, 0),$$
$$R_0(U) = {}^t(0, 0, 1).$$

Setting $\epsilon = \frac{\gamma-1}{2}$, we can write the *forward 1-rarefaction curve* $\widehat{\mathcal{R}}_1^F(U_0)$ and the *backward 2-rarefaction curve* $\widehat{\mathcal{R}}_2^B(U_0)$ issuing from U_0 as

$$\begin{aligned} \widehat{\mathcal{R}}_1^F(U_0): &\quad \begin{cases} u - u_0 &= -\frac{\sqrt{\gamma} a^{\frac{1}{\gamma}}}{\epsilon} e^{\frac{\epsilon}{\gamma}\eta_0} (p^{\frac{\epsilon}{\gamma}} - p_0^{\frac{\epsilon}{\gamma}}), \\ \eta - \eta_0 &= 0, \end{cases} \quad p \le p_0, \\ \widehat{\mathcal{R}}_2^B(U_0): &\quad \begin{cases} u - u_0 &= \frac{\sqrt{\gamma} a^{\frac{1}{\gamma}}}{\epsilon} e^{\frac{\epsilon}{\gamma}\eta_0} (p^{\frac{\epsilon}{\gamma}} - p_0^{\frac{\epsilon}{\gamma}}), \\ \eta - \eta_0 &= 0. \end{cases} \quad p \le p_0. \end{aligned} \tag{2.1}$$

These curves are integral curves of $R_j(U)$ $(j = 1, 2)$, respectively. If $U \in \widehat{\mathcal{R}}_1^F(U_0)$ there is a 1-rarefaction wave connecting U_0 and U; if $U \in \widehat{\mathcal{R}}_2^B(U_0)$ there is a 2-rarefaction wave connecting U and U_0.

A self-similar jump discontinuity having the form

$$U(x,t) = \begin{cases} U_- & \text{for} \quad x < st, \\ U_+ & \text{for} \quad x > st, \end{cases} \tag{2.2}$$

is a *weak solution* of (1.1) if and only if the constant states U_- and U_+ satisfy the *Rankine-Hugoniot condition*

$$\begin{cases} \mathcal{E}_+ - \mathcal{E}_- + \frac{1}{2}(p_+ + p_-)(v_+ - v_-) = 0, \\ (u_+ - u_-)^2 = -(p_+ - p_-)(v_+ - v_-), \end{cases}$$

where the shock speed s satisfies $s^2 = -\frac{p_+ - p_-}{v_+ - v_-}$. For a polytropic gas, the Rankine-Hugoniot condition is equivalent to

$$\begin{cases} e^{(\gamma-1)(\eta_+ - \eta_-)} & = & \left(\dfrac{p_+}{p_-}\right)\left\{\dfrac{(\gamma-1)p_+ + (\gamma+1)p_-}{(\gamma+1)p_+ + (\gamma-1)p_-}\right\}^{\gamma}, \\ (u_+ - u_-)^2 & = & \dfrac{2v_-(p_+ - p_-)^2}{(\gamma+1)p_+ + (\gamma-1)p_-}. \end{cases} \tag{2.3}$$

If $p_+ \neq p_-$, we have two branches of solutions to (2.3):

$$u_+ - u_- = \pm\sqrt{\frac{2v_-}{(\gamma+1)p_+ + (\gamma-1)p_-}}(p_+ - p_-), \quad s = \pm\sqrt{-\frac{p_+ - p_-}{v_+ - v_-}}. \tag{2.4}$$

Let us fix p_- and η_-. When considering η_+ as a function of p_+, we have

$$\frac{d\eta_+}{dp_+} = \frac{(\gamma+1)(p_+ - p_-)^2}{p_+\{(\gamma-1)p_+ + (\gamma+1)p_-\}\{(\gamma+1)p_+ + (\gamma-1)p_-\}} > 0, \tag{2.5}$$

which shows that $\eta_+ > \eta_-$ if and only if $p_+ > p_-$. Since the entropy must increase as time goes on, the physically relevant branches are

$$p_+ > p_- \text{ for } s < 0 \quad \text{and} \quad p_+ < p_- \text{ for } s > 0. \tag{2.6}$$

A jump discontinuity (2.2) lying on a Hugoniot branch satisfying (2.6) is called a *shock wave* and the line of discontinuity $x = st$ is referred to as a *shock front*. More precisely, by setting

$$\begin{aligned} \mathcal{G}(p_0, p; \eta_0) &= \frac{a^{\frac{1}{\gamma}} e^{\frac{\epsilon}{\gamma}\eta_0}(p - p_0)}{p_0^{\frac{1}{2\gamma}}\{(1+\epsilon)p + \epsilon p_0\}^{\frac{1}{2}}}, \\ \mathcal{H}(p_0, p) &= \frac{1}{2\epsilon}\log\left[\left(\frac{p}{p_0}\right)\left\{\frac{\epsilon p + (1+\epsilon)p_0}{(1+\epsilon)p + \epsilon p_0}\right\}^{\gamma}\right], \end{aligned}$$

we define the *forward* 1-*shock curve* $\widehat{\mathcal{S}}_1^F(U_0)$ and the *backward* 2-*shock curve* $\widehat{\mathcal{S}}_2^B(U_0)$ issuing from U_0 as

$$\begin{aligned} \widehat{\mathcal{S}}_1^F(U_0) : & \begin{cases} u - u_0 = -\mathcal{G}(p_0, p; \eta_0), \\ \eta - \eta_0 = \mathcal{H}(p_0, p), \end{cases} \quad p > p_0, \\ \widehat{\mathcal{S}}_2^B(U_0) : & \begin{cases} u - u_0 = \mathcal{G}(p_0, p; \eta_0), \\ \eta - \eta_0 = \mathcal{H}(p_0, p), \end{cases} \quad p > p_0. \end{aligned} \tag{2.7}$$

At last, if $p_+ = p_-$ we have an *entropy wave*

$$u_+ = u_-, \quad p_+ = p_-, \quad \eta_+ \neq \eta_-, \quad s = 0, \tag{2.8}$$

which coincides with the integral curve $R_0(U_-)$ when η_+ varies. This type of discontinuity is also called a *contact discontinuity.*

We define the *forward* 1*-wave curve* $\widehat{\mathcal{W}}_1^F(U_L) = \widehat{\mathcal{R}}_1^F(U_L) \cup \widehat{\mathcal{S}}_1^F(U_L)$ and the *backward* 2*-wave curve* $\widehat{\mathcal{W}}_2^B(U_R) = \widehat{\mathcal{R}}_2^B(U_R) \cup \widehat{\mathcal{S}}_2^B(U_R)$; the 0*-wave curves* are just the integral curves R_0. Each wave curve constitutes a C^2-curve with Lipschitz-continuous second derivative; they represent all realizable rarefaction, shock and entropy waves. More precisely, if $(p, u, \eta) \in \widehat{\mathcal{W}}_1^F(U_L)$, then there is a 1-rarefaction or shock wave connecting (p_L, u_L, η_L) and (p, u, η); if $(p, u, \eta) \in \widehat{\mathcal{W}}_2^B(U_R)$, there is a 2-rarefaction or shock wave connecting (p, u, η) and (p_R, u_R, η_R). The projections of the curves $\widehat{\mathcal{W}}_1^F(U_L)$ and $\widehat{\mathcal{W}}_2^B(U_R)$ on the pu-plane are denote by $\mathcal{W}_1^F(U_L)$ and $\mathcal{W}_2^B(U_R)$, respectively.

The Riemann problem, i.e., the initial-value problem for (1.1) with piecewise constant initial data

$$U(x, 0) = \begin{cases} U_L & \text{if } x < 0, \\ U_R & \text{if } x > 0, \end{cases} \tag{2.9}$$

is solved in the following way. Let (p_L, u_L, η_L) and (p_R, u_R, η_R) be given Riemann data. If the curves $\mathcal{W}_1^F(U_L)$ and $\mathcal{W}_2^B(U_R)$, have an intersection point (p_m, u_m), then the state $(p_m, u_m, \eta_m^-) \in \widehat{\mathcal{W}}_1^F(U_L)$ and $(p_m, u_m, \eta_m^+) \in \widehat{\mathcal{W}}_2^B(U_R)$ are connected by an entropy wave. Since the sound speed is expressed as

$$c = \sqrt{\gamma p v} = \sqrt{\gamma} a^{\frac{1}{\gamma}} e^{\frac{\epsilon}{\gamma}\eta} p^{\frac{\epsilon}{\gamma}},$$

we have the following well-known theorem, where uniqueness is understood in the sense of Smoller [20, Theorem 18.6].

Theorem 2.1. *The Riemann problem for* (1.1) *with data* (2.9) *has a unique solution if*

$$u_R - u_L < \frac{c_L + c_R}{\epsilon}. \tag{2.10}$$

3 The Wave-Front Tracking Scheme

The wave-front tracking scheme provides a way for constructing approximate solutions to the system (1.1) with initial data

$$U(x, 0) = \overline{U}(x) \tag{3.1}$$

in the class $\mathrm{BV}(\mathbb{R})$. We quickly sketch here the main steps, referring the reader to [7] for details. We define

$$\eta_* = \inf \overline{\eta}(x). \tag{3.2}$$

Let h be a positive number. We approximate the initial data $\overline{U}(x)$ by a step function $\overline{U}^h(x)$ having finitely many jumps; moreover, we may assume $\eta_* \leq \min \overline{\eta}^h(x)$.

Let $x_1 < \cdots < x_M$ be the points of discontinuity of $\overline{U}^h(x)$. At each x_m, we solve the Riemann problem with initial data $U_L = \overline{U}^h(x_m - 0)$, $U_R = \overline{U}^h(x_m + 0)$

in an approximate way: every rarefaction wave is substituted by several small fans consisting of constant states and jump discontinuities separating them. This is called the *accurate Riemann solver*.

The issue is how to extend the approximate solution U^h after a wave interaction. To avoid the breakdown due to the possible divergence of the jump discontinuities within a finite time, a *simplified Riemann solver* must be introduced. It consists in prolonging each interacting wave, say θ' and θ'', with a wave of the same family and size; if the fronts belong to the same family they are prolonged as a single front of size $\theta' + \theta''$. Since the waves in general do not commute, a *nonphysical front* is introduced with a sufficiently high speed $\hat{\lambda}$.

Now, we describe how the two solvers are used in the construction of U^h. Recall that the amount of waves emerging from an interaction is estimated by the product $|\theta'\theta''|$ of the strengths of the incoming waves. Then, we fix a threshold $\rho > 0$ and extend U^h past an interaction of two waves θ' and θ'' at (x,t) as follows: if $|\theta'\theta''| \geq \rho$ then we use the accurate Riemann solver with data $U^h(x, t-0)$, else, if $|\theta'\theta''| < \rho$, we use the simplified Riemann solver, with the proviso that if one of the waves is an entropy wave, then we use the former solver if $M_0|\theta'\theta''| \geq \rho$ and the latter if $M_0|\theta'\theta''| \leq \rho$. Here, M_0 is the constant defined in (5.28).

The above procedure gives an approximate solution up to some time $T > 0$; in order to prove that U^h is defined for $t \in (0, +\infty)$ we shall show that the number of interactions remains finite for any time.

4 Interaction of Two Incoming Waves

We define $\overline{p}_* = \inf \overline{p}(x)$. We shall show in Lemma 5.2 that there exist p_*, p^* and H such that all waves under consideration are in the region

$$0 < p_* \leq p \leq p^*, \quad 0 \leq \eta - \eta_* \leq H, \tag{4.1}$$

if $\epsilon\mathrm{TV}(\overline{p}, \overline{u}, \overline{\eta})$ is sufficiently small. Therefore, in the following we assume that (4.1) holds. We introduce the Riemann invariants with respect to η_*

$$w = u - \frac{\sqrt{\gamma}a^{\frac{1}{\gamma}}}{\epsilon} e^{\frac{\epsilon}{\gamma}\eta_*}(p^{\frac{\epsilon}{\gamma}} - \overline{p}_*^{\frac{\epsilon}{\gamma}}), \quad z = u + \frac{\sqrt{\gamma}a^{\frac{1}{\gamma}}}{\epsilon} e^{\frac{\epsilon}{\gamma}\eta_*}(p^{\frac{\epsilon}{\gamma}} - \overline{p}_*^{\frac{\epsilon}{\gamma}})$$

and set

$$\tau = \frac{z - w}{2} = \frac{\sqrt{\gamma}a^{\frac{1}{\gamma}}}{\epsilon} e^{\frac{\epsilon}{\gamma}\eta_*}(p^{\frac{\epsilon}{\gamma}} - \overline{p}_*^{\frac{\epsilon}{\gamma}}). \tag{4.2}$$

Remark 4.1. *In Liu [15], the Riemann invariants are chosen with respect to a different entropy level, denoted there as s_*, which is an* upper *bound for the values of the entropy (see [15, Lemma 5.1]). Our choice of η_* is related instead to the* lower *bound of entropy. As a consequence, the slopes of the shock-rarefaction curves in the wz-plane defined below are positive while in [15] they are negative.*

The strengths of the shock and rarefaction waves will be measured by w and z. The pressure is considered to be a function of $\tau : p = p(\tau)$ and we set

$$g(\tau_0, \tau; \eta_0) = \mathcal{G}\big(p(\tau_0), p(\tau); \eta_0\big), \quad h(\tau_0, \tau) = \mathcal{H}\big(p(\tau_0), p(\tau)\big). \tag{4.3}$$

Since $w+z\pm e^{\frac{\epsilon}{\gamma}(\eta_0-\eta_*)}(z-w)$ are constant along rarefaction curves, we find that the forward 1-rarefaction curve $\widehat{\mathcal{R}}_1^F(U_0)$ and the backward 2-rarefaction curve $\widehat{\mathcal{R}}_2^B(U_0)$ are expressed as

$$\begin{array}{llll} \widehat{\mathcal{R}}_1^F(U_0): & z-z_0 & = & \dfrac{e^{\frac{\epsilon}{\gamma}(\eta_0-\eta_*)}-1}{e^{\frac{\epsilon}{\gamma}(\eta_0-\eta_*)}+1}(w-w_0), \quad w\geq w_0, \\ \widehat{\mathcal{R}}_2^B(U_0): & w-w_0 & = & \dfrac{e^{\frac{\epsilon}{\gamma}(\eta_0-\eta_*)}-1}{e^{\frac{\epsilon}{\gamma}(\eta_0-\eta_*)}+1}(z-z_0), \quad\quad z\leq z_0. \end{array}$$

Note that $0<\frac{e^{\frac{\epsilon}{\gamma}(\eta_0-\eta_*)}-1}{e^{\frac{\epsilon}{\gamma}(\eta_0-\eta_*)}+1}<1$.

The forward 1-shock curve $\widehat{\mathcal{S}}_1^F(U_0)$ and the backward 2-shock curve $\widehat{\mathcal{S}}_2^B(U_0)$ issuing from U_0 are also represented by using the Riemann invariant coordinates. Remark that by (4.3) the Hugoniot curves through U_0 are expressed as

$$u-u_0=\mp g(\tau_0,\tau;\eta_0), \quad \eta-\eta_0=h(\tau_0,\tau),$$

which define in particular the function $\eta(\tau)=\eta_0+h(\tau_0,\tau)$. Note that $0<\epsilon\leq\frac{1}{3}$ is equivalent to $1<\gamma\leq\frac{5}{3}$.

The following results were proved by Liu [15] (see also Asakura [4]).

Lemma 4.1. *If $0<\epsilon\leq\frac{1}{3}$, then there are functions $z_1=z_1(w;\eta_0)$ and $w_2=w_2(z;\eta_0)$ such that*

$$\begin{array}{lll} \widehat{\mathcal{S}}_1^F(U_0) & = & \{(w,z,\eta);\ z=z_1(w;\eta_0),\ \eta=\eta(\tau),\ w<w_0\}, \\ \widehat{\mathcal{S}}_2^B(U_0) & = & \{(w,z,\eta);\ w=w_2(z;\eta_0),\ \eta=\eta(\tau),\ z>z_0\}. \end{array}$$

Moreover, there is a constant $B>1$, which depends on p_*, p^*, H, such that

$$\begin{array}{lllll} \dfrac{e^{\frac{\epsilon}{\gamma}(\eta_0-\eta_*)}-1}{e^{\frac{\epsilon}{\gamma}(\eta_0-\eta_*)}+1} & = z_1'(w_0;\eta_0) & \leq & z_1'(w;\eta_0) & \leq \dfrac{Be^{\frac{\epsilon}{\gamma}(\eta_0-\eta_*)}-1}{Be^{\frac{\epsilon}{\gamma}(\eta_0-\eta_*)}+1}, \\ \dfrac{e^{\frac{\epsilon}{\gamma}(\eta_0-\eta_*)}-1}{e^{\frac{\epsilon}{\gamma}(\eta_0-\eta_*)}+1} & = w_2'(z_0;\eta_0) & \leq & w_2'(z;\eta_0) & \leq \dfrac{Be^{\frac{\epsilon}{\gamma}(\eta_0-\eta_*)}-1}{Be^{\frac{\epsilon}{\gamma}(\eta_0-\eta_*)}+1}. \end{array}$$

At last,

$$\begin{array}{ccccccc} & & z_1''(w;\eta_0) & < & 0 & < & w_2''(z;\eta_0), \\ \eta_1'(w) & < & \eta_1'(w_0) & = & 0 & = & \eta_2'(z_0) \ < \ \eta_2'(z). \end{array}$$

Lemma 4.2. *If $\eta_1>\eta_0$ then*

$$z_1(w;\eta_1)\leq z_1(w;\eta_0), \quad w_2(z;\eta_1)\leq w_2(z;\eta_0). \tag{4.4}$$

We denote by α, β, ξ, π, respectively, the strengths of 1-shock waves, 2-shock waves, 1-rarefaction waves, 2-rarefaction waves, respectively. They are defined by

$$\begin{array}{lll} \alpha=w_0-w & \text{if} & (p,u,\eta)\in\widehat{\mathcal{S}}_1^F(p_0,u_0,\eta_0), \\ \beta=z-z_0 & \text{if} & (p,u,\eta)\in\widehat{\mathcal{S}}_2^B(p_0,u_0,\eta_0), \\ \xi=w-w_0 & \text{if} & (p,u,\eta_0)\in\widehat{\mathcal{R}}_1^F(p_0,u_0,\eta_0), \\ \pi=z_0-z & \text{if} & (p,u,\eta_0)\in\widehat{\mathcal{R}}_2^B(p_0,u_0,\eta_0). \end{array}$$

We also define for entropy waves

$$\delta = \eta_L - \eta_R \quad \text{if} \quad (p_0, u_0, \eta_L), (p_0, u_0, \eta_R) \text{ are the side states.}$$

The strength of an entropy wave will be defined later at (4.9).

In order to measure the increase of the entropy across shock waves, we define the quantities $\delta_\alpha, \delta_\beta$ as

$$\begin{aligned} \delta_\alpha &= \eta - \eta_0 \quad \text{if} \quad (p, u, \eta) \in \widehat{\mathbb{S}}_1^F(p_0, u_0, \eta_0), \\ \delta_\beta &= \eta - \eta_0 \quad \text{if} \quad (p, u, \eta) \in \widehat{\mathbb{S}}_2^B(p_0, u_0, \eta_0). \end{aligned}$$

Remark that all these quantities are positive.

From now on, we also denote by α, β, δ, ξ, π, the corresponding waves. Suppose that U_L, U_M are connected by a 2-wave θ_2 (or an entropy wave) and U_M, U_R by a 1-wave θ_1(or an entropy wave); these waves are assumed to interact. Then, under the assumption (2.10), the Riemann problem has a unique solution connecting the states U_L and U_R. It is composed of a 1-wave θ_1' connecting U_L to U^-, an entropy wave δ' connecting U^- to U^+ and a 2-wave θ_2' connecting U^+ to U_R. This interaction is simply denoted by

$$\theta_2 + \theta_1 \to \theta_1' + \delta' + \theta_2'.$$

The local interaction estimates are gathered in the following lemma, which was first proved by Liu [15]; we also refer to Asakura [4] for more details. Notation below is as in [4] and slightly differs from that in [15].

Lemma 4.3. *Assume that* $0 < \epsilon \leq \frac{1}{3}$. *Assume also that* p *and* η *satisfy* (4.1) *and* ϵH *is sufficiently small. Then there are constants* $0 < D_0 < 1$ *and* $D, D_1, D_2 > 0$, *depending only on* p_*, p^*, H, *such that the following estimates hold.*

(1) $\beta + \alpha \to \alpha' + \delta' + \beta'$; *we have* $\delta_{\alpha'} \geq \delta_\alpha - \epsilon D_2 \alpha\beta$, $\delta_{\beta'} \geq \delta_\beta - \epsilon D_2 \alpha\beta$,

$$\alpha' \leq \alpha + \epsilon D \alpha\beta, \quad |\delta'| \leq \epsilon D_2 \alpha\beta, \quad \beta' \leq \beta + \epsilon D \alpha\beta.$$

(2) $\pi + \alpha \to \alpha' + \delta' + \pi'$; *we have* $\delta_{\alpha'} \geq \delta_\alpha - \epsilon D_2 \alpha\pi$,

$$\alpha' \leq \alpha + \epsilon D \alpha\pi, \quad |\delta'| \leq \epsilon D_2 \alpha\pi, \quad \pi' \leq \pi + \epsilon D \alpha\pi.$$

(3) $\alpha_1 + \alpha_2 \to \alpha' + \delta' + \pi'$; *we have* $\delta_{\alpha'} \geq \delta_{\alpha_1} + \delta_{\alpha_2} - \epsilon D_2 \alpha_1\alpha_2$,

$$\alpha_1 + \alpha_2 \leq \alpha' \leq \alpha_1 + \alpha_2 + \epsilon D \alpha_1\alpha_2, \quad |\delta'| \leq D_2 \alpha_1\alpha_2, \quad \pi' \leq D \alpha_1\alpha_2.$$

(4) $\delta + \alpha \to \alpha' + \delta' + \theta'$; *we have* $\delta_{\alpha'} \geq \delta_\alpha - \epsilon D \alpha|\delta|$,

$$\alpha' \leq \alpha + \epsilon D_1 \alpha|\delta|, \quad |\delta'| \leq |\delta| + \epsilon D \alpha|\delta|, \quad \theta' \leq \epsilon D_1 \alpha|\delta|.$$

(5) $\delta + \xi \to \xi' + \delta' + \theta'$; *we have*

$$\xi' \leq \xi + \epsilon D_1 \xi|\delta|, \quad |\delta'| \leq |\delta| + \epsilon D \xi|\delta|, \quad \theta' \leq \epsilon D_1 \xi|\delta|.$$

(6) $\xi + \alpha \to \alpha' + \delta' + \beta'$; *we have* $\delta_{\alpha'} \geq \delta_\alpha - D(\alpha - \alpha') - \epsilon D_2 \alpha\xi$, $\delta_{\beta'} \geq 0$,

$$\alpha' \leq \alpha - \xi, \quad |\delta'| \leq D(\alpha - \alpha') + \epsilon D_2 \alpha\xi, \quad \beta' \leq D_0(\alpha - \alpha') + \epsilon D \alpha'\xi.$$

(7) $\xi + \alpha \to \xi' + \delta' + \beta'$; *we have* $\delta_{\beta'} \geq \delta_\alpha - \epsilon D_2 \alpha\xi$,

$$\xi' \leq \xi, \quad |\delta'| \leq D\alpha, \quad \beta' \leq D_0\alpha..$$

(8) $\alpha + \xi \to \alpha' + \delta' + \beta'$; *we have* $\delta_{\alpha'} \geq \delta_\alpha - D(\alpha - \alpha') - \epsilon D_2 \alpha\xi$, $\delta_{\beta'} \geq 0$,

$$\alpha' \leq \alpha - \xi, \quad |\delta'| \leq D(\alpha - \alpha'), \quad \beta' \leq D_0(\alpha - \alpha').$$

(9) $\alpha + \xi \to \xi' + \delta' + \beta'$; *we have* $\delta_{\beta'} \geq \delta_\alpha - \epsilon D_2 \alpha\xi$,

$$\xi' \leq \xi, \quad |\delta'| \leq D\alpha, \quad \beta' \leq D_0\alpha.$$

(10) $\alpha + \xi \to \xi' + \delta' + \pi'$; *we have*

$$\xi' \leq \xi - \alpha, \quad |\delta'| = \delta_\alpha \leq D\alpha, \quad \pi' \leq \epsilon D\alpha\xi.$$

(11) $\pi + \xi \to \xi' + \pi'$; *then* $\xi' = \xi$ *and* $\pi' = \pi$.

Note that

$$\delta' = \begin{cases} \delta_{\alpha'} - \delta_\alpha - (\delta_{\beta'} - \delta_\beta) & \text{in case (1),} \\ \delta_{\alpha'} - \delta_{\alpha_1} - \delta_{\alpha_2} & \text{in case (3).} \end{cases} \tag{4.5}$$

Remark also that, because of Remark 4.1, the estimate for α' in Case (3) is the converse of that given in [15].

Let us denote by $\mathrm{P}_L = \mathrm{P}(w_L, z_L), \mathrm{P}_R = \mathrm{P}(w_R, z_R)$ etc. points of the wz-plane and by $|\mathrm{PQ}|$ the Euclidean distance between two points P and Q. The projections of the forward and backward shock curves on the wz-plane are denoted by $\mathcal{S}_j^F(w_0, z_0; \eta_0)$ and $\mathcal{S}_j^B(w_0, z_0; \eta_0)$, $j = 1, 2$, respectively.

In addition to Lemma 4.3 we also need the following result.

Lemma 4.4. *Consider Case* (3) *under the assumptions of Lemma 4.3. Then there is a constant* B_0*, which only depends on* p_**,* p^* *and* H*, such that* $0 < B_0 < 1$ *and*

$$\pi' \leq B_0 \min\{\alpha_1, \alpha_2\}. \tag{4.6}$$

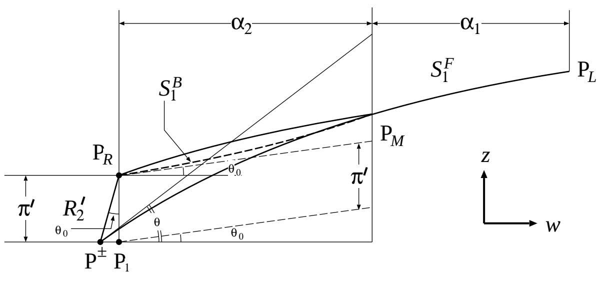

Figure 1: Proof of the inequality $\pi' \leq B_0\alpha_2$ in Lemma 4.4.

Proof. First, we prove the inequality $\pi' \leq B_0\alpha_2$. Consider the backward 1-shock curve and the backward 2-rarefaction curve issuing from P_R (see Figure 1). The angle θ_0 formed by the tangent to $\mathcal{S}_1^B(w_R, z_R; \eta_R)$ at P_R and the horizontal line equals the one formed by $\mathcal{R}_2'^B = \mathcal{R}_2^B(w_R, z_R; \eta_R)$ and the vertical line $\mathrm{P}_R\mathrm{P}_1$.

Here P_1 is intersection point $\{z = z_\pm\} \cap \{w = w_R\}$. The slope of $\mathcal{S}_1^F(w_L, z_L; \eta_L)$ at $\mathrm{P}_\pm$ is denoted by $\tan\theta$.

We have $|\mathrm{P}_\pm\mathrm{P}_1| = \pi' \tan\theta_0$ and hence

$$\pi' \;\le\; (\alpha_2 + |\mathrm{P}_\pm\mathrm{P}_1|)\tan\theta - \alpha_2\tan\theta_0 \;=\; \pi'\tan\theta\tan\theta_0 + \alpha_2(\tan\theta - \tan\theta_0).$$

Thus we have

$$\pi' \le \frac{\alpha_2(\tan\theta - \tan\theta_0)}{1 - \tan\theta\tan\theta_0}. \tag{4.7}$$

By Lemma 4.1 and (4.1) we deduce

$$0 < \tan\theta \le \frac{Be^{\frac{\epsilon}{\gamma}H} - 1}{Be^{\frac{\epsilon}{\gamma}H} + 1} < 1.$$

Moreover, for θ fixed, the function $\phi(\xi) = \frac{\tan\theta - \xi}{1 - (\tan\theta)\xi}$ is a decreasing function of ξ in $[0, \tan\theta]$. Then, by (4.7) we conclude that

$$\pi' \le \alpha_2\tan\theta \le \frac{Be^{\frac{\epsilon}{\gamma}H} - 1}{Be^{\frac{\epsilon}{\gamma}H} + 1}\,\alpha_2 =: B_0\alpha_2,$$

and we have proved the desired inequality.

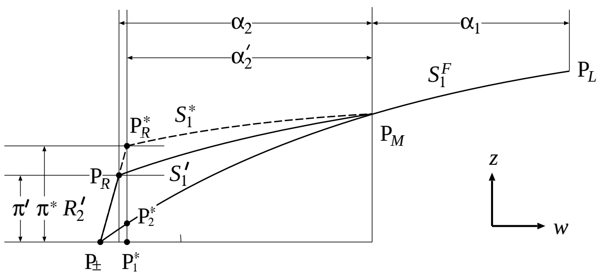

Figure 2: Proof of the inequality $\pi' \le B_0\alpha_1$ in Lemma 4.4.

Now, we prove the inequality $\pi' \le B_0\alpha_1$; we refer to Figure 2. We first notice that $\eta_M > \eta_L$; hence, by Lemma 4.2 we have

$$z_1(w; \eta_M) < z_1(w; \eta_L).$$

This shows that the shock curve $\mathcal{S}_1^* = \mathcal{S}_1^F(w_M, z_M; \eta_L)$ is located above the curve $\mathcal{S}_1' = \mathcal{S}_1^F(w_M, z_M; \eta_M)$. Hence we have

$$\begin{aligned}\pi^* \;&=\; |\mathrm{P}_R^*\mathrm{P}_2^*| + |\mathrm{P}_2^*\mathrm{P}_1^*| \;\le\; |\mathrm{P}_R^*\mathrm{P}_2^*| + \frac{Be^{\frac{\epsilon}{\gamma}H} - 1}{Be^{\frac{\epsilon}{\gamma}H} + 1}|\mathrm{P}_\pm\mathrm{P}_1^*| \\ &\le\; |\mathrm{P}_R^*\mathrm{P}_2^*| + \frac{Be^{\frac{\epsilon}{\gamma}H} - 1}{Be^{\frac{\epsilon}{\gamma}H} + 1} \cdot \frac{e^{\frac{\epsilon}{\gamma}H} - 1}{e^{\frac{\epsilon}{\gamma}H} + 1} \cdot \pi^*.\end{aligned}$$

Thus we obtain

$$\pi' < \pi^* \le \frac{(Be^{\frac{\epsilon}{\gamma}H} + 1)(e^{\frac{\epsilon}{\gamma}H} + 1)}{2(B+1)e^{\frac{\epsilon}{\gamma}H}} \cdot |\mathrm{P}_R^*\mathrm{P}_2^*|.$$

Since $\pi'' = |P_R^* P_2^*|$ is the strength of the rarefaction wave generated by the interaction

$$\alpha_1 + \alpha_2' \to \alpha'' + \pi'' \quad (\alpha'' = \alpha_1 + \alpha_2')$$

in the isentropic gas dynamics, we find some $0 < D_0' < 1$ so that $\pi'' \le D_0'\alpha_1$, [18]. With a slight abuse of notation set

$$B_0 := \frac{(Be^{\frac{\epsilon}{\gamma}H} + 1)(e^{\frac{\epsilon}{\gamma}H} + 1)}{2(B+1)e^{\frac{\epsilon}{\gamma}H}} D_0'. \tag{4.8}$$

Then $B_0 < 1$ if ϵH sufficiently small [1] and we obtain the desired estimate. □

In Lemma 4.4, the assumption $0 < \epsilon \le \frac{1}{3}$ and the smallness of ϵH were only used for the inequality $\pi' \le B_0\alpha_1$ and not for the inequality $\pi' \le B_0\alpha_2$.

Remark 4.2. *We refer to the cases listed in Lemma 4.3. In Cases (1)–(5) the waves outgoing from an interaction are estimated in a quadratic way by ϵ times the product of the incoming waves; namely, by $\epsilon D\alpha\beta$, $\epsilon D\alpha\pi$, $\epsilon D\alpha_1\alpha_2$, $\epsilon\alpha|\delta|$, $\epsilon D\xi|\delta|$, respectively. An exception is the case of δ' and π' in Case (3), where ϵ is missing. This case requires a special consideration below; in particular for π' we shall use Lemma 4.4. The interaction estimates in Cases (6)–(10) are of a different nature and are considered separately.*

Therefore the local interactions are subdivided into the following 4 groups:

A: (1), (2), (11);	*B:* (3);
C: (4), (5);	*D:* (6), (7), (8), (9), (10).

Since the estimates are similar within each group, we shall discuss only Cases (1), (3), (4), (6) as typical ones.

Now, we introduce a small number $M_0 > 0$ to be fixed later on at (5.28) and define the *strength of an entropy wave* δ as

$$M_0|\delta|. \tag{4.9}$$

This definition aims at controlling the outgoing waves, as we now show.

- Consider Case (3); by Lemma 4.4, we have

$$\begin{aligned} \pi' + M_0|\delta'| &\le B_0 \min\{\alpha_1, \alpha_2\} + M_0 D_2 \alpha_1\alpha_2 \\ &\le \{B_0 + M_0 D_2 \sup_\alpha \alpha\} \min\{\alpha_1, \alpha_2\}. \end{aligned}$$

  Then, the number M_0 is chosen so small that

$$B_0 + M_0 D_2 \sup_\alpha \alpha \le D_* < 1, \tag{4.10}$$

  where $D_* \in (D_0, 1)$ is an arbitrary number fixed once for all (we warn the reader that, for typographical reasons, from Section 7 on, we shall denote $D_* = c$, see (7.3)). The requirement that $D_* > D_0$ is going to be used in (4.11) and (4.15).

[1] By setting $B \to \infty$, a sufficient condition is $\epsilon H < \gamma \log\left(\frac{1}{D_0'} - 1\right)$.

- Consider Case (6) and assume $D_2 M_0 < D$, see (5.14). Since $\alpha' \le \alpha - \xi$, we find that

$$\begin{aligned} \beta' + M_0|\delta'| &\le (D_0 + DM_0)(\alpha - \alpha') + \epsilon(D\alpha' + D_2 M_0 \alpha)\xi \\ &\le \{D_0 + DM_0 + 2\epsilon D\alpha\}(\alpha - \alpha'). \end{aligned}$$

Let $\epsilon \sup \alpha$ and M_0 be so small that

$$D_0 + DM_0 + 2\epsilon D \sup_\alpha \alpha \le D_* < 1. \tag{4.11}$$

Then we have the following important estimate:

$$\beta' + M_0|\delta'| \le D_*(\alpha - \alpha'). \tag{4.12}$$

Moreover, unless of replacing D with $D + \epsilon D_2 \sup_\alpha \alpha$, we may assume $\delta_{\alpha'} \ge \delta_\alpha - D(\alpha - \alpha')$. We denote by C_H a strictly positive constant, which depends on p_*, p^* and H, such that $\delta_\alpha \le C_H \alpha$ and analogously for β. We also choose M_0 so small that

$$M_0 C_H \le \tfrac{1}{2}. \tag{4.13}$$

In Cases (7), (9) and (10) the quantity α' in (4.12) is missing; moreover, in Case (10) β' is replaced by π'. If we denote both β' and π' with θ', then we easily see that the estimate

$$\theta' + M_0|\delta'| \le D_*\alpha, \tag{4.14}$$

holds for all Cases (6)–(10), provided that (4.11) is replaced by

$$D_0 + DM_0 + 2\epsilon D \sup_{\alpha,\beta,\xi,\pi} \{\alpha, \beta, \xi, \pi\} \le D_* < 1. \tag{4.15}$$

5 Global Interaction Estimates

We consider an approximate solution $U^h = (p^h, u^h, \eta^h)$ defined for $t \in [0, T]$ as in Section 3; for simplicity we often drop the superscript h. Then, we denote $w(x,t) = w\big(p(x,t), u(x,t)\big)$, $z(x,t) = z\big(p(x,t), u(x,t)\big)$ and $\overline{w}(x) = w(x,0)$, $\overline{z}(x) = z(x,0)$.

Let P, Q be two arbitrary points in the wz-plane. By denoting $w_1 = z$, $w_2 = w$, we define a distance in that plane by

$$|\mathrm{P} - \mathrm{Q}| = \max_{j=1,2}\{|w_j(\mathrm{P}) - w_j(\mathrm{Q})|\}. \tag{5.1}$$

Since $\mathrm{TV}\big(\overline{p}(x), \overline{u}(x), \overline{\eta}(x)\big)$ is finite, there are limit states

$$U_{\pm\infty} = (p_{\pm\infty}, u_{\pm\infty}, \eta_{\pm\infty}) = \lim_{x \to \pm\infty} \big(\overline{p}(x), \overline{u}(x), \overline{\eta}(x)\big),$$

whose projections on the wz-plane are simply denoted by $\mathrm{P}_{\pm\infty} = \mathrm{P}(w_{\pm\infty}, z_{\pm\infty})$. Consider a sequence of states $U_{-\infty} = U_0, U_1, U_2, \ldots, U_n = U_\infty$, connected by j-waves ($1 \le j \le 2$) and denote by $\mathrm{P}_{-\infty} = \mathrm{P}_0, \mathrm{P}_1, \mathrm{P}_2, \ldots, \mathrm{P}_n = \mathrm{P}_\infty$, their projections in the wz-plane. We define

$$\begin{aligned} \mathcal{L}_j^+ &= \{l : \ w_j(\mathrm{P}_l) \ge w_j(\mathrm{P}_{l-1}), \ 1 \le l \le n\}, \\ \mathcal{L}_j^- &= \{l : \ w_j(\mathrm{P}_l) < w_j(\mathrm{P}_{l-1}), \ 1 \le l \le n\} \end{aligned}$$

and then

$$\mathcal{L}^- = \{l: \ U_{l-1} \text{ and } U_l \text{ are connected by a shock wave}, \ 1 \le l \le n\}.$$

Then we have (see Asakura and Corli [5, Theorem A.4])

$$\sum_{1\le l\le n} |\mathrm{P}_l - \mathrm{P}_{l-1}| \le 2|\mathrm{P}_\infty - \mathrm{P}_{-\infty}| + 3\sum_{l\in\mathcal{L}^-} |\mathrm{P}_l - \mathrm{P}_{l-1}|. \tag{5.2}$$

For any space-like curve J containing no interaction points (this condition is assumed in the following for any space-like curve, without any further mention), we define the global interaction functional F by

$$\begin{aligned} F(J) &= L_F(J) + \epsilon Q(J), && (5.3)\\ L_F(J) &= \sum_J \big\{(\alpha - M_0\delta_\alpha) + (\beta - M_0\delta_\beta) + M_0|\delta|\big\}, \\ Q(J) &= M_0M_1\sum_{J:\mathcal{A}}(\alpha+\beta+\xi+\pi)|\delta| && (5.4)\\ &\quad + M_1\sum_{J:\mathcal{A}}(\xi\alpha+\xi\beta+\pi\alpha+\pi\beta) + M_2\sum_{J:\mathcal{A}}(\alpha_1\alpha_2+\alpha\beta+\beta_1\beta_2). \end{aligned}$$

Here M_1 and M_2 are positive parameters to be fixed later on. Moreover, $\sum_J$ ($\sum_{J:\mathcal{A}}$) denotes the summation of all waves crossing J (and *approaching*, respectively, [7]). We also define

$$G(J) = 2|\mathrm{P}_\infty - \mathrm{P}_{-\infty}| + 8F(J). \tag{5.5}$$

Above, we defined the functionals in terms of space-like curves, instead of t, in view of further applications, see [12], [7, §7.5]. We denote

$$\begin{aligned} \mathrm{TV}(J) &= TV_D(J) + M_0 TV_H(J), \\ TV_D(J) &= \mathrm{TV}\big(w(x,t), z(x,t)\big)|_{(x,t)\in J}, && (5.6)\\ TV_H(J) &= \mathrm{TV}\eta(x,t)|_{(x,t)\in J}, && (5.7) \end{aligned}$$

where $TV_D(J)$ is computed with respect to the metric (5.1). We also define

$$M_3 = M_1 + M_2. \tag{5.8}$$

Proposition 5.1. *If the constants M_0, M_1, M_2 satisfy* (4.13) *and*

$$\epsilon M_3 G(J) \le 1, \tag{5.9}$$

then

$$\tfrac{1}{2}F(J) \le \mathrm{TV}(J) \le G(J). \tag{5.10}$$

Proof. By (4.13) we deduce

$$\alpha - M_0\delta_\alpha \ge \alpha/2, \quad \beta - M_0\delta_\beta \ge \beta/2. \tag{5.11}$$

Then, by (5.2),

$$\begin{aligned} \mathrm{TV}(J) &\le 2|\mathrm{P}_\infty - \mathrm{P}_{-\infty}| + (3 + C_H M_0)\sum_J(\alpha+\beta) + M_0\sum_J|\delta| \\ &\le 2|\mathrm{P}_\infty - \mathrm{P}_{-\infty}| + 8L_F(J), \end{aligned}$$

that proves the inequality on the right of (5.10); remark that condition (5.9) has not been used. By this inequality and (5.9) we have

$$\begin{aligned} F(J) &\leq TV_D(J) + M_0 TV_H(J) \\ &\quad + \epsilon M_0 M_1 TV_D(J) TV_H(J) + \epsilon M_3 TV_D(J)^2 \\ &\leq \mathrm{TV}(J) + \epsilon M_3 G(J) \mathrm{TV}(J) \ \leq\ 2\mathrm{TV}(J), \end{aligned}$$

that proves the inequality on the left of (5.10). □

Let O be a space-like curve such that there are no points of interaction between O and $t = 0$. The global interaction estimates consist in showing

$$F(J) \leq F(O), \tag{5.12}$$

for any arbitrary space-like curve J, provided that $\epsilon\mathrm{TV}(O)$ is sufficiently small and M_0, M_1, M_2 satisfy a set of constraints. We assume, analogously to (4.1),

$$G(O) \leq G^*, \tag{5.13}$$

for some fixed constant G^* (defined in the proof of Lemma (5.2)). About M_0, besides the conditions (4.11), (4.13) we further require that

$$M_0 D \leq D_1, \quad M_0 D_2 \leq D. \tag{5.14}$$

To prove (5.12) we argue as follows. Suppose that (5.12) holds for J and for every space-like curve lying between J and $t = 0$. Let J' be a space-like curve lying between J and $t = \infty$. If there are no interaction points between J and J', then obviously $F(J') = F(J) \leq F(O)$. Let us assume that there is a single interaction point P between J' and J.

If θ', θ'' are the two incoming waves at P, we define

$$Q(\mathrm{P}) = \begin{cases} D\theta'\theta'' & \text{if both } \theta' \text{ and } \theta'' \text{ are 1 or 2 waves,} \\ D_1|\theta'\theta''| & \text{if either } \theta' \text{ or } \theta'' \text{ is an entropy wave.} \end{cases} \tag{5.15}$$

We shall show a stronger estimate, which is needed later on (Lemma 5.3),

$$F(J') - F(J) \leq -3\epsilon Q(\mathrm{P}), \tag{5.16}$$

for any case of Lemma 4.3. Then, by an inductive argument, we have (5.12) for all J.

According to the cases we are dealing with, we impose some conditions on the parameters; Lemma 5.2 shall prove that a choice of the parameters can be done once for all. As we mentioned in Remark 4.2, we shall carry out the estimates only for the most complicated cases (1), (3), (4) and (6). Below, we use Lemma 4.3 several times.

Case (1): $\beta + \alpha \to \alpha' + \delta' + \beta'$. We have

$$\begin{aligned} & L_F(J') - L_F(J) \\ = \ & (\alpha' - \alpha) - M_0(\delta_{\alpha'} - \delta_\alpha) + (\beta' - \beta) - M_0(\delta_{\beta'} - \delta_\beta) + M_0|\delta'| \\ \leq \ & 2\epsilon D\alpha\beta + M_0\big\{|\delta'| - (\delta_{\alpha'} - \delta_\alpha) - (\delta_{\beta'} - \delta_\beta)\big\}. \end{aligned}$$

By (4.5), if $\delta' \geq 0$ we have $|\delta'| = \delta_{\alpha'} - \delta_\alpha - (\delta_{\beta'} - \delta_\beta)$; if $\delta' < 0$, we have $|\delta'| = \delta_{\beta'} - \delta_\beta - (\delta_{\alpha'} - \delta_\alpha)$. In both cases by (5.14) we obtain

$$L_F(J') - L_F(J) \leq 4\epsilon D\alpha\beta.$$

Moreover, by (5.10) and (5.14) we have

$$\begin{aligned}
& Q(J') - Q(J) \\
\leq\ & M_0 M_1 \Big(\sum (\alpha_J + \beta_J + \xi_J + \pi_J)|\delta'| + \sum \big([\alpha' - \alpha]_+ + [\beta' - \beta]_+ \big) |\delta_J| \Big) \\
& + M_1 \sum (\xi_J + \pi_J) \big([\alpha' - \alpha]_+ + [\beta' - \beta]_+ \big) \\
& + M_2 \sum \big(\alpha_J [\alpha' - \alpha]_+ + \beta_J [\beta' - \beta]_+ \big) - M_2 \alpha\beta \\
\leq\ & M_0 M_1 TV_D(J)\epsilon D_2 \alpha\beta + 2M_0 M_1 TV_H(J)\epsilon D\alpha\beta + 2M_1 TV_D(J)\epsilon D\alpha\beta \\
& + 2M_2 TV_D(J)\epsilon D\alpha\beta - M_2\alpha\beta \\
\leq\ & \big\{ 3\epsilon D M_3 G(J) - M_2 \big\} \alpha\beta. \qquad (5.17)
\end{aligned}$$

Here, $\sum(\alpha_J + \beta_J)$ denotes the total amount of shock waves crossing J and so on; we used the notation $[x]_+ = \max\{x, 0\}$. Hence, by (5.12) and assuming

$$7D + 3\epsilon D M_3 G(O) \leq M_2, \qquad (5.18)$$

we have $Q(J') - Q(J) \leq -7D\alpha\beta$. Then (5.16) follows.

Case (3): $\alpha_1 + \alpha_2 \to \alpha' + \delta' + \pi'$. We have

$$\begin{aligned}
L_F(J') - L_F(J) &= (\alpha' - \alpha_1 - \alpha_2) - M_0(\delta_{\alpha'} - \delta_{\alpha_1} - \delta_{\alpha_2}) + M_0|\delta'| \\
&\leq \epsilon D\alpha_1\alpha_2 + M_0 \big\{ |\delta'| - (\delta_{\alpha'} - \delta_{\alpha_1} - \delta_{\alpha_2}) \big\}.
\end{aligned}$$

By (4.5) and (5.14), if $\delta_{\alpha'} \geq \delta_{\alpha_1} + \delta_{\alpha_2}$ we have $|\delta'| = \delta_{\alpha'} - \delta_{\alpha_1} - \delta_{\alpha_2}$ and $L_F(J') - L_F(J) \leq \epsilon D\alpha_1\alpha_2$. If $\delta_{\alpha'} < \delta_{\alpha_1} + \delta_{\alpha_2}$, we have $|\delta'| = \delta_{\alpha_1} + \delta_{\alpha_2} - \delta_{\alpha'} \leq \epsilon D_2 \alpha_1\alpha_2$ and hence $L_F(J') - L_F(J) \leq 3\epsilon D\alpha_1\alpha_2$ by (5.14). In any case we have

$$L_F(J') - L_F(J) \leq 3\epsilon D\alpha_1\alpha_2.$$

We find by (5.14)

$$\begin{aligned}
& Q(J') - Q(J) \\
\leq\ & M_0 M_1 \Big(\sum (\alpha_J + \beta_J + \xi_J + \pi_J)|\delta'| + \sum \big([\alpha' - \alpha_1 - \alpha_2]_+ + \pi' \big) |\delta_J| \Big) \\
& + M_1 \sum (\alpha_J + \beta_J)\pi' + M_1 \sum (\xi_J + \pi_J)[\alpha' - \alpha_1 - \alpha_2]_+ \\
& + M_2 \sum (\alpha_J + \beta_J)[\alpha' - \alpha_1 - \alpha_2]_+ - M_2\alpha_1\alpha_2 \\
\leq\ & M_0 M_1 TV_D(J) D_2 \alpha_1\alpha_2 + (\epsilon + 1) M_0 M_1 TV_H(J) D\alpha_1\alpha_2 \\
& + M_1 TV_D(J) D\alpha_1\alpha_2 + M_1 TV_D(J)\epsilon D\alpha_1\alpha_2 \\
& + M_2 TV_D(J)\epsilon D\alpha_1\alpha_2 - M_2\alpha_1\alpha_2 \\
\leq\ & \big\{ 2DM_1 G(J) + \epsilon D M_3 G(J) - M_2 \big\} \alpha_1\alpha_2.
\end{aligned}$$

Hence, by the hypothesis (5.12) and assuming

$$6D + D\big\{ 2M_1 + \epsilon M_3 \big\} G(O) \leq M_2, \qquad (5.19)$$

we obtain $Q(J') - Q(J) \leq -6D\alpha_1\alpha_2$. As a consequence (5.16) follows.

Case (4): $\delta + \alpha \to \alpha' + \delta' + \theta'$. First, letting $\delta_{\pi'} = 0$,

$$\begin{aligned} L_F(J') - L_F(J) &= (\alpha' - \alpha) - M_0(\delta_{\alpha'} - \delta_\alpha) + \theta' - M_0\delta_{\theta'} + M_0(|\delta'| - |\delta|) \\ &\leq 3\epsilon D_1|\delta|\alpha + M_0 C_H \epsilon D|\delta|\alpha + M_0\epsilon D|\delta|\alpha \\ &\leq 5\epsilon D_1|\delta|\alpha, \end{aligned}$$

by (4.13). Second,

$$\begin{aligned} & Q(J') - Q(J) \\ \leq\ & M_0M_1\Big(\sum(\alpha_J + \beta_J + \xi_J + \pi_J)[|\delta'| - |\delta|]_+ + \sum\big([\alpha' - \alpha]_+ + \theta'\big)|\delta_J|\Big) \\ & -M_0M_1|\delta|\alpha + M_1\sum(\alpha_J + \beta_J + \xi_J + \pi_J)\theta' \\ & +M_2\sum(\alpha_J + \beta_J)\big([\alpha' - \alpha]_+ + \theta'\big) \\ \leq\ & M_0M_1TV_D(J)\epsilon D|\delta|\alpha + 2M_0M_1TV_H(J)\epsilon D_1|\delta|\alpha - \ M_0M_1|\delta|\alpha \\ & +M_1TV_D(J)\epsilon D_1|\delta|\alpha + 2M_2TV_D(J)\epsilon D_1|\delta|\alpha \\ \leq\ & \big\{2\epsilon D_1M_3G(J) - M_0M_1\big\}|\delta|\alpha. \end{aligned}$$

Hence, by using the hypothesis (5.12) and assuming

$$8D_1 + 2\epsilon D_1M_3G(O) \leq M_0M_1, \tag{5.20}$$

we obtain $Q(J') - Q(J) \leq -8D_1|\delta|\alpha$. Then (5.16) follows.

Case (6): $\xi + \alpha \to \alpha' + \delta' + \beta'$. We have

$$L_F(J') - L_F(J) \quad = \quad (\alpha' - \alpha) - M_0(\delta_{\alpha'} - \delta_\alpha) + \beta' - M_0\delta_{\beta'} + M_0|\delta'|.$$

Note that, by (5.11),

$$\begin{aligned} \alpha' - \alpha + \beta' &\leq -(1 - D_0)(\alpha - \alpha') + \epsilon D\alpha\xi \\ &\leq -\big\{1 - D_0 - 2\epsilon F(J)D\big\}(\alpha - \alpha'). \end{aligned}$$

We conclude that

$$L_F(J') - L_F(J) = -\big\{1 - D_0 - 2M_0D - 2D\epsilon F(J)\big\}(\alpha - \alpha') + 2\epsilon M_0D_2\alpha\xi.$$

Moreover,

$$\begin{aligned} & Q(J') - Q(J) \\ \leq\ & M_0M_1\Big(\sum(\alpha_J + \beta_J + \xi_J + \pi_J)|\delta'| + \sum\beta'|\delta_J|\Big) \\ & +M_1\sum(\xi_J + \pi_J)\beta' - M_1\alpha\xi + M_2\sum(\alpha_J + \beta_J)\beta' \\ \leq\ & M_0M_1TV_D(J)|\delta'| + M_0M_1\beta'TV_H(J) + M_1\beta'TV_D(J) \\ & -M_1\alpha\xi + M_2TV_D(J)\beta' \\ \leq\ & M_0M_1G(J)|\delta'| + M_3G(J)\beta' - M_1\alpha\xi \qquad (5.21) \\ \leq\ & (2M_1 + M_2)D_1G(J)(\alpha - \alpha') + \big\{\epsilon(2M_1 + M_2)DG(J) - M_1\big\}\alpha\xi. \end{aligned}$$

Remark that we may replace the constants D, D_1, D_2 in Lemma 4.3 by larger ones: the result still holds. Then, we assume $D_1 \geq D_0$; by (5.12), (5.14) we obtain

$$\begin{aligned} & F(J') - F(J) \\ \leq \;& -\big\{1 - D_0 - 2M_0 D - 2\epsilon D F(O) - \epsilon^2 (2M_1 + M_2) D_1 G(O)\big\}(\alpha - \alpha') \\ & + \epsilon\big\{\epsilon(2M_1 + M_2) D G(O) - (M_1 - 2M_0 D_2)\big\}\alpha\xi. \end{aligned}$$

By assuming

$$2\epsilon D F(O) + \epsilon^2 (2M_1 + M_2) D_1 G(O) \leq 1 - D_0 - 2M_0 D, \tag{5.22}$$

$$\epsilon(2M_1 + M_2) D G(O) \leq M_1 - 2M_0 D_2 - 3D, \tag{5.23}$$

we obtain (5.16) also in this case.

Lemma 5.1. *Consider any approximate solution U^h, define*

$$k = \tfrac{1}{2}\sqrt{\gamma}\, a^{\frac{1}{\gamma}} e^{\frac{\epsilon}{\gamma}\eta_*} p_{-\infty}^{\frac{\epsilon}{\gamma}} \tag{5.24}$$

and assume that for some constant $K > 0$ and any J we have

$$TV_D(J) \leq K, \qquad \epsilon TV_D(J) \leq k.$$

Then, there exist p_*, p^* and H, which depend on the initial data (3.1) *and on K, such that U^h is valued in the region* (4.1).

Proof. By (4.2) we have

$$\frac{\left(p^h\right)^{\frac{\epsilon}{\gamma}} - \overline{p}_*^{\frac{\epsilon}{\gamma}}}{\epsilon} = \frac{z^h - w^h}{2\sqrt{\gamma}\, a^{\frac{1}{\gamma}} e^{\frac{\epsilon}{\gamma}\eta_*}}.$$

By (5.10) we deduce that, for any t,

$$\mathrm{TV}\Big[\tfrac{\left(p^h\right)^{\frac{\epsilon}{\gamma}} - \overline{p}_*^{\frac{\epsilon}{\gamma}}}{\epsilon}\Big] \leq \frac{\mathrm{TV}_D(t)}{\sqrt{\gamma}\, a^{\frac{1}{\gamma}} e^{\frac{\epsilon}{\gamma}\eta_*}},$$

where the total variation on the left-hand side is the usual total variation with respect to the variable x and $\mathrm{TV}_D(t)$ is referred to the horizontal space-like curve at time t. Then,

$$\Big|\tfrac{\left(p^h\right)^{\frac{\epsilon}{\gamma}} - p_{-\infty}^{\frac{\epsilon}{\gamma}}}{\epsilon}\Big| = \Big|\tfrac{\left(p^h\right)^{\frac{\epsilon}{\gamma}} - \overline{p}_*^{\frac{\epsilon}{\gamma}}}{\epsilon} - \tfrac{p_{-\infty}^{\frac{\epsilon}{\gamma}} - \overline{p}_*^{\frac{\epsilon}{\gamma}}}{\epsilon}\Big| \leq \frac{\mathrm{TV}_D(t)}{\sqrt{\gamma}\, a^{\frac{1}{\gamma}} e^{\frac{\epsilon}{\gamma}\eta_*}}. \tag{5.25}$$

As a consequence,

$$\frac{\Big|\left(\frac{p^h}{p_{-\infty}}\right)^{\frac{\epsilon}{\gamma}} - 1\Big|}{\epsilon} \leq \frac{K}{2k}. \tag{5.26}$$

We first look for the upper bound p^* of p^h. If $p^h \leq p_{-\infty}$ we are done. Otherwise, let E^* be the set of points (t, x) where $p^h(t, x) > p_{-\infty}$ and only consider points in E^*. By (5.26) and introducing the increasing function $\phi(y) = (e^y - 1)/y$ we deduce

$$\gamma \log \tfrac{p^h}{p_{-\infty}} = \gamma \log \tfrac{p^h}{p_{-\infty}}\, \phi(0) \leq \gamma \log \tfrac{p^h}{p_{-\infty}}\, \phi\left(\tfrac{\epsilon}{\gamma} \log \tfrac{p^h}{p_{-\infty}}\right) \leq \frac{K}{2k},$$

whence

$$p^h \le p_{-\infty} e^{K/(2\gamma k)} \doteq p^*.$$

We now look for a lower bound p_* of p^h by arguing as above. If $p^h \ge p_{-\infty}$ we are done. Otherwise, let E_* be the set of points (t,x) where $p^h(t,x) < p_{-\infty}$ and only consider points in E_*; we denote $A = \left(p^h\right)^{\frac{\epsilon}{\gamma}} / p_{-\infty}^{\frac{\epsilon}{\gamma}} < 1$. By assumption we have $\epsilon \mathrm{TV}_D(t) < k$. Then (5.25) implies $A \ge \frac{1}{2}$. By the elementary identity $A - 1 = \log A \int_0^1 A^s\, ds$ we deduce

$$\left| \frac{\left(p^h\right)^{\frac{\epsilon}{\gamma}} - p_{-\infty}^{\frac{\epsilon}{\gamma}}}{\epsilon} \right| \ge \frac{1}{2\gamma \log 2}\, p_{-\infty}^{\frac{\epsilon}{\gamma}} \log \frac{p_{-\infty}}{p^h}$$

and then, by (5.25),

$$p^h \ge p_{-\infty}\, e^{-\frac{K}{k}\gamma \log 2} \doteq p_*.$$

At last, we are concerned with the definition of H. First, consider a Riemann problem with initial data (2.9), giving rise to waves α, δ and β. By the definitions of δ_α and δ_β it follows that $|\delta| \le |\eta_L - \eta_R| + |\delta_\alpha| + |\delta_\beta|$. If the solution of the Riemann problem contains rarefaction waves, the estimate is even simpler, since in that case the entropy does not change. As a consequence,

$$\begin{aligned} 0 \;\le\; \eta^h - \eta_* \;&\le\; \mathrm{TV}\eta^h(t) \;\le\; \mathrm{TV}\overline{\eta} + \sum \left(|\delta_\alpha| + |\delta_\beta|\right) \\ &\le\; \mathrm{TV}\overline{\eta} + C(p_*, p^*) TV_D(t) \le \mathrm{TV}\overline{\eta} + K C(p_*, p^*) \doteq H, \end{aligned}$$

by (2.5), for some constant $C(p_*, p^*)$. □

Lemma 5.2 (Global Interaction Estimates)**.** *Suppose that* $\mathrm{TV}(\overline{U}) \le K_0$ *for some positive constant* K_0 *and that* $\epsilon\mathrm{TV}(\overline{U})$ *is sufficiently small. Then, it follows that* (5.16) *holds for every pair of space-like curves* J *and* J' *as above and*

$$F(J') \le F(J) \le F(O). \tag{5.27}$$

Moreover, the approximate solution U^h *is contained in the region* (4.1) *for some* p_*, p^* *and* H.

Proof. We choose $K = 18K_0$ in Lemma 5.1; then we have p_*, p^* and H. The choice of p_*, p^* and H determines the constants D_0, D, D_1, D_2 in Lemma 4.3; in turn, the choice of D_0, D, D_1, D_2 fixes D_* and C_H. Moreover, we define $G^* = K_0$. At last, we define the constants M_0, M_1, M_2 by

$$0 < M_0 < \min\left\{ \frac{1}{2C_H}, \frac{D_* - D_0}{D}, \frac{D_1}{D}, \frac{D}{D_2}, \frac{1 - D_0}{2D} \right\}, \tag{5.28}$$

$$M_1 = \frac{9D_1}{M_0}, \qquad M_2 = 9D + 2M_1 D G^*. \tag{5.29}$$

In turn, M_0, M_1 and M_2 completely define the functionals F and G.

We justify formulas (5.28) and (5.29) by looking for constants M_0, M_1, M_2 satisfying (4.11) (or (4.15)), (4.13), (5.9), (5.14), (5.18), (5.19) (5.20), (5.22), (5.23), provided $\epsilon\mathrm{TV}(\overline{p}, \overline{u}, \overline{\eta})$ is sufficiently small. Indeed, we impose bounds on $\epsilon G(O)$, since $G(O) \le 18\mathrm{TV}(\overline{p}, \overline{u}, \overline{\eta})$ by (5.5) and (5.10).

We start by looking for necessary conditions about M_0, M_1, M_2. In order that conditions (4.11), (5.14), (5.22) hold we require that $D_0 < D_* < 1$ and choose M_0 satisfying (5.28).

Since $1 - D_0 < 1$ and we may assume $D \geq C_H$ by possibly enlarging D, then (4.13) is satisfied. About M_1, by considering the extreme case $\epsilon G(O) \to 0$, conditions (5.20), (5.23) require that $M_1 > \max\{8D_1/M_0, 2M_0D_2 + 3D\} = 8D_1/M_0$ by (5.28). About M_2 and conditions (5.18), (5.19), we need $M_2 > 7D + 2M_1DG^*$. Then, we choose M_1 and M_2 satisfying (5.29). Of course, other choices of these parameters are possible, leading to slightly different bounds on $\epsilon G(O)$; however, we are not aiming at optimal bounds.

We are then left to impose that $\epsilon G(O)$ is small in order that (4.11), (5.9), (5.18), (5.19) (5.20), (5.22), (5.23) are satisfied. Condition (5.20) reads

$$\epsilon M_3 G(O) \leq \tfrac{1}{2}.$$

This implies (5.9), (5.18) and (5.19). At last we are left with (4.11), (5.22), (5.23); we conclude by requiring

$$\begin{aligned} &\epsilon G(O) \\ &\leq \min\left\{2\left(\frac{D_* - D_0}{D} - M_0\right), \frac{1}{2M_3}, \frac{1 - D_0 - 2M_0D}{D + 2M_3D_1}, \frac{M_1 - 2M_0D_2 - 3D}{2M_3D}\right\}. \end{aligned}$$

At last, the inequalities $\mathrm{TV}_D(J) \leq K = 18K_0$ and $G(O) \leq G^* = K_0$ follow by Proposition 5.1. □

Lemma 5.3. *Under the same assumptions of Lemma 5.2, we have*

$$3\epsilon \sum_{\mathrm{P}} Q(\mathrm{P}) \leq F(O),$$

where the summation is done over all the interaction points.

Proof. Let J' and J be space-like curves as above. By (5.16) we deduce

$$3\epsilon Q(\mathrm{P}) \leq F(J) - F(J').$$

By summing up over all interaction points located between O and J' we have

$$3\epsilon \sum_{\mathrm{P}\ \text{between}\ O\ \text{and}\ J'} Q(\mathrm{P}) \leq F(O) - F(J') \leq F(O).$$

Since J' is an arbitrary space-like curve, we have proved the lemma. □

6 Decomposition by Paths

In this section we introduce of the notion of path in an approximate solution $U^h(x,t)$. Roughly speaking, a *path* is a sequence $\mathrm{P}(x_0,t_0)$, $\mathrm{P}(x_1,t_1)$, $\ldots$, $\mathrm{P}(x_n,t_n)$ of interaction points (apart from the case $t_0 = 0$) in the xt-plane, with $0 \leq t_0 < t_1 < \ldots < t_n$, which are connected by shock or entropy waves. By writing $\mathrm{P}_j = \mathrm{P}(x_j,t_j)$, a path is represented by

$$\Gamma:\ \mathrm{P}_0 \to \mathrm{P}_1 \to \cdots \to \mathrm{P}_n$$

and the segment $\mathrm{P}_{j-1}\mathrm{P}_j$ is called a *front.* If a shock or an entropy front starts at P_{n-1} and propagates without interacting with other waves, we continue the path as a half-line following that front and denote $\mathrm{P}_n = \mathrm{P}_\infty = \infty$.

The *index* (c_j, k_j) of the front $\mathrm{P}_{j-1}\mathrm{P}_j$ was defined by Asakura [3], Temple and Young [21]. We recall such definition below; the numbers c_j and k_j are called the *type* and the *generation order* of the front. First, a suitable positive integer, to be defined later on, is assigned to k_1. Then,

$$c_j = \begin{cases} 1 & \text{if } \mathrm{P}_{j-1}\mathrm{P}_j \text{ is a 1-shock wave,} \\ 2 & \text{if } \mathrm{P}_{j-1}\mathrm{P}_j \text{ is a 2-shock wave,} \\ 0 & \text{if } \mathrm{P}_{j-1}\mathrm{P}_j \text{ is an entropy wave,} \end{cases} \qquad \text{for } j \geq 1,$$

$$k_j = \begin{cases} k_{j-1} & \text{if } c_j = c_{j-1}, \\ k_{j-1}+1 & \text{if } c_j \neq c_{j-1}, \end{cases} \qquad \text{for } j \geq 2.$$

The sequence (c_1, k_1), (c_2, k_2), $\ldots$, (c_n, k_n) is called the *index* of the path. If $\mathrm{P}_0 \in \{t = 0\}$, then we set $k_1 = 1$ and the path is called a *primary path*; it is denoted by $\Gamma^P : \mathrm{P}_0 \to \mathrm{P}_1 \to \cdots \to \mathrm{P}_n$ and each $\mathrm{P}_{j-1}\mathrm{P}_j$ is a *primary front*.

Shock and entropy waves interact with other waves and generate new waves as described in Lemma 4.3. If an interaction $\theta' + \theta''$ occurs at some point P_0 and generates a shock or an entropy wave of amplitude $O(1)\epsilon\theta'\theta''$, the front of that wave is called a *secondary front* and a *secondary path* starts at P_0; it is denoted by $\Gamma^S : \mathrm{P}_0 \to \mathrm{P}_1 \to \cdots \to \mathrm{P}_n$. In this case, the initial generation order k_1 is at least 2. Secondary paths also arise when the strength of an outgoing wave is larger than that of the ingoing wave of the same family; in that case the secondary path accounts for the difference of the strengths. In any case, secondary paths are generated only in Cases (1)–(5). The outgoing entropy wave appearing in Case (3) always gives rise to a primary path, see Remark 4.2.

The construction of the paths along with the definition of their strengths is done iteratively as follows. Below, we denote waves $\alpha, \beta, \ldots$ with front $\mathrm{P}_{j-1}\mathrm{P}_j$ by $\alpha : \mathrm{P}_{j-1}\mathrm{P}_j$, $\beta : \mathrm{P}_{j-1}\mathrm{P}_j$ and so on. We assume for the moment that all waves involved in the interactions under consideration are *physical*, see Section 3; nonphysical waves are considered at the end of the construction.

First, we focus on the time interval ranging from $t = 0$ to the first interaction time t_1. Consider the case of a shock or entropy wave θ issuing from $\mathrm{P}_0 \in \{t = 0\}$ and interacting with another wave at P_1. The front $\mathrm{P}_0\mathrm{P}_1$ forms a primary path $\Gamma^P : \mathrm{P}_0 \to \mathrm{P}_1$. The strength of Γ^P is α if $\theta = \alpha$ (or β if $\theta = \beta$) and $M_0|\delta|$ if $\theta = \delta$; the generation order is 1.

Next, $t = t_1$. Suppose first that a *shock wave* $\alpha : \mathrm{P}_0\mathrm{P}_1$ interacts with another wave at P_1 and generates $\theta'_1 : \mathrm{P}_1\mathrm{P}_2$, $\theta'_2 : \mathrm{P}_1\mathrm{P}'_2$ and $\delta' : \mathrm{P}_1\mathrm{P}''_2$. We only consider Cases (1), (3), (4), (6) in Lemma 4.3. Below, we denote some generic constants which may change from line to line by

$$D', D'' \in [0, D], \quad D'_1 \in [0, D_1], \quad D''_2 \in [0, D_2], \quad D'_* \in [0, D_*]$$

(1): $\beta + \alpha \to \alpha' + \delta' + \beta'$. If $\alpha' \leq \alpha$, we decompose Γ^P into the paths Γ^P_1 and Γ^P_2, which have the same index of Γ^P but strengths α' and $\alpha - \alpha'$, respectively. The path Γ^P_1 is extended to P_2 with unchanged generation order and strength; the path Γ^P_2 stops at P_1. If $\alpha' > \alpha$, we have $\alpha' - \alpha = \epsilon D'\alpha\beta$; we extend Γ^P in the same direction with strength α and generate a secondary path $\Gamma^S_1 : \mathrm{P}_1 \to \mathrm{P}_2$ with index $(1, 2)$ and strength $\epsilon D'\alpha\beta$. The paths related to β are dealt analogously.
An entropy wave δ' with strength $M_0|\delta'| = \epsilon D''_2\alpha\beta$ is generated; we define a secondary path $\Gamma^S_0 : \mathrm{P}_1 \to \mathrm{P}''_2$ with index $(0, 2)$ and strength $\epsilon M_0 D''_2\alpha\beta$.

(3): $\alpha_1 + \alpha_2 \to \alpha' + \delta' + \pi'$. Assume for instance that $\alpha = \alpha_1$. We have $\alpha' - (\alpha_1 + \alpha_2) = \epsilon D' \alpha_1 \alpha_2$; then we extend Γ^P in the same direction with strength α_1 and generate a secondary path $\Gamma_1^S : \ \mathrm{P}_1 \to \mathrm{P}_2$ with index $(1,2)$ and strength $\epsilon D' \alpha_1 \alpha_2$.
An entropy wave δ' with strength $|\delta'| = D_2'' \alpha_1 \alpha_2$ is generated. We define a primary path $\Gamma_0^P : \ \mathrm{P}_1 \to \mathrm{P}_2''$ with index $(0,2)$ and strength $M_0 D_2'' \alpha_1 \alpha_2$.

(4): $\delta + \alpha \to \alpha' + \delta' + \theta'$. If $\alpha' \le \alpha$, we proceed exactly as in the first part of Case (1). If $\alpha' > \alpha$, we have $\alpha' - \alpha = \epsilon D_1' \alpha |\delta|$. Then we extend Γ^P in the same direction with strength α and generate a secondary path $\Gamma_1^S : \ \mathrm{P}_1 \to \mathrm{P}_2$ with index $(1,2)$ and strength $\epsilon D_1' \alpha |\delta|$.
If $\theta' = \beta'$, then $\beta' = \epsilon D_1' \alpha |\delta|$ and we have another secondary path $\Gamma_2^S : \ \mathrm{P}_1 \to \mathrm{P}_2'$ with index $(2,2)$ and strength $\epsilon D_1' \alpha |\delta|$. The entropy wave δ constitutes a primary path $\Gamma_0^P : \ \mathrm{P}_0' \to \mathrm{P}_1$. If $|\delta'| \le |\delta|$ we decompose Γ_0^P into the paths $\Gamma_{0,1}^P$ and $\Gamma_{0,2}^P$, which have the same index of Γ_0^P but strengths $M_0|\delta'|$ and $M_0(|\delta'| - |\delta|)$, respectively. The path $\Gamma_{0,1}^P$ is extended to P_2'' with unchanged generation order and strength, the path $\Gamma_{0,2}^P$ is stopped. If $|\delta'| > |\delta|$ and then $|\delta'| - |\delta| = \epsilon D'' \alpha |\delta|$, we have a secondary path $\Gamma_0^S : \ \mathrm{P}_1 \to \mathrm{P}_2''$ with index $(0,2)$ and strength $\epsilon M_0 D'' \alpha |\delta|$.

(6): $\xi + \alpha \to \alpha' + \delta' + \beta'$. By Lemma 4.3 and (4.12) we find $\zeta \ge \xi$ and $D_*' \le D_*$ such that

$$\alpha' = \alpha - \zeta, \quad \beta' + M_0|\delta'| = D_*' \zeta. \tag{6.1}$$

We define ζ_2, ζ_0 by $D_*' \zeta_2 = \beta'$, $D_*' \zeta_0 = M_0 |\delta'|$ so that

$$\zeta = \zeta_2 + \zeta_0, \quad \zeta_2 : \zeta_0 = \beta' : M_0 |\delta'|. \tag{6.2}$$

The path Γ^P is decomposed into three paths Γ_1^P, Γ_2^P and Γ_0^P, whose strengths are $\alpha - \zeta$, ζ_2 and ζ_0, respectively. The path Γ_1^P is extended to P_2 with index $(1,1)$ and strength $\alpha - \zeta$. The path Γ_2^P is extended to P_2' with index $(2,2)$ and strength $\beta' = D_*' \zeta_2$. Finally, Γ_0^P is extended to $\mathrm{P}_0 \to \mathrm{P}_1 \to \mathrm{P}_2''$, where $\mathrm{P}_1 \mathrm{P}_2''$ has index $(0,2)$ and strength $M_0 |\delta'| = D_*' \zeta_0$. No secondary path shows up in this case.

Remark 6.1. *In Cases (1) and (4) both possibilities $\alpha' \le \alpha$ and $\alpha' > \alpha$ may occur [4].*

Now, suppose that an *entropy wave* $\delta : \mathrm{P}_0 \mathrm{P}_1$ interacts with another wave at P_1; in this case, we already have a primary path $\Gamma_0^P : \ \mathrm{P}_0 \to \mathrm{P}_1$. The paths due to the interaction of δ with a 1-shock wave have been already defined in Case (4). In the case δ interacts with a 1-rarefaction wave, the definitions are analogous (see Lemma 4.3, Case (5)).

Then, $t = t_n$. We suppose that the paths have been constructed up to the interaction time $t = t_n$. First, we define the generation order of a shock and entropy wave γ. The front of γ belongs to a finite number N of paths, which can be ordered by their increasing generation order as follows:

$$\Gamma_1^P, \ \dots \ \Gamma_p^P, \ \Gamma_{p+1}^S, \ \dots \ \Gamma_N^S, \quad \text{for} \quad k_1^P \le \ldots \le k_p^P, \ k_{p+1}^S \le \ldots \le k_N^S. \tag{6.3}$$

Here k_m^P, k_m^S denote the generation orders of the front; we drop the indexes P and S in the following. The *generation order of* γ is defined by by

$$k_\gamma = \min_{1 \le l \le N} \{k_l\} . \tag{6.4}$$

We now accomplish the construction of the decomposition by paths. Consider a path $\Gamma : \ \mathrm{P}_0 \to \mathrm{P}_1 \to \cdots \to \mathrm{P}_n$ to $t = t_n$; we first assume that P_{n-1} and P_n are connected by a shock α. The front $\mathrm{P}_{n-1}\mathrm{P}_n$ belongs to some paths Γ_l, for $1 \le l \le N$; the path Γ_l has index $(1, k_l)$ and strength α_l. Of course, the path Γ is one of the Γ_l's. By (6.3), the strength α is decomposed into

$$\alpha = \sum_{l=1}^{N} \alpha_l .$$

Denote by $\alpha' : \mathrm{P}_n\mathrm{P}_{n+1}$, $\beta' : \mathrm{P}_n\mathrm{P}'_{n+1}$, $\delta' : \ \mathrm{P}_n\mathrm{P}''_{n+1}$ the waves produced by the interaction of α with another wave at P_n. Let k_* denote either $\max\{k_\alpha, k_\beta\}$ or $\max\{k_\alpha, k_\delta\}$. Below, we omit for brevity Cases (3) and (4), since they are analogous to Case (1); see [5] for complete details on the algorithm.

(1): $\beta + \alpha \to \alpha' + \delta' + \beta'$. If $\alpha' \le \alpha$, there exist $1 \le r \le N$ and $0 \le \overline{\alpha}_r < \alpha_r$ such that $\alpha - \alpha' = \overline{\alpha}_r + \sum_{l=r+1}^{N} \alpha_l$. We split Γ_r into the paths $\Gamma_r^{(1)}$ and $\Gamma_r^{(2)}$ so that the orders of the paths are unchanged while their strengths $\alpha_r^{(1)}$ and $\alpha_r^{(2)}$ are decomposed according to the proportion

$$\alpha_r^{(2)} : \alpha_r^{(1)} \quad = \quad \overline{\alpha}_r : (\alpha_r - \overline{\alpha}_r) \ . \tag{6.5}$$

Then, we extend Γ_l, $1 \le l \le r-1$, and $\Gamma_r^{(1)}$ to P_{n+1} with orders and strengths unchanged. The paths $\Gamma_r^{(2)}$ and Γ_l, $r+1 \le l \le N$, stop.

If $\alpha' > \alpha$, we have $\alpha' - \alpha = \epsilon D'\alpha\beta$; we extend each Γ_l in the same direction with the same strength and generate a secondary path $\Gamma_1^S : \ \mathrm{P}_n \to \mathrm{P}_{n+1}$ with index $(1, k_* + 1)$ and strength $\epsilon D'\alpha\beta$.
If $\beta' > \beta$, we have $\beta' - \beta = \epsilon D'\alpha\beta$; we generate a secondary path Γ_2^S : $\mathrm{P}_n \to \mathrm{P}'_{n+1}$ with index $(2, k_*+1)$ and strength $\epsilon D'\alpha\beta$. At last, an entropy wave of strength $|\delta'| = \epsilon D''_2\alpha\beta$ is generated and we have a secondary path $\Gamma_0^S : \ \mathrm{P}_n \to \mathrm{P}''_{n+1}$ with index $(0, k_* + 1)$ and strength $\epsilon M_0 D''_2 \alpha\beta$.

(6): $\xi + \alpha \to \alpha' + \delta' + \beta'$. No secondary path is generated in this case. As in the first step, there are positive quantities ζ and D'_* satisfying (6.1); we also fix ζ_2 and ζ_0 satisfying (6.2). Then, there are $1 \le N_1 \le N_2 \le N$ and $\overline{\alpha}_{N_1}$, $\overline{\alpha}_{N_2}$ with $0 \le \overline{\alpha}_{N_1} < \alpha_{N_1}$, $0 \le \overline{\alpha}_{N_2} < \alpha_{N_2}$ such that

$$\alpha' = \sum_{l=1}^{N_1} \alpha_l - \overline{\alpha}_{N_1}, \quad \zeta_2 = \overline{\alpha}_{N_1} + \sum_{l=N_1+1}^{N_2} \alpha_l - \overline{\alpha}_{N_2}, \quad \zeta_0 = \overline{\alpha}_{N_2} + \sum_{l=N_2+1}^{N} \alpha_l . \tag{6.6}$$

(a) We extend every Γ_l with $1 \le l \le N_1 - 1$ to P_{n+1} with indices and strengths unchanged. Primary paths remain primary and so secondary.

Then we split Γ_{N_1} into $\Gamma_{N_1}^{(1)}$ and $\Gamma_{N_1}^{(2)}$ so that

$$\alpha_{N_1}^{(1)} + \alpha_{N_1}^{(2)} = \alpha_{N_1}, \quad \alpha_{N_1}^{(1)} : \alpha_{N_1}^{(2)} = (\alpha_{N_1} - \overline{\alpha}_{N_1}) \ : \ \overline{\alpha}_{N_1} .$$

Therefore $\alpha_{N_1}^{(1)} = \alpha_{N_1} - \overline{\alpha}_{N_1}$ and $\alpha_{N_1}^{(2)} = \overline{\alpha}_{N_1}$. Let k_{N_1} be the order of Γ_{N_1} in $\mathrm{P}_{n-1}\mathrm{P}_n$. We extend $\Gamma_{N_1}^{(1)}$ to P_{n+1} with index $(2, k_{N_1})$ and strength $\alpha_{N_1}^{(1)}$. Then we extend $\Gamma_{N_1}^{(2)}$ to P'_{n+1} with index $(2, k_{N_1}+1)$ and strength $D'_*\alpha_{N_1}^{(2)} = D'_*\overline{\alpha}_{N_1}$.

(b) Similarly, Γ_{N_2} is split into $\Gamma_{N_2}^{(1)}$ and $\Gamma_{N_2}^{(2)}$ so that

$$\alpha_{N_2}^{(1)} + \alpha_{N_2}^{(2)} = \alpha_{N_2}, \quad \alpha_{N_2}^{(1)} : \alpha_{N_2}^{(2)} = (\alpha_{N_2} - \overline{\alpha}_{N_2}) : \overline{\alpha}_{N_2}.$$

We extend the paths Γ_l for $N_1+1 \le l \le N_2-1$ to P'_{n+1} with index $(2, k_l+1)$ and strength $D'_*\alpha_l$. We also extend $\Gamma_{N_2}^{(1)}$ to P_{n+1} with index $(2, k_{N_2}+1)$ and strength $D'_*\alpha_{N_2}^{(1)}$ as well as $\Gamma_{N_2}^{(2)}$ to P''_{n+1} with index $(0, k_{N_2}+1)$ and strength $D'_*\alpha_{N_2}^{(2)}$ (entropy wave).

(c) At last we extend the paths Γ_l for $l \ge N_2+1$ to P''_{n+1} with index $(0, k_l+1)$ and strength $D'_*\alpha_l$.

Now, we assume that P_{n-1} and P_n are connected by an entropy wave δ, which interacts with another wave at P_n and generates $\alpha' : \mathrm{P}_n\mathrm{P}_{n+1}$, $\beta' : \mathrm{P}_n\mathrm{P}'_{n+1}$ and $\delta' : \mathrm{P}_n\mathrm{P}''_{n+1}$. In this case, there is a path $\Gamma_0 : \mathrm{P}_{n-1} \to \mathrm{P}_n$. If δ interacts with a 1-shock wave, we refer to the Case (4) above. If $\delta : \mathrm{P}_{n-1}\mathrm{P}_n$ interacts with a 1-rarefaction wave we define the paths in a completely analogous way. Therefore, we have completed the construction of paths involving only *physical waves*.

At last, we discuss the case of *non-physical waves*. The generation order of a non-physical wave is defined as in [7]. About paths, if a shock wave α or an entropy wave δ belongs to a path Γ_l and the interaction generates a non-physical wave, then the path Γ_l is simply extended in the previous direction with the same index and strength.

A collection of paths $\mathbf{\Gamma} = \{\Gamma_l\}$, which are divided into primary paths $\mathbf{\Gamma}^{\boldsymbol{P}} = \{\Gamma_l^P\}$ and secondary paths $\mathbf{\Gamma}^{\boldsymbol{S}} = \{\Gamma_l^S\}$, is then defined up to the next interaction time t_{n+1} and hence as long as the approximate solution exists. This concludes the definition of the paths. Remark that the construction above implicitly define the initial generation order k_1 of a path, a quantity that was not previously fixed.

We now introduce analogous definitions for rarefaction waves, which however are not related to paths.

We assign a *generation order* to the approximate rarefaction waves as follows; our definition slightly differs from that given in [7]. Recall that in the approximate solutions a rarefaction wave of size θ is split into $N = [\theta/h] + 1$ fronts, each of them having strength $\theta/N < h$, [7]. We assign order 1 to any of these rarefaction fronts issuing from $t = 0$.

Suppose that a rarefaction front ξ with generation order k interacts with a wave θ with generation order k' and the waves θ'_1, δ', θ'_2 are produced. If $\theta'_1 = \xi'$, the generation order of ξ' is defined to be k. If $\theta'_2 = \pi'$ and $\theta \ne \pi$, its generation order is $\max\{k, k'\} + 1$. Lemma 4.3 states that the interaction of two shock waves of the same family, Case (3), and possibly that of a shock wave and an entropy wave, Case (4), also generates a rarefaction wave. In these cases, by denoting k, k' the generation orders of α_1, α_2 or δ, α, respectively, the generation order of π' is defined to be $\max\{k, k'\} + 1$.

When a rarefaction wave ξ of generation order k interacts with another wave and produces ξ', the amplitude of ξ' can exceed h. In this case, we have to divide ξ' into $\xi'_1 + \xi'_2$ so that $\xi_j \le h$, $j = 1, 2$. That is the reason why we do not construct paths for rarefaction waves. We will assign the same generation order k to both ξ'_j, $j = 1, 2$.

We call *secondary rarefaction waves* those rarefactions that are *generated* in Cases (4), (5) and (10). Secondary rarefaction waves were called *reflected* in [5] because of the special interaction patterns; we do not use this terminology here because of the reflected rarefaction waves of Case (3), which need a particular treatment.

By the above construction and definitions we deduce the following proposition, where we use the definition (5.15) of $Q(\mathrm{P})$.

Proposition 6.1. *The amount of secondary rarefaction waves or secondary paths which are generated at an interaction point* P *is estimated, for each family, by* $\epsilon Q(\mathrm{P})$.

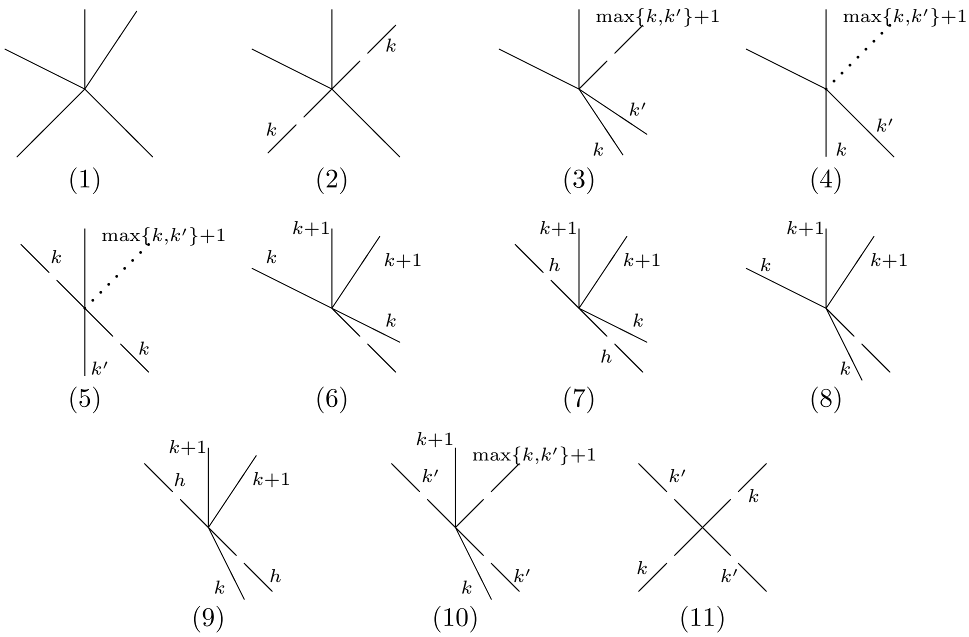

Figure 3: Generation orders for the cases in Lemma 4.3. Solid lines indicate shock or entropy waves; broken lines represent rarefaction waves; dotted lines denote either shock or rarefaction waves.

7 Estimates along paths

In this section we prove several results about the decay of waves along paths. As we saw in the previous section, paths only involve shock or entropy waves; we denote either of these waves by σ. By the estimates along paths we provide a bound on the total amount of rarefaction waves.

Let $\Gamma: \ \mathrm{P}_0 \to \mathrm{P}_1 \to \cdots \to \mathrm{P}_n$ be a path, with P_i occurring at time t_i. We denote by P_Γ ($t_{0,\Gamma}$) the point (respectively, the time) where the path Γ is generated. We denote by c_m, k_m and σ_m the type, generation order and strength of the front $\mathrm{P}_{m-1}\mathrm{P}_m$ in Γ, respectively; the sequences $c_\Gamma = \{c_m\}$, $k_\Gamma = \{k_m\}$ and $\sigma_\Gamma = \{\sigma_m\}$ are the type, generation order and strength of Γ. We consider the path Γ as a Lipschitz curve $x = \Gamma(t)$; then, the type, order and strength of Γ are piecewise constant functions of t, denoted by $c_\Gamma(t)$, $k_\Gamma(t)$, $\sigma_\Gamma(t)$, respectively. Here we set $\sigma_\Gamma(t_i) = \sigma_\Gamma(t_i-)$ for $i = 1, 2, \ldots, n$, $\sigma_\Gamma(t_{0,\Gamma}) = \sigma_\Gamma(t_1)$ and so on for c_Γ and k_Γ. We also denote with a slight abuse of notation, for $i = 1, \ldots, n$,

$$\sigma_\Gamma(\mathrm{P}_i) = \sigma_\Gamma(t_i-). \tag{7.1}$$

We denote by $\mathbf{\Gamma}$ the collection of all paths. Moreover, for every shock or entropy wave σ, we denote by $\mathbf{\Gamma}_\sigma$ the collection of paths which the front of σ belongs to. We now state a result about the decomposition of waves, whose proof is obvious by the construction of paths.

Lemma 7.1. *Consider any approximate solution, a time t where no interaction occurs, a shock or an entropy wave σ at time t. Then*

$$\sum_{\Gamma \in \mathbf{\Gamma}_\sigma} \sigma_\Gamma(t) = \begin{cases} \sigma & \text{if } \sigma \text{ is a shock,} \\ M_0|\sigma| & \text{if } \sigma \text{ is an entropy wave.} \end{cases} \ . \tag{7.2}$$

For typographical reasons we denote

$$c = D_*. \tag{7.3}$$

Lemma 7.2. *Consider any approximate solution and any path $\Gamma: \ \mathrm{P}_0 \to \mathrm{P}_1 \to \cdots \to \mathrm{P}_n$. Let k_m be the generation order and σ_m the strength of the front $\mathrm{P}_{m-1}\mathrm{P}_m$ in Γ. Then*

$$\begin{array}{lll} k_{m+1} = k_m & \Rightarrow & \sigma_{m+1} = \sigma_m, \\ k_{m+1} = k_m + 1 & \Rightarrow & \sigma_{m+1} \le c\,\sigma_m\,. \end{array}$$

Proof. The paths are constructed in such a way to satisfy the condition above. In particular, formula (6.1) shows that σ_{m+1} does not exceed $c\,\sigma_m$. □

We now introduce the *total amount of shock (and entropy) waves at time t*:

$$\begin{aligned} L^-(t) &= \sum\{\alpha(t) + \beta(t)\}, \\ L(t) &= \sum\{\alpha(t) + \beta(t) + M_0|\delta(t)|\}, \end{aligned} \tag{7.4}$$

where the sum is performed over all shock (and entropy, resp.) waves occurring in the approximate solution at time t. Lemma 7.2 clearly implies the following result.

Lemma 7.3 (Pathwise version of Nishida's lemma, [16])**.** *For any approximate solution and t different from interaction times, we have:*

1. $\sigma_\Gamma(t) \le c^{k_\Gamma(t) - k_\Gamma(t')}\sigma_\Gamma(t')$ *for any $\Gamma \in \mathbf{\Gamma}$ and $0 \le t' \le t$,*
2. $L(t) = \sum_{\Gamma \in \mathbf{\Gamma}} \sigma_\Gamma(t)$.

For any $\Gamma \in \mathbf{\Gamma}$ we denote by $t_{0,\Gamma}$, $P_{0,\Gamma}$ the time, resp. the point, at which Γ is generated. Then by the above lemma

$$\sigma_\Gamma(t) \le c^{k_\Gamma(t)-k_\Gamma(t_{0,\Gamma})}\sigma_\Gamma(t_{0,\Gamma})\,. \tag{7.5}$$

Let t be different from interaction times. For $k = 1, 2, \dots,$ we denote the total amount of the strengths at time t of *all primary paths* whose generation order at time t is k (resp., k and more) by

$$L_k^P(t) = \sum_{\Gamma\in\mathbf{\Gamma}^P,\, k_\Gamma(t)=k} \sigma_\Gamma(t)\,, \quad V_k^P(t) = \sum_{l\ge k} L_l^P(t)\,.$$

We also define for $k \ge 2$

$V_k^S(t)$**:** the total amount at time t of the strengths of both *secondary paths* and *secondary rarefaction waves* generated at times $t' \le t$, whose generation orders are larger than or equal to k.

We denote $F(0) = F(O)$ and observe that by Lemma 5.3 we have

$$V_k^S(t) \le F(0), \tag{7.6}$$

for every $k \ge 2$ and every t.

Proposition 7.1. *For every approximate solution and* $k \ge 1$, *we have*

$$L_k^P(t) \le c^{k-1}L(0), \tag{7.7}$$

$$V_k^P(t) \le \frac{c^{k-1}}{1-c}L(0). \tag{7.8}$$

Proof. If $\Gamma \in \mathbf{\Gamma}^P$ then $t_{0,\Gamma} = 0$. Since $k_\Gamma(0) = 1$ we deduce $\sigma_\Gamma(t) \le c^{k_\Gamma(t)-1}\sigma_\Gamma(0)$ by (7.5). Then for any $k \ge 1$,

$$\sum_{\substack{\Gamma\in\mathbf{\Gamma}^P\\ k_\Gamma(t)=k}} \sigma_\Gamma(t) \le c^{k-1}\sum_{\Gamma\in\mathbf{\Gamma}^P}\sigma_\Gamma(0) = c^{k-1}L(0).$$

Formula (7.8) follows by $\sum_{l\ge k} c^{l-1} = \frac{c^{k-1}}{1-c}$. □

Let us denote

$$\mathcal{F}(0) = L(0) + F(0).$$

Proposition 7.2. *For every approximate solution we have*

$$L^-(t) \le \mathcal{F}(0). \tag{7.9}$$

Proof. The amount of primary fronts for shock waves at $t = 0$ equals $L^-(0) \le L(0)$. If a secondary front Γ^S of a shock wave is generated at point P, then the strength of Γ^S in that segment is less than $\epsilon Q(\mathrm{P})$. As a consequence, the result follows by Lemmas 7.1 and 5.3. □

We now prove a lemma that is stronger than Lemma 5.3 because the estimate is independent of ϵ. This result will be used to prove Proposition 8.1, which shows that the approximate solution is defined for all $0 \le t < \infty$.

Lemma 7.4. *Under the same assumptions of Lemma 5.2, we have*

$$\sum_{\mathrm{P}} Q(\mathrm{P}) \le \frac{1}{6} Q(0) + \frac{cM_3}{6(1-c)} G^* \mathcal{F}(0), \tag{7.10}$$

where the summation is done over all the interaction points.

Proof. Let J' and J be space-like curves and assume that there is a single interaction point P between J' and J.

In Cases (1), (3) and (4), by virtue of (5.18), (5.19) and (5.20), we deduce

$$6Q(\mathrm{P}) \le Q(J) - Q(J'). \tag{7.11}$$

In Case (6), we proved in (5.21) that

$$M_1 \alpha \xi \le Q(J) - Q(J') + 2M_3 \left(\beta' + M_0|\delta'|\right) G(J),$$

where β', δ' are outgoing waves generated by the interaction at P and denoted from now on by $\beta'(\mathrm{P})$, $\delta'(\mathrm{P})$, respectively. By (5.28), (5.29) and Lemma 5.2, we obtain

$$9Q(\mathrm{P}) \le Q(J) - Q(J') + 2M_3 \left\{\beta'(\mathrm{P}) + M_0|\delta'(\mathrm{P})|\right\} G(O). \tag{7.12}$$

About Case (10), the outgoing 2-wave generated by the interaction is a rarefaction wave π'; in the following, we denote both a shock and a rarefaction wave by θ'. Summing up over all interaction points P between O and J', by (7.11) and (7.12) we deduce

$$\begin{aligned} &\sum_{\mathrm{P \ between}\ O\ \mathrm{and}\ J'} 6Q(\mathrm{P}) \\ \le\ & Q(O) - Q(J') + 2M_3 G(O) \sum_{\substack{\mathrm{P \ between}\ O\ \mathrm{and}\ J' \\ \mathrm{Cases\ (6)–(10)}}} \left\{\theta'(\mathrm{P}) + M_0|\delta'(\mathrm{P})|\right\}. \end{aligned}$$

Next, we claim that

$$\sum_{\substack{\mathrm{P \ between}\ O\ \mathrm{and}\ J' \\ \mathrm{Cases\ (6)–(10)}}} \left\{\theta'(\mathrm{P}) + M_0|\delta'(\mathrm{P})|\right\} \le \frac{c}{1-c} \mathcal{F}(0). \tag{7.13}$$

Indeed, in Cases (6)–(10) we have $\theta' + M_0|\delta'| \le c\alpha$ by (4.14) and (7.3). Then, by (7.5) we have

$$\begin{aligned} &\sum_{\substack{\mathrm{P \ between}\ O\ \mathrm{and}\ J' \\ \mathrm{Cases\ (6)–(10)}}} \left\{\theta'(\mathrm{P}) + M_0|\delta'(\mathrm{P})|\right\} \\ \le\ & c \sum_{\Gamma \in \mathbf{\Gamma}} \sum_{\mathrm{P} \in \Gamma} \sigma_\Gamma(\mathrm{P}) \\ \le\ & c \Big(\sum_{\Gamma \in \mathbf{\Gamma}^P} + \sum_{\Gamma \in \mathbf{\Gamma}^S} \Big) \Big(\sigma_\Gamma(\mathrm{P}_{0,\Gamma}) \sum_{\mathrm{P} \in \Gamma} c^{k_\Gamma(\mathrm{P}) - k_\Gamma(\mathrm{P}_{0,\Gamma})} \Big). \end{aligned} \tag{7.14}$$

In all summations above, we clearly understand that only interactions of type (6)–(10) are involved. Let m_Γ be the number of points $\mathrm{P} \in \Gamma$; we notice that m_Γ is finite by Proposition 8.1.

If Γ is a primary path, then $k_\Gamma(\mathrm{P}_{0,\Gamma}) = 1$, $t_{0,\Gamma} = 0$; by (7.5), (7.7) we deduce

$$\begin{aligned} \sum_{\mathrm{P}\in\Gamma} c^{k_\Gamma(\mathrm{P})-k_\Gamma(\mathrm{P}_{0,\Gamma})} &\leq \sum_{j=1}^{m_\Gamma} c^{j-1} \leq \frac{1}{1-c}, \\ \sum_{\Gamma\in\mathbf{\Gamma}^P} \sigma_\Gamma(t_{0,\Gamma}) &\leq L(0). \end{aligned} \tag{7.15}$$

If Γ is a secondary path, by Lemma 5.3 we have

$$\begin{aligned} \sum_{\mathrm{P}\in\Gamma} c^{k_\Gamma(\mathrm{P})-k_\Gamma(\mathrm{P}_{0,\Gamma})} &\leq \sum_{j=0}^{m_\Gamma} c^{j} \leq \frac{1}{1-c}, \\ \sum_{\Gamma\in\mathbf{\Gamma}^S} \sigma_\Gamma(t_{0,\Gamma}) &\leq 3\epsilon \sum_{\mathrm{P}} Q(\mathrm{P}) \leq F(O). \end{aligned}$$

The claim (7.13) follows. Since J' is an arbitrary space-like curve, we proved the lemma. □

By Lemma 4.3, the total amount of rarefaction waves generated by interactions of type (3) is bounded from above by

$$D \sum_{\mathrm{P}:\,\text{Case (3)}} \alpha_1(\mathrm{P})\alpha_2(\mathrm{P}) = \sum_{\mathrm{P}:\,\text{Case (3)}} Q(\mathrm{P}). \tag{7.16}$$

Condition (5.14) implies $M_0 D_2 \leq D$; then, also the total amount of entropy paths generated by an interaction of type (3) is estimated by the right-hand side of (7.16). Thus we have proved the following result.

Proposition 7.3. *The total amount of rarefaction waves and entropy paths generated by an interaction of type (3) is less than the right-hand side of* (7.10).

Remark 7.1. *The strength of a rarefaction wave may increase only in Cases (2) or (5). Otherwise, either it does not change (Case (11)) or decreases (Cases (7), (9), (10)).*

New rarefaction waves are only generated in Cases (3), (4), (5) and (10). We recall that rarefactions generated in Cases (4), (5) and (10) are called secondary rarefactions. We observe that in Case (3) the generated rarefaction wave is only estimated by $Q(\mathrm{P})$ *and not by* $\epsilon Q(\mathrm{P})$.

We first analyze Case (3); the next proposition provides a result that is slightly stronger than that of Proposition 7.3 and Lemma 6.1 in [3]. Notice that we estimate as well the amount of paths generated in the interaction and associated to entropy waves.

Proposition 7.4. *Let* $k \geq 2$ *and* t *not an interaction time. The total amount of rarefaction waves of order larger than or equal to* k*, which are generated by an interaction of type (3) in the time interval* $[0, t]$*, is less than*

$$\frac{D\mathcal{F}(0)}{2(1-c)^2} \left\{ \frac{c^{k-2}}{1-c} L(0) + V^S_{k-1}(t) \right\}. \tag{7.17}$$

The total amount of entropy paths whose generation order is larger than or equal to k *and are generated by an interaction of type (3), is less than* (7.17)*, with* D_2 *replacing* D.

Proof. We denote by $\{\mathrm{P}_m\}$ the collection of all interaction points in the time interval $[0,t]$ where

$$\text{Case (3) occurs} \quad \text{and} \quad \max\{k_{\alpha_1(\mathrm{P}_m)}, k_{\alpha_2(\mathrm{P}_m)}\} \geq k-1. \tag{7.18}$$

Here, $\alpha_j(\mathrm{P}_m)$, for $j=1,2$, are the strengths of the shock waves incoming at P_m, analogously to the notation above. The rarefaction waves (and the entropy paths) of the statement are precisely those generated at such points P_m. By Lemma 4.3 and the definition (6.4), the total amount of these rarefaction waves (resp., entropy paths) is bounded from above by

$$D\sum_{\mathrm{P}_m} \alpha_1(\mathrm{P}_m)\alpha_2(\mathrm{P}_m). \tag{7.19}$$

In the case of entropy paths, simply replace D above with D_2. Then, we must prove that (7.19) is less than (7.17). The same argument also covers the case of the entropy paths.

Let $\mathbf{\Gamma}_{\alpha_j}(\mathrm{P}_m)$ denote the collection of the paths composing $\alpha_j(\mathrm{P}_m)$, for $j=1,2$; the strength of a path $\Gamma_j \in \mathbf{\Gamma}_{\alpha_j}(\mathrm{P}_m)$ is denoted by $\sigma_{\Gamma_j}(\mathrm{P}_m)$. By Lemma 7.1 we have

$$\begin{aligned}\sum_{\mathrm{P}_m} \alpha_1(\mathrm{P}_m)\alpha_2(\mathrm{P}_m) &\leq \sum_{\mathrm{P}_m} \sum_{\Gamma_1\in\mathbf{\Gamma}_{\alpha_1}(\mathrm{P}_m)} \sum_{\Gamma_2\in\mathbf{\Gamma}_{\alpha_2}(\mathrm{P}_m)} \sigma_{\Gamma_1}(\mathrm{P}_m)\sigma_{\Gamma_2}(\mathrm{P}_m)\\ &\leq \frac{1}{2}\sum_{\Gamma\in\mathbf{\Gamma}} \sum_{\mathrm{P}_m\in\Gamma} \sigma_\Gamma(\mathrm{P}_m) \sum_{\Gamma^*\in\mathbf{\Gamma}^*(\Gamma,\mathrm{P}_m)} \sigma_{\Gamma^*}(\mathrm{P}_m),\end{aligned} \tag{7.20}$$

where $\mathbf{\Gamma}^*(\Gamma,\mathrm{P}_m)$ is the set of the paths Γ^* interacting with the path Γ at P_m. Fix a path Γ; since clearly $\bigcup_{\mathrm{P}_m\in\Gamma}\mathbf{\Gamma}^*(\Gamma,\mathrm{P}_m) \subset \mathbf{\Gamma}\setminus\{\Gamma\}$, by reversing the order of summations we find that

$$\sum_{\mathrm{P}_m\in\Gamma} \sigma_\Gamma(\mathrm{P}_m) \sum_{\Gamma^*\in\mathbf{\Gamma}^*(\Gamma,\mathrm{P}_m)} \sigma_{\Gamma^*}(\mathrm{P}_m) \leq \sum_{\Gamma^*\in\mathbf{\Gamma}\setminus\{\Gamma\}} \sum_{\mathrm{P}_m\in\Gamma\cap\Gamma^*} \sigma_\Gamma(\mathrm{P}_m)\sigma_{\Gamma^*}(\mathrm{P}_m). \tag{7.21}$$

Now, we consider any two points $\mathrm{P}_m, \mathrm{P}'_m \in \Gamma\cap\Gamma^*$, with $\mathrm{P}_m \neq \mathrm{P}'_m$, and suppose that P'_m is consecutive to P_m. By this we mean there is no point of the collection $\{\mathrm{P}_m\}$ belonging to $\Gamma\cap\Gamma^*$ and lying between P_m and P'_m; moreover, P'_m lies after P_m. It is impossible that both $k_\Gamma(\mathrm{P}_m) = k_\Gamma(\mathrm{P}'_m)$ and $k_{\Gamma^*}(\mathrm{P}_m) = k_{\Gamma^*}(\mathrm{P}'_m)$ occur. Then, if we move from P_m to P'_m along the path Γ or Γ^*, either the generation order of Γ or Γ^* increases by at least two or those of both Γ and Γ^* increase by at least one. Hence, by Lemma 7.2 we have

$$\begin{aligned}\sum_{\mathrm{P}_m\in\Gamma\cap\Gamma^*} \sigma_\Gamma(\mathrm{P}_m)\sigma_{\Gamma^*}(\mathrm{P}_m) &\leq \sum_{j\geq 1} c^{2(j-1)}\sigma_\Gamma(\mathrm{P}_0)\sigma_{\Gamma^*}(\mathrm{P}_0)\\ &\leq \frac{1}{1-c^2}\sigma_\Gamma(\mathrm{P}_0)\sigma_{\Gamma^*}(\mathrm{P}_0),\end{aligned} \tag{7.22}$$

where, with a slight abuse of notation, we denoted by P_0 the first point of the P_m's in $\Gamma\cap\Gamma^*$. By (7.18), we have either $k_\Gamma(\mathrm{P}_0) \geq k-1$ or $k_{\Gamma^*}(\mathrm{P}_0) \geq k-1$ (of course both possibilities may occur). In order to simplify the notation in the proof, we assume that the first possibility always occurs, namely that $k_\Gamma(\mathrm{P}_0) \geq$

$k-1$; otherwise, simply replace Γ with Γ^*. Under the notation above we find the estimate

$$\sum_{\Gamma,\Gamma^*\in\mathbf{\Gamma}} \sigma_\Gamma(\mathrm{P}_0)\sigma_{\Gamma^*}(\mathrm{P}_0) \quad \leq \quad \frac{\mathcal{F}(0)}{1-c}\left\{\frac{c^{k-2}}{1-c}L(0)+\mathcal{F}(0)\right\}.$$

Above, we split the sums into primary and secondary paths associated to shock waves; the total amount of the former is bounded by $L^-(0)$, that of the latter by $F(0)$, because of (7.6). Together, they are bounded by $\mathcal{F}(0)$. The second factor is deduced by (7.8) and the definition of $V^S_{k-1}(t)$.

This proves that (7.19) is less than (7.17) and concludes the proof. □

Remark 7.2. *We notice that* (7.17) *can be bounded as follows, by using* (7.6)*:*

$$\frac{D\mathcal{F}(0)}{2(1-c)^2}\left\{\frac{c^{k-2}}{1-c}L(0)+V^S_{k-1}(t)\right\} \leq \frac{D\mathcal{F}(0)^2}{2(1-c)^3}. \tag{7.23}$$

A byproduct of Proposition 7.4 concerns the total amount of rarefaction waves produced by an interaction of type (3): it suffices to take $k=2$ *in* (7.17) *or simply the right-hand side of* (7.23)*. These bounds must be considered together with that already provided in Proposition 7.3.*

Cases (2) and (5) are considered below. We denote by $\xi(t)$ the continuation of the rarefaction ξ for times t beyond an interaction time t_1. Therefore, for instance, in Case (5) we have that $\xi(t)=\xi'$ just after the interaction and so on.

Proposition 7.5. *Suppose that a* 1*-rarefaction wave* ξ *interacts at* $t=t_1$ *with either a* 2*-shock or an entropy wave* σ*. Let* Γ *be a path to which the front of* σ *belongs. Then, no front of* 2*-shock or entropy wave interacting with* $\xi(t)$ *at* $t>t_1$ *belongs to* Γ*.*

Proof. Let $t_2>t_1$ be the first time at which $\xi(t)$ interacts with a wave σ' whose front belongs to Γ; the wave σ' necessarily is a 1-shock wave.

If the 1-wave that outcomes from the interaction at t_2 is a shock, then $\xi(t)$ no longer exists for $t>t_2$. If it is a rarefaction, then the path Γ is continued along the 2-shock front or entropy front; hence, the path does not cross $\xi(t)$. □

We denote

$$\bar{D}=\frac{D_1}{M_0}\left(1+\frac{DG^*}{8(1-c^3)}\right).$$

Proposition 7.6. *Consider a rarefaction wave* ξ *at* $t=t_1$*. For* $t>t_1$ *we have*

$$\xi(t)\leq e^{\epsilon\bar{D}\mathcal{F}(0)}\xi. \tag{7.24}$$

Proof. For either a shock or an entropy wave σ we introduce the notation

$$D_\sigma=\begin{cases} D & \text{if } \sigma=\alpha,\beta,\\ D_1 & \text{if } \sigma=\delta.\end{cases}$$

Let the 1-rarefaction ξ' be the continuation of ξ after the interaction of ξ with either a 2-shock wave (Case (2)) or an entropy wave (Case (5)) σ, see Remark 7.1. By Lemma 4.3 we find that

$$\xi'\leq(1+\epsilon D_\sigma|\sigma|)\xi.$$

Hence, recalling what we pointed out before Proposition 7.5, we have

$$\xi(t) \ \le \ \prod_{\sigma=\beta,\delta} \bigl(1+\epsilon D_\sigma|\sigma|\bigr)\xi \ \le \ e^{\epsilon\sum_{\sigma=\beta,\delta} D_\sigma|\sigma|}\xi, \tag{7.25}$$

where the product and the sum are performed over all waves β or δ which interact with $\xi(t')$ at times $t' < t$ in an interaction of type (2) or (5), respectively. We observe that every shock or entropy wave in that sum occurs there only once, because of Proposition 7.5.

We now consider the sum in (7.25). First, we observe that $D \le \frac{D_1}{M_0}$, because of (5.28). We decompose the shock waves into primary and secondary paths; we do the same for entropy waves but, in this case, we also need to take into account the rarefaction waves generated in Case (3), see Proposition 7.4. We obtain, by (7.23),

$$\sum_{\sigma=\beta,\delta} D_\sigma|\sigma| \le \frac{D_1}{M_0}\left\{\mathcal{F}(0)+\frac{D\mathcal{F}(0)^2}{2(1-c)^3}\right\} \le \frac{D_1}{M_0}\mathcal{F}(0)\left\{1+\frac{DG^*}{8(1-c^3)}\right\}$$

and then (7.24) follows by (7.25). □

We denote by $\mathcal{R}_k(t)$ ($\mathcal{W}_k(t)$) the total amount of the strengths of all rarefaction waves (of all waves, respectively) having generation order larger than or equal to k at time t.

Proposition 7.7. *If $k \ge 2$ we have*

$$\mathcal{R}_k(t) \ \le \ \frac{D\mathcal{F}(0)e^{\epsilon\bar{D}\mathcal{F}(0)}}{2(1-c)(1-c^2)}\left\{\frac{c^{k-2}}{1-c}L(0)+V^S_{k-1}(t)\right\}+V^S_k(t), \tag{7.26}$$

$$\begin{aligned} \mathcal{W}_k(t) \ \le \ & \frac{c^{k-1}}{1-c}L(0)+V^S_k(t) \\ & + \frac{D\mathcal{F}(0)e^{\epsilon\bar{D}\mathcal{F}(0)}}{2(1-c)(1-c^2)}\left\{\frac{c^{k-2}}{1-c}L(0)+V^S_{k-1}(t)\right\}. \end{aligned} \tag{7.27}$$

Proof. We recall that the generation order of a rarefaction wave does not increase in time after an interaction; as a consequence, the rarefaction waves generated at time 0+ keep their generation order 1. Then (7.26) follows by Propositions 7.4, 7.6 and the definition of $V^S_k(t)$.

In order to prove (7.27) we also need to take into account shock and entropy waves. Then, (7.27) follows by (7.6) and (7.26). □

8 Stability of Wave-Front Tracking Scheme

First, following Bressan [7], we prove that the approximate solution constructed according to the algorithm in Section 3 is defined for all $0 \le t < \infty$.

Proposition 8.1. *Let $\mathcal{I}$ be the set of interaction points of an approximate solution. Then, the accurate Riemann solver is used at most a finite number of times in $\mathcal{I}$. As a consequence, the approximate solution is defined for all $0 \le t < \infty$.*

Proof. The accurate Riemann solver is used when the strengths of the interacting waves satisfy $|\theta'\theta''| \geq \rho$. Lemma 7.4 shows that the number of the interaction points where this occurs is less than C/ρ, for a suitable C independent of ϵ. Then, the number of physical fronts is finite. We emphasize that also Lemma 7.4 gives a bound on the number of interaction points, which however depends on ϵ.

Non-physical fronts are possibly generated when two physical fronts interact; moreover, any two physical fronts interact only once. Thus, also the number of non-physical fronts is finite. □

Now, we estimate the total amount of secondary paths and secondary rarefaction waves in terms of the quadratic functionals. Let us denote for simplicity

$$\sum_{\geq k;\mathcal{A}} \gamma\theta \quad = \quad \sum_{\substack{\max\{k_\gamma,k_\theta\}\geq k \\ \gamma,\theta \text{ approaching}}} \gamma\theta,$$

for $k \geq 1$. We define, analogously to (5.4),

$$\begin{aligned} Q_k(t) \quad = \quad & M_0M_1 \sum_{\geq k;\mathcal{A}} (\alpha+\beta+\xi+\pi)|\delta| \\ & + M_1 \sum_{\geq k;\mathcal{A}} (\xi\alpha+\xi\beta+\pi\alpha+\pi\beta) + \; M_2 \sum_{\geq k;\mathcal{A}} (\alpha_1\alpha_2+\alpha\beta+\beta_1\beta_2), \end{aligned}$$

where the summation is over the waves at time t. Then, we denote $Q(t) = Q_1(t)$, similarly to (5.4). In a complete analogous way we define $TV_D(t)$ and $TV_H(t)$ as in (5.6), (5.7). We also define

$$\mathrm{TV}(t) = TV_D(t) + M_0 TV_H(t), \quad \mathrm{Q}_k^{\pm}(t) = \sum_{0<\tau<t} [\Delta Q_k(\tau)]_{\pm},$$

where $\Delta Q_k(\tau) = Q_k(\tau+) - Q_k(\tau-)$ and τ runs over all interaction times; we denoted by $[x]_\pm$ the positive and negative parts of x. At last, we denote

$$\widetilde{\mathrm{TV}} = \sup_{t>0} \mathrm{TV}(t-), \quad \widetilde{\mathrm{V}}_k^S = \sup_{t>0} V_k^S(t), \quad \widetilde{\mathcal{W}}_k = \sup_{t>0} \mathcal{W}_k(t), \tag{8.1}$$

$$\widetilde{\mathrm{Q}}_k^{\pm} = \textstyle\sum_{\tau>0} [\Delta Q_k(\tau)]_{\pm}. \tag{8.2}$$

We notice that

$$\mathrm{Q}_k^+(t) - \mathrm{Q}_k^-(t) = \sum_{0<\tau\leq t} \{[\Delta Q_k(\tau)]_+ - [\Delta Q_k(\tau)]_-\} = Q_k(t) \geq 0 \tag{8.3}$$

for $k \geq 2$ and then $\mathrm{Q}_k^-(t) \leq \mathrm{Q}_k^+(t)$. The following lemma is a refinement of Lemmas 5.2 and 5.3.

Lemma 8.1. *Assume that a secondary rarefaction wave or a secondary path of strength θ and order $l \geq 2$ is generated at time τ. Then*

$$\theta \leq \tfrac{1}{3}\epsilon[\Delta Q_{l-1}(\tau)]_- \, . \tag{8.4}$$

Proof. Consider for instance Case (1) and suppose that a secondary path of order l is generated; then either α or β has generation order $l-1$. Proceeding as in (5.17) but replacing $\sum$ with $\sum_{\geq l-1;\mathcal{A}}$ we find, together with (5.18), that $\Delta Q_{l-1}(\tau) \leq -7D\alpha\beta$. It follows from Proposition 6.1 that

$$\theta \leq \epsilon D\alpha\beta \leq -\tfrac{1}{7}\epsilon\Delta Q_{l-1}(\tau).$$

The proof in Cases (2)–(5) and (10) is completely analogous. □

Proposition 8.2. *For $k \geq 2$ we have*

$$V_k^S(t) \leq e^{\epsilon\bar{D}\mathcal{F}(0)}\frac{\epsilon}{3}\sum_{h=0}^{k-2} c^h \mathrm{Q}_{k-h-1}^-(t). \tag{8.5}$$

Proof. First, suppose that a secondary path Γ^S of order l is generated at time $\tau < t$. We claim that the contribution of Γ^S to $V_k^S(t)$ is less than

$$\begin{cases} \frac{1}{3}\epsilon c^h[\Delta Q_{k-h-1}(\tau)]_- & \text{if } l = k-h \leq k\,, \\ \frac{1}{3}\epsilon[\Delta Q_{k-1}(\tau)]_- & \text{if } l > k\,. \end{cases} \tag{8.6}$$

If $l \leq k$, then $l = k - h$ for some $0 \leq h \leq k-2$. The path Γ^S contributes to $V_k^S(t)$ only if it is continued in the time interval (τ, t) through at least h interactions. By (7.5) and (8.4) we obtain the first estimate in (8.6).

If $l > k$, the amount of Γ^S contributing to $V_k^S(t)$ is less than $\frac{\epsilon}{3}[\Delta Q_{l-1}(\tau)]_- \leq \frac{\epsilon}{3}[\Delta Q_{k-1}(\tau)]_-$ again by (8.4). This proves (8.6).

Then, the contribution of the secondary paths to $V_k^S(t)$ is bounded by

$$\frac{\epsilon}{3}\sum_{h=0}^{k-2}\sum_{\substack{0<\tau<t:\\ \Gamma^S \text{ is generated}}} c^h[\Delta Q_{k-h-1}(\tau)]_- \leq \frac{\epsilon}{3}\sum_{h=0}^{k-2} c^h\mathrm{Q}_{k-h-1}^-(t). \tag{8.7}$$

Next, suppose that at time $\tau < t$ a secondary rarefaction wave θ of order l is generated; it contributes to $V_k^S(t)$ if and only if $l \geq k$. In this case we have, by (8.4),

$$\theta \leq \tfrac{1}{3}\epsilon[\Delta Q_{l-1}(\tau)]_- \leq \tfrac{1}{3}\epsilon[\Delta Q_{k-1}(\tau)]_-\,. \tag{8.8}$$

By (7.24) and (8.8), the contribution of the secondary rarefaction waves (2nd RW) to $V_k^S(t)$ is bounded by

$$e^{\epsilon\bar{D}\mathcal{F}(0)}\frac{\epsilon}{3}\sum_{\substack{0<\tau<t:\\ \text{2nd RW is generated}}}[\Delta Q_{k-1}(\tau)]_- \leq e^{\epsilon\bar{D}\mathcal{F}(0)}\frac{\epsilon}{3}\mathrm{Q}_{k-1}^-(t). \tag{8.9}$$

Above, the sum is performed over all times $0 < \tau < t$ where a secondary rarefaction wave is generated. Thus (8.5) follows by (8.7) and (8.9). □

Next, we study the variation of $Q_k(t)$. For $k \geq 1$, let I_k denote the set of times where two waves γ, θ with $\max\{k_\gamma, k_\theta\} = k$ interact. The interaction patterns in Lemma 4.3 are denoted by

W_1W_2: a 1-wave and a 2-wave, Cases (1), (2);

W_0W: a 1 or 2-wave and an entropy wave, Cases (4), (5);

S_iS_i: two shock waves of the same family, Case (3);

S_iR_i**:** a shock wave and a rarefaction wave of the same family, Cases (6)–(10).

We notice that Case (11) is trivial. In the following Lemma 8.2, we denote $\theta' = \beta'$ in Cases (6)–(9) and $\theta' = \pi'$ in Case (10), as in the previous section. The estimates (8.10)–(8.12) are obtained by summing up the estimates obtained in the possible interactions and must be read as follows: referring to (8.10), for example, in an interaction W_1W_2 or W_0W (S_iS_i, S_iR_i) only the first (second, third, respectively) summand in braces appears. The notation $S_iR_i\,(k_\alpha < k-1)$ in (8.11) and (8.12) refers to the case S_iR_i when the order of the incoming shock wave is less then $k-1$.

Lemma 8.2. *Let t be an interaction time and* P *the point where the interaction occurs. We have the following estimates:*

(i) if $t \in I_1 \cup \cdots \cup I_{k-2}$, for $k \geq 3$, then

$$\Delta Q_k(t) \leq M_3 \left\{3\epsilon Q(\mathrm{P}) + 3D\alpha_1\alpha_2 + \theta' + M_0|\delta'|\right\} \mathcal{W}_k(t-); \tag{8.10}$$

(ii) if $t \in I_{k-1}$, for $k \geq 2$, then

$$\Delta Q_k(t) \leq \\ \leq \begin{cases} M_3\left\{\epsilon[\Delta Q_{k-1}(t)]_- + 3D\alpha_1\alpha_2\right\} \mathrm{TV}(t-) & \\ \qquad + M_3\left(\theta' + M_0|\delta'|\right) \mathcal{W}_k(t-) & S_iR_i\,(k_\alpha < k-1) \\ M_3\left\{\epsilon[\Delta Q_{k-1}(t)]_- + 3D\alpha_1\alpha_2\right\} \mathrm{TV}(t-) & \\ \qquad + M_3\left(\theta' + M_0|\delta'|\right) \mathrm{TV}(t-) & \textit{otherwise}; \end{cases} \tag{8.11}$$

(iii) if $t \in I_k \cup I_{k+1} \cup \cdots$, for $k \geq 1$, then

$$\Delta Q_k(t) \leq \begin{cases} M_3\left(\theta' + M_0|\delta'|\right) \mathcal{W}_k(t-) & \\ \quad + \alpha\xi\left(\epsilon M_3 D\mathrm{TV}(t-) - M_1\right) & S_iR_i\,(k_\alpha < k-1) \\ M_3\left(\theta' + M_0|\delta'|\right) \mathrm{TV}(t-) & \textit{otherwise.} \end{cases} \tag{8.12}$$

Proof. If $t \in I_1 \cup \cdots \cup I_{k-2}$, $k \geq 3$, by Lemma 4.3, (5.14), (7.6) and the notation introduced before (7.27) we have

$$\Delta Q_k(t) \quad \leq \quad \begin{cases} 3\epsilon M_3 Q(\mathrm{P})\mathcal{W}_k(t-) & : \ W_1W_2,\ W_0W \\ 3DM_3\alpha_1\alpha_2\mathcal{W}_k(t-) & : \ S_iS_i \\ M_3(\theta' + M_0|\delta'|)\mathcal{W}_k(t-) & : \ S_iR_i. \end{cases}$$

Hence, the estimate (8.10) follows by summing up the lines above.

If $t \in I_{k-1}$, $k \geq 2$, by (4.6) we have

$$\Delta Q_k(t) \quad \leq \quad \begin{cases} \epsilon M_3[\Delta Q_{k-1}(t)]_-\mathrm{TV}(t-) & : \ W_1W_2,\ W_0W \\ 3DM_3\alpha_1\alpha_2\mathrm{TV}(t-) & : \ S_iS_i \\ M_3(\beta' + M_0|\delta'|)\mathcal{W}_k(t-) & \\ \quad + \epsilon M_3 D\alpha\xi\mathrm{TV}(t-) & : \ S_iR_i\,(k_\alpha < k-1) \\ M_3(\theta' + M_0|\delta'|)\mathrm{TV}(t-) & : \ S_iR_i\,(k_\alpha = k-1), \end{cases}$$

whence (8.11). We notice that the additional summand $\epsilon M_3 D\alpha\xi \mathrm{TV}(t-)$ is only due to Case (10).

Finally, we consider the case $t \in I_k \cup I_{k+1} \cup \cdots$, $k \geq 1$. Recalling the proof of Lemma 5.2 we deduce that

$$\Delta Q_k(t) \quad \leq \quad \begin{cases} 0 & : \; W_1W_2,\, W_0W,\, S_iS_i \\ (M_3\theta' + M_1M_0|\delta'|)\mathcal{W}_k(t-) & \\ \qquad + \alpha\xi\left(\epsilon M_3 D \mathrm{TV}(t-) - M_1\right) & : \; S_iR_i\,(k_\alpha < k-1) \\ (M_3\theta' + M_1M_0|\delta'|)\mathrm{TV}(t-) & : \; S_iR_i\,(k_\alpha = k-1) \end{cases}$$

The proof is complete. □

Now, we carry out an iterative estimate for the quadratic functional $\widetilde{\mathrm{Q}}_k^+$, see (8.2). To this aim we define

$$\kappa = \epsilon M_3 \widetilde{\mathrm{TV}} < 1 \qquad \text{and} \qquad \kappa' = \epsilon M_3 \mathcal{F}(0) < 1. \tag{8.13}$$

Proposition 8.3. *For $k \geq 2$ we have*

$$\begin{aligned} \widetilde{\mathrm{Q}}_k^+ \quad \leq \quad & \kappa \widetilde{\mathrm{Q}}_{k-1}^+ + \left(\frac{1+c}{1-c} + \frac{3D}{2}\mathcal{F}(0)\right) M_3 \widetilde{\mathcal{W}}_k \mathcal{F}(0) \\ & + \left(c^{k-2}L(0) + \widetilde{\mathrm{V}}_{k-1}^S\right)\left(c + \frac{3D}{2}\mathcal{F}(0)\right)\frac{M_3}{1-c}\widetilde{\mathrm{TV}}. \end{aligned} \tag{8.14}$$

Proof. We may assume that at each interaction time there is precisely one interaction [7]; then each interaction time τ uniquely corresponds to an interaction point P_τ. It follows from Lemma 8.2 that, for $k \geq 3$,

$$\begin{aligned} \widetilde{\mathrm{Q}}_k^+ \quad \leq \quad & M_3 \widetilde{\mathcal{W}}_k \sum_{I_1 \cup \cdots \cup I_{k-2}} \left\{3\epsilon Q(\mathrm{P}_\tau) + 3D\alpha_1\alpha_2 + \theta' + M_0|\delta'|\right\} \\ & + M_3 \widetilde{\mathrm{TV}} \sum_{I_{k-1}} \left\{\epsilon[\Delta Q_{k-1}(\tau)]_- + 3D\alpha_1\alpha_2\right\} \\ & + M_3 \widetilde{\mathrm{TV}} \sum_{I_{k-1}\cup I_k \cup \cdots}^{**} \left\{\theta' + M_0|\delta'|\right\} + M_3 \widetilde{\mathcal{W}}_k \sum_{I_{k-1}\cup I_k \cup \cdots}^{*} \left\{\theta' + M_0|\delta'|\right\}. \end{aligned}$$

The summations are made over all interacting times τ and the estimates must be read as we explained above Lemma 8.2. Moreover, we denoted with $*$ the summation related to the case $S_iR_i(k_\alpha < k-1)$ and with $**$ otherwise, see (8.11), (8.12).

Now, we estimate the sums appearing in the above formula. First, by Lemma 5.3 we deduce

$$3\epsilon \sum_{I_1 \cup \cdots \cup I_{k-2}} Q(\mathrm{P}_\tau) \leq F(0).$$

On the other hand, by (8.3) we find that

$$\sum_{I_{k-1}} [\Delta Q_{k-1}(\tau)]_- \leq \sum_{\tau} [\Delta Q_{k-1}(\tau)]_- \leq \sum_{\tau} [\Delta Q_{k-1}(\tau)]_+ = \widetilde{\mathrm{Q}}_{k-1}^+.$$

Next, we claim that

$$\sum_{I_{k-1}\cup I_k\cup\cdots}^{**} (\theta' + M_0|\delta'|) \le \frac{c}{1-c}\left(c^{k-2}L(0) + \widetilde{\mathrm{V}}_{k-1}^S\right), \tag{8.15}$$

$$\sum_{I_1\cup\cdots\cup I_{k-2}} (\theta' + M_0|\delta'|) + \sum_{I_{k-1}\cup I_k\cup\cdots}^{*} (\theta' + M_0|\delta'|) \le \frac{c}{1-c}\mathcal{F}(0), \tag{8.16}$$

and also that

$$\sum_{I_1\cup\cdots\cup I_{k-2}} \alpha_1\alpha_2 \quad\le\quad \frac{\mathcal{F}(0)^2}{2}, \tag{8.17}$$

$$\sum_{I_{k-1}} \alpha_1\alpha_2 \quad\le\quad \frac{\mathcal{F}(0)}{2}\left(\frac{c^{k-2}}{1-c}L(0) + \widetilde{\mathrm{V}}_{k-1}^S\right). \tag{8.18}$$

The estimate (8.14) shall follow by the above estimates.

We first consider (8.15). We define $\mathcal{I}_h = \{\mathrm{P}_\tau;\ \tau \in I_h\}$ and proceed almost exactly as in proving (7.14) in Lemma 7.4. We have

$$\sum_{\mathrm{P}\in\mathcal{I}_{k-1}\cup\mathcal{I}_k\cup\cdots}^{**} \left\{\theta'(\mathrm{P}) + M_0|\delta'(\mathrm{P})|\right\} \le c\sum_{\Gamma\in\boldsymbol{\Gamma}}\ \sum_{\mathrm{P}\in\Gamma\cap(\mathcal{I}_{k-1}\cup\mathcal{I}_k\cup\cdots)}^{**} \sigma_\Gamma(\mathrm{P}).$$

For every $\mathrm{P} \in \Gamma\cap(\mathcal{I}_{k-1}\cup\mathcal{I}_k\cup\cdots)$, we have $k_\Gamma(\mathrm{P}) \ge k-1$ by the definition of strength of a path given above (7.1). Let m_Γ be the number of points $\mathrm{P} \in \Gamma\cap(\mathcal{I}_{k-1}\cup\mathcal{I}_k\cup\cdots)$. For the collection of primary paths we have

$$\begin{aligned}\sum_{\Gamma\in\boldsymbol{\Gamma}^P}\ \sum_{\mathrm{P}\in\Gamma\cap(\mathcal{I}_{k-1}\cup\mathcal{I}_k\cup\cdots)}^{**} \sigma_\Gamma(\mathrm{P}) \quad&\le\quad \sum_{\Gamma\in\boldsymbol{\Gamma}^P}\sigma_\Gamma(\mathrm{P}_{0,\Gamma}) \sum_{\mathrm{P}\in\Gamma\cap(\mathcal{I}_{k-1}\cup\mathcal{I}_k\cup\cdots)}^{**} c^{k_\Gamma(\mathrm{P})-k_\Gamma(\mathrm{P}_{0,\Gamma})}\\ &\le\quad \sum_{\Gamma\in\boldsymbol{\Gamma}^P}\sigma_\Gamma(\mathrm{P}_{0,\Gamma}) \sum_{j=k-1}^{m_\Gamma} c^{j-1} \quad\le\quad \frac{L(0)c^{k-2}}{1-c}.\end{aligned} \tag{8.19}$$

For secondary paths we have

$$\sum_{\Gamma\in\boldsymbol{\Gamma}^S}\ \sum_{\mathrm{P}\in\Gamma\cap(\mathcal{I}_{k-1}\cup\mathcal{I}_k\cup\cdots)} \sigma_\Gamma(\mathrm{P}) \le \widetilde{\mathrm{V}}_{k-1}^S.$$

Then, estimate (8.15) follows.

The proof of (8.16) is analogous: the first sum is estimated as in (7.14), (7.15) and for the second one we exploit (7.8) with $k = 2$ and (7.6): the first sum is estimated as in (7.14), (7.15) and for the second one we exploit (7.8) with $k = 2$ and (7.6). More precisely,

$$\begin{aligned}&\sum_{I_1\cup\cdots\cup I_{k-2}} (\theta' + M_0|\delta'|) + \sum_{I_{k-1}\cup I_k\cup\cdots}^{*} (\theta' + M_0|\delta'|)\\ \le\quad& \sum_{\mathrm{P}\in\Gamma}\left\{\theta'(\mathrm{P}) + M_0|\delta'(\mathrm{P})|\right\}\\ \le\quad& c\Big(\sum_{\Gamma\in\boldsymbol{\Gamma}^P} + \sum_{\Gamma\in\boldsymbol{\Gamma}^S}\Big)\Big(\sigma_\Gamma(\mathrm{P}_{0,\Gamma})\sum_{\mathrm{P}\in\Gamma} c^{k_\Gamma(\mathrm{P})-k_\Gamma(\mathrm{P}_{0,\Gamma})}\Big).\end{aligned}$$

If Γ is a primary path, then

$$\sum_{\mathrm{P}\in\Gamma} c^{k_\Gamma(\mathrm{P})-k_\Gamma(\mathrm{P}_{0,\Gamma})} \le \sum_{j=1}^{\infty} c^{j-1} \le \frac{1}{1-c} \quad \text{and} \quad \sum_{\Gamma\in\mathbf{\Gamma}^P} \sigma_\Gamma(t_{0,\Gamma}) \le L(0).$$

If Γ is a secondary path, then

$$\sum_{\mathrm{P}\in\Gamma} c^{k_\Gamma(\mathrm{P})-k_\Gamma(\mathrm{P}_{0,\Gamma})} \le \sum_{j=0}^{\infty} c^{j} \le \frac{1}{1-c} \quad \text{and} \quad \sum_{\Gamma\in\mathbf{\Gamma}^S} \sigma_\Gamma(t_{0,\Gamma}) \le \widetilde{\mathrm{V}}_2^S \le F(0).$$

This proves (8.16).

Next, we prove the estimate (8.18); the proof of (8.17) is analogous. The proof is analogous to that of Proposition 7.4. We have, by Proposition 7.2,

$$\begin{aligned}
\sum_{P\in\mathcal{I}_{k-1}} |\alpha_1(\mathrm{P})\alpha_2(\mathrm{P})| &\le \tfrac{1}{2}\sum_{\Gamma\in\mathbf{\Gamma}} \sum_{\mathrm{P}\in\Gamma\cap\mathcal{I}_{k-1}} \sigma_\Gamma(\mathrm{P}) \sum_{\Gamma'\in\mathbf{\Gamma}(\mathrm{P})\setminus\Gamma} \sigma_{\Gamma'}(\mathrm{P}) \\
&\le \tfrac{1}{2}\sum_{\Gamma\in\mathbf{\Gamma}} \sum_{\Gamma'\in\mathbf{\Gamma}\setminus\Gamma} \sum_{\mathrm{P}\in\Gamma\cap\Gamma'\cap\mathcal{I}_{k-1}} \sigma_\Gamma(\mathrm{P})\sigma_{\Gamma'}(\mathrm{P}) \\
&\le \frac{\mathcal{F}(0)}{2}\Big(\sum_{\Gamma\in\mathbf{\Gamma}^P} + \sum_{\Gamma\in\mathbf{\Gamma}^S}\Big) \sum_{\mathrm{P}\in\Gamma\cap\mathcal{I}_{k-1}} \sigma_\Gamma(\mathrm{P}) \\
&\le \frac{\mathcal{F}(0)}{2}\left(\frac{c^{k-2}}{1-c}L(0) + \widetilde{\mathrm{V}}_{k-1}^S\right),
\end{aligned}$$

by arguing as in proving (8.15). This proves (8.18). □

In the following we assume that $\epsilon\mathrm{TV}(0)$ is sufficiently small, and then κ and κ' in (8.13) do. As a consequence, we assume that

$$e^{\epsilon\bar{D}\mathcal{F}(0)} \le 2. \tag{8.20}$$

Proposition 8.4. *If $\epsilon\mathrm{TV}(0)$ is sufficiently small, then, for some $c < \lambda_2 < 1$ we have*

$$\widetilde{\mathrm{V}}_k^S = O(1)\lambda_2^k. \tag{8.21}$$

Proof. We denote $\mathrm{Z}_k = \epsilon(\widetilde{\mathrm{Q}}_k^+ + c\widetilde{\mathrm{Q}}_{k-1}^+ + \cdots + c^{k-1}\widetilde{\mathrm{Q}}_1^+)$. By (8.21) we deduce

$$\epsilon\widetilde{\mathrm{Q}}_k^+ = \mathrm{Z}_k - c\mathrm{Z}_{k-1} \tag{8.22}$$

and by (8.5) and (8.20) we have

$$\widetilde{\mathrm{V}}_k^S \le \frac{2}{3}\mathrm{Z}_{k-1}. \tag{8.23}$$

Therefore it is sufficient to prove (8.21) with $\widetilde{\mathrm{V}}_k^S$ replaced by Z_{k-1}. We plug (8.23) into (7.27) to get

$$\widetilde{\mathcal{W}}_k \le \frac{c^{k-2}}{1-c}L(0)\left(c + \frac{D\mathcal{F}(0)}{(1-c)(1-c^2)}\right) + \frac{2}{3}\mathrm{Z}_{k-1} + \frac{2D\mathcal{F}(0)}{3(1-c)(1-c^2)}\mathrm{Z}_{k-2}.$$

By multiplying (8.14) by ϵ and taking again into account (8.22) we deduce

$$\begin{aligned} Z_k - cZ_{k-1} \quad &\le \quad \left(\kappa + \frac{2}{3}B\kappa'\right) Z_{k-1} + \left(-c\kappa + BE\kappa' + \frac{2}{3}\frac{\kappa C}{1-c}\right) Z_{k-2} \\ &\quad + \left(AB\kappa' + \frac{L(0)C}{1-c}\kappa\right) c^{k-2}, \end{aligned} \tag{8.24}$$

where

$$\begin{aligned} A \quad &= \quad \tfrac{L(0)}{1-c}\left(c + \tfrac{D\mathcal{F}(0)}{(1-c)(1-c^2)}\right), \quad & B \quad &= \quad \tfrac{1+c}{1-c} + \tfrac{3D\mathcal{F}(0)}{2}, \\ C \quad &= \quad c + \tfrac{3D\mathcal{F}(0)}{2}, & E \quad &= \quad \tfrac{2D\mathcal{F}(0)}{3(1-c)(1-c^2)}. \end{aligned}$$

We rewrite inequality (8.24) as

$$Z_k - aZ_{k-1} + bZ_{k-2} \le dc^{k-2}, \tag{8.25}$$

for

$$a = c + \kappa + \tfrac{2}{3}B\kappa', \quad b = c\kappa - BE\kappa' - \tfrac{2}{3}\tfrac{\kappa}{1-c}C, \quad d = AB\kappa' + \tfrac{L(0)}{1-c}C\kappa.$$

If both κ and κ' are sufficiently small, then the equation $\lambda^2 - a\lambda + b = 0$ has two real roots $|\lambda_1| < \lambda_2$ with $0 < c < \lambda_2 < 1$. Since $Z_k \ge 0$, we deduce from the inequality (8.25)

$$Z_k = O(1)\lambda_2^{k-1}$$

which proves the proposition. □

Proof of Theorem 1.3: The proof is analogous to that of Theorem 2.1 in Asakura-Corli [5]; then, only a sketch is reported here. Estimates of *physical* waves are obtained by Proposition 7.1 and Proposition 8.4. About estimates of non-physical waves, let ϵ denote an arbitrary non-physical wave. As in Proposition 11.10 in Asakura-Corli [5] it follows that $|\epsilon| \le C_1\rho$ and

$$\sum_{\substack{\epsilon\in\mathcal{NP} \\ k_\epsilon \ge k}} |\epsilon| \le C_2 \sup_{t\ge 0}\left\{V_k^P(t) + V_k^S(t)\right\}$$

for some positive constants C_1 and C_2.

Proposition 8.5. *For given $h > 0$, there exists $\rho > 0$ such that the approximate solution constructed by the front tracking scheme satisfies*

$$\sum_{\epsilon\in\mathcal{NP}} |\epsilon| \le h. \tag{8.26}$$

Proof. The proof goes as in Bressan [7]. Let N_0 be the number of shock waves at $t = 0$. Then there exists a polynomial $P(\xi, \eta)$ such that, by exploiting the above inequalities, we have

$$\sum_{\epsilon\in\mathcal{NP}} |\epsilon| = \sum_{\substack{\epsilon\in\mathcal{NP} \\ k_\epsilon \le k-1}} |\epsilon| + \sum_{\substack{\epsilon\in\mathcal{NP} \\ k_\epsilon \ge k}} |\epsilon| = O(1)P(N_0, h^{-1})\rho + O(1)\lambda_2^k.$$

Hence, we choose k such that $O(1)\lambda_2^k \le \frac{h}{2}$ and then ρ so that (8.26) holds. □

By Proposition 8.26 we have a uniform bound of non-physical waves and hence of $\mathrm{TV}U^h(\cdot, t)$. The existence of a global solution is proved by the usual argument in Bressan [7] and Smoller [20].

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