Università degli Studi di Ferrara

DOCTORAL COURSE IN MATHEMATICS

CYCLE XXXVIII

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A COMPARATIVE STUDY OF SELECTED BOUNDS FOR POLYNOMIAL CONSTRAINTS

Scientific/Disciplinary Sector (SDS) INFO-01/A Informatics

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Years 2022/23-2024/2025

Abstract (English)

This Doctoral Thesis focuses on the comparative study of bounds for polynomial functions over real multi-intervals within the framework of logic programming with polynomial constraints.

The Thesis begins by introducing bounds for real polynomial functions defined over a bounded real interval with non-negative values. Three types of bounds are analyzed: the first is based on the maximum and minimum of the Bernstein polynomial coefficients; the second relies on interval arithmetic; the third consists of simple bounds obtained by considering the absolute values and the signs of the polynomial function coefficients and the right bound of the interval. The aim of the research is to identify, among the proposed bounds, the one that best approximates the behavior of the polynomial function. The study first compares the simple bounds and the one derived from interval arithmetic, which is the most accurate on a non-negative interval, because it takes into account both the left and right bounds. Then, the bounds based on Bernstein polynomials are compared with the others. At this stage, several difficulties arise, as Bernstein polynomials are expressed in a specific basis, whereas the other bounds use the canonical basis. This difference makes the expressions difficult to compare. However, by employing appropriate upper bounds—specifically, Newton's binomial formula and the properties of binomial coefficients—these difficulties can be overcome. It is thus shown that the bounds based on the Bernstein polynomial coefficients is the most accurate, although also the most computationally expensive.

Once the problem has been addressed in one dimension, the three types of bounds are extended to multi-intervals. A comparison is then made among the various approximations over multi-intervals. By carefully handling the needed multi-indices, it is shown that the bounds based on Bernstein polynomials provide the best approximations, but the worst computational time.

A further extension explored in the Thesis is the study of bounds related to the sampling of polynomial functions. The analysis begins with the unit interval and then considers a non-negative real interval. In the unidimensional case, by suitably adapting the absolute values of the three types of bounds to first and second derivatives, new approximations can be defined and compared. The study is then extended to higher dimensions, first to the unit multi-interval and then to a box with non-negative values. While in the unit multi-interval the results are consistent with the unidimensional case, the box scenario for non-negative multi-interval is more complex, since greater attention must be paid to the interval bounds when establishing upper limits. The situation has become more complicated, but for non-negative multi-intervals, we initially managed to understand the behavior in two specific cases: the first when all the right bounds are greater than or equal to one; the second when all the right bounds are less than or equal to one. We then achieved results for any non-negative multi-interval, also using the previously obtained bounds.

The major result of this Thesis is a set of orderings among different bounds for polynomial functions, both for non-negative real intervals and for non-negative real multi-intervals.

Keywords: Constraint logic programming, Constraint propagation, Polynomial constraints, Bernstein polynomials, Function bounds

Abstract (Italiano)

Questa tesi di dottorato riguarda lo studio comparativo di stime per funzioni polinomiali su multi-intervalli reali nell'ambito della programmazione logica con vincoli polinomiali. La trattazione si apre con l'introduzione di stime per funzioni polinomiali reali definite su un intervallo reale limitato a valori non negativi. Le stime analizzate sono di tre tipologie: la prima si basa sul massimo e sul minimo dei coefficienti dei polinomi di Bernstein; la seconda si fonda sull'aritmetica degli intervalli; la terza comprende semplici stime ottenute considerando i valori assoluti ed i segni dei coefficienti della funzione polinomiale e l'estremo superiore dell'intervallo.

L'obiettivo del lavoro è identificare, tra le stime studiate, quella che meglio approssima il comportamento della funzione polinomiale. Inizialmente si confrontano le stime semplici e quella ottenuta dall'aritmetica degli intervalli, la quale, su un intervallo a valori non negativi, risulta la più efficace perché tiene conto sia dell'estremo inferiore sia di quello superiore. Successivamente, si mette a confronto anche la stima basata sui polinomi di Bernstein. In quest'ultima fase emergono alcune difficoltà, poiché i polinomi di Bernstein sono espressi in una base specifica, mentre le altre stime utilizzano la base canonica. Questa differenza rende le espressioni difficilmente confrontabili. Tuttavia, sfruttando opportune maggiorazioni, in particolare basate sul binomio di Newton e sulle proprietà dei coefficienti binomiali, è possibile superare tali ostacoli. Si giunge così alla conclusione che la stima basata sui coefficienti del polinomio di Bernstein è la più precisa, sebbene risulti anche la più onerosa dal punto di vista computazionale.

Una volta analizzato il problema in una dimensione, le tre tipologie di stime vengono estese a multi-intervalli in più variabili. Si effettua quindi un confronto tra le diverse approssimazioni nei multi-intervalli. Prestando particolare attenzione all'uso dei multi-indici, si dimostra che le stime basate sui polinomi di Bernstein offrono le migliori approssimazioni, ma presentano le peggiori prestazioni nei termini del tempo di calcolo.

Un'ulteriore estensione trattata nella tesi riguarda lo studio delle stime associate al campionamento di funzioni polinomiali. L'analisi prende in considerazione dapprima l'intervallo unitario e successivamente un intervallo reale a valori non negativi. Nel caso unidimensionale, adattando opportunamente i valori assoluti delle tre tipologie di stime alle derivate prime e seconde, è possibile definire e confrontare queste nuove approssimazioni. Lo studio viene poi esteso al caso multidimensionale, considerando prima il multi-intervallo unitario e poi multi-intervalli con valori non negativi. Mentre nel multi-intervallo unitario si ottengono risultati coerenti con il caso unidimensionale, nel caso di multi-intervalli non negativi la situazione diventa più delicata, poiché nelle maggiorazioni occorre prestare particolare attenzione agli estremi degli intervalli. Il quadro si è complicato, ma per multi-intervalli non negativi si è arrivati inizialmente a comprendere il comportamento in due casi specifici: il primo quando tutti gli estremi superiori sono maggiori o uguali di uno; il secondo quando tutti gli estremi superiori sono minori o uguali di uno. Si sono poi ottenuti risultati generali per qualsiasi multi-intervallo non negativo, anche grazie all'uso delle stime precedentemente ottenute.

Il risultato principale di questa tesi consiste in un insieme di ordinamenti tra diverse stime per funzioni polinomiali, sia per intervalli reali non negativi, sia per multi-intervalli reali non negativi.

Parole chiave: Programmazione logica con vincoli, Propagazione di vincoli, Vincoli polinomiali, Polinomi di Bernstein, Stime di funzioni

Contents

Chapter 1

Introduction

This Doctoral Thesis focuses on the comparative study of bounds for polynomial functions over real multi-intervals within the framework of logic programming with polynomial constraints. In fact, numerous relationships and challenges, both practical and theoretical, can be naturally modeled through polynomial constraints.

The concept of polynomials is fundamental and pervasive, with significant applications across nearly all fields of science and engineering. The extensive body of literature on computer algebra highlights the extensive study of polynomials from a computational perspective [29]. Polynomial constraints, which include equalities and inequalities involving polynomials with real variables, have been extensively explored in the domains of constraint satisfaction [22] and constraint optimization [23]. A particular subclass of polynomial constraints, referred to as polynomial constraints on finite domains, restricts the variables to take values from finite subsets of integers. These constraints also encompass disequalities of polynomials, a feature often overlooked in the context of real variables. Initially investigated within the framework of constraint logic programming [2, 5, 6, 7], polynomial constraints over finite domains can also be analyzed within the broader scope of constraint satisfaction problems, as explored in earlier works [3, 4].

A critical aspect of effectively supporting constraint logic programming is the computation of tight bounds for real polynomial functions over real boxes, particularly when the constraint language under consideration permits the expression of polynomial constraints over finite domains.

One of the most essential tasks in reasoning about polynomial constraints on finite domains involves computing the precise bounds for real polynomial functions over real boxes [4]. This Thesis presents methods for bounding real polynomial functions over real boxes and compares them in terms of the tightness of their bounds. All proofs presented in this Thesis are original and have not been copied from any source.

The primary contribution of this work is the comparative analysis of various bounds to identify the tightest ones, independent of the specific function and box under consideration. Thus, this Thesis provides quantitative insights into the range of polynomials over intervals and boxes. The results presented herein can be regarded as a preliminary step toward the development of a hierarchy of bounding techniques tailored to reasoning with polynomial constraints on finite domains. It should be noted that the study emphasizes the computational aspects of these bounding methods.

The second part of this Thesis focuses on bounds derived from function values of polynomials. Initially, it reviews existing results for univariate real functions over the unit interval [9, 20]. New methods are then introduced for one-dimensional functions over the unit square and non-negative real intervals. All bounds are compared both in terms of their tightness and computational efficiency. The Thesis subsequently presents the known bounds [12] and extends the newly proposed bounds for univariate functions to bounds for multivariate functions. Again, all bounds are assessed for both their tightness and computational performance.

This Thesis is organized as follows.

Chapter 2 introduces four distinct methods for bounding univariate real polynomial functions over real intervals. These methods are subsequently compared in terms of the tightness of the bounds they generate. The primary contribution of this chapter is the proof that one of the four methods consistently provides tighter bounds. This result serves as a preliminary foundation for exploring a hierarchy of bounding methods for real polynomial functions over real boxes.

Chapter 3 extends the univariate methods discussed in Chapter 2 to the multivariate case, focusing on polynomial functions over real boxes. This chapter builds upon the one-dimensional results, generalizing them to higher dimensions.

Chapter 4 examines the limitations of bounding polynomial functions via their values. Building on the work of Cargo and Shisha [9] and Rivlin [20] on the unit interval, it shows how combining classical polynomial bounding methods with value-based techniques leads to new bounds. These results are first derived for univariate polynomials on the unit interval and then extended to non-negative real intervals.

Chapter 5 extends the concepts discussed in Chapter 4 to several dimensions, focusing on bounding polynomial functions based on their values. The approach begins with the framework presented by Garloff [12] for the unit box in multiple dimensions, and proceeds by generalizing the unidimensional case to approximate the function with new bounds. This methodology is then adapted and expanded to cover multivariate polynomial functions over non-negative multi-intervals.

Ultimately, the conclusions of this Thesis summarize the findings and suggest potential avenues for future research and development arising from this study.

1.1 Motivations

Following consolidated nomenclature [1], a constraint C with variables $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ whose domains are $D_1, D_2, \ldots, D_n$,

$$C \subseteq D_1 \times D_2 \times \cdots \times D_n, \tag{1.1}$$

is *satisfiable* if $C \neq \emptyset$, and *unsatisfiable* otherwise. A constraint C is *hyper-arc consistent* (sometimes called *generalized arc consistent* or *domain consistent*) [1] if

$$\forall i \in [1..n]\ \forall a \in D_i\ \exists \mathbf{t} = (t_1, t_2, \ldots, t_n) \in C\ :\ t_i = a. \tag{1.2}$$

A set of $m \geq 0$ constraints $\{C_i\}_{i=1}^{m}$ is satisfiable if it is empty or all constraints are satisfiable, it is unsatisfiable if at least one constraint is unsatisfiable, and it is hyper-arc consistent if it is empty or all constraints are hyper-arc consistent. Hyper-arc consistency is a property of constraints as well as domains and is occasionally described using the term hyper-arc consistent domains.

The proposed Thesis deals with a set of $m \geq 0$ primitive constraints $\{C_i\}_{i=1}^{m}$ with variables $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ together with a nonempty set of initial approximations of domains, one for each variable, expressed as finite intervals

$$[\underline{d}_i..\overline{d}_i] \quad 1 \leq i \leq n \tag{1.3}$$

where $\{\underline{d}_i\}_{i=1}^{n}$ are the initial lower bounds of the domains of variables, and $\{\overline{d}_i\}_{i=1}^{n}$ are the initial upper bounds. Such bounds form a box $D \subset \mathbb{R}^n$ which is an initial approximation of the domains of variables, and the thesis deals with the inclusion-maximal subset of D which ensures that all constraints are hyper-arc consistent, if it exists. The goal is to treat all primitive constraints in canonical form $p(\mathbf{x}) \geq 0$ by employing an algorithm that recursively partitions D into boxes and tests the satisfiability of each constraint in the current box $B \subseteq D$, as [4].

If it is not possible to prove either satisfiability or unsatisfiability of the constraint set, a variable is selected and its current domain is partitioned into two disjoint subdomains, thereby generating two separate boxes. Each of these boxes is then processed recursively by applying the same reasoning procedure. This splitting and analysis process continues iteratively until, for every generated box, the set of constraints can be conclusively classified as either satisfiable or unsatisfiable. This algorithm always terminates because, in the worst case, boxes collapse to singleton subsets of D, $\{\mathbf{v}\} \subseteq D$ for some $\mathbf{v} \in D$, for which the check for satisfiability is trivially $p(\mathbf{v}) \geq 0$. When boxes are not singleton sets, in order to check for the satisfiability of a constraint $p(\mathbf{x}) \geq 0$, the idea is to assume that suitable lower bound $l \in \mathbb{R}$ and upper bound $u \in \mathbb{R}$ for the polynomial p in box B can be computed. Such values can be used to verify if the currently analyzed constraint is satisfiable in B, as follows

1. If $u < 0$ then the constraint is unsatisfiable in B;

2. If $l \geq 0$ then the constraint is satisfiable in B;

3. Otherwise, B should be split and analyzed recursively.

It is worth noting that, since constraints are in canonical form, we are not interested in the actual values of l and u, but only in their signs. Any algorithm that allows the effective computation of such signs can be used to support the proposed algorithm. In the following, we assume that modified Bernstein coefficients are used to compute the signs of l and u. In particular, using the notation introduced in previous section, define

$$\tilde{l} = \min\{\tilde{b}_I^{(D)}\}_{I \leq L} \qquad \tilde{u} = \max\{\tilde{b}_I^{(D)}\}_{I \leq L}, \tag{1.4}$$

where L is the multi-degree of p. From literature [4] the sign remains the same

$$\operatorname{sgn} l = \operatorname{sgn} \tilde{l} \qquad \operatorname{sgn} u = \operatorname{sgn} \tilde{u}. \tag{1.5}$$

The literature provides various algorithms to compute Bernstein coefficients [11, 13, 24], which are likely to be more effective than direct computation in practical situations. Most of such algorithms can be adapted to compute modified Bernstein coefficients, and their expected performance improvements over direct computation is currently under investigation. The performance of the proposed algorithm also depends on the initial bounds of each domain, and on the details of the subdivision into disjoint boxes. A significant number of alternatives has been studied in the literature to address the problem of effectively subdividing a domain into disjoint boxes [16, 17], and all available techniques can be adopted in the proposed algorithm. The simplest of such techniques assumes that variables are selected using a fixed total order, and that, once a variable x_i is selected, its current domain $[\underline{x}_i..\overline{x}_i]$ is split into two intervals $[\underline{x}_i..s_i]$ and $[(s_i+1)..\overline{x}_i]$ at

$$s_i = \left\lfloor \frac{(\underline{x}_i + \overline{x}_i)}{2} \right\rfloor. \tag{1.6}$$

This is the simplest choice and further research on this topic is ongoing to possibly improve the performance of the algorithm.

1.2 Related Work

The study of bounds for polynomial functions over real intervals and boxes is a central topic that connects classical approximation theory, numerical analysis, global optimization, and, more recently, constraint logic programming. Its origins can be traced to the development of Bernstein polynomials in the early 20th century, introduced by Bernstein as a constructive proof of the Weierstrass approximation theorem and later systematized by Lorentz [15]. The Bernstein basis offers a numerically stable representation of polynomials over a compact interval, with coefficients that are closely related to the values of the polynomial itself. This property has made Bernstein polynomials a cornerstone of computer-aided geometric design, curve modeling, and robust numerical computation [10, 11, 24].

Parallel to this line of development, interval arithmetic emerged as a rigorous framework for bounding function values. Moore formalized arithmetic operations over intervals, providing a guarantee that the true function value lies within the computed interval. However, standard interval arithmetic suffers from the *dependency problem*: when a variable appears multiple times, naive interval evaluation often overestimates the true range of a function. To address these limitations, affine arithmetic and Taylor models were proposed, providing tighter enclosures and higher-order approximations of nonlinear dependencies [14]. Hickey et al. demonstrated that integrating these techniques into symbolic-numeric systems allows both reliability and efficiency, though interval-based methods alone are often insufficient for precise polynomial bounding in high-dimensional spaces.

A significant alternative to interval-based methods arises from the Bernstein polynomial representation. Early insights by Cargo and Shisha [9] and rigorous formalizations by Garloff [12, 13] established that the range of a polynomial over a box is contained in the convex hull of its Bernstein coefficients. This convex-hull property immediately yields both upper and lower bounds and is generally sharper than those obtained through interval evaluation. Garloff further explored the computational and theoretical aspects of this representation, providing results on convergence, coefficient transformation, and multidimensional subdivision. His work clarified the theoretical underpinnings of Bernstein bounds and offered practical algorithms for computing enclosures of multivariate polynomials efficiently.

The univariate case, while conceptually simpler, also benefits from Bernstein representations. Farouki and Rajan [11] presented algorithms for various basic polynomial procedures in Bernstein form: the arithmetic operations, substitution of polynomials, elimination of variables, determination of greatest common divisors, emphasizing numerical stability and efficiency.

Farouki's retrospective review [10] highlights the enduring relevance of the Bernstein basis in geometric design and numerical computation, noting that its geometric intuition often provides better insight than purely algebraic or interval-based techniques. Complementary studies by Rokne [21] and Rivlin [20] further establish the Bernstein basis as a robust tool for estimating polynomial extrema with tight bounds and controlled numerical error.

The multivariate extension of Bernstein polynomial bounds introduces additional challenges. The number of coefficients grows combinatorially with both the degree of the polynomial and the number of variables, which can make naive computation infeasible. Garloff [12] analyzed the structure of these multivariate expansions, proving convergence of bounds under recursive box subdivision and showing that adaptive subdivision can efficiently reduce overestimation. Ray and Nataraj [18, 19] later developed matrix-based methods for efficiently computing the Bernstein coefficients of multivariate polynomials, relying on matrix operations such as multiplication, inversion, and reshaping to convert a polynomial from its power form to its Bernstein form, thereby providing a scalable and efficient alternative to other computational techniques for large polynomial systems. Additionally, Smith [25] introduced approaches for constructing constant bound functions for sparse multivariate polynomials, which significantly reduces computational cost by exploiting sparsity patterns.

In parallel, the study of constraints with polynomial bounds has become increasingly significant. Constraint logic programming (CLP) provides a declarative paradigm for reasoning over polynomial constraints, with the potential to combine symbolic and numerical methods seamlessly [22]. SWI-Prolog [27, 28, 30], like other Prolog systems, supports the integration of external arithmetic/constraint solvers, notably through its finite-domain (CLP(FD)) library. Bergenti, Monica, and Rossi [2, 3, 4, 5, 6, 7] contributed significantly to this research by applying a modified Bernstein form and presenting algorithms capable of effectively solving polynomial constraints expressed as inequalities of polynomials with integer coefficients over finite domains. Beyond finite-domain problems, these methods have been extended to handle higher-degree polynomials using hyper-arc consistency techniques [2], highlighting the potential of Bernstein-based propagation in symbolic-numeric reasoning.

Other approaches to polynomial constraint solving, such as satisfiability modulo theories (SMT), complement these techniques. Borralleras et al. [8] proposed a method for solving non-linear constraints by encoding the problem as an SMT problem that considers only linear arithmetic. Their approach primarily focuses on proving the satisfiability of constraints, which is more relevant in many applications. Nevertheless, they also introduced techniques based on the analysis of unsatisfiable cores that enable the efficient proof of unsatisfiability for a broad class of problems.

The literature also situates these advances in the broader context of polynomial algebra and non-linear optimization. Von zur Gathen and Gerhard [29] provide a thorough treatment of symbolic computation, including polynomial arithmetic, factorization, and transformation. Ruszczyński [23] Integrates convex analysis, the theory of optimality conditions, duality theory, and numerical methods for solving unconstrained and constrained optimization problems.

Recent advancements in computational mathematics and uncertainty quantification have increasingly concentrated on the development and application of hybrid approaches that integrate different mathematical techniques. Specifically, there has been a growing emphasis on combining Bernstein bounds—methods for estimating rigorous upper and lower limits on a function's behavior—with interval analysis or Taylor model evaluations. Both of these methods are used to represent and manage the uncertainty inherent in computational systems. These hybrid approaches aim to leverage the strengths of each technique, enabling more accurate and dependable solutions in fields such as global optimization, verification of complex systems, and the analysis of nonlinear dynamics.

In conclusion, the study of polynomial bounds over intervals and boxes represents a rich convergence of ideas. Interval arithmetic provides foundational, rigorous enclosures but can be conservative. Bernstein polynomial representations, as formalized and extended by Garloff and others, offer tight geometric bounds applicable in both univariate and multivariate settings. Constraint logic programming presents a declarative framework for embedding these techniques into broader symbolic-numeric reasoning systems. The present Thesis builds on these complementary strands by systematically comparing interval bounds, Bernstein bounds, and coefficient-based simple bounds, extending the analysis to derivatives, multidimensional boxes, and sampling-based methods. This investigation not only highlights relative precision and computational cost but also bridges theoretical insight and practical applicability, contributing to a comprehensive understanding of polynomial bounding techniques.

Chapter 2

Bounds in the Univariate Case

The theory of approximation ensures that any sufficiently smooth function in an interval can be approximated with arbitrary precision by a polynomial function, while also providing effective tools for constructing the desired approximating functions.

One such tool is the Bernstein polynomial basis, developed over more than a century as a means to support the approximation of sufficiently smooth functions within intervals. Bernstein polynomials have been widely used in various applications, but their use has been almost entirely restricted to the study of real-valued functions, with the exception of global optimization in mixed-integer problems.

This Chapter addresses some polynomial bounds. Initially, elementary bounds are introduced through simple upper bounds, before focusing on interval arithmetic. Subsequently, Bernstein polynomials are discussed, along with the bounds that can be derived from them. In the second part, various bounds are compared in order to determine which is the most accurate among those defined.

2.1 Definitions in the Univariate Case

Let Π_l be the vector space of (real) polynomial functions in one real variable with real coefficients and degree less than or equal to $l \in \mathbb{N}$ (note that $\mathbb{N}$ includes zero)

$$\Pi_l = \mathrm{span}_{\mathbb{R}}\{P_i\}_{i=0}^{l} \qquad P_i(x) = x^i \qquad 0 \leqslant i \leqslant l, \tag{2.1}$$

where $\mathrm{span}_{\mathbb{R}}(\cdot)$ generates a vector space using real numbers, and the set $\{P_i\}_{i=0}^{l}$ is normally called *monomial basis* of Π_l. The studied bounding methods for polynomial functions in Π_l work on closed bounded intervals in $\mathbb{R}$ but, with no loss of generality, these intervals are assumed to be in $\mathbb{R}_{\geqslant 0}$ in the rest of this Thesis. The monomial basis is the most straightforward choice to express polynomial functions because it naturally maps polynomial functions to the common algebraic characterization of polynomials, and vice versa.

The following definition characterizes the first considered bounding method for univariate polynomial functions over closed bounded intervals.

Definition 2.1. *Given a closed bounded interval $[\underline{x},\overline{x}] \subset \mathbb{R}_{\geqslant 0}$ and a polynomial function $p \in \Pi_l$ expressed as*

$$p(x) = \sum_{i=0}^{l} a_i P_i(x), \tag{2.2}$$

where $\{a_i\}_{i=0}^{l} \subset \mathbb{R}$, let

$$\overline{p}_A([\underline{x},\overline{x}]) = \sum_{i=0}^{l} |a_i| \overline{x}^i \tag{2.3}$$

and

$$\underline{p}_A([\underline{x},\overline{x}]) = -\sum_{i=0}^{l} |a_i| \overline{x}^i, \tag{2.4}$$

where, as usual, the sums evaluate to zero if no addend is usable. The real value $\underline{p}_A([\underline{x},\overline{x}])$ is called univariate absolute lower bound *of p over $[\underline{x},\overline{x}]$ and the real value $\overline{p}_A([\underline{x},\overline{x}])$ is called* univariate absolute upper bound *of p over $[\underline{x},\overline{x}]$.*

Note that, independently of the considered function and interval, $\underline{p}_A([\underline{x},\overline{x}]) \leqslant 0$ because it is the sum of non-positive addends. Notably, it is zero only if all the terms of the polynomial function are zero, i.e. p is the zero polynomial. Similarly, $\overline{p}_A([\underline{x},\overline{x}]) \geqslant 0$ because it is the sum of non-negative addends, and it is zero only if all the terms of the polynomial function are zero.

The following proposition motivates the names, absolute bounds, chosen for the two bounds discussed in Definition 2.1.

Proposition 2.1. *Given a closed bounded interval $[\underline{x},\overline{x}] \subset \mathbb{R}_{\geqslant 0}$ and a polynomial function $p \in \Pi_l$ expressed as in (2.2), for all $x \in [\underline{x},\overline{x}]$*

$$\underline{p}_A([\underline{x},\overline{x}]) \leqslant p(x) \leqslant \overline{p}_A([\underline{x},\overline{x}]). \tag{2.5}$$

Proof. The proof of (2.5) follows from the fact that, for all $x \in [\underline{x},\overline{x}] \subset \mathbb{R}_{\geqslant 0}$,

$$p(x) = \sum_{i=0}^{l} a_i x^i \leqslant \sum_{i=0}^{l} |a_i| \overline{x}^i \tag{2.6}$$

since $[\underline{x},\overline{x}] \subset \mathbb{R}_{\geqslant 0}$, which proves the upper bound inequality. Similarly, for the lower bound inequality, the following inequality holds for all $x \in [\underline{x},\overline{x}]$

$$p(x) = \sum_{i=0}^{l} a_i x^i \geqslant -\sum_{i=0}^{l} |a_i| \overline{x}^i, \tag{2.7}$$

which proves the proposition. □

Definition 2.2. *Given a closed bounded interval* $[\underline{x},\overline{x}] \subset \mathbb{R}_{\geqslant 0}$ *and a polynomial function* $p \in \Pi_l$ *expressed as*

$$p(x) = \sum_{i=0}^{l} a_i P_i(x), \tag{2.8}$$

where $\{a_i\}_{i=0}^{l} \subset \mathbb{R}$, *let*

$$\overline{p}_S([\underline{x},\overline{x}]) = \sum_{i=0;a_i>0}^{l} a_i \overline{x}^i \tag{2.9}$$

and

$$\underline{p}_S([\underline{x},\overline{x}]) = \sum_{i=0;a_i<0}^{l} a_i \overline{x}^i, \tag{2.10}$$

where, as usual, the sums evaluate to zero if no addend is usable. The real value $\underline{p}_S([\underline{x},\overline{x}])$ *is called* univariate simple lower bound *of* p *over* $[\underline{x},\overline{x}]$ *and the real value* $\overline{p}_S([\underline{x},\overline{x}])$ *is called* univariate simple upper bound *of* p *over* $[\underline{x},\overline{x}]$.

Note that, independently of the considered function and interval, $\underline{p}_S([\underline{x},\overline{x}]) \leqslant 0$ because it is the sum of non-positive addends. Notably, it is zero only if all the terms of the polynomial function are non-negative. Similarly, $\overline{p}_S([\underline{x},\overline{x}]) \geqslant 0$ because it is the sum of non-negative addends, and it is zero only if all the terms of the polynomial function are non-positive.

The following proposition motivates the names, simple bounds, chosen for the two bounds discussed in Definition 2.2.

Proposition 2.2. *Given a closed bounded interval* $[\underline{x},\overline{x}] \subset \mathbb{R}_{\geqslant 0}$ *and a polynomial function* $p \in \Pi_l$ *expressed as in (2.2), for all* $x \in [\underline{x},\overline{x}]$

$$\underline{p}_S([\underline{x},\overline{x}]) \leqslant p(x) \leqslant \overline{p}_S([\underline{x},\overline{x}]). \tag{2.11}$$

Proof. The proof of (2.11) follows from the fact that, for all $x \in [\underline{x},\overline{x}]$,

$$p(x) = \sum_{i=0}^{l} a_i x^i \leqslant \sum_{i=0;a_i>0}^{l} a_i \overline{x}^i \tag{2.12}$$

since $[\underline{x},\overline{x}] \subset \mathbb{R}_{\geqslant 0}$, which proves the upper bound inequality. Similarly, for the lower bound inequality, the following inequality holds for all $x \in [\underline{x},\overline{x}]$

$$p(x) = \sum_{i=0}^{l} a_i x^i \geqslant \sum_{i=0;a_i<0}^{l} a_i \overline{x}^i, \tag{2.13}$$

which proves the proposition. □

The following definition characterizes the third considered bounding method for univariate polynomial functions over closed bounded intervals.

Definition 2.3. *Given a closed bounded interval $[\underline{x},\overline{x}] \subset \mathbb{R}_{\geqslant 0}$ and a polynomial function $p \in \Pi_l$ expressed as in (2.2), let*

$$\overline{p}_I([\underline{x},\overline{x}]) = \sum_{i=0}^{l} \max\{a_i\underline{x}^i, a_i\overline{x}^i\} \tag{2.14}$$

$$\underline{p}_I([\underline{x},\overline{x}]) = \sum_{i=0}^{l} \min\{a_i\underline{x}^i, a_i\overline{x}^i\}. \tag{2.15}$$

The value $\underline{p}_I([\underline{x},\overline{x}])$ is called univariate interval lower bound *of p over $[\underline{x},\overline{x}]$ and the value $\overline{p}_I([\underline{x},\overline{x}])$ is called* univariate interval upper bound *of p over $[\underline{x},\overline{x}]$.*

The following proposition motivates the names, interval bounds, chosen for the two bounds discussed in Definition 2.3.

Proposition 2.3. *Given a closed bounded interval $[\underline{x},\overline{x}] \subset \mathbb{R}_{\geqslant 0}$ and a polynomial function $p \in \Pi_l$ expressed as in (2.2), then for all $x \in [\underline{x},\overline{x}]$*

$$\underline{p}_I([\underline{x},\overline{x}]) \leqslant p(x) \leqslant \overline{p}_I([\underline{x},\overline{x}]). \tag{2.16}$$

Proof. Consider the $i-$th addend a_ix^i of the polynomial function p, with $x \in [\underline{x},\overline{x}]$ and $0 \leqslant i \leqslant l$. The following three cases can be studied:

- If $a_i = 0$ then
$$a_ix^i = 0 = \min\{a_i\underline{x}^i, a_i\overline{x}^i\}; \tag{2.17}$$
- If $a_i > 0$, since $x \geqslant \underline{x}$ and $a_ix^i \geqslant a_i\underline{x}^i$, then
$$a_ix^i \geqslant \min\{a_i\underline{x}^i, a_i\overline{x}^i\} = a_i\underline{x}^i; \text{ and} \tag{2.18}$$
- If $a_i < 0$, since $x \leqslant \overline{x}$ and $a_ix^i \geqslant a_i\overline{x}^i$, then
$$a_ix^i \geqslant \min\{a_i\underline{x}^i, a_i\overline{x}^i\} = a_i\overline{x}^i. \tag{2.19}$$

Therefore, the following lower bound inequality is proved for all $x \in [\underline{x},\overline{x}]$

$$p(x) = \sum_{i=0}^{l} a_ix^i \geqslant \sum_{i=0}^{l} \min\{a_i\underline{x}^i, a_i\overline{x}^i\} = \underline{p}_I([\underline{x},\overline{x}]). \tag{2.20}$$

The validity of the upper bound inequality, as follows

$$p(x) = \sum_{i=0}^{l} a_ix^i \leqslant \sum_{i=0}^{l} \max\{a_i\underline{x}^i, a_i\overline{x}^i\} = \overline{p}_I([\underline{x},\overline{x}]), \tag{2.21}$$

proves the proposition. □

The following proposition provides an alternative characterization of the interval bounds.

Proposition 2.4. *Given a closed bounded interval $[\underline{x}, \overline{x}] \subset \mathbb{R}_{\geqslant 0}$ and a polynomial function $p \in \Pi_l$ expressed as in (2.2), the following are alternative forms for $\underline{p}_I([\underline{x}, \overline{x}])$ and $\overline{p}_I([\underline{x}, \overline{x}])$*

$$\overline{p}_I([\underline{x}, \overline{x}]) = \sum_{i=0; a_i > 0}^{l} a_i \overline{x}^i + \sum_{i=0; a_i < 0}^{l} a_i \underline{x}^i \tag{2.22}$$

$$\underline{p}_I([\underline{x}, \overline{x}]) = \sum_{i=0; a_i < 0}^{l} a_i \overline{x}^i + \sum_{i=0; a_i > 0}^{l} a_i \underline{x}^i. \tag{2.23}$$

Proof. Consider the $i-$th term $a_i x^i$ of the polynomial function p, with $x \in [\underline{x}, \overline{x}]$ and $0 \leqslant i \leqslant l$. The following three cases can be studied:

- If $a_i > 0$, then
$$\max\{a_i \underline{x}^i, a_i \overline{x}^i\} = a_i \overline{x}^i; \tag{2.24}$$
- If $a_i = 0$, then
$$\max\{a_i \underline{x}^i, a_i \overline{x}^i\} = 0; \text{ and} \tag{2.25}$$
- If $a_i < 0$, then
$$\max\{a_i \underline{x}^i, a_i \overline{x}^i\} = a_i \underline{x}^i. \tag{2.26}$$

These cases easily prove (2.22). Similar considerations can be used to prove (2.23). □

The following observations are required in order to introduce the definition of the fourth considered bounding method for univariate polynomial functions over closed bounded intervals.

The monomial basis is not the only basis of Π_l and to ease the study of the signs of polynomial functions in bounded intervals, the interest is in an alternative basis. Such a basis uses so called *Bernstein polynomials* (sometimes called *Bernstein basis polynomials*) of degree l

$$B_i^{(l)}(x) = \binom{l}{i} x^i (1-x)^{l-i} \quad 0 \leqslant i \leqslant l. \tag{2.27}$$

It is easy to show that $\{B_i^{(l)}\}_{i=0}^{l} \subset \Pi_l$ and the following proposition shows that Bernstein polynomials form a basis for Π_l, called *Bernstein basis*. The proof of the proposition is constructive and it provides a means to express a polynomial function in the Bernstein basis once its expression in the monomial basis is known.

Proposition 2.5. *Given a polynomial function $p \in \Pi_l$ expressed in the monomial basis as*

$$p(x) = \sum_{i=0}^{l} a_i P_i(x), \tag{2.28}$$

where $\{a_i\}_{i=0}^{l} \subset \mathbb{R}$, then

$$p(x) = \sum_{i=0}^{l} b_i B_i^{(l)}(x), \tag{2.29}$$

where $\{b_i\}_{i=0}^l \subset \mathbb{R}$*, and*

$$b_i = \sum_{j=0}^{i} \frac{\binom{i}{j}}{\binom{l}{j}} a_j \quad 0 \leqslant i \leqslant l. \tag{2.30}$$

Proof. First, note that

$$p(x) = \sum_{j=0}^{l} a_j P_j(x) = \sum_{j=0}^{l} a_j x^j = \sum_{j=0}^{l} a_j x^j [x + (1-x)]^{l-j}, \tag{2.31}$$

which can be rewritten, using the binomial identity, as

$$\sum_{j=0}^{l} a_j x^j [x + (1-x)]^{l-j} = \sum_{j=0}^{l} a_j x^j \sum_{k=0}^{l-j} \binom{l-j}{k} x^{l-j-k} (1-x)^k. \tag{2.32}$$

Moreover,

$$\begin{aligned} \sum_{j=0}^{l} a_j x^j \sum_{k=0}^{l-j} \binom{l-j}{k} x^{l-j-k} (1-x)^k &= \sum_{j=0}^{l} \sum_{k=0}^{l-j} \binom{l-j}{k} a_j x^{l-k} (1-x)^k & (2.33) \\ &= \sum_{k=0}^{l} \sum_{j=0}^{l-k} \binom{l-j}{k} a_j x^{l-k} (1-x)^k, & (2.34) \end{aligned}$$

where the following identity, which holds for any $\{t_{j,k}\}_{j,k=0}^l \subset \mathbb{R}$, is used to obtain (2.34)

$$\sum_{j=0}^{l} \sum_{k=0}^{l-j} t_{j,k} = \sum_{k=0}^{l} \sum_{j=0}^{l-k} t_{j,k}. \tag{2.35}$$

Finally, introducing $i = l - k$

$$\begin{aligned} \sum_{k=0}^{l} \sum_{j=0}^{l-k} \binom{l-j}{k} a_j x^{l-k} (1-x)^k &= \sum_{i=0}^{l} \sum_{j=0}^{i} \binom{l-j}{l-i} a_j x^i (1-x)^{l-i} & (2.36) \\ &= \sum_{i=0}^{l} \sum_{j=0}^{i} \frac{\binom{l-j}{l-i}}{\binom{l}{i}} \binom{l}{i} a_j x^i (1-x)^{l-i} & (2.37) \\ &= \sum_{i=0}^{l} \left[\sum_{j=0}^{i} \frac{\binom{i}{j}}{\binom{l}{j}} a_j \right] B_i^{(l)}(x) & (2.38) \\ &= \sum_{i=0}^{l} b_i B_i^{(l)}(x), & (2.39) \end{aligned}$$

which prove the proposition. □

Proposition 2.5 relies solely on the definition of Bernstein polynomials and elementary algebraic manipulations, without invoking any of their specific properties. Furthermore, equation (2.36) expresses $p(x)$ as a linear combination of the following polynomials

$$x^i (1-x)^{l-i} = \frac{B_i^{(l)}(x)}{\binom{l}{i}} \quad 0 \leqslant i \leqslant l \tag{2.40}$$

with coefficients

$$\sum_{j=0}^{i} \binom{l-j}{i-j} a_j \quad 0 \leqslant i \leqslant l, \tag{2.41}$$

which ensures that the definition of Bernstein polynomials and ordinary algebraic manipulations are sufficient to express $p(x)$ in an alternative basis, closely related to the Bernstein basis.

The following proposition states two of the most important properties of Bernstein polynomials, which are then used to prove Proposition 2.6.

Lemma 2.1. *Given the set of Bernstein polynomials of degree l, $\{B_i^{(l)}\}_{i=0}^{l}$, the following hold*

$$\forall x \in [0,1], \forall i : 0 \leqslant i \leqslant l \qquad B_i^{(l)}(x) \geqslant 0 \tag{2.42}$$

$$\forall x \in \mathbb{R} \qquad \sum_{i=0}^{l} B_i^{(l)}(x) = 1. \tag{2.43}$$

Proof. The proof of (2.42) follows from the fact that, for all $x \in [0,1]$, Bernstein polynomials are sums of non-negative terms. The proof of (2.43) is an application of the binomial identity

$$\sum_{i=0}^{l} B_i^{(l)}(x) = \sum_{i=0}^{l} \binom{l}{i} x^i (1-x)^{l-i} = (x+1-x)^l = 1, \tag{2.44}$$

which prove the proposition. □

The following is a central property of polynomial functions expressed in the Bernstein basis, and it justifies the importance of the Bernstein basis in the study of the signs of polynomial functions in bounded intervals.

Proposition 2.6. *Let $p \in \Pi_l$ be a polynomial function expressed in the Bernstein basis as*

$$p(x) = \sum_{i=0}^{l} b_i B_i^{(l)}(x) \tag{2.45}$$

and let

$$\underline{p}_B([0,1]) = \min\{b_i\}_{i=0}^{l} \qquad \overline{p}_B([0,1]) = \max\{b_i\}_{i=0}^{l} \tag{2.46}$$

then

$$\forall x \in [0,1] \quad \underline{p}_B([0,1]) \leqslant p(x) \leqslant \overline{p}_B([0,1]). \tag{2.47}$$

Proof. Using only Lemma 2.1, for all $x \in [0,1]$

$$p(x) = \sum_{i=0}^{l} b_i B_i^{(l)}(x) \leqslant \overline{p}_B([0,1]) \sum_{i=0}^{l} B_i^{(l)}(x) = \overline{p}_B([0,1]) \tag{2.48}$$

$$\underline{p}_B([0,1]) = \underline{p}_B([0,1]) \sum_{i=0}^{l} B_i^{(l)}(x) \leqslant \sum_{i=0}^{l} b_i B_i^{(l)}(x) = p(x), \tag{2.49}$$

which prove the proposition. □

The proposition 2.6 shows an important property of the Bernstein basis, which can be used to study polynomial functions in the interval $[0,1]$. In order to study a polynomial function in a generic bounded interval $[\underline{x},\overline{x}] \subset \mathbb{R}$, the following affine change of variable is used

$$x = (\overline{x} - \underline{x})t + \underline{x} \tag{2.50}$$

to introduce a new variable t, and to turn the study of the polynomial function for $x \in [\underline{x}, \overline{x}]$ into the study of a related function for $t \in [0,1]$.

Proposition 2.7. *Given a bounded interval $D = [\underline{x}, \overline{x}] \subset \mathbb{R}$ and a polynomial function $p \in \Pi_l$ expressed in the monomial basis as*

$$p(x) = \sum_{i=0}^{l} a_i P_i(x), \tag{2.51}$$

where $\{a_i\}_{i=0}^{l} \subset \mathbb{R}$, let $p^{(D)} \in \Pi_l$ be the polynomial function obtained from p using the change of variable (2.50). $p^{(D)}$ can be expressed in the Bernstein basis using the following coefficients

$$b_i^{(D)} = \sum_{j=0}^{i} \frac{\binom{i}{j}}{\binom{l}{j}} (\overline{x} - \underline{x})^j \sum_{k=0}^{l} \binom{k}{j} \underline{x}^{k-j} a_k \quad 0 \leqslant i \leqslant l. \tag{2.52}$$

Proof (sketch). Starting from the expression of p in the monomial basis, the change of variables (2.50) is applied to obtain

$$p^{(D)}(t) = \sum_{i=0}^{l} a_i P_i((\overline{x} - \underline{x})t + \underline{x}) = \sum_{i=0}^{l} a_i [(\overline{x} - \underline{x})t + \underline{x}]^i. \tag{2.53}$$

Using the binomial identity to expand $[(\overline{x} - \underline{x})t + \underline{x}]^i$, the coefficients in the monomial basis of $p^{(D)}$ can be computed, which can be used to express $\{b_i^{(D)}\}_{i=0}^{l}$ using (2.30). $\square$

The coefficients $\{b_i^{(D)}\}_{i=0}^{l}$ appearing in Proposition 2.7 are commonly referred to as the *Bernstein coefficients* of the polynomial p on the interval $D = [\underline{x}, \overline{x}]$. One of their most important properties is that they can be directly used to bound the range of p in D, as follows.

Corollary 2.1. *Given a polynomial function $p \in \Pi_l$, a bounded interval $D = [\underline{x}, \overline{x}] \subset \mathbb{R}$, and $\{b_i^{(D)}\}_{i=0}^{l}$ as in Proposition 2.7*

$$\forall x \in D \quad \min\{b_i^{(D)}\}_{i=0}^{l} \leqslant p(x) \leqslant \max\{b_i^{(D)}\}_{i=0}^{l}. \tag{2.54}$$

Proof. The proof follows immediately from Proposition 2.6 and Proposition 2.7. $\square$

Let

$$\underline{p}_B([\underline{x}, \overline{x}]) = \min\{b_i^{(D)}\}_{i=0}^{l} \tag{2.55}$$

$$\overline{p}_B([\underline{x}, \overline{x}]) = \max\{b_i^{(D)}\}_{i=0}^{l} \tag{2.56}$$

then, by Corollary 2.1, it follows that

$$\forall x \in [\underline{x}, \overline{x}] \quad \underline{p}_B([\underline{x}, \overline{x}]) \leqslant p(x) \leqslant \overline{p}_B([\underline{x}, \overline{x}]). \tag{2.57}$$

2.2 Comparison in the Univariate Case

The following proposition ensures that, given a closed bounded interval $[\underline{x},\overline{x}] \subset \mathbb{R}_{\geqslant 0}$, the simple bounds are tighter than the absolute bounds.

Proposition 2.8. *Given a closed bounded interval $[\underline{x},\overline{x}] \subset \mathbb{R}_{\geqslant 0}$ and a polynomial function $p \in \Pi_l$ expressed as in (2.2), the simple bounds $\underline{p}_S([\underline{x},\overline{x}])$ and $\overline{p}_S([\underline{x},\overline{x}])$ are tighter than the absolute bounds $\underline{p}_A([\underline{x},\overline{x}])$ and $\overline{p}_A([\underline{x},\overline{x}])$, so for all $x \in [\underline{x},\overline{x}]$ the following inequalities hold*

$$\underline{p}_A([\underline{x},\overline{x}]) \leqslant \underline{p}_S([\underline{x},\overline{x}]) \leqslant p(x) \leqslant \overline{p}_S([\underline{x},\overline{x}]) \leqslant \overline{p}_A([\underline{x},\overline{x}]). \tag{2.58}$$

Proof. Given the definition of $\overline{p}_A([\underline{x},\overline{x}])$, in (2.3), and the definition of $\overline{p}_S([\underline{x},\overline{x}])$, in (2.9), therefore

$$\begin{aligned}
\overline{p}_A([\underline{x},\overline{x}]) &= \sum_{i=0}^{l} |a_i|\overline{x}^i && (2.59)\\
&\geqslant \sum_{i=0;a_i>0}^{l} a_i\overline{x}^i && (2.60)\\
&= \overline{p}_S([\underline{x},\overline{x}]) && (2.61)
\end{aligned}$$

because the number of positive addends in $\overline{p}_A([\underline{x},\overline{x}])$ is greater or equal than the one of $\overline{p}_S([\underline{x},\overline{x}])$ and so the bound $\overline{p}_S([\underline{x},\overline{x}])$ is tighter than the bound $\overline{p}_A[\underline{x},\overline{x}])$.

Similarly,

$$\begin{aligned}
\underline{p}_A([\underline{x},\overline{x}]) &= -\sum_{i=0}^{l} |a_i|\overline{x}^i && (2.62)\\
&\leqslant \sum_{i=0;a_i<0}^{l} a_i\overline{x}^i && (2.63)\\
&= \underline{p}_S([\underline{x},\overline{x}]) && (2.64)
\end{aligned}$$

because the number of negative addends in $\underline{p}_A([\underline{x},\overline{x}])$ is greater or equal than the one of $\underline{p}_S([\underline{x},\overline{x}])$ and so the bound $\underline{p}_S([\underline{x},\overline{x}])$ is tighter than the bound $\underline{p}_A[\underline{x},\overline{x}])$, which prove the proposition. □

The following proposition ensures that, given a closed bounded interval $[\underline{x},\overline{x}] \subset \mathbb{R}_{\geqslant 0}$, the interval bounds are tighter than the simple bounds.

Proposition 2.9. *Given a closed bounded interval $[\underline{x},\overline{x}] \subset \mathbb{R}_{\geqslant 0}$ and a polynomial function $p \in \Pi_l$ expressed as in (2.2), the interval bounds $\underline{p}_I([\underline{x},\overline{x}])$ and $\overline{p}_I([\underline{x},\overline{x}])$ are tighter than the simple bounds $\underline{p}_S([\underline{x},\overline{x}])$ and $\overline{p}_S([\underline{x},\overline{x}])$ because for all $x \in [\underline{x},\overline{x}]$ the following inequalities hold*

$$\underline{p}_S([\underline{x},\overline{x}]) \leqslant \underline{p}_I([\underline{x},\overline{x}]) \leqslant p(x) \leqslant \overline{p}_I([\underline{x},\overline{x}]) \leqslant \overline{p}_S([\underline{x},\overline{x}]). \tag{2.65}$$

Proof. Given the definition of $\overline{p}_S([\underline{x},\overline{x}])$, in (2.9), if $\overline{p}_S([\underline{x},\overline{x}]) = 0$, then $a_i \leqslant 0$ for all $0 \leqslant i \leqslant l$. Therefore, for Proposition 2.4,

$$
\begin{aligned}
\overline{p}_S([\underline{x},\overline{x}]) &= 0 && (2.66)\\
&\geqslant \sum_{i=0;a_i<0}^{l} a_i \underline{x}^i && (2.67)\\
&= \sum_{i=0}^{l} \max\{a_i\underline{x}^i, a_i\overline{x}^i\} && (2.68)\\
&= \overline{p}_I([\underline{x},\overline{x}]). && (2.69)
\end{aligned}
$$

Now, if $\overline{p}_S([\underline{x},\overline{x}]) \neq 0$, then there exists some $0 \leqslant j \leqslant l$ such that $a_j > 0$. If $a_i \leqslant 0$, then the contribution of $i-$th coefficient of the polynomial function is zero for $\overline{p}_S([\underline{x},\overline{x}])$ but it equals

$$\max\{a_i\underline{x}^i, a_i\overline{x}^i\} = a_i\underline{x}^i \leqslant 0 \tag{2.70}$$

for $\overline{p}_I([\underline{x},\overline{x}])$. Similarly, if $a_i > 0$, then

$$a_i\overline{x}^i = \max\{a_i\underline{x}^i, a_i\overline{x}^i\}. \tag{2.71}$$

Therefore,

$$\overline{p}_S([\underline{x},\overline{x}]) \geqslant \overline{p}_I([\underline{x},\overline{x}]), \tag{2.72}$$

and so the bound $\overline{p}_I([\underline{x},\overline{x}])$ is tighter than the bound $\overline{p}_S[\underline{x},\overline{x}])$. The proposition is proved using similar considerations for $\underline{p}_I([\underline{x},\overline{x}])$ and $\underline{p}_S([\underline{x},\overline{x}])$. □

The following proposition ensures that, given a closed bounded interval $[\underline{x},\overline{x}] \subset \mathbb{R}_{\geqslant 0}$, the Bernstein bounds are tighter than the interval bounds.

Proposition 2.10. *Given a closed bounded interval $[\underline{x},\overline{x}] \subset \mathbb{R}_{\geqslant 0}$ and a polynomial function $p \in \Pi_l$ expressed as in (2.2), the Bernstein bounds $\underline{p}_B([\underline{x},\overline{x}])$ and $\overline{p}_B([\underline{x},\overline{x}])$ are tighter than the interval bounds $\underline{p}_I([\underline{x},\overline{x}])$ and $\overline{p}_I([\underline{x},\overline{x}])$ because for all $x \in [\underline{x},\overline{x}]$ the following inequalities hold*

$$\underline{p}_I([\underline{x},\overline{x}]) \leqslant \underline{p}_B([\underline{x},\overline{x}]) \leqslant p(x) \leqslant \overline{p}_B([\underline{x},\overline{x}]) \leqslant \overline{p}_I([\underline{x},\overline{x}]). \tag{2.73}$$

Proof. Consider the j-th term of the Bernstein polynomial, with $0 \leqslant j \leqslant l$.

$$b_j^{(l)} = \sum_{i=0}^{j} \frac{\binom{j}{i}}{\binom{l}{i}} a_i, \tag{2.74}$$

for the equality

$$\frac{\binom{j}{0}}{\binom{l}{0}} = 1, \tag{2.75}$$

therefore,

$$b_j^{(l)} = \sum_{i=0}^{j} \frac{\binom{j}{i}}{\binom{l}{i}} a_i \tag{2.76}$$

$$= a_0 + \sum_{i=1}^{j} \frac{\binom{j}{i}}{\binom{l}{i}} a_i \tag{2.77}$$

$$\leqslant a_0 + \sum_{i=1;a_i>0}^{j} \frac{\binom{j}{i}}{\binom{l}{i}} a_i \tag{2.78}$$

$$\leqslant a_0 + \sum_{i=1;a_i>0}^{j} a_i \tag{2.79}$$

$$\leqslant a_0 + \sum_{i=1;a_i>0}^{l} a_i. \tag{2.80}$$

Now, the proof continues in two parts: the first on the interval $[0,1]$, the second on the generic interval $[\underline{x}, \overline{x}] \subset \mathbb{R}_{\geqslant 0}$. In $[0,1]$ the extrema of the interval are $\overline{x} = 1$ and $\underline{x} = 0$, so $\overline{x}^i = 1^i = 1$, therefore for the Definition 2.3 and for the Proposition 2.4

$$\overline{p}_I([0,1]) = \sum_{i=0;a_i>0}^{l} a_i \cdot \overline{x}^i + \sum_{i=0;a_i<0}^{l} a_i \cdot \underline{x}^i \tag{2.81}$$

$$= \sum_{i=0;a_i>0}^{l} a_i \cdot 1^i + \sum_{i=0;a_i<0}^{l} a_i \cdot 0^i \tag{2.82}$$

the second sum is equal to zero if $a_0 \geqslant 0$ and is equal to a_0 if $a_0 < 0$. So,

$$\overline{p}_I([0,1]) = \sum_{i=0;a_i>0}^{l} a_i \cdot 1^i + \sum_{i=0;a_i<0}^{l} a_i \cdot 0^i = \sum_{i=1;a_i>0}^{l} a_i + a_0 \tag{2.83}$$

which is the expression of the last chain of inequalities (2.76). Therefore,

$$b_j^{(l)} \leqslant \overline{p}_I([0,1]) \tag{2.84}$$

since j is arbitrary, calculate the maximum for j between 0 and l

$$\max_j b_j^{(l)} \leqslant \overline{p}_I([0,1]) \tag{2.85}$$

$$\overline{p}_B([0,1]) \leqslant \overline{p}_I([0,1]). \tag{2.86}$$

Similar considerations can be used to prove

$$\underline{p}_B([0,1]) \geqslant \underline{p}_I([0,1]). \tag{2.87}$$

Let $\overline{x} > \underline{x} > 0$, $D = [\underline{x}, \overline{x}]$ and $0 \leqslant i \leqslant l$. Consider the $i-$th coefficient of the Bernstein polynomials, for the expression (2.52) of the Proposition 2.7

$$b_i^{(D)} = \sum_{j=0}^{i} \frac{\binom{i}{j}}{\binom{l}{j}} (\overline{x} - \underline{x})^j \sum_{k=j}^{l} \binom{k}{j} \underline{x}^{k-j} a_k \quad 0 \leqslant i \leqslant l \tag{2.88}$$

So, under the assumption $\overline{x} > \underline{x} > 0$

$$b_i^{(D)} = \sum_{j=0}^{i} \frac{\binom{i}{j}}{\binom{l}{j}} (\overline{x}-\underline{x})^j \sum_{k=j}^{l} \binom{k}{j} \underline{x}^{k-j} a_k \tag{2.89}$$

$$= \sum_{j=0}^{0} \frac{\binom{i}{j}}{\binom{l}{j}} (\overline{x}-\underline{x})^j \sum_{k=j}^{l} \binom{k}{j} \underline{x}^{k-j} a_k + \sum_{j=1}^{i} \frac{\binom{i}{j}}{\binom{l}{j}} (\overline{x}-\underline{x})^j \sum_{k=j}^{l} \binom{k}{j} \underline{x}^{k-j} a_k \tag{2.90}$$

$$= \sum_{k=0}^{l} \underline{x}^k a_k + \sum_{k=1}^{l} a_k \sum_{j=1;j\leqslant k}^{i} \frac{\binom{i}{j}}{\binom{l}{j}} (\overline{x}-\underline{x})^j \binom{k}{j} \underline{x}^{k-j} \tag{2.91}$$

$$= \sum_{k=0;a_k<0}^{l} \underline{x}^k a_k + \sum_{k=0;a_k>0}^{l} \underline{x}^k a_k \tag{2.92}$$

$$+ \sum_{k=1;a_k<0}^{l} a_k \sum_{j=1;j\leqslant k}^{i} \frac{\binom{i}{j}}{\binom{l}{j}} (\overline{x}-\underline{x})^j \binom{k}{j} \underline{x}^{k-j} \tag{2.93}$$

$$+ \sum_{k=1;a_k>0}^{l} a_k \sum_{j=1;j\leqslant k}^{i} \frac{\binom{i}{j}}{\binom{l}{j}} (\overline{x}-\underline{x})^j \binom{k}{j} \underline{x}^{k-j} \tag{2.94}$$

$$= \sum_{k=0;a_k<0}^{l} \underline{x}^k a_k \tag{2.95}$$

$$+ \sum_{k=1;a_k<0}^{l} a_k \sum_{j=1;j\leqslant k}^{i} \frac{\binom{i}{j}}{\binom{l}{j}} (\overline{x}-\underline{x})^j \binom{k}{j} \underline{x}^{k-j} \tag{2.96}$$

$$+ \sum_{k=0;a_k>0}^{l} a_k \sum_{j=0;j\leqslant k}^{i} \frac{\binom{i}{j}}{\binom{l}{j}} (\overline{x}-\underline{x})^j \binom{k}{j} \underline{x}^{k-j} \tag{2.97}$$

$$\leqslant \sum_{k=0;a_k<0}^{l} \underline{x}^k a_k \tag{2.98}$$

$$+ \sum_{k=1;a_k<0}^{l} a_k \sum_{j=1;j\leqslant k}^{i} \frac{\binom{i}{j}}{\binom{l}{j}} (\overline{x}-\underline{x})^j \binom{k}{j} \underline{x}^{k-j} \tag{2.99}$$

$$+ \sum_{k=0;a_k>0}^{l} a_k \sum_{j=0;j\leqslant k}^{i} (\overline{x}-\underline{x})^j \binom{k}{j} \underline{x}^{k-j} \tag{2.100}$$

because the ratio between the binomial coefficients is less than or equal to 1. Moreover,

$$\sum_{k=0;a_k<0}^{l} \underline{x}^k a_k \tag{2.101}$$

$$+ \sum_{k=1;a_k<0}^{l} a_k \sum_{j=1;j\leqslant k}^{i} \frac{\binom{i}{j}}{\binom{l}{j}} (\overline{x}-\underline{x})^j \binom{k}{j} \underline{x}^{k-j} \tag{2.102}$$

$$+ \sum_{k=0;a_k>0}^{l} a_k \sum_{j=0;j\leqslant k}^{i} (\overline{x}-\underline{x})^j \binom{k}{j} \underline{x}^{k-j} \tag{2.103}$$

(2.104)

$$\leqslant \sum_{k=0;a_k<0}^{l} \underline{x}^k a_k \tag{2.105}$$

$$+ \sum_{k=1;a_k<0}^{l} a_k \sum_{j=1;j\leqslant k}^{i} \frac{\binom{i}{j}}{\binom{l}{j}} (\overline{x}-\underline{x})^j \binom{k}{j} \underline{x}^{k-j} \tag{2.106}$$

$$+ \sum_{k=0;a_k>0}^{l} a_k \sum_{j=0}^{k} (\overline{x}-\underline{x})^j \binom{k}{j} \underline{x}^{k-j} \tag{2.107}$$

because the number of addends of the sum with index j is less than or equal to i and k, so it is less than or equal to k. So,

$$\sum_{k=0;a_k<0}^{l} \underline{x}^k a_k \tag{2.108}$$

$$+ \sum_{k=1;a_k<0}^{l} a_k \sum_{j=1;j\leqslant k}^{i} \frac{\binom{i}{j}}{\binom{l}{j}} (\overline{x}-\underline{x})^j \binom{k}{j} \underline{x}^{k-j} \tag{2.109}$$

$$+ \sum_{k=0;a_k>0}^{l} a_k \sum_{j=0}^{k} (\overline{x}-\underline{x})^j \binom{k}{j} \underline{x}^{k-j} \tag{2.110}$$

$$= \sum_{k=0;a_k<0}^{l} \underline{x}^k a_k \tag{2.111}$$

$$+ \sum_{k=1;a_k<0}^{l} a_k \sum_{j=1;j\leqslant k}^{i} \frac{\binom{i}{j}}{\binom{l}{j}} (\overline{x}-\underline{x})^j \binom{k}{j} \underline{x}^{k-j} \tag{2.112}$$

$$+ \sum_{k=0;a_k>0}^{l} a_k \overline{x}^k \tag{2.113}$$

for the binomial theorem. In addition,

$$\sum_{k=0;a_k<0}^{l} \underline{x}^k a_k \tag{2.114}$$

$$+ \sum_{k=1;a_k<0}^{l} a_k \sum_{j=1;j\leqslant k}^{i} \frac{\binom{i}{j}}{\binom{l}{j}} (\overline{x}-\underline{x})^j \binom{k}{j} \underline{x}^{k-j} \tag{2.115}$$

$$+ \sum_{k=0;a_k>0}^{l} a_k \overline{x}^k \tag{2.116}$$

$$\leqslant \sum_{k=0;a_k<0}^{l} \underline{x}^k a_k + \sum_{k=0;a_k>0}^{l} a_k \overline{x}^k \tag{2.117}$$

$$= \overline{p}_I([\underline{x},\overline{x}]). \tag{2.118}$$

Therefore,

$$b_i^{(D)} \leqslant \overline{p}_I([\underline{x},\overline{x}]) \tag{2.119}$$

since i is arbitrary, calculate the maximum for i between 0 and l

$$\max_i b_i^{(D)} \leqslant \overline{p}_I([\underline{x},\overline{x}]). \tag{2.120}$$

State with $\overline{p}_B([\underline{x},\overline{x}])$ the quantity $\max_i b_i^{(D)}$, so

$$\overline{p}_B([\underline{x},\overline{x}]) \leqslant \overline{p}_I([\underline{x},\overline{x}]). \tag{2.121}$$

Similarly, the comparison between the minimum $\min_i b_i^{(D)}$ and $\underline{p}_I([\underline{x},\overline{x}])$, so

$$\min_i b_i^{(D)} = \underline{p}_B([\underline{x},\overline{x}]) \geqslant \underline{p}_I([\underline{x},\overline{x}]). \tag{2.122}$$

Similar considerations can be used to prove in case of $\underline{x} = 0$. □

Summarizing, in a closed interval $[\underline{x},\overline{x}] \subset \mathbb{R}_{\geqslant 0}$

$$\underline{p}_A([\underline{x},\overline{x}]) \leqslant \underline{p}_S([\underline{x},\overline{x}]) \leqslant \underline{p}_I([\underline{x},\overline{x}]) \leqslant \underline{p}_B([\underline{x},\overline{x}]) \tag{2.123}$$

$$\leqslant p(x) \leqslant \overline{p}_B([\underline{x},\overline{x}]) \leqslant \overline{p}_I([\underline{x},\overline{x}]) \leqslant \overline{p}_S([\underline{x},\overline{x}]) \leqslant \overline{p}_A([\underline{x},\overline{x}]). \tag{2.124}$$

2.3 Illustrative Examples

The following examples are provided solely to illustrate the results. They do not constitute exhaustive tests.

2.3.1 Example with a polynomial of second degree

Consider the closed bounded interval $[1,10] \subset \mathbb{R}$ and the polynomial function

$$f(x) = -x^2 + 10x + 5. \tag{2.125}$$

The considered polynomial function represents a parabola with the vertex at $(5,30)$ in the Cartesian plane. At the extrema of the considered interval, the considered function evaluates as follows

$$f(1) = 14 \qquad f(10) = 5. \tag{2.126}$$

The parabola opens downward, so in $[1,10]$ the function reaches its maximum value 30 and its minimum value 5.

The upper bound $\overline{f}_A([1,10])$ and the lower bound $\underline{f}_A([1,10])$ are

$$\overline{f}_A([1,10]) = |-1| \cdot (10)^2 + |10| \cdot 10 + |5| = 205 \tag{2.127}$$

and

$$\underline{f}_A([1,10]) = -(|-1| \cdot (10)^2 + |10| \cdot 10 + |5|) = -205 \tag{2.128}$$

Figure 2.1(a) shows the plot of the considered function in the considered interval. The figure also shows the simple lower bound $\overline{f}_A([1,10])$ and the simple upper bound $\underline{f}_A([1,10])$.

From the definitions of the simple bounds, since there exist $0 \leq i \leq l$ and $0 \leq j \leq l$ such that $a_i \geq 0$ and $a_j \leq 0$, it follows that the bounds $\overline{f}_S([1,10])$ and $\underline{f}_S([1,10])$ are

$$\overline{f}_S([1,10]) = 10 \cdot 10 + 5 = 105 \tag{2.129}$$

and

$$\underline{f}_S([1,10]) = -1 \cdot (10)^2 = -100. \tag{2.130}$$

Figure 2.1(b) shows the plot of the considered function in the considered interval. The figure also shows the simple lower bound $\overline{f}_S([1,10])$ and the simple upper bound $\underline{f}_S([1,10])$.

The upper bound $\overline{f}_I([1,10])$ and the lower bound $\underline{f}_I([1,10])$ are

$$\overline{f}_I([1,10]) = -1 \cdot (1)^2 + 10 \cdot 10 + 5 = 104 \tag{2.131}$$

and

$$\underline{f}_I([1,10]) = -1 \cdot (10)^2 + 10 \cdot 1 + 5 = -85. \tag{2.132}$$

Figure 2.1(c) shows the plot of the considered function in the considered interval. The figure also shows the interval lower bound $\overline{f}_I([1,10])$ and the interval upper bound $\underline{f}_I([1,10])$. Note that, as expected, the interval bounds are tighter than the simple bounds.

From the definitions of the Bernstein bounds, the coefficients are

$$b_0^{(2)} = f(1) = 14 \tag{2.133}$$

$$b_1^{(2)} = 50 \tag{2.134}$$

$$b_2^{(2)} = f(10) = 5 \tag{2.135}$$

therefore,

$$\overline{f}_B([1,10]) = max_i b_i^{(D)} = 5 \tag{2.136}$$

$$\underline{f}_B([1,10]) = min_i b_i^{(D)} = 50. \tag{2.137}$$

Figure 2.1(d) shows the plot of the considered function in the considered interval. The figure also shows the interval lower bound $\overline{f}_B([1,10])$ and the interval upper bound $\underline{f}_B([1,10])$. Note that, as expected, the Bernstein bounds are tighter than the interval bounds.

Table 2.1: Report of the computation times in seconds obtained in MATLAB for the different bounds.

Absolute	Simple	Interval	Bernstein
$9.3 \cdot 10^{-5}$ s	$1.09 \cdot 10^{-4}$ s	$1.39 \cdot 10^{-4}$ s	$9.66 \cdot 10^{-4}$ s

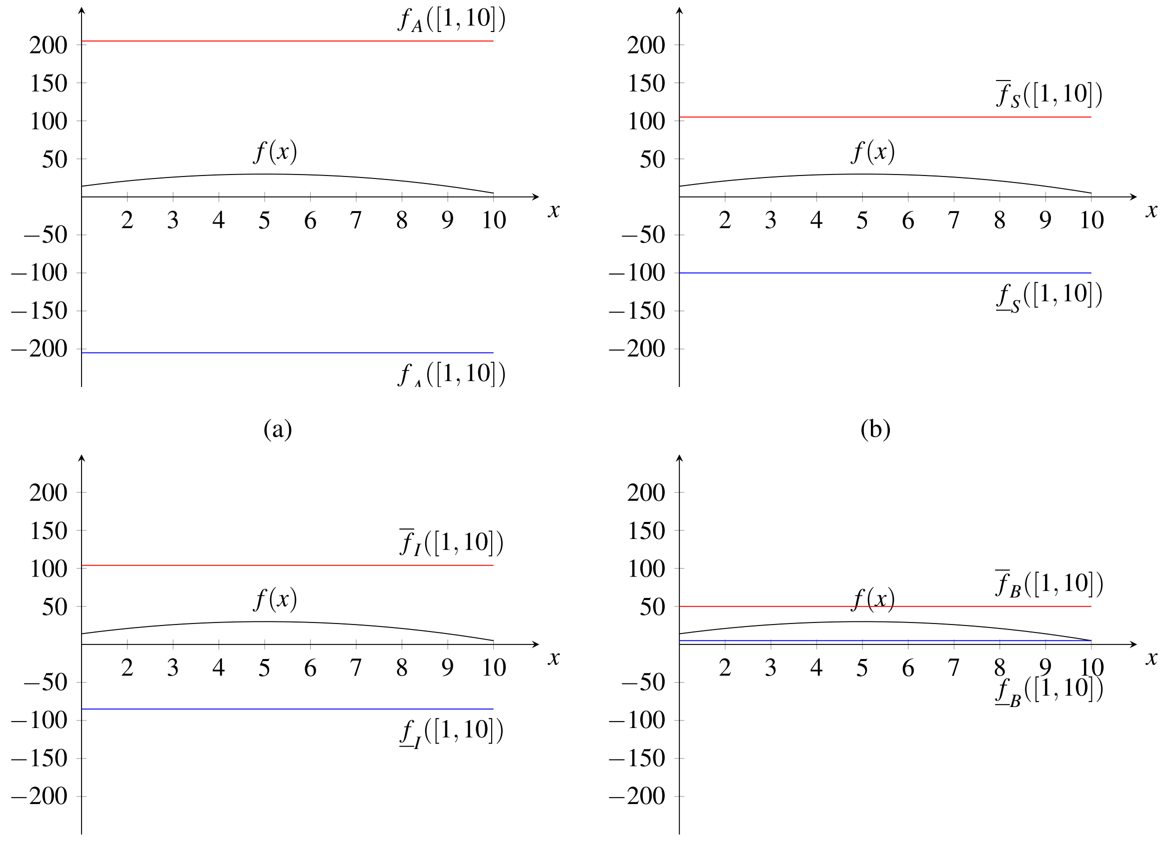


Figure 2.1: For $f(x) = -x^2 + 10x + 5$, (a) absolute bounds over $[1,10]$ (red and blue lines), (b) simple bounds over $[1,10]$ (red and blue lines), (c) interval bounds over $[1,10]$ (red and blue lines) and (d) Bernstein bounds over $[1,10]$ (red and blue lines).

2.3.2 Example with a polynomial of forth degree

Consider the closed bounded interval $[5,6] \subset \mathbb{R}$ and the polynomial function

$$f(x) = (x-4)(x-5)(x-6)(x-7) = x^4 - 22x^3 + 179x^2 - 638x + 840. \tag{2.138}$$

The considered polynomial function represents a quartic function. At the extrema of the considered interval, the considered function evaluates as follows

$$f(5) = 0 \qquad f(6) = 0. \tag{2.139}$$

In the considered interval, so in $[5,6]$ the function reaches its maximum value $9/16 = 0.5625$ and its minimum value 0.

The upper bound $\overline{f}_A([5,6])$ and the lower bound $\underline{f}_A([5,6])$ are

$$\overline{f}_A([5,6]) = (6)^4 + |-22| \cdot (6)^3 + |179| \cdot (6)^2 + |-638| \cdot (6) + |840| = 17160 \tag{2.140}$$

and

$$\underline{f}_A([5,6]) = -((6)^4 + |-22| \cdot (6)^3 + 179 \cdot (6)^2 + |-638| \cdot (6) + |840|) = -17160 \qquad (2.141)$$

Figure 2.2(a) shows the plot of the considered function in the considered interval. The figure also shows the simple lower bound $\overline{f}_A([5,6])$ and the simple upper bound $\underline{f}_A([5,6])$.

From the definitions of the simple bounds, since there exist $0 \le i \le l$ and $0 \le j \le l$ such that $a_i \ge 0$ and $a_j \le 0$, it follows that the bounds $\overline{f}_S([5,6])$ and $\underline{f}_S([5,6])$ are

$$\overline{f}_S([5,6]) = (6)^4 + 179 \cdot (6)^2 + 840 = 8580 \qquad (2.142)$$

and

$$\underline{f}_S([5,6]) = -22 \cdot (6)^3 - 638 \cdot (6) = -8.580. \qquad (2.143)$$

Figure 2.2(b) shows the plot of the considered function in the considered interval. The figure also shows the simple lower bound $\overline{f}_S([5,6])$ and the simple upper bound $\underline{f}_S([5,6])$.

The upper bound $\overline{f}_I([5,6])$ and the lower bound $\underline{f}_I([5,6])$ are

$$\overline{f}_I([5,6]) = (6)^4 - 22 \cdot (5)^3 + 179 \cdot (6)^2 - 638 \cdot (5) + 840 = 2640 \qquad (2.144)$$

and

$$\underline{f}_I([5,6]) = (5)^4 - 22 \cdot (6)^3 + 179 \cdot (5)^2 - 638 \cdot (6) + 840 = -2640. \qquad (2.145)$$

Figure 2.2(c) shows the plot of the considered function in the considered interval. The figure also shows the interval lower bound $\overline{f}_I([5,6])$ and the interval upper bound $\underline{f}_I([5,6])$. Note that, as expected, the interval bounds are tighter than the simple bounds.

From the definitions of the Bernstein bounds, the coefficients are

$$b_0^{(4)} = f(5) = 0 \qquad (2.146)$$

$$b_1^{(4)} = 0.5 \qquad (2.147)$$

$$b_2^{(4)} = \frac{5}{6} = 0.8\overline{3} \qquad (2.148)$$

$$b_3^{(4)} = 0.5 \qquad (2.149)$$

$$b_4^{(4)} = f(6) = 0 \qquad (2.150)$$

therefore, the bounds are

$$\overline{f}_B([5,6]) = max_i b_i^{(D)} = \frac{5}{6} = 0.8\overline{3} \qquad (2.151)$$

$$\underline{f}_B([5,6]) = min_i b_i^{(D)} = 0. \qquad (2.152)$$

Figure 2.2(d) shows the plot of the considered function in the considered interval. The figure also shows the interval lower bound $\overline{f}_B([5,6])$ and the interval upper bound $\underline{f}_B([5,6])$. Note that, as expected, the Bernstein bounds are tighter than the interval bounds.

Table 2.2: Report of the computation times in seconds obtained in MATLAB for the different bounds.

Absolute	Simple	Interval	Bernstein
$1.977 \cdot 10^{-3}$ s	$1.918 \cdot 10^{-3}$ s	$2.395 \cdot 10^{-3}$ s	$2.649 \cdot 10^{-3}$ s

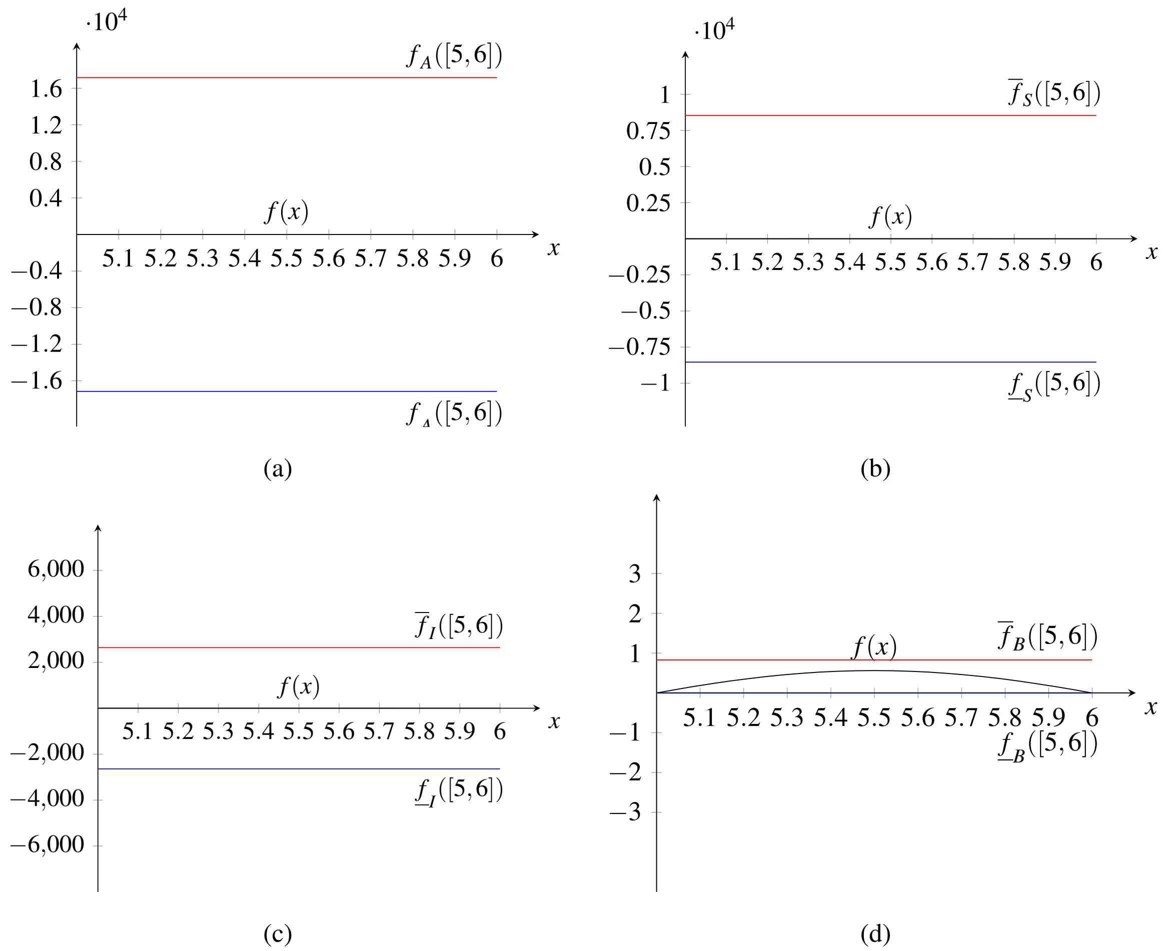


Figure 2.2: For $f(x) = x^4 - 22x^3 + 179x^2 - 638x + 840$, (a) absolute bounds over $[5,6]$ with $\overline{f}_A([5,6]) = 17160$ and $\underline{f}_A([5,6]) = -17160$ (red and blue lines), (b) simple bounds over $[5,6]$ with $\overline{f}_S([5,6]) = 8540$ and $\underline{f}_S([5,6]) = -8580$ (red and blue lines), (c) interval bounds over $[5,6]$ with $\overline{f}_I([5,6]) = 2640$ and $\underline{f}_I([5,6]) = -2640$ (red and blue lines) and (d) Bernstein bounds over $[5,6]$ with $\overline{f}_B([5,6]) = 0.8\overline{3}$ and $\underline{f}_B([5,6]) = 0$ (red and blue lines).

Chapter 3

Bounds in the Multivariate Case

The purpose of this chapter is to extend the results obtained on $\mathbb{R}_{\geqslant 0}$ to the closed boxes in $\mathbb{R}^n_{\geqslant 0}$. To do this, it will first be necessary to introduce a new notation, then provide the definitions of the bounds and compare these bounds with each other. Finally, the asymptotic computation times of the various discussed bounds will be calculated.

3.1 Definitions in the Multivariate Case

The extension of the results obtained in the previous chapter for real polynomial functions of one variable to real polynomial functions of $n \in \mathbb{N}_+$ real variables $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ requires a specific notation [15]. A *multi-index* $I \in \mathbb{N}^n$ is an n-tuple of indices

$$I = (i_1, i_2, \ldots, i_n), \tag{3.1}$$

where the common notation of using an uppercase letter to denote a multi-index, and the same letter, in lowercase with a subscript, for its components is adopted in the rest of this Thesis. Given two multi-indices $I \in \mathbb{N}^n$ and $J \in \mathbb{N}^n$, they can be partially ordered as follows

$$J \leqslant I \iff j_1 \leqslant i_1 \wedge j_2 \leqslant i_2 \wedge \cdots \wedge j_n \leqslant i_n. \tag{3.2}$$

Two multi-indices $I \in \mathbb{N}^n$ and $J \in \mathbb{N}^n$ can be added and subtracted

$$\begin{aligned} I + J &= (i_1 + j_1, i_2 + j_2, \ldots, i_n + j_n) && (3.3)\\ I - J &= (i_1 - j_1, i_2 - j_2, \ldots, i_n - j_n) \quad \text{with } J \leqslant I. && (3.4) \end{aligned}$$

Finally, the following abbreviations are allowed for multi-indices $I \in \mathbb{N}^n$ and $J \in \mathbb{N}^n$, and for $\mathbf{v} \in \mathbb{R}^n$ with $\mathbf{v} = (v_1, v_2, \dots, v_n)$

$$\mathbf{v}^I = \prod_{k=1}^{n} v_k^{i_k} \tag{3.5}$$

$$\sum_{I \leqslant J} (\cdot) = \sum_{i_1=0}^{j_1} \sum_{i_2=0}^{j_2} \cdots \sum_{i_n=0}^{j_n} (\cdot). \tag{3.6}$$

Using the notation of multi-indices, a real polynomial function $p(\mathbf{x})$ of several real variables can be expressed as

$$p(\mathbf{x}) = \sum_{I \leqslant L} a_I \mathbf{x}^I, \tag{3.7}$$

where $\{a_I\}_{I \leqslant L} \subset \mathbb{R}$ and multi-index L is called *multi-degree* of $p(\mathbf{x})$. With the notation of multi-indices, Π_L, the vector space of real polynomial functions of n real variables with real coefficients and multi-degree less than or equal to multi-index L, is

$$\Pi_L = \mathrm{span}_{\mathbb{R}} \{P_I\}_{I \leqslant L} \qquad P_I(\mathbf{x}) = \mathbf{x}^I \qquad I \leqslant L. \tag{3.8}$$

Note that $\{P_I\}_{I \leqslant L}$ is the monomial basis of Π_L.

The notation of multi-indices is introduced to compactly express real polynomial functions. The following notation for boxes is adopted to compactly generalize absolute, simple, interval and Bernstein bounds, as defined in the previous section for the univariate case. Given $\underline{\mathbf{x}} \in \mathbb{R}^n$ and $\overline{\mathbf{x}} \in \mathbb{R}^n$, the closed bounded box $[\underline{\mathbf{x}}, \overline{\mathbf{x}}]$ is defined as

$$[\underline{\mathbf{x}}, \overline{\mathbf{x}}] = \bigtimes_{i=1}^{n} [\min\{\underline{x}_i, \overline{x}_i\}, \max\{\underline{x}_i, \overline{x}_i\}]. \tag{3.9}$$

With no loss of generality, in the rest of this Thesis it is assumed that $\underline{\mathbf{x}}$ and $\overline{\mathbf{x}}$ are chosen such that $\underline{x}_i \leqslant \overline{x}_i$ for all $1 \leqslant i \leqslant n$. So,

$$[\underline{\mathbf{x}}, \overline{\mathbf{x}}] = \bigtimes_{i=1}^{n} [\overline{x}_i, \underline{x}_i]. \tag{3.10}$$

With the notation introduced, it is now possible to define the bounds for the closed bounded boxes in $\mathbb{R}^n_{\geqslant 0}$.

Definition 3.1. *Given a closed bound box* $[\underline{\mathbf{x}}, \overline{\mathbf{x}}] \subset \mathbb{R}^n_{\geqslant 0}$ *and a polynomial function* $p \in \Pi_L$ *expressed as in (3.7), let*

$$\overline{p}_A([\underline{\mathbf{x}}, \overline{\mathbf{x}}]) = \sum_{I \leqslant L} |a_I| \overline{\mathbf{x}}^I \tag{3.11}$$

and

$$\underline{p}_A([\underline{\mathbf{x}}, \overline{\mathbf{x}}]) = -\sum_{I \leqslant L} |a_I| \overline{\mathbf{x}}^I, \tag{3.12}$$

where, as usual, the sums evaluate to zero if no addend is usable. The value $\underline{p}_A([\underline{\mathbf{x}}, \overline{\mathbf{x}}])$ *is called* (multivariate) absolute lower bound *of p over* $[\underline{\mathbf{x}}, \overline{\mathbf{x}}]$ *and the value* $\overline{p}_A([\underline{\mathbf{x}}, \overline{\mathbf{x}}])$ *is called* (multivariate) absolute upper bound *of p over* $[\underline{\mathbf{x}}, \overline{\mathbf{x}}]$.

Note that, independently of the considered function and interval, $\underline{p}_A([\underline{\mathbf{x}},\overline{\mathbf{x}}]) \leqslant 0$ because it is the sum of non-positive addends. Notably, it is zero only if all the terms of the polynomial function are zero, i.e. p is the zero polynomial. Similarly, $\overline{p}_A([\underline{\mathbf{x}},\overline{\mathbf{x}}]) \geqslant 0$ because it is the sum of non-negative addends, and it is zero only if all the terms of the polynomial function are zero.

The following proposition motivates the names, absolute bounds, chosen for the two bounds discussed in Definition 3.1.

Proposition 3.1. *Given a closed bounded box $[\underline{\mathbf{x}},\overline{\mathbf{x}}] \subset \mathbb{R}^n_{\geqslant 0}$ and a polynomial function $p \in \Pi_L$ expressed as in (3.7), then for all $\mathbf{x} \in [\underline{\mathbf{x}},\overline{\mathbf{x}}]$*

$$\underline{p}_A([\underline{\mathbf{x}},\overline{\mathbf{x}}]) \leqslant p(\mathbf{x}) \leqslant \overline{p}_A([\underline{\mathbf{x}},\overline{\mathbf{x}}]). \tag{3.13}$$

Proof. Note that for $\mathbf{x} \in [\underline{\mathbf{x}},\overline{\mathbf{x}}]$, $\mathbf{x}^I \geqslant 0$ for all multi-indices $I \in \mathbb{N}^n$ because $[\underline{\mathbf{x}},\overline{\mathbf{x}}] \subset \mathbb{R}^n_{\geqslant 0}$. Therefore, the proposition is proved because of Proposition 2.1. □

Definition 3.2. *Given a closed bound box $[\underline{\mathbf{x}},\overline{\mathbf{x}}] \subset \mathbb{R}^n_{\geqslant 0}$ and a polynomial function $p \in \Pi_L$ expressed as in (3.7), let*

$$\overline{p}_S([\underline{\mathbf{x}},\overline{\mathbf{x}}]) = \sum_{I \leqslant L; a_I > 0}^{l} a_I \overline{\mathbf{x}}^I \tag{3.14}$$

and

$$\underline{p}_S([\underline{\mathbf{x}},\overline{\mathbf{x}}]) = \sum_{I \leqslant L; a_I < 0}^{l} a_I \overline{\mathbf{x}}^I, \tag{3.15}$$

where, as usual, the sums evaluate to zero if no addend is usable. The value $\underline{p}_S([\underline{\mathbf{x}},\overline{\mathbf{x}}])$ is called (multivariate) simple lower bound *of p over $[\underline{\mathbf{x}},\overline{\mathbf{x}}]$ and the value $\overline{p}_S([\underline{\mathbf{x}},\overline{\mathbf{x}}])$ is called* (multivariate) simple upper bound *of p over $[\underline{\mathbf{x}},\overline{\mathbf{x}}]$.*

Note that, independently of the considered function and interval, $\underline{p}_S([\underline{\mathbf{x}},\overline{\mathbf{x}}]) \leqslant 0$ because it is the sum of non-positive addends. Notably, it is zero only if all the terms of the polynomial function are non-negative. Similarly, $\overline{p}_S([\underline{\mathbf{x}},\overline{\mathbf{x}}]) \geqslant 0$ because it is the sum of non-negative addends, and it is zero only if all the terms of the polynomial function are non-positive.

The following proposition motivates the names, simple bounds, chosen for the two bounds discussed in Definition 3.2.

Proposition 3.2. *Given a closed bounded box $[\underline{\mathbf{x}},\overline{\mathbf{x}}] \subset \mathbb{R}^n_{\geqslant 0}$ and a polynomial function $p \in \Pi_L$ expressed as in (3.7), then for all $\mathbf{x} \in [\underline{\mathbf{x}},\overline{\mathbf{x}}]$*

$$\underline{p}_S([\underline{\mathbf{x}},\overline{\mathbf{x}}]) \leqslant p(\mathbf{x}) \leqslant \overline{p}_S([\underline{\mathbf{x}},\overline{\mathbf{x}}]). \tag{3.16}$$

Proof. Note that for $\mathbf{x} \in [\underline{\mathbf{x}},\overline{\mathbf{x}}]$, $\mathbf{x}^I \geqslant 0$ for all multi-indices $I \in \mathbb{N}^n$ because $[\underline{\mathbf{x}},\overline{\mathbf{x}}] \subset \mathbb{R}^n_{\geqslant 0}$. Therefore, the proposition is proved due to Proposition 2.2. □

Definition 3.3. *Given a closed bounded box* $[\underline{\mathbf{x}}, \overline{\mathbf{x}}] \subset \mathbb{R}^n_{\geqslant 0}$ *and a polynomial function* $p \in \Pi_L$ *expressed as in (3.7), let*

$$\overline{p}_I([\underline{\mathbf{x}}, \overline{\mathbf{x}}]) = \sum_{I \leqslant L} \max\{a_I \underline{\mathbf{x}}^I, a_I \overline{\mathbf{x}}^I\} \tag{3.17}$$

$$\underline{p}_I([\underline{\mathbf{x}}, \overline{\mathbf{x}}]) = \sum_{I \leqslant L} \min\{a_I \underline{\mathbf{x}}^I, a_I \overline{\mathbf{x}}^I\} \tag{3.18}$$

The value $\underline{p}_I([\underline{\mathbf{x}}, \overline{\mathbf{x}}])$ *is called* (multivariate) interval lower bound *of p over* $[\underline{\mathbf{x}}, \overline{\mathbf{x}}]$ *and the value* $\overline{p}_I([\underline{\mathbf{x}}, \overline{\mathbf{x}}])$ *is called* (multivariate) interval upper bound *of p over* $[\underline{\mathbf{x}}, \overline{\mathbf{x}}]$.

The following proposition provides an alternative way to compute the interval bounds.

Proposition 3.3. *Given a closed bounded box* $[\underline{\mathbf{x}}, \overline{\mathbf{x}}] \subset \mathbb{R}^n_{\geqslant 0}$ *and a polynomial function* $p \in \Pi_L$ *expressed as in (3.7), the following hold*

$$\overline{p}_I([\underline{\mathbf{x}}, \overline{\mathbf{x}}]) = \sum_{I \leqslant L; a_I > 0} a_I \overline{\mathbf{x}}^I + \sum_{I \leqslant L; a_I < 0} a_I \underline{\mathbf{x}}^I \tag{3.19}$$

$$\underline{p}_I([\underline{\mathbf{x}}, \overline{\mathbf{x}}]) = \sum_{I \leqslant L; a_I < 0} a_I \overline{\mathbf{x}}^I + \sum_{I \leqslant L; a_I > 0} a_I \underline{\mathbf{x}}^I. \tag{3.20}$$

Proof. The right-hand side of (3.19) is

$$\sum_{I \leqslant L; a_I > 0} a_I \overline{\mathbf{x}}^I + \sum_{I \leqslant L; a_I < 0} a_I \underline{\mathbf{x}}^I, \tag{3.21}$$

while the right-hand side of the definition of $\overline{p}_I([\underline{\mathbf{x}}, \overline{\mathbf{x}}])$ is

$$\sum_{I \leqslant L} \max\{a_I \underline{\mathbf{x}}^I, a_I \overline{\mathbf{x}}^I\}. \tag{3.22}$$

Now, given $I \leqslant L$, the study of the coefficient a_I of the polynomial function p degenerates in these three cases:

- If $a_I > 0$, then
$$\max\{a_I \underline{\mathbf{x}}^I, a_I \overline{\mathbf{x}}^I\} = a_I \overline{\mathbf{x}}^I; \tag{3.23}$$

- If $a_I = 0$, then
$$\max\{a_I \underline{\mathbf{x}}^I, a_I \overline{\mathbf{x}}^I\} = 0; \text{ and} \tag{3.24}$$

- If $a_I < 0$, then
$$\max\{a_I \underline{\mathbf{x}}^I, a_I \overline{\mathbf{x}}^I\} = a_I \underline{\mathbf{x}}^I. \tag{3.25}$$

Therefore, the I-th coefficient a_I of the polynomial function gives the same contribution to both sides of (3.19), which proves the proposition for $\overline{p}_I([\underline{\mathbf{x}}, \overline{\mathbf{x}}])$. Similar considerations can be used to prove the proposition for $\underline{p}_I([\underline{\mathbf{x}}, \overline{\mathbf{x}}])$. □

Note that $\{P_I\}_{I\leqslant L}$ is the monomial basis of Π_L. In order to introduce the Bernstein basis of Π_L, Bernstein polynomials with n variables are defined as follows

$$B_I^{(L)}(\mathbf{x}) = \prod_{k=0}^{n} B_{i_k}^{(l_k)}(x_k) \quad I \leqslant L, \tag{3.26}$$

where I and L are multi-indices. It is easy to show that $\{B_I^{(L)}\}_{I\leqslant L} \subset \Pi_L$. The following proposition, which generalizes Proposition 2.5, shows that Bernstein polynomials are a basis for Π_L, normally called the Bernstein basis.

Proposition 3.4. *Given a polynomial function $p \in \Pi_L$ expressed in the monomial basis as*

$$p(\mathbf{x}) = \sum_{I\leqslant L} a_I P_I(\mathbf{x}), \tag{3.27}$$

where $\{a_I\}_{I\leqslant L} \subset \mathbb{R}$, then

$$p(\mathbf{x}) = \sum_{I\leqslant L} b_I B_I^{(L)}(\mathbf{x}), \tag{3.28}$$

where $\{b_I\}_{I\leqslant L} \subset \mathbb{R}$, and

$$b_I = \sum_{J\leqslant I} \frac{\binom{I}{J}}{\binom{L}{J}} a_J \quad I \leqslant L. \tag{3.29}$$

Proof (sketch). The proof uses structural induction on $p(\mathbf{x})$ by considering it as a polynomial function in variable x_1 with polynomial coefficients in the remaining variables. Each polynomial coefficient is then considered as a polynomial function in one variable with polynomial coefficients in the remaining variables, until coefficients are real numbers, which is the base case of induction. In the base case, Proposition 2.5 can be used to rewrite the coefficients of polynomial functions with one variable. Once the base case is settled, induction proceeds backward until the coefficients of the expression of $p(\mathbf{x})$ written as a polynomial function in variable x_1 are finally found. $\square$

Notably, Proposition 3.4 and Lemma 2.1 enable the proof of the following proposition, which ultimately justifies the use of Bernstein polynomials for studying the range of polynomial functions over box domains.

Proposition 3.5. *Let $p \in \Pi_L$ be a polynomial function expressed in the Bernstein basis as*

$$p(\mathbf{x}) = \sum_{I\leqslant L} b_I B_I^{(L)}(\mathbf{x}) \tag{3.30}$$

and let

$$\underline{p}_B([0,1]^n) = \min\{b_I\}_{I\leqslant L} \qquad \overline{p}_B([0,1]^n) = \max\{b_I\}_{I\leqslant L} \tag{3.31}$$

then

$$\forall \mathbf{x} \in [0,1]^n \quad \underline{p}_B([0,1]^n) \leqslant p(\mathbf{x}) \leqslant \overline{p}_B([0,1]^n). \tag{3.32}$$

Proof. First, note that for all $1 \leqslant r \leqslant n$, and for all $\mathbf{x} \in \mathbb{R}^n$

$$\sum_{i=0}^{l_r} \left(B_i^{(l_r)}(\mathbf{x}) \prod_{k=0,k\neq r}^{n} B_{i_k}^{(l_k)}(\mathbf{x}) \right) = \prod_{k=0,k\neq r}^{n} B_{i_k}^{(l_k)}(\mathbf{x}) \sum_{i=0}^{l_r} B_i^{(l_r)}(\mathbf{x}) = \prod_{k=0,k\neq r}^{n} B_{i_k}^{(l_k)}(\mathbf{x}), \tag{3.33}$$

where (2.43) of Lemma 2.1 is used in the last equality. Notably (3.33) ensures that for all $\mathbf{x} \in \mathbb{R}^n$

$$\sum_{I \leqslant L} B_I^{(L)}(\mathbf{x}) = 1. \tag{3.34}$$

Then, using (2.42) of Lemma 2.1, for all $\mathbf{x} \in [0,1]^n$

$$p(\mathbf{x}) = \sum_{I \leqslant L} b_I B_I^{(L)}(\mathbf{x}) \quad \leqslant \quad \overline{p}_B([0,1]^n) \sum_{I \leqslant L} B_I^{(L)}(\mathbf{x}) = \overline{p}_B([0,1]^n) \tag{3.35}$$

$$\underline{p}_B([0,1]^n) = \underline{p}_B([0,1]^n) \sum_{I \leqslant L} B_I^{(L)}(\mathbf{x}) \quad \leqslant \quad \sum_{I \leqslant L} b_I B_I^{(L)}(\mathbf{x}) = p(\mathbf{x}), \tag{3.36}$$

which prove the proposition. □

Finally, one of the most important properties of Bernstein coefficients generalizes to polynomial functions in a box $D \subset \mathbb{R}^n$. First, a notation to express boxes in $\mathbb{R}^n$ is needed once two vectors $\underline{\mathbf{x}} \in \mathbb{R}^n$ and $\overline{\mathbf{x}} \in \mathbb{R}^n$ are known

$$D = [\underline{\mathbf{x}}, \overline{\mathbf{x}}] = [\underline{x}_1, \overline{x}_1] \times [\underline{x}_2, \overline{x}_2] \times \cdots \times [\underline{x}_n, \overline{x}_n] \subset \mathbb{R}^n, \tag{3.37}$$

then an appropriate change of variables from $\mathbf{x} \in D$ to $\mathbf{t} \in [0,1]^n$, with $\mathbf{t} = (t_1, t_2, \ldots, t_n)$, is needed

$$x_i = (\overline{x}_i - \underline{x}_i) t_i + \underline{x}_i \quad 0 \leqslant i \leqslant n. \tag{3.38}$$

Proposition 3.6. *Given a polynomial function $p \in \Pi_L$ with coefficients $\{a_I\}_{I=0}^{L}$ in the monomial basis, and a box D as in (3.37), let $p^{(D)} \in \Pi_L$ be the polynomial function obtained from p using the change of variables (3.38). $p^{(D)}$ can be expressed in the Bernstein basis using the following coefficients*

$$b_I^{(D)} = \sum_{J \leqslant I} \frac{\binom{I}{J}}{\binom{L}{J}} (\overline{\mathbf{x}} - \underline{\mathbf{x}})^J \sum_{K \leqslant L} \binom{K}{J} \underline{\mathbf{x}}^{K-J} a_K \quad I \leqslant L. \tag{3.39}$$

Proof (sketch). The proof follows the same structural induction on the polynomial function p as in the proof of the Proposition 3.4. □

As for the case of one variable, $\{b_I^{(D)}\}_{I \leqslant L}$ are normally called Bernstein coefficients of p in D, and (3.39) provides a way to compute them using the coefficients $\{a_I\}_{I \leqslant L}$ of p in the monomial basis. Notably, the literature provides other algorithms to compute Bernstein coefficients [11, 13, 24], which are likely to be more effective in practical situations. Corollary 2.1, which is the most important property of Bernstein coefficients for polynomial functions in one variable, generalizes to the multivariate case, as follows.

Corollary 3.1. *Given a polynomial function $p \in \Pi_L$, a box $D = [\underline{\mathbf{x}}, \overline{\mathbf{x}}] \subset \mathbb{R}^n$, and $\{b_I^{(D)}\}_{I \leqslant L}$ as in Proposition 3.6*

$$\forall \mathbf{x} \in D \quad \min\{b_I^{(D)}\}_{I \leqslant L} \leqslant p(\mathbf{x}) \leqslant \max\{b_I^{(D)}\}_{I \leqslant L}. \tag{3.40}$$

Proof. The proof follows immediately from Proposition 3.5 and Proposition 3.6. □

Let

$$\underline{p}_B([\underline{\mathbf{x}}, \overline{\mathbf{x}}]) = \min\{b_I^{(D)}\}_{I \leqslant L} \quad \overline{p}_B([\underline{\mathbf{x}}, \overline{\mathbf{x}}]) = \max\{b_I^{(D)}\}_{I \leqslant L} \tag{3.41}$$

then

$$\forall \mathbf{x} \in D = [\underline{\mathbf{x}}, \overline{\mathbf{x}}] \quad \underline{p}_B([\underline{\mathbf{x}}, \overline{\mathbf{x}}]) \leqslant p(\mathbf{x}) \leqslant \overline{p}_B([\underline{\mathbf{x}}, \overline{\mathbf{x}}]). \tag{3.42}$$

3.2 Comparison in the Multivariate Case

The following proposition ensures that, given a closed bounded box $[\underline{\mathbf{x}}, \overline{\mathbf{x}}] \subset \mathbb{R}^n_{\geqslant 0}$, the simple bounds defined are tighter than the absolute bounds.

Proposition 3.7. *Given a closed bounded box $[\underline{\mathbf{x}}, \overline{\mathbf{x}}] \subset \mathbb{R}^n_{\geqslant 0}$ and a polynomial function $p \in \Pi_L$ expressed as in (3.7), the simple bounds $\underline{p}_S([\underline{\mathbf{x}}, \overline{\mathbf{x}}])$ and $\overline{p}_S([\underline{\mathbf{x}}, \overline{\mathbf{x}}])$ are tighter than the absolute bounds $\underline{p}_A([\underline{\mathbf{x}}, \overline{\mathbf{x}}])$ and $\overline{p}_A([\underline{\mathbf{x}}, \overline{\mathbf{x}}])$, respectively. So, for all $\mathbf{x} \in [\underline{\mathbf{x}}, \overline{\mathbf{x}}]$ the following inequalities hold*

$$\underline{p}_A([\underline{\mathbf{x}}, \overline{\mathbf{x}}]) \leqslant \underline{p}_S([\underline{\mathbf{x}}, \overline{\mathbf{x}}]) \leqslant p(\mathbf{x}) \leqslant \overline{p}_S([\underline{\mathbf{x}}, \overline{\mathbf{x}}]) \leqslant \overline{p}_A([\underline{\mathbf{x}}, \overline{\mathbf{x}}]). \tag{3.43}$$

Proof. The definition of $\overline{p}_A([\underline{\mathbf{x}}, \overline{\mathbf{x}}])$ in Definition 3.1 ensures that, if $\overline{p}_A([\underline{\mathbf{x}}, \overline{\mathbf{x}}]) = 0$, then p is the zero polynomial and, so, $\overline{p}_S([\underline{\mathbf{x}}, \overline{\mathbf{x}}]) = 0$ for Definition 3.2. Similar considerations can be used in case of $\underline{p}_A([\underline{\mathbf{x}}, \overline{\mathbf{x}}]) = 0$ to prove $\underline{p}_A([\underline{\mathbf{x}}, \overline{\mathbf{x}}]) = \underline{p}_S([\underline{\mathbf{x}}, \overline{\mathbf{x}}])$.

Instead, if $\overline{p}_A([\underline{\mathbf{x}}, \overline{\mathbf{x}}]) \neq 0$, there exists some $J \leqslant L$ such as $a_J \neq 0$ and

$$\begin{aligned} \overline{p}_A([\underline{\mathbf{x}}, \overline{\mathbf{x}}]) &= \sum_{I \leqslant L} |a_I| \overline{\mathbf{x}}^I && (3.44) \\ &\geqslant \sum_{I \leqslant L; a_I > 0} a_I \overline{\mathbf{x}}^I && (3.45) \\ &= \overline{p}_S([\underline{\mathbf{x}}, \overline{\mathbf{x}}]). && (3.46) \end{aligned}$$

Similarly,

$$\begin{aligned} \underline{p}_A([\underline{\mathbf{x}}, \overline{\mathbf{x}}]) &= -\sum_{I \leqslant L} |a_I| \overline{\mathbf{x}}^I && (3.47) \\ &\leqslant \sum_{I \leqslant L; a_I < 0} a_I \overline{\mathbf{x}}^I && (3.48) \\ &= \underline{p}_S([\underline{\mathbf{x}}, \overline{\mathbf{x}}]) && (3.49) \end{aligned}$$

because the number of negative addends in $\underline{p}_A([\underline{\mathbf{x}}, \overline{\mathbf{x}}])$ is greater or equal than the one of $\underline{p}_S([\underline{\mathbf{x}}, \overline{\mathbf{x}}])$ and so the bound $\underline{p}_S([\underline{\mathbf{x}}, \overline{\mathbf{x}}])$ is tighter than the bound $\underline{p}_A[\underline{\mathbf{x}}, \overline{\mathbf{x}}])$, which prove the proposition. □

The following proposition ensures that, given a closed bounded box $[\underline{\mathbf{x}},\overline{\mathbf{x}}] \subset \mathbb{R}^n_{\geqslant 0}$, the interval bounds defined are tighter than the simple bounds.

Proposition 3.8. *Given a closed bounded box $[\underline{\mathbf{x}},\overline{\mathbf{x}}] \subset \mathbb{R}^n_{\geqslant 0}$ and a polynomial function $p \in \Pi_L$ expressed as in (3.7), the interval bounds $\underline{p}_I([\underline{\mathbf{x}},\overline{\mathbf{x}}])$ and $\overline{p}_I([\underline{\mathbf{x}},\overline{\mathbf{x}}])$ are tighter than the simple bounds $\underline{p}_S([\underline{\mathbf{x}},\overline{\mathbf{x}}])$ and $\overline{p}_S([\underline{\mathbf{x}},\overline{\mathbf{x}}])$, respectively. So, for all $\mathbf{x} \in [\underline{\mathbf{x}},\overline{\mathbf{x}}]$ the following inequalities hold*

$$\underline{p}_S([\underline{\mathbf{x}},\overline{\mathbf{x}}]) \leqslant \underline{p}_I([\underline{\mathbf{x}},\overline{\mathbf{x}}]) \leqslant p(\mathbf{x}) \leqslant \overline{p}_I([\underline{\mathbf{x}},\overline{\mathbf{x}}]) \leqslant \overline{p}_S([\underline{\mathbf{x}},\overline{\mathbf{x}}]). \tag{3.50}$$

Proof. The definition of $\overline{p}_S([\underline{\mathbf{x}},\overline{\mathbf{x}}])$ in Definition 3.2 ensures that if $\overline{p}_S([\underline{\mathbf{x}},\overline{\mathbf{x}}]) = 0$, then

- p is the zero polynomial and $\overline{p}_I([\underline{\mathbf{x}},\overline{\mathbf{x}}]) = 0$; or
- For Proposition 3.3

$$\begin{aligned} \overline{p}_S([\underline{\mathbf{x}},\overline{\mathbf{x}}]) &= 0 && (3.51)\\ &\geqslant \sum_{I\leqslant L; a_I<0} a_I \underline{\mathbf{x}}^I && (3.52)\\ &= \sum_{I\leqslant L} \max\{a_I\underline{\mathbf{x}}^I, a_I\overline{\mathbf{x}}^I\} && (3.53)\\ &= \overline{p}_I([\underline{\mathbf{x}},\overline{\mathbf{x}}]). && (3.54) \end{aligned}$$

Instead, if $\overline{p}_S([\underline{\mathbf{x}},\overline{\mathbf{x}}]) \neq 0$, there exists some $J \leqslant L$ such as $a_J > 0$. Consider the $I-$th coefficient a_I of the polynomial function, if $a_I \leqslant 0$, then for $\overline{p}_S([\underline{\mathbf{x}},\overline{\mathbf{x}}])$ the contribution of $I-$th coefficient is zero, but for $\overline{p}_I([\underline{\mathbf{x}},\overline{\mathbf{x}}])$ it equals

$$\max\{a_I\underline{\mathbf{x}}^I, a_I\overline{\mathbf{x}}^I\} = a_I\underline{\mathbf{x}}^I \leqslant 0. \tag{3.55}$$

Similarly, if $a_I > 0$, then

$$a_I\overline{\mathbf{x}}^I = \max\{a_I\underline{\mathbf{x}}^I, a_I\overline{\mathbf{x}}^I\}. \tag{3.56}$$

In summary, the contribution of $I-$th coefficient of the sum of in $\overline{p}_S([\underline{\mathbf{x}},\overline{\mathbf{x}}])$ is greater or equal to the sum of $\overline{p}_I([\underline{\mathbf{x}},\overline{\mathbf{x}}])$. Therefore,

$$\overline{p}_S([\underline{\mathbf{x}},\overline{\mathbf{x}}]) \geqslant \overline{p}_I([\underline{\mathbf{x}},\overline{\mathbf{x}}]), \tag{3.57}$$

which ensures that the bound $\overline{p}_I([\underline{\mathbf{x}},\overline{\mathbf{x}}])$ is tighter than the bound $\overline{p}_S([\underline{\mathbf{x}},\overline{\mathbf{x}}])$. Similar considerations for $\underline{p}_I([\underline{\mathbf{x}},\overline{\mathbf{x}}])$ and $\underline{p}_S([\underline{\mathbf{x}},\overline{\mathbf{x}}])$ complete the proof of the proposition. □

The following proposition ensures that, given a closed bounded box $[\underline{\mathbf{x}},\overline{\mathbf{x}}] \subset \mathbb{R}^n_{\geqslant 0}$, the Bernstein bounds are tighter than the interval bounds.

Proposition 3.9. *Given a closed bounded box $D = [\underline{\mathbf{x}},\overline{\mathbf{x}}] \subset \mathbb{R}^n_{\geqslant 0}$ and a polynomial function $p \in \Pi_L$ expressed as in (3.7), the Bernstein bounds $\underline{p}_B([\underline{\mathbf{x}},\overline{\mathbf{x}}])$ and $\overline{p}_B([\underline{\mathbf{x}},\overline{\mathbf{x}}])$ are tighter than the interval bounds $\underline{p}_I([\underline{\mathbf{x}},\overline{\mathbf{x}}])$ and $\overline{p}_I([\underline{\mathbf{x}},\overline{\mathbf{x}}])$, respectively. So, for all $\mathbf{x} \in [\underline{\mathbf{x}},\overline{\mathbf{x}}]$ the following inequalities hold*

$$\underline{p}_I([\underline{\mathbf{x}},\overline{\mathbf{x}}]) \leqslant \underline{p}_B([\underline{\mathbf{x}},\overline{\mathbf{x}}]) \leqslant p(x) \leqslant \overline{p}_B([\underline{\mathbf{x}},\overline{\mathbf{x}}]) \leqslant \overline{p}_I([\underline{\mathbf{x}},\overline{\mathbf{x}}]). \tag{3.58}$$

Proof. Given the dimension $n \in \mathbb{N}_+$, $l_1, l_2, \dots, l_n$ positive integer numbers, $i_1, i_2, \dots, i_n$ between 0 and $l_1, l_2, \dots, l_n$, respectively, and $\overline{x}_1 > \underline{x}_1 > 0, \overline{x}_2 > \underline{x}_2 > 0, \dots, \overline{x}_n > \underline{x}_n > 0$, set I the multi-index with components $(i_1, i_2, \dots, i_n)$. Consider $b_I^{(D)}$ the $I-$th coefficient of the Bernstein polynomials (using the multi-index notation):

$$
\begin{aligned}
b_I^{(D)} &= \sum_{J \leqslant I} \frac{\binom{I}{J}}{\binom{L}{J}} (\overline{\mathbf{x}} - \underline{\mathbf{x}})^J \sum_{K \leqslant L} \binom{K}{J} \underline{\mathbf{x}}^{K-J} a_K & (3.59)\\
&= \sum_{J \leqslant 0} \frac{\binom{I}{J}}{\binom{L}{J}} (\overline{\mathbf{x}} - \underline{\mathbf{x}})^J \sum_{K \leqslant L} \binom{K}{J} \underline{\mathbf{x}}^{K-J} a_K + \sum_{J \leqslant I; J \neq 0} \frac{\binom{I}{J}}{\binom{L}{J}} (\overline{\mathbf{x}} - \underline{\mathbf{x}})^J \sum_{K \leqslant L} \binom{K}{J} \underline{\mathbf{x}}^{K-J} a_K & (3.60)\\
&= \sum_{K \leqslant L} \underline{\mathbf{x}}^K a_K + \sum_{K \leqslant L; K \neq 0} a_K \sum_{J \leqslant I; J \leqslant K; J \neq 0} \frac{\binom{I}{J}}{\binom{L}{J}} (\overline{\mathbf{x}} - \underline{\mathbf{x}})^J \binom{K}{J} \underline{\mathbf{x}}^{K-J} & (3.61)\\
&= \sum_{K \leqslant L; a_K < 0} \underline{\mathbf{x}}^K a_K + \sum_{K \leqslant L; a_K > 0} \underline{\mathbf{x}}^K a_K & (3.62)\\
&+ \sum_{K \leqslant L; K \neq 0; a_K < 0} a_K \sum_{J \leqslant I; J \leqslant K; J \neq 0} \frac{\binom{I}{J}}{\binom{L}{J}} (\overline{\mathbf{x}} - \underline{\mathbf{x}})^J \binom{K}{J} \underline{\mathbf{x}}^{K-J} & (3.63)\\
&+ \sum_{K \leqslant L; K \neq 0; a_K > 0} a_K \sum_{J \leqslant I; J \leqslant K; J \neq 0} \frac{\binom{I}{J}}{\binom{L}{J}} (\overline{\mathbf{x}} - \underline{\mathbf{x}})^J \binom{K}{J} \underline{\mathbf{x}}^{K-J} & (3.64)\\
&= \sum_{K \leqslant L; a_K < 0} \underline{\mathbf{x}}^K a_K + \sum_{K \leqslant L; K \neq 0; a_K < 0} a_K \sum_{J \leqslant I; J \leqslant K; J \neq 0} \frac{\binom{I}{J}}{\binom{L}{J}} (\overline{\mathbf{x}} - \underline{\mathbf{x}})^J \binom{K}{J} \underline{\mathbf{x}}^{K-J} & (3.65)\\
&+ \sum_{K \leqslant L; a_K > 0} a_K \sum_{J \leqslant I; J \leqslant K} \frac{\binom{I}{J}}{\binom{L}{J}} (\overline{\mathbf{x}} - \underline{\mathbf{x}})^J \binom{K}{J} \underline{\mathbf{x}}^{K-J} & (3.66)\\
&\leqslant \sum_{K \leqslant L; a_K < 0} \underline{\mathbf{x}}^K a_K + \sum_{K \leqslant L; K \neq 0; a_K < 0} a_K \sum_{J \leqslant I; J \leqslant K; J \neq 0} \frac{\binom{I}{J}}{\binom{L}{J}} (\overline{\mathbf{x}} - \underline{\mathbf{x}})^J \binom{K}{J} \underline{\mathbf{x}}^{K-J} & (3.67)\\
&+ \sum_{K \leqslant L; a_K > 0} a_K \sum_{J \leqslant I; J \leqslant K} (\overline{\mathbf{x}} - \underline{\mathbf{x}})^J \binom{K}{J} \underline{\mathbf{x}}^{K-J} & (3.68)
\end{aligned}
$$

because the ratio between two binomial coefficients is less than or equal to 1. Moreover,

$$
\begin{aligned}
&\sum_{K \leqslant L; a_K < 0} \underline{\mathbf{x}}^K a_K + \sum_{K \leqslant L; K \neq 0; a_K < 0} a_K \sum_{J \leqslant I; J \leqslant K; J \neq 0} \frac{\binom{I}{J}}{\binom{L}{J}} (\overline{\mathbf{x}} - \underline{\mathbf{x}})^J \binom{K}{J} \underline{\mathbf{x}}^{K-J} & (3.69)\\
&+ \sum_{K \leqslant L; a_K > 0} a_K \sum_{J \leqslant I; J \leqslant K} (\overline{\mathbf{x}} - \underline{\mathbf{x}})^J \binom{K}{J} \underline{\mathbf{x}}^{K-J} & (3.70)\\
&\leqslant \sum_{K \leqslant L; a_K < 0} \underline{\mathbf{x}}^K a_K + \sum_{K \leqslant L; K \neq 0; a_K < 0} a_K \sum_{J \leqslant I; J \leqslant K; J \neq 0} \frac{\binom{I}{J}}{\binom{L}{J}} (\overline{\mathbf{x}} - \underline{\mathbf{x}})^J \binom{K}{J} \underline{\mathbf{x}}^{K-J} & (3.71)\\
&+ \sum_{K \leqslant L; a_K > 0} a_K \sum_{J \leqslant K} (\overline{\mathbf{x}} - \underline{\mathbf{x}})^J \binom{K}{J} \underline{\mathbf{x}}^{K-J} & (3.72)
\end{aligned}
$$

because in the sum with $J \leqslant I$ and $J \leqslant K$ the number of addends is less than or equal to the number of addends in the sum with $J \leqslant K$. So,

$$\sum_{K \leqslant L; a_K < 0} \underline{\mathbf{x}}^K a_K + \sum_{K \leqslant L; K \neq 0; a_K < 0} a_K \sum_{J \leqslant I; J \leqslant K; J \neq 0} \frac{\binom{I}{J}}{\binom{L}{J}} (\overline{\mathbf{x}} - \underline{\mathbf{x}})^J \binom{K}{J} \underline{\mathbf{x}}^{K-J} \tag{3.73}$$

$$+ \sum_{K \leqslant L; a_K > 0} a_K \sum_{J \leqslant K} (\overline{\mathbf{x}} - \underline{\mathbf{x}})^J \binom{K}{J} \underline{\mathbf{x}}^{K-J} \tag{3.74}$$

$$= \sum_{K \leqslant L; a_K < 0} \underline{\mathbf{x}}^K a_K + \sum_{K \leqslant L; K \neq 0; a_K < 0} a_K \sum_{J \leqslant I; J \leqslant K; J \neq 0} \frac{\binom{I}{J}}{\binom{L}{J}} (\overline{\mathbf{x}} - \underline{\mathbf{x}})^J \binom{K}{J} \underline{\mathbf{x}}^{K-J} \tag{3.75}$$

$$+ \sum_{K \leqslant L; a_K > 0} a_K \overline{\mathbf{x}}^K \tag{3.76}$$

for the binomial theorem. In addition,

$$\sum_{K \leqslant L; a_K < 0} \underline{\mathbf{x}}^K a_K + \sum_{K \leqslant L; K \neq 0; a_K < 0} a_K \sum_{J \leqslant I; J \leqslant K; J \neq 0} \frac{\binom{I}{J}}{\binom{L}{J}} (\overline{\mathbf{x}} - \underline{\mathbf{x}})^J \binom{K}{J} \underline{\mathbf{x}}^{K-J} \tag{3.77}$$

$$+ \sum_{K \leqslant L; a_K > 0} a_K \overline{\mathbf{x}}^K \tag{3.78}$$

$$\leqslant \sum_{K \leqslant L; a_K < 0} \underline{\mathbf{x}}^K a_K + \sum_{K \leqslant L; a_K > 0} a_K \overline{\mathbf{x}}^K \tag{3.79}$$

$$= \overline{p}_I([\underline{\mathbf{x}}, \overline{\mathbf{x}}]). \tag{3.80}$$

Therefore,

$$b_I^{(D)} \leqslant \overline{p}_I([\underline{\mathbf{x}}, \overline{\mathbf{x}}]) \tag{3.81}$$

since I is arbitrary, calculate the maximum for I between 0 and L

$$\max_I b_I^{(D)} \leqslant \overline{p}_I([\underline{\mathbf{x}}, \overline{\mathbf{x}}]). \tag{3.82}$$

State with $\overline{p}_B([\underline{\mathbf{x}}, \overline{\mathbf{x}}])$ the quantity $\max_I b_I^{(D)}$, so

$$\overline{p}_B([\underline{\mathbf{x}}, \overline{\mathbf{x}}]) \leqslant \overline{p}_I([\underline{\mathbf{x}}, \overline{\mathbf{x}}]). \tag{3.83}$$

Similarly, the comparison between the minimum $\min_I b_I^{(D)}$ and $\underline{p}_I([\underline{\mathbf{x}}, \overline{\mathbf{x}}])$, so

$$\min_I b_I^{(D)} = \underline{p}_B([\underline{\mathbf{x}}, \overline{\mathbf{x}}]) \geqslant \underline{p}_I([\underline{\mathbf{x}}, \overline{\mathbf{x}}]). \tag{3.84}$$

Similar considerations can be used to prove in case of $\underline{x}_I = 0$ for some I. □

Summarizing, in a closed bounded box $[\underline{\mathbf{x}}, \overline{\mathbf{x}}] \subset \mathbb{R}^n_{\geqslant 0}$

$$\underline{p}_A([\underline{\mathbf{x}}, \overline{\mathbf{x}}]) \leqslant \underline{p}_S([\underline{\mathbf{x}}, \overline{\mathbf{x}}]) \leqslant \underline{p}_I([\underline{\mathbf{x}}, \overline{\mathbf{x}}]) \leqslant \underline{p}_B([\underline{\mathbf{x}}, \overline{\mathbf{x}}]) \tag{3.85}$$

$$\leqslant p(x) \leqslant \overline{p}_B([\underline{\mathbf{x}}, \overline{\mathbf{x}}]) \leqslant \overline{p}_I([\underline{\mathbf{x}}, \overline{\mathbf{x}}]) \leqslant \overline{p}_S([\underline{\mathbf{x}}, \overline{\mathbf{x}}]) \leqslant \overline{p}_A([\underline{\mathbf{x}}, \overline{\mathbf{x}}]). \tag{3.86}$$

3.3 Note on the Computational Complexity

Set n the number of variables (the dimension) and l the maximum degree of the single variable. For the absolute, simple and interval bounds, using Horner's rule to evaluate a polynomial in a point, the time computational complexity is $O(l^n)$.

To calculate Bernstein bound, as [19], with the calculus of the definition (3.39), the computational time complexity is $O(l^{2n})$. With Garloff's, instead, method the computational time becomes $O(l^{n+1})$ for the unit box and for a general box domain it is $O(l^{2n})$. With the Ray and Nataraj's method the computational time becomes $O(l^{n+1})$ for a general box domain.

3.4 Illustrative Example

The following examples are provided solely to illustrate the results. They do not constitute exhaustive tests.

3.4.1 Example with a polynomial of second degree

Consider the closed bounded box $[0,4]\times[0,2]$ and the polynomial function of two variables

$$f(x,y) = x^2 - 2xy + 4y^2 - 4x - 2y + 24. \tag{3.87}$$

The study of the extrema of the considered function requires computing its partial derivatives

$$\frac{\partial f}{\partial x} = 2x - 2y - 4 \qquad \frac{\partial f}{\partial y} = -2x + 8y - 2. \tag{3.88}$$

These partial derivatives allows computing the critical point $\mathbf{u} = (1,3)$. Note that $\mathbf{u}$ is contained in the considered box. The second order partial derivatives are

$$\frac{\partial^2 f}{\partial x^2} = 2 \qquad \frac{\partial^2 f}{\partial xy} = \frac{\partial^2 f}{\partial yx} = -2 \qquad \frac{\partial^2 f}{\partial y^2} = 8 \tag{3.89}$$

and the determinant of the Hessian matrix of the considered function is

$$H(x,y) = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial xy}\right)^2 = 2\cdot 8 - (-2)^2 = 12. \tag{3.90}$$

Since $H(\mathbf{u}) > 0$, $\mathbf{u}$ is a minimum point. The point $\mathbf{u}$ is on the boundary of the domain and the value of the function at this point is 17.

To complete the study of the extrema of f, it is necessary to consider the boundary of considered box, whose vertices are $\mathbf{b} = (0,0)$, $\mathbf{c} = (0,2)$, $\mathbf{d} = (4,2)$, and $\mathbf{a} = (4,0)$:

- Segment **b**-**c**, whose equation is $x = 0$ with $0 \le y \le 2$. By substituting $x = 0$ into the considered function f, one obtains the function $h(y) = 4y^2 - 2y + 24$, which has a global minimum for $y = 0$ and a global maximum for $y = 2$ in $0 \le y \le 2$, with respective values 24 and 36;

- Segment **c**-**d**, whose equation is $y = 2$ with $0 \le x \le 4$. By substituting $y = 2$ into the considered function f, one obtains the function $h(x) = x^2 - 8x + 36$, which decreases in $0 \le x \le 4$ and has a global minimum for $x = 4$ and a global maximum for $x = 0$ in $0 \le x \le 4$, with respective values 36 and 20;

- Segment **a**-**d**, whose equation is $x = 4$ with $0 \le y \le 2$. By substituting $x = 4$ into the considered function f, one obtains the function $h(y) = 4y^2 - 10y + 24$, which has a global minimum for $y = \frac{5}{4} = 1.25$ and a global maximum for $y = 0$ in $0 \le y \le 2$, with respective values 17.75 and 24; and

- Segment **a**-**b**, whose equation is $y = 0$ with $0 \le x \le 4$. By substituting $y = 0$ into the considered function f, one obtains the function $h(x) = x^2 - 4x + 24$, which has a global minimum for $x = 2$ and a global maximum for $x = 0$ in $0 \le x \le 4$ with respective values 20 and 24.

By comparing the obtained values, it is evident that, in the considered box, 36 is the global maximum of the function and 17 is the global minimum of the function.

Now, consider the absolute bounds. The bounds $\overline{f}_A([0,4] \times [0,2])$ and $\underline{f}_A([0,4] \times [0,2])$ are

$$\overline{f}_A([0,4] \times [0,2]) = 92, \tag{3.91}$$

and

$$\underline{f}_A([0,4] \times [0,2]) = -92. \tag{3.92}$$

Figure 3.1(a) shows the plot of the considered function in the considered box. The figure also shows the absolute lower bound $\overline{f}_A([0,4] \times [0,2])$ and the absolute upper bound $\underline{f}_A([0,4] \times [0,2])$.

Consider the simple bounds. Since there exist $I \le L$ and $J \le L$ such that $a_I \ge 0$ and $a_J \le 0$, it follows that the bounds $\overline{f}_S([0,4] \times [0,2])$ and $\underline{f}_S([0,4] \times [0,2])$ are

$$\overline{f}_S([0,4] \times [0,2]) = 56, \tag{3.93}$$

and

$$\underline{f}_S([0,4] \times [0,2]) = -36. \tag{3.94}$$

Figure 3.1(b) shows the plot of the considered function in the considered box. The figure also shows the simple lower bound $\overline{f}_S([0,4] \times [0,2])$ and the simple upper bound $\underline{f}_S([0,4] \times [0,2])$.

The interval bounds are

$$\overline{f}_I([0,4]\times[0,2]) = 56 \tag{3.95}$$

and

$$\underline{f}_I([0,4]\times[0,2]) = -12. \tag{3.96}$$

Figure 3.1(c) shows the plot of the considered function in the considered box. The figure also shows the interval lower bound and the interval upper bound. Note that, as expected, the interval bounds are tighter than the simple bounds.

The Bernstein coefficients are

$$\begin{pmatrix} 24 & 22 & 36 \\ 16 & 10 & 20 \\ 24 & 14 & 20 \end{pmatrix}$$

so the Bernstein bounds are

$$\overline{f}_B([0,4]\times[0,2]) = 36. \tag{3.97}$$

and

$$\underline{f}_B([0,4]\times[0,2]) = 10 \tag{3.98}$$

In this case, the maximum value of the function coincides with $\overline{f}_B([0,4]\times[0,2])$, whereas the minimum value of the function is strictly greater than $\underline{f}_B([0,4]\times[0,2])$.

Figure 3.1(d) shows the plot of the considered function in the considered box. The figure also shows the Bernstein lower bound and the Bernstein upper bound. Note that, as expected, the Bernstein bounds are tighter than the interval bounds.

Table 3.1: Report of the computation times in seconds obtained in MATLAB for the different bounds.

Absolute	Simple	Interval	Bernstein
$9.96\cdot 10^{-4}$ s	$8.85\cdot 10^{-4}$ s	$1.617\cdot 10^{-3}$ s	$1.6082\cdot 10^{-2}$ s

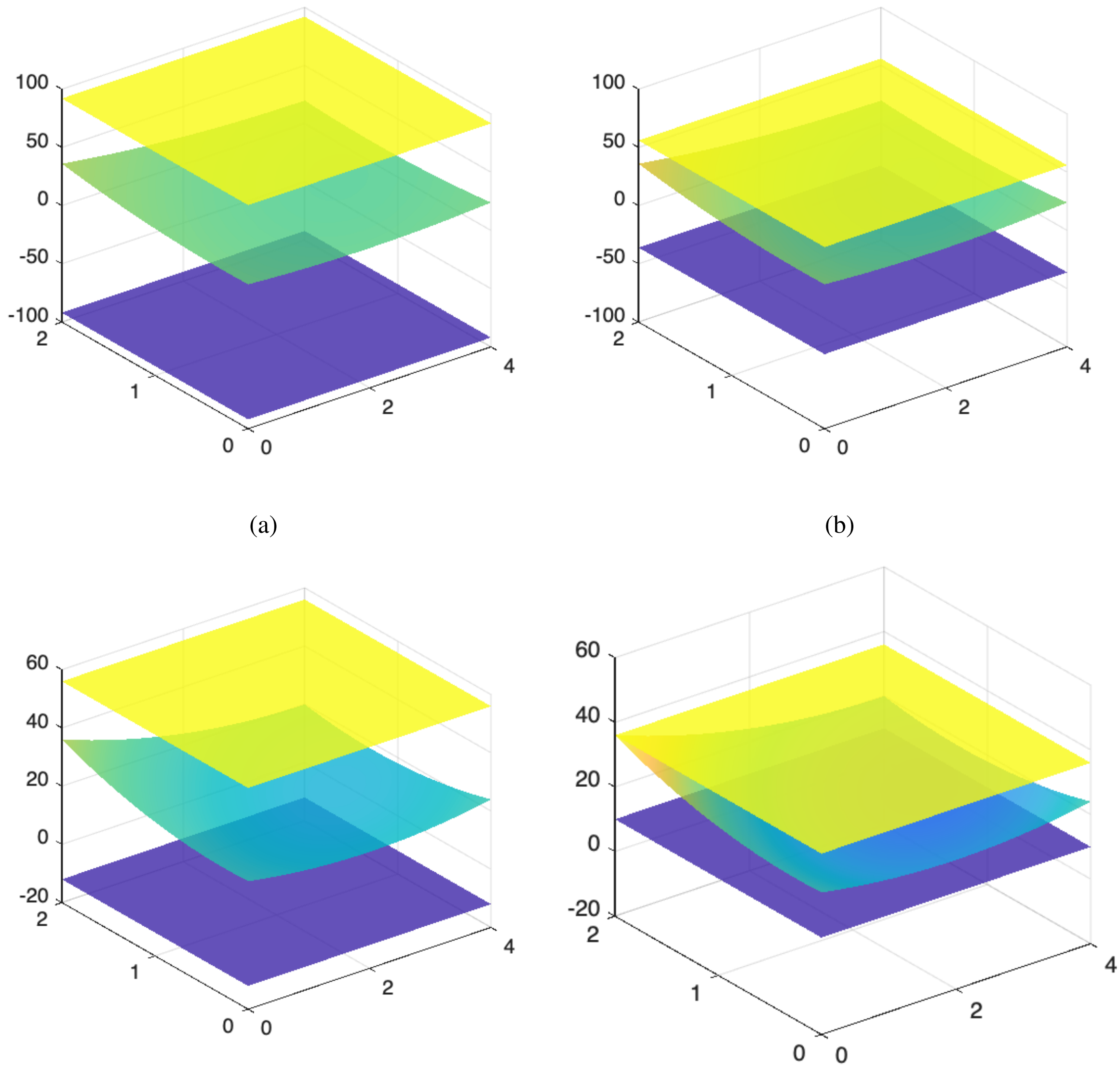
3.4.2 Example with a polynomial of third degree

Consider the closed bounded box $[4,8]\times[4,8]$ and the polynomial function of two variables

$$g(x,y) = \frac{1}{2}x^2 - 2xy + \frac{1}{3}y^3. \tag{3.99}$$

The study of the extrema of the considered function requires computing its partial derivatives

$$\frac{\partial g}{\partial x} = x - 2y \qquad \frac{\partial g}{\partial y} = -2x + y^2. \tag{3.100}$$



(a) (b) (c) (d)

Figure 3.1: For $f(x,y)=x^2-2x*y+4y^2-4x-2y+24$, (a) absolute bounds over $[0,4]\times[0,2]$ (yellow and blue planes) and (b) simple bounds over $[0,4]\times[0,2]$ (yellow and blue planes), (c) interval bounds over $[0,4]\times[0,2]$ (yellow and blue planes) and (d) Bernstein bounds over $[0,4]\times[0,2]$ (yellow and blue planes).

These partial derivatives allows computing the critical points $\mathbf{u}=(0,0)$ and $\mathbf{a}=(8,4)$. Note that $\mathbf{u}$ is not contained in the considered box while $\mathbf{a}$ is contained. The second order partial derivatives are

$$\frac{\partial^2 g}{\partial x^2}=1 \qquad \frac{\partial^2 g}{\partial xy}=\frac{\partial^2 g}{\partial yx}=-2 \qquad \frac{\partial^2 g}{\partial y^2}=2y \tag{3.101}$$

and the determinant of the Hessian matrix of the considered function is

$$H(x,y)=\frac{\partial^2 g}{\partial x^2}\frac{\partial^2 g}{\partial y^2}-\left(\frac{\partial^2 g}{\partial xy}\right)^2=2y-4. \tag{3.102}$$

Since $H(\mathbf{a}) = 4 > 0$, $\mathbf{a}$ is a minimum point. The point $\mathbf{a}$ is on the boundary of the domain and the value of the function at this point is $-10.\overline{6}$.

To complete the study of the extrema of g, it is necessary to consider the boundary of considered box, whose vertices are $\mathbf{b} = (4,4)$, $\mathbf{c} = (4,8)$, $\mathbf{d} = (8,8)$, and $\mathbf{a} = (8,4)$:

- Segment **b**-**c**, whose equation is $x = 4$ with $4 \leq y \leq 8$. By substituting $x = 4$ into the considered function g, one obtains the function $h(y) = \frac{1}{3}y^3 - 8y + 8$, which has a global minimum for $y = 4$ and a global maximum for $y = 8$ in $4 \leq y \leq 8$, with respective values $-2.\overline{6}$ and $114.\overline{6}$;

- Segment **c**-**d**, whose equation is $y = 8$ with $4 \leq x \leq 8$. By substituting $y = 8$ into the considered function g, one obtains the function $h(x) = \frac{1}{2}x^2 - 16x + \frac{512}{3}$, which has a global minimum for $x = 8$ and a global maximum for $x = 4$ in $4 \leq x \leq 8$, with respective values $74.\overline{6}$ and $114.\overline{6}$;

- Segment **a**-**d**, whose equation is $x = 8$ with $4 \leq y \leq 8$. By substituting $x = 8$ into the considered function g, one obtains the function $h(y) = \frac{1}{3}y^3 - 16y + 32$, which has a global minimum for $y = 4$ and a global maximum for $y = 8$ in $4 \leq y \leq 8$, with respective values $-10.\overline{6}$ and $74.\overline{6}$; and

- Segment **a**-**b**, whose equation is $y = 4$ with $4 \leq x \leq 8$. By substituting $y = 4$ into the considered function g, one obtains the function $h(x) = \frac{1}{2}x^2 - 8x + \frac{64}{3}$, which has a global minimum for $x = 8$ and a global maximum in for $x = 4$ $4 \leq x \leq 8$ with respective values $-10.\overline{6}$ and $-2.\overline{6}$.

By comparing the obtained values, it is evident that, in the considered box, $114.\overline{6}$ is the global maximum of the function and $-10.\overline{6}$ is the global minimum of the function.

Now, consider the absolute bounds. The bounds $\overline{g}_A([4,8] \times [4,8])$ and $\underline{g}_A([4,8] \times [4,8])$ are

$$\overline{g}_A([4,8] \times [4,8]) = 330.\overline{6}, \tag{3.103}$$

and

$$\underline{g}_A([4,8] \times [4,8]) = -330.\overline{6}. \tag{3.104}$$

Figure 3.2(a) shows the plot of the considered function in the considered box. The figure also shows the absolute lower bound $\overline{g}_A([4,8] \times [4,8])$ and the absolute upper bound $\underline{g}_A([4,8] \times [4,8])$.

Consider the simple bounds. Since there exist $I \leq L$ and $J \leq L$ such that $a_I \geq 0$ and $a_J \leq 0$, it follows that the bounds $\overline{g}_S([4,8] \times [4,8])$ and $\underline{g}_S([4,8] \times [4,8])$ are

$$\overline{g}_S([4,8] \times [4,8]) = 202.\overline{6}, \tag{3.105}$$

and

$$\underline{g}_S([4,8] \times [4,8]) = -128. \tag{3.106}$$

Figure 3.2(b) shows the plot of the considered function in the considered box. The figure also shows the simple lower bound $\overline{g}_S([4,8]\times[4,8])$ and the simple upper bound $\underline{g}_S([4,8]\times[4,8])$.

The interval bounds are

$$\overline{g}_I([4,8]\times[4,8]) = 170.\overline{6} \tag{3.107}$$

and

$$\underline{g}_I([4,8]\times[4,8]) = -98.\overline{6}. \tag{3.108}$$

Figure 3.2(c) shows the plot of the considered function in the considered box. The figure also shows the interval lower bound and the interval upper bound. Note that, as expected, the interval bounds are tighter than the simple bounds.

The Bernstein coefficients are

$$\begin{pmatrix} -2.\overline{6} & 8 & 40 & 114.\overline{6} \\ -10.\overline{6} & -5.\overline{3} & 21.\overline{3} & 90.\overline{6} \\ -10.\overline{6} & -10.\overline{6} & 10.\overline{6} & 74.\overline{6} \end{pmatrix}$$

so the Bernstein bounds are

$$\overline{g}_B([4,8]\times[4,8]) = 114.\overline{6}. \tag{3.109}$$

and

$$\underline{g}_B([4,8]\times[4,8]) = -10.\overline{6} \tag{3.110}$$

In this case the maximum and the minimum of the function are the same as $\overline{g}_B([4,8]\times[4,8])$ and $\underline{g}_B([4,8]\times[4,8])$, respectively.

Figure 3.2(d) shows the plot of the considered function in the considered box. The figure also shows the Bernstein lower bound and the Bernstein upper bound. Note that, as expected, the Bernstein bounds are tighter than the interval bounds.

Table 3.2: Report of the computation times in seconds obtained in MATLAB for the different bounds.

Absolute	Simple	Interval	Bernstein
$1.198\cdot10^{-3}$ s	$1.415\cdot10^{-3}$ s	$1.656\cdot10^{-3}$ s	$3.005\cdot10^{-3}$ s

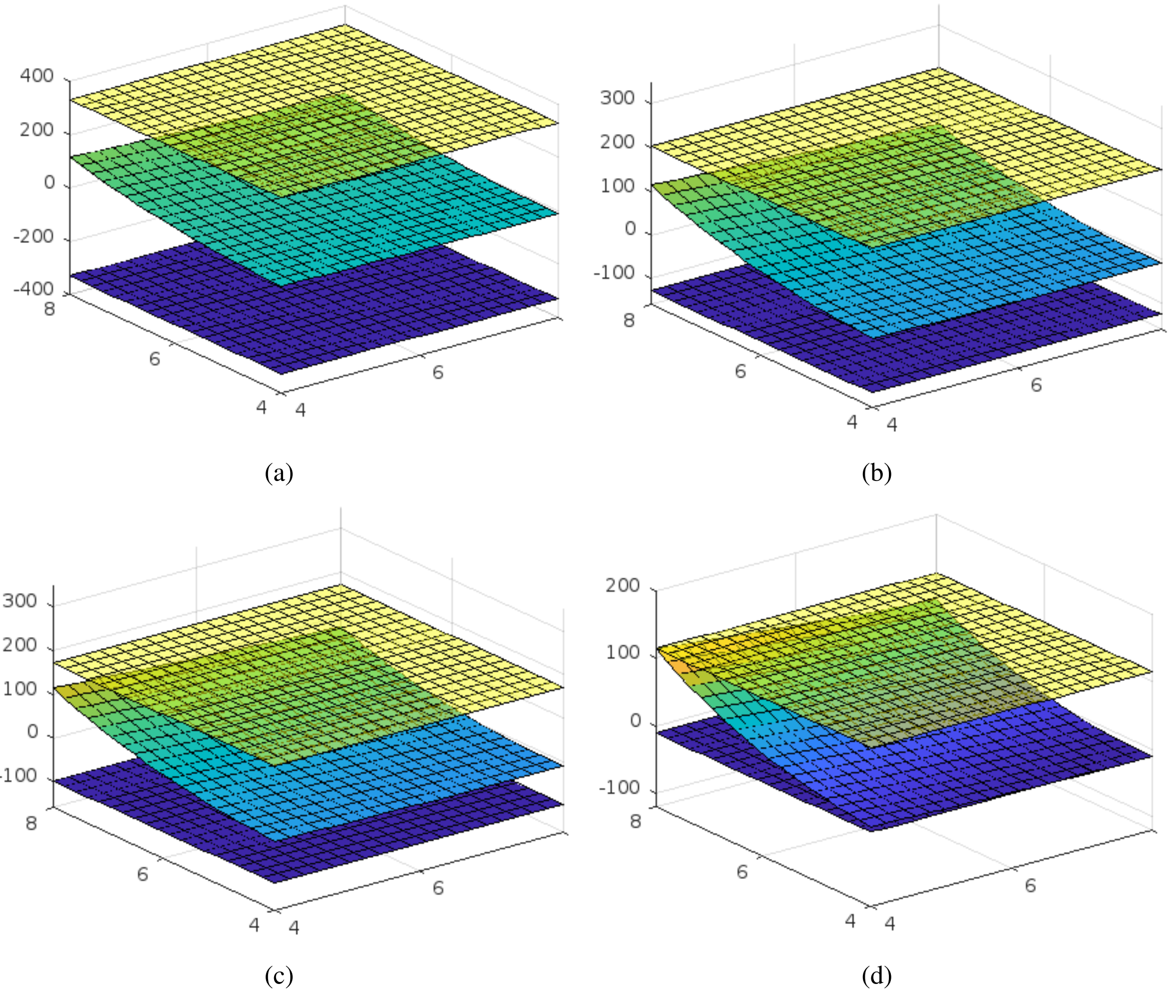


Figure 3.2: For $g(x,y) = \frac{1}{2}x^2 - 2xy + \frac{1}{3}y^3$, (a) absolute bounds over $[4,8] \times [4,8]$ (yellow and blue planes) and (b) simple bounds over $[4,8] \times [4,8]$ (yellow and blue planes), (c) interval bounds over $[4,8] \times [4,8]$ (yellow and blue planes) and (d) Bernstein bounds over $[4,8] \times [4,8]$ (yellow and blue planes).

Chapter 4

Bounds with Univariate Function Values

This Chapter begins with the known bounds using polynomial function values in the interval $[0,1]$ such as the Cargo and Shisha bound, found in 1965, as [9]; followed by the Rivlin bound, presented five years later as [20]. Then, starting from these, new bounds are presented. This study is extended to a closed interval in $\mathbb{R}_{\geqslant 0}$. In the last section, the computational times of these bounds are shown.

4.1 Bounds in the Interval [0,1]

Cargo-Shisha bound. Following the article of Cargo and Shisha [9], for a polynomial P with real coefficients, it is possible, given $\varepsilon > 0$, to compute, by means of arithmetical operations whose number can be readily determined, upper and lower bounds for $\{p(x) : 0 \leqslant x \leqslant 1\}$ which differ from the corresponding sharp bounds by less than ε. This is an immediate consequence of the simple observation that, if $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ $(n \geqslant 1)$ is a polynomial with real coefficients, then, for each positive integer m, for all x such that $0 \leqslant x \leqslant 1$

$$\min_{1\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\} - \frac{1}{m}\sum_{k=1}^{n} k|a_k| \leqslant p(x) \leqslant \max_{1\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\} + \frac{1}{m}\sum_{k=1}^{n} k|a_k| \tag{4.1}$$

follows from the Mean Value Theorem together with the inequality $|p'(x)| \leqslant \sum_{k=1}^{n} k|a_k|$ which holds for all x in the interval $0 \leqslant x \leqslant 1$.

Rivlin bound. Rivlin enunciated the following theorem, which improves the bound of Cargo and Shisha, as [20].

Theorem 4.1. *Suppose* $0 = t_0 < t_1 < \ldots < t_k = 1$, *and* $d_k = \max(t_{j+1} - t_j)$, $0 \leqslant j \leqslant k-1$, *then, if* $K = \{0, \ldots, k\}$

$$\min_{i\in K} p(t_i) - \frac{d_k^2}{8}\max_{x\in[0,1]}|p''(x)| \leqslant \min_{x\in[0,1]} p(x) \leqslant \max_{x\in[0,1]} p(x) \leqslant \max_{i\in K} p(t_i) + \frac{d_k^2}{8}\max_{x\in[0,1]}|p''(x)|. \tag{4.2}$$

Proof. Suppose $p(\xi) = \max_{x\in[0,1]} p(x)$ and

$$|t_j - \xi| \leqslant |t_i - \xi| \quad 0 \leqslant i \leqslant k, \tag{4.3}$$

so that

$$|t_j - \xi| \leqslant \frac{d_k}{2}. \tag{4.4}$$

According to Taylor's formula, it is possible to obtain

$$p(t_j) = p(\xi) + (t_j - \xi)p'(\xi) + \frac{(t_j - \xi)^2}{2} p''(\eta), \tag{4.5}$$

where $\eta \in [0,1]$. If $\xi = 0$ or $\xi = 1$, the right-most inequality in (4.2) is trivially true. If $0 < \xi < 1$ then $p'(\xi) = 0$ and (4.5) implies that

$$\max_{x\in[0,1]} p(x) \leqslant \max_{j\in K} p(t_j) + \frac{(t_j - \xi)^2}{2} |p''(\eta)| \tag{4.6}$$

$$\leqslant \max_{j\in K} p(t_j) + \frac{d_k^2}{8} \max_{\eta\in[0,1]} |p''(\eta)| \tag{4.7}$$

An entirely analogous argument establishes the lower bound on $\min_{x\in[0,1]} p(x)$ in (4.2). □

Corollary 4.1. *If $t_i = i/k$ then*

$$\min_{j\in K} p\left(\frac{j}{k}\right) - \frac{1}{8k^2} \sum_{j=0}^{n} (j-1)j|a_j| \leqslant \min_{x\in[0,1]} p(x) \tag{4.8}$$

$$\leqslant \max_{x\in[0,1]} p(x) \leqslant \max_{j\in K} p\left(\frac{j}{k}\right) + \frac{1}{8k^2} \sum_{j=0}^{n} (j-1)j|a_j|. \tag{4.9}$$

Proof. From hypothesis $t_i = i/k$ then $d_k = 1/k$, and since

$$\max_{x\in[0,1]} |p''(x)| \leqslant \sum_{j=0}^{n} (j-1)j|a_j| \tag{4.10}$$

(4.2) becomes

$$\min_{j\in K} p\left(\frac{j}{k}\right) - \frac{1}{8k^2} \sum_{j=0}^{n} (j-1)j|a_j| \leqslant \min_{x\in[0,1]} p(x) \tag{4.11}$$

$$\leqslant \max_{x\in[0,1]} p(x) \leqslant \max_{j\in K} p\left(\frac{j}{k}\right) + \frac{1}{8k^2} \sum_{j=0}^{n} (j-1)j|a_j| \tag{4.12}$$

which is the thesis. □

Building on these results, one can rigorously establish new bounds that further strengthen and generalize the existing theoretical framework.

4.1.1 New Cargo-Shisha bounds

Suppose $p(x) = \sum_{k=0}^{n} a_k x^k$, observing that

$$\overline{p}'_A = \sum_{k=1}^{n} k|a_k| \tag{4.13}$$

and starting from the inequalities (2.123) for the derivative, the derivative of p in Bernstein basis is:

$$\begin{aligned} p'(x) &= \sum_{k=1}^{n} k a_k x^{k-1} && (4.14)\\ &= \sum_{i=0}^{n-1} (i+1) a_{i+1} x^i \quad \text{where } i = k-1 && (4.15)\\ &= \sum_{j=0}^{n-1} b_j^{'(n-1)} B_j^{n-1}(x), && (4.16) \end{aligned}$$

where the Bernstein coefficients are

$$b_j^{'(n-1)} = \sum_{i=0}^{j} (i+1) a_{i+1} \frac{\binom{j}{i}}{\binom{n-1}{i}}. \tag{4.17}$$

Now, a new bound is obtained using the absolute value.

$$\begin{aligned} |p'(x)| &= \left| \sum_{k=0}^{n} k a_k x^{k-1} \right| && (4.18)\\ &= \left| \sum_{j=0}^{n-1} b_j^{'(n-1)} B_j^{n-1}(x) \right| && (4.19)\\ &\leqslant \sum_{j=0}^{n-1} |b_j^{'(n-1)} B_j^{n-1}(x)| && (4.20)\\ &= \sum_{j=0}^{n-1} |b_j^{'(n-1)}| B_j^{n-1}(x) && (4.21) \end{aligned}$$

because $B_j^{n-1}(x) \geqslant 0$. Moreover,

$$\begin{aligned} \sum_{j=0}^{n-1} |b_j^{'(n-1)}| B_j^{n-1}(x) &\leqslant \sum_{j=0}^{n-1} \left(\max_{0 \leqslant j \leqslant n-1} |b_j^{'(n-1)}| \right) B_j^{n-1}(x) && (4.22)\\ &= \max_{0 \leqslant j \leqslant n-1} |b_j^{'(n-1)}| \sum_{j=0}^{n-1} B_j^{n-1}(x) && (4.23)\\ &= \max_{0 \leqslant j \leqslant n-1} |b_j^{'(n-1)}| && (4.24) \end{aligned}$$

because $\sum_{j=0}^{n-1} B_j^{n-1}(x) = 1$.

So, using the idea of Cargo and Shisha, it is possible to define new bounds, by calculating Bernstein coefficients for the derivative of the polynomial p

$$\begin{aligned} &\min_{1 \leqslant k \leqslant m} \left\{ p\left(\frac{k}{m}\right) \right\} - \frac{1}{m} \max_{0 \leqslant j \leqslant n-1} |b_j^{'(n-1)}| && (4.25)\\ &\leqslant p(x) && (4.26)\\ &\leqslant \max_{1 \leqslant k \leqslant m} \left\{ p\left(\frac{k}{m}\right) \right\} + \frac{1}{m} \max_{0 \leqslant j \leqslant n-1} |b_j^{'(n-1)}|. && (4.27) \end{aligned}$$

This is the comparison for the derivative between the Cargo-Shisha bound and this new bound: set j such that $0 \leqslant j \leqslant n-1$

$$|b_j^{'(n-1)}| = \left|\sum_{i=0}^{j}(i+1)a_{i+1}\frac{\binom{j}{i}}{\binom{n-1}{i}}\right| \tag{4.28}$$

$$\leqslant \sum_{i=0}^{j}(i+1)|a_{i+1}|\frac{\binom{j}{i}}{\binom{n-1}{i}} \tag{4.29}$$

$$\leqslant \sum_{i=0}^{j}(i+1)|a_{i+1}| \tag{4.30}$$

because $\frac{\binom{j}{i}}{\binom{n-1}{i}} \leqslant 1$. In addition,

$$\sum_{i=0}^{j}(i+1)|a_{i+1}| \leqslant \sum_{k=1}^{n} k|a_k|. \tag{4.31}$$

Since j is arbitrary, calculating the maximum for j between 0 and l, using the bound (4.1) and the idea of Cargo and Shisha, the comparison between the bounds becomes

$$\min_{1\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\} - \frac{1}{m}\sum_{k=1}^{n} k|a_k| \tag{4.32}$$

$$\leqslant \min_{1\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\} - \frac{1}{m}\max_{0\leqslant j\leqslant n-1}|b_j^{'(n-1)}| \tag{4.33}$$

$$\leqslant p(x) \tag{4.34}$$

$$\leqslant \max_{1\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\} + \frac{1}{m}\max_{0\leqslant j\leqslant n-1}|b_j^{'(n-1)}| \tag{4.35}$$

$$\leqslant \max_{1\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\} + \frac{1}{m}\sum_{k=1}^{n} k|a_k|. \tag{4.36}$$

Using the bounds presented in Chapter 2 and their comparison, it is possible to obtain further bounds. Similarly, from inequalities (2.123), set $x \in [0,1]$.

$$\underline{p}_A([0,1]) \leqslant \underline{p}_S([0,1]) \leqslant \underline{p}_I([0,1]) \leqslant \underline{p}_B([0,1]) \leqslant p(x) \tag{4.37}$$

$$\leqslant \overline{p}_B([0,1]) \leqslant \overline{p}_I([0,1]) \leqslant \overline{p}_S([0,1]) \leqslant \overline{p}_A([0,1]) \tag{4.38}$$

it is possible to define an extension of the bounds on the interval $[0,1]$ to the absolute values.

$$|p(x)| \leqslant \max\{|\underline{p}_B([0,1])|, |\overline{p}_B([0,1])|\} \leqslant \max\{|\underline{p}_I([0,1])|, |\overline{p}_I([0,1])|\} \tag{4.39}$$

$$\leqslant \max\{|\underline{p}_S([0,1])|, |\overline{p}_S([0,1])|\} \leqslant \max\{|\underline{p}_A([0,1])|, \overline{p}_A([0,1])|\} \tag{4.40}$$

since $\max\{|\underline{p}_B([0,1])|, |\overline{p}_B([0,1])|\} = \max_{0\leqslant j\leqslant n}|b_j^{(n)}|$ and $\max\{|\underline{p}_A([0,1])|, |\overline{p}_A([0,1])|\} = \overline{p}_A([0,1])$, therefore

$$|p(x)| \leqslant \max_{0\leqslant j\leqslant n}|b_j^{(n)}| \leqslant \max\{|\underline{p}_I([0,1])|, |\overline{p}_I([0,1])|\} \tag{4.41}$$

$$\leqslant \max\{|\underline{p}_S([0,1])|, |\overline{p}_S([0,1])|\} \leqslant \overline{p}_A([0,1]). \tag{4.42}$$

Applying the law to the derivative, the following inequalities can be obtained.

$$|p'(x)| \leqslant \max_{0\leqslant j\leqslant n-1} |b_j^{'(n-1)}| \leqslant \max\{|\underline{p}_I'([0,1])|, |\overline{p}_I'([0,1])|\} \tag{4.43}$$

$$\leqslant \max\{|\underline{p}_S'([0,1])|, |\overline{p}_S'([0,1])|\} \leqslant \overline{p}_A'([0,1]). \tag{4.44}$$

So, inequalities (4.32)-(4.36) can be improved.

$$\min_{1\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\} - \frac{1}{m}\overline{p}_A'([0,1]) \leqslant \min_{1\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\} - \frac{1}{m}\max\{|\underline{p}_S'([0,1])|, |\overline{p}_S'([0,1])|\} \tag{4.45}$$

$$\leqslant \min_{1\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\} - \frac{1}{m}\max\{|\underline{p}_I'([0,1])|, |\overline{p}_I'([0,1])|\} \tag{4.46}$$

$$\leqslant \min_{1\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\} - \frac{1}{m}\max_{0\leqslant j\leqslant n-1} |b_j^{'(n-1)}| \tag{4.47}$$

$$\leqslant p(x) \tag{4.48}$$

$$\leqslant \max_{1\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\} + \frac{1}{m}\max_{0\leqslant j\leqslant n-1} |b_j^{'(n-1)}| \tag{4.49}$$

$$\leqslant \max_{1\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\} + \frac{1}{m}\max\{|\underline{p}_I'([0,1])|, |\overline{p}_I'([0,1])|\} \tag{4.50}$$

$$\leqslant \max_{1\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\} + \frac{1}{m}\max\{|\underline{p}_S'([0,1])|, |\overline{p}_S'([0,1])|\} \tag{4.51}$$

$$\leqslant \max_{1\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\} + \frac{1}{m}\overline{p}_A'([0,1]). \tag{4.52}$$

Rewriting with the respective definitions, the previous inequalities become as follows.

$$\min_{1\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\} - \frac{1}{m}\sum_{k=1}^{n} k|a_k| \tag{4.53}$$

$$\leqslant \min_{1\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\} - \frac{1}{m}\max\left\{\left|\sum_{k=1;a_k>0}^{n} ka_k\right|, \left|\sum_{k=1;a_k<0}^{n} ka_k\right|\right\} \tag{4.54}$$

$$\leqslant \min_{1\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\} - \frac{1}{m}\max\left\{\left|a_1 + \sum_{k=2;a_k>0}^{n} ka_k\right|, \left|a_1 + \sum_{k=2;a_k<0}^{n} ka_k\right|\right\} \tag{4.55}$$

$$\leqslant \min_{1\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\} - \frac{1}{m}\max_{0\leqslant j\leqslant n-1} |b_j^{'(n-1)}| \tag{4.56}$$

$$\leqslant p(x) \tag{4.57}$$

$$\leqslant \max_{1\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\} + \frac{1}{m}\max_{0\leqslant j\leqslant n-1} |b_j^{'(n-1)}| \tag{4.58}$$

$$\leqslant \max_{1\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\} + \frac{1}{m}\max\{\left|a_1 + \sum_{k=2;a_k>0}^{n} ka_k\right|, \left|a_1 + \sum_{k=2;a_k<0}^{n} ka_k\right|\right\} \tag{4.59}$$

$$\leqslant \max_{1\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\} + \frac{1}{m}\max\left\{\left|\sum_{k=1;a_k>0}^{n} ka_k\right|, \left|\sum_{k=1;a_k<0}^{n} ka_k\right|\right\} \tag{4.60}$$

$$\leqslant \max_{1\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\} + \frac{1}{m}\sum_{k=1}^{n} k|a_k|. \tag{4.61}$$

4.1.2 New Rivlin bounds

Suppose $p(x) = \sum_{k=0}^{n} a_k x^k$, observing that

$$p''_A = \sum_{j=2}^{n} (j-1)j|a_j| \tag{4.62}$$

and starting from the inequalities (2.123) for the derivative, suppose $p(x) = \sum_{k=0}^{n} a_k x^k$, so the second derivative of p in Bernstein basis is:

$$p''(x) = \left(\sum_{k=1}^{n} k a_k x^{k-1}\right)' \tag{4.63}$$

$$= \sum_{k=2}^{n} (k-1) k a_k x^{k-2} \tag{4.64}$$

$$= \sum_{i=0}^{n-2} (i+1)(i+2) a_{i+2} x^i, \tag{4.65}$$

where $i = k-2$. In addition,

$$\sum_{i=0}^{n-2} (i+1)(i+2) a_{i+2} x^i = \sum_{j=0}^{n-2} b_j^{''(n-2)} B_j^{n-2}(x), \tag{4.66}$$

where the Bernstein coefficients are

$$b_j^{''(n-2)} = \sum_{i=0}^{j} (i+1)(i+2) a_{i+2} \frac{\binom{j}{i}}{\binom{n-2}{i}}. \tag{4.67}$$

Now, a new bound is obtained using the absolute value.

$$|p''(x)| = \left|\sum_{k=0}^{n} (k-1) k a_k x^{k-2}\right| \tag{4.68}$$

$$= \left|\sum_{j=0}^{n-2} b_j^{''(n-2)} B_j^{n-2}(x)\right| \tag{4.69}$$

$$\leqslant \sum_{j=0}^{n-2} |b_j^{''(n-2)} B_j^{n-2}(x)| \tag{4.70}$$

$$= \sum_{j=0}^{n-2} |b_j^{''(n-2)}| B_j^{n-2}(x) \tag{4.71}$$

because $B_j^{n-2}(x) \geqslant 0$. Moreover,

$$\sum_{j=0}^{n-2} |b_j^{''(n-2)}| B_j^{n-2}(x) \leqslant \sum_{j=0}^{n-2} \left(\max_{0 \leqslant j \leqslant n-2} |b_j^{''(n-2)}|\right) B_j^{n-2}(x) \tag{4.72}$$

$$= \max_{0 \leqslant j \leqslant n-2} |b_j^{''(n-2)}| \sum_{j=0}^{n-2} B_j^{n-2}(x) \tag{4.73}$$

$$= \max_{0 \leqslant j \leqslant n-2} |b_j^{''(n-2)}| \tag{4.74}$$

because $\sum_{j=0}^{n-2} B_j^{n-2}(x) = 1$.

So, using a similar proof of Rivlin, it is possible to define a new bound.

$$\min_{j\in K} p\left(\frac{j}{k}\right) - \frac{1}{8k^2}\max_{0\leqslant j\leqslant n-2}|b_j^{''(n-2)}| \leqslant \min_{x\in[0,1]} p(x) \tag{4.75}$$

$$\leqslant \max_{x\in[0,1]} p(x) \tag{4.76}$$

$$\leqslant \max_{j\in K} p\left(\frac{j}{k}\right) + \frac{1}{8k^2}\max_{0\leqslant j\leqslant n-2}|b_j^{''(n-2)}|. \tag{4.77}$$

This is the comparison between the Rivlin bound and this new bound: set j such that $0 \leqslant j \leqslant n-2$,

$$|b_j^{''(n-2)}| = \left|\sum_{i=0}^{j}(i+1)(i+2)a_{i+2}\frac{\binom{j}{i}}{\binom{n-2}{i}}\right| \tag{4.78}$$

$$\leqslant \sum_{i=0}^{j}(i+1)(i+2)|a_{i+2}|\frac{\binom{j}{i}}{\binom{n-2}{i}} \tag{4.79}$$

$$\leqslant \sum_{i=0}^{j}(i+1)(i+2)|a_{i+2}| \tag{4.80}$$

because $\frac{\binom{j}{i}}{\binom{n-2}{i}} \leqslant 1$. Additionally,

$$\sum_{i=0}^{j}(i+1)(i+2)|a_{i+2}| \leqslant \sum_{i=0}^{n-2}(i+1)(i+2)|a_{i+2}| \tag{4.81}$$

$$= \sum_{k=2}^{n}(k-1)k|a_k| \tag{4.82}$$

So, using the bound (4.2) and the proofs of Theorem 4.1 and of Corollary 4.1 of Rivlin, the bounds become

$$\min_{j\in K} p\left(\frac{j}{k}\right) - \frac{1}{8k^2}\sum_{j=2}^{n}(j-1)j|a_j| \leqslant \min_{j\in K} p\left(\frac{j}{k}\right) - \frac{1}{8k^2}\max_{0\leqslant j\leqslant n-2}|b_j^{''(n-2)}| \tag{4.83}$$

$$\leqslant \min_{x\in[0,1]} p(x) \tag{4.84}$$

$$\leqslant \max_{x\in[0,1]} p(x) \tag{4.85}$$

$$\leqslant \max_{j\in K} p\left(\frac{j}{k}\right) + \frac{1}{8k^2}\max_{0\leqslant j\leqslant n-2}|b_j^{''(n-2)}| \tag{4.86}$$

$$\leqslant \max_{j\in K} p\left(\frac{j}{k}\right) + \frac{1}{8k^2}\sum_{j=2}^{n}(j-1)j|a_j|. \tag{4.87}$$

Using the bounds established in Chapter 2 and their comparison, additional bounds can be derived. Following the approach of Section 4.1.1, inequalities (4.41)-(4.42) are applied to the second derivative on $x \in [0,1]$, which leads to the estimate.

$$|p''(x)| \leqslant \max_{0\leqslant j\leqslant n-2}|b_j^{''(n-2)}| \leqslant \max\{|\underline{p}_I''([0,1])|, |\overline{p}_I''([0,1])|\} \tag{4.88}$$

$$\leqslant \max\{|\underline{p}_S''([0,1])|, |\overline{p}_S''([0,1])|\} \leqslant \overline{p}_A''([0,1]). \tag{4.89}$$

So, inequalities (4.83)-(4.87) can be improved.

$$
\begin{aligned}
\min_{0\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\}-\frac{1}{8k^2}\overline{p}''_A([0,1]) &\leqslant \min_{0\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\}-\frac{1}{8k^2}\max\{|\underline{p}''_S([0,1])|,|\overline{p}''_S([0,1])|\} && (4.90)\\
&\leqslant \min_{0\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\}-\frac{1}{8k^2}\max\{|\underline{p}''_I([0,1])|,|\overline{p}''_I([0,1])|\} && (4.91)\\
&\leqslant \min_{0\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\}-\frac{1}{8k^2}\max_{0\leqslant j\leqslant n-2}|b_j^{''(n-2)}| && (4.92)\\
&\leqslant \min_{x\in[0,1]} p(x) && (4.93)\\
&\leqslant \max_{x\in[0,1]} p(x) && (4.94)\\
&\leqslant \max_{0\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\}+\frac{1}{8k^2}\max_{0\leqslant j\leqslant n-2}|b_j^{''(n-2)}| && (4.95)\\
&\leqslant \max_{0\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\}+\frac{1}{8k^2}\max\{|\underline{p}''_I([0,1])|,|\overline{p}''_I([0,1])|\} && (4.96)\\
&\leqslant \max_{0\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\}+\frac{1}{8k^2}\max\{|\underline{p}''_S([0,1])|,|\overline{p}''_S([0,1])|\} && (4.97)\\
&\leqslant \max_{0\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\}+\frac{1}{8k^2}\overline{p}''_A([0,1]). && (4.98)
\end{aligned}
$$

Rewriting with the respective definitions, the previous inequalities become as follows.

$$
\begin{aligned}
&\min_{0\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\}-\frac{1}{8k^2}\sum_{j=2}^{n}(j-1)j|a_j| && (4.99)\\
&\leqslant \min_{0\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\}-\frac{1}{8k^2}\max\left\{\left|\sum_{j=2;a_j>0}^{n}(j-1)ja_j\right|,\left|\sum_{j=2;a_j<0}^{n}(j-1)ja_j\right|\right\} && (4.100)\\
&\leqslant \min_{0\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\}-\frac{1}{8k^2}\max\left\{\left|2a_2+\sum_{j=3;a_j>0}^{n}(j-1)ja_j\right|,\left|2a_2+\sum_{j=3;a_j<0}^{n}(j-1)ja_j\right|\right\} && (4.101)\\
&\leqslant \min_{0\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\}-\frac{1}{8k^2}\max_{0\leqslant j\leqslant n-2}|b_j^{''(n-2)}| && (4.102)\\
&\leqslant p(x) && (4.103)\\
&\leqslant \max_{0\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\}+\frac{1}{8k^2}\max_{0\leqslant j\leqslant n-2}|b_j^{''(n-2)}| && (4.104)\\
&\leqslant \max_{0\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\}+\frac{1}{8k^2}\max\left\{\left|2a_2+\sum_{j=3;a_j>0}^{n}(j-1)ja_j\right|,\left|2a_2+\sum_{j=3;a_j<0}^{n}(j-1)ja_j\right|\right\} && (4.105)\\
&\leqslant \max_{0\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\}+\frac{1}{8k^2}\max\left\{\left|\sum_{j=2;a_j>0}^{n}(j-1)ja_j\right|,\left|\sum_{j=2;a_j<0}^{n}(j-1)ja_j\right|\right\} && (4.106)\\
&\leqslant \max_{0\leqslant k\leqslant m}\left\{p\left(\frac{k}{m}\right)\right\}+\frac{1}{8k^2}\sum_{j=2}^{n}(j-1)j|a_j|. && (4.107)
\end{aligned}
$$

4.2 Bounds in the Interval $[\underline{x},\overline{x}] \subset \mathbb{R}_{\geqslant 0}$

4.2.1 Extension of the Cargo-Shisha bound

Given a closed bounded interval $D = [\underline{x},\overline{x}] \subset \mathbb{R}_{\geqslant 0}$ and a polynomial function $p \in \Pi_l$ expressed as in (2.2),

$$\min_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}-\frac{\overline{x}-\underline{x}}{m}\sum_{j=1}^{n} j|a_j|\overline{x}^{j-1} \tag{4.108}$$

$$\leqslant p(x) \tag{4.109}$$

$$\leqslant \max_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}+\frac{\overline{x}-\underline{x}}{m}\sum_{j=1}^{n} j|a_j|\overline{x}^{j-1} \tag{4.110}$$

which follows from the law of the mean and the inequality for x such that $\underline{x} \leqslant x \leqslant \overline{x}$ and from

$$|p'(x)| = \left|\sum_{j=1}^{n} ja_jx^{j-1}\right| \tag{4.111}$$

$$\leqslant \sum_{j=1}^{n} j|a_j|x^{j-1} \tag{4.112}$$

$$\leqslant \sum_{j=1}^{n} j|a_j|\overline{x}^{j-1}. \tag{4.113}$$

4.2.2 Extension of the Rivlin bound

Let $D = [\underline{x},\overline{x}] \subset \mathbb{R}_{\geqslant 0}$ be a closed bounded interval, and let $p \in \Pi_l$ be a polynomial function as defined in (2.2).

Theorem 4.2. *Suppose* $\underline{x} = t_0 < t_1 < ... < t_k = \overline{x}$*, and* $d_k = \max(t_{j+1}-t_j)$*,* $0 \leqslant j \leqslant k-1$*, then, if* $K = \{0,...,k\}$

$$\min_{i\in K} p(t_i) - \frac{d_k^2}{8}\max_{x\in[\underline{x},\overline{x}]}|p''(x)| \leqslant \min_{x\in[\underline{x},\overline{x}]} p(x) \leqslant \max_{x\in[\underline{x},\overline{x}]} p(x) \leqslant \max_{i\in K} p(t_i) + \frac{d_k^2}{8}\max_{x\in[\underline{x},\overline{x}]}|p''(x)|. \tag{4.114}$$

Proof. Suppose $p(\xi) = \max_{x\in[\underline{x},\overline{x}]} p(x)$ and

$$|t_j-\xi| \leqslant |t_i-\xi| \quad i = 0,\dots,k, \tag{4.115}$$

so that

$$|t_j-\xi| \leqslant \frac{d_k}{2}. \tag{4.116}$$

According to Taylor's formula, it is possible to obtain

$$p(t_j) = p(\xi) + (t_j-\xi)p'(\xi) + \frac{(t_j-\xi)^2}{2}p''(\eta), \tag{4.117}$$

where $\eta \in [\underline{x},\overline{x}]$. If $\xi = \underline{x},\overline{x}$, the right-most inequality in (4.114) is trivially true. If $\underline{x} < \xi < \overline{x}$ then $p'(\xi) = 0$ and (4.117) implies that

$$\max_{x\in[\underline{x},\overline{x}]} p(x) \leqslant \max_{j\in K} p(t_j) + \frac{(t_j-\xi)^2}{2}|p''(\eta)| \tag{4.118}$$

$$\leqslant \max_{j\in K} p(t_j) + \frac{d_k^2}{8}\max_{j\in[\underline{x},\overline{x}]}|p''(\eta)| \tag{4.119}$$

An entirely analogous argument establishes the lower bound on $\min_{x\in[\underline{x},\overline{x}]} p(x)$ in (4.114). □

Corollary 4.2. *If* $t_i = (\overline{x}-\underline{x})\frac{i}{k}+\underline{x}$ *then*

$$\min_{j\in K} p\left((\overline{x}-\underline{x})\frac{j}{k}+\underline{x}\right) - \frac{(\overline{x}-\underline{x})^2}{8k^2}\sum_{j=2}^{n}(j-1)j|a_j|\overline{x}^{j-2} \tag{4.120}$$

$$\leqslant \min_{x\in[\underline{x},\overline{x}]} p(x) \tag{4.121}$$

$$\leqslant \max_{x\in[\underline{x},\overline{x}]} p(x) \tag{4.122}$$

$$\leqslant \max_{j\in K} p\left((\overline{x}-\underline{x})\frac{j}{k}+\underline{x}\right) + \frac{(\overline{x}-\underline{x})^2}{8k^2}\sum_{j=2}^{n}(j-1)j|a_j|\overline{x}^{j-2}. \tag{4.123}$$

Proof. From hypothesis $t_i = (\overline{x}-\underline{x})\frac{i}{k}+\underline{x}$ then $d_k = (\overline{x}-\underline{x})/k$, and since

$$\max_{x\in[\underline{x},\overline{x}]}|p''(x)| \leqslant \sum_{j=2}^{n}(j-1)j|a_j|\overline{x}^{j-2} \tag{4.124}$$

(4.114) becomes

$$\min_{j\in K} p\left((\overline{x}-\underline{x})\frac{j}{k}+\underline{x}\right) - \frac{(\overline{x}-\underline{x})^2}{8k^2}\sum_{j=2}^{n}(j-1)j|a_j|\overline{x}^{j-2} \tag{4.125}$$

$$\leqslant \min_{x\in[\underline{x},\overline{x}]} p(x) \tag{4.126}$$

$$\leqslant \max_{x\in[\underline{x},\overline{x}]} p(x) \tag{4.127}$$

$$\leqslant \max_{j\in K} p\left((\overline{x}-\underline{x})\frac{j}{k}+\underline{x}\right) + \frac{(\overline{x}-\underline{x})^2}{8k^2}\sum_{j=2}^{n}(j-1)j|a_j|\overline{x}^{j-2} \tag{4.128}$$

which is the thesis. □

4.2.3 New bounds of the Cargo-Shisha extension

Observe that

$$\overline{x} \geqslant \overline{x}-\underline{x} \quad \text{and} \quad -\overline{x} \leqslant -(\overline{x}-\underline{x}) \tag{4.129}$$

and so,

$$(\overline{x}-\underline{x})\sum_{j=1}^{n} j|a_j|\overline{x}^{j-1} \leqslant \sum_{j=1}^{n} j|a_j|\overline{x}^{j}. \tag{4.130}$$

It is possible to define a new bound, which is better than the extension of the Cargo-Shisha bound.

$$\min_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}-\frac{1}{m}\sum_{j=1}^{n}j|a_j|\overline{x}^j \tag{4.131}$$

for the minimum and

$$\max_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}+\frac{1}{m}\sum_{j=1}^{n}j|a_j|\overline{x}^j \tag{4.132}$$

for the maximum. Therefore,

$$\min_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}-\frac{1}{m}\sum_{j=1}^{n}j|a_j|\overline{x}^j \tag{4.133}$$

$$\leqslant \min_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}-\frac{\overline{x}-\underline{x}}{m}\sum_{j=1}^{n}j|a_j|\overline{x}^{j-1} \tag{4.134}$$

$$\leqslant p(x) \tag{4.135}$$

$$\leqslant \max_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}+\frac{\overline{x}-\underline{x}}{m}\sum_{j=1}^{n}j|a_j|\overline{x}^{j-1} \tag{4.136}$$

$$\leqslant \max_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}+\frac{1}{m}\sum_{j=1}^{n}j|a_j|\overline{x}^j. \tag{4.137}$$

Suppose $p(x)=\sum_{k=0}^{n}a_kx^k$ and observe that

$$p'_A=\sum_{k=1}^{n}k|a_k|\overline{x}^k. \tag{4.138}$$

Starting from the inequalities (2.123) for the derivative, using the affine change as (2.50)

$$x=(\overline{x}-\underline{x})t+\underline{x} \tag{4.139}$$

and the Proposition 2.7, so the derivative of p in Bernstein basis is:

$$p'(x)=\sum_{k=0}^{n}ka_kx^{k-1} \tag{4.140}$$

$$=\sum_{k=1}^{n}ka_kx^{k-1} \tag{4.141}$$

$$=\sum_{i=0}^{n-1}(i+1)a_{i+1}x^i, \tag{4.142}$$

where $i=k-1$. In addition,

$$\sum_{i=0}^{n-1}(i+1)a_{i+1}x^i=\sum_{i=0}^{n-1}b_i^{'(D)}B_i^{n-1}(t), \tag{4.143}$$

where the Bernstein coefficients are

$$b_i^{'(D)}=\sum_{j=0}^{i}\frac{\binom{i}{j}}{\binom{n-1}{j}}(\overline{x}-\underline{x})^j\sum_{k=j}^{n-1}\binom{k}{j}\underline{x}^{k-j}(k+1)a_{k+1}\quad 0\leqslant i\leqslant n-1. \tag{4.144}$$

Now, a new bound is obtained using the absolute value

$$|p'(x)| = \left|\sum_{i=0}^{n-1} b_i^{'(D)} B_i^{n-1}(t)\right| \tag{4.145}$$

$$\leqslant \sum_{i=0}^{n-1} |b_i^{'(D)}| B_i^{n-1}(t) \tag{4.146}$$

because $B_i^{n-1}(t) \geqslant 0$. Furthermore,

$$\sum_{i=0}^{n-1} |b_i^{'(D)}| B_i^{n-1}(t) \leqslant \sum_{i=0}^{n-1} \max_{0\leqslant i\leqslant n-1} |b_i^{'(D)}| B_i^{n-1}(t) \tag{4.147}$$

$$= \max_{0\leqslant i\leqslant n-1} |b_i^{'(D)}| \sum_{i=0}^{n-1} B_i^{n-1}(t) \tag{4.148}$$

$$= \max_{0\leqslant i\leqslant n-1} |b_i^{'(D)}| \tag{4.149}$$

because $\sum_{i=0}^{n-1} B_i^{n-1}(t) = 1$.

So, using the idea of Cargo and Shisha, it is possible to define new bounds, calculating Bernstein coefficients for the derivative of the polynomial p

$$\min_{0\leqslant k\leqslant m} \left\{ p\left((\overline{x}-\underline{x})\frac{k}{m} + \underline{x} \right) \right\} - \frac{\overline{x}-\underline{x}}{m} \max_{0\leqslant i\leqslant n-1} |b_i^{'(D)}| \tag{4.150}$$

$$\leqslant p(x) \tag{4.151}$$

$$\leqslant \max_{0\leqslant k\leqslant m} \left\{ p\left((\overline{x}-\underline{x})\frac{k}{m} + \underline{x} \right) \right\} + \frac{\overline{x}-\underline{x}}{m} \max_{0\leqslant i\leqslant n-1} |b_i^{'(D)}|. \tag{4.152}$$

This is a comparison for the derivative between the extension of the Cargo-Shisha bound and this new bound. Let $\overline{x} > \underline{x} > 0$, $D = [\underline{x},\overline{x}]$ and $0 \leqslant i \leqslant n-1$. Consider the $i-$th coefficient of the Bernstein polynomials, which has expression (4.144)

So, under the assumption $\overline{x} > \underline{x} > 0$

$$|b_i^{'(D)}| = \left| \sum_{j=0}^{i} \frac{\binom{i}{j}}{\binom{n-1}{j}} (\overline{x}-\underline{x})^j \sum_{k=j}^{n-1} \binom{k}{j} \underline{x}^{k-j}(k+1)a_{k+1} \right| \tag{4.153}$$

$$= \left| \sum_{j=0}^{0} \frac{\binom{i}{j}}{\binom{n-1}{j}} (\overline{x}-\underline{x})^j \sum_{k=j}^{n-1} \binom{k}{j} \underline{x}^{k-j}(k+1)a_{k+1} \right. \tag{4.154}$$

$$\left. + \sum_{j=1}^{i} \frac{\binom{i}{j}}{\binom{n-1}{j}} (\overline{x}-\underline{x})^j \sum_{k=j}^{n-1} \binom{k}{j} \underline{x}^{k-j}(k+1)a_{k+1} \right| \tag{4.155}$$

$$= \left| \sum_{k=0}^{n-1} \underline{x}^k (k+1)a_{k+1} + \sum_{k=1}^{n-1} (k+1)a_{k+1} \sum_{j=1; j\leqslant k}^{i} \frac{\binom{i}{j}}{\binom{n-1}{j}} (\overline{x}-\underline{x})^j \binom{k}{j} \underline{x}^{k-j} \right| \tag{4.156}$$

$$= \left| \sum_{k=0; a_{k+1}<0}^{n-1} \underline{x}^k (k+1)a_{k+1} + \sum_{k=0; a_{k+1}>0}^{n-1} \underline{x}^k (k+1)a_{k+1} \right. \tag{4.157}$$

$$+ \sum_{k=1; a_{k+1}<0}^{n-1} (k+1)a_{k+1} \sum_{j=1; j\leqslant k}^{i} \frac{\binom{i}{j}}{\binom{n-1}{j}} (\overline{x}-\underline{x})^j \binom{k}{j} \underline{x}^{k-j} \tag{4.158}$$

$$\left. + \sum_{k=1; a_{k+1}>0}^{n-1} (k+1)a_{k+1} \sum_{j=1; j\leqslant k}^{i} \frac{\binom{i}{j}}{\binom{n-1}{j}} (\overline{x}-\underline{x})^j \binom{k}{j} \underline{x}^{k-j} \right| \tag{4.159}$$

$$= \left| \sum_{k=0; a_{k+1}<0}^{n-1} (k+1)a_{k+1} \sum_{j=0; j\leqslant k}^{i} \frac{\binom{i}{j}}{\binom{n-1}{j}} (\overline{x}-\underline{x})^j \binom{k}{j} \underline{x}^{k-j} \right. \tag{4.160}$$

$$\left. + \sum_{k=0; a_{k+1}>0}^{n-1} (k+1)a_{k+1} \sum_{j=0; j\leqslant k}^{i} \frac{\binom{i}{j}}{\binom{n-1}{j}} (\overline{x}-\underline{x})^j \binom{k}{j} \underline{x}^{k-j} \right| \tag{4.161}$$

$$\leqslant \left| \sum_{k=0; a_{k+1}<0}^{n-1} (k+1)a_{k+1} \sum_{j=0; j\leqslant k}^{i} \frac{\binom{i}{j}}{\binom{n-1}{j}} (\overline{x}-\underline{x})^j \binom{k}{j} \underline{x}^{k-j} \right| \tag{4.162}$$

$$+ \left| \sum_{k=0; a_{k+1}>0}^{n-1} (k+1)a_{k+1} \sum_{j=0; j\leqslant k}^{i} \frac{\binom{i}{j}}{\binom{n-1}{j}} (\overline{x}-\underline{x})^j \binom{k}{j} \underline{x}^{k-j} \right| \tag{4.163}$$

$$\leqslant \left| \sum_{k=0; a_{k+1}<0}^{n-1} (k+1)a_{k+1} \sum_{j=0; j\leqslant k}^{i} (\overline{x}-\underline{x})^j \binom{k}{j} \underline{x}^{k-j} \right| \tag{4.164}$$

$$+ \left| \sum_{k=0; a_{k+1}>0}^{n-1} (k+1)a_{k+1} \sum_{j=0; j\leqslant k}^{i} (\overline{x}-\underline{x})^j \binom{k}{j} \underline{x}^{k-j} \right| \tag{4.165}$$

because the ratio between the two binomial coefficients is less than or equal to 1. Additionally,

$$\left| \sum_{k=0; a_{k+1}<0}^{n-1} (k+1)a_{k+1} \sum_{j=0; j\leqslant k}^{i} (\overline{x}-\underline{x})^j \binom{k}{j} \underline{x}^{k-j} \right| \tag{4.166}$$

$$+ \left| \sum_{k=0; a_{k+1}>0}^{n-1} (k+1)a_{k+1} \sum_{j=0; j\leqslant k}^{i} (\overline{x}-\underline{x})^j \binom{k}{j} \underline{x}^{k-j} \right| \tag{4.167}$$

$$\leqslant \left| \sum_{k=0; a_{k+1}<0}^{n-1} (k+1)a_{k+1} \sum_{j=0}^{k} (\overline{x}-\underline{x})^j \binom{k}{j} \underline{x}^{k-j} \right| \tag{4.168}$$

$$+ \left| \sum_{k=0; a_{k+1}>0}^{n-1} (k+1)a_{k+1} \sum_{j=0}^{k} (\overline{x}-\underline{x})^j \binom{k}{j} \underline{x}^{k-j} \right| \tag{4.169}$$

because the number of addends in the sums with index j is less than or equal to i and k, and therefore it is less than or equal to k. Moreover,

$$\left|\sum_{k=0;a_{k+1}<0}^{n-1}(k+1)a_{k+1}\sum_{j=0}^{k}(\overline{x}-\underline{x})^j\binom{k}{j}\underline{x}^{k-j}\right| \tag{4.170}$$

$$+\left|\sum_{k=0;a_{k+1}>0}^{n-1}(k+1)a_{k+1}\sum_{j=0}^{k}(\overline{x}-\underline{x})^j\binom{k}{j}\underline{x}^{k-j}\right| \tag{4.171}$$

$$=\left|\sum_{k=0;a_{k+1}<0}^{n-1}(k+1)a_{k+1}\overline{x}^k\right|+\left|\sum_{k=0;a_{k+1}>0}^{n-1}(k+1)a_{k+1}\overline{x}^k\right| \tag{4.172}$$

for the binomial theorem. In addition,

$$\left|\sum_{k=0;a_{k+1}<0}^{n-1}(k+1)a_{k+1}\overline{x}^k\right|+\left|\sum_{k=0;a_{k+1}>0}^{n-1}(k+1)a_{k+1}\overline{x}^k\right| \tag{4.173}$$

$$=\sum_{k=0;a_{k+1}<0}^{n-1}(k+1)|a_{k+1}|\overline{x}^k+\sum_{k=0;a_{k+1}>0}^{n-1}(k+1)|a_{k+1}|\overline{x}^k \tag{4.174}$$

$$=\sum_{k=0}^{n-1}(k+1)|a_{k+1}|\overline{x}^k \tag{4.175}$$

Therefore,

$$|b_i^{'(D)}| \leqslant \sum_{k=0}^{n-1}(k+1)|a_{k+1}|\overline{x}^k \tag{4.176}$$

since i is arbitrary, calculate the maximum for i between 0 and $n-1$

$$\max_{0\leqslant i\leqslant n-1}|b_i^{'(D)}| \leqslant \sum_{k=0}^{n-1}(k+1)|a_{k+1}|\overline{x}^k. \tag{4.177}$$

Similar considerations can be used to prove the statement in the case $\underline{x}=0$. Therefore, using the bound (4.108) and the proof present in the article of Cargo and Shisha, the comparison between the bounds becomes as follows.

$$\min_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}-\frac{\overline{x}-\underline{x}}{m}\sum_{j=1}^{n}j|a_j|\overline{x}^{j-1} \tag{4.178}$$

$$\min_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}-\frac{\overline{x}-\underline{x}}{m}\max_{0\leqslant i\leqslant n-1}|b_i^{'(D)}| \tag{4.179}$$

$$\leqslant p(x) \tag{4.180}$$

$$\leqslant \max_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}+\frac{\overline{x}-\underline{x}}{m}\max_{0\leqslant i\leqslant n-1}|b_i^{'(D)}| \tag{4.181}$$

$$\leqslant \max_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}+\frac{\overline{x}-\underline{x}}{m}\sum_{j=1}^{n}j|a_j|\overline{x}^{j-1}. \tag{4.182}$$

Using the bounds presented in Chapter 2 and their comparison, it is possible to obtain the above inequalities, and also further bounds. Similarly as 4.1.1, from inequalities (2.123), set $x\in[\underline{x},\overline{x}]\subset\mathbb{R}_{\geqslant 0}$,

$$\underline{p}_A([\underline{x},\overline{x}]) \leqslant \underline{p}_S([\underline{x},\overline{x}]) \leqslant \underline{p}_I([\underline{x},\overline{x}]) \leqslant \underline{p}_B([\underline{x},\overline{x}]) \leqslant p(x) \tag{4.183}$$

$$\leqslant \overline{p}_B([\underline{x},\overline{x}]) \leqslant \overline{p}_I([\underline{x},\overline{x}]) \leqslant \overline{p}_S([\underline{x},\overline{x}]) \leqslant \overline{p}_A([\underline{x},\overline{x}]). \tag{4.184}$$

It is possible to define an extension of the bounds on the interval $[\underline{x},\overline{x}]$ to the absolute values.

$$|p(x)| \leqslant \max\{|\underline{p}_B([\underline{x},\overline{x}])|, |\overline{p}_B([\underline{x},\overline{x}])|\} \leqslant \max\{|\underline{p}_I([\underline{x},\overline{x}])|, |\overline{p}_I([\underline{x},\overline{x}])|\} \tag{4.185}$$

$$\leqslant \max\{|\underline{p}_S([\underline{x},\overline{x}])|, |\overline{p}_S([\underline{x},\overline{x}])|\} \leqslant \max\{|\underline{p}_A([\underline{x},\overline{x}])|, \overline{p}_A([\underline{x},\overline{x}])|\} \tag{4.186}$$

since $\max\{|\underline{p}_B([\underline{x},\overline{x}])|, |\overline{p}_B([\underline{x},\overline{x}])|\} = \max_{0\leqslant j\leqslant n} |b_j^{(n)}|$ and $\max\{|\underline{p}_A([\underline{x},\overline{x}])|, |\overline{p}_A([\underline{x},\overline{x}])|\} = \overline{p}_A([\underline{x},\overline{x}])$ then

$$|p(x)| \leqslant \max_{0\leqslant j\leqslant n} |b_j^{(n)}| \leqslant \max\{|\underline{p}_I([\underline{x},\overline{x}])|, |\overline{p}_I([\underline{x},\overline{x}])|\} \tag{4.187}$$

$$\leqslant \max\{|\underline{p}_S([\underline{x},\overline{x}])|, |\overline{p}_S([\underline{x},\overline{x}])|\} \leqslant \overline{p}_A([\underline{x},\overline{x}]). \tag{4.188}$$

Applying the law to the derivative, the following inequalities can be obtained.

$$|p'(x)| \leqslant \max_{0\leqslant i\leqslant n-1} |b_i^{'(D)}| \leqslant \max\{|\underline{p}'_I([\underline{x},\overline{x}])|, |\overline{p}'_I([\underline{x},\overline{x}])|\} \tag{4.189}$$

$$\leqslant \max\{|\underline{p}'_S([\underline{x},\overline{x}])|, |\overline{p}'_S([\underline{x},\overline{x}])|\} \leqslant \overline{p}'_A([\underline{x},\overline{x}]). \tag{4.190}$$

So, inequalities (4.178)-(4.182) can be improved.

$$\min_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\} - \frac{\overline{x}-\underline{x}}{m}\overline{p}'_A([\underline{x},\overline{x}]) \tag{4.191}$$

$$\leqslant \min_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\} - \frac{\overline{x}-\underline{x}}{m}\max\{|\underline{p}'_S([\underline{x},\overline{x}])|, |\overline{p}'_S([\underline{x},\overline{x}])|\} \tag{4.192}$$

$$\leqslant \min_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\} - \frac{\overline{x}-\underline{x}}{m}\max\{|\underline{p}'_I([\underline{x},\overline{x}])|, |\overline{p}'_I([\underline{x},\overline{x}])|\} \tag{4.193}$$

$$\leqslant \min_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\} - \frac{\overline{x}-\underline{x}}{m}\max_{0\leqslant i\leqslant n-1}|b_i^{'(D)}| \tag{4.194}$$

$$\leqslant p(x) \tag{4.195}$$

$$\leqslant \max_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\} + \frac{\overline{x}-\underline{x}}{m}\max_{0\leqslant i\leqslant n-1}|b_i^{'(D)}| \tag{4.196}$$

$$\leqslant \max_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\} + \frac{\overline{x}-\underline{x}}{m}\max\{|\underline{p}'_I([\underline{x},\overline{x}])|, |\overline{p}'_I([\underline{x},\overline{x}])|\} \tag{4.197}$$

$$\leqslant \max_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\} + \frac{\overline{x}-\underline{x}}{m}\max\{|\underline{p}'_S([\underline{x},\overline{x}])|, |\overline{p}'_S([\underline{x},\overline{x}])|\} \tag{4.198}$$

$$\leqslant \max_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\} + \frac{\overline{x}-\underline{x}}{m}\overline{p}'_A([\underline{x},\overline{x}]). \tag{4.199}$$

Rewriting with the respective definitions, the previous inequalities become as follows.

$$\min_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}-\frac{\overline{x}-\underline{x}}{m}\sum_{j=1}^{n} j|a_j|\overline{x}^{j-1} \tag{4.200}$$

$$\leqslant \min_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}-\frac{\overline{x}-\underline{x}}{m}\max\left\{\left|\sum_{k=1;a_k>0}^{n} ka_k\overline{x}^{k-1}\right|,\left|\sum_{k=1;a_k<0}^{n} ka_k\overline{x}^{k-1}\right|\right\} \tag{4.201}$$

$$\leqslant \min_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}-\frac{\overline{x}-\underline{x}}{m}\max\left\{\left|\sum_{k=1;a_k>0}^{n} \max\{ka_k\underline{x}^{k-1},ka_k\overline{x}^{k-1}\}\right|,\right. \tag{4.202}$$

$$\left.\left|\sum_{k=1;a_k<0}^{n} \min\{ka_k\underline{x}^{k-1},ka_k\overline{x}^{k-1}\}\right|\right\} \tag{4.203}$$

$$\leqslant \min_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}-\frac{\overline{x}-\underline{x}}{m}\max_{0\leqslant i\leqslant n-1}|b_i^{'(D)}| \tag{4.204}$$

$$\leqslant p(x) \tag{4.205}$$

$$\leqslant \max_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}+\frac{\overline{x}-\underline{x}}{m}\max_{0\leqslant i\leqslant n-1}|b_i^{'(D)}| \tag{4.206}$$

$$\leqslant \max_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}+\frac{\overline{x}-\underline{x}}{m}\max\left\{\left|\sum_{k=1;a_k>0}^{n} \max\{ka_k\underline{x}^{k-1},ka_k\overline{x}^{k-1}\}\right|,\right. \tag{4.207}$$

$$\left.\left|\sum_{k=1;a_k<0}^{n} \min\{ka_k\underline{x}^{k-1},ka_k\overline{x}^{k-1}\}\right|\right\} \tag{4.208}$$

$$\leqslant \max_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}+\frac{\overline{x}-\underline{x}}{m}\max\left\{\left|\sum_{k=1;a_k>0}^{n} ka_k\overline{x}^{k-1}\right|,\left|\sum_{k=1;a_k<0}^{n} ka_k\overline{x}^{k-1}\right|\right\} \tag{4.209}$$

$$\leqslant \max_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}+\frac{\overline{x}-\underline{x}}{m}\sum_{j=1}^{n} j|a_j|\overline{x}^{j-1}. \tag{4.210}$$

4.2.4 New bounds for the Rivlin extension

Let $D=[\underline{x},\overline{x}]\subset\mathbb{R}_{\geqslant 0}$, observe that

$$\overline{x}\geqslant\overline{x}-\underline{x} \tag{4.211}$$

$$\overline{x}^2\geqslant(\overline{x}-\underline{x})^2 \tag{4.212}$$

$$-\overline{x}^2\leqslant-(\overline{x}-\underline{x})^2 \tag{4.213}$$

and so,

$$(\overline{x}-\underline{x})^2\sum_{j=2}^{n}(j-1)j|a_j|\overline{x}^{j-2}\leqslant\sum_{j=2}^{n}(j-1)j|a_j|\overline{x}^{j}. \tag{4.214}$$

It is possible to define a new bound, which is better than the extension of the Rivlin bound.

$$\min_{j\in K} p\left((\overline{x}-\underline{x})\frac{j}{k}+\underline{x}\right)-\frac{1}{8k^2}\sum_{j=2}^{n}(j-1)j|a_j|\overline{x}^{j} \tag{4.215}$$

for the minimum and

$$\max_{j\in K} p\left((\overline{x}-\underline{x})\frac{j}{k}+\underline{x}\right)+\frac{1}{8k^2}\sum_{j=2}^{n}(j-1)j|a_j|\overline{x}^{j} \tag{4.216}$$

for the maximum. Therefore,

$$\min_{j\in K} p\left((\overline{x}-\underline{x})\frac{j}{k}+\underline{x}\right)-\frac{1}{8k^2}\sum_{j=2}^{n}(j-1)j|a_j|\overline{x}^j \tag{4.217}$$

$$\leqslant \min_{j\in K} p\left((\overline{x}-\underline{x})\frac{j}{k}+\underline{x}\right)-\frac{(\overline{x}-\underline{x})^2}{8k^2}\sum_{j=2}^{n}(j-1)j|a_j|\overline{x}^{j-2} \tag{4.218}$$

$$\leqslant \min_{x\in[\underline{x},\overline{x}]} p(x) \tag{4.219}$$

$$\leqslant \max_{x\in[\underline{x},\overline{x}]} p(x) \tag{4.220}$$

$$\leqslant \max_{j\in K} p\left((\overline{x}-\underline{x})\frac{j}{k}+\underline{x}\right)+\frac{(\overline{x}-\underline{x})^2}{8k^2}\sum_{j=2}^{n}(j-1)j|a_j|\overline{x}^{j-2} \tag{4.221}$$

$$\leqslant \max_{j\in K} p\left((\overline{x}-\underline{x})\frac{j}{k}+\underline{x}\right)+\frac{1}{8k^2}\sum_{j=2}^{n}(j-1)j|a_j|\overline{x}^j. \tag{4.222}$$

Suppose $p(x)=\sum_{k=0}^{n} a_k x^k$, observing that

$$p''_A=\sum_{j=2}^{n}(j-1)j|a_j|\overline{x}^{j-2} \tag{4.223}$$

using the affine change as (2.50)

$$x=(\overline{x}-\underline{x})t+\underline{x} \tag{4.224}$$

and the Proposition 2.7, so the second derivative of p in Bernstein basis is for the expression (4.63)

$$p''(x)=\sum_{j=0}^{n-2} b_j^{''(n-2)}B_j^{n-2}(t), \tag{4.225}$$

where the Bernstein coefficients are

$$b_i^{''(D)}=\sum_{j=0}^{i}\frac{\binom{i}{j}}{\binom{n-2}{j}}(\overline{x}-\underline{x})^j\sum_{k=j}^{n-2}\binom{k}{j}\underline{x}^{k-j}(k+1)(k+2)a_{k+2} \quad \text{with } 0\leqslant i\leqslant n-2,. \tag{4.226}$$

Now, a new bound is obtained using the absolute value

$$|p''(x)|=\left|\sum_{k=0}^{n}(k-1)ka_kx^{k-2}\right| \tag{4.227}$$

$$=\left|\sum_{j=0}^{n-2} b_j^{''(D)}B_j^{n-2}(t)\right| \tag{4.228}$$

$$\leqslant \sum_{j=0}^{n-2}|b_j^{''(D)}B_j^{n-2}(t)| \tag{4.229}$$

$$=\sum_{j=0}^{n-2}|b_j^{''(D)}|B_j^{n-2}(t) \quad \text{because } B_j^{n-2}(t)\geqslant 0 \tag{4.230}$$

$$\leqslant \sum_{j=0}^{n-2}\left(\max_{0\leqslant j\leqslant n-2}|b_j^{''(D)}|\right)B_j^{n-2}(t) \tag{4.231}$$

$$=\max_{0\leqslant j\leqslant n-2}|b_j^{''(D)}|\sum_{j=0}^{n-2}B_j^{n-2}(t) \tag{4.232}$$

$$=\max_{0\leqslant j\leqslant n-2}|b_j^{''(D)}| \quad \text{because } \sum_{j=0}^{n-2}B_j^{n-2}(t)=1. \tag{4.233}$$

So, using a similar proof of Rivlin, it is possible to define a new bound

$$\min_{j \in K} p\left((\overline{x}-\underline{x})\frac{j}{k}+\underline{x}\right)-\frac{(\overline{x}-\underline{x})^2}{8k^2}\max_{0\leqslant j\leqslant n-2}|b_j^{''(D)}| \tag{4.234}$$

$$\leqslant \min_{x\in[\underline{x},\overline{x}]} p(x) \tag{4.235}$$

$$\leqslant \max_{x\in[\underline{x},\overline{x}]} p(x) \tag{4.236}$$

$$\leqslant \max_{j \in K} p\left((\overline{x}-\underline{x})\frac{j}{k}+\underline{x}\right)+\frac{(\overline{x}-\underline{x})^2}{8k^2}\max_{0\leqslant j\leqslant n-2}|b_j^{''(D)}|. \tag{4.237}$$

This presents a comparison between the extension of the Rivlin bound and the newly derived bound: set i such that $0 \leqslant i \leqslant n-2$

$$|b_i^{''(D)}| = \left|\sum_{j=0}^{i}\frac{\binom{i}{j}}{\binom{n-2}{j}}(\overline{x}-\underline{x})^j\sum_{k=j}^{n-2}\binom{k}{j}\underline{x}^{k-j}(k+1)(k+2)a_{k+2}\right| \tag{4.238}$$

$$= \left|\sum_{j=0}^{0}\frac{\binom{i}{j}}{\binom{n-2}{j}}(\overline{x}-\underline{x})^j\sum_{k=j}^{n-2}\binom{k}{j}\underline{x}^{k-j}(k+1)(k+2)a_{k+2}\right. \tag{4.239}$$

$$\left.+\sum_{j=1}^{i}\frac{\binom{i}{j}}{\binom{n-2}{j}}(\overline{x}-\underline{x})^j\sum_{k=j}^{n-2}\binom{k}{j}\underline{x}^{k-j}(k+1)(k+2)a_{k+2}\right| \tag{4.240}$$

$$= \left|\sum_{k=0}^{n-2}\underline{x}^k(k+1)(k+2)a_{k+2}+\sum_{k=1}^{n-2}(k+1)(k+2)a_{k+2}\sum_{j=1;j\leqslant k}^{i}\frac{\binom{i}{j}}{\binom{n-2}{j}}(\overline{x}-\underline{x})^j\binom{k}{j}\underline{x}^{k-j}\right| \tag{4.241}$$

$$= \left|\sum_{k=0;a_{k+2}<0}^{n-2}\underline{x}^k(k+1)(k+2)a_{k+2}+\sum_{k=0;a_{k+2}>0}^{n-2}\underline{x}^k(k+1)(k+2)a_{k+2}\right. \tag{4.242}$$

$$+\sum_{k=1;a_{k+2}<0}^{n-2}(k+1)(k+2)a_{k+2}\sum_{j=1;j\leqslant k}^{i}\frac{\binom{i}{j}}{\binom{n-2}{j}}(\overline{x}-\underline{x})^j\binom{k}{j}\underline{x}^{k-j} \tag{4.243}$$

$$\left.+\sum_{k=1;a_{k+2}>0}^{n-2}(k+1)(k+2)a_{k+2}\sum_{j=1;j\leqslant k}^{i}\frac{\binom{i}{j}}{\binom{n-2}{j}}(\overline{x}-\underline{x})^j\binom{k}{j}\underline{x}^{k-j}\right| \tag{4.244}$$

$$= \left|\sum_{k=0;a_{k+2}<0}^{n-2}(k+1)(k+2)a_{k+2}\sum_{j=0;j\leqslant k}^{i}\frac{\binom{i}{j}}{\binom{n-2}{j}}(\overline{x}-\underline{x})^j\binom{k}{j}\underline{x}^{k-j}\right. \tag{4.245}$$

$$\left.+\sum_{k=0;a_{k+2}>0}^{n-2}(k+1)(k+2)a_{k+2}\sum_{j=0;j\leqslant k}^{i}\frac{\binom{i}{j}}{\binom{n-2}{j}}(\overline{x}-\underline{x})^j\binom{k}{j}\underline{x}^{k-j}\right| \tag{4.246}$$

$$\leqslant \left|\sum_{k=0;a_{k+2}<0}^{n-2}(k+1)(k+2)a_{k+2}\sum_{j=0;j\leqslant k}^{i}\frac{\binom{i}{j}}{\binom{n-2}{j}}(\overline{x}-\underline{x})^j\binom{k}{j}\underline{x}^{k-j}\right| \tag{4.247}$$

$$+\left|\sum_{k=0;a_{k+2}>0}^{n-2}(k+1)(k+2)a_{k+2}\sum_{j=0;j\leqslant k}^{i}\frac{\binom{i}{j}}{\binom{n-2}{j}}(\overline{x}-\underline{x})^j\binom{k}{j}\underline{x}^{k-j}\right| \tag{4.248}$$

$$\leqslant \left|\sum_{k=0;a_{k+2}<0}^{n-2}(k+1)(k+2)a_{k+2}\sum_{j=0;j\leqslant k}^{i}(\overline{x}-\underline{x})^j\binom{k}{j}\underline{x}^{k-j}\right| \tag{4.249}$$

$$+\left|\sum_{k=0;a_{k+2}>0}^{n-2}(k+1)(k+2)a_{k+2}\sum_{j=0;j\leqslant k}^{i}(\overline{x}-\underline{x})^j\binom{k}{j}\underline{x}^{k-j}\right| \tag{4.250}$$

because the two ratios between the binomial coefficients are less than or equal to 1. Furthermore,

$$\Big|\sum_{k=0;a_{k+2}<0}^{n-2}(k+1)(k+2)a_{k+2}\sum_{j=0;j\leqslant k}^{i}(\overline{x}-\underline{x})^j\binom{k}{j}\underline{x}^{k-j}\Big| \tag{4.251}$$

$$+\Big|\sum_{k=0;a_{k+2}>0}^{n-2}(k+1)(k+2)a_{k+2}\sum_{j=0;j\leqslant k}^{i}(\overline{x}-\underline{x})^j\binom{k}{j}\underline{x}^{k-j}\Big| \tag{4.252}$$

$$\leqslant\Big|\sum_{k=0;a_{k+2}<0}^{n-2}(k+1)(k+2)a_{k+2}\sum_{j=0}^{k}(\overline{x}-\underline{x})^j\binom{k}{j}\underline{x}^{k-j}\Big| \tag{4.253}$$

$$+\Big|\sum_{k=0;a_{k+2}>0}^{n-2}(k+1)(k+2)a_{k+2}\sum_{j=0}^{k}(\overline{x}-\underline{x})^j\binom{k}{j}\underline{x}^{k-j}\Big| \tag{4.254}$$

because the number of addends of the sums with index j are less than or equal to i and to k, so they are less than or equal to k. In addition,

$$\Big|\sum_{k=0;a_{k+2}<0}^{n-2}(k+1)(k+2)a_{k+2}\sum_{j=0}^{k}(\overline{x}-\underline{x})^j\binom{k}{j}\underline{x}^{k-j}\Big| \tag{4.255}$$

$$+\Big|\sum_{k=0;a_{k+2}>0}^{n-2}(k+1)(k+2)a_{k+2}\sum_{j=0}^{k}(\overline{x}-\underline{x})^j\binom{k}{j}\underline{x}^{k-j}\Big| \tag{4.256}$$

$$=\Big|\sum_{k=0;a_{k+2}<0}^{n-2}(k+1)(k+2)a_{k+2}\overline{x}^k\Big|+\Big|\sum_{k=0;a_{k+2}>0}^{n-2}(k+1)(k+2)a_{k+2}\overline{x}^k\Big| \tag{4.257}$$

for the binomial theorem. Moreover,

$$\Big|\sum_{k=0;a_{k+2}<0}^{n-2}(k+1)(k+2)a_{k+2}\overline{x}^k\Big|+\Big|\sum_{k=0;a_{k+2}>0}^{n-2}(k+1)(k+2)a_{k+2}\overline{x}^k\Big| \tag{4.258}$$

$$=\sum_{k=0;a_{k+2}<0}^{n-2}(k+1)(k+2)|a_{k+2}|\overline{x}^k+\sum_{k=0;a_{k+2}>0}^{n-2}(k+1)(k+2)|a_{k+2}|\overline{x}^k \tag{4.259}$$

$$=\sum_{k=0}^{n-2}(k+1)(k+2)|a_{k+2}|\overline{x}^k \tag{4.260}$$

$$=\sum_{j=2}^{n}(j-1)j|a_j|\overline{x}^{j-2}. \tag{4.261}$$

Therefore,

$$|b_i^{''(D)}|\leqslant\sum_{j=2}^{n}(j-1)j|a_j|\overline{x}^{j-2} \tag{4.262}$$

since i is arbitrary, calculate the maximum for i between 0 and $n-2$

$$\max_i |b_i^{''(D)}|\leqslant\sum_{j=2}^{n}(j-1)j|a_j|\overline{x}^{j-2}. \tag{4.263}$$

Similar considerations can be used to prove in case of $\underline{x}=0$. So, using the bounds (4.234) and (4.237) and the proof present in the article of Rivlin, the bounds become

$$\min_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}-\frac{1}{8k^2}\sum_{j=2}^{n}(j-1)j|a_j|\overline{x}^j \tag{4.264}$$

$$\leqslant \min_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}-\frac{(\overline{x}-\underline{x})^2}{8k^2}\sum_{j=2}^{n}(j-1)j|a_j|\overline{x}^{j-2} \tag{4.265}$$

$$\leqslant \min_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}-\frac{(\overline{x}-\underline{x})^2}{8k^2}\max_{0\leqslant j\leqslant n-2}|b_j^{''(D)}| \tag{4.266}$$

$$\leqslant \min_{x\in[\underline{x},\overline{x}]} p(x) \tag{4.267}$$

$$\leqslant \max_{x\in[\underline{x},\overline{x}]} p(x) \tag{4.268}$$

$$\leqslant \max_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}+\frac{(\overline{x}-\underline{x})^2}{8k^2}\max_{0\leqslant j\leqslant n-2}|b_j^{''(D)}| \tag{4.269}$$

$$\leqslant \max_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}+\frac{(\overline{x}-\underline{x})^2}{8k^2}\sum_{j=2}^{n}(j-1)j|a_j|\overline{x}^{j-2} \tag{4.270}$$

$$\leqslant \max_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}+\frac{1}{8k^2}\sum_{j=2}^{n}(j-1)j|a_j|\overline{x}^j. \tag{4.271}$$

So, using the maximum of the Bernstein coefficients, the bound is better than the extension of Rivlin.

Using the bounds presented in Chapter 2 and their comparison, it is possible to obtain the above inequalities, and also further bounds. Similarly as 4.2.3, from inequalities (4.183), set $x \in [\underline{x},\overline{x}] \subset \mathbb{R}_{\geqslant 0}$, it is possible to extend the bounds on the interval $[\underline{x},\overline{x}]$ to obtain bounds on the absolute value of the second derivative.

$$|p''(x)| \leqslant \max_{0\leqslant i\leqslant n-2}|b_i^{''(D)}| \leqslant \max\{|\underline{p}_I''([\underline{x},\overline{x}])|, |\overline{p}_I''([\underline{x},\overline{x}])|\} \tag{4.272}$$

$$\leqslant \max\{|\underline{p}_S''([\underline{x},\overline{x}])|, |\overline{p}_S''([\underline{x},\overline{x}])|\} \leqslant \overline{p}_A''([\underline{x},\overline{x}]). \tag{4.273}$$

So, inequalities (4.264)-(4.271) can be improved.

$$\min_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}-\frac{1}{8k^2}\sum_{j=2}^{n}(j-1)j|a_j|\overline{x}^j \tag{4.274}$$

$$\leqslant \min_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}-\frac{(\overline{x}-\underline{x})^2}{8k^2}\overline{p}_A''([\underline{x},\overline{x}]) \tag{4.275}$$

$$\leqslant \min_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}-\frac{(\overline{x}-\underline{x})^2}{8k^2}\max\{|\underline{p}_S''([\underline{x},\overline{x}])|, |\overline{p}_S''([\underline{x},\overline{x}])|\} \tag{4.276}$$

$$\leqslant \min_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}-\frac{(\overline{x}-\underline{x})^2}{8k^2}\max\{|\underline{p}_I''([\underline{x},\overline{x}])|, |\overline{p}_I''([\underline{x},\overline{x}])|\} \tag{4.277}$$

$$\leqslant \min_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}-\frac{(\overline{x}-\underline{x})^2}{8k^2}\max_{0\leqslant i\leqslant n-2}|b_i^{''(D)}| \tag{4.278}$$

$$\leqslant \min_{x\in[\underline{x},\overline{x}]} p(x) \tag{4.279}$$

$$\leqslant \max_{x\in[\underline{x},\overline{x}]} p(x) \tag{4.280}$$

$$\leqslant \max_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}+\frac{(\overline{x}-\underline{x})^2}{8k^2}\max_{0\leqslant i\leqslant n-2}|b_i^{''(D)}| \tag{4.281}$$

$$\leqslant \max_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}+\frac{(\overline{x}-\underline{x})^2}{8k^2}\max\{|\underline{p}_I''([\underline{x},\overline{x}])|,|\overline{p}_I''([\underline{x},\overline{x}])|\} \tag{4.282}$$

$$\leqslant \max_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}+\frac{(\overline{x}-\underline{x})^2}{8k^2}\max\{|\underline{p}_S''([\underline{x},\overline{x}])|,|\overline{p}_S''([\underline{x},\overline{x}])|\} \tag{4.283}$$

$$\leqslant \max_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}+\frac{(\overline{x}-\underline{x})^2}{8k^2}\overline{p}_A''([\underline{x},\overline{x}]). \tag{4.284}$$

Rewriting with the respective definitions, the previous inequalities become as follows.

$$\min_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}-\frac{1}{8k^2}\sum_{j=2}^{n}(j-1)j|a_j|\overline{x}^j \tag{4.285}$$

$$\leqslant \min_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}-\frac{(\overline{x}-\underline{x})^2}{8k^2}\sum_{j=2}^{n}(j-1)j|a_j|\overline{x}^{j-2} \tag{4.286}$$

$$\leqslant \min_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}-\frac{(\overline{x}-\underline{x})^2}{8k^2}\max\left\{\left|\sum_{j=2;a_j>0}^{n}(j-1)ja_j\overline{x}^{j-2}\overline{x}^k\right|,\right. \tag{4.287}$$

$$\left.\left|\sum_{j=2;a_j<0}^{n}(j-1)ja_j\overline{x}^{j-2}\right|\right\} \tag{4.288}$$

$$\leqslant \min_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}-\frac{(\overline{x}-\underline{x})^2}{8k^2}\max\left\{\left|\sum_{j=2;a_j>0}^{n}\max\{(j-1)ja_j\underline{x}^{j-2},\right.\right. \tag{4.289}$$

$$\left.\left.(j-1)ja_j\overline{x}^{j-2}\}\right|,\left|\sum_{j=2;a_j<0}^{n}\min\{(j-1)ja_j\underline{x}^{j-2},(j-1)ja_j\overline{x}^{j-2}\}\right|\right\} \tag{4.290}$$

$$\leqslant \min_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}-\frac{(\overline{x}-\underline{x})^2}{8k^2}\max_{0\leqslant i\leqslant n-2}|b_i^{''(D)}| \tag{4.291}$$

$$\leqslant \min_{x\in[\underline{x},\overline{x}]}p(x) \tag{4.292}$$

$$\leqslant \max_{x\in[\underline{x},\overline{x}]}p(x) \tag{4.293}$$

$$\leqslant \max_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}+\frac{(\overline{x}-\underline{x})^2}{8k^2}\max_{0\leqslant i\leqslant n-2}|b_i^{''(D)}| \tag{4.294}$$

$$\leqslant \max_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}+\frac{(\overline{x}-\underline{x})^2}{8k^2}\max\left\{\left|\sum_{j=2;a_j>0}^{n}\max\{(j-1)ja_j\underline{x}^{j-2},\right.\right. \tag{4.295}$$

$$\left.\left.(j-1)ja_j\overline{x}^{j-2}\}\right|,\left|\sum_{j=2;a_j<0}^{n}\min\{(j-1)ja_j\underline{x}^{j-2},(j-1)ja_j\overline{x}^{k}\}\right|\right\} \tag{4.296}$$

$$\leqslant \max_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}+\frac{(\overline{x}-\underline{x})^2}{8k^2}\max\left\{\left|\sum_{k=1;a_k>0}^{n}(j-1)ja_j\overline{x}^{j-2}\right|,\right. \tag{4.297}$$

$$\left.\left|\sum_{k=1;a_k<0}^{n}(j-1)ja_j\overline{x}^{j-2}\right|\right\} \tag{4.298}$$

$$\leqslant \max_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}+\frac{(\overline{x}-\underline{x})^2}{8k^2}\sum_{j=2}^{n}(j-1)j|a_j|\overline{x}^{j-2} \tag{4.299}$$

$$\leqslant \max_{0\leqslant k\leqslant m}\left\{p\left((\overline{x}-\underline{x})\frac{k}{m}+\underline{x}\right)\right\}+\frac{1}{8k^2}\sum_{j=2}^{n}(j-1)j|a_j|\overline{x}^j. \tag{4.300}$$

4.3 Note on the Computational Complexity

Set k the number of points and l the maximum degree of the variable. For the Cargo-Shisha bound and also for its extension, using Horner's rule to evaluate a polynomial at a point, the computational time complexity is $O(k \cdot l)$.

For the Rivlin bound and also for its extension, the computational time is the same, that is $O(k \cdot l)$.

Also, for bounds using polynomial function values linked to the arithmetic interval and simple bounds, the computational time is the same, i.e. $O(k \cdot l)$.

Instead, to compute Bernstein polynomials as in the article of Ray and Nataraj, with the calculus of the deifnition, or with Garloff's method, or with the Ray and Nataraj method, the computational time is always $O(l^2)$ for both the unit box and a general box domain. Therefore, for the new bounds related to Bernstein polynomials, defined in (4.25) and (4.27), (4.75) and (4.77), (4.150) and (4.152),(4.234) and (4.237), the computational time is $O(l^2 + k \cdot l)$.

Chapter 5

Bounds with Multivariate Function Values

The focus of this Chapter is to define and expand the bounds using polynomial function values to the intervals to $\mathbb{R}^n_{\geqslant 0}$. At the beginning, the interval $[0,1]^2$ is examined, starting from the bound of Garloff, who in 1985 extended the bound of Rivlin, followed by new improving bounds. The previous bounds are extended before on $[0,1]^n$ and then on a closed interval of $\mathbb{R}^n_{\geqslant 0}$. In the last section, the computational times of the bounds in $\mathbb{R}^n_{\geqslant 0}$ are shown.

5.1 Bounds in the Interval $[0,1]^2$

Garloff bound. Garloff extended the method developed by Theorem 4.1 of Rivlin for the univariate case, as [12]. For a positive integer k define $K = \{(i,j)|0 \leqslant i,j \leqslant k\}$. The bounds involve the function values of the polynomial on the grid in the unit square $I = [0,1]^2$ given by $\mathbf{t}_{ij} = (\frac{i}{k}, \frac{j}{k})$, with $(i,j) \in K$.

Theorem 5.1. *Let p be given by*

$$p(x,y) = \sum_{i,j=0}^{n} a_{ij} x^i y^j. \tag{5.1}$$

Then

$$\max_{(x,y)\in I} p(x,y) \leqslant \max_{(i,j)\in K} p(\mathbf{t}_{ij}) + \alpha_k \tag{5.2}$$

$$\min_{(x,y)\in I} p(x,y) \geqslant \min_{(i,j)\in K} p(\mathbf{t}_{ij}) - \alpha_k, \tag{5.3}$$

where

$$\alpha_k = \frac{1}{8k^2} \sum_{i,j=0}^{n} (i+j)(i+j-1)|a_{ij}|. \tag{5.4}$$

Garloff added that the generalization to the higher-dimensional case is obvious [12]. This is the proof non-presented in 1985.

Proof. Suppose $p(\boldsymbol{\xi}) = \max_{\mathbf{x}\in[0,1]^2} p(\mathbf{x}) = p(\xi_x, \xi_y)$ and, given a norm n on $\mathbb{R}^2$,

$$\|\mathbf{t}_{ij} - \boldsymbol{\xi}\|_n \leqslant \|\mathbf{t}_{lm} - \boldsymbol{\xi}\|_n \quad \text{for all } 0 \leqslant l, m \leqslant k; \tag{5.5}$$

so that

$$\|\mathbf{t}_{ij} - \boldsymbol{\xi}\|_\infty \leqslant \frac{1}{2k}. \tag{5.6}$$

According to Taylor's formula, it is possible to obtain

$$p(\mathbf{t}_{ij}) = p\left(\frac{i}{k}, \frac{j}{k}\right) \tag{5.7}$$

$$= p(\boldsymbol{\xi}) + \left(\frac{i}{k} - \xi_x\right) p_x(\boldsymbol{\xi}) + \left(\frac{j}{k} - \xi_y\right) p_y(\boldsymbol{\xi}) \tag{5.8}$$

$$+ \left(\frac{i}{k} - \xi_x\right)^2 \frac{p_{xx}(\boldsymbol{\eta})}{2} + \left(\frac{i}{k} - \xi_x\right)\left(\frac{j}{k} - \xi_y\right) p_{xy}(\boldsymbol{\eta}) + \left(\frac{j}{k} - \xi_y\right)^2 \frac{p_{yy}(\boldsymbol{\eta})}{2}, \tag{5.9}$$

where $\boldsymbol{\eta} \in [0,1]^2$. If $\boldsymbol{\xi} \in \{(0,0), (1,0), (0,1), (1,1)\}$, inequality (5.2) is trivially true. If $\boldsymbol{\xi}$ belongs to the border of the interval $[0,1]^2$, that is, $\boldsymbol{\xi} \in \partial[0,1]^2$. Then, it is possible to reduce to a Taylor expansion with one variable by parameterizing the part of the border with $\boldsymbol{\xi}$, making the function dependent on a parameter. So, the dimension is one and for Corollary 4.1, the inequality (5.2) is true.

Otherwise, if $\boldsymbol{\xi}$ does not belong to the border, $p_x(\boldsymbol{\xi}) = p_y(\boldsymbol{\xi}) = 0$, and (5.7)-(5.9) imply that

$$\max_{\mathbf{x}\in[0,1]^2} p(\mathbf{x}) \leqslant \max_{i,j\in K} p(\mathbf{t}_{ij}) \tag{5.10}$$

$$+ \left(\frac{i}{k} - \xi_x\right)^2 \frac{|p_{xx}(\boldsymbol{\eta})|}{2} + \left|\frac{i}{k} - \xi_x\right| \left|\frac{j}{k} - \xi_y\right| |p_{xy}(\boldsymbol{\eta})| + \left(\frac{j}{k} - \xi_y\right)^2 \frac{|p_{yy}(\boldsymbol{\eta})|}{2} \tag{5.11}$$

$$\leqslant \max_{i,j\in K} p(\mathbf{t}_{ij}) + \frac{1}{8k^2}|p_{xx}(\boldsymbol{\eta})| + \frac{1}{4k^2}|p_{xy}(\boldsymbol{\eta})| + \frac{1}{8k^2}|p_{yy}(\boldsymbol{\eta})|. \tag{5.12}$$

Observe for the second derivative of p

$$p_{xx}(x,y) = \sum_{i,j=0}^{n} (i-1) i a_{ij} x^{i-2} y^j \tag{5.13}$$

$$p_{xy}(x,y) = \sum_{i,j=0}^{n} ij a_{ij} x^{i-1} y^{j-1} \tag{5.14}$$

$$p_{yy}(x,y) = \sum_{i,j=0}^{n} (j-1) j a_{ij} x^i y^{j-2}. \tag{5.15}$$

In the unit square $x, y \in [0,1]$, and for the triangle inequality, so

$$|p_{xx}(x,y)| \leqslant \sum_{i,j=0}^{n} (i-1) i |a_{ij}| \tag{5.16}$$

$$|p_{xy}(x,y)| \leqslant \sum_{i,j=0}^{n} ij |a_{ij}| \tag{5.17}$$

$$|p_{yy}(x,y)| \leqslant \sum_{i,j=0}^{n} (j-1) j |a_{ij}|. \tag{5.18}$$

For the inequalities (5.10)-(5.12)

$$\max_{\mathbf{x}\in[0,1]^2} p(\mathbf{x}) \leqslant \max_{i,j\in K} p(\mathbf{t}_{ij}) + \frac{1}{8k^2}\left(\sum_{i,j=0}^{n}(i-1)i|a_{ij}| + 2\sum_{i,j=0}^{n} ij|a_{ij}| + \sum_{i,j=0}^{n}(j-1)j|a_{ij}|\right) \tag{5.19}$$

$$= \max_{i,j\in K} p(\mathbf{t}_{ij}) + \frac{1}{8k^2}\sum_{i,j=0}^{n}\Big((i-1)i + 2ij + (j-1)j\Big)|a_{ij}| \tag{5.20}$$

$$= \max_{i,j\in K} p(\mathbf{t}_{ij}) + \frac{1}{8k^2}\sum_{i,j=0}^{n}\Big(i^2 - i + 2ij + j^2 - j\Big)|a_{ij}| \tag{5.21}$$

$$= \max_{i,j\in K} p(\mathbf{t}_{ij}) + \frac{1}{8k^2}\sum_{i,j=0}^{n}(i+j)(i+j-1)|a_{ij}| \tag{5.22}$$

$$= \max_{i,j\in K} p(\mathbf{t}_{ij}) + \alpha_k, \tag{5.23}$$

where α_k is defined in (5.4). This proves the first part of the theorem. An entirely analogous argument establishes the lower bound on $\min_{\mathbf{x}\in[0,1]^2} p(\mathbf{x})$ in (5.2). □

5.1.1 New bounds for the Garloff bound

The addends a_k present in the Theorem 5.1 of Garloff can be improved using the idea in one dimension for the extension of Rivlin in Section 4.1.2. In the same hypothesis of Garloff theorem, for a positive integer k define $K = \{(i,j)|0 \leqslant i,j \leqslant k\}$; on the unit square $I = [0,1]^2$ given by $\mathbf{t}_{ij} = (\frac{i}{k}, \frac{j}{k})$, with $(i,j) \in K$.

Proposition 5.1. *Let p be given by*

$$p(x,y) = \sum_{i,j=0}^{n} a_{ij}x^iy^j. \tag{5.24}$$

Then

$$\max_{(x,y)\in I} p(x,y) \leqslant \max_{(i,j)\in K} p(\mathbf{t}_{ij}) + \beta_k \leqslant \max_{(i,j)\in K} p(\mathbf{t}_{ij}) + \alpha_k \tag{5.25}$$

$$\min_{(x,y)\in I} p(x,y) \geqslant \min_{(i,j)\in K} p(\mathbf{t}_{ij}) - \beta_k \geqslant \min_{(i,j)\in K} p(\mathbf{t}_{ij}) - \alpha_k, \tag{5.26}$$

where α_k are defined in (5.4) and

$$\beta_k = \frac{1}{8k^2}(|b_{xx}| + 2|b_{xy}| + |b_{yy}|), \tag{5.27}$$

where $|b_{xx}|$, $|b_{xy}|$, $|b_{yy}|$ are the maxima of absolute values for the Bernstein coefficients for the partial derivatives of p, respectively for p_{xx}, p_{xy}, p_{yy}.

Proof. Starting from the equalities (5.7)-(5.9), observe for the second derivatives of p in Bernstein basis

$$p_{xx}(x,y) = \sum_{i,j=0}^{n} (i-1)ia_{ij}x^{i-2}y^{j} \tag{5.28}$$

$$= \sum_{k=0}^{n-2}\sum_{j=0}^{n}(k+1)(k+2)a_{k+2,j}x^{k}y^{j} \quad \text{where } k = i-2 \tag{5.29}$$

$$= \sum_{m=0}^{n-2}\sum_{p=0}^{n} b_{xxmp}^{(n-2,n)} B_{mp}^{n-2,n}(x,y), \tag{5.30}$$

where the Bernstein coefficients are

$$b_{xxmp} = \sum_{k=0}^{m}\sum_{j=0}^{p}(k+1)(k+2)a_{k+2,j}\frac{\binom{m}{k}}{\binom{n-2}{k}}\frac{\binom{p}{j}}{\binom{n}{j}} \quad \text{with } 0 \leqslant m \leqslant n-2; 0 \leqslant p \leqslant n \tag{5.31}$$

and

$$p_{xy}(x,y) = \sum_{i,j=0}^{n} ija_{ij}x^{i-1}y^{j-1} \tag{5.32}$$

$$= \sum_{k=0}^{n-1}\sum_{l=0}^{n-1}(k+1)(l+1)a_{k+1,l+1}x^{k}y^{l} \quad \text{where } k = i-1 \text{ and } l = j-1 \tag{5.33}$$

$$= \sum_{m=0}^{n-1}\sum_{p=0}^{n-1} b_{xymp}^{(n-1,n-1)} B_{mp}^{n-1,n-1}(x,y), \tag{5.34}$$

where the Bernstein coefficients are

$$b_{xymp} = \sum_{k=0}^{m}\sum_{l=0}^{p}(k+1)(l+1)a_{k+1,l+1}\frac{\binom{m}{k}}{\binom{n-1}{k}}\frac{\binom{p}{l}}{\binom{n-1}{l}} \quad \text{with } 0 \leqslant m \leqslant n-1; 0 \leqslant p \leqslant n-1 \tag{5.35}$$

and

$$p_{yy}(x,y) = \sum_{i,j=0}^{n} (j-1)ja_{ij}x^{i}y^{j-2} \tag{5.36}$$

$$= \sum_{i=0}^{n}\sum_{l=0}^{n-2}(l+1)(l+2)a_{i,l+2}x^{i}y^{l} \quad \text{where } l = j-2 \tag{5.37}$$

$$= \sum_{m=0}^{n}\sum_{p=0}^{n-2} b_{yymp}^{(n,n-2)} B_{mp}^{n,n-2}(x,y), \tag{5.38}$$

where the Bernstein coefficients are

$$b_{yymp} = \sum_{i=0}^{m}\sum_{l=0}^{p}(l+1)(l+2)a_{i,l+2}\frac{\binom{m}{i}}{\binom{n}{i}}\frac{\binom{p}{l}}{\binom{n-2}{l}} \quad \text{with } 0 \leqslant m \leqslant n; 0 \leqslant p \leqslant n-2. \tag{5.39}$$

So the inequalities (5.10)-(5.12) present in the Proof of the Theorem 5.1 become

$$\max_{\mathbf{x}\in[0,1]^2} p(\mathbf{x}) \leqslant \max_{i,j\in K} p(\mathbf{t}_{ij}) \tag{5.40}$$

$$+\left(\frac{i}{k}-\xi_x\right)^2\frac{|p_{xx}(\boldsymbol{\eta})|}{2}+\left|\frac{i}{k}-\xi_x\right|\left|\frac{j}{k}-\xi_y\right||p_{xy}(\boldsymbol{\eta})|+\left(\frac{j}{k}-\xi_y\right)^2\frac{|p_{yy}(\boldsymbol{\eta})|}{2} \tag{5.41}$$

$$\leqslant \max_{i,j\in K} p(\mathbf{t}_{ij})+\frac{1}{8k^2}|p_{xx}(\boldsymbol{\eta})|+\frac{1}{4k^2}|p_{xy}(\boldsymbol{\eta})|+\frac{1}{8k^2}|p_{yy}(\boldsymbol{\eta})| \tag{5.42}$$

$$= \max_{i,j\in K} p(\mathbf{t}_{ij})+\frac{1}{8k^2}\Big(|p_{xx}(\boldsymbol{\eta})|+2|p_{xy}(\boldsymbol{\eta})|+|p_{yy}(\boldsymbol{\eta})|\Big) \tag{5.43}$$

$$= \max_{i,j\in K} p(\mathbf{t}_{ij})+\frac{1}{8k^2}\Bigg(\Bigg|\sum_{m=0}^{n-2}\sum_{p=0}^{n} b_{xxmp}^{(n-2,n)}B_{mp}^{n-2,n}(\boldsymbol{\eta})\Bigg| \tag{5.44}$$

$$+2\Bigg|\sum_{m=0}^{n-1}\sum_{p=0}^{n-1} b_{xymp}^{(n-1,n-1)}B_{mp}^{n-1,n-1}(\boldsymbol{\eta})\Bigg|+\Bigg|\sum_{m=0}^{n}\sum_{p=0}^{n-2} b_{yymp}^{(n,n-2)}B_{mp}^{n,n-2}(\boldsymbol{\eta})\Bigg|\Bigg) \tag{5.45}$$

$$\leqslant \max_{i,j\in K} p(\mathbf{t}_{ij})+\frac{1}{8k^2}\Bigg(\sum_{m=0}^{n-2}\sum_{p=0}^{n} |b_{xxmp}^{(n-2,n)}|B_{mp}^{n-2,n}(\boldsymbol{\eta}) \tag{5.46}$$

$$+2\sum_{m=0}^{n-1}\sum_{p=0}^{n-1} |b_{xymp}^{(n-1,n-1)}|B_{mp}^{n-1,n-1}(\boldsymbol{\eta})+\sum_{m=0}^{n}\sum_{p=0}^{n-2} |b_{yymp}^{(n,n-2)}|B_{mp}^{n,n-2}(\boldsymbol{\eta})\Bigg) \tag{5.47}$$

$$\leqslant \max_{i,j\in K} p(\mathbf{t}_{ij})+\frac{1}{8k^2}\Bigg(\max_{m,p}|b_{xxmp}^{(n-2,n)}|\sum_{m=0}^{n-2}\sum_{p=0}^{n} B_{mp}^{n-2,n}(\boldsymbol{\eta}) \tag{5.48}$$

$$+2\max_{m,p}|b_{xymp}^{(n-1,n-1)}|\sum_{m=0}^{n-1}\sum_{p=0}^{n-1} B_{mp}^{n-1,n-1}(\boldsymbol{\eta})+\max_{m,p}|b_{yymp}^{(n,n-2)}|\sum_{m=0}^{n}\sum_{p=0}^{n-2} B_{mp}^{n,n-2}(\boldsymbol{\eta})\Bigg) \tag{5.49}$$

$$= \max_{i,j\in K} p(\mathbf{t}_{ij})+\frac{1}{8k^2}\Bigg(\max_{m,p}|b_{xxmp}^{(n-2,n)}|+2\max_{m,p}|b_{xymp}^{(n-1,n-1)}|+\max_{m,p}|b_{yymp}^{(n,n-2)}|\Bigg) \tag{5.50}$$

because $\sum_{m,p} B_{mp} = 1$. In addition,

$$\max_{i,j\in K} p(\mathbf{t}_{ij})+\frac{1}{8k^2}\Bigg(\max_{m,p}|b_{xxmp}^{(n-2,n)}|+2\max_{m,p}|b_{xymp}^{(n-1,n-1)}|+\max_{m,p}|b_{yymp}^{(n,n-2)}|\Bigg) \tag{5.51}$$

$$= \max_{(i,j)\in K} p(\mathbf{t}_{ij})+\beta_k \tag{5.52}$$

because for (5.27) $\beta_k = \frac{1}{8k^2}(|b_{xx}|+2|b_{xy}|+|b_{yy}|)$. Furthermore,

$$\max_{(i,j)\in K} p(\mathbf{t}_{ij})+\beta_k \tag{5.53}$$

$$= \max_{i,j\in K} p(\mathbf{t}_{ij})+\frac{1}{8k^2}\Bigg(\max_{m,p}\Bigg|\sum_{k=0}^{m}\sum_{j=0}^{p}(k+1)(k+2)a_{k+2,j}\frac{\binom{m}{k}}{\binom{n-2}{k}}\frac{\binom{p}{j}}{\binom{n}{j}}\Bigg| \tag{5.54}$$

$$+2\max_{m,p}\Bigg|\sum_{k=0}^{m}\sum_{l=0}^{p}(k+1)(l+1)a_{k+1,l+1}\frac{\binom{m}{k}}{\binom{n-1}{k}}\frac{\binom{p}{l}}{\binom{n-1}{l}}\Bigg| \tag{5.55}$$

$$+\max_{m,p}\Bigg|\sum_{i=0}^{m}\sum_{l=0}^{p}(l+1)(l+2)a_{i,l+2}\frac{\binom{m}{i}}{\binom{n}{i}}\frac{\binom{p}{l}}{\binom{n-2}{l}}\Bigg|\Bigg) \tag{5.56}$$

$$\leqslant \max_{i,j\in K} p(\mathbf{t}_{ij}) + \frac{1}{8k^2}\left(\sum_{k=0}^{n-2}\sum_{j=0}^{n}(k+1)(k+2)|a_{k+2,j}|\right. \tag{5.57}$$

$$\left. +2\sum_{k=0}^{n-1}\sum_{l=0}^{n-1}(k+1)(l+1)|a_{k+1,l+1}| + \sum_{i=0}^{n}\sum_{l=0}^{n-2}(l+1)(l+2)|a_{i,l+2}|\right) \tag{5.58}$$

for the triangle inequality and because the sums are among all coefficients. Moreover,

$$\max_{i,j\in K} p(\mathbf{t}_{ij}) + \frac{1}{8k^2}\left(\sum_{k=0}^{n-2}\sum_{j=0}^{n}(k+1)(k+2)|a_{k+2,j}|\right. \tag{5.59}$$

$$\left. +2\sum_{k=0}^{n-1}\sum_{l=0}^{n-1}(k+1)(l+1)|a_{k+1,l+1}| + \sum_{i=0}^{n}\sum_{l=0}^{n-2}(l+1)(l+2)|a_{i,l+2}|\right) \tag{5.60}$$

$$= \max_{i,j\in K} p(\mathbf{t}_{ij}) + \frac{1}{8k^2}\left(\sum_{i,j=0}^{n}(i-1)i|a_{i,j}| + 2\sum_{i,j=0}^{n} ij|a_{i,j}| + \sum_{i,j=0}^{n}(j-1)j|a_{i,j}|\right) \tag{5.61}$$

$$= \max_{i,j\in K} p(\mathbf{t}_{ij}) + \frac{1}{8k^2}\left(\sum_{i,j=0}^{n}(i+j)(i+j-1)|a_{i,j}|\right) \tag{5.62}$$

$$= \max_{i,j\in K} p(\mathbf{t}_{ij}) + \alpha_k, \tag{5.63}$$

where α_k is defined in (5.4). So,

$$\max_{\mathbf{x}\in[0,1]^2} p(\mathbf{x}) \leqslant \max_{(i,j)\in K} p(\mathbf{t}_{ij}) + \beta_k \leqslant \max_{i,j\in K} p(\mathbf{t}_{ij}) + \alpha_k. \tag{5.64}$$

This proves the first part of the proposition. An entirely analogous argument establishes the lower bound on $\min_{\mathbf{x}\in[0,1]^2} p(\mathbf{x})$ in (5.26). □

Using the bounds presented in Chapter 3 and their comparison, it is possible to obtain the inequalities of the Proposition 5.1, and also further bounds. Suppose $p(\mathbf{x}) = \sum_{i,j=0}^{n} a_{i,j}x^i y^j$, observing that in the interval $[0,1]^2$

$$p_{xxA} = \sum_{i,j=0}^{n}(i-1)i|a_{i,j}| \tag{5.65}$$

$$p_{xyA} = p_{yxA} = \sum_{i,j=0}^{n} ij|a_{i,j}| \tag{5.66}$$

$$p_{yyA} = \sum_{i,j=0}^{n}(i+j)(i+j-1)|a_{i,j}| \tag{5.67}$$

Similarly as 4.1.2, from the inequalities (3.85)-(3.86), the extension of the bounds (4.41)-(4.42) to the interval $[0,1]^2$ for the absolute values can be defined. Set $\mathbf{x}\in[0,1]^2$,

$$|p(\mathbf{x})| \leqslant \max_{0\leqslant m\leqslant n; 0\leqslant p\leqslant n} |b_{mp}^{([0,1]^2)}| \leqslant \max\{|\underline{p}_I([0,1]^2)|, |\overline{p}_I([0,1]^2)|\} \tag{5.68}$$

$$\leqslant \max\{|\underline{p}_S([0,1]^2)|, |\overline{p}_S([0,1]^2)|\} \leqslant \overline{p}_A([0,1]^2). \tag{5.69}$$

Applying the inequalities (5.68)-(5.69) for the second derivatives, as follows.

$$
\begin{aligned}
&|p_{xx}(\mathbf{x})|+|p_{xy}(\mathbf{x})|+|p_{yx}(\mathbf{x})|+|p_{yy}(\mathbf{x})| && (5.70)\\
&=|p_{xx}(\mathbf{x})|+2|p_{xy}(\mathbf{x})|+|p_{yy}(\mathbf{x})| && (5.71)\\
&\leqslant |b_{xx}|+2|b_{xy}|+|b_{yy}| && (5.72)\\
&\leqslant \max\{|\underline{p}_{xxI}([0,1]^2)|,|\overline{p}_{xxI}([0,1]^2)|\}+2\max\{|\underline{p}_{xyI}([0,1]^2)|,|\overline{p}_{xyI}([0,1]^2)|\} && (5.73)\\
&+\max\{|\underline{p}_{yyI}([0,1]^2)|,|\overline{p}_{yyI}([0,1]^2)|\} && (5.74)\\
&\leqslant \max\{|\underline{p}_{xxS}([0,1]^2)|,|\overline{p}_{xxS}([0,1]^2)|\}+2\max\{|\underline{p}_{xyS}([0,1]^2)|,|\overline{p}_{xyS}([0,1]^2)|\} && (5.75)\\
&+\max\{|\underline{p}_{yyS}([0,1]^2)|,|\overline{p}_{yyS}([0,1]^2)|\} && (5.76)\\
&\leqslant \overline{p}_{xxA}([0,1]^2)+2\overline{p}_{xyA}([0,1]^2)+\overline{p}_{yyA}([0,1]^2). && (5.77)
\end{aligned}
$$

So, since $\mathbf{x}$ is arbitrary in $[0,1]^2$, the inequalities (5.25)-(5.26) can be improved.

$$
\begin{aligned}
&\min_{(i,j)\in K} p(\mathbf{t}_{ij})-\alpha_k && (5.78)\\
&=\min_{i,j\in K} p(\mathbf{t}_{ij})-\frac{1}{8k^2}\left(\overline{p}_{xxA}([0,1]^2)+2\overline{p}_{xyA}([0,1]^2)+\overline{p}_{yyA}([0,1]^2)\right) && (5.79)\\
&\leqslant \min_{i,j\in K} p(\mathbf{t}_{ij})-\frac{1}{8k^2}\Big(\max\{|\underline{p}_{xxS}([0,1]^2)|,|\overline{p}_{xxS}([0,1]^2)|\} && (5.80)\\
&+2\max\{|\underline{p}_{xyS}([0,1]^2)|,|\overline{p}_{xyS}([0,1]^2)|\}+\max\{|\underline{p}_{yyS}([0,1]^2)|,|\overline{p}_{yyS}([0,1]^2)|\}\Big) && (5.81)\\
&\leqslant \min_{i,j\in K} p(\mathbf{t}_{ij})-\frac{1}{8k^2}\Big(\max\{|\underline{p}_{xxI}([0,1]^2)|,|\overline{p}_{xxI}([0,1]^2)|\} && (5.82)\\
&+2\max\{|\underline{p}_{xyI}([0,1]^2)|,|\overline{p}_{xyI}([0,1]^2)|\}+\max\{|\underline{p}_{yyI}([0,1]^2)|,|\overline{p}_{yyI}([0,1]^2)|\}\Big) && (5.83)\\
&\leqslant \min_{(i,j)\in K} p(\mathbf{t}_{ij})-\beta_k && (5.84)\\
&=\min_{i,j\in K} p(\mathbf{t}_{ij})-\frac{1}{8k^2}\left(|b_{xx}|+2|b_{xy}|+|b_{yy}|\right) && (5.85)\\
&\leqslant \min_{\mathbf{x}\in[0,1]^2} p(\mathbf{x}) && (5.86)\\
&\leqslant \max_{\mathbf{x}\in[0,1]^2} p(\mathbf{x}) && (5.87)\\
&\leqslant \max_{(i,j)\in K} p(\mathbf{t}_{ij})+\beta_k && (5.88)\\
&=\max_{i,j\in K} p(\mathbf{t}_{ij})+\frac{1}{8k^2}\left(|b_{xx}|+2|b_{xy}|+|b_{yy}|\right) && (5.89)\\
&\leqslant \max_{i,j\in K} p(\mathbf{t}_{ij})+\frac{1}{8k^2}\Big(\max\{|\underline{p}_{xxI}([0,1]^2)|,|\overline{p}_{xxI}([0,1]^2)|\} && (5.90)\\
&+2\max\{|\underline{p}_{xyI}([0,1]^2)|,|\overline{p}_{xyI}([0,1]^2)|\}+\max\{|\underline{p}_{yyI}([0,1]^2)|,|\overline{p}_{yyI}([0,1]^2)|\}\Big) && (5.91)
\end{aligned}
$$

$$\leqslant \max_{i,j\in K} p(\mathbf{t}_{ij}) + \frac{1}{8k^2}\Bigg(\max\{|\underline{p}_{xxS}([0,1]^2)|, |\overline{p}_{xxS}([0,1]^2)|\} \tag{5.92}$$

$$+2\max\{|\underline{p}_{xyS}([0,1]^2)|, |\overline{p}_{xyS}([0,1]^2)|\} + \max\{|\underline{p}_{yyS}([0,1]^2)|, |\overline{p}_{yyS}([0,1]^2)|\}\Bigg) \tag{5.93}$$

$$\leqslant \max_{(i,j)\in K} p(\mathbf{t}_{ij}) + \alpha_k \tag{5.94}$$

$$= \max_{i,j\in K} p(\mathbf{t}_{ij}) + \frac{1}{8k^2}\Bigg(\overline{p}_{xxA}([0,1]^2) + 2\overline{p}_{xyA}([0,1]^2) + \overline{p}_{yyA}([0,1]^2)\Bigg). \tag{5.95}$$

Rewriting with the respective definitions, the previous inequalities become as follows.

$$\min_{i,j\in K} p(\mathbf{t}_{ij}) - \frac{1}{8k^2}\left(\sum_{i,j=0}^{n}(i+j)(i+j-1)|a_{ij}|\right) \tag{5.96}$$

$$\leqslant \min_{i,j\in K} p(\mathbf{t}_{ij}) - \frac{1}{8k^2}\Bigg(\max\left\{\left|\sum_{i,j=0;a_{ij}>0}^{n} i(i-1)a_{ij}\right|, \left|\sum_{i,j=0;a_{ij}<0}^{n} i(i-1)a_{ij}\right|\right\} \tag{5.97}$$

$$+2\max\left\{\left|\sum_{i,j=0;a_{ij}>0}^{n} ija_{ij}\right|, \left|\sum_{i,j=0;a_{ij}<0}^{n} ija_{ij}\right|\right\} \tag{5.98}$$

$$+\max\left\{\left|\sum_{i,j=0;a_{ij}>0}^{n} j(j-1)a_{ij}\right|, \left|\sum_{i,j=0;a_{ij}<0}^{n} j(j-1)a_{ij}\right|\right\}\Bigg) \tag{5.99}$$

$$\leqslant \min_{i,j\in K} p(\mathbf{t}_{ij}) - \frac{1}{8k^2}\Bigg(\max\Bigg\{\left|2a_{2,0} + \sum_{i,j=0;(i,j)\neq(2,0);a_{ij}>0}^{n} i(i-1)a_{ij}\right|, \tag{5.100}$$

$$\left|2a_{2,0} + \sum_{i,j=0;(i,j)\neq(2,0);a_{ij}<0}^{n} i(i-1)a_{ij}\right|\Bigg\} \tag{5.101}$$

$$+2\max\left\{\left|a_{1,1} + \sum_{i,j=0;(i,j)\neq(1,1);a_{ij}>0}^{n} ija_{ij}\right|, \left|a_{1,1} + \sum_{i,j=0;(i,j)\neq(1,1);a_{ij}<0}^{n} ija_{ij}\right|\right\} \tag{5.102}$$

$$+\max\left\{\left|2a_{0,2} + \sum_{i,j=0;(i,j)\neq(0,2);a_{ij}>0}^{n} j(j-1)a_{ij}\right|, \left|2a_{0,2} + \sum_{i,j=0;(i,j)\neq(0,2);a_{ij}<0}^{n} j(j-1)a_{ij}\right|\right\}\Bigg) \tag{5.103}$$

$$\leqslant \min_{i,j\in K} p(\mathbf{t}_{ij}) - \frac{1}{8k^2}\left(\max_{m,p}|b_{xxmp}^{(n-2,n)}| + 2\max_{m,p}|b_{xymp}^{(n-1,n-1)}| + \max_{m,p}|b_{yymp}^{(n,n-2)}|\right) \tag{5.104}$$

$$\leqslant \min_{\mathbf{x}\in[0,1]^2} p(\mathbf{x}) \tag{5.105}$$

$$\leqslant \max_{\mathbf{x}\in[0,1]^2} p(\mathbf{x}) \tag{5.106}$$

$$\leqslant \max_{i,j\in K} p(\mathbf{t}_{ij}) + \frac{1}{8k^2}\left(\max_{m,p}|b_{xxmp}^{(n-2,n)}| + 2\max_{m,p}|b_{xymp}^{(n-1,n-1)}| + \max_{m,p}|b_{yymp}^{(n,n-2)}|\right) \tag{5.107}$$

$$\leqslant \max_{i,j\in K} p(\mathbf{t}_{ij}) + \frac{1}{8k^2}\Bigg(\max\Bigg\{\left|2a_{2,0} + \sum_{i,j=0;(i,j)\neq(2,0);a_{ij}>0}^{n} i(i-1)a_{ij}\right|, \tag{5.108}$$

$$\left|2a_{2,0} + \sum_{i,j=0;(i,j)\neq(2,0);a_{ij}<0}^{n} i(i-1)a_{ij}\right|\Bigg\} \tag{5.109}$$

$$+2\max\left\{\left|a_{1,1} + \sum_{i,j=0;(i,j)\neq(1,1);a_{ij}>0}^{n} ija_{ij}\right|, \left|a_{1,1} + \sum_{i,j=0;(i,j)\neq(1,1);a_{ij}<0}^{n} ija_{ij}\right|\right\} \tag{5.110}$$

$$+\max\left\{\left|2a_{0,2}+\sum_{i,j=0;(i,j)\neq(0,2);a_{ij}>0}^{n} j(j-1)a_{ij}\right|,\left|2a_{0,2}+\sum_{i,j=0;(i,j)\neq(0,2);a_{ij}<0}^{n} j(j-1)a_{ij}\right|\right\}\Bigg) \quad (5.111)$$

$$\leqslant \max_{i,j\in K} p(\mathbf{t}_{ij})+\frac{1}{8k^2}\Bigg(\max\left\{\left|\sum_{i,j=0;a_{ij}>0}^{n} i(i-1)a_{ij}\right|,\left|\sum_{i,j=0;a_{ij}<0}^{n} i(i-1)a_{ij}\right|\right\} \quad (5.112)$$

$$+2\max\left\{\left|\sum_{i,j=0;a_{ij}>0}^{n} ija_{ij}\right|,\left|\sum_{i,j=0;a_{ij}<0}^{n} ija_{ij}\right|\right\} \quad (5.113)$$

$$+\max\left\{\left|\sum_{i,j=0;a_{ij}>0}^{n} j(j-1)a_{ij}\right|,\left|\sum_{i,j=0;a_{ij}<0}^{n} j(j-1)a_{ij}\right|\right\}\Bigg) \quad (5.114)$$

$$\leqslant \max_{i,j\in K} p(\mathbf{t}_{ij})+\frac{1}{8k^2}\left(\sum_{i,j=0}^{n}(i+j)(i+j-1)|a_{ij}|\right). \quad (5.115)$$

5.2 Bounds in the Interval $[0,1]^n$

The generalization to the n-dimensional case is as follows. Let k be a positive integer and define $K=\{(i_1,i_2,\ldots,i_n)|0\leqslant i_1,i_2,\ldots,i_n\leqslant k\}$. The bounds involve the function values of the polynomial on the grid in the unit square $[0,1]^n$ given by $\mathbf{t}_I=\mathbf{t}_{i_1,i_2,\ldots,i_n}=(\frac{i_1}{k},\frac{i_2}{k},\ldots,\frac{i_n}{k})$, with $(i_1,i_2,\ldots,i_n)\in K$.

Proposition 5.2. *Let p be given by*

$$p(\mathbf{x})=\sum_{I\leqslant L} a_I\mathbf{x}^I. \quad (5.116)$$

Then

$$\max_{\mathbf{x}\in[0,1]^n} p(\mathbf{x})\leqslant \max_{I\in K} p(\mathbf{t}_I)+\beta_K\leqslant \max_{I\in K} p(\mathbf{t}_I)+\alpha_K \quad (5.117)$$

$$\min_{\mathbf{x}\in[0,1]^n} p(\mathbf{x})\geqslant \min_{I\in K} p(\mathbf{t}_I)-\beta_K\geqslant \min_{I\in K} p(\mathbf{t}_I)-\alpha_K, \quad (5.118)$$

where α_K and β_K are defined as

$$\alpha_K=\frac{1}{8k^2}\sum_{i_1,i_2,\ldots,i_n=0}^{l_1,l_2,\ldots,l_n}(i_1+i_2+\cdots+i_n)(i_1+i_2+\cdots+i_n-1)|a_{i_1,i_2,\ldots,i_n}| \quad (5.119)$$

$$=\frac{1}{8k^2}\sum_{I=(i_1,i_2,\ldots,i_n)\leqslant L}(i_1+i_2+\cdots+i_n)(i_1+i_2+\cdots+i_n-1)|a_I| \quad (5.120)$$

and

$$\beta_K=\frac{1}{8k^2}\sum_{i,j=1;j>i}^{n}\left(|b_{x_ix_i}|+2|b_{x_ix_j}|\right), \quad (5.121)$$

where $|b_{x_ix_j}|$ is the maximum of the absolute values of the Bernstein coefficients for the partial derivatives $p_{\mathbf{x}_i\mathbf{x}_j}$ of p.

Proof. The proof is the extension of the proof in two dimensions. The proof is by induction on the dimension n. The base case is $n=1$. In this case, the inequalities (4.83)-(4.87) are exactly the inequalities (5.117)-(5.118), so it proves the base case.

For the inductive step, we assume that the inequalities (5.117)-(5.118) are satisfied for every integer d such that $1 \leqslant d < n$. The objective is to show that the statement holds for n. Suppose that $p(\boldsymbol{\xi}) = \max_{\mathbf{x}\in[0,1]^n} p(\mathbf{x}) = p(\xi_1, \xi_2, \ldots, \xi_n)$ and, given a norm s on $\mathbb{R}^n$,

$$\|\mathbf{t} - \boldsymbol{\xi}\|_s = \|\mathbf{t}_{i_1,i_2,\ldots,i_n} - (\xi_1, \xi_2, \ldots, \xi_n)\|_s \leqslant \|\mathbf{t}_{j_1,j_2,\ldots,j_n} - \boldsymbol{\xi}\|_s \quad 0 \leqslant j_1, j_2, \ldots, j_n \leqslant k; \tag{5.122}$$

so that

$$\|\mathbf{t}_{i_1,i_2,\ldots,i_n} - \boldsymbol{\xi}\|_\infty \leqslant \frac{1}{2k}. \tag{5.123}$$

According to Taylor's formula, it is possible to obtain

$$p(\mathbf{t}) = p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) \tag{5.124}$$

$$= p\left(\frac{i_1}{k}, \frac{i_2}{k}, \ldots, \frac{i_n}{k}\right) \tag{5.125}$$

$$= p(\boldsymbol{\xi}) + \sum_{j=1}^{n} \left(\frac{i_j}{k} - \xi_j\right) p_{x_j}(\boldsymbol{\xi}) + \sum_{j=1}^{n} \left(\frac{i_j}{k} - \xi_j\right)^2 \frac{p_{x_j x_j}(\boldsymbol{\eta})}{2} + \tag{5.126}$$

$$\sum_{j,r=1;r>j}^{n} \left(\frac{i_j}{k} - \xi_j\right)\left(\frac{i_r}{k} - \xi_r\right) p_{x_j x_r}(\boldsymbol{\eta}), \tag{5.127}$$

where $\boldsymbol{\eta} \in [0,1]^n$.

If $\boldsymbol{\xi} \in \{(0,0,\ldots,0), (1,0,\ldots,0), (0,1,0,\ldots,0), \ldots, (0,0,\ldots,0,1), (1,1,\ldots,1)\}$, or, in general, $\boldsymbol{\xi}$ belongs to the border of the interval $[0,1]^n$, that is, $\boldsymbol{\xi} \in \partial[0,1]^n$, then $\xi_j = 0$ or $\xi_j = 1$ for some j such that $1 \leqslant j \leqslant n$. It is possible to reduce to a Taylor expansion with d variables by parameterizing the part of the border with $\boldsymbol{\xi}$, with $d < n$, making the function dependent on d parameters. So, the dimension is less than n and for the inductive hypothesis, the inequalities (5.117)-(5.118) are true.

Otherwise, $p_{x_j}(\boldsymbol{\xi}) = 0$, and

$$\max_{\mathbf{x}\in[0,1]^n} p(\mathbf{x}) \leqslant \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) \tag{5.128}$$

$$+ \sum_{j=1}^{n} \left(\frac{i_j}{k} - \xi_j\right)^2 \frac{|p_{x_j x_j}(\boldsymbol{\eta})|}{2} + \sum_{j,r=1;r>j}^{n} \left|\frac{i_j}{k} - \xi_j\right| \left|\frac{i_r}{k} - \xi_r\right| |p_{x_j x_r}(\boldsymbol{\eta})| \tag{5.129}$$

$$\leqslant \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \frac{1}{8k^2} \sum_{j=1}^{n} |p_{x_j x_j}(\boldsymbol{\eta})| + \frac{1}{4k^2} \sum_{j,r=1;r>j}^{n} |p_{x_j x_r}(\boldsymbol{\eta})|. \tag{5.130}$$

Now, the second derivatives can be expressed in Bernstein basis. Set s such that $1 \leqslant s \leqslant n$, the pure second derivatives of p in Bernstein basis are

$$p_{x_s x_s}(\mathbf{x}) = \sum_{i_1,i_2,\ldots,i_n=0}^{l_1,l_2,\ldots,l_n} (i_s - 1) i_s a_{i_1,i_2,\ldots,i_n} x_1^{i_1} x_2^{i_2} \cdots x_{s-1}^{i_{s-1}} x_{s+1}^{i_{s+1}} \cdots x_n^{i_n} x_s^{i_s - 2} \tag{5.131}$$

$$= \sum_{i_1,i_2,\ldots,i_{s-1},i_{s+1},\ldots,i_n=0}^{l_1,l_2,\ldots,l_{s-1},l_{s+1},\ldots,l_n} \sum_{j_s=0}^{l_s-2} (j_s+1)(j_s+2) a_{i_1,i_2,\ldots,i_{s-1},j_s+2,i_{s+1},\ldots,i_n} \tag{5.132}$$

$$\cdot x_1^{i_1} x_2^{i_2} \cdots x_{s-1}^{i_{s-1}} x_s^{j_s} x_{s+1}^{i_{s+1}} \cdots x_n^{i_n} \quad \text{where } j_s = i_s - 2 \tag{5.133}$$

$$= \sum_{m_1,m_2,\ldots,m_{s-1},m_s,m_{s+1},\ldots,m_n=0}^{l_1,l_2,\ldots,l_{s-1},l_s-2,l_{s+1},\ldots,l_n} b_{x_sx_sm_1,m_2,\ldots,m_{s-1},m_s,m_{s+1},\ldots,m_n}^{(l_1,l_2,\ldots,l_{s-1},l_s-2,l_{s+1},\ldots,l_n)} B_{m_1,m_2,\ldots,m_{s-1},m_s,m_{s+1},\ldots,m_n}^{l_1,l_2,\ldots,l_{s-1},l_s-2,l_{s+1},\ldots,l_n}(\mathbf{x}) \tag{5.134}$$

$$= \sum_{M\leqslant L_{ss}} b_{x_sx_sM} B_{ssM}(\mathbf{x}), \tag{5.135}$$

where the multi-indexes $M = (m_1, m_2, \ldots, \ldots, m_n)$ and $L_{ss} = (l_1, l_2, \ldots, l_{s-1}, l_s-2, l_{s+1}, \ldots, l_n)$. Set $\tilde{a}_{J_{ss}} = (a_{j_1}, a_{j_2}, \ldots, a_{j_{s-1}}, (j_s+1)(j_s+2)a_{j_s+2}, a_{j_{s+1}}, \ldots, a_{j_n})$, the Bernstein basis can be called shortly B_{ssM}, instead the Bernstein coefficients, called shortly $b_{x_sx_sM}$, are

$$b_{x_sx_sM} = b_{x_sx_sm_1,m_2,\ldots,m_{s-1},m_s,m_{s+1},\ldots,m_n}^{(l_1,l_2,\ldots,l_{s-1},l_s-2,l_{s+1},\ldots,l_n)} = \tag{5.136}$$

$$= \sum_{k_1,k_2,\ldots,k_{s-1},k_s,k_{s+1},\ldots,k_n=0}^{m_1,m_2,\ldots,m_{s-1},m_s,m_{s+1},\ldots,m_n} (k_s+1)(k_s+2)a_{k_1,k_2,\ldots,k_{s-1},k_s+2,k_{s+1},\ldots,k_n} \tag{5.137}$$

$$\cdot \frac{\binom{m_1}{k_1}}{\binom{l_1}{k_1}} \cdot \frac{\binom{m_2}{k_2}}{\binom{l_2}{k_2}} \cdots \frac{\binom{m_{s-1}}{k_{s-1}}}{\binom{l_{s-1}}{k_{s-1}}} \cdot \frac{\binom{m_s}{k_s}}{\binom{l_s-2}{k_s}} \cdot \frac{\binom{m_{s+1}}{k_{s+1}}}{\binom{l_{s+1}}{k_{s+1}}} \cdots \frac{\binom{m_n}{k_n}}{\binom{l_n}{k_n}} \tag{5.138}$$

$$= \sum_{K\leqslant M} \frac{\binom{M}{K}}{\binom{L_{ss}}{K}} \tilde{a}_{Kss} \tag{5.139}$$

with $0 \leqslant m_1 \leqslant l_1; 0 \leqslant m_2 \leqslant l_2; \ldots; 0 \leqslant m_{s-1} \leqslant l_{s-1}; 0 \leqslant m_s \leqslant l_s-2; 0 \leqslant m_{s+1} \leqslant l_{s+1}; \ldots; 0 \leqslant m_n \leqslant l_n$.

Instead, set s,u such that $1 \leqslant s,u \leqslant n$, without loss of generality, assume $u > s$, and the mixed second derivatives of p on the Bernstein basis are

$$p_{x_sx_u}(\mathbf{x}) = \sum_{i_1,i_2,\ldots,i_n=0}^{l_1,l_2,\ldots,l_n} i_s i_u a_{i_1,i_2,\ldots,i_n} x_1^{i_1} x_2^{i_2} \cdots x_{s-1}^{i_{s-1}} x_{s+1}^{i_{s+1}} \cdots x_{u-1}^{i_{u-1}} x_{u+1}^{i_{u+1}} \cdots x_n^{i_n} x_s^{i_s-1} x_u^{i_u-1} \tag{5.140}$$

$$= \sum_{i_1,i_2,\ldots,i_{s-1},i_{s+1},\ldots,i_{u-1},i_{u+1},\ldots,i_n=0}^{l_1,l_2,\ldots,l_{s-1},l_{s+1},\ldots,l_{u-1},l_{u+1},\ldots,l_n} \sum_{j_s=0}^{l_s-1} \sum_{j_u=0}^{l_u-1} (j_s+1)(j_u+1) \tag{5.141}$$

$$\cdot a_{i_1,i_2,\ldots,i_{s-1},j_s+1,i_{s+1},\ldots,i_{u-1},j_u+1,i_{u+1},\ldots,i_n} \cdot x_1^{i_1} x_2^{i_2} \cdots x_{s-1}^{i_{s-1}} x_s^{j_s} x_{s+1}^{i_{s+1}} \cdots x_{u-1}^{i_{u-1}} x_u^{j_u} x_{u+1}^{i_{u+1}} \cdots x_n^{i_n} \tag{5.142}$$

where $j_s = i_s - 1$ and $j_u = i_u - 1$. In addition,

$$\sum_{i_1,i_2,\ldots,i_{s-1},i_{s+1},\ldots,i_{u-1},i_{u+1},\ldots,i_n=0}^{l_1,l_2,\ldots,l_{s-1},l_{s+1},\ldots,l_{u-1},l_{u+1},\ldots,l_n} \sum_{j_s=0}^{l_s-1} \sum_{j_u=0}^{l_u-1} (j_s+1)(j_u+1) \tag{5.143}$$

$$\cdot a_{i_1,i_2,\ldots,i_{s-1},j_s+1,i_{s+1},\ldots,i_{u-1},j_u+1,i_{u+1},\ldots,i_n} \cdot x_1^{i_1} x_2^{i_2} \cdots x_{s-1}^{i_{s-1}} x_s^{j_s} x_{s+1}^{i_{s+1}} \cdots x_{u-1}^{i_{u-1}} x_u^{j_u} x_{u+1}^{i_{u+1}} \cdots x_n^{i_n} \tag{5.144}$$

$$= \sum_{m_1,m_2,\ldots,m_{s-1},m_s,m_{s+1},\ldots,m_{u-1},m_u,m_{u+1},\ldots,m_n=0}^{l_1,l_2,\ldots,l_{s-1},l_s-1,l_{s+1},\ldots,l_{u-1},l_u-1,l_{u+1},\ldots,l_n} b_{x_sx_um_1,m_2,\ldots,m_{s-1},m_s,m_{s+1},\ldots,m_{u-1},m_u,m_{u+1},\ldots,m_n}^{(l_1,l_2,\ldots,l_{s-1},l_s-1,l_{s+1},\ldots,l_{u-1},l_u-1,l_{u+1},\ldots,l_n)} \tag{5.145}$$

$$\cdot B_{m_1,m_2,\ldots,m_{s-1},m_s,m_{s+1},\ldots,m_{u-1},m_u,m_{u+1},\ldots,m_n}^{l_1,l_2,\ldots,l_{s-1},l_s-1,l_{s+1},\ldots,l_{u-1},l_u-1,l_{u+1},\ldots,l_n}(\mathbf{x}) \tag{5.146}$$

$$= \sum_{M\leqslant L_{su}} b_{x_sx_uM} B_{suM}(\mathbf{x}), \tag{5.147}$$

where the multi-indexes $M = (m_1, m_2, \ldots, \ldots, m_n)$ and $L_{su} = (l_1, l_2, \ldots, l_{s-1}, l_s-1, l_{s+1}, \ldots, l_{u-1}, l_u-1, l_{u+1}, \ldots, l_n)$. Set $\bar{a}_{J_{su}} = (a_{j_1}, a_{j_2}, \ldots, a_{j_{s-1}}, (j_s+1)a_{j_s+1}, a_{j_{s+1}}, \ldots, a_{j_{u-1}}, (j_u+1)a_{j_u+1}, a_{j_{u+1}}, \ldots, a_{j_n})$, the Bernstein basis can be called shortly B_{suM}, and the Bernstein coefficients, called shortly $b_{x_sx_uM}$, are

$$b_{x_s x_u M} = b^{(l_1,l_2,\dots,l_{s-1},l_s-1,l_{s+1},\dots,l_{u-1},l_u-1,l_{u+1},\dots,l_n)}_{x_s x_u m_1,m_2,\dots,m_{s-1},m_s,m_{s+1},\dots,m_{u-1},m_u,m_{u+1},\dots,m_n} = \tag{5.148}$$

$$= \sum_{k_1,k_2,\dots,k_{s-1},k_s,k_{s+1},\dots,k_{u-1},k_u,k_{u+1},\dots,k_n=0}^{m_1,m_2,\dots,m_{s-1},m_s,m_{s+1},\dots,m_{u-1},m_u,m_{u+1},\dots,m_n} (k_s+1)(k_u+1) \tag{5.149}$$

$$\cdot\, a_{k_1,k_2,\dots,k_{s-1},k_s+1,k_{s+1},\dots,k_{u-1},k_u+1,k_{u+1},\dots,k_n} \tag{5.150}$$

$$\cdot \frac{\binom{m_1}{k_1}}{\binom{l_1}{k_1}} \cdot \frac{\binom{m_2}{k_2}}{\binom{l_2}{k_2}} \cdots \frac{\binom{m_{s-1}}{k_{s-1}}}{\binom{l_{s-1}}{k_{s-1}}} \cdot \frac{\binom{m_s}{k_s}}{\binom{l_s-1}{k_s}} \cdot \frac{\binom{m_{s+1}}{k_{s+1}}}{\binom{l_{s+1}}{k_{s+1}}} \cdots \frac{\binom{m_{u-1}}{k_{u-1}}}{\binom{l_{u-1}}{k_{u-1}}} \cdot \frac{\binom{m_u}{k_u}}{\binom{l_u-1}{k_u}} \cdot \frac{\binom{m_{u+1}}{k_{u+1}}}{\binom{l_{u+1}}{k_{u+1}}} \cdots \frac{\binom{m_n}{k_n}}{\binom{l_n}{k_n}} \tag{5.151}$$

$$= \sum_{K \leqslant M} \frac{\binom{M}{K}}{\binom{L_{su}}{K}} \bar{a}_{Ksu} \tag{5.152}$$

where $0 \leqslant m_1 \leqslant l_1;\ 0 \leqslant m_2 \leqslant l_2;\ \dots;\ 0 \leqslant m_{s-1} \leqslant l_{s-1};\ 0 \leqslant m_s \leqslant l_s - 1;\ 0 \leqslant m_{s+1} \leqslant l_{s+1};\ \dots;$ $0 \leqslant m_{u-1} \leqslant l_{u-1}; 0 \leqslant m_u \leqslant l_u - 1; 0 \leqslant m_{u+1} \leqslant l_{u+1}; \dots; 0 \leqslant m_n \leqslant l_n$.

So, the inequalities (5.128)-(5.130) become

$$\max_{\mathbf{x}\in[0,1]^n} p(\mathbf{x}) \leqslant \max_{(i_1,i_2,\dots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\dots,i_n}) \tag{5.153}$$

$$+ \sum_{j=1}^{n} \left(\frac{i_j}{k} - \xi_j\right)^2 \frac{|p_{x_j x_j}(\boldsymbol{\eta})|}{2} + \sum_{j,r=1;r>j}^{n} \left|\frac{i_j}{k} - \xi_j\right| \left|\frac{i_r}{k} - \xi_r\right| |p_{x_j x_r}(\boldsymbol{\eta})| \tag{5.154}$$

$$\leqslant \max_{(i_1,i_2,\dots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\dots,i_n}) + \frac{1}{8k^2}\sum_{j=1}^{n} |p_{x_j x_j}(\boldsymbol{\eta})| + \frac{1}{4k^2}\sum_{j,r=1;r>j}^{n} |p_{x_j x_r}(\boldsymbol{\eta})| \tag{5.155}$$

$$= \max_{(i_1,i_2,\dots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\dots,i_n}) + \frac{1}{8k^2}\left(\sum_{j=1}^{n} (|p_{x_j x_j}(\boldsymbol{\eta})| + 2\sum_{j,r=1;r>j}^{n} |p_{x_j x_r}(\boldsymbol{\eta})|\right) \tag{5.156}$$

$$= \max_{(i_1,i_2,\dots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\dots,i_n}) + \frac{1}{8k^2}\sum_{j,r=1;r>j}^{n} (|p_{x_j x_j}(\boldsymbol{\eta})| + 2|p_{x_j x_r}(\boldsymbol{\eta})|) \tag{5.157}$$

$$= \max_{(i_1,i_2,\dots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\dots,i_n}) + \frac{1}{8k^2}\sum_{j,r=1;r>j}^{n} \left(\left|\sum_{M\leqslant L_{jj}} b_{x_j x_j M} B_{jjM}(\boldsymbol{\eta})\right|\right. \tag{5.158}$$

$$\left. + 2\left|\sum_{M\leqslant L_{jr}} b_{x_j x_r M} B_{jrM}(\boldsymbol{\eta})\right|\right) \tag{5.159}$$

$$\leqslant \max_{(i_1,i_2,\dots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\dots,i_n}) + \frac{1}{8k^2}\sum_{j,r=1;r>j}^{n} \left(\sum_{M\leqslant L_{jj}} |b_{x_j x_j M}| B_{jjM}(\boldsymbol{\eta})\right. \tag{5.160}$$

$$\left. + 2\sum_{M\leqslant L_{jr}} |b_{x_j x_r M}| B_{jrM}(\boldsymbol{\eta})\right) \tag{5.161}$$

$$\leqslant \max_{(i_1,i_2,\dots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\dots,i_n}) + \frac{1}{8k^2}\sum_{j,r=1;r>j}^{n} \left(\max_{M\leqslant L_{jj}} |b_{x_j x_j M}| \sum_{M\leqslant L_{jj}} B_{jjM}(\boldsymbol{\eta})\right. \tag{5.162}$$

$$\left. + 2\max_{M\leqslant L_{jr}} |b_{x_j x_r M}| \sum_{M\leqslant L_{jr}} B_{jrM}(\boldsymbol{\eta})\right) \tag{5.163}$$

$$= \max_{(i_1,i_2,\dots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\dots,i_n}) + \frac{1}{8k^2}\sum_{j,r=1;r>j}^{n} \left(\max_{M\leqslant L_{jj}} |b_{x_j x_j M}| + 2\max_{M\leqslant L_{jr}} |b_{x_j x_r M}|\right) \tag{5.164}$$

because $\sum_{M\leqslant L_{jj}} B_{jjM} = 1$ and $\sum_{M\leqslant L_{jr}} B_{jrM} = 1$. In addition,

$$\max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \frac{1}{8k^2}\sum_{j,r=1;r>j}^{n}\left(\max_{M\leqslant L_{jj}}|b_{x_jx_jM}| + 2\max_{M\leqslant L_{jr}}|b_{x_jx_rM}|\right) \tag{5.165}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \beta_K \tag{5.166}$$

because for (5.121) $\beta_K = \dfrac{1}{8k^2}\sum_{i,j=1;j>i}^{n}\left(|b_{x_ix_i}| + 2|b_{x_ix_j}|\right)$. Moreover,

$$\max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \beta_K \tag{5.167}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \frac{1}{8k^2}\sum_{j,r=1;r>j}^{n}\left(\max_{M\leqslant L_{jj}}\left|\sum_{K\leqslant M}\frac{\binom{M}{K}}{\binom{L_{jj}}{K}}\tilde{a}_{Kjj}\right| + 2\max_{M\leqslant L_{jr}}\left|\sum_{K\leqslant M}\frac{\binom{M}{K}}{\binom{L_{jr}}{K}}\bar{a}_{Kjr}\right|\right) \tag{5.168}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) \tag{5.169}$$

$$+\frac{1}{8k^2}\sum_{j,r=1;r>j}^{n}\left(\max_{M=(m_1,m_2,\ldots,m_{j-1},m_j,m_{j+1},\ldots,m_n)\leqslant L_{jj}}\left|\sum_{k_1,k_2,\ldots,k_{j-1},k_j,k_{j+1},\ldots,k_n=0}^{m_1,m_2,\ldots,m_{j-1},m_j,m_{j+1},\ldots,m_n}(k_j+1)(k_j+2)\right.\right. \tag{5.170}$$

$$\left.\cdot\, a_{k_1,k_2,\ldots,k_{j-1},k_j+2,k_{j+1},\ldots,k_n}\cdot\frac{\binom{m_1}{k_1}}{\binom{l_1}{k_1}}\cdot\frac{\binom{m_2}{k_2}}{\binom{l_2}{k_2}}\cdots\frac{\binom{m_{j-1}}{k_{j-1}}}{\binom{l_{j-1}}{k_{j-1}}}\cdot\frac{\binom{m_j}{k_j}}{\binom{l_j-2}{k_j}}\cdot\frac{\binom{m_{j+1}}{k_{j+1}}}{\binom{l_{j+1}}{k_{j+1}}}\cdots\frac{\binom{m_n}{k_n}}{\binom{l_n}{k_n}}\right|+2 \tag{5.171}$$

$$\cdot\max_{M=(m_1,m_2,\ldots,m_{j-1},m_j,m_{j+1},\ldots,m_{r-1},m_r,m_{r+1},\ldots,m_n)\leqslant L_{jr}}\left|\sum_{k_1,k_2,\ldots,k_{j-1},k_j,k_{j+1},\ldots,k_{r-1},k_r,k_{r+1},\ldots,k_n=0}^{m_1,m_2,\ldots,m_{j-1},m_j,m_{j+1},\ldots,m_{r-1},m_r,m_{r+1},\ldots,m_n}(k_j+1)\right. \tag{5.172}$$

$$\cdot\,(k_r+1)a_{k_1,k_2,\ldots,k_{j-1},k_j+1,k_{j+1},\ldots,k_{r-1},k_r+1,k_{r+1},\ldots,k_n} \tag{5.173}$$

$$\left.\left.\cdot\frac{\binom{m_1}{k_1}}{\binom{l_1}{k_1}}\cdot\frac{\binom{m_2}{k_2}}{\binom{l_2}{k_2}}\cdots\frac{\binom{m_{j-1}}{k_{j-1}}}{\binom{l_{j-1}}{k_{j-1}}}\cdot\frac{\binom{m_j}{k_j}}{\binom{l_j-1}{k_j}}\cdot\frac{\binom{m_{j+1}}{k_{j+1}}}{\binom{l_{j+1}}{k_{j+1}}}\cdots\frac{\binom{m_{r-1}}{k_{r-1}}}{\binom{l_{r-1}}{k_{r-1}}}\cdot\frac{\binom{m_r}{k_r}}{\binom{l_r-1}{k_r}}\cdot\frac{\binom{m_{r+1}}{k_{r+1}}}{\binom{l_{r+1}}{k_{r+1}}}\cdots\frac{\binom{m_n}{k_n}}{\binom{l_n}{k_n}}\right|\right) \tag{5.174}$$

$$\leqslant \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) \tag{5.175}$$

$$+\frac{1}{8k^2}\sum_{j,r=1;r>j}^{n}\left(\max_{M=(m_1,m_2,\ldots,m_{j-1},m_j,m_{j+1},\ldots,m_n)\leqslant L_{jj}}\sum_{k_1,k_2,\ldots,k_{j-1},k_j,k_{j+1},\ldots,k_n=0}^{m_1,m_2,\ldots,m_{j-1},m_j,m_{j+1},\ldots,m_n}(k_j+1)(k_j+2)\right. \tag{5.176}$$

$$\cdot\,|a_{k_1,k_2,\ldots,k_{j-1},k_j+2,k_{j+1},\ldots,k_n}|\cdot\frac{\binom{m_1}{k_1}}{\binom{l_1}{k_1}}\cdot\frac{\binom{m_2}{k_2}}{\binom{l_2}{k_2}}\cdots\frac{\binom{m_{j-1}}{k_{j-1}}}{\binom{l_{j-1}}{k_{j-1}}}\cdot\frac{\binom{m_j}{k_j}}{\binom{l_j-2}{k_j}}\cdot\frac{\binom{m_{j+1}}{k_{j+1}}}{\binom{l_{j+1}}{k_{j+1}}}\cdots\frac{\binom{m_n}{k_n}}{\binom{l_n}{k_n}}+2 \tag{5.177}$$

$$\cdot\max_{M=(m_1,m_2,\ldots,m_{j-1},m_j,m_{j+1},\ldots,m_{r-1},m_r,m_{r+1},\ldots,m_n)\leqslant L_{jr}}\sum_{k_1,k_2,\ldots,k_{j-1},k_j,k_{j+1},\ldots,k_{r-1},k_r,k_{r+1},\ldots,k_n=0}^{m_1,m_2,\ldots,m_{j-1},m_j,m_{j+1},\ldots,m_{r-1},m_r,m_{r+1},\ldots,m_n}(k_j+1) \tag{5.178}$$

$$\cdot\,(k_r+1)|a_{k_1,k_2,\ldots,k_{j-1},k_j+1,k_{j+1},\ldots,k_{r-1},k_r+1,k_{r+1},\ldots,k_n}| \tag{5.179}$$

$$\left.\cdot\frac{\binom{m_1}{k_1}}{\binom{l_1}{k_1}}\cdot\frac{\binom{m_2}{k_2}}{\binom{l_2}{k_2}}\cdots\frac{\binom{m_{j-1}}{k_{j-1}}}{\binom{l_{j-1}}{k_{j-1}}}\cdot\frac{\binom{m_j}{k_j}}{\binom{l_j-1}{k_j}}\cdot\frac{\binom{m_{j+1}}{k_{j+1}}}{\binom{l_{j+1}}{k_{j+1}}}\cdots\frac{\binom{m_{r-1}}{k_{r-1}}}{\binom{l_{r-1}}{k_{r-1}}}\cdot\frac{\binom{m_r}{k_r}}{\binom{l_r-1}{k_r}}\cdot\frac{\binom{m_{r+1}}{k_{r+1}}}{\binom{l_{r+1}}{k_{r+1}}}\cdots\frac{\binom{m_n}{k_n}}{\binom{l_n}{k_n}}\right) \tag{5.180}$$

for the triangle inequality. Additionally,

$$\max_{(i_1,i_2,\dots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\dots,i_n}) \tag{5.181}$$

$$+\frac{1}{8k^2}\sum_{j,r=1;r>j}^{n}\Bigg(\max_{M=(m_1,m_2,\dots,m_{j-1},m_j,m_{j+1},\dots,m_n)\leqslant L_{jj}}\Bigg|\sum_{k_1,k_2,\dots,k_{j-1},k_j,k_{j+1},\dots,k_n=0}^{m_1,m_2,\dots,m_{j-1},m_j,m_{j+1},\dots,m_n}(k_j+1)(k_j+2) \tag{5.182}$$

$$\cdot a_{k_1,k_2,\dots,k_{j-1},k_j+2,k_{j+1},\dots,k_n}\cdot\frac{\binom{m_1}{k_1}}{\binom{l_1}{k_1}}\cdot\frac{\binom{m_2}{k_2}}{\binom{l_2}{k_2}}\cdots\frac{\binom{m_{j-1}}{k_{j-1}}}{\binom{l_{j-1}}{k_{j-1}}}\cdot\frac{\binom{m_j}{k_j}}{\binom{l_j-2}{k_j}}\cdot\frac{\binom{m_{j+1}}{k_{j+1}}}{\binom{l_{j+1}}{k_{j+1}}}\cdots\frac{\binom{m_n}{k_n}}{\binom{l_n}{k_n}}\Bigg|+2 \tag{5.183}$$

$$\cdot\max_{M=(m_1,m_2,\dots,m_{j-1},m_j,m_{j+1},\dots,m_{r-1},m_r,m_{r+1},\dots,m_n)\leqslant L_{jr}}\Bigg|\sum_{k_1,k_2,\dots,k_{j-1},k_j,k_{j+1},\dots,k_{r-1},k_r,k_{r+1},\dots,k_n=0}^{m_1,m_2,\dots,m_{j-1},m_j,m_{j+1},\dots,m_{r-1},m_r,m_{r+1},\dots,m_n}(k_j+1) \tag{5.184}$$

$$\cdot(k_r+1)a_{k_1,k_2,\dots,k_{j-1},k_j+1,k_{j+1},\dots,k_{r-1},k_r+1,k_{r+1},\dots,k_n} \tag{5.185}$$

$$\cdot\frac{\binom{m_1}{k_1}}{\binom{l_1}{k_1}}\cdot\frac{\binom{m_2}{k_2}}{\binom{l_2}{k_2}}\cdots\frac{\binom{m_{j-1}}{k_{j-1}}}{\binom{l_{j-1}}{k_{j-1}}}\cdot\frac{\binom{m_j}{k_j}}{\binom{l_j-1}{k_j}}\cdot\frac{\binom{m_{j+1}}{k_{j+1}}}{\binom{l_{j+1}}{k_{j+1}}}\cdots\frac{\binom{m_{r-1}}{k_{r-1}}}{\binom{l_{r-1}}{k_{r-1}}}\cdot\frac{\binom{m_r}{k_r}}{\binom{l_r-1}{k_r}}\cdot\frac{\binom{m_{r+1}}{k_{r+1}}}{\binom{l_{r+1}}{k_{r+1}}}\cdots\frac{\binom{m_n}{k_n}}{\binom{l_n}{k_n}}\Bigg|\Bigg) \tag{5.186}$$

$$\leqslant\max_{(i_1,i_2,\dots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\dots,i_n}) \tag{5.187}$$

$$+\frac{1}{8k^2}\sum_{j,r=1;r>j}^{n}\Bigg(\max_{M=(m_1,m_2,\dots,m_{j-1},m_j,m_{j+1},\dots,m_n)\leqslant L_{jj}}\sum_{k_1,k_2,\dots,k_{j-1},k_j,k_{j+1},\dots,k_n=0}^{m_1,m_2,\dots,m_{j-1},m_j,m_{j+1},\dots,m_n}(k_j+1)(k_j+2) \tag{5.188}$$

$$\cdot|a_{k_1,k_2,\dots,k_{j-1},k_j+2,k_{j+1},\dots,k_n}|\cdot\frac{\binom{m_1}{k_1}}{\binom{l_1}{k_1}}\cdot\frac{\binom{m_2}{k_2}}{\binom{l_2}{k_2}}\cdots\frac{\binom{m_{j-1}}{k_{j-1}}}{\binom{l_{j-1}}{k_{j-1}}}\cdot\frac{\binom{m_j}{k_j}}{\binom{l_j-2}{k_j}}\cdot\frac{\binom{m_{j+1}}{k_{j+1}}}{\binom{l_{j+1}}{k_{j+1}}}\cdots\frac{\binom{m_n}{k_n}}{\binom{l_n}{k_n}}+2 \tag{5.189}$$

$$\cdot\max_{M=(m_1,m_2,\dots,m_{j-1},m_j,m_{j+1},\dots,m_{r-1},m_r,m_{r+1},\dots,m_n)\leqslant L_{jr}}\sum_{k_1,k_2,\dots,k_{j-1},k_j,k_{j+1},\dots,k_{r-1},k_r,k_{r+1},\dots,k_n=0}^{m_1,m_2,\dots,m_{j-1},m_j,m_{j+1},\dots,m_{r-1},m_r,m_{r+1},\dots,m_n}(k_j+1) \tag{5.190}$$

$$\cdot(k_r+1)|a_{k_1,k_2,\dots,k_{j-1},k_j+1,k_{j+1},\dots,k_{r-1},k_r+1,k_{r+1},\dots,k_n}| \tag{5.191}$$

$$\cdot\frac{\binom{m_1}{k_1}}{\binom{l_1}{k_1}}\cdot\frac{\binom{m_2}{k_2}}{\binom{l_2}{k_2}}\cdots\frac{\binom{m_{j-1}}{k_{j-1}}}{\binom{l_{j-1}}{k_{j-1}}}\cdot\frac{\binom{m_j}{k_j}}{\binom{l_j-1}{k_j}}\cdot\frac{\binom{m_{j+1}}{k_{j+1}}}{\binom{l_{j+1}}{k_{j+1}}}\cdots\frac{\binom{m_{r-1}}{k_{r-1}}}{\binom{l_{r-1}}{k_{r-1}}}\cdot\frac{\binom{m_r}{k_r}}{\binom{l_r-1}{k_r}}\cdot\frac{\binom{m_{r+1}}{k_{r+1}}}{\binom{l_{r+1}}{k_{r+1}}}\cdots\frac{\binom{m_n}{k_n}}{\binom{l_n}{k_n}}\Bigg) \tag{5.192}$$

$$\leqslant\max_{(i_1,i_2,\dots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\dots,i_n}) \tag{5.193}$$

$$+\frac{1}{8k^2}\sum_{j,r=1;r>j}^{n}\Bigg(\max_{M=(m_1,m_2,\dots,m_{j-1},m_j,m_{j+1},\dots,m_n)\leqslant L_{jj}}\sum_{k_1,k_2,\dots,k_{j-1},k_j,k_{j+1},\dots,k_n=0}^{m_1,m_2,\dots,m_{j-1},m_j,m_{j+1},\dots,m_n}(k_j+1)(k_j+2) \tag{5.194}$$

$$\cdot|a_{k_1,k_2,\dots,k_{j-1},k_j+2,k_{j+1},\dots,k_n}|+2 \tag{5.195}$$

$$\cdot\max_{M=(m_1,m_2,\dots,m_{j-1},m_j,m_{j+1},\dots,m_{r-1},m_r,m_{r+1},\dots,m_n)\leqslant L_{jr}}\sum_{k_1,k_2,\dots,k_{j-1},k_j,k_{j+1},\dots,k_{r-1},k_r,k_{r+1},\dots,k_n=0}^{m_1,m_2,\dots,m_{j-1},m_j,m_{j+1},\dots,m_{r-1},m_r,m_{r+1},\dots,m_n}(k_j+1) \tag{5.196}$$

$$\cdot(k_r+1)|a_{k_1,k_2,\dots,k_{j-1},k_j+1,k_{j+1},\dots,k_{r-1},k_r+1,k_{r+1},\dots,k_n}| \tag{5.197}$$

$$=\max_{(i_1,i_2,\dots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\dots,i_n}) \tag{5.198}$$

$$+\frac{1}{8k^2}\sum_{j,r=1;r>j}^{n}\Bigg(\sum_{k_1,k_2,\dots,k_{j-1},k_j,k_{j+1},\dots,k_n=0}^{l_1,l_2,\dots,l_{j-1},l_j-2,l_{j+1},\dots,l_n}(k_j+1)(k_j+2)|a_{k_1,k_2,\dots,k_{j-1},k_j+2,k_{j+1},\dots,k_n}|+2 \tag{5.199}$$

$$\cdot\sum_{k_1,k_2,\dots,k_{j-1},k_j,k_{j+1},\dots,k_{r-1},k_r,k_{r+1},\dots,k_n=0}^{l_1,l_2,\dots,l_{j-1},l_j-1,l_{j+1},\dots,l_{r-1},l_r-1,l_{r+1},\dots,l_n}(k_j+1)(k_r+1)|a_{k_1,k_2,\dots,k_{j-1},k_j+1,k_{j+1},\dots,k_{r-1},k_r+1,k_{r+1},\dots,k_n}|\Bigg) \tag{5.200}$$

because the sums are among all coefficients. Furthermore,

$$\max_{(i_1,i_2,\dots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\dots,i_n}) \tag{5.201}$$

$$+\frac{1}{8k^2}\sum_{j,r=1;r>j}^{n}\Bigg(\sum_{k_1,k_2,\dots,k_{j-1},k_j,k_{j+1},\dots,k_n=0}^{l_1,l_2,\dots,l_{j-1},l_j-2,l_{j+1},\dots,l_n}(k_j+1)(k_j+2)|a_{k_1,k_2,\dots,k_{j-1},k_j+2,k_{j+1},\dots,k_n}|+2 \tag{5.202}$$

$$\cdot\sum_{k_1,k_2,\dots,k_{j-1},k_j,k_{j+1},\dots,k_{r-1},k_r,k_{r+1},\dots,k_n=0}^{l_1,l_2,\dots,l_{j-1},l_j-1,l_{j+1},\dots,l_{r-1},l_r-1,l_{r+1},\dots,l_n}(k_j+1)(k_r+1)|a_{k_1,k_2,\dots,k_{j-1},k_j+1,k_{j+1},\dots,k_{r-1},k_r+1,k_{r+1},\dots,k_n}|\Bigg) \tag{5.203}$$

$$=\max_{(i_1,i_2,\dots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\dots,i_n}) \tag{5.204}$$

$$+\frac{1}{8k^2}\sum_{j,r=1;r>j}^{n}\Bigg(\sum_{i_1,i_2,\dots,i_{j-1},i_j,i_{j+1},\dots,i_n=0}^{l_1,l_2,\dots,l_{j-1},l_j,l_{j+1},\dots,l_n} i_j(i_j-1)|a_{i_1,i_2,\dots,i_{j-1},i_j,i_{j+1},\dots,i_n}| \tag{5.205}$$

$$+2\sum_{i_1,i_2,\dots,i_{j-1},i_j,i_{j+1},\dots,i_{r-1},i_r,i_{r+1},\dots,i_n=0}^{l_1,l_2,\dots,l_{j-1},l_j,l_{j+1},\dots,l_{r-1},l_r,l_{r+1},\dots,l_n} i_j i_r|a_{i_1,i_2,\dots,i_{j-1},i_j,i_{j+1},\dots,i_{r-1},i_r,i_{r+1},\dots,i_n}|\Bigg) \tag{5.206}$$

$$=\max_{(i_1,i_2,\dots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\dots,i_n})+\frac{1}{8k^2}\sum_{i_1,i_2,\dots,i_n=0}^{l_1,l_2,\dots,l_n}(i_1+i_2+\cdots+i_n)(i_1+i_2+\cdots+i_n-1)|a_{i_1,i_2,\dots,i_n}| \tag{5.207}$$

$$=\max_{(i_1,i_2,\dots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\dots,i_n})+\alpha_K, \tag{5.208}$$

where α_K is defined in (5.119). So,

$$\max_{\mathbf{x}\in[0,1]^n} p(\mathbf{x}) \leqslant \max_{(i_1,i_2,\dots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\dots,i_n})+\beta_K \leqslant \max_{(i_1,i_2,\dots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\dots,i_n})+\alpha_K. \tag{5.209}$$

This proves the first part of the theorem. An entirely analogous argument establishes the lower bound on $\min_{\mathbf{x}\in[0,1]^n} p(\mathbf{x})$ in (5.118). $\square$

Using the bounds presented in Chapter 3 and their comparison, it is possible to obtain the inequalities of the Proposition 5.2, and also further bounds. Suppose $p(\mathbf{x})=\sum_{I\leqslant L} a_I\mathbf{x}^I$, observing that in the interval $[0,1]^n$, set s,u such that $1\leqslant s,u\leqslant n$, with no loss of generality, assume $u>s$.

$$p_{x_sx_sA}=\sum_{i_1,i_2,\dots,i_n=0}^{l_1,l_2,\dots,l_n}(i_s-1)i_s|a_{i_1,i_2,\dots,i_n}| \tag{5.210}$$

$$p_{x_sx_uA}=\sum_{i_1,i_2,\dots,i_n=0}^{l_1,l_2,\dots,l_n} i_s i_u|a_{i_1,i_2,\dots,i_n}| \tag{5.211}$$

Similarly as in Section 5.1.1, from the inequalities (3.85)-(3.86), the extension of the bounds (5.68)-(5.69) to the interval $[0,1]^n$ for the absolute values can be defined. Set $\mathbf{x}\in[0,1]^n$,

$$|p(\mathbf{x})| \leqslant \max_{M\leqslant L}|b_M^{([0,1]^n)}| \leqslant \max\{|\underline{p}_I([0,1]^n)|,|\overline{p}_I([0,1]^n)|\} \tag{5.212}$$

$$\leqslant \max\{|\underline{p}_S([0,1]^n)|,|\overline{p}_S([0,1]^n)|\} \leqslant \overline{p}_A([0,1]^n). \tag{5.213}$$

Applying the inequalities (4.41)-(4.42) for the second derivatives, as follows.

$$\sum_{j=1}^{n} |p_{x_jx_j}(\mathbf{x})| + 2\sum_{j,r=1;r>j}^{n} |p_{x_jx_r}(\mathbf{x})| = \sum_{j,r=1;r>j}^{n} \Big(|p_{x_jx_j}(\mathbf{x})| + 2|p_{x_jx_r}(\mathbf{x})| \Big) \tag{5.214}$$

$$\leqslant \sum_{j,r=1;r>j}^{n} \Big(|b_{x_jx_j}| + 2|b_{x_jx_r}| \Big) \tag{5.215}$$

$$\leqslant \sum_{j,r=1;r>j}^{n} \Big(\max\{|\underline{p}_{x_jx_jI}([0,1]^n)|, |\overline{p}_{x_jx_jI}([0,1]^n)|\} \tag{5.216}$$

$$+ 2\max\{|\underline{p}_{x_jx_rI}([0,1]^n)|, |\overline{p}_{x_jx_rI}([0,1]^n)|\} \Big) \tag{5.217}$$

$$\leqslant \sum_{j,r=1;r>j}^{n} \Big(\max\{|\underline{p}_{x_jx_jS}([0,1]^n)|, |\overline{p}_{x_jx_jS}([0,1]^n)|\} \tag{5.218}$$

$$+ 2\max\{|\underline{p}_{x_jx_rS}([0,1]^n)|, |\overline{p}_{x_jx_rS}([0,1]^n)|\} \Big) \tag{5.219}$$

$$\leqslant \sum_{j,r=1;r>j}^{n} \Big(\overline{p}_{x_jx_jA}([0,1]^n) + 2\overline{p}_{x_jx_rA}([0,1]^n) \Big). \tag{5.220}$$

So, since $\mathbf{x}$ is arbitrary in $[0,1]^n$, the inequalities (5.25)-(5.26) can be improved.

$$\min_{I\in K} p(\mathbf{t}_I) - \alpha_K \tag{5.221}$$

$$= \min_{I\in K} p(\mathbf{t}_I) - \frac{1}{8k^2} \sum_{j,r=1;r>j}^{n} \Big(\overline{p}_{x_jx_jA}([0,1]^n) + 2\overline{p}_{x_jx_rA}([0,1]^n) \Big) \tag{5.222}$$

$$\leqslant \min_{I\in K} p(\mathbf{t}_I) - \frac{1}{8k^2} \sum_{j,r=1;r>j}^{n} \Big(\max\{|\underline{p}_{x_jx_jS}([0,1]^n)|, |\overline{p}_{x_jx_jS}([0,1]^n)|\} \tag{5.223}$$

$$+ 2\max\{|\underline{p}_{x_jx_rS}([0,1]^n)|, |\overline{p}_{x_jx_rS}([0,1]^n)|\} \Big) \tag{5.224}$$

$$\leqslant \min_{I\in K} p(\mathbf{t}_I) - \frac{1}{8k^2} \sum_{j,r=1;r>j}^{n} \Big(\max\{|\underline{p}_{x_jx_jI}([0,1]^n)|, |\overline{p}_{x_jx_jI}([0,1]^n)|\} \tag{5.225}$$

$$+ 2\max\{|\underline{p}_{x_jx_rI}([0,1]^n)|, |\overline{p}_{x_jx_rI}([0,1]^n)|\} \Big) \tag{5.226}$$

$$\leqslant \min_{I\in K} p(\mathbf{t}_I) - \beta_K \tag{5.227}$$

$$= \min_{I\in K} p(\mathbf{t}_I) - \frac{1}{8k^2} \sum_{j,r=1;r>j}^{n} \Big(|b_{x_jx_j}| + 2|b_{x_jx_r}| \Big) \tag{5.228}$$

$$\leqslant \min_{\mathbf{x}\in[0,1]^n} p(\mathbf{x}) \tag{5.229}$$

$$\leqslant \max_{\mathbf{x}\in[0,1]^n} p(\mathbf{x}) \tag{5.230}$$

$$\leqslant \max_{I\in K} p(\mathbf{t}_I) + \beta_K \tag{5.231}$$

$$= \max_{I\in K} p(\mathbf{t}_I) + \frac{1}{8k^2} \sum_{j,r=1;r>j}^{n} \Big(|b_{x_jx_j}| + 2|b_{x_jx_r}| \Big) \tag{5.232}$$

$$\leqslant \max_{I\in K} p(\mathbf{t}_I) + \frac{1}{8k^2}\sum_{j,r=1;r>j}^{n}\Bigg(\max\{|\underline{p}_{x_jx_jI}([0,1]^n)|,|\overline{p}_{x_jx_jI}([0,1]^n)|\} \tag{5.233}$$

$$+2\max\{|\underline{p}_{x_jx_rI}([0,1]^n)|,|\overline{p}_{x_jx_rI}([0,1]^n)|\}\Bigg) \tag{5.234}$$

$$\leqslant \max_{I\in K} p(\mathbf{t}_I) + \frac{1}{8k^2}\sum_{j,r=1;r>j}^{n}\Bigg(\max\{|\underline{p}_{x_jx_jS}([0,1]^n)|,|\overline{p}_{x_jx_jS}([0,1]^n)|\} \tag{5.235}$$

$$+2\max\{|\underline{p}_{x_jx_rS}([0,1]^n)|,|\overline{p}_{x_jx_rS}([0,1]^n)|\}\Bigg) \tag{5.236}$$

$$\leqslant \max_{I\in K} p(\mathbf{t}_I) + \alpha_K \tag{5.237}$$

$$= \max_{I\in K} p(\mathbf{t}_I) + \frac{1}{8k^2}\sum_{j,r=1;r>j}^{n}\Bigg(\overline{p}_{x_jx_jA}([0,1]^n) + 2\overline{p}_{x_jx_rA}([0,1]^n)\Bigg). \tag{5.238}$$

Rewriting with the respective definitions, the previous inequalities become as follows.

$$\min_{I\in K} p(\mathbf{t}_I) - \frac{1}{8k^2}\Bigg(\sum_{I=(i_1,i_2,\ldots,i_n)\leqslant L}(i_1+i_2+\cdots+i_n)(i_1+i_2+\cdots+i_n-1)|a_I|\Bigg) \tag{5.239}$$

$$\leqslant \min_{I\in K} p(\mathbf{t}_I) - \frac{1}{8k^2}\sum_{j,r=1;r>j}^{n}\Bigg(\max\Bigg\{\Bigg|\sum_{I=(i_1,i_2,\ldots,i_j,\ldots,i_n)\leqslant L;a_I>0} i_j(i_j-1)a_I\Bigg|, \tag{5.240}$$

$$\Bigg|\sum_{I=(i_1,i_2,\ldots,i_j,\ldots,i_n)\leqslant L;a_I<0} i_j(i_j-1)a_I\Bigg|\Bigg\} \tag{5.241}$$

$$+2\max\Bigg\{\Bigg|\sum_{I=(i_1,i_2,\ldots,i_j,\ldots,i_r,\ldots,i_n)\leqslant L;a_I>0} i_ji_ra_I\Bigg|,\Bigg|\sum_{I=(i_1,i_2,\ldots,i_j,\ldots,i_r,\ldots,i_n)\leqslant L;a_I<0} i_ji_ra_I\Bigg|\Bigg\}\Bigg) \tag{5.242}$$

$$\leqslant \min_{I\in K} p(\mathbf{t}_I) - \frac{1}{8k^2}\sum_{j,r=1;r>j}^{n}\Bigg(\max\Bigg\{\Bigg|\sum_{I=(i_1,i_2,\ldots,i_j,\ldots,i_n)\leqslant L;a_I<0} i_j(i_j-1)a_I \tag{5.243}$$

$$+\sum_{I=(i_1,i_2,\ldots,i_j,\ldots,i_n)\leqslant L;i_1=0,i_2=0,\ldots,i_{j-1}=0,i_j=2,i_{j+1}=0,\ldots,i_n=0;a_I>0} i_j(i_j-1)a_I\Bigg|, \tag{5.244}$$

$$\Bigg|\sum_{I=(i_1,i_2,\ldots,i_j,\ldots,i_n)\leqslant L;a_I>0} i_j(i_j-1)a_I \tag{5.245}$$

$$+\sum_{I=(i_1,i_2,\ldots,i_j,\ldots,i_n)\leqslant L;i_1=0,i_2=0,\ldots,i_{j-1}=0,i_j=2,i_{j+1}=0,\ldots,i_n=0;a_I<0} i_j(i_j-1)a_I\Bigg|\Bigg\} \tag{5.246}$$

$$+2\max\Bigg\{\Bigg|\sum_{I=(i_1,i_2,\ldots,i_j,\ldots,i_r,\ldots,i_n)\leqslant L;a_I<0} i_ji_ra_I \tag{5.247}$$

$$+\sum_{I=(i_1,i_2,\ldots,i_j,\ldots,i_n)\leqslant L;i_1=0,i_2=0,\ldots,i_{j-1}=0,i_j=1,i_{j+1}=0,\ldots,i_{r-1}=0,i_r=1,i_{jr+1}=0,\ldots,i_n=0;a_I>0} i_ji_ra_I\Bigg|, \tag{5.248}$$

$$\Bigg|\sum_{I=(i_1,i_2,\ldots,i_j,\ldots,i_r,\ldots,i_n)\leqslant L;a_I>0} i_ji_ra_I \tag{5.249}$$

$$+\sum_{I=(i_1,i_2,\ldots,i_j,\ldots,i_n)\leqslant L;i_1=0,i_2=0,\ldots,i_{j-1}=0,i_j=1,i_{j+1}=0,\ldots,i_{r-1}=0,i_r=1,i_{jr+1}=0,\ldots,i_n=0;a_I<0} i_ji_ra_I\Bigg|\Bigg\}\Bigg) \tag{5.250}$$

$$\leqslant \min_{I\in K} p(\mathbf{t}_I) - \frac{1}{8k^2}\sum_{j,r=1;r>j}^{n}\Bigg(\max_{M\leqslant L_{jj}}|b_{x_jx_jM}| + 2\max_{M\leqslant L_{jr}}|b_{x_jx_rM}|\Bigg) \tag{5.251}$$

$$\leqslant \min_{\mathbf{x}\in[0,1]^n} p(\mathbf{x}) \tag{5.252}$$

$$\leqslant \max_{\mathbf{x}\in[0,1]^n} p(\mathbf{x}) \tag{5.253}$$

$$\leqslant \max_{I\in K} p(\mathbf{t}_I) + \frac{1}{8k^2} \sum_{j,r=1;r>j}^{n} \left(\max_{M\leqslant L_{jj}} |b_{x_j x_j M}| + 2 \max_{M\leqslant L_{jr}} |b_{x_j x_r M}| \right) \tag{5.254}$$

$$\leqslant \max_{I\in K} p(\mathbf{t}_I) + \frac{1}{8k^2} \sum_{j,r=1;r>j}^{n} \left(\max\left\{ \left| \sum_{I=(i_1,i_2,\dots,i_j,\dots,i_n)\leqslant L;a_I<0} i_j(i_j-1)a_I \right.\right.\right. \tag{5.255}$$

$$+ \left.\sum_{I=(i_1,i_2,\dots,i_j,\dots,i_n)\leqslant L;i_1=0,i_2=0,\dots,i_{j-1}=0,i_j=2,i_{j+1}=0,\dots,i_n=0;a_I>0} i_j(i_j-1)a_I \right|, \tag{5.256}$$

$$\left| \sum_{I=(i_1,i_2,\dots,i_j,\dots,i_n)\leqslant L;a_I>0} i_j(i_j-1)a_I \right. \tag{5.257}$$

$$+ \left.\left.\sum_{I=(i_1,i_2,\dots,i_j,\dots,i_n)\leqslant L;i_1=0,i_2=0,\dots,i_{j-1}=0,i_j=2,i_{j+1}=0,\dots,i_n=0;a_I<0} i_j(i_j-1)a_I \right| \right\} \tag{5.258}$$

$$+ 2\max\left\{ \left| \sum_{I=(i_1,i_2,\dots,i_j,\dots,i_r,\dots,i_n)\leqslant L;a_I<0} i_j i_r a_I \right.\right. \tag{5.259}$$

$$+ \left.\sum_{I=(i_1,i_2,\dots,i_j,\dots,i_n)\leqslant L;i_1=0,i_2=0,\dots,i_{j-1}=0,i_j=1,i_{j+1}=0,\dots,i_{r-1}=0,i_r=1,i_{jr+1}=0,\dots,i_n=0;a_I>0} i_j i_r a_I \right|, \tag{5.260}$$

$$\left| \sum_{I=(i_1,i_2,\dots,i_j,\dots,i_r,\dots,i_n)\leqslant L;a_I>0} i_j i_r a_I \right. \tag{5.261}$$

$$+ \left.\left.\left.\sum_{I=(i_1,i_2,\dots,i_j,\dots,i_n)\leqslant L;i_1=0,i_2=0,\dots,i_{j-1}=0,i_j=1,i_{j+1}=0,\dots,i_{r-1}=0,i_r=1,i_{jr+1}=0,\dots,i_n=0;a_I<0} i_j i_r a_I \right| \right\} \right) \tag{5.262}$$

$$\leqslant \max_{I\in K} p(\mathbf{t}_I) + \frac{1}{8k^2} \sum_{j,r=1;r>j}^{n} \left(\max\left\{ \left| \sum_{I=(i_1,i_2,\dots,i_j,\dots,i_n)\leqslant L;a_I>0} i_j(i_j-1)a_I \right|, \right.\right. \tag{5.263}$$

$$\left.\left| \sum_{I=(i_1,i_2,\dots,i_j,\dots,i_n)\leqslant L;a_I<0} i_j(i_j-1)a_I \right| \right\} \tag{5.264}$$

$$\left. + 2\max\left\{ \left| \sum_{I=(i_1,i_2,\dots,i_j,\dots,i_r,\dots,i_n)\leqslant L;a_I>0} i_j i_r a_I \right|, \left| \sum_{I=(i_1,i_2,\dots,i_j,\dots,i_r,\dots,i_n)\leqslant L;a_I<0} i_j i_r a_I \right| \right\} \right) \tag{5.265}$$

$$\leqslant \max_{I\in K} p(\mathbf{t}_I) + \frac{1}{8k^2} \left(\sum_{I=(i_1,i_2,\dots,i_n)\leqslant L} (i_1+i_2+\cdots+i_n)(i_1+i_2+\cdots+i_n-1)|a_I| \right). \tag{5.266}$$

5.3 Bounds in the Interval $[\underline{\mathbf{x}}, \overline{\mathbf{x}}] \subset \mathbb{R}^n_{\geqslant 0}$

Lemma 5.1. *Let n be a natural number greater than zero, $I = (i_1, \dots, i_n)$ be a multi-index. Then*

$$\sum_{j,r=1;r>j}^{n} i_j(i_j-1) + 2i_j i_r = (i_1+\cdots+i_n)(i_1+\cdots+i_n-1) \tag{5.267}$$

Proof. The proof uses induction on n. When $n = 1$, then the left-hand side of equation (5.267) becomes $(i_1)(i_1-1)$, which is the same as the right-hand side. When $n = 2$ left-hand side of equation

(5.267) is

$$i_1(i_1-1)+2i_1i_2+i_2(i_2-1) = i_1^2 - i_1 + 2i_1i_2 + i_2^2 - i_2 \tag{5.268}$$

$$= (i_1+i_2)(i_1+i_2-1) \tag{5.269}$$

which is the right-hand side of (5.267).

Assuming (5.267) holds for $n-1$, to prove for n calculate the left-hand side of equation (5.267)

$$\sum_{j,r=1;r>j}^{n} i_j(i_j-1)+2i_ji_r \tag{5.270}$$

$$= \sum_{j,r=1;r>j}^{n-1} \Big(i_j(i_j-1)+2i_ji_r\Big) + i_n(i_n-1)+2i_1i_n+2i_2i_n+\cdots+2i_{n-1}i_n \tag{5.271}$$

$$= (i_1+\cdots+i_{n-1})(i_1+\cdots+i_{n-1}-1)+i_n(i_n-1)+i_1i_n+i_2i_n+\cdots+i_{n-1}i_n \tag{5.272}$$

$$+ i_n(i_n-1)+i_1i_n+i_2i_n+\cdots+i_{n-1}i_n \tag{5.273}$$

$$= (i_1+\cdots+i_{n-1})(i_1+\cdots+i_{n-1}-1)+i_n(i_n-1)+i_n(i_1+i_2+\cdots+i_{n-1}) \tag{5.274}$$

$$+ i_n(i_1+i_2+\cdots+i_{n-1}) \tag{5.275}$$

$$= (i_1+\cdots+i_{n-1})(i_1+\cdots+i_{n-1}+i_n-1)+i_n(i_1+i_2+\cdots+i_{n-1}+i_n-1) \tag{5.276}$$

$$= (i_1+\cdots+i_n)(i_1+\cdots+i_n-1) \tag{5.277}$$

which is the right-hand side of (5.267), so it proves the lemma. □

The generalization to closed bounded boxes in n dimensions is given as follows. Let k be a positive integer and define $K = \{(i_1, i_2, \ldots, i_n) | 0 \leqslant i_1, i_2, \ldots, i_n \leqslant k\}$; consider the closed bound box $[\underline{\mathbf{x}}, \overline{\mathbf{x}}] \subset \mathbb{R}^n_{\geqslant 0}$ given by

$$\mathbf{t}_I = \mathbf{t}_{i_1, i_2, \ldots, i_n} \tag{5.278}$$

$$= \left((\overline{x}_1 - \underline{x}_1)\frac{i_1}{k} + \underline{x}_1, (\overline{x}_2 - \underline{x}_2)\frac{i_2}{k} + \underline{x}_2, \ldots, (\overline{x}_n - \underline{x}_n)\frac{i_n}{k} + \underline{x}_n \right), \tag{5.279}$$

with $(i_1, i_2, \ldots, i_n) \in K$.

Proposition 5.3. *Let p be given by*

$$p(\mathbf{x}) = \sum_{I \leqslant L} a_I \mathbf{x}^I. \tag{5.280}$$

Under the assumption $\overline{x}_j \geqslant 1$ *for all* j *such that* $1 \leqslant j \leqslant n$, *then*

$$\max_{\mathbf{x} \in [\underline{\mathbf{x}}, \overline{\mathbf{x}}]} p(\mathbf{x}) \leqslant \max_{I \in K} p(\mathbf{t}_I) + \beta_K \leqslant \max_{I \in K} p(\mathbf{t}_I) + \alpha_K \tag{5.281}$$

$$\min_{\mathbf{x} \in [\underline{\mathbf{x}}, \overline{\mathbf{x}}]} p(\mathbf{x}) \geqslant \min_{I \in K} p(\mathbf{t}_I) - \beta_K \geqslant \min_{I \in K} p(\mathbf{t}_I) - \alpha_K, \tag{5.282}$$

where α_K *and* β_K *are defined as*

$$\alpha_K = \frac{\|\overline{\mathbf{x}} - \underline{\mathbf{x}}\|_\infty^2}{8k^2} \sum_{i_1,i_2,\ldots,i_n=0}^{l_1,l_2,\ldots,l_n} (i_1+i_2+\cdots+i_n)(i_1+i_2+\cdots+i_n-1)|a_{i_1,i_2,\ldots,i_n}| \tag{5.283}$$

$$\cdot \overline{\mathbf{x}}_1^{i_1} \overline{\mathbf{x}}_2^{i_2} \cdots \overline{\mathbf{x}}_n^{i_n} \tag{5.284}$$

$$= \frac{\|\overline{\mathbf{x}} - \underline{\mathbf{x}}\|_\infty^2}{8k^2} \sum_{I=(i_1,i_2,\ldots,i_n)\leqslant L} (i_1+i_2+\cdots+i_n)(i_1+i_2+\cdots+i_n-1)|a_I|\overline{\mathbf{x}}^I \tag{5.285}$$

and

$$\beta_K = \frac{\|\overline{\mathbf{x}} - \underline{\mathbf{x}}\|_\infty^2}{8k^2} \sum_{i,j=1;j>i}^{n} \left(|b_{x_i x_i}| + 2|b_{x_i x_j}| \right) \tag{5.286}$$

$$= \|\overline{\mathbf{x}} - \underline{\mathbf{x}}\|_\infty^2 \cdot \beta_K^{[0,1]^n}, \tag{5.287}$$

where $|b_{x_i x_j}|$ *is the maximum of the absolute values of the Bernstein coefficients for the partial derivatives* $p_{\mathbf{x}_i \mathbf{x}_j}$ *of* p *and* $\beta_K^{[0,1]^n} = \frac{1}{8k^2} \sum_{i,j=1;j>i}^{n} (|b_{x_i x_i}| + 2|b_{x_i x_j}|)$ *is the same expression of the case* $[0,1]^n$*, defined in (5.121).*

Under the assumption $\overline{x}_j \leqslant 1$ *for all* j *such that* $1 \leqslant j \leqslant n$*,*

$$\max_{\mathbf{x}\in[\underline{\mathbf{x}},\overline{\mathbf{x}}]} p(\mathbf{x}) \leqslant \max_{I\in K} p(\mathbf{t}_I) + \beta_K \leqslant \max_{I\in K} p(\mathbf{t}_I) + \|\overline{\mathbf{x}} - \underline{\mathbf{x}}\|_\infty^2 \cdot \alpha_K^{[0,1]^n} \leqslant \max_{I\in K} p(\mathbf{t}_I) + \alpha_K^{[0,1]^n} \tag{5.288}$$

$$\min_{\mathbf{x}\in[\underline{\mathbf{x}},\overline{\mathbf{x}}]} p(\mathbf{x}) \geqslant \min_{I\in K} p(\mathbf{t}_I) - \beta_K \geqslant \min_{I\in K} p(\mathbf{t}_I) - \|\overline{\mathbf{x}} - \underline{\mathbf{x}}\|_\infty^2 \cdot \alpha_K^{[0,1]^n} \geqslant \min_{I\in K} p(\mathbf{t}_I) - \alpha_K^{[0,1]^n}, \tag{5.289}$$

where $\alpha_K^{[0,1]^n} = \frac{1}{8k^2} \sum_{i_1,i_2,\ldots,i_n=0}^{l_1,l_2,\ldots,l_n} (i_1+i_2+\cdots+i_n)(i_1+i_2+\cdots+i_n-1)|a_{i_1,i_2,\ldots,i_n}|$ *is the same value of the case* $[0,1]^n$*, defined in (5.119), and* β_K *is defined in (5.286).*

Proof. The proof is an extension of the proof in one dimension and of the proof for the case $[0,1]^n$. The proof proceeds by induction on the dimension n. In the base case $n = 1$. obtain, from the inequalities (4.285)-(4.300), that

$$\min_{I\in K} p(t_I) - \frac{1}{8k^2} \sum_{j=0}^{n} (j-1)j|a_j|\overline{x}^j \tag{5.290}$$

$$\leqslant \min_{I\in K} p(t_I) - \frac{(\overline{x}-\underline{x})^2}{8k^2} \sum_{j=0}^{n} (j-1)j|a_j|\overline{x}^{j-2} \tag{5.291}$$

$$\leqslant \min_{I\in K} p(t_I) - \frac{(\overline{x}-\underline{x})^2}{8k^2} \max_{0\leqslant i\leqslant n-2} |b_i^{''([\underline{x},\overline{x}])}| \tag{5.292}$$

$$\leqslant \min_{x\in[\underline{x},\overline{x}]} p(x) \tag{5.293}$$

$$\leqslant \max_{x \in [\underline{x}, \overline{x}]} p(x) \tag{5.294}$$

$$\leqslant \max_{I \in K} p(t_I) + \frac{(\overline{x}-\underline{x})^2}{8k^2} \max_{0 \leqslant i \leqslant n-2} |b_i^{''([\underline{x},\overline{x}])}| \tag{5.295}$$

$$\leqslant \max_{I \in K} p(t_I) + \frac{(\overline{x}-\underline{x})^2}{8k^2} \sum_{j=0}^{n} (j-1)j|a_j|\overline{x}^{j-2} \tag{5.296}$$

$$\leqslant \max_{I \in K} p(t_I) + \frac{1}{8k^2} \sum_{j=0}^{n} (j-1)j|a_j|\overline{x}^{j}. \tag{5.297}$$

Under the assumption $\overline{x} \geqslant 1$, then

$$\min_{I \in K} p(t_I) - \frac{(\overline{x}-\underline{x})^2}{8k^2} \sum_{j=0}^{n} (j-1)j|a_j|\overline{x}^{j} \tag{5.298}$$

$$\leqslant \min_{I \in K} p(t_I) - \frac{(\overline{x}-\underline{x})^2}{8k^2} \sum_{j=0}^{n} (j-1)j|a_j|\overline{x}^{j-2} \tag{5.299}$$

$$\leqslant \min_{I \in K} p(t_I) - \frac{(\overline{x}-\underline{x})^2}{8k^2} \max_{0 \leqslant i \leqslant n-2} |b_i^{''([\underline{x},\overline{x}])}| \tag{5.300}$$

$$\leqslant \min_{x \in [\underline{x}, \overline{x}]} p(x) \tag{5.301}$$

$$\leqslant \max_{x \in [\underline{x}, \overline{x}]} p(x) \tag{5.302}$$

$$\leqslant \max_{I \in K} p(t_I) + \frac{(\overline{x}-\underline{x})^2}{8k^2} \max_{0 \leqslant i \leqslant n-2} |b_i^{''([\underline{x},\overline{x}])}| \tag{5.303}$$

$$\leqslant \max_{I \in K} p(t_I) + \frac{(\overline{x}-\underline{x})^2}{8k^2} \sum_{j=0}^{n} (j-1)j|a_j|\overline{x}^{j-2} \tag{5.304}$$

$$\leqslant \max_{I \in K} p(t_I) + \frac{(\overline{x}-\underline{x})^2}{8k^2} \sum_{j=0}^{n} (j-1)j|a_j|\overline{x}^{j}. \tag{5.305}$$

So, the inequalities (5.281)-(5.282) are true; therefore, it proves the base case. Instead, if $\overline{x} \leqslant 1$, then

$$\min_{I \in K} p(t_I) - \sum_{j=0}^{n} (j-1)j|a_j| \tag{5.306}$$

$$\leqslant \min_{I \in K} p(t_I) - \frac{(\overline{x}-\underline{x})^2}{8k^2} \sum_{j=0}^{n} (j-1)j|a_j| \tag{5.307}$$

$$\leqslant \min_{I \in K} p(t_I) - \frac{(\overline{x}-\underline{x})^2}{8k^2} \sum_{j=0}^{n} (j-1)j|a_j|\overline{x}^{j-2} \tag{5.308}$$

$$\leqslant \min_{I \in K} p(t_I) - \frac{(\overline{x}-\underline{x})^2}{8k^2} \max_{0 \leqslant i \leqslant n-2} |b_i^{''([\underline{x},\overline{x}])}| \tag{5.309}$$

$$\leqslant \min_{x \in [\underline{x}, \overline{x}]} p(x) \tag{5.310}$$

$$\leqslant \max_{x \in [\underline{x}, \overline{x}]} p(x) \tag{5.311}$$

$$\leqslant \max_{I\in K} p(t_I) + \frac{(\overline{x}-\underline{x})^2}{8k^2} \max_{0\leqslant i\leqslant n-2} |b_i^{''([\underline{x},\overline{x}])}| \tag{5.312}$$

$$\leqslant \max_{I\in K} p(t_I) + \frac{(\overline{x}-\underline{x})^2}{8k^2} \sum_{j=0}^{n} (j-1)j|a_j|\overline{x}^{j-2} \tag{5.313}$$

$$\leqslant \max_{I\in K} p(t_I) + \frac{(\overline{x}-\underline{x})^2}{8k^2} \sum_{j=0}^{n} (j-1)j|a_j| \tag{5.314}$$

$$\leqslant \max_{I\in K} p(t_I) + \sum_{j=0}^{n} (j-1)j|a_j|. \tag{5.315}$$

So, the inequalities (5.288)-(5.289) are true; therefore, it proves the base case.

For the inductive hypothesis, in the respective hypothesis, assume that the inequalities (5.281)-(5.282) and (5.288)-(5.289) are true for all integers d such that $1 \leqslant d < n$. The goal is to prove that the thesis is true for n. Suppose $p(\boldsymbol{\xi}) = \max_{\mathbf{x}\in[\underline{\mathbf{x}},\overline{\mathbf{x}}]} p(\mathbf{x}) = p(\xi_1,\xi_2,\ldots,\xi_n)$ and, given a norm s on $\mathbb{R}^n$,

$$\|\mathbf{t}_I - \boldsymbol{\xi}\|_s = \|\mathbf{t}_{i_1,i_2,\ldots,i_n} - (\xi_1,\xi_2,\ldots,\xi_n)\|_s \leqslant \|\mathbf{t}_{j_1,j_2,\ldots,j_n} - \boldsymbol{\xi}\|_s \quad 0 \leqslant j_1, j_2, \ldots, j_n \leqslant k; \tag{5.316}$$

so that

$$\|\mathbf{t}_{i_1,i_2,\ldots,i_n} - \boldsymbol{\xi}\|_\infty \leqslant \max_{1\leqslant r\leqslant n} \frac{|\overline{x}_r - \underline{x}_r|}{2k} \tag{5.317}$$

$$= \frac{\|\overline{\mathbf{x}} - \underline{\mathbf{x}}\|_\infty}{2k}. \tag{5.318}$$

According to Taylor's formula, it is possible to obtain

$$p(\mathbf{t}_I) = p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) \tag{5.319}$$

$$= p\left((\overline{x}_1 - \underline{x}_1)\frac{i_1}{k} + \underline{x}_1, (\overline{x}_2 - \underline{x}_2)\frac{i_2}{k} + \underline{x}_2, \ldots, (\overline{x}_n - \underline{x}_n)\frac{i_n}{k} + \underline{x}_n \right) \tag{5.320}$$

$$= p(\boldsymbol{\xi}) + \sum_{j=1}^{n} \left((\overline{x}_j - \underline{x}_j)\frac{i_j}{k} + \underline{x}_j - \xi_j \right) p_{x_j}(\boldsymbol{\xi}) \tag{5.321}$$

$$+ \sum_{j=1}^{n} \left((\overline{x}_j - \underline{x}_j)\frac{i_j}{k} + \underline{x}_j - \xi_j \right)^2 \frac{p_{x_j x_j}(\boldsymbol{\eta})}{2} \tag{5.322}$$

$$+ \sum_{j,r=1;r>j}^{n} \left((\overline{x}_j - \underline{x}_j)\frac{i_j}{k} + \underline{x}_j - \xi_j \right)\left((\overline{x}_r - \underline{x}_r)\frac{i_r}{k} + \underline{x}_r - \xi_r \right) p_{x_j x_r}(\boldsymbol{\eta}), \tag{5.323}$$

where $\boldsymbol{\eta} \in [\underline{\mathbf{x}},\overline{\mathbf{x}}]$.

If the maximum $\boldsymbol{\xi}$ is such that $\boldsymbol{\xi} = \underline{\mathbf{x}}$ or $\boldsymbol{\xi} = \overline{\mathbf{x}}$ or, in general, $\boldsymbol{\xi}$ belongs to the border of the interval $[\underline{\mathbf{x}},\overline{\mathbf{x}}]$, that is, $\boldsymbol{\xi} \in \partial[\underline{\mathbf{x}},\overline{\mathbf{x}}]$, then $\xi_j = \underline{\mathbf{x}}_j$ or $\xi_j = \overline{\mathbf{x}}_j$ for some j such that $1 \leqslant j \leqslant n$. It is possible to reduce to a Taylor expansion with d variables by parameterizing the part of the border with $\boldsymbol{\xi}$, with $d < n$, making the function dependent on d parameters. So, the dimension is less than n and for the inductive hypothesis, the inequalities (5.281)-(5.282) and (5.288)-(5.289) are true.

Otherwise, $p_{x_j}(\boldsymbol{\xi})=0$ for all j such that $1\leqslant j\leqslant n$, and

$$\max_{\mathbf{x}\in[\underline{\mathbf{x}},\overline{\mathbf{x}}]} p(\mathbf{x})\leqslant \max_{(i_1,i_2,\dots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\dots,i_n})+\sum_{j=1}^{n}\left((\overline{x}_j-\underline{x}_j)\frac{i_j}{k}+\underline{x}_j-\xi_j\right)^2\frac{|p_{x_jx_j}(\boldsymbol{\eta})|}{2}+ \tag{5.324}$$

$$\sum_{j,r=1;r>j}^{n}\left|(\overline{x}_j-\underline{x}_j)\frac{i_j}{k}+\underline{x}_j-\xi_j\right|\left|(\overline{x}_r-\underline{x}_r)\frac{i_r}{k}+\underline{x}_r-\xi_r\right||p_{x_jx_r}(\boldsymbol{\eta})| \tag{5.325}$$

$$\leqslant \max_{(i_1,i_2,\dots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\dots,i_n})+\frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2}\sum_{j=1}^{n}|p_{x_jx_j}(\boldsymbol{\eta})| \tag{5.326}$$

$$+\frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{4k^2}\sum_{j,r=1;r>j}^{n}|p_{x_jx_r}(\boldsymbol{\eta})|. \tag{5.327}$$

Now, the second derivatives can be expressed in Bernstein basis. Using an appropriate change of variables, as (3.38), from $\boldsymbol{\eta}\in D=[\underline{\mathbf{x}},\overline{\mathbf{x}}]$ to $\mathbf{v}\in[0,1]^n$, with $\mathbf{v}=(v_1,v_2,\dots,v_n)$, is needed

$$\eta_i=(\overline{x}_i-\underline{x}_i)v_i+\underline{x}_i \quad 1\leqslant i\leqslant n. \tag{5.328}$$

Set s such that $1\leqslant s\leqslant n$, the pure second derivatives of p in Bernstein basis are

$$p_{x_sx_s}(\mathbf{x})=\sum_{i_1,i_2,\dots,i_n=0}^{l_1,l_2,\dots,l_n}(i_s-1)i_s a_{i_1,i_2,\dots,i_n}x_1^{i_1}x_2^{i_2}\cdots x_{s-1}^{i_{s-1}}x_{s+1}^{i_{s+1}}\cdots x_n^{i_n}x_s^{i_s-2} \tag{5.329}$$

$$=\sum_{i_1,i_2,\dots,i_{s-1},i_{s+1},\dots,i_n=0}^{l_1,l_2,\dots,l_{s-1},l_{s+1},\dots,l_n}\sum_{j_s=0}^{l_s-2}(j_s+1)(j_s+2)a_{i_1,i_2,\dots,i_{s-1},j_s+2,i_{s+1},\dots,i_n} \tag{5.330}$$

$$\cdot x_1^{i_1}x_2^{i_2}\cdots x_{s-1}^{i_{s-1}}x_s^{j_s}x_{s+1}^{i_{s+1}}\cdots x_n^{i_n}\quad \text{where } j_s=i_s-2 \tag{5.331}$$

$$=\sum_{m_1,m_2,\dots,m_{s-1},m_s,m_{s+1},\dots,m_n=0}^{l_1,l_2,\dots,l_{s-1},l_s-2,l_{s+1},\dots,l_n} b_{x_sx_sm_1,m_2,\dots,m_{s-1},m_s,m_{s+1},\dots,m_n}^{(l_1,l_2,\dots,l_{s-1},l_s-2,l_{s+1},\dots,l_n)}B_{m_1,m_2,\dots,m_{s-1},m_s,m_{s+1},\dots,m_n}^{l_1,l_2,\dots,l_{s-1},l_s-2,l_{s+1},\dots,l_n}(\mathbf{v}) \tag{5.332}$$

$$=\sum_{M\leqslant L_{ss}} b_{x_sx_sM}B_{ssM}(\mathbf{v}), \tag{5.333}$$

where the multi-indexes $M=(m_1,m_2,\dots,\dots,\dots,m_n)$ and $L_{ss}=(l_1,l_2,\dots,l_{s-1},l_s-2,l_{s+1},\dots,l_n)$. Set $\tilde{a}_{J_{ss}}=(a_{j_1},a_{j_2},\dots,a_{j_{s-1}},(j_s+1)(j_s+2)a_{j_s+2},a_{j_{s+1}},\dots,a_{j_n})$, the Bernstein basis can be denoted by B_{ssM}, instead the Bernstein coefficients, denoted by $b_{x_sx_sM}$, are

$$b_{x_sx_sM}=b_{x_sx_sm_1,m_2,\dots,m_{s-1},m_s,m_{s+1},\dots,m_n}^{(l_1,l_2,\dots,l_{s-1},l_s-2,l_{s+1},\dots,l_n)} \tag{5.334}$$

$$=\sum_{k_1,k_2,\dots,k_{s-1},k_s,k_{s+1},\dots,k_n=0}^{m_1,m_2,\dots,m_{s-1},m_s,m_{s+1},\dots,m_n}\frac{\binom{m_1}{k_1}}{\binom{l_1}{k_1}}\cdot\frac{\binom{m_2}{k_2}}{\binom{l_2}{k_2}}\cdots\frac{\binom{m_{s-1}}{k_{s-1}}}{\binom{l_{s-1}}{k_{s-1}}}\cdot\frac{\binom{m_s}{k_s}}{\binom{l_s-2}{k_s}}\cdot\frac{\binom{m_{s+1}}{k_{s+1}}}{\binom{l_{s+1}}{k_{s+1}}}\cdots\frac{\binom{m_n}{k_n}}{\binom{l_n}{k_n}} \tag{5.335}$$

$$\cdot(\overline{x}_1-\underline{x}_1)^{k_1}\cdots(\overline{x}_n-\underline{x}_n)^{k_n} \tag{5.336}$$

$$\cdot\sum_{j_1=k_1,j_2=k_2,\dots,j_{s-1}=k_{s-1},j_s=k_s,j_{s+1}=k_{s+1},\dots,j_n=k_n}^{l_1,l_2,\dots,l_{s-1},l_s-2,l_{s+1},\dots,l_n}\binom{j_1}{k_1}\cdots\binom{j_n}{k_n}\left(\prod_{p=1,p\neq s}^{n}\underline{x}^{j_p-k_p}a_{j_p}\right) \tag{5.337}$$

$$\cdot\underline{x}^{j_s-k_s}(j_s+1)(j_s+2)a_{j_s+2} \tag{5.338}$$

$$=\sum_{K\leqslant M}\frac{\binom{M}{K}}{\binom{L_{ss}}{K}}(\overline{\mathbf{x}}-\underline{\mathbf{x}})^K\sum_{J\leqslant L_{ss}}\binom{J}{K}\underline{\mathbf{x}}^{J-K}\tilde{a}_{J_{ss}} \tag{5.339}$$

with $0 \leqslant m_1 \leqslant l_1; 0 \leqslant m_2 \leqslant l_2; \ldots; 0 \leqslant m_{s-1} \leqslant l_{s-1}; 0 \leqslant m_s \leqslant l_s - 2; 0 \leqslant m_{s+1} \leqslant l_{s+1}; \ldots; 0 \leqslant m_n \leqslant l_n$. Instead, set s,u such that $1 \leqslant s,u \leqslant n$, without loss of generality, assume $u > s$, and the mixed second derivatives of p on the Bernstein basis are

$$p_{x_s x_u}(\mathbf{x}) = \sum_{i_1,i_2,\ldots,i_n=0}^{l_1,l_2,\ldots,l_n} i_s i_u a_{i_1,i_2,\ldots,i_n} x_1^{i_1} x_2^{i_2} \cdots x_{s-1}^{i_{s-1}} x_{s+1}^{i_{s+1}} \cdots x_{u-1}^{i_{u-1}} x_{u+1}^{i_{u+1}} \cdots x_n^{i_n} x_s^{i_s-1} x_u^{i_u-1} \tag{5.340}$$

$$= \sum_{i_1,i_2,\ldots,i_{s-1},i_{s+1},\ldots,i_{u-1},i_{u+1},\ldots,i_n=0}^{l_1,l_2,\ldots,l_{s-1},l_{s+1},\ldots,l_{u-1},l_{u+1},\ldots,l_n} \sum_{j_s=0}^{l_s-1} \sum_{j_u=0}^{l_u-1} (j_s+1)(j_u+1) \tag{5.341}$$

$$\cdot a_{i_1,i_2,\ldots,i_{s-1},j_s+1,i_{s+1},\ldots,i_{u-1},j_u+1,i_{u+1},\ldots,i_n} \cdot x_1^{i_1} x_2^{i_2} \cdots x_{s-1}^{i_{s-1}} x_s^{j_s} x_{s+1}^{i_{s+1}} \cdots x_{u-1}^{i_{u-1}} x_u^{j_u} x_{u+1}^{i_{u+1}} \cdots x_n^{i_n} \tag{5.342}$$

$$\text{where } j_s = i_s - 1 \text{ and } j_u = i_u - 1 \tag{5.343}$$

$$= \sum_{m_1,m_2,\ldots,m_{s-1},m_s,m_{s+1},\ldots,m_{u-1},m_u,m_{u+1},\ldots,m_n=0}^{l_1,l_2,\ldots,l_{s-1},l_s-1,l_{s+1},\ldots,l_{u-1},l_u-1,l_{u+1},\ldots,l_n} b^{(l_1,l_2,\ldots,l_{s-1},l_s-1,l_{s+1},\ldots,l_{u-1},l_u-1,l_{u+1},\ldots,l_n)}_{x_s x_u m_1,m_2,\ldots,m_{s-1},m_s,m_{s+1},\ldots,m_{u-1},m_u,m_{u+1},\ldots,m_n} \tag{5.344}$$

$$\cdot B^{l_1,l_2,\ldots,l_{s-1},l_s-1,l_{s+1},\ldots,l_{u-1},l_u-1,l_{u+1},\ldots,l_n}_{m_1,m_2,\ldots,m_{s-1},m_s,m_{s+1},\ldots,m_{u-1},m_u,m_{u+1},\ldots,m_n}(\mathbf{v}) \tag{5.345}$$

$$= \sum_{M \leqslant L_{su}} b_{x_s x_u M} B_{suM}(\mathbf{v}), \tag{5.346}$$

where the multi-indexes $M = (m_1, m_2, \ldots, \ldots, m_n)$ and $L_{su} = (l_1, l_2, \ldots, l_{s-1}, l_s - 1, l_{s+1}, \ldots, l_{u-1}, l_u - 1, l_{u+1}, \ldots, l_n)$. Set $\bar{a}_{J_{su}} = (a_{j_1}, a_{j_2}, \ldots, a_{j_{s-1}}, (j_s+1)a_{j_s+1}, a_{j_{s+1}}, \ldots, a_{j_{u-1}}, (j_u+1)a_{j_u+1}, a_{j_{u+1}}, \ldots, a_{j_n})$, the Bernstein basis can be denoted by B_{suM}, instead the Bernstein coefficients, denoted by $b_{x_s x_u M}$, are

$$b_{x_s x_u M} = b^{(l_1,l_2,\ldots,l_{s-1},l_s-1,l_{s+1},\ldots,l_{u-1},l_u-1,l_{u+1},\ldots,l_n)}_{x_s x_u m_1,m_2,\ldots,m_{s-1},m_s,m_{s+1},\ldots,m_{u-1},m_u,m_{u+1},\ldots,m_n} \tag{5.347}$$

$$= \sum_{k_1,k_2,\ldots,k_{s-1},k_s,k_{s+1},\ldots,k_{u-1},k_u,k_{u+1},\ldots,k_n=0}^{m_1,m_2,\ldots,m_{s-1},m_s,m_{s+1},\ldots,m_{u-1},m_u,m_{u+1},\ldots,m_n} \frac{\binom{m_1}{k_1}}{\binom{l_1}{k_1}} \cdot \frac{\binom{m_2}{k_2}}{\binom{l_2}{k_2}} \cdots \frac{\binom{m_{s-1}}{k_{s-1}}}{\binom{l_{s-1}}{k_{s-1}}} \cdot \frac{\binom{m_s}{k_s}}{\binom{l_s-1}{k_s}} \cdot \frac{\binom{m_{s+1}}{k_{s+1}}}{\binom{l_{s+1}}{k_{s+1}}} \tag{5.348}$$

$$\cdots \frac{\binom{m_{u-1}}{k_{u-1}}}{\binom{l_{u-1}}{k_{u-1}}} \cdot \frac{\binom{m_u}{k_u}}{\binom{l_u-1}{k_u}} \cdot \frac{\binom{m_{u+1}}{k_{u+1}}}{\binom{l_{u+1}}{k_{u+1}}} \cdots \frac{\binom{m_n}{k_n}}{\binom{l_n}{k_n}} (\overline{x}_1 - \underline{x}_1)^{k_1} \cdots (\overline{x}_n - \underline{x}_n)^{k_n} \tag{5.349}$$

$$\cdot \sum_{j_1=k_1,j_2=k_2,\ldots,j_{s-1}=k_{s-1},j_s=k_s,j_{s+1}=k_{s+1},\ldots,j_n=k_n}^{l_1,l_2,\ldots,l_{s-1},l_s-1,l_{s+1},\ldots,l_{u-1},l_u-1,l_{u+1},\ldots,l_n} \binom{j_1}{k_1} \cdots \binom{j_n}{k_n} \left(\prod_{p=1,p\neq s,p\neq u}^{n} \underline{x}^{j_p-k_p} a_{j_p} \right) \tag{5.350}$$

$$\cdot \underline{x}^{j_s-k_s}(j_s+1)a_{j_s+1} \cdot \underline{x}^{j_u-k_u}(j_u+1)a_{j_u+1} \tag{5.351}$$

$$= \sum_{K \leqslant M} \frac{\binom{M}{K}}{\binom{L_{su}}{K}} (\overline{\mathbf{x}} - \underline{\mathbf{x}})^K \sum_{J \leqslant L_{su}} \binom{J}{K} \underline{\mathbf{x}}^{J-K} \bar{a}_{Jsu} \tag{5.352}$$

with $0 \leqslant m_1 \leqslant l_1$; $0 \leqslant m_2 \leqslant l_2$; $\ldots$; $0 \leqslant m_{s-1} \leqslant l_{s-1}$; $0 \leqslant m_s \leqslant l_s - 1$; $0 \leqslant m_{s+1} \leqslant l_{s+1}$; $\ldots$; $0 \leqslant m_{u-1} \leqslant l_{u-1}$; $0 \leqslant m_u \leqslant l_u - 1$; $0 \leqslant m_{u+1} \leqslant l_{u+1}; \ldots; 0 \leqslant m_n \leqslant l_n$.

So, inequalities (5.324)-(5.327) become

$$\max_{\mathbf{x}\in[\underline{\mathbf{x}},\overline{\mathbf{x}}]} p(\mathbf{x}) \leqslant \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \sum_{j=1}^{n}\left((\overline{x}_j-\underline{x}_j)\frac{i_j}{k}+\underline{x}_j-\xi_j\right)^2\frac{|p_{x_jx_j}(\boldsymbol{\eta})|}{2} \tag{5.353}$$

$$+\sum_{j,r=1;r>j}^{n}\left|(\overline{x}_j-\underline{x}_j)\frac{i_j}{k}+\underline{x}_j-\xi_j\right|\left|(\overline{x}_r-\underline{x}_r)\frac{i_r}{k}+\underline{x}_r-\xi_r\right||p_{x_jx_r}(\boldsymbol{\eta})| \tag{5.354}$$

$$\leqslant \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2}\sum_{j=1}^{n}|p_{x_jx_j}(\boldsymbol{\eta})| \tag{5.355}$$

$$+\frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{4k^2}\sum_{j,r=1;r>j}^{n}|p_{x_jx_r}(\boldsymbol{\eta})| \tag{5.356}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2}\left(\sum_{j=1}^{n}|p_{x_jx_j}(\boldsymbol{\eta})|+2\sum_{j,r=1;r>j}^{n}|p_{x_jx_r}(\boldsymbol{\eta})|\right) \tag{5.357}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2}\sum_{j,r=1;r>j}^{n}(|p_{x_jx_j}(\boldsymbol{\eta})|+2|p_{x_jx_r}(\boldsymbol{\eta})|) \tag{5.358}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2}\sum_{j,r=1;r>j}^{n}\left(\left|\sum_{M\leqslant L_{jj}} b_{x_jx_jM}B_{jjM}(\mathbf{v})\right|\right. \tag{5.359}$$

$$\left.+2\left|\sum_{M\leqslant L_{jr}} b_{x_jx_rM}B_{jrM}(\mathbf{v})\right|\right) \tag{5.360}$$

$$\leqslant \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2}\sum_{j,r=1;r>j}^{n}\left(\sum_{M\leqslant L_{jj}}|b_{x_jx_jM}|B_{jjM}(\mathbf{v})\right. \tag{5.361}$$

$$\left.+2\sum_{M\leqslant L_{jr}}|b_{x_jx_rM}|B_{jrM}(\mathbf{v})\right) \tag{5.362}$$

$$\leqslant \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2}\sum_{j,r=1;r>j}^{n}\left(\max_{M\leqslant L_{jj}}|b_{x_jx_jM}|\sum_{M\leqslant L_{jj}}B_{jjM}(\mathbf{v})\right. \tag{5.363}$$

$$\left.+2\max_{M\leqslant L_{jr}}|b_{x_jx_rM}|\sum_{M\leqslant L_{jr}}B_{jrM}(\mathbf{v})\right) \tag{5.364}$$

$$\leqslant \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2}\sum_{j,r=1;r>j}^{n}\left(\max_{M\leqslant L_{jj}}|b_{x_jx_jM}|+2\max_{M\leqslant L_{jr}}|b_{x_jx_rM}|\right) \tag{5.365}$$

because $\sum_{M\leqslant L_{jj}} B_{jjM}(\mathbf{v}) = 1$ and $\sum_{M\leqslant L_{jr}} B_{jrM}(\mathbf{v}) = 1$. In addition,

$$\max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2}\sum_{j,r=1;r>j}^{n}\left(\max_{M\leqslant L_{jj}}|b_{x_jx_jM}|+2\max_{M\leqslant L_{jr}}|b_{x_jx_rM}|\right) \tag{5.366}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \beta_K \tag{5.367}$$

because for (5.286) $\beta_K = \dfrac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2}\sum_{i,j=1;j>i}^{n}\left(|b_{x_ix_i}|+2|b_{x_ix_j}|\right)$. Moreover,

$$\max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \beta_K \tag{5.368}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) \tag{5.369}$$

$$+\frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2}\sum_{j,r=1;r>j}^{n}\left(\max_{M\leqslant L_{jj}}\left|\sum_{N\leqslant M}\frac{\binom{M}{N}}{\binom{L_{jj}}{N}}(\overline{\mathbf{x}}-\underline{\mathbf{x}})^N\sum_{K\leqslant L_{jj}}\binom{K}{N}\underline{\mathbf{x}}^{K-N}\tilde{a}_{Kjj}\right|\right. \tag{5.370}$$

$$\left.+2\max_{M\leqslant L_{jr}}\left|\sum_{N\leqslant M}\frac{\binom{M}{N}}{\binom{L_{jr}}{N}}(\overline{\mathbf{x}}-\underline{\mathbf{x}})^N\sum_{K\leqslant L_{jr}}\binom{K}{N}\underline{\mathbf{x}}^{K-N}\tilde{a}_{Kjr}\right|\right) \tag{5.371}$$

Starting from the idea of the proof of Proposition 3.9, it is possible to bound Bernstein coefficients for the second derivative of the polynomial p.

$$|b_{x_jx_jM}| = \left|\sum_{N\leqslant M}\frac{\binom{M}{N}}{\binom{L_{jj}}{N}}(\overline{\mathbf{x}}-\underline{\mathbf{x}})^N\sum_{K\leqslant L_{jj}}\binom{K}{N}\underline{\mathbf{x}}^{K-N}\tilde{a}_{Kjj}\right| \tag{5.372}$$

$$= \left|\sum_{N\leqslant 0}\frac{\binom{M}{N}}{\binom{L_{jj}}{N}}(\overline{\mathbf{x}}-\underline{\mathbf{x}})^N\sum_{K\leqslant L_{jj}}\binom{K}{N}\underline{\mathbf{x}}^{K-N}\tilde{a}_{Kjj}\right. \tag{5.373}$$

$$\left.+\sum_{N\leqslant M;N\neq 0}\frac{\binom{M}{N}}{\binom{L_{jj}}{N}}(\overline{\mathbf{x}}-\underline{\mathbf{x}})^N\sum_{K\leqslant L_{jj}}\binom{K}{N}\underline{\mathbf{x}}^{K-N}\tilde{a}_{Kjj}\right| \tag{5.374}$$

$$= \left|\sum_{K\leqslant L_{jj}}\underline{\mathbf{x}}^K\tilde{a}_{Kjj}+\sum_{K\leqslant L_{jj};K\neq 0}\tilde{a}_{Kjj}\sum_{N\leqslant M;N\leqslant K;N\neq 0}\frac{\binom{M}{N}}{\binom{L_{jj}}{N}}(\overline{\mathbf{x}}-\underline{\mathbf{x}})^N\binom{K}{N}\underline{\mathbf{x}}^{K-N}\right| \tag{5.375}$$

$$= \left|\sum_{K\leqslant L_{jj};\tilde{a}_{Kjj}<0}\underline{\mathbf{x}}^K\tilde{a}_{Kjj}+\sum_{K\leqslant L_{jj};\tilde{a}_{Kjj}>0}\underline{\mathbf{x}}^K\tilde{a}_{Kjj}\right. \tag{5.376}$$

$$+\sum_{K\leqslant L_{jj};K\neq 0;\tilde{a}_{Kjj}<0}\tilde{a}_{Kjj}\sum_{N\leqslant M;N\leqslant K;N\neq 0}\frac{\binom{M}{N}}{\binom{L_{jj}}{N}}(\overline{\mathbf{x}}-\underline{\mathbf{x}})^N\binom{K}{N}\underline{\mathbf{x}}^{K-N} \tag{5.377}$$

$$\left.+\sum_{K\leqslant L_{jj};K\neq 0;\tilde{a}_{Kjj}>0}\tilde{a}_{Kjj}\sum_{N\leqslant M;N\leqslant K;N\neq 0}\frac{\binom{M}{N}}{\binom{L_{jj}}{N}}(\overline{\mathbf{x}}-\underline{\mathbf{x}})^N\binom{K}{N}\underline{\mathbf{x}}^{K-N}\right| \tag{5.378}$$

$$= \left|\sum_{K\leqslant L_{jj};\tilde{a}_{Kjj}<0}\tilde{a}_{Kjj}\sum_{N\leqslant M;N\leqslant K}\frac{\binom{M}{N}}{\binom{L_{jj}}{N}}(\overline{\mathbf{x}}-\underline{\mathbf{x}})^N\binom{K}{N}\underline{\mathbf{x}}^{K-N}\right. \tag{5.379}$$

$$\left.+\sum_{K\leqslant L_{jj};\tilde{a}_{Kjj}>0}\tilde{a}_{Kjj}\sum_{N\leqslant M;N\leqslant K}\frac{\binom{M}{N}}{\binom{L_{jj}}{N}}(\overline{\mathbf{x}}-\underline{\mathbf{x}})^N\binom{K}{N}\underline{\mathbf{x}}^{K-N}\right| \tag{5.380}$$

$$\leqslant \left|\sum_{K\leqslant L_{jj};\tilde{a}_{Kjj}<0}\tilde{a}_{Kjj}\sum_{N\leqslant M;N\leqslant K}\frac{\binom{M}{N}}{\binom{L_{jj}}{N}}(\overline{\mathbf{x}}-\underline{\mathbf{x}})^N\binom{K}{N}\underline{\mathbf{x}}^{K-N}\right| \tag{5.381}$$

$$+\left|\sum_{K\leqslant L_{jj};\tilde{a}_{Kjj}>0}\tilde{a}_{Kjj}\sum_{N\leqslant M;N\leqslant K}\frac{\binom{M}{N}}{\binom{L_{jj}}{N}}(\overline{\mathbf{x}}-\underline{\mathbf{x}})^N\binom{K}{N}\underline{\mathbf{x}}^{K-N}\right| \tag{5.382}$$

$$\leqslant \left|\sum_{K\leqslant L_{jj};\tilde{a}_{Kjj}<0}\tilde{a}_{Kjj}\sum_{N\leqslant M;N\leqslant K}(\overline{\mathbf{x}}-\underline{\mathbf{x}})^N\binom{K}{N}\underline{\mathbf{x}}^{K-N}\right| \tag{5.383}$$

$$+\left|\sum_{K\leqslant L_{jj};\tilde{a}_{Kjj}>0}\tilde{a}_{Kjj}\sum_{N\leqslant M;N\leqslant K}(\overline{\mathbf{x}}-\underline{\mathbf{x}})^N\binom{K}{N}\underline{\mathbf{x}}^{K-N}\right| \tag{5.384}$$

because the ratio between two binomial coefficients is less than or equal to 1. Additionally,

$$\left| \sum_{K \leqslant L_{jj}; \tilde{a}_{Kjj} < 0} \tilde{a}_{Kjj} \sum_{N \leqslant M; N \leqslant K} (\overline{\mathbf{x}} - \underline{\mathbf{x}})^N \binom{K}{N} \underline{\mathbf{x}}^{K-N} \right| \tag{5.385}$$

$$+ \left| \sum_{K \leqslant L_{jj}; \tilde{a}_{Kjj} > 0} \tilde{a}_{Kjj} \sum_{N \leqslant M; N \leqslant K} (\overline{\mathbf{x}} - \underline{\mathbf{x}})^N \binom{K}{N} \underline{\mathbf{x}}^{K-N} \right| \tag{5.386}$$

$$\leqslant \left| \sum_{K \leqslant L_{jj}; \tilde{a}_{Kjj} < 0} \tilde{a}_{Kjj} \sum_{N \leqslant K} (\overline{\mathbf{x}} - \underline{\mathbf{x}})^N \binom{K}{N} \underline{\mathbf{x}}^{K-N} \right| \tag{5.387}$$

$$+ \left| \sum_{K \leqslant L_{jj}; \tilde{a}_{Kjj} > 0} \tilde{a}_{Kjj} \sum_{N \leqslant K} (\overline{\mathbf{x}} - \underline{\mathbf{x}})^N \binom{K}{N} \underline{\mathbf{x}}^{K-N} \right| \tag{5.388}$$

because in the sums with $N \leqslant M$ and $N \leqslant K$ the number of addends is less than or equal to the number of addends in the sum with $N \leqslant K$. Moreover,

$$\left| \sum_{K \leqslant L_{jj}; \tilde{a}_{Kjj} < 0} \tilde{a}_{Kjj} \sum_{N \leqslant K} (\overline{\mathbf{x}} - \underline{\mathbf{x}})^N \binom{K}{N} \underline{\mathbf{x}}^{K-N} \right| \tag{5.389}$$

$$+ \left| \sum_{K \leqslant L_{jj}; \tilde{a}_{Kjj} > 0} \tilde{a}_{Kjj} \sum_{N \leqslant K} (\overline{\mathbf{x}} - \underline{\mathbf{x}})^N \binom{K}{N} \underline{\mathbf{x}}^{K-N} \right| \tag{5.390}$$

$$= \left| \sum_{K \leqslant L_{jj}; \tilde{a}_{Kjj} < 0} \tilde{a}_{Kjj} \overline{\mathbf{x}}^K \right| + \left| \sum_{K \leqslant L_{jj}; \tilde{a}_{Kjj} > 0} \tilde{a}_{Kjj} \overline{\mathbf{x}}^K \right| \tag{5.391}$$

for the binomial theorem. Furthermore,

$$\left| \sum_{K \leqslant L_{jj}; \tilde{a}_{Kjj} < 0} \tilde{a}_{Kjj} \overline{\mathbf{x}}^K \right| + \left| \sum_{K \leqslant L_{jj}; \tilde{a}_{Kjj} > 0} \tilde{a}_{Kjj} \overline{\mathbf{x}}^K \right| = \sum_{K \leqslant L_{jj}} |\tilde{a}_{Kjj}| \overline{\mathbf{x}}^K. \tag{5.392}$$

So, using similar considerations about $b_{x_j x_r M}$, it is possible to bound these coefficients.

$$|b_{x_j x_r M}| \leqslant \sum_{K \leqslant L_{jr}} |\bar{a}_{Kjr}| \overline{\mathbf{x}}^K. \tag{5.393}$$

So, the inequalities (5.353)-(5.371) become

$$\max_{\mathbf{x} \in [\underline{\mathbf{x}}, \overline{\mathbf{x}}]} p(\mathbf{x}) \leqslant \max_{(i_1, i_2, \ldots, i_n) \in K} p(\mathbf{t}_{i_1, i_2, \ldots, i_n}) + \beta_K \tag{5.394}$$

$$\leqslant \max_{(i_1, i_2, \ldots, i_n) \in K} p(\mathbf{t}_{i_1, i_2, \ldots, i_n}) \tag{5.395}$$

$$+ \frac{\|\overline{\mathbf{x}} - \underline{\mathbf{x}}\|_\infty^2}{8k^2} \sum_{j,r=1; r>j}^{n} \left(\sum_{K \leqslant L_{jj}} |\bar{a}_{Kjj}| \overline{\mathbf{x}}^K + 2 \sum_{K \leqslant L_{jr}} |\bar{a}_{Kjr}| \overline{\mathbf{x}}^K \right) \tag{5.396}$$

$$= \max_{(i_1, i_2, \ldots, i_n) \in K} p(\mathbf{t}_{i_1, i_2, \ldots, i_n}) \tag{5.397}$$

$$+ \frac{\|\overline{\mathbf{x}} - \underline{\mathbf{x}}\|_\infty^2}{8k^2} \sum_{j,r=1; r>j}^{n} \left(\sum_{k_1, k_2, \ldots, k_{j-1}, k_j, k_{j+1}, \ldots, k_n = 0}^{l_1, l_2, \ldots, l_{j-1}, l_j - 2, l_{j+1}, \ldots, l_n} |(k_j + 1)(k_j + 2) \right. \tag{5.398}$$

$$\cdot a_{k_1, k_2, \ldots, k_{j-1}, k_j + 2, k_{j+1}, \ldots, k_n} | \overline{x}_1^{k_1} \cdots \overline{x}_{j-1}^{k_{j-1}} \overline{x}_j^{k_j} \overline{x}_{j+1}^{k_{j+1}} \cdots \overline{x}_n^{k_n} \tag{5.399}$$

$$+ 2 \sum_{k_1, k_2, \ldots, k_{j-1}, k_j, k_{j+1}, \ldots, k_{r-1}, k_r, k_{r+1}, \ldots, k_n = 0}^{l_1, l_2, \ldots, l_{j-1}, l_j - 1, l_{j+1}, \ldots, l_{r-1}, l_r - 1, l_{r+1}, \ldots, l_n} |(k_j + 1)(k_r + 1) \tag{5.400}$$

$$\left. \cdot a_{k_1, k_2, \ldots, k_{j-1}, k_j + 1, k_{j+1}, \ldots, k_{r-1}, k_r + 1, k_{r+1}, \ldots, k_n} | \overline{x}_1^{k_1} \cdots \overline{x}_{j-1}^{k_{j-1}} \overline{x}_j^{k_j} \overline{x}_{j+1}^{k_{j+1}} \cdots \overline{x}_{r-1}^{k_{r-1}} \overline{x}_r^{k_r} \overline{x}_{r+1}^{k_{r+1}} \cdots \overline{x}_n^{k_n} \right) \tag{5.401}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) \tag{5.402}$$

$$+ \frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2} \sum_{j,r=1;r>j}^{n} \left(\sum_{k_1,k_2,\ldots,k_{j-1},k_j,k_{j+1},\ldots,k_n=0}^{l_1,l_2,\ldots,l_{j-1},l_j-2,l_{j+1},\ldots,l_n} (k_j+1)(k_j+2) \right. \tag{5.403}$$

$$\cdot |a_{k_1,k_2,\ldots,k_{j-1},k_j+2,k_{j+1},\ldots,k_n}| \overline{x}_1^{k_1} \cdots \overline{x}_{j-1}^{k_{j-1}} \overline{x}_j^{k_j} \overline{x}_{j+1}^{k_{j+1}} \cdots \overline{x}_n^{k_n} \tag{5.404}$$

$$+2 \sum_{k_1,k_2,\ldots,k_{j-1},k_j,k_{j+1},\ldots,k_{r-1},k_r,k_{r+1},\ldots,k_n=0}^{l_1,l_2,\ldots,l_{j-1},l_j-1,l_{j+1},\ldots,l_{r-1},l_r-1,l_{r+1},\ldots,l_n} (k_j+1)(k_r+1) \tag{5.405}$$

$$\left. \cdot |a_{k_1,k_2,\ldots,k_{j-1},k_j+1,k_{j+1},\ldots,k_{r-1},k_r+1,k_{r+1},\ldots,k_n}| \overline{x}_1^{k_1} \cdots \overline{x}_{j-1}^{k_{j-1}} \overline{x}_j^{k_j} \overline{x}_{j+1}^{k_{j+1}} \cdots \overline{x}_{r-1}^{k_{r-1}} \overline{x}_r^{k_r} \overline{x}_{r+1}^{k_{r+1}} \cdots \overline{x}_n^{k_n} \right) \tag{5.406}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) \tag{5.407}$$

$$+ \frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2} \sum_{j,r=1;r>j}^{n} \left(\sum_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_n=0}^{l_1,l_2,\ldots,l_{j-1},l_j,l_{j+1},\ldots,l_n} i_j(i_j-1)|a_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_n}| \right. \tag{5.408}$$

$$\cdot \overline{x}_1^{i_1} \cdots \overline{x}_{j-1}^{i_{j-1}} \overline{x}_j^{i_j-2} \overline{x}_{j+1}^{i_{j+1}} \cdots \overline{x}_n^{i_n} \tag{5.409}$$

$$+2 \sum_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_{r-1},i_r,i_{r+1},\ldots,i_n=0}^{l_1,l_2,\ldots,l_{j-1},l_j,l_{j+1},\ldots,l_{r-1},l_r,l_{r+1},\ldots,l_n} i_j i_r |a_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_{r-1},i_r,i_{r+1},\ldots,i_n}| \tag{5.410}$$

$$\left. \cdot \overline{x}_1^{i_1} \cdots \overline{x}_{j-1}^{i_{j-1}} \overline{x}_j^{i_j-1} \overline{x}_{j+1}^{i_{j+1}} \cdots \overline{x}_{r-1}^{i_{r-1}} \overline{x}_r^{i_r-1} \overline{x}_{r+1}^{i_{r+1}} \cdots \overline{x}_n^{i_n} \right) \tag{5.411}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) \tag{5.412}$$

$$+ \frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2} \sum_{j,r=1;r>j}^{n} \left(\sum_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_{r-1},i_r,i_{r+1},\ldots,i_n=0}^{l_1,l_2,\ldots,l_{j-1},l_j,l_{j+1},\ldots,l_{r-1},l_r,l_{r+1},\ldots,l_n} i_j(i_j-1)|a_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_n}| \right. \tag{5.413}$$

$$\cdot + \overline{x}_1^{i_1} \cdots \overline{x}_{j-1}^{i_{j-1}} \overline{x}_j^{i_j-2} \overline{x}_{j+1}^{i_{j+1}} \cdots \overline{x}_n^{i_n} \tag{5.414}$$

$$\left. +2 i_j i_r |a_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_{r-1},i_r,i_{r+1},\ldots,i_n}| \cdot \overline{x}_1^{i_1} \cdots \overline{x}_{j-1}^{i_{j-1}} \overline{x}_j^{i_j-1} \overline{x}_{j+1}^{i_{j+1}} \cdots \overline{x}_{r-1}^{i_{r-1}} \overline{x}_r^{i_r-1} \overline{x}_{r+1}^{i_{r+1}} \cdots \overline{x}_n^{i_n} \right) \tag{5.415}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) \tag{5.416}$$

$$+ \frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2} \sum_{j,r=1;r>j}^{n} \left(\sum_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_{r-1},i_r,i_{r+1},\ldots,i_n=0}^{l_1,l_2,\ldots,l_{j-1},l_j,l_{j+1},\ldots,l_{r-1},l_r,l_{r+1},\ldots,l_n} [(i_j-1)\overline{x}_r \right. \tag{5.417}$$

$$\left. +2 i_r \overline{x}_j] i_j |a_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_n}| \cdot \overline{x}_1^{i_1} \cdots \overline{x}_{j-1}^{i_{j-1}} \overline{x}_j^{i_j-2} \overline{x}_{j+1}^{i_{j+1}} \cdots \overline{x}_{r-1}^{i_{r-1}} \overline{x}_r^{i_r-1} \overline{x}_{r+1}^{i_{r+1}} \cdots \overline{x}_n^{i_n} \right) \tag{5.418}$$

Under the assumption $\overline{x}_j \geqslant 1$ for all j such that $1 \leqslant s \leqslant n$: set j, r such that $1 \leqslant j, r \leqslant n$,

$$(i_j-1)\overline{x}_r + 2i_r\overline{x}_j \leqslant (i_j-1)\overline{x}_r + 2i_r\overline{x}_r\overline{x}_j \tag{5.419}$$

$$\leqslant (i_j-1)\overline{x}_r\overline{x}_j + 2i_r\overline{x}_r\overline{x}_j \tag{5.420}$$

$$= [(i_j-1)+2i_r]\overline{x}_r\overline{x}_j \tag{5.421}$$

$$\leqslant [(i_j-1)+2i_r]\overline{x}_r\overline{x}_j^2 \tag{5.422}$$

The inequalities above (5.353)-(5.371) become

$$\max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) \tag{5.423}$$

$$+\frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2}\sum_{j,r=1;r>j}^{n}\left(\sum_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_{r-1},i_r,i_{r+1},\ldots,i_n=0}^{l_1,l_2,\ldots,l_{j-1},l_j,l_{j+1},\ldots,l_{r-1},l_r,l_{r+1},\ldots,l_n}[(i_j-1)\overline{x}_r\right. \tag{5.424}$$

$$\left.+2i_r\overline{x}_j]i_j|a_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_n}|\cdot\overline{x}_1^{i_1}\cdots\overline{x}_{j-1}^{i_{j-1}}\overline{x}_j^{i_j-2}\overline{x}_{j+1}^{i_{j+1}}\cdots\overline{x}_{r-1}^{i_{r-1}}\overline{x}_r^{i_r-1}\overline{x}_{r+1}^{i_{r+1}}\cdots\overline{x}_n^{i_n}\right) \tag{5.425}$$

$$\leqslant \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) \tag{5.426}$$

$$+\frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2}\sum_{j,r=1;r>j}^{n}\left(\sum_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_{r-1},i_r,i_{r+1},\ldots,i_n=0}^{l_1,l_2,\ldots,l_{j-1},l_j,l_{j+1},\ldots,l_{r-1},l_r,l_{r+1},\ldots,l_n}[(i_j-1)+2i_r]\overline{x}_r\overline{x}_j^2 i_j\right. \tag{5.427}$$

$$\left.\cdot|a_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_n}|\overline{x}_1^{i_1}\cdots\overline{x}_{j-1}^{i_{j-1}}\overline{x}_j^{i_j-2}\overline{x}_{j+1}^{i_{j+1}}\cdots\overline{x}_{r-1}^{i_{r-1}}\overline{x}_r^{i_r-1}\overline{x}_{r+1}^{i_{r+1}}\cdots\overline{x}_n^{i_n}\right) \tag{5.428}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) \tag{5.429}$$

$$+\frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2}\sum_{j,r=1;r>j}^{n}\left(\sum_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_{r-1},i_r,i_{r+1},\ldots,i_n=0}^{l_1,l_2,\ldots,l_{j-1},l_j,l_{j+1},\ldots,l_{r-1},l_r,l_{r+1},\ldots,l_n}[i_j(i_j-1)+2i_ji_r]|a_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_n}|\right. \tag{5.430}$$

$$\left.\cdot\overline{x}_1^{i_1}\cdots\overline{x}_{j-1}^{i_{j-1}}\overline{x}_j^{i_j}\overline{x}_{j+1}^{i_{j+1}}\cdots\overline{x}_{r-1}^{i_{r-1}}\overline{x}_r^{i_r}\overline{x}_{r+1}^{i_{r+1}}\cdots\overline{x}_n^{i_n}\right) \tag{5.431}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n})+\frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2}\sum_{j,r=1;r>j}^{n}\left(\sum_{I\leqslant L}[i_j(i_j-1)+2i_ji_r]|a_I|\overline{\mathbf{x}}^I\right) \tag{5.432}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n})+\frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2}\sum_{I\leqslant L}\left(\sum_{j,r=1;r>j}^{n}[i_j(i_j-1)+2i_ji_r]\right)|a_I|\overline{\mathbf{x}}^I \tag{5.433}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n})+\frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2}\sum_{I=(i_1,i_2,\ldots,i_n)\leqslant L}(i_1+\cdots+i_n)(i_1+\cdots+i_n-1)|a_I|\overline{\mathbf{x}}^I \tag{5.434}$$

for Lemma 5.1. Moreover,

$$\max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n})+\frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2}\sum_{I=(i_1,i_2,\ldots,i_n)\leqslant L}(i_1+\cdots+i_n)(i_1+\cdots+i_n-1)|a_I|\overline{\mathbf{x}}^I \tag{5.435}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n})+\alpha_K, \tag{5.436}$$

where α_K is defined in (5.283).

So, under the assumption $\overline{x}_j \geqslant 1$ for all j such that $1 \leqslant j \leqslant n$,

$$\max_{\mathbf{x}\in[\underline{\mathbf{x}},\overline{\mathbf{x}}]} p(\mathbf{x}) \leqslant \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n})+\beta_K \tag{5.437}$$

$$\leqslant \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n})+\alpha_K. \tag{5.438}$$

This proves the first part of the theorem. An entirely analogous argument establishes the lower bound on $\min_{\mathbf{x}\in[\underline{\mathbf{x}},\overline{\mathbf{x}}]} p(\mathbf{x})$.

Instead under the assumption $\overline{x}_j \leqslant 1$ for all j such that $1 \leqslant j \leqslant n$: set j such that $1 \leqslant j \leqslant n$, if $i_j = 0$ then $[(i_j-1)\overline{x}_r + 2i_r\overline{x}_j]i_j = 0$, if $i_j = 1$ then $i_j - 1 = 0$. Set r such that $1 \leqslant r \leqslant n$, if $i_r = 0$ then $2i_r = 0$. So, if $i_j \geqslant 2$ or if $i_r \geqslant 1$ then

$$[(i_j-1)\overline{x}_r + 2i_r\overline{x}_j]i_j \cdot \overline{x}_1^{i_1} \cdots \overline{x}_{j-1}^{i_{j-1}} \overline{x}_j^{i_j-2} \overline{x}_{j+1}^{i_{j+1}} \cdots \overline{x}_{r-1}^{i_{r-1}} \overline{x}_r^{i_r-1} \overline{x}_{r+1}^{i_{r+1}} \cdots \overline{x}_n^{i_n} \leqslant [(i_j-1)\overline{x}_r + 2i_r\overline{x}_j]i_j \tag{5.439}$$

$$\leqslant [(i_j-1) + 2i_r]i_j \tag{5.440}$$

The inequalities above (5.353)-(5.371) become

$$\max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) \tag{5.441}$$

$$+ \frac{\|\overline{\mathbf{x}} - \underline{\mathbf{x}}\|_\infty^2}{8k^2} \sum_{j,r=1;r>j}^{n} \Bigg(\sum_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_{r-1},i_r,i_{r+1},\ldots,i_n=0}^{l_1,l_2,\ldots,l_{j-1},l_j,l_{j+1},\ldots,l_{r-1},l_r,l_{r+1},\ldots,l_n} [(i_j-1)\overline{x}_r + 2i_r\overline{x}_j]i_j \tag{5.442}$$

$$\cdot |a_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_n}| \overline{x}_1^{i_1} \cdots \overline{x}_{j-1}^{i_{j-1}} \overline{x}_j^{i_j-2} \overline{x}_{j+1}^{i_{j+1}} \cdots \overline{x}_{r-1}^{i_{r-1}} \overline{x}_r^{i_r-1} \overline{x}_{r+1}^{i_{r+1}} \cdots \overline{x}_n^{i_n} \Bigg) \tag{5.443}$$

$$\leqslant \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \frac{\|\overline{\mathbf{x}} - \underline{\mathbf{x}}\|_\infty^2}{8k^2} \tag{5.444}$$

$$\cdot \sum_{j,r=1;r>j}^{n} \Bigg(\sum_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_{r-1},i_r,i_{r+1},\ldots,i_n=0}^{l_1,l_2,\ldots,l_{j-1},l_j,l_{j+1},\ldots,l_{r-1},l_r,l_{r+1},\ldots,l_n} [(i_j-1) + 2i_r]i_j |a_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_n}| \Bigg) \tag{5.445}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \frac{\|\overline{\mathbf{x}} - \underline{\mathbf{x}}\|_\infty^2}{8k^2} \tag{5.446}$$

$$\cdot \sum_{j,r=1;r>j}^{n} \Bigg(\sum_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_{r-1},i_r,i_{r+1},\ldots,i_n=0}^{l_1,l_2,\ldots,l_{j-1},l_j,l_{j+1},\ldots,l_{r-1},l_r,l_{r+1},\ldots,l_n} [i_j(i_j-1) + 2i_ji_r] |a_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_n}| \Bigg) \tag{5.447}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \frac{\|\overline{\mathbf{x}} - \underline{\mathbf{x}}\|_\infty^2}{8k^2} \sum_{j,r=1;r>j}^{n} \Bigg(\sum_{I\leqslant L} [i_j(i_j-1) + 2i_ji_r] |a_I| \Bigg) \tag{5.448}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \frac{\|\overline{\mathbf{x}} - \underline{\mathbf{x}}\|_\infty^2}{8k^2} \sum_{I\leqslant L} \Bigg(\sum_{j,r=1;r>j}^{n} [i_j(i_j-1) + 2i_ji_r] \Bigg) |a_I| \tag{5.449}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \frac{\|\overline{\mathbf{x}} - \underline{\mathbf{x}}\|_\infty^2}{8k^2} \sum_{I=(i_1,i_2,\ldots,i_n)\leqslant L} (i_1 + \cdots + i_n)(i_1 + \cdots + i_n - 1)|a_I| \tag{5.450}$$

for Lemma 5.1. Furthermore,

$$\max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \frac{\|\overline{\mathbf{x}} - \underline{\mathbf{x}}\|_\infty^2}{8k^2} \sum_{I=(i_1,i_2,\ldots,i_n)\leqslant L} (i_1 + \cdots + i_n)(i_1 + \cdots + i_n - 1)|a_I| \tag{5.451}$$

$$\leqslant \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \frac{1}{8k^2} \sum_{i_1,i_2,\ldots,i_n=0}^{l_1,l_2,\ldots,l_n} (i_1 + i_2 + \cdots + i_n)(i_1 + i_2 + \cdots + i_n - 1)|a_{i_1,i_2,\ldots,i_n}| \tag{5.452}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \alpha_K^{[0,1]^n}, \tag{5.453}$$

which is the same α_K in $[0,1]^n$ defined in (5.119).

So, under the assumption $\overline{x}_j \leqslant 1$ for all j such that $1 \leqslant j \leqslant n$,

$$\max_{\mathbf{x}\in[\underline{\mathbf{x}},\overline{\mathbf{x}}]} p(\mathbf{x}) \leqslant \max_{(i_1,i_2,\dots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\dots,i_n}) + \beta_K \tag{5.454}$$

$$\leqslant \max_{(i_1,i_2,\dots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\dots,i_n}) + \|\overline{\mathbf{x}} - \underline{\mathbf{x}}\|_\infty^2 \cdot \alpha_K^{[0,1]^n} \tag{5.455}$$

$$\leqslant \max_{(i_1,i_2,\dots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\dots,i_n}) + \alpha_K^{[0,1]^n}. \tag{5.456}$$

This proves the second part of the theorem. An entirely analogous argument establishes the lower bound on $\min_{\mathbf{x}\in[\underline{\mathbf{x}},\overline{\mathbf{x}}]} p(\mathbf{x})$. □

In Proposition 5.3, the quantity $\alpha_K^{[0,1]^n}$ is identical to the one obtained in the particular case where the domain is $[0,1]^n$. That is, neither its definition nor its numerical value is affected by the considerations introduced in the proposition. On the other hand, the constant $\beta_K^{[0,1]^n}$ behaves differently. While it is defined by the same analytical expression as in the case of the domain $[0,1]^n$, its actual value is not the same. This discrepancy stems from the fact that $\beta_K^{[0,1]^n}$ depends explicitly on the Bernstein coefficients associated with the polynomial under consideration. These coefficients, in turn, are computed using formulas that involve the parameters $\underline{\mathbf{x}}$ and $\overline{\mathbf{x}}$. As a result, changes in these parameters modify the Bernstein coefficients and consequently affect the value of $\beta_K^{[0,1]^n}$, even though its formal expression remains unchanged.

Proposition 5.4. *Let p be given by*

$$p(\mathbf{x}) = \sum_{I\leqslant L} a_I \mathbf{x}^I. \tag{5.457}$$

Then

$$\max_{\mathbf{x}\in[\underline{\mathbf{x}},\overline{\mathbf{x}}]} p(\mathbf{x}) \leqslant \max_{I\in K} p(\mathbf{t}_I) + \tilde{\beta}_K \leqslant \max_{I\in K} p(\mathbf{t}_I) + \tilde{\alpha}_K \tag{5.458}$$

$$\min_{\mathbf{x}\in[\underline{\mathbf{x}},\overline{\mathbf{x}}]} p(\mathbf{x}) \geqslant \min_{I\in K} p(\mathbf{t}_I) - \tilde{\beta}_K \geqslant \min_{I\in K} p(\mathbf{t}_I) - \tilde{\alpha}_K, \tag{5.459}$$

where $\tilde{\alpha}_K$ and $\tilde{\beta}_K$ are defined as

$$\tilde{\alpha}_K = \frac{1}{8k^2} \sum_{i_1,i_2,\dots,i_n=0}^{l_1,l_2,\dots,l_n} (i_1+i_2+\dots+i_n)(i_1+i_2+\dots+i_n-1)|a_{i_1,i_2,\dots,i_n}|\overline{\mathbf{x}}_1^{i_1}\overline{\mathbf{x}}_2^{i_2}\cdots\overline{\mathbf{x}}_n^{i_n} \tag{5.460}$$

$$= \frac{1}{8k^2} \sum_{I=(i_1,i_2,\dots,i_n)\leqslant L} (i_1+i_2+\dots+i_n)(i_1+i_2+\dots+i_n-1)|a_I|\overline{\mathbf{x}}^I \tag{5.461}$$

and

$$\tilde{\beta}_K = \frac{1}{8k^2} \sum_{i,j=1;j>i}^{n} \Big(|b_{x_ix_i}|(\overline{x}_i - \underline{x}_i)^2 + 2|b_{x_ix_j}|(\overline{x}_i - \underline{x}_i)(\overline{x}_j - \underline{x}_j) \Big), \tag{5.462}$$

where $|b_{x_ix_j}|$ is the maximum of the absolute values of the Bernstein coefficients for the partial derivatives $p_{\mathbf{x}_i\mathbf{x}_j}$ of p.

Proof. The proof is by induction on the dimension n. The base case is $n = 1$. In this case, from the inequalities (4.285)-(4.300)

$$\min_{I\in K} p(t_I) - \frac{1}{8k^2}\sum_{j=0}^{n}(j-1)j|a_j|\overline{x}^j \tag{5.463}$$

$$\leqslant \min_{I\in K} p(t_I) - \frac{(\overline{x}-\underline{x})^2}{8k^2}\max_{0\leqslant i\leqslant n-2}|b_i^{''([\underline{x},\overline{x}])}| \tag{5.464}$$

$$\leqslant \min_{x\in[\underline{x},\overline{x}]} p(x) \tag{5.465}$$

$$\leqslant \max_{x\in[\underline{x},\overline{x}]} p(x) \tag{5.466}$$

$$\leqslant \max_{I\in K} p(t_I) + \frac{(\overline{x}-\underline{x})^2}{8k^2}\max_{0\leqslant i\leqslant n-2}|b_i^{''([\underline{x},\overline{x}])}| \tag{5.467}$$

$$\leqslant \max_{I\in K} p(t_I) + \frac{1}{8k^2}\sum_{j=0}^{n}(j-1)j|a_j|\overline{x}^j. \tag{5.468}$$

So, the inequalities (5.458)-(5.459) are true; therefore, it proves the base case.

For the inductive hypothesis, assume that the inequalities (5.458)-(5.459) are true for all integers d such that $1 \leqslant d < n$. The goal is to prove that the thesis is true for n.

The idea is similar to the proof of Proposition 5.3, but instead of using the norm infinity in (5.317), it is possible to think in one single dimension. Let k a natural number, j such that $1 \leqslant j \leqslant n$, i_j such that $0 \leqslant i_j \leqslant k$. If $i_j = 0$, then

$$\left|(\overline{x}_j-\underline{x}_j)\frac{i_j}{k}+\underline{x}_j-\xi_j\right| \leqslant \frac{1}{2}\left|(\overline{x}_j-\underline{x}_j)\frac{i_j}{k}+\underline{x}_j-\left[(\overline{x}_j-\underline{x}_j)\frac{i_j+1}{k}+\underline{x}_j\right]\right| \tag{5.469}$$

$$= \frac{\overline{x}_j-\underline{x}_j}{2k} \tag{5.470}$$

else, if i_j is such that $1 \leqslant i_j \leqslant k$

$$\left|(\overline{x}_j-\underline{x}_j)\frac{i_j}{k}+\underline{x}_j-\xi_j\right| \leqslant \frac{1}{2}\left|(\overline{x}_j-\underline{x}_j)\frac{i_j}{k}+\underline{x}_j-\left[(\overline{x}_j-\underline{x}_j)\frac{i_j-1}{k}+\underline{x}_j\right]\right| \tag{5.471}$$

$$= \frac{\overline{x}_j-\underline{x}_j}{2k}. \tag{5.472}$$

According to Taylor's formula, it is possible to obtain

$$p(\mathbf{t}_I) = p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) \tag{5.473}$$

$$= p\left((\overline{x}_1-\underline{x}_1)\frac{i_1}{k}+\underline{x}_1,(\overline{x}_2-\underline{x}_2)\frac{i_2}{k}+\underline{x}_2,\ldots,(\overline{x}_n-\underline{x}_n)\frac{i_n}{k}+\underline{x}_n\right) \tag{5.474}$$

$$= p(\boldsymbol{\xi}) + \sum_{j=1}^{n}\left((\overline{x}_j-\underline{x}_j)\frac{i_j}{k}+\underline{x}_j-\xi_j\right)p_{x_j}(\boldsymbol{\xi}) + \sum_{j=1}^{n}\left((\overline{x}_j-\underline{x}_j)\frac{i_j}{k}+\underline{x}_j-\xi_j\right)^2\frac{p_{x_jx_j}(\boldsymbol{\eta})}{2} \tag{5.475}$$

$$+ \sum_{j,r=1;r>j}^{n}\left((\overline{x}_j-\underline{x}_j)\frac{i_j}{k}+\underline{x}_j-\xi_j\right)\left((\overline{x}_r-\underline{x}_r)\frac{i_r}{k}+\underline{x}_r-\xi_r\right)p_{x_jx_r}(\boldsymbol{\eta}), \tag{5.476}$$

where $\boldsymbol{\eta} \in [\underline{\mathbf{x}}, \overline{\mathbf{x}}]$.

If the maximum $\boldsymbol{\xi}$ is such that $\boldsymbol{\xi} = \underline{\mathbf{x}}$ or $\boldsymbol{\xi} = \overline{\mathbf{x}}$ or, in general, $\boldsymbol{\xi}$ belongs to the border of the interval $[\underline{\mathbf{x}}, \overline{\mathbf{x}}]$, that is, $\boldsymbol{\xi} \in \partial[\underline{\mathbf{x}}, \overline{\mathbf{x}}]$, then $\xi_j = \underline{\mathbf{x}}_j$ or $\xi_j = \overline{\mathbf{x}}_j$ for some j such that $1 \leqslant j \leqslant n$. It is possible to reduce to a Taylor expansion with d variables by parameterizing the part of the border with $\boldsymbol{\xi}$, with $d < n$, making the function dependent on d parameters. So, the dimension is less than n and for the inductive hypothesis, the inequalities (5.458)-(5.459) are true.

Otherwise, $p_{x_j}(\boldsymbol{\xi}) = 0$ for all j such that $1 \leqslant j \leqslant n$, so, the equalities (5.473)-(5.476) become

$$\max_{\mathbf{x}\in[\underline{\mathbf{x}},\overline{\mathbf{x}}]} p(\mathbf{x}) \leqslant \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \sum_{j=1}^{n}\left((\overline{x}_j - \underline{x}_j)\frac{i_j}{k} + \underline{x}_j - \xi_j\right)^2 \frac{|p_{x_jx_j}(\boldsymbol{\eta})|}{2} \tag{5.477}$$

$$+ \sum_{j,r=1;r>j}^{n} \left|(\overline{x}_j - \underline{x}_j)\frac{i_j}{k} + \underline{x}_j - \xi_j\right| \left|(\overline{x}_r - \underline{x}_r)\frac{i_r}{k} + \underline{x}_r - \xi_r\right| |p_{x_jx_r}(\boldsymbol{\eta})| \tag{5.478}$$

$$\leqslant \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \frac{1}{8k^2}\sum_{j=1}^{n}(\overline{x}_j - \underline{x}_j)^2 |p_{x_jx_j}(\boldsymbol{\eta})| \tag{5.479}$$

$$+ \frac{1}{4k^2}\sum_{j,r=1;r>j}^{n}(\overline{x}_j - \underline{x}_j)(\overline{x}_r - \underline{x}_r)|p_{x_jx_r}(\boldsymbol{\eta})| \tag{5.480}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \frac{1}{8k^2}\left(\sum_{j=1}^{n}((\overline{x}_j - \underline{x}_j)^2 |p_{x_jx_j}(\boldsymbol{\eta})|\right. \tag{5.481}$$

$$\left. + 2\sum_{j,r=1;r>j}^{n}(\overline{x}_j - \underline{x}_j)(\overline{x}_r - \underline{x}_r)|p_{x_jx_r}(\boldsymbol{\eta})|\right) \tag{5.482}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \frac{1}{8k^2}\sum_{j,r=1;r>j}^{n}\left((\overline{x}_j - \underline{x}_j)^2 |p_{x_jx_j}(\boldsymbol{\eta})|\right. \tag{5.483}$$

$$\left. + 2(\overline{x}_j - \underline{x}_j)(\overline{x}_r - \underline{x}_r)|p_{x_jx_r}(\boldsymbol{\eta})|\right) \tag{5.484}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \frac{1}{8k^2}\sum_{j,r=1;r>j}^{n}\left((\overline{x}_j - \underline{x}_j)^2 \left|\sum_{M\leqslant L_{jj}} b_{x_jx_jM}B_{jjM}(\mathbf{v})\right|\right. \tag{5.485}$$

$$\left. + 2(\overline{x}_j - \underline{x}_j)(\overline{x}_r - \underline{x}_r)\left|\sum_{M\leqslant L_{jr}} b_{x_jx_rM}B_{jrM}(\mathbf{v})\right|\right) \tag{5.486}$$

$$\leqslant \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \frac{1}{8k^2}\sum_{j,r=1;r>j}^{n}\left((\overline{x}_j - \underline{x}_j)^2 \sum_{M\leqslant L_{jj}} |b_{x_jx_jM}|B_{jjM}(\mathbf{v})\right. \tag{5.487}$$

$$\left. + 2(\overline{x}_j - \underline{x}_j)(\overline{x}_r - \underline{x}_r)\sum_{M\leqslant L_{jr}} |b_{x_jx_rM}|B_{jrM}(\mathbf{v})\right) \tag{5.488}$$

$$\leqslant \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \frac{1}{8k^2}\sum_{j,r=1;r>j}^{n}\left((\overline{x}_j - \underline{x}_j)^2\right. \tag{5.489}$$

$$\left. \cdot \max_{M\leqslant L_{jj}} |b_{x_jx_jM}| \sum_{M\leqslant L_{jj}} B_{jjM}(\mathbf{v}) + 2(\overline{x}_j - \underline{x}_j)(\overline{x}_r - \underline{x}_r)\max_{M\leqslant L_{jr}} |b_{x_jx_rM}| \sum_{M\leqslant L_{jr}} B_{jrM}(\mathbf{v})\right) \tag{5.490}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) \tag{5.491}$$

$$+ \frac{1}{8k^2}\sum_{j,r=1;r>j}^{n}\left((\overline{x}_j - \underline{x}_j)^2 \max_{M\leqslant L_{jj}} |b_{x_jx_jM}| + 2(\overline{x}_j - \underline{x}_j)(\overline{x}_r - \underline{x}_r)\max_{M\leqslant L_{jr}} |b_{x_jx_rM}|\right) \tag{5.492}$$

because $\sum_{M \leqslant L_{jj}} B_{jjM}(\mathbf{v}) = 1$ and $\sum_{M \leqslant L_{jr}} B_{jrM}(\mathbf{v}) = 1$. In addition,

$$\max_{(i_1,i_2,\dots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\dots,i_n}) \tag{5.493}$$

$$+\frac{1}{8k^2}\sum_{j,r=1;r>j}^{n}\left((\overline{x}_j-\underline{x}_j)^2\max_{M\leqslant L_{jj}}|b_{x_jx_jM}|+2(\overline{x}_j-\underline{x}_j)(\overline{x}_r-\underline{x}_r)\max_{M\leqslant L_{jr}}|b_{x_jx_rM}|\right) \tag{5.494}$$

$$=\max_{(i_1,i_2,\dots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\dots,i_n})+\tilde{\beta}_K \tag{5.495}$$

because for (5.286). $\tilde{\beta}_K = \dfrac{1}{8k^2}\sum_{i,j=1;j>i}^{n}\left((\overline{x}_i-\underline{x}_i)^2|b_{x_ix_i}|+2(\overline{x}_i-\underline{x}_i)(\overline{x}_j-\underline{x}_j)|b_{x_ix_j}|\right)$. Moreover,

$$\max_{(i_1,i_2,\dots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\dots,i_n})+\tilde{\beta}_K \tag{5.496}$$

$$=\max_{(i_1,i_2,\dots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\dots,i_n}) \tag{5.497}$$

$$+\frac{1}{8k^2}\sum_{j,r=1;r>j}^{n}\left((\overline{x}_j-\underline{x}_j)^2\max_{M\leqslant L_{jj}}\left|\sum_{N\leqslant M}\frac{\binom{M}{N}}{\binom{L_{jj}}{N}}(\overline{\mathbf{x}}-\underline{\mathbf{x}})^N\sum_{K\leqslant L_{jj}}\binom{K}{N}\underline{\mathbf{x}}^{K-N}\tilde{a}_{Kjj}\right|\right. \tag{5.498}$$

$$\left.+2(\overline{x}_j-\underline{x}_j)(\overline{x}_r-\underline{x}_r)\max_{M\leqslant L_{jr}}\left|\sum_{N\leqslant M}\frac{\binom{M}{N}}{\binom{L_{jr}}{N}}(\overline{\mathbf{x}}-\underline{\mathbf{x}})^N\sum_{K\leqslant L_{jr}}\binom{K}{N}\underline{\mathbf{x}}^{K-N}\bar{a}_{Kjr}\right|\right) \tag{5.499}$$

Using the bound (5.372) for Bernstein coefficients $b_{x_jx_jM}$ of the second derivative of the polynomial p

$$|b_{x_jx_jM}| \leqslant \sum_{K\leqslant L_{jj}} |\tilde{a}_{Kjj}|\overline{\mathbf{x}}^K \tag{5.500}$$

and, using similar considerations for $b_{x_jx_rM}$, it is possible to bound these coefficients

$$|b_{x_jx_rM}| \leqslant \sum_{K\leqslant L_{jr}} |\bar{a}_{Kjr}|\overline{\mathbf{x}}^K. \tag{5.501}$$

So, inequalities (5.477)-(5.499) become

$$\max_{\mathbf{x}\in[\underline{\mathbf{x}},\overline{\mathbf{x}}]} p(\mathbf{x}) \leqslant \max_{(i_1,i_2,\dots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\dots,i_n})+\tilde{\beta}_K \tag{5.502}$$

$$\leqslant \max_{(i_1,i_2,\dots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\dots,i_n}) \tag{5.503}$$

$$+\frac{1}{8k^2}\sum_{j,r=1;r>j}^{n}\left((\overline{x}_j-\underline{x}_j)^2\sum_{K\leqslant L_{jj}}|\bar{a}_{Kjj}|\overline{\mathbf{x}}^K+2(\overline{x}_j-\underline{x}_j)(\overline{x}_r-\underline{x}_r)\sum_{K\leqslant L_{jr}}|\bar{a}_{Kjr}|\overline{\mathbf{x}}^K\right) \tag{5.504}$$

$$=\max_{(i_1,i_2,\dots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\dots,i_n}) \tag{5.505}$$

$$+\frac{1}{8k^2}\sum_{j,r=1;r>j}^{n}\left((\overline{x}_j-\underline{x}_j)^2\sum_{k_1,k_2,\dots,k_{j-1},k_j,k_{j+1},\dots,k_n=0}^{l_1,l_2,\dots,l_{j-1},l_j-2,l_{j+1},\dots,l_n}|(k_j+1)(k_j+2)\right. \tag{5.506}$$

$$\cdot\, a_{k_1,k_2,\dots,k_{j-1},k_j+2,k_{j+1},\dots,k_n}|\overline{x}_1^{k_1}\cdots\overline{x}_{j-1}^{k_{j-1}}\overline{x}_j^{k_j}\overline{x}_{j+1}^{k_{j+1}}\cdots\overline{x}_n^{k_n} \tag{5.507}$$

$$+2(\overline{x}_j-\underline{x}_j)(\overline{x}_r-\underline{x}_r)\sum_{k_1,k_2,\dots,k_{j-1},k_j,k_{j+1},\dots,k_{r-1},k_r,k_{r+1},\dots,k_n=0}^{l_1,l_2,\dots,l_{j-1},l_j-1,l_{j+1},\dots,l_{r-1},l_r-1,l_{r+1},\dots,l_n}|(k_j+1)(k_r+1) \tag{5.508}$$

$$\left.\cdot\, a_{k_1,k_2,\dots,k_{j-1},k_j+1,k_{j+1},\dots,k_{r-1},k_r+1,k_{r+1},\dots,k_n}|\overline{x}_1^{k_1}\cdots\overline{x}_{j-1}^{k_{j-1}}\overline{x}_j^{k_j}\overline{x}_{j+1}^{k_{j+1}}\cdots\overline{x}_{r-1}^{k_{r-1}}\overline{x}_r^{k_r}\overline{x}_{r+1}^{k_{r+1}}\cdots\overline{x}_n^{k_n}\right) \tag{5.509}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) \tag{5.510}$$

$$+ \frac{1}{8k^2} \sum_{j,r=1;r>j}^{n} \Bigg((\overline{x}_j - \underline{x}_j)^2 \sum_{k_1,k_2,\ldots,k_{j-1},k_j,k_{j+1},\ldots,k_n=0}^{l_1,l_2,\ldots,l_{j-1},l_j-2,l_{j+1},\ldots,l_n} (k_j+1)(k_j+2) \tag{5.511}$$

$$\cdot |a_{k_1,k_2,\ldots,k_{j-1},k_j+2,k_{j+1},\ldots,k_n}| \overline{x}_1^{k_1} \cdots \overline{x}_{j-1}^{k_{j-1}} \overline{x}_j^{k_j} \overline{x}_{j+1}^{k_{j+1}} \cdots \overline{x}_n^{k_n} \tag{5.512}$$

$$+ 2(\overline{x}_j - \underline{x}_j)(\overline{x}_r - \underline{x}_r) \sum_{k_1,k_2,\ldots,k_{j-1},k_j,k_{j+1},\ldots,k_{r-1},k_r,k_{r+1},\ldots,k_n=0}^{l_1,l_2,\ldots,l_{j-1},l_j-1,l_{j+1},\ldots,l_{r-1},l_r-1,l_{r+1},\ldots,l_n} (k_j+1)(k_r+1) \tag{5.513}$$

$$\cdot |a_{k_1,k_2,\ldots,k_{j-1},k_j+1,k_{j+1},\ldots,k_{r-1},k_r+1,k_{r+1},\ldots,k_n}| \overline{x}_1^{k_1} \cdots \overline{x}_{j-1}^{k_{j-1}} \overline{x}_j^{k_j} \overline{x}_{j+1}^{k_{j+1}} \cdots \overline{x}_{r-1}^{k_{r-1}} \overline{x}_r^{k_r} \overline{x}_{r+1}^{k_{r+1}} \cdots \overline{x}_n^{k_n} \Bigg) \tag{5.514}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) \tag{5.515}$$

$$+ \frac{1}{8k^2} \sum_{j,r=1;r>j}^{n} \Bigg((\overline{x}_j - \underline{x}_j)^2 \sum_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_n=0}^{l_1,l_2,\ldots,l_{j-1},l_j,l_{j+1},\ldots,l_n} i_j(i_j-1)|a_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_n}| \tag{5.516}$$

$$\cdot \overline{x}_1^{i_1} \cdots \overline{x}_{j-1}^{i_{j-1}} \overline{x}_j^{i_j-2} \overline{x}_{j+1}^{i_{j+1}} \cdots \overline{x}_n^{i_n} \tag{5.517}$$

$$+ 2(\overline{x}_j - \underline{x}_j)(\overline{x}_r - \underline{x}_r) \sum_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_{r-1},i_r,i_{r+1},\ldots,i_n=0}^{l_1,l_2,\ldots,l_{j-1},l_j,l_{j+1},\ldots,l_{r-1},l_r,l_{r+1},\ldots,l_n} i_j i_r |a_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_{r-1},i_r,i_{r+1},\ldots,i_n}| \tag{5.518}$$

$$\cdot \overline{x}_1^{i_1} \cdots \overline{x}_{j-1}^{i_{j-1}} \overline{x}_j^{i_j-1} \overline{x}_{j+1}^{i_{j+1}} \cdots \overline{x}_{r-1}^{i_{r-1}} \overline{x}_r^{i_r-1} \overline{x}_{r+1}^{i_{r+1}} \cdots \overline{x}_n^{i_n} \Bigg) \tag{5.519}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) \tag{5.520}$$

$$+ \frac{1}{8k^2} \sum_{j,r=1;r>j}^{n} (\overline{x}_j - \underline{x}_j) \cdot \Bigg(\sum_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_{r-1},i_r,i_{r+1},\ldots,i_n=0}^{l_1,l_2,\ldots,l_{j-1},l_j,l_{j+1},\ldots,l_{r-1},l_r,l_{r+1},\ldots,l_n} i_j(i_j-1)|a_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_n}| \tag{5.521}$$

$$\cdot \overline{x}_1^{i_1} \cdots \overline{x}_{j-1}^{i_{j-1}} \overline{x}_j^{i_j-2} \overline{x}_{j+1}^{i_{j+1}} \cdots \overline{x}_n^{i_n} (\overline{x}_j - \underline{x}_j) + 2 i_j i_r |a_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_{r-1},i_r,i_{r+1},\ldots,i_n}| \tag{5.522}$$

$$\cdot \overline{x}_1^{i_1} \cdots \overline{x}_{j-1}^{i_{j-1}} \overline{x}_j^{i_j-1} \overline{x}_{j+1}^{i_{j+1}} \cdots \overline{x}_{r-1}^{i_{r-1}} \overline{x}_r^{i_r-1} \overline{x}_{r+1}^{i_{r+1}} \cdots \overline{x}_n^{i_n} (\overline{x}_r - \underline{x}_r) \Bigg) \tag{5.523}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) \tag{5.524}$$

$$+ \frac{1}{8k^2} \sum_{j,r=1;r>j}^{n} (\overline{x}_j - \underline{x}_j) \cdot \Bigg(\sum_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_{r-1},i_r,i_{r+1},\ldots,i_n=0}^{l_1,l_2,\ldots,l_{j-1},l_j,l_{j+1},\ldots,l_{r-1},l_r,l_{r+1},\ldots,l_n} [(i_j-1)\overline{x}_r(\overline{x}_j - \underline{x}_j) \tag{5.525}$$

$$+ 2 i_r \overline{x}_j (\overline{x}_r - \underline{x}_r)] i_j |a_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_n}| \overline{x}_1^{i_1} \cdots \overline{x}_{j-1}^{i_{j-1}} \overline{x}_j^{i_j-2} \overline{x}_{j+1}^{i_{j+1}} \cdots \overline{x}_{r-1}^{i_{r-1}} \overline{x}_r^{i_r-1} \overline{x}_{r+1}^{i_{r+1}} \cdots \overline{x}_n^{i_n} \Bigg). \tag{5.526}$$

Because $(\overline{x}_j - \underline{x}_j) \leqslant \overline{x}_j$ for all j such that $1 \leqslant j \leqslant n$,

$$(i_j-1)\overline{x}_r(\overline{x}_j - \underline{x}_j) + 2 i_r \overline{x}_j (\overline{x}_r - \underline{x}_r) \leqslant (i_j-1)\overline{x}_r\overline{x}_j + 2 i_r \overline{x}_j (\overline{x}_r - \underline{x}_r) \tag{5.527}$$

$$\leqslant (i_j-1)\overline{x}_r\overline{x}_j + 2 i_r \overline{x}_j \overline{x}_r \tag{5.528}$$

$$= [(i_j-1) + 2 i_r]\overline{x}_j\overline{x}_r \tag{5.529}$$

The inequalities above (5.502)-(5.526) become

$$\max_{\mathbf{x}\in[\underline{\mathbf{x}},\overline{\mathbf{x}}]} p(\mathbf{x}) \tag{5.530}$$

$$\leqslant \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \tilde{\beta}_K \tag{5.531}$$

$$\leqslant \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) \tag{5.532}$$

$$+\frac{1}{8k^2}\sum_{j,r=1;r>j}^{n}(\overline{x}_j-\underline{x}_j)\cdot\Bigg(\sum_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_{r-1},i_r,i_{r+1},\ldots,i_n=0}^{l_1,l_2,\ldots,l_{j-1},l_j,l_{j+1},\ldots,l_{r-1},l_r,l_{r+1},\ldots,l_n}[(i_j-1)\overline{x}_r(\overline{x}_j-\underline{x}_j) \tag{5.533}$$

$$+2i_r\overline{x}_j(\overline{x}_r-\underline{x}_r)]i_j|a_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_n}|\overline{x}_1^{i_1}\cdots\overline{x}_{j-1}^{i_{j-1}}\overline{x}_j^{i_j-2}\overline{x}_{j+1}^{i_{j+1}}\cdots\overline{x}_{r-1}^{i_{r-1}}\overline{x}_r^{i_r-1}\overline{x}_{r+1}^{i_{r+1}}\cdots\overline{x}_n^{i_n}\Bigg) \tag{5.534}$$

$$\leqslant \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) \tag{5.535}$$

$$+\frac{1}{8k^2}\sum_{j,r=1;r>j}^{n}(\overline{x}_j-\underline{x}_j)\cdot\Bigg(\sum_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_{r-1},i_r,i_{r+1},\ldots,i_n=0}^{l_1,l_2,\ldots,l_{j-1},l_j,l_{j+1},\ldots,l_{r-1},l_r,l_{r+1},\ldots,l_n}[(i_j-1)+2i_r]\overline{x}_j\overline{x}_r \tag{5.536}$$

$$\cdot\, i_j|a_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_n}|\overline{x}_1^{i_1}\cdots\overline{x}_{j-1}^{i_{j-1}}\overline{x}_j^{i_j-2}\overline{x}_{j+1}^{i_{j+1}}\cdots\overline{x}_{r-1}^{i_{r-1}}\overline{x}_r^{i_r-1}\overline{x}_{r+1}^{i_{r+1}}\cdots\overline{x}_n^{i_n}\Bigg) \tag{5.537}$$

$$\leqslant \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) \tag{5.538}$$

$$+\frac{1}{8k^2}\sum_{j,r=1;r>j}^{n}\overline{x}_j\cdot\Bigg(\sum_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_{r-1},i_r,i_{r+1},\ldots,i_n=0}^{l_1,l_2,\ldots,l_{j-1},l_j,l_{j+1},\ldots,l_{r-1},l_r,l_{r+1},\ldots,l_n}[(i_j-1)+2i_r]\overline{x}_j\overline{x}_r \tag{5.539}$$

$$\cdot\, i_j|a_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_n}|\overline{x}_1^{i_1}\cdots\overline{x}_{j-1}^{i_{j-1}}\overline{x}_j^{i_j-2}\overline{x}_{j+1}^{i_{j+1}}\cdots\overline{x}_{r-1}^{i_{r-1}}\overline{x}_r^{i_r-1}\overline{x}_{r+1}^{i_{r+1}}\cdots\overline{x}_n^{i_n}\Bigg) \tag{5.540}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) \tag{5.541}$$

$$+\frac{1}{8k^2}\sum_{j,r=1;r>j}^{n}\Bigg(\sum_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_{r-1},i_r,i_{r+1},\ldots,i_n=0}^{l_1,l_2,\ldots,l_{j-1},l_j,l_{j+1},\ldots,l_{r-1},l_r,l_{r+1},\ldots,l_n}[i_j(i_j-1)+2i_ji_r]|a_{i_1,i_2,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_n}| \tag{5.542}$$

$$\cdot\,\overline{x}_1^{i_1}\cdots\overline{x}_{j-1}^{i_{j-1}}\overline{x}_j^{i_j}\overline{x}_{j+1}^{i_{j+1}}\cdots\overline{x}_{r-1}^{i_{r-1}}\overline{x}_r^{i_r}\overline{x}_{r+1}^{i_{r+1}}\cdots\overline{x}_n^{i_n}\Bigg) \tag{5.543}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \frac{1}{8k^2}\sum_{j,r=1;r>j}^{n}\Bigg(\sum_{I\leqslant L}[i_j(i_j-1)+2i_ji_r]|a_I|\overline{\mathbf{x}}^I\Bigg) \tag{5.544}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \frac{1}{8k^2}\sum_{I\leqslant L}\Bigg(\sum_{j,r=1;r>j}^{n}[i_j(i_j-1)+2i_ji_r]\Bigg)|a_I|\overline{\mathbf{x}}^I \tag{5.545}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \frac{1}{8k^2}\sum_{I=(i_1,i_2,\ldots,i_n)\leqslant L}(i_1+\cdots+i_n)(i_1+\cdots+i_n-1)|a_I|\overline{\mathbf{x}}^I \tag{5.546}$$

for Lemma 5.1. Furthermore,

$$\max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \frac{1}{8k^2}\sum_{I=(i_1,i_2,\ldots,i_n)\leqslant L}(i_1+\cdots+i_n)(i_1+\cdots+i_n-1)|a_I|\overline{\mathbf{x}}^I \tag{5.547}$$

$$= \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \tilde{\alpha}_K, \tag{5.548}$$

where $\tilde{\alpha}_K$ is defined in (5.460).

So,

$$\max_{\mathbf{x}\in[\underline{\mathbf{x}},\overline{\mathbf{x}}]} p(\mathbf{x}) \leqslant \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \tilde{\beta}_K \leqslant \max_{(i_1,i_2,\ldots,i_n)\in K} p(\mathbf{t}_{i_1,i_2,\ldots,i_n}) + \tilde{\alpha}_K. \tag{5.549}$$

This proves the first part of the theorem. An entirely analogous argument establishes the lower bound on $\min_{\mathbf{x}\in[\underline{\mathbf{x}},\overline{\mathbf{x}}]} p(\mathbf{x})$ in (5.459). □

Using the bounds presented in Chapter 3 and their comparison, and the inequalities of Proposition 5.4, it is possible to obtain further bounds. Suppose $p(\mathbf{x}) = \sum_{I\leqslant L} a_I \mathbf{x}^I$, similarly to Section 5.2, from the inequalities (3.85)-(3.86), the extension of the bounds (5.212) to the interval $[\underline{\mathbf{x}}, \overline{\mathbf{x}}] \subset \mathbb{R}^n_{\geqslant 0}$ for the absolute values can be defined. Set $\mathbf{x} \in [\underline{\mathbf{x}}, \overline{\mathbf{x}}] \subset \mathbb{R}^n_{\geqslant 0}$,

$$|p(\mathbf{x})| \leqslant \max_{M\leqslant L} |b_M^{([\underline{\mathbf{x}},\overline{\mathbf{x}}])}| \leqslant \max\{|\underline{p}_I([\underline{\mathbf{x}},\overline{\mathbf{x}}])|, |\overline{p}_I([\underline{\mathbf{x}},\overline{\mathbf{x}}])|\} \tag{5.550}$$

$$\leqslant \max\{|\underline{p}_S([\underline{\mathbf{x}},\overline{\mathbf{x}}])|, |\overline{p}_S([\underline{\mathbf{x}},\overline{\mathbf{x}}])|\} \leqslant \overline{p}_A([\underline{\mathbf{x}},\overline{\mathbf{x}}]). \tag{5.551}$$

Applying the inequalities (5.550)-(5.551) for the second derivatives, as follows.

$$\sum_{j=1}^{n} |p_{x_jx_j}(\mathbf{x})| + 2\sum_{j,r=1;r>j}^{n} |p_{x_jx_r}(\mathbf{x})| \leqslant \sum_{j=1}^{n} |b_{x_jx_j}| + 2\sum_{j,r=1;r>j}^{n} |b_{x_jx_r}| \tag{5.552}$$

$$\leqslant \sum_{j=1}^{n} \max\{|\underline{p}_{x_jx_jI}([\underline{\mathbf{x}},\overline{\mathbf{x}}])|, |\overline{p}_{x_jx_jI}([\underline{\mathbf{x}},\overline{\mathbf{x}}])|\} \tag{5.553}$$

$$+2\sum_{j,r=1;r>j}^{n} \max\{|\underline{p}_{x_jx_rI}([\underline{\mathbf{x}},\overline{\mathbf{x}}])|, |\overline{p}_{x_jx_rI}([\underline{\mathbf{x}},\overline{\mathbf{x}}])|\} \tag{5.554}$$

$$\leqslant \sum_{j=1}^{n} \max\{|\underline{p}_{x_jx_jS}([\underline{\mathbf{x}},\overline{\mathbf{x}}])|, |\overline{p}_{x_jx_jS}([\underline{\mathbf{x}},\overline{\mathbf{x}}])|\} \tag{5.555}$$

$$+2\sum_{j,r=1;r>j}^{n} \max\{|\underline{p}_{x_jx_rS}([\underline{\mathbf{x}},\overline{\mathbf{x}}])|, |\overline{p}_{x_jx_rS}([\underline{\mathbf{x}},\overline{\mathbf{x}}])|\} \tag{5.556}$$

$$\leqslant \sum_{j=1}^{n} \overline{p}_{x_jx_jA}([\underline{\mathbf{x}},\overline{\mathbf{x}}]) + 2\sum_{j,r=1;r>j}^{n} \overline{p}_{x_jx_rA}([\underline{\mathbf{x}},\overline{\mathbf{x}}]). \tag{5.557}$$

From Definition 5.462 of $\tilde{\beta}_K$, the value $\tilde{\beta}_K$ can be bounded as follows.

$$\tilde{\beta}_K = \frac{1}{8k^2} \sum_{j,r=1;r>j}^{n} \left(|b_{x_jx_j}|(\overline{x}_j - \underline{x}_j)^2 + 2|b_{x_jx_r}|(\overline{x}_j - \underline{x}_j)(\overline{x}_r - \underline{x}_r) \right) \tag{5.558}$$

$$\leqslant \frac{1}{8k^2} \sum_{j,r=1;r>j}^{n} \left(|b_{x_jx_j}| \|\overline{\mathbf{x}} - \underline{\mathbf{x}}\|_\infty^2 + 2|b_{x_jx_r}| \|\overline{\mathbf{x}} - \underline{\mathbf{x}}\|_\infty^2 \right) \tag{5.559}$$

$$= \frac{\|\overline{\mathbf{x}} - \underline{\mathbf{x}}\|_\infty^2}{8k^2} \sum_{j,r=1;r>j}^{n} \left(|b_{x_jx_j}| + 2|b_{x_jx_r}| \right) \tag{5.560}$$

$$\leqslant \frac{\|\overline{\mathbf{x}} - \underline{\mathbf{x}}\|_\infty^2}{8k^2} \sum_{j,r=1;r>j}^{n} \Big(\max\{|\underline{p}_{x_jx_jI}([\underline{\mathbf{x}},\overline{\mathbf{x}}])|, |\overline{p}_{x_jx_jI}([\underline{\mathbf{x}},\overline{\mathbf{x}}])|\} \tag{5.561}$$

$$+2\max\{|\underline{p}_{x_jx_rI}([\underline{\mathbf{x}},\overline{\mathbf{x}}])|, |\overline{p}_{x_jx_rI}([\underline{\mathbf{x}},\overline{\mathbf{x}}])|\} \Big) \tag{5.562}$$

$$\tag{5.563}$$

$$\leqslant \frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2} \sum_{j,r=1;r>j}^{n} \Big(\max\{|\underline{p}_{x_jx_jS}([\underline{\mathbf{x}},\overline{\mathbf{x}}])|, |\overline{p}_{x_jx_jS}([\underline{\mathbf{x}},\overline{\mathbf{x}}])|\} \tag{5.564}$$

$$+2\max\{|\underline{p}_{x_jx_rS}([\underline{\mathbf{x}},\overline{\mathbf{x}}])|, |\overline{p}_{x_jx_rS}([\underline{\mathbf{x}},\overline{\mathbf{x}}])|\} \Big) \tag{5.565}$$

$$\leqslant \frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2} \sum_{j,r=1;r>j}^{n} \Big(\overline{p}_{x_jx_jA}([\underline{\mathbf{x}},\overline{\mathbf{x}}]) + 2\overline{p}_{x_jx_rA}([\underline{\mathbf{x}},\overline{\mathbf{x}}]) \Big). \tag{5.566}$$

Therefore, since $\mathbf{x}$ is arbitrary in $[\underline{\mathbf{x}},\overline{\mathbf{x}}]$, from the inequalities (5.458), (5.459) of the Proposition 5.4 further bounds can be obtained.

$$\min_{I\in K} p(\mathbf{t}_I) - \frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2} \sum_{j,r=1;r>j}^{n} \Big(\overline{p}_{x_jx_jA}([\underline{\mathbf{x}},\overline{\mathbf{x}}]) + 2\overline{p}_{x_jx_rA}([\underline{\mathbf{x}},\overline{\mathbf{x}}]) \Big) \tag{5.567}$$

$$\leqslant \min_{I\in K} p(\mathbf{t}_I) - \frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2} \sum_{j,r=1;r>j}^{n} \Big(\max\{|\underline{p}_{x_jx_jS}([\underline{\mathbf{x}},\overline{\mathbf{x}}])|, |\overline{p}_{x_jx_jS}([\underline{\mathbf{x}},\overline{\mathbf{x}}])|\} \tag{5.568}$$

$$+2\max\{|\underline{p}_{x_jx_rS}([\underline{\mathbf{x}},\overline{\mathbf{x}}])|, |\overline{p}_{x_jx_rS}([\underline{\mathbf{x}},\overline{\mathbf{x}}])|\} \Big) \tag{5.569}$$

$$\leqslant \min_{I\in K} p(\mathbf{t}_I) - \frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2} \sum_{j,r=1;r>j}^{n} \Big(\max\{|\underline{p}_{x_jx_jI}([\underline{\mathbf{x}},\overline{\mathbf{x}}])|, |\overline{p}_{x_jx_jI}([\underline{\mathbf{x}},\overline{\mathbf{x}}])|\} \tag{5.570}$$

$$+2\max\{|\underline{p}_{x_jx_rI}([\underline{\mathbf{x}},\overline{\mathbf{x}}])|, |\overline{p}_{x_jx_rI}([\underline{\mathbf{x}},\overline{\mathbf{x}}])|\} \Big) \tag{5.571}$$

$$\leqslant \min_{I\in K} p(\mathbf{t}_I) - \frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2} \sum_{j,r=1;r>j}^{n} \Big(|b_{x_jx_j}| + 2|b_{x_jx_r}| \Big) \tag{5.572}$$

$$\leqslant \min_{I\in K} p(\mathbf{t}_I) - \tilde{\beta}_K \tag{5.573}$$

$$= \min_{I\in K} p(\mathbf{t}_I) - \frac{1}{8k^2} \sum_{j,r=1;r>j}^{n} \Big(|b_{x_jx_j}|(\overline{x}_j-\underline{x}_j)^2 + 2|b_{x_jx_r}|(\overline{x}_j-\underline{x}_j)(\overline{x}_r-\underline{x}_r) \Big) \tag{5.574}$$

$$\leqslant \min_{\mathbf{x}\in[\underline{\mathbf{x}},\overline{\mathbf{x}}]} p(\mathbf{x}) \tag{5.575}$$

$$\leqslant \max_{\mathbf{x}\in[\underline{\mathbf{x}},\overline{\mathbf{x}}]} p(\mathbf{x}) \tag{5.576}$$

$$\leqslant \max_{I\in K} p(\mathbf{t}_I) + \tilde{\beta}_K \tag{5.577}$$

$$= \max_{I\in K} p(\mathbf{t}_I) + \frac{1}{8k^2} \sum_{j,r=1;r>j}^{n} \Big(|b_{x_jx_j}|(\overline{x}_j-\underline{x}_j)^2 + 2|b_{x_jx_r}|(\overline{x}_j-\underline{x}_j)(\overline{x}_r-\underline{x}_r) \Big) \tag{5.578}$$

$$\leqslant \max_{I\in K} p(\mathbf{t}_I) + \frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2} \sum_{j,r=1;r>j}^{n} \Big(|b_{x_jx_j}| + 2|b_{x_jx_r}| \Big) \tag{5.579}$$

$$\leqslant \max_{I\in K} p(\mathbf{t}_I) + \frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2} \sum_{j,r=1;r>j}^{n} \Big(\max\{|\underline{p}_{x_jx_jI}([\underline{\mathbf{x}},\overline{\mathbf{x}}])|, |\overline{p}_{x_jx_jI}([\underline{\mathbf{x}},\overline{\mathbf{x}}])|\} \tag{5.580}$$

$$+2\max\{|\underline{p}_{x_jx_rI}([\underline{\mathbf{x}},\overline{\mathbf{x}}])|, |\overline{p}_{x_jx_rI}([\underline{\mathbf{x}},\overline{\mathbf{x}}])|\} \Big) \tag{5.581}$$

$$\leqslant \max_{I\in K} p(\mathbf{t}_I) + \frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2} \sum_{j,r=1;r>j}^{n} \Big(\max\{|\underline{p}_{x_jx_jS}([\underline{\mathbf{x}},\overline{\mathbf{x}}])|, |\overline{p}_{x_jx_jS}([\underline{\mathbf{x}},\overline{\mathbf{x}}])|\} \tag{5.582}$$

$$+2\max\{|\underline{p}_{x_jx_rS}([\underline{\mathbf{x}},\overline{\mathbf{x}}])|, |\overline{p}_{x_jx_rS}([\underline{\mathbf{x}},\overline{\mathbf{x}}])|\} \Big) \tag{5.583}$$

$$\leqslant \max_{I\in K} p(\mathbf{t}_I) + \frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2} \sum_{j,r=1;r>j}^{n} \Big(\overline{p}_{x_jx_jA}([\underline{\mathbf{x}},\overline{\mathbf{x}}]) + 2\overline{p}_{x_jx_rA}([\underline{\mathbf{x}},\overline{\mathbf{x}}])\Big). \tag{5.584}$$

Rewriting with the respective definitions, the previous inequalities become as follows.

$$\min_{I\in K} p(\mathbf{t}_I) - \frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2} \sum_{j,r=1;r>j}^{n} \Bigg(\sum_{I=(i_1,i_2,\dots,i_j,\dots,i_n)\leqslant L} i_j(i_j-1)|a_I| \overline{x}_1^{i_1}\overline{x}_2^{i_2}\cdots\overline{x}_{j-1}^{i_{j-1}}\overline{x}_{j+1}^{i_{j+1}}\cdots\overline{x}_n^{i_n}\overline{x}_j^{i_j-2} \tag{5.585}$$

$$+2 \sum_{I=(i_1,i_2,\dots,i_j,\dots,i_r,\dots,i_n)\leqslant L} i_ji_r|a_I|\overline{x}_1^{i_1}\overline{x}_2^{i_2}\cdots\overline{x}_{j-1}^{i_{j-1}}\overline{x}_{j+1}^{i_{j+1}}\cdots\overline{x}_{r-1}^{i_{r-1}}\overline{x}_{r+1}^{i_{r+1}}\cdots\overline{x}_n^{i_n}\overline{x}_j^{i_j-1}\overline{x}_r^{i_r-1}\Bigg) \tag{5.586}$$

$$\leqslant \min_{I\in K} p(\mathbf{t}_I) - \frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2} \sum_{j,r=1;r>j}^{n} \Bigg(\max\Bigg\{\Bigg| \sum_{I=(i_1,i_2,\dots,i_j,\dots,i_n)\leqslant L;a_I>0} i_j(i_j-1)a_I\overline{x}_1^{i_1}\overline{x}_2^{i_2}\cdots\overline{x}_{j-1}^{i_{j-1}} \tag{5.587}$$

$$\cdot\overline{x}_{j+1}^{i_{j+1}}\cdots\overline{x}_n^{i_n}\overline{x}_j^{i_j-2}\Bigg|, \Bigg| \sum_{I=(i_1,i_2,\dots,i_j,\dots,i_n)\leqslant L;a_I<0} i_j(i_j-1)a_I\overline{x}_1^{i_1}\overline{x}_2^{i_2}\cdots\overline{x}_{j-1}^{i_{j-1}}\overline{x}_{j+1}^{i_{j+1}}\cdots\overline{x}_n^{i_n}\overline{x}_j^{i_j-2}\Bigg|\Bigg\} \tag{5.588}$$

$$+2\max\Bigg\{\Bigg| \sum_{I=(i_1,i_2,\dots,i_j,\dots,i_r,\dots,i_n)\leqslant L;a_I>0} i_ji_ra_I\overline{x}_1^{i_1}\overline{x}_2^{i_2}\cdots\overline{x}_{j-1}^{i_{j-1}}\overline{x}_{j+1}^{i_{j+1}}\cdots\overline{x}_{r-1}^{i_{r-1}}\overline{x}_{r+1}^{i_{r+1}}\cdots\overline{x}_n^{i_n}\overline{x}_j^{i_j-1}\overline{x}_r^{i_r-1}\Bigg|, \tag{5.589}$$

$$\Bigg| \sum_{I=(i_1,i_2,\dots,i_j,\dots,i_r,\dots,i_n)\leqslant L;a_I<0} i_ji_ra_I\overline{x}_1^{i_1}\overline{x}_2^{i_2}\cdots\overline{x}_{j-1}^{i_{j-1}}\overline{x}_{j+1}^{i_{j+1}}\cdots\overline{x}_{r-1}^{i_{r-1}}\overline{x}_{r+1}^{i_{r+1}}\cdots\overline{x}_n^{i_n}\overline{x}_j^{i_j-1}\overline{x}_r^{i_r-1}\Bigg|\Bigg\}\Bigg) \tag{5.590}$$

$$\leqslant \min_{I\in K} p(\mathbf{t}_I) - \frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2} \sum_{j,r=1;r>j}^{n} \Bigg(\max\Bigg\{\Bigg| \sum_{I=(i_1,i_2,\dots,i_j,\dots,i_n)\leqslant L;a_I>0} i_j(i_j-1)a_I\overline{x}_1^{i_1}\overline{x}_2^{i_2}\cdots\overline{x}_{j-1}^{i_{j-1}} \tag{5.591}$$

$$\cdot\overline{x}_{j+1}^{i_{j+1}}\cdots\overline{x}_n^{i_n}\overline{x}_j^{i_j-2} + \sum_{I=(i_1,i_2,\dots,i_j,\dots,i_n)\leqslant L;a_I<0} i_j(i_j-1)a_I\underline{x}_1^{i_1}\underline{x}_2^{i_2}\cdots\underline{x}_{j-1}^{i_{j-1}}\underline{x}_{j+1}^{i_{j+1}}\cdots\underline{x}_n^{i_n}\underline{x}_j^{i_j-2}\Bigg|, \tag{5.592}$$

$$\Bigg| \sum_{I=(i_1,i_2,\dots,i_j,\dots,i_n)\leqslant L;a_I<0} i_j(i_j-1)a_I\overline{x}_1^{i_1}\overline{x}_2^{i_2}\cdots\overline{x}_{j-1}^{i_{j-1}}\overline{x}_{j+1}^{i_{j+1}}\cdots\overline{x}_n^{i_n}\overline{x}_j^{i_j-2} \tag{5.593}$$

$$+ \sum_{I=(i_1,i_2,\dots,i_j,\dots,i_n)\leqslant L;a_I>0} i_j(i_j-1)a_I\underline{x}_1^{i_1}\underline{x}_2^{i_2}\cdots\underline{x}_{j-1}^{i_{j-1}}\underline{x}_{j+1}^{i_{j+1}}\cdots\underline{x}_n^{i_n}\underline{x}_j^{i_j-2}\Bigg|\Bigg\} \tag{5.594}$$

$$+2\max\Bigg\{\Bigg| \sum_{I=(i_1,i_2,\dots,i_j,\dots,i_r,\dots,i_n)\leqslant L;a_I>0} i_ji_ra_I\overline{x}_1^{i_1}\overline{x}_2^{i_2}\cdots\overline{x}_{j-1}^{i_{j-1}}\overline{x}_{j+1}^{i_{j+1}}\cdots\overline{x}_{r-1}^{i_{r-1}}\overline{x}_{r+1}^{i_{r+1}}\cdots\overline{x}_n^{i_n}\overline{x}_j^{i_j-1}\overline{x}_r^{i_r-1} \tag{5.595}$$

$$+ \sum_{I=(i_1,i_2,\dots,i_j,\dots,i_r,\dots,i_n)\leqslant L;a_I<0} i_ji_ra_I\underline{x}_1^{i_1}\underline{x}_2^{i_2}\cdots\underline{x}_{j-1}^{i_{j-1}}\underline{x}_{j+1}^{i_{j+1}}\cdots\underline{x}_{r-1}^{i_{r-1}}\underline{x}_{r+1}^{i_{r+1}}\cdots\underline{x}_n^{i_n}\underline{x}_j^{i_j-1}\underline{x}_r^{i_r-1}\Bigg|, \tag{5.596}$$

$$\Bigg| \sum_{I=(i_1,i_2,\dots,i_j,\dots,i_r,\dots,i_n)\leqslant L;a_I<0} i_ji_ra_I\overline{x}_1^{i_1}\overline{x}_2^{i_2}\cdots\overline{x}_{j-1}^{i_{j-1}}\overline{x}_{j+1}^{i_{j+1}}\cdots\overline{x}_{r-1}^{i_{r-1}}\overline{x}_{r+1}^{i_{r+1}}\cdots\overline{x}_n^{i_n}\overline{x}_j^{i_j-1}\overline{x}_r^{i_r-1} \tag{5.597}$$

$$+ \sum_{I=(i_1,i_2,\dots,i_j,\dots,i_r,\dots,i_n)\leqslant L;a_I>0} i_ji_ra_I\underline{x}_1^{i_1}\underline{x}_2^{i_2}\cdots\underline{x}_{j-1}^{i_{j-1}}\underline{x}_{j+1}^{i_{j+1}}\cdots\underline{x}_{r-1}^{i_{r-1}}\underline{x}_{r+1}^{i_{r+1}}\cdots\underline{x}_n^{i_n}\underline{x}_j^{i_j-1}\underline{x}_r^{i_r-1}\Bigg|\Bigg\}\Bigg) \tag{5.598}$$

$$\leqslant \min_{I\in K} p(\mathbf{t}_I) - \frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2} \sum_{j,r=1;r>j}^{n} \Big(\max_{M\leqslant L_{jj}} |b_{x_jx_jM}| + 2\max_{M\leqslant L_{jr}} |b_{x_jx_rM}| \Big) \tag{5.599}$$

$$\leqslant \min_{I\in K} p(\mathbf{t}_I) - \frac{1}{8k^2} \sum_{j,r=1;r>j}^{n} \Big(\max_{M\leqslant L_{jj}} |b_{x_jx_jM}|(\overline{x}_j-\underline{x}_j)^2 + 2\max_{M\leqslant L_{jr}} |b_{x_jx_rM}|(\overline{x}_j-\underline{x}_j)(\overline{x}_r-\underline{x}_r) \Big) \tag{5.600}$$

$$\leqslant \min_{\mathbf{x}\in[\underline{\mathbf{x}},\overline{\mathbf{x}}]} p(\mathbf{x}) \tag{5.601}$$

$$\leqslant \max_{\mathbf{x}\in[\underline{\mathbf{x}},\overline{\mathbf{x}}]} p(\mathbf{x}) \tag{5.602}$$

$$\leqslant \max_{I\in K} p(\mathbf{t}_I)+\frac{1}{8k^2}\sum_{j,r=1;r>j}^{n}\left(\max_{M\leqslant L_{jj}}|b_{x_jx_jM}|(\overline{x}_j-\underline{x}_j)^2+2\max_{M\leqslant L_{jr}}|b_{x_jx_rM}|(\overline{x}_j-\underline{x}_j)(\overline{x}_r-\underline{x}_r)\right) \tag{5.603}$$

$$\leqslant \max_{I\in K} p(\mathbf{t}_I)+\frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2}\sum_{j,r=1;r>j}^{n}\left(\max_{M\leqslant L_{jj}}|b_{x_jx_jM}|+2\max_{M\leqslant L_{jr}}|b_{x_jx_rM}|\right) \tag{5.604}$$

$$\leqslant \max_{I\in K} p(\mathbf{t}_I)+\frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2}\sum_{j,r=1;r>j}^{n}\left(\max\left\{\left|\sum_{I=(i_1,i_2,\ldots,i_j,\ldots,i_n)\leqslant L;a_I>0} i_j(i_j-1)a_I\overline{x}_1^{i_1}\overline{x}_2^{i_2}\cdots\overline{x}_{j-1}^{i_{j-1}}\right.\right.\right. \tag{5.605}$$

$$\left.\cdot\overline{x}_{j+1}^{i_{j+1}}\cdots\overline{x}_n^{i_n}\overline{x}_j^{i_j-2}+\sum_{I=(i_1,i_2,\ldots,i_j,\ldots,i_n)\leqslant L;a_I<0} i_j(i_j-1)a_I\underline{x}_1^{i_1}\underline{x}_2^{i_2}\cdots\underline{x}_{j-1}^{i_{j-1}}\underline{x}_{j+1}^{i_{j+1}}\cdots\underline{x}_n^{i_n}\underline{x}_j^{i_j-2}\right|, \tag{5.606}$$

$$\left|\sum_{I=(i_1,i_2,\ldots,i_j,\ldots,i_n)\leqslant L;a_I<0} i_j(i_j-1)a_I\overline{x}_1^{i_1}\overline{x}_2^{i_2}\cdots\overline{x}_{j-1}^{i_{j-1}}\overline{x}_{j+1}^{i_{j+1}}\cdots\overline{x}_n^{i_n}\overline{x}_j^{i_j-2}\right. \tag{5.607}$$

$$\left.\left.+\sum_{I=(i_1,i_2,\ldots,i_j,\ldots,i_n)\leqslant L;a_I>0} i_j(i_j-1)a_I\underline{x}_1^{i_1}\underline{x}_2^{i_2}\cdots\underline{x}_{j-1}^{i_{j-1}}\underline{x}_{j+1}^{i_{j+1}}\cdots\underline{x}_n^{i_n}\underline{x}_j^{i_j-2}\right|\right\} \tag{5.608}$$

$$+2\max\left\{\left|\sum_{I=(i_1,i_2,\ldots,i_j,\ldots,i_r,\ldots,i_n)\leqslant L;a_I>0} i_ji_ra_I\overline{x}_1^{i_1}\overline{x}_2^{i_2}\cdots\overline{x}_{j-1}^{i_{j-1}}\overline{x}_{j+1}^{i_{j+1}}\cdots\overline{x}_{r-1}^{i_{r-1}}\overline{x}_{r+1}^{i_{r+1}}\cdots\overline{x}_n^{i_n}\overline{x}_j^{i_j-1}\overline{x}_r^{i_r-1}\right.\right. \tag{5.609}$$

$$\left.+\sum_{I=(i_1,i_2,\ldots,i_j,\ldots,i_r,\ldots,i_n)\leqslant L;a_I<0} i_ji_ra_I\underline{x}_1^{i_1}\underline{x}_2^{i_2}\cdots\underline{x}_{j-1}^{i_{j-1}}\underline{x}_{j+1}^{i_{j+1}}\cdots\underline{x}_{r-1}^{i_{r-1}}\underline{x}_{r+1}^{i_{r+1}}\cdots\underline{x}_n^{i_n}\underline{x}_j^{i_j-1}\underline{x}_r^{i_r-1}\right|, \tag{5.610}$$

$$\left|\sum_{I=(i_1,i_2,\ldots,i_j,\ldots,i_r,\ldots,i_n)\leqslant L;a_I<0} i_ji_ra_I\overline{x}_1^{i_1}\overline{x}_2^{i_2}\cdots\overline{x}_{j-1}^{i_{j-1}}\overline{x}_{j+1}^{i_{j+1}}\cdots\overline{x}_{r-1}^{i_{r-1}}\overline{x}_{r+1}^{i_{r+1}}\cdots\overline{x}_n^{i_n}\overline{x}_j^{i_j-1}\overline{x}_r^{i_r-1}\right. \tag{5.611}$$

$$\left.\left.\left.+\sum_{I=(i_1,i_2,\ldots,i_j,\ldots,i_r,\ldots,i_n)\leqslant L;a_I>0} i_ji_ra_I\underline{x}_1^{i_1}\underline{x}_2^{i_2}\cdots\underline{x}_{j-1}^{i_{j-1}}\underline{x}_{j+1}^{i_{j+1}}\cdots\underline{x}_{r-1}^{i_{r-1}}\underline{x}_{r+1}^{i_{r+1}}\cdots\underline{x}_n^{i_n}\underline{x}_j^{i_j-1}\underline{x}_r^{i_r-1}\right|\right\}\right) \tag{5.612}$$

$$\leqslant \max_{I\in K} p(\mathbf{t}_I)+\frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2}\sum_{j,r=1;r>j}^{n}\left(\max\left\{\left|\sum_{I=(i_1,i_2,\ldots,i_j,\ldots,i_n)\leqslant L;a_I>0} i_j(i_j-1)a_I\overline{x}_1^{i_1}\overline{x}_2^{i_2}\cdots\overline{x}_{j-1}^{i_{j-1}}\right.\right.\right. \tag{5.613}$$

$$\left.\left.\cdot\overline{x}_{j+1}^{i_{j+1}}\cdots\overline{x}_n^{i_n}\overline{x}_j^{i_j-2}\right|,\left|\sum_{I=(i_1,i_2,\ldots,i_j,\ldots,i_n)\leqslant L;a_I<0} i_j(i_j-1)a_I\overline{x}_1^{i_1}\overline{x}_2^{i_2}\cdots\overline{x}_{j-1}^{i_{j-1}}\overline{x}_{j+1}^{i_{j+1}}\cdots\overline{x}_n^{i_n}\overline{x}_j^{i_j-2}\right|\right\} \tag{5.614}$$

$$+2\max\left\{\left|\sum_{I=(i_1,i_2,\ldots,i_j,\ldots,i_r,\ldots,i_n)\leqslant L;a_I>0} i_ji_ra_I\overline{x}_1^{i_1}\overline{x}_2^{i_2}\cdots\overline{x}_{j-1}^{i_{j-1}}\overline{x}_{j+1}^{i_{j+1}}\cdots\overline{x}_{r-1}^{i_{r-1}}\overline{x}_{r+1}^{i_{r+1}}\cdots\overline{x}_n^{i_n}\overline{x}_j^{i_j-1}\overline{x}_r^{i_r-1}\right|,\right. \tag{5.615}$$

$$\left.\left.\left|\sum_{I=(i_1,i_2,\ldots,i_j,\ldots,i_r,\ldots,i_n)\leqslant L;a_I<0} i_ji_ra_I\overline{x}_1^{i_1}\overline{x}_2^{i_2}\cdots\overline{x}_{j-1}^{i_{j-1}}\overline{x}_{j+1}^{i_{j+1}}\cdots\overline{x}_{r-1}^{i_{r-1}}\overline{x}_{r+1}^{i_{r+1}}\cdots\overline{x}_n^{i_n}\overline{x}_j^{i_j-1}\overline{x}_r^{i_r-1}\right|\right\}\right) \tag{5.616}$$

$$\leqslant \max_{I\in K} p(\mathbf{t}_I)+\frac{\|\overline{\mathbf{x}}-\underline{\mathbf{x}}\|_\infty^2}{8k^2}\sum_{j,r=1;r>j}\left(\sum_{I=(i_1,i_2,\ldots,i_j,\ldots,i_n)\leqslant L} i_j(i_j-1)|a_I|\overline{x}_1^{i_1}\overline{x}_2^{i_2}\cdots\overline{x}_{j-1}^{i_{j-1}}\overline{x}_{j+1}^{i_{j+1}}\cdots\overline{x}_n^{i_n}\right. \tag{5.617}$$

$$\left.\cdot\overline{x}_j^{i_j-2}+2\sum_{I=(i_1,i_2,\ldots,i_j,\ldots,i_r,\ldots,i_n)\leqslant L} i_ji_r|a_I|\overline{x}_1^{i_1}\overline{x}_2^{i_2}\cdots\overline{x}_{j-1}^{i_{j-1}}\overline{x}_{j+1}^{i_{j+1}}\cdots\overline{x}_{r-1}^{i_{r-1}}\overline{x}_{r+1}^{i_{r+1}}\cdots\overline{x}_n^{i_n}\overline{x}_j^{i_j-1}\overline{x}_r^{i_r-1}\right). \tag{5.618}$$

5.4 Note on the Computational Complexity

Set n the number of variables (dimension), k^n the number of points, and l the maximum degree of the single variable. Extending Horner's rule to evaluate a polynomial in a point for a multivariate polynomial, the computational time is $O(k^n \cdot l^n)$. So, for the Garloff bounds defined in Theorem 5.1 the computational time is $O(k^n \cdot l^n)$.

Also, for bounds using polynomial function values, linked to the arithmetic interval and simple bounds, the computational time is the same, that is $O(k^n \cdot l^n)$.

Instead, to compute Bernstein polynomials, as [19], using the definition (3.39), the computational time complexity is $O(l^{2n})$. With Garloff's method, instead, the computational time becomes $O(l^{n+1})$ for the unit box and for a general box domain it is $O(l^{2n})$. With the Ray and Nataraj method, the computational time becomes $O(l^{n+1})$ for a general box domain. So, for the new bounds defined in Propositions 5.1, 5.2, 5.3, and 5.4, the computational time is $O(k^n \cdot l^n + l^{n+1} \cdot n^2)$. This time is the sum between $O(k^n \cdot l^n)$, to calculate the polynomial in the k^n points, and $O(l^{n+1})$, to calculate Bernstein bounds multiplied by the number of partial second derivatives $\binom{n}{2}$.

Conclusions

This Doctoral Thesis has presented a comprehensive comparative analysis of bounds for polynomial functions on real multi-intervals, formulated within the framework of logic programming with polynomial constraints.

The investigation began with the analysis of polynomial functions defined over a bounded real interval with non-negative values. Three types of bounds were considered:

1. Bounds derived from the maximum and minimum of the Bernstein polynomial coefficients;
2. Bounds obtained through interval arithmetic; and
3. Simple bounds based on the absolute values and signs of the polynomial coefficients in conjunction with the right endpoint of the interval.

The primary objective was to determine which of these bounds best approximates the behavior of the polynomial function.

Initially, the study compared the simple bounds with those derived from interval arithmetic. The latter was found to offer greater accuracy over non-negative intervals, as it accounts for both interval endpoints. Subsequently, the Bernstein-based bounds were compared with the others. This phase introduced certain challenges due to the distinct polynomial bases involved: Bernstein polynomials are expressed in their own basis, whereas the other bounds rely on the canonical basis, complicating direct comparison. Nevertheless, by leveraging appropriate upper bounds—particularly Newton's binomial theorem and properties of binomial coefficients—these difficulties were resolved. It was demonstrated that while the Bernstein-based bounds provide the highest accuracy, they are also the most computationally demanding.

Building on the one-dimensional analysis, the three categories of bounds were subsequently generalized to multi-interval domains. This extension required addressing additional structural and computational challenges inherent to higher-dimensional settings. A comparative assessment of the resulting bounds demonstrated that, consistent with the unidimensional case, those derived from Bernstein representations provide the most accurate approximations. However, this improved precision comes at the expense of significantly higher computational complexity. In particular, the efficient handling and organization of multi-indices emerged as a crucial aspect of this stage of the study, as they directly influenced both the feasibility of the implementation and the overall computational performance.

An additional extension examined in the Thesis concerned the bounds derived from sampled polynomial functions. The analysis began on the unit interval and was subsequently generalized to non-negative real intervals. In the one-dimensional case, new approximations were developed by adapting the three original bounds to the first and second derivatives of polynomial functions. The study then progressed to higher dimensions, initially considering the unit multi-interval, and later extending to general boxes with non-negative values. While the results over the unit multi-interval mirrored those of the unidimensional case, the more general case involving non-negative boxes presented additional complexity. In particular, greater care was required in managing the interval endpoints when establishing upper bounds.

Two specific cases were initially examined: one in which all right endpoints of the intervals are greater than or equal to one, and another in which they are less than or equal to one. These cases facilitated the development of a broader understanding, ultimately enabling the derivation of results for arbitrary non-negative multi-intervals using the previously established bounds.

The principal contribution of this Thesis is the establishment of a hierarchy of bounds for polynomial functions, applicable both to non-negative real intervals and to non-negative real multi-intervals. This hierarchy provides valuable insights into the trade-offs between accuracy and computational efficiency, and it offers a solid foundation for further research in the domain of logic programming with polynomial constraints.

This research lays the foundation for a range of exciting future directions that promise to enhance both theoretical understanding and practical applications in the realm of polynomial reasoning. One significant area for future exploration is the development of new bounds for closed bounded boxes, both for closed intervals on $\mathbb{R}_{\geqslant 0}$ and for higher-dimensional spaces like $\mathbb{R}^n_{\geqslant 0}$. Refining these bounds and evaluating them against existing alternatives, particularly with respect to computational efficiency and accuracy, can enhance the precision and performance of algorithms that depend on them.

A particularly promising avenue lies in exploiting the algebraic properties of polynomials to derive more accurate bounds. This approach could lead to more efficient methods for bounding the behavior of polynomials over real domains, potentially reducing the computational overhead associated with polynomial constraint reasoning. Furthermore, improving the computational complexity of algorithms that utilize these bounds could have wide-ranging implications for various applications in computational mathematics, optimization, and verification.

Extending the focus to simplexes offers another potential avenue for development [26]. Although considerable work has been conducted on bounding the behavior of polynomials within finite, bounded boxes, applying these techniques to simplexes could yield new insights, particularly in optimization problems where simplexes play a central role.

Moreover, a key area for further research is the design of innovative algorithms capable of efficiently reasoning about the satisfiability of polynomial constraints over finite domains. This is especially critical in fields such as constraint satisfaction problems, where polynomial constraints often form a significant part of the problem structure. Developing algorithms specifically tailored to the characteristics of finite domains enables the effective addressing of challenges in a range of fields, including automated theorem proving and artificial intelligence.

Finally, it is worth noting that the study of real roots of polynomials has been a well-explored area in mathematical literature. A substantial body of work exists on methods for finding and characterizing real roots, and these results can be leveraged to enhance current polynomial reasoning techniques. For instance, applying existing methods to more precisely narrow the locations of real zeros could significantly improve the efficiency of root-finding algorithms, leading to faster and more reliable solutions in applications that require high precision.

Furthermore, an extension consists of an experimental study of computational complexity, using examples drawn from real-world problems.

In conclusion, the scope for future work in this domain is broad, and each of these proposed directions offers exciting opportunities to advance both the theoretical foundations and practical tools for reasoning with polynomials. Ongoing research is expected to lead to significant advancements in the ability to handle polynomial constraints, with potential benefits across a wide range of disciplines, including mathematics, engineering, and computer science.

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Acknowledgements

I thank Professor Federico Bergenti because he guided me and helped me along this journey.
I would also like to thank Stefania Monica, Davide Beretta, Simone Dallospedale, Gloria Tamboroni, Mattia Alex Leoni, the other Ph.D. students: Alessandro Minuzzo, Alessandra Migliaccio, Ettore Logiudice, Victor Gustavo May Custodio and Cristian Sopio.
I want to thank all my friends.
I would like to thank my parents Agostino and Paola, my brother Gino, and my uncle Enzo because they are a wonderful family and because, thanks to them, I was able to follow this path.
Last but not least, I would like to thank my partner Lucia, who has supported, encouraged, and advised me throughout this journey.