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Spectral Analysis of Schrödinger Operators with Quadratic Confinement Potentials in Electric and Magnetic Fields

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Dedicated to the memory of my mother, Marina,
whose love, guidance, and encouragement continue to inspire me.

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Abstract

This thesis investigates a class of Schrödinger operators with scalar confinement potentials of quadratic growth. The study focuses on the spectral properties of these operators and on how the geometry of the scalar potential affects the behavior of eigenfunctions. In the purely electric case, it is shown that the operator's spectrum is discrete due to the compact embedding of the relevant Hilbert space. Explicit lower bounds for the ground-state energy, spectral stability estimates, and a priori uniform-norm decay estimates for the eigenfunctions at infinity are established. Moreover, convergence results are provided, showing that both the spectrum and eigenspaces approach those of the classical quantum harmonic oscillator. The analysis is then extended to include the presence of a magnetic field, under suitable regularity and growth assumptions on the vector potential. These assumptions are general enough to cover the case of a non-trivial uniform magnetic field. The use of Kato-type inequalities and perturbative arguments allows the main spectral and regularity results of the electric case to be extended to the magnetic framework.

Sunto

Questa tesi analizza una classe di operatori di Schrödinger con potenziali di confinamento scalari a crescita quadratica. Lo studio è incentrato sulle proprietà spettrali di tali operatori e su come la geometria del potenziale scalare influenzi il comportamento delle autofunzioni. Nel caso puramente elettrico, si dimostra che lo spettro dell'operatore è discreto grazie all'immersione compatta dello spazio di Hilbert pertinente. Vengono inoltre stabiliti limiti inferiori espliciti per l'energia dello stato fondamentale, stime di stabilità spettrale e stime di decadimento a priori della norma uniforme delle autofunzioni all'infinito. Inoltre, si ottengono risultati di convergenza che mostrano come lo spettro e gli autospazi si avvicinino a quelli dell'oscillatore armonico quantistico classico.

L'analisi viene poi estesa alla presenza di un campo magnetico, sotto opportune ipotesi di regolarità e crescita sul potenziale vettore. Tali ipotesi sono sufficientemente generali da includere il caso di un campo magnetico uniforme non banale. L'impiego di disuguaglianze di tipo Kato e di argomenti perturbativi consente di estendere al contesto magnetico i principali risultati spettrali e di regolarità ottenuti nel caso elettrico.

Key References

The thesis builds primarily on the following works:

[A_1] C. Alessi. "Some Spectral Results for Magnetic Schrödinger Operators with Quadratic Confinement". in preparation. 2025 (cit. on p. 4).

[A_2] C. Alessi, L. Brasco, and M. Miranda Jr. "A Schrödinger operator with confining potential having quadratic growth". In: *Anal. Math. Phys.* 15 (2025), p. 85 (cit. on p. 4).

Contents

Chapter 1

Introduction

This chapter offers an overview of the scientific context and theoretical foundations of the thesis, highlighting the motivation for the research and the principal aims of the work.

1.1 General Overview and Motivation

The subject of this thesis is a class of Schrödinger operators with a scalar confining potential exhibiting quadratic growth, which physically localizes particles within a given region of space. The analysis focuses on establishing spectral properties of the operator and on a qualitative study of the associated eigenfunctions. We first consider the purely electric case and subsequently introduce a uniform magnetic field, extending the study to the full electromagnetic setting. In this work, a purely variational approach to the problem is adopted, with appropriate adaptations for the electromagnetic case.

The study of the Schrödinger operator

$$\mathcal{H} = -\Delta + V(x)$$

with domain in $L^2(\mathbb{R}^N)$, has attracted sustained attention from the mathematical community from the 1950s to the present day. This enduring interest stems from a wide range of fundamental questions that continue to motivate significant developments in the field. Among the most prominent issues are the definition of $\mathcal{H}$ as a self-adjoint operator on a suitable Hilbert space and the qualitative analysis of its spectral properties [44, 49, 52].

Over the decades, many of these problems have been addressed through a progressively more detailed investigation of the potential $V(x)$. In particular, the regularity assumptions originally imposed on the potential have been systematically weakened, thereby extending the theoretical framework to encompass increasingly broad classes of potentials [44]. A natural further step in this progression has been the transition from the study of operators associated with regular potentials to the analysis of operators involving singular potentials, a development that has introduced substantial new mathematical challenges [46].

Subsequent research has also focused on Schrödinger operators in the presence not only of electric fields, but also of magnetic fields, giving rise to the magnetic Schrödinger operator

$$\mathcal{H}_A = -\Delta_A + V(x)$$

with domain in $L^2(\mathbb{R}^N)$. Although the electric case may be regarded as a special instance of the magnetic one, the general analysis conventionally begins with the former, for which the theoretical foundations are more firmly established, before being extended to the electromagnetic setting.

Within this framework, the theory of semigroups plays a fundamental role, providing a natural and powerful tool for the study of self-adjoint operators and their associated time evolutions [28, 44, 52]. In parallel with this approach, an alternative method is based on the theory of quadratic forms [18, 20, 50]. This formulation allows one to treat operators associated with potentials of low regularity: it requires substantially weaker assumptions on both the scalar and vector potentials, and it enables the rigorous definition of the operator as well as the analysis of its spectral properties through functional-analytic techniques.

In extension to the electromagnetic setting [4, 19, 20, 25, 27, 53], the diamagnetic inequality plays a key role, exploiting intrinsic diamagnetic phenomena [51]. This inequality yields a variety of important estimates and has been the subject of extensive study, together with the Kato inequality from which it originates [26].

This research is, also, deeply rooted in physics and was originally inspired by the work of Lebeuf and Pavloff [34], "Bose–Einstein beams: coherent propagation through a guide". They investigate the dynamics of a Bose–Einstein condensate (BEC) beam in a waveguide, described by a time-dependent, nonlinear Schrödinger equation, known as the *Gross–Pitaevskii equation*, formulated independently by E. Gross between 1961 and 1963 and by L. Pitaevskii in 1961.

The model in [34] can be written as

$$i\,\partial_t\varphi(t,x) = -\Delta\varphi(t,x) - \omega^2 V_\Sigma(x)\varphi(t,x) + a\,|\varphi(t,x)|^2\varphi(t,x), \tag{1.1.1}$$

for $t > 0$ and $x = (x_1, \dots, x_N) \in \mathbb{R}^N$, where i denote the imaginary unit, $\varphi(t,x)$ is the condensate wavefunction, $V_\Sigma(x)$ is the confining potential, and $\omega \in \mathbb{R}\setminus\{0\}$, $a \in \mathbb{R}$ are physical constants. We observe that a is related to $g = \frac{4\pi\hbar^2 a_{sc}}{m}$, which is proportional to the s-wave scattering length a_{sc} of two interacting bosons. In [34], Σ is a one-dimensional curve of small curvature, but sharper bends and two-dimensional surfaces are also experimentally relevant [15].

1.2 Object of the Thesis

In this thesis, we interpreted the potential $V(x)$ as the squared distance from a non-empty compact set $\Sigma \subseteq \mathbb{R}^N$ and we focus on the *linear stationary case*, obtained by reducing the model (1.1.1) to a linear eigenvalue problem: the *stationary Schrödinger equation*

$$[-\Delta + V_\Sigma(x)]\varphi = \lambda\,\varphi, \quad \lambda \in \mathbb{R} \tag{1.2.1}$$

with associated elliptic operator

$$\mathcal{H}_V[\varphi] := -\Delta\varphi + V_\Sigma\varphi, \quad \varphi \in L^2(\mathbb{R}^N).$$

This provides the starting point for our analysis, which is primarily concerned with the fundamental spectral properties of the operator, including spectral stability estimates, integrability results, exponential decay at infinity and regularity for the eigenfunctions.

Building on this framework, we then consider a magnetic Schrödinger operator

$$\mathcal{H}_{A,V}[\varphi] := -\Delta_A\varphi + V_\Sigma\varphi, \quad \varphi \in L^2(\mathbb{R}^N).$$

with the same scalar potential $V_\Sigma(x)$ and a vector potential $A(x)$ generating a uniform magnetic field $B(x)$.

Although several arguments can be adapted from the electric case, the presence of a magnetic potential $A(x)$ introduces structural modifications that affect both the analytical framework and the nature of the estimates. The inclusion of a magnetic field $B(x)$ replaces the standard Laplacian with the magnetic Laplacian Δ_A, a second-order differential operator with variable complex coefficients, thereby generating additional technical difficulties. A further challenge stems from the lack of translation invariance of the magnetic gradient ∇_A. Due to these structural features of magnetic operators, some techniques employed in the purely electric case are no longer directly applicable and must be appropriately adapted. The properties of the vector potential assumed in this work, together with Kato's inequality, the diamagnetic inequality and the use of perturbative arguments provide the key tools for overcoming these difficulties.

It should be noted that, in extending the analysis to the electromagnetic case and in establishing that the operator has a discrete spectrum, several more general results are available in the literature [4, 31, 11], which impose weaker conditions on the vector potential. In the present work, however, stronger assumptions on the vector potential are adopted, in order to control the mixed terms arising in the expansion of the magnetic Laplacian and to apply certain technical tools, such as Kato's inequality.

We briefly comment on the extension to the electromagnetic case. In some instances, the problem can be reduced to the purely electric case, yielding results on compactness, integrability, and exponential decay of the eigenfunctions. In other cases, techniques analogous to those used in the electric case can be applied, with appropriate adaptations, leading, for example, to higher regularity results. Finally, in further situations, the same techniques may be employed, suitably adapted to magnetic setting, resulting in spectral and eigenspaces stability. Regarding the results obtained, in the electromagnetic case one generally observes a generalization of those obtained in the electric case, such as the dependence on the vector potential $A(x)$ and on geometric parameters related to the potential well.

An important and original aspect of this thesis is the employment of classical variational methods in both setting. Although these techniques are well-established, their use in the study of Schrödinger operators is neither straightforward nor trivial.

In this research, we also pay attention to the influence of the confining potential's geometry on spectral properties and eigenfunctions behavior. Such considerations are physically motivated, since an understanding of the geometry of the potential well is instrumental in the study of waveguides and associated physical phenomena.

This work does not aim to be exhaustive in this field, but rather represents a preliminary study that can serve as a basis for further developments. Of particular interest would be a return to the study of the original non linear, time-dependent case in order to analyze dispersive-type estimates or other phenomena typical of the electromagnetic context. The study of different geometries of the potential well is also of particular interest, especially for obtaining results relevant to physics, as it extends the research to non-compact manifolds.

The results in the electric framework have been published in [A_2] and a detailed account of the results for the electromagnetic case will appear shortly in [A_1].

1.3 Physical Background

From a physical standpoint, for example, our work is strongly motivated by the dynamics of coherent atomic beams in curved or imperfect guides. The propagation of coherent atomic beams in confined geometries presents both rich theoretical challenges and significant relevance for modern quantum technologies. In systems such as *ring traps, microtraps*, or *atom chips*, both the shape of the guide and the external fields strongly influence how atoms propagate, localize, or remain stable. In particular, the interplay between confinement, curvature, and magnetic flux is critical for *atomtronic devices, matter-wave interferometers*, and *quantum sensors*. Curvature-induced geometric potentials can create *bound states* [32, 41], and the presence of a uniform magnetic field may further enrich these phenomena [41, 40].

Bose–Einstein condensates (BECs) provide a unique platform for observing quantum phenomena on a macroscopic scale. They are widely studied in fundamental physics, for instance in the dynamics of *coherent matter waves, solitons*, and *vortices*. BECs are also relevant to emerging *quantum technologies*, including *atomtronic circuits, interferometric* *sensors*, and *precision measurements* of gravitational and magnetic fields. For studies of atomic Bose–Einstein condensates, such as superfluid behavior, we refer the reader to [37, 42], and for the so-called *atomic waveguide configurations* to [3]. Important steps in the theoretical description of these configurations were reported in the pioneering work [34], whose physical insight motivated the need for a rigorous mathematical formulation of the problem. From a mathematical perspective, BECs motivate the analysis of nonlinear Schrödinger equations and spectral properties of associated operators, especially in confined or curved geometries.

1.4 Mathematical Background

The Schrödinger equation provides the fundamental framework for the description of microscopic phenomena. Its associated operator, the Schrödinger operator, encodes the essential features of the physical system and has therefore become one of the central objects of study in Analysis and Mathematical Physics, both in the electric and the electromagnetic setting. Over the years, a substantial body of mathematical tools and theories has been developed to provide a rigorous formulation of the Schrödinger operator. These developments also address its various generalizations and have led to the investigation of foundational questions, including the proper definition of self-adjoint realizations and a detailed analysis of their spectral properties.

What follows is a very brief overview of some key results developed over time, encompassing both those closely related to the specific case considered here and more general findings within this area of research.

The Spectral Theory for the Schrödinger operator with a magnetic field has been extensively studied in various settings. Early and influential contributions include the works of Avron, Herbst, and Simon (1978) [4], who developed the theory for general vector and scalar potentials. Significant further results were obtained by Kato in two fundamental works. In the first, published in 1972 [26], he analyzed the Schrödinger operator with singular scalar potentials, establishing conditions on the potential $V(x)$ that ensure the self-adjointness of the operator. In his 1978 work [27], he extended these results by introducing a vector potential $A(x)$ and formulating suitable assumptions on $A(x)$ to guarantee that the operator remains self-adjoint.

Another notable contribution to the study of singular potentials is the work of Rozenblum and Melgaard (2005) [46], who offer a rigorous overview of Schrödinger operators with highly irregular potentials, effectively bridging classical results (Kato, Birman, Simon) with more recent developments and generalizations.

A comprehensive review of recent developments concerning rigorous mathematical results on the magnetic Schrödinger operator can be found in the work of Erdős (2005) [10], while a survey on the spectral effects of the magnetic potential is provided in Helffer (1994) [19].

More recent advances were achieved by Kondratiev, Maz'ya, and Shubin (2002, 2009) [31, 30], who established several criteria for the discreteness of the spectrum and for strict positivity in $L^2(\mathbb{R}^N)$ of the magnetic Schrödinger operator with a positive scalar potential. Broader results, valid on compact Riemannian manifolds, were later obtained by Kondratiev and Shubin (2002) [31], and by Shigekawa (1987) [47].

We also highlight a very recent paper by Frank (2024) [13], addressing a broad class of confining potentials (including the class we consider). In that work, a sharp lower bound on the ground state energy $\lambda_1(V)$ is derived, together with a stability estimate.

For more general results, we refer to the work of Kieffer and Loss (2021), in which they study the time-dependent, nonlinear Schrödinger equation in the presence of a magnetic field [29].

1.5 Structure of the Thesis

This section provides an overview of the thesis structure, highlighting the organization and main topics of each chapter. The thesis is organized as follows: after reviewing the relevant physical and mathematical background, we present the linear electric analysis, then proceed to the electromagnetic generalization.

1.5.1 Chapter 1

The first chapter of this thesis 1 provides a general introduction. In Section 1.1, the mathematical and physical motivations underlying this research are briefly presented; Section 1.3 focuses on the physical aspects and the most relevant application contexts; Section 1.4 offers an overview of the main results in the spectral theory of Schrödinger operators, including some recent developments. The chapter concludes with a summary of the overall structure of the thesis, presented chapter by chapter.

1.5.2 Chapter 2

In Chapter 2 we introduce the analytical preliminaries required in this thesis, and we also present the magnetic framework. In particular, Section 2.1 fixes the notation used throughout the work; Section 2.2 is devoted to Hilbert spaces. Basic definitions are given in Subsection 2.2.1; Subsection 2.2.2 recalls the notion of compact operators in Hilbert spaces; Subsection 2.2.3 defines the resolvent set and the spectrum of an operator; in Subsection 2.2.4 we introduce the resolvent operator. The section concludes with the Spectral Theorem 2.2.9, stated in Subsection 2.2.5. Section 2.3 briefly reviews the theory of Sobolev spaces. After recalling the main definitions, Section 2.3.2 presents several classical inequalities adapted to the specific setting: the Sobolev inequality (2.3.3), the Ladyzhenskaya inequality (2.3.4), the Morrey inequality (2.3.5), together with its proof, and an interpolation inequality in L^p spaces (2.3.8). Section 2.3.3 introduces translations of Sobolev functions and briefly recalls their main properties, together with some useful estimates (2.3.9), (2.3.10); the Section ends with a review of weighted Sobolev spaces in Subsection 2.3.4. Section 2.4 presents the variational formulation of some classical differential problems. Section 2.5 is devoted to a classical H^2-regularity result. In Subsection 2.5.1, the Calderón–Zygmund Theorem 2.5.1 is stated, and a proof adapted to the purely electric setting is provided. Section 2.6 introduces the Hausdorff distance between two potential wells. In Section 2.7 the electromagnetic framework is introduced. In Subsection 2.7.1 we review the existing literature on magnetic Sobolev spaces; in Subsection 2.7.2 the magnetic differential operators are introduced, highlighting the properties that are critical for this work in Remark 2.7.1. In Subsection 2.7.3, the vector potential $A(x)$ and the magnetic field $B(x)$ are defined; Subsection 2.7.4 recalls gauge invariance. Subsection 2.7.5 is devoted to introducing several tools useful for handling the extension to the electromagnetic case, including the diamagnetic inequality stated in Proposition 2.7.3 and a particular form of Kato's inequality stated in Lemma 2.7.4. The chapter concludes with an important physically relevant example presented in Subection 2.7.6.

1.5.3 Chapter 3

Chapter 3 is devoted to the study of the Schrödinger operator in the purely electric case. In Section 3.2 we introduce the operator $\mathcal{H}_V$ together with the confining electric potential $V(x)$. Section 3.3 states the basic properties of the scalar potential $V(x)$ and a Poincaré-type inequality adapted to the electric case is proved in Proposition 3.3.5. Section 3.4 is devoted to the spectral analysis of the Schrödinger operator: we introduce the associated quadratic form and determine its finiteness domain in Definition 3.4.1; we then define the Schrödinger operator induced by this form and establish the compactness of the embedding

$$H^1(\mathbb{R}^N; V) \hookrightarrow L^2(\mathbb{R}^N)$$

in Theorem 3.4.3, which provides a result of broader generality. Lemma 3.4.4 provides a proof of the resolvent operator satisfying the assumptions required for the application of the Spectral Theorem. The section concludes with the variational characterization of the eigenvalues presented in Corollary 3.4.5, and with an explicit lower bound for the ground-state energy (depending only on the dimension N and the radius R) given in Remark 3.4.6 (and see Proposition 3.3.5). Section 3.5 is dedicated to spectral stability, obtained by proving two-sided bounds on the difference $\lambda_k(V_1) - \lambda_k(V_2)$, where $V_1 = V_{\Sigma_1}$ and $V_2 = V_{\Sigma_2}$ are two potentials belonging to our class. Through an explicit estimate, we show that this difference can be controlled by the *Hausdorff distance* between the potential wells Σ_1 and Σ_2, as established in Proposition 3.5.3. In Section 3.6 we prove integrability estimates for the eigenstates: a global L^∞–bound in Proposition 3.6.1, and a weighted integrability against polynomial weights with arbitrary order of growth in Proposition 3.6.3. Section 3.7 contains Theorem 3.7.2, which establishes higher regularity for the eigenstates and yields a rigorous description of the domain of the electric Schrödinger operator, up to the optimal threshold given in Proposition 3.7.3 and Remark 3.7.4. The section concludes with a classical regularity result for the eigenstates, proved in Proposition 3.7.3. Section 3.8 presents the exponential decay of the eigenstates at infinity in the uniform norm, with an explicit a priori bound provided in Theorem 3.8.3. The chapter concludes with Section 3.9, dedicated to the stability of the eigenspaces. In Theorem 3.9.1 and Corollary 3.9.2 we analyze the regime where R is close to 0, i.e., when the potential well Σ collapses toward the origin so that the operator should converge to the classical quantum harmonic oscillator. We rigorously confirm this behavior by providing an explicit estimate for the distance between the spectra and eigenspaces of the two operators, depending only on the dimension N, the order n of the eigenvalue, and the geometric parameter R.

1.5.4 Chapter 4

Chapter 4 is devoted to the extension of the results obtained in the purely electric case to the electromagnetic setting. The chapter follows the same structure as the previous one, and some proofs are omitted since they can be deduced from the corresponding arguments in the electric case. The magnetic Schrödinger operator is introduced in Section 4.2, while the properties of the vector potential are discussed in Section 4.3 and Proposition 4.3.2 provides an electromagnetic version of the Poincaré inequality. Subsequently, Section 4.4 establishes the main spectral properties of the operator $\mathcal{H}_{A,V}$. After introducing the associated quadratic form and its domain of finiteness in Definition 4.4.1, the magnetic Schrödinger operator is defined, and the focus is placed on the discreteness of its spectrum in Section 4.4.1. Unlike in the electric case, it is first shown in Lemma 4.4.3 that the inclusion

$$H^1_A(\mathbb{R}^N;V) \hookrightarrow H^1(\mathbb{R}^N;V)$$

is continuous, thanks to the assumptions on the vector potential. This result allows the analysis to be reduced to the electric case, yielding the compact embedding

$$H^1_A(\mathbb{R}^N;V) \hookrightarrow L^2(\mathbb{R}^N).$$

Corollary 4.4.6 provides the variational characterization of the eigenvalues via the min-max argument, and, as in the electric case, an explicit lower bound for the first eigenvalue is established in Remark 4.4.7. Section 4.5 is devoted to spectral stability results. Proposition 4.5.2, exploiting the properties of the scalar potential, extends to the magnetic case the two-sided bounds on the differences $\lambda_{k,A}(V_1) - \lambda_{k,A}(V_2)$, controlled by the *Hausdorff distance* between the potential wells Σ_1 and Σ_2. In Section 4.6, integrability results and the exponential decay in the sup norm of the eigenstates are established, relying on Kato's inequality. Lemma 4.6.1 ensures the validity of the hypotheses required for the application of this inequality. Proposition 4.6.2 provides the global L^∞–bound, while Proposition 4.6.4 addresses weighted integrability against polynomial weights of arbitrary order. The section concludes with Theorem 4.6.6, which proves the exponential decay at infinity of the eigenstates. Section 4.7 contains Theorem 4.7.1, which provides an H^2–regularity result for the eigenstates and, consequently, a rigorous definition of the operator domain. These results are obtained by applying the Calderón-Zygmund Theorem, which can be used thanks to a perturbative argument based on the standard Laplacian. In Section 4.8, the stability results for eigenspaces are extended to the electromagnetic case through Theorem 4.8.1 and Corollary 4.8.2, and it is again shown that the operator converges to the magnetic quantum harmonic oscillator.

1.5.5 Chapter 5

Chapter 5 considers a particular geometric configuration of the potential well, namely the $\mathbb{S}^1$-type well defined in (3.2.6), and analyzes the behavior as the radius R becomes increasingly large in the electric setting.

Chapter 2

The Analytical Framework

2.1 Basic Notations

For the reader's convenience, we collect here the main notations and conventions that will be used throughout the thesis.

We use the notation $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ to denote the set of *positive integers*, $\Re$ to denote the *real part* of a complex number and $(\,\cdot\,)_+$ to denote the *positive part.* For every $t \in \mathbb{R}$, we denoted its *integer part* by

$$\lfloor t \rfloor = \max \Big\{ n \in \mathbb{Z} \,:\, t \geq n \Big\}.$$

To denote the *unit sphere* in the Lebesgue space $L^2(\mathbb{R}^N)$, we use the notation

$$S_2(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) \;:\; \|u\|_{L^2(\mathbb{R}^N)} = 1 \right\}.$$

2.2 Introduction to Hilbert Spaces and Operators

Hilbert spaces provide the natural setting for quantum mechanics, where physical states are represented by vectors and observables by linear operators, and form the foundational framework for the rigorous analysis of linear operators.

In what follows, we outline the key notions that are essential for establishing the central result of this section: the Spectral Theorem. For a more detailed introduction to Hilbert spaces, we refer the reader to classical references such as [9, 28, 36, 43, 45].

2.2.1 Hilbert Spaces

Definition 2.2.1 (Inner Product)**.** Let H be a vector space over $\mathbb{C}$. An inner product on H is a map

$$\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$$

satisfying, for all $x, y, z \in H$ and $\alpha \in \mathbb{C}$:

1. $\langle x, y\rangle = \overline{\langle y, x\rangle}$ (conjugate symmetry),
2. $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ (linearity in the first argument),
3. $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$,
4. $\langle x, x\rangle \geq 0$ (positive semidefinite),
5. $\langle x, x\rangle = 0 \iff x = 0$ (positive definite).

The norm induced by the inner product is $\|x\|_H := \langle x, x\rangle^{1/2}$, for all $x \in H$. Standard results include the Cauchy–Schwarz inequality, $|\langle x, y\rangle| \leq \|x\|\,\|y\|$, and the triangle inequality, $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in H$.

Definition 2.2.2 (Hilbert Space)**.** A vector space H equipped with an inner product $\langle\cdot,\cdot\rangle$ is a Hilbert space if it is complete with respect to the norm induced by the inner product.

2.2.2 Operators on Hilbert Spaces

In the context of the study of Schrödinger operators, the assumptions imposed on the operators are motivated by physical considerations and ensure that the mathematical objects reflect the expected physical behavior. In particular, requiring the operator to have a discrete spectrum (i.e., to be compact) corresponds mathematically to the quantization of energy levels, a fundamental phenomenon in quantum mechanics. Moreover, assuming that the operator is self-adjoint guarantees that its spectrum is real, as energy values must necessarily be real to be physically meaningful.

Definition 2.2.3 (Adjoint Operator)**.** Let $T : H \to H$ be bounded and linear. The adjoint of T is the unique operator $T^* : H \to H$ such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \quad \forall x, y \in H.$$

Definition 2.2.4 (Self-Adjoint Operator)**.** A bounded linear operator $T : H \to H$ is self-adjoint if $T = T^*$.

Definition 2.2.5 (Compact Operator)**.** A bounded linear operator $T : H \to H$ is compact if the image of the unit ball

$$B_H := \{x \in H : \|x\| \leq 1\}$$

is relatively compact in H. Equivalently, T maps bounded sequences to sequences admitting convergent subsequences.

2.2.3 Resolvent and Spectrum

Given a bounded and linear operator $T : H \to H$, and given $\lambda \in \mathbb{R}$ and $f \in H$, one naturally asks for which values of λ and f the equation

$$-\Delta u(x) + \lambda u(x) = f \tag{2.2.1}$$

can be solved. Many problems in partial differential equations can be reduced to an equation of the form (2.2.1). From a physical perspective, the values of λ often correspond to observable quantities (such as energy levels), and the solvability of the equation reflects whether a given measurement is physically meaningful. For this reason, it is natural to define the resolvent set as the set of all $\lambda \in \mathbb{R}$ for which the equation (2.2.1) admits a unique solution for every given right-hand side $f \in H$. Conversely, the spectrum of the operator is defined as the set of all $\lambda \in \mathbb{R}$ for which such a solution does not exist.

Definition 2.2.6 (Resolvent Set and Spectrum)**.** Let $T : H \to H$ be bounded and linear. For $\lambda \in \mathbb{R}$, set $T_\lambda := T - \lambda I$.

1. The resolvent set $\rho(T)$ consists of λ for which T_λ is bijective; its inverse is $(T - \lambda I)^{-1}$.
2. The spectrum is the complement of the resolvent set in $\mathbb{R}$: $\sigma(T) := \mathbb{R} \setminus \rho(T)$.

2.2.4 Resolvent Operators

At this point, we introduce the definition of the resolvent operator corresponding to the problem (2.2.1) previously discussed.

Definition 2.2.7 (Resolvent Operator)**.** For $\lambda \in \mathbb{R}$, the resolvent operator $R(f) : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ maps f to the unique weak solution $u = R(f)$ of

$$(\lambda - \Delta)u = f.$$

Equivalently, $R(f) = (\lambda - \Delta)^{-1}$ whenever the inverse exists.

2.2.5 Hilbert Basis and the Spectral Theorem

We next present a classical and standard formulation of the Spectral Theorem, as given in [9, Theorem VI.11].

As a preliminary step towards the formulation of the Spectral Theorem, we introduce the following

Definition 2.2.8 (Hilbert Basis)**.** A sequence $(e_n)_{n\in\mathbb{N}}$ in a separable Hilbert space H is a Hilbert basis if it is orthonormal and its linear span is dense in H.

Theorem 2.2.9 (Spectral Theorem)**.** *Let T be compact and self-adjoint on a separable Hilbert space H. Then H admits a Hilbert basis consisting of eigenvectors of T.*

Corollary 2.2.10 (Spectral Decomposition)**.** *If T is compact and self-adjoint, there exists an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of eigenvectors with eigenvalues $\lambda_n \to 0$ such that*

$$Tx = \sum_n \lambda_n \langle x, e_n\rangle e_n, \quad \forall x \in H.$$

2.3 Introduction to Sobolev Spaces

We briefly recall the main definitions and properties of *Sobolev spaces*. In particular, we focus on the case $p = 2$, the space $W^{1,2}(\Omega)$, usually denoted by $H^1(\Omega)$, which acquires the structure of a *Hilbert space* and represents the natural setting for the study of Schrödinger equation solutions.

For further details and proofs, the reader is referred, for example, to [7, 8, 16, 36].

2.3.1 The Sobolev Space $W^{1,2}$

To begin with, we introduce the definitions of *weak derivative* and *weak gradient*.

Definition 2.3.1. Let $\Omega \subseteq \mathbb{R}^N$ be an open set, $p \in [1, +\infty]$, and $u \in L^1_{\mathrm{loc}}(\Omega)$. If there exists $v_i \in L^p_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega u \, \frac{\partial \varphi}{\partial x_i} \, dx = -\int_\Omega v_i \, \varphi \, dx \quad \forall \varphi \in C_c^\infty(\Omega),$$

then the function v_i is uniquely determined and is called the *i-th weak derivative* of the function $u(x)$:

$$D_i u = \frac{\partial u}{\partial x_i} = v_i.$$

In addition, the vector

$$Du = \nabla u = \Big(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N}\Big)$$

is called the *weak gradient* of the function $u(x)$.

The definition of weak derivative is instrumental in introducing the concept of *weak solution* to a differential equation, which is essential for obtaining the weak formulation of classical differential problems.

As an example, we recall the problem previously mentioned (2.2.1). Let $\Omega \subseteq \mathbb{R}^N$ be an open set, and let $\lambda \in \mathbb{R}$ with $\lambda > 0$:

$$-\Delta u(x) + \lambda u(x) = f \tag{2.3.1}$$

where $\Delta u(x) = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$ denotes the Laplacian of $u(x)$. A function $u(x)$, such that both u and its gradient ∇u belong to $L^2(\Omega)$, is said to be a *weak solution* of (2.3.1) if

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx + \lambda \int_\Omega u\varphi \, dx = \int_\Omega f\varphi \, dx \quad \forall \varphi \in C_c^\infty(\Omega). \tag{2.3.2}$$

The previous identity is obtained by multiplying (2.3.1) by φ, integrating, and then integrating by parts the first term. The weak solution is sought in an appropriate functional space: the *Sobolev space* $W^{1,2}(\Omega)$.

Definition 2.3.2 (Sobolev Space)**.** Let $p \in [1, +\infty]$ and let $\Omega \subseteq \mathbb{R}^N$ be an open set. The *Sobolev space* $W^{1,p}(\Omega)$ is defined as

$$W^{1,p}(\Omega) = \big\{ u \in L^p(\Omega) : \nabla u \in L^p(\Omega) \big\}.$$

This space is endowed with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)},$$

which is equivalent to the norm

$$\|u\|_{W^{1,p}(\Omega)} = \Big(\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega)} \Big)^{\frac{1}{p}}.$$

The space $W^{1,p}(\Omega)$ is a Banach space. In the particular case $p = 2$, we define

$$H^1(\Omega) = W^{1,2}(\Omega).$$

The space $H^1(\Omega)$ is a Hilbert space, equipped with the following inner product:

$$\langle u, v \rangle_{H^1(\Omega)} = \langle u, v \rangle_{L^2(\Omega)} + \sum_i \langle \partial_i u, \partial_i v \rangle_{L^2(\Omega)},$$

which induces the norm

$$\|u\|_{H^1(\Omega)} = \Big(\|u\|^2_{L^2(\Omega)} + \sum_i \|\partial_i u\|^2_{L^2(\Omega)} \Big)^{\frac{1}{2}}.$$

2.3.2 Sobolev-type Inequalities

At this point, we recall some well-known *Sobolev-type inequalities* from the literature, suitably adapted to the specific framework considered in this thesis, which will be particularly relevant in the analysis of the electrical case.

The following inequalities hold for functions in $C_c^\infty(\mathbb{R}^N)$ and thus, by density, for the whole space $H^1(\mathbb{R}^N)$, as well:

- for $N \geq 3$ the *Sobolev inequality*

$$\mathcal{T}_N \left(\int_{\mathbb{R}^N} |u|^{2^*} \, dx \right)^{\frac{2}{2^*}} \leq \int_{\mathbb{R}^N} |\nabla u|^2 \, dx; \tag{2.3.3}$$

  where

$$2^* = \frac{2\,N}{N-2} \qquad \text{and} \qquad \mathcal{T}_N = N\,(N-2)\,\pi \left(\frac{\Gamma(N/2)}{\Gamma(N)} \right)^{\frac{2}{N}},$$

see for example [39, Chapter 2, Section 3.5];

- for $N = 2$ the *Ladyzhenskaya inequality*

$$\mathcal{L}_q \int_{\mathbb{R}^2} |u|^q \, dx \le \left(\int_{\mathbb{R}^2} |\nabla u|^2 \, dx \right)^{\frac{q-2}{2}} \left(\int_{\mathbb{R}^2} |u|^2 \, dx \right), \tag{2.3.4}$$

for every $2 < q < \infty$. For the special case $q = 4$, which is actually the original Ladyzhenskaya inequality (see [33]), we know that we can take $\mathcal{L}_4 = \pi$ (see [35, equation (1.11)]);

- finally, for $N = 1$ the *Morrey inequality*

$$\frac{1}{2} \|u\|^2_{L^\infty(\mathbb{R})} \le \left(\int_{\mathbb{R}} |u'|^2 \, dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |u|^2 \, dx \right)^{\frac{1}{2}}. \tag{2.3.5}$$

This result can be readily obtained from the Fundamental Theorem of Calculus. We include a brief proof for completeness.

Proof. Let $u \in C_c^\infty(\mathbb{R})$. By the Fundamental Theorem of Calculus, one has

$$u(x)^2 = \int_{-\infty}^{x} \frac{d}{dt}\big(u(t)^2\big) \, dt \tag{2.3.6}$$

$$= 2 \int_{-\infty}^{x} u(t) \, u'(t) \, dt. \tag{2.3.7}$$

Taking absolute values, we obtain

$$|u(x)|^2 \le 2 \int_{-\infty}^{x} |u(t)| \, |u'(t)| \, dt.$$

Applying the Cauchy–Schwarz inequality to the right-hand side yields

$$|u(x)|^2 \le 2 \, \|u\|_{L^2(\mathbb{R})} \, \|u'\|_{L^2(\mathbb{R})}.$$

Taking the supremum over $x \in \mathbb{R}$ gives the desired estimate. □

Finally we recall a simple interpolation result in L^p spaces. For further details see for example [7, Problem 1.7.22]

Lemma 2.3.3. *Let $u \in L^r(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$, with $1 \le r < q \le \infty$. Then $u \in L^p(\mathbb{R}^N)$ for every $r < p < q$, and*

$$\|u\|_{L^p(\mathbb{R}^N)} \le \|u\|^\theta_{L^r(\mathbb{R}^N)} \, \|u\|^{1-\theta}_{L^q(\mathbb{R}^N)} \tag{2.3.8}$$

for some $\theta \in (0, 1)$.

Proof. We restrict our attention to the case of finite exponents, i.e., $q < \infty$. Let $r < p < q$. Then there exists $\alpha \in (0,1)$ such that

$$p = \alpha q + (1-\alpha)r.$$

From this, one easily obtains

$$\alpha = \frac{p-r}{q-r}.$$

Applying Hölder's inequality with the conjugate pair of exponents $\left(\frac{1}{\alpha}, \frac{1}{1-\alpha}\right)$, we get

$$\begin{aligned}\|u\|_{L^p(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |u|^p\,dx\right)^{1/p} &= \left(\int_{\mathbb{R}^N} |u|^{\alpha q+(1-\alpha)r}\,dx\right)^{1/p}\\ &\leq \left(\int_{\mathbb{R}^N} |u|^q\,dx\right)^{\alpha/p} \left(\int_{\mathbb{R}^N} |u|^r\,dx\right)^{(1-\alpha)/p}.\end{aligned}$$

This can be rewritten as

$$\|u\|_{L^p(\mathbb{R}^N)} \leq \|u\|_{L^r(\mathbb{R}^N)}^{\frac{r}{p}\frac{p-r}{q-r}} \|u\|_{L^q(\mathbb{R}^N)}^{\frac{q}{p}\frac{q-p}{q-r}}.$$

Define

$$\theta := \frac{r}{p}\frac{p-r}{q-r},$$

so that

$$1-\theta = \frac{q}{p}\frac{q-p}{q-r}.$$

Substituting, we obtain the desired inequality (2.3.8). □

Remark 2.3.4. *In the specific setting of this thesis, we consider the exponents*

$$(r,q) = (2, 2^*),$$

yielding

$$\|u\|_{L^p(\mathbb{R}^N)} \leq \|u\|_{L^2(\mathbb{R}^N)}^{1-\theta} \|u\|_{L^{2^*}(\mathbb{R}^N)}^{\theta}.$$

2.3.3 Translations of Sobolev Functions

Now, we recall some basic properties of *translations* for Sobolev functions, which are a useful tool in several results in the purely electric case. In particular, the results reported here will be used to prove the compactness result of Theorem 3.4.3 and the H^2–regularity of eigenfunctions in Theorem 3.7.2. A notable application appears in the proof of the Calderón–Zygmund Theorem 2.5.1.

Let $u \in L^1_{\text{loc}}(\mathbb{R}^N)$. For any vector $h \in \mathbb{R}^N$, we denote the *translation* of u as

$$\tau_h u(x) = u(x+h), \quad x \in \mathbb{R}^N,$$

and the corresponding *difference* by

$$\delta_h u(x) = \tau_h u(x) - u(x) = u(x+h) - u(x).$$

Translations are continuous in the L^2-norm, i.e.,

$$\lim_{|h|\to 0} \|\tau_h u - u\|_{L^2(\mathbb{R}^N)} = 0.$$

The derivatives of u behave naturally under translations:

$$\partial_i(\tau_h u) = \tau_h(\partial_i u), \quad \forall i = 1, \dots, N.$$

This is also valid for the weak derivatives. For $u \in W^{1,2}(\mathbb{R}^n)$, a difference can be estimated via the gradient:

$$\|\tau_h u - u\|_{L^2(\mathbb{R}^N)} = \|\delta_h u\|_{L^2(\mathbb{R}^N)} \leq |h| \, \|\nabla u\|_{L^2(\mathbb{R}^N)}. \tag{2.3.9}$$

The last tool we report rely on the result presented, for example, in [38, Lemma 4.5], which we rewrite here adapted to the context of this thesis. We take $\mathbf{h} = h\,\mathbf{e}_k$ for $h \in \mathbb{R} \setminus \{0\}$ and $k \in \{1, \dots, N\}$, $F \in L^2(\mathbb{R}^N)$ then introduce the following semi-discrete integration by parts formula

$$\int_{\mathbb{R}^N} \delta_{h\,\mathbf{e}_k} F \, \delta_{h\,\mathbf{e}_k} u \, dx = -h \int_{\mathbb{R}^N} \frac{\partial}{\partial x_k} \delta_{h\,\mathbf{e}_k} u \left(\int_0^1 F(x + t\,h\,\mathbf{e}_k)\, dt \right) dx. \tag{2.3.10}$$

Let us set for simplicity

$$W(x) = \int_0^1 F(x + t\,h\,\mathbf{e}_k)\, dt,$$

by using that $F \in L^2(\mathbb{R}^N)$ we get that $W \in L^2(\mathbb{R}^N)$, as well. Moreover, by Jensen's inequality and the translation invariance of L^p norms, we have

$$\begin{aligned}
\|W\|^2_{L^2(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} \left| \int_0^1 F(x + t\,h\,\mathbf{e}_k)\, dt \right|^2 dx \\
&\leq \int_{\mathbb{R}^N} \int_0^1 |F(x + t\,h\,\mathbf{e}_k)|^2 \, dt\, dx = \int_0^1 \left(\int_{\mathbb{R}^N} |F(x + t\,h\,\mathbf{e}_k)|^2 \, dx \right) dt = \|F\|^2_{L^2(\mathbb{R}^N)}.
\end{aligned}$$

2.3.4 Weighted Sobolev Spaces

In this subsection we introduce the concept of weighted Lebesgue spaces, which will serve as a preliminary tool for defining the corresponding weighted Sobolev spaces. The latter play a central role in the analysis developed in this thesis, since the Schrödinger operator under consideration is perturbed by the presence of an electric scalar potential. From a mathematical point of view, the domain of the quadratic form associated with the operator will therefore be characterized as the *intersection* between the Hilbert space H^1 and a suitable weighted Lebesgue space, as will be shown later. This intersection gives the weighted Sobolev space. For a more detailed introduction to weighted Sobolev space, we refer the reader to classical references such as [1, 39].

To this end, we begin by recalling the definition of a weighted Lebesgue space.

Definition 2.3.5 (Weighted Lebesgue Space)**.** Let $w : \mathbb{R}^N \to [0, \infty)$ be a continuous function such that $w(x) > 0$ everywhere. For $1 \leq p < \infty$, the *weighted Lebesgue space* is defined by

$$L^p(\mathbb{R}^N; w) = \left\{ u \text{ measurable on } \mathbb{R}^N : \int_{\mathbb{R}^N} |u(x)|^p w(x)\, dx < \infty \right\}.$$

The associated norm is given by

$$\|u\|_{L^p(\mathbb{R}^N;w)} = \left(\int_{\mathbb{R}^N} |u(x)|^p w(x)\, dx \right)^{1/p}.$$

For $p = \infty$, we set $L^\infty(\mathbb{R}^N; w) = L^\infty(\mathbb{R}^N)$.

Here measurability will always be intended with respect to the $N-$dimensional Lebesgue measure.

2.4 The Variational Formulation

In this type of research, variational methods are employed both in the purely electric and in the electromagnetic case. The variational method offers a natural framework for the study of self-adjoint differential operators, particularly those of elliptic type. It is based on associating to a classical differential problem an energy functional whose stationary points coincide with the solutions of the Euler–Lagrange equation of the system.

Let H be a real or complex Hilbert space, and let

$$a : H \times H \to \mathbb{R}$$

be a continuous and coercive bilinear (or sesquilinear) form. The associated quadratic form is defined by $\mathcal{Q}(u) = a(u, u)$, and the corresponding energy functional is

$$J(u) = \tfrac{1}{2} a(u, u) - l(u),$$

where $l(u) = \langle f, u \rangle_H$ for every $f \in L^2(\mathbb{R}^N)$. The variational problem consists in finding $u \in H$ such that

$$J'(u)(v) = 0 \quad \forall v \in H,$$

or equivalently,

$$a(u, v) = l(v) \quad \forall v \in H.$$

This weak formulation allows one to extend the notion of solution to functions of lower regularity and, under standard assumptions, to ensure existence and uniqueness via the Lax–Milgram theorem ([16, 1]).

For the stationary Schrödinger equation

$$\mathcal{H}_V[u] = -\Delta u + V(x)u = \lambda u, \quad \lambda \in R$$

with scalar potential $V : \mathbb{R}^N \to \mathbb{R}$, the associated bilinear form is

$$a(u, v) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla \overline{v}\, dx + \int_{\mathbb{R}^N} V u \overline{v}\, dx,$$

and the corresponding quadratic form is

$$\mathcal{Q}(u) = \int_{\mathbb{R}^N} |\nabla u|^2\, dx + \int_{\mathbb{R}^N} V|u|^2\, dx.$$

Let $J : H^1(\mathbb{R}^N; V) \to \mathbb{R}$ be the energy functional associated with the form

$$J(u) := a(u, u) = \int_{\mathbb{R}^N} |\nabla u|^2\, dx + 2\int_{\mathbb{R}^N} V(x)\, |u|^2\, dx - \lambda \int_{\mathbb{R}^N} |u|^2\, dx.$$

To obtain the eigenvalues λ, we use the normalization condition

$$\|u\|^2_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |u|^2\, dx = 1. \tag{2.4.1}$$

Introducing a *Lagrange multiplier* λ for this constraint, one can define the Lagrangian functional

$$\mathcal{L}(u) := a(u, u) - \lambda \int_{\mathbb{R}^N} |u|^2\, dx, \tag{2.4.2}$$

whose stationarity condition

$$\delta\mathcal{L}(u) = 0 \quad \Longleftrightarrow \quad a(u, v) = \lambda \int_{\mathbb{R}^N} u\overline{v}\, dx \quad \forall v \in H^1(\mathbb{R}^N), \tag{2.4.3}$$

exactly reproduces the variational formulation of the stationary Schrödinger equation. In this way, the energy functional can be written in the general form

$$J(u) = a(u, u) - l(u), \tag{2.4.4}$$

where $l(u) := \lambda \int_{\mathbb{R}^N} |u|^2 dx$ plays the role of the linear term in the theory of bilinear forms.

The stationary points of J are precisely the weak solutions of the corresponding Euler–Lagrange equation, namely

$$-\Delta u + V(x)\, u = \lambda u,$$

that is, the eigenfunctions of the Schrödinger operator with potential V associated with the eigenvalue λ. The operator $\mathcal{H}_V$ is self-adjoint with respect to this form, a fact guaranteed by the theory of closed, semi-bounded forms ([28, 43]), and its analysis can be entirely conducted in variational terms. The addition of a vector potential $A : \mathbb{R}^N \to \mathbb{R}^N$ leads to the magnetic Schrödinger operator

$$\mathcal{H}_{A,V} = (\nabla + iA)^2 + V,$$

which can be interpreted as a variational perturbation of $\mathcal{H}_V$ through the covariant derivative $\nabla_A = \nabla + iA$. In both cases, the variational formulation is a fundamental tool for studying the existence, regularity, and spectral properties of solutions.

The eigenvalues of such operators admit a *variational characterization* in terms of the associated quadratic form and can be obtained via a *min–max principle*, ([43, 28]). In particular, if the discrete spectrum is non-empty, the eigenvalues arise as critical values of the energy functional restricted to subspaces of increasing dimension, without the need to impose explicit orthogonality conditions among the eigenfunctions. The min–max principle is expressed as

$$\lambda_n = \inf_{\substack{W \subset V \\ \dim W = n}} \sup_{\substack{u \in W \\ u \neq 0}} \frac{\mathcal{Q}(u)}{\|u\|^2_{L^2(\mathbb{R}^N)}}.$$

The first eigenvalue follows from an associated minimization problem, as will be shown in Remark 3.4.6.

This characterization provides a unified description of the discrete spectrum and constitutes a powerful tool for the study of Schrödinger-type operators, in the electric and magnetic case.

2.5 Regularity Estimates

In this section, we recall a classical result that is instrumental in establishing the second order regularity of solutions to elliptic equations.

2.5.1 Calderon–Zygmund Theorem

To prove the H^2-regularity of the eigenfunctions, we rely on a fundamental elliptic regularity theorem due to Calderón and Zygmund. This result ensures that, under suitable assumptions on the data, weak solutions to elliptic equations enjoy higher regularity, which in the electric setting also leads to the classical regularity of the eigenfunctions. In what follows, we present a proof of the Theorem in the case $p = 2$, which is the setting relevant for our purposes. This argument, developed for the purely electric case, will be recalled in Section 3.7. In the electromagnetic setting, a closely related line of reasoning is adopted, applied to the standard Laplacian. Indeed, the magnetic Laplacian can be understood as a perturbation of the standard one, which allows the use of the Theorem, although the resulting procedure differs in certain aspects. For the clasical Calderon–Zygmund estimates see, for example, [16, Theorem 9.9].

Theorem 2.5.1 (Calderon–Zygmund)**.** *Let $u \in W^{1,2}(\mathbb{R}^N)$ be such that*

$$\int_{\mathbb{R}^N} \langle \nabla u \nabla \varphi \rangle \, dx = \int_{\mathbb{R}^N} F\varphi \, dx \quad \forall \varphi \in W^{1,2}(\mathbb{R}^N)$$

where $F \in L^2(\mathbb{R}^N)$. Then $u \in W^{2,2}(\mathbb{R}^N)$.

Proof. The proof employs the classical *Nirenberg–Stampacchia method*, using a discrete differentiation of the equation. We set

$$-\Delta u = F \in L^2(\mathbb{R}^N),$$

thus the function u weakly solves the Poisson equation, i.e.

$$\int_{\mathbb{R}^N} \langle \nabla u, \nabla \varphi \rangle \, dx = \int_{\mathbb{R}^N} F \, \varphi \, dx, \qquad \text{for every } \varphi \in\in W^{1,2}(\mathbb{R}^N).$$

By a simple change of variable, for every $\mathbf{h} \in \mathbb{R}^N \setminus \{0\}$ we have that $\tau_{\mathbf{h}} u$ satisfies

$$\int_{\mathbb{R}^N} \langle \nabla(\tau_{\mathbf{h}} u), \nabla \varphi \rangle \, dx = \int_{\mathbb{R}^N} \tau_{\mathbf{h}} F \, \varphi \, dx, \qquad \text{for every } \varphi \in H^1(\mathbb{R}^N; V).$$

We subtract from this equation the one satisfied by u and then take the test function $\varphi = \delta_{\mathbf{h}} u$. This yields

$$\int_{\mathbb{R}^N} |\delta_{\mathbf{h}} \nabla u|^2 \, dx = \int_{\mathbb{R}^N} \delta_{\mathbf{h}} F \, \delta_{\mathbf{h}} u \, dx. \tag{2.5.1}$$

From equation (2.5.1) with $\mathbf{h} = h \, \mathbf{e}_k$, using the semi–discrete integration by parts formula (2.3.10) we thus get

$$\int_{\mathbb{R}^N} |\delta_{h \, \mathbf{e}_k} \nabla u|^2 \, dx = -h \int_{\mathbb{R}^N} W \, \delta_{h \, \mathbf{e}_k} \left(\frac{\partial u}{\partial x_k} \right) \, dx.$$

By applying Hölder's inequality, this in turn gives

$$\int_{\mathbb{R}^N} |\delta_{h \, \mathbf{e}_k} \nabla u|^2 \, dx \le |h| \, \|W\|_{L^2(\mathbb{R}^N)} \left\| \delta_{h \, \mathbf{e}_k} \frac{\partial u}{\partial x_k} \right\|_{L^2(\mathbb{R}^N)}.$$

The Hessian term on the right-hand side can now be absorbed in the left-hand side: indeed, by Young's inequality, we have

$$\int_{\mathbb{R}^N} |\delta_{h \, \mathbf{e}_k} \nabla u|^2 \, dx \le \frac{|h|^2}{2} \, \|W\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{2} \left\| \delta_{h \, \mathbf{e}_k} \frac{\partial u}{\partial x_k} \right\|^2_{L^2(\mathbb{R}^N)}, \tag{2.5.2}$$

and then we notice that

$$\begin{aligned}\left\|\delta_{h\,\mathbf{e}_k}\frac{\partial u}{\partial x_k}\right\|^2_{L^2(\mathbb{R}^N)} &= \int_{\mathbb{R}^N}\left|\frac{\partial u}{\partial x_k}(x+h\,\mathbf{e}_k)-\frac{\partial u}{\partial x_k}(x)\right|^2 dx\\ &\le \int_{\mathbb{R}^N}|\nabla u(x+h\,\mathbf{e}_k)-\nabla u(x)|^2\,dx = \int_{\mathbb{R}^N}|\delta_{h\,\mathbf{e}_k}\nabla u|^2\,dx.\end{aligned}$$

Thus, from (2.5.2) we get

$$\int_{\mathbb{R}^N}|\delta_{h\,\mathbf{e}_k}\nabla u|^2\,dx \le |h|^2\,\|W\|^2_{L^2(\mathbb{R}^N)} \le |h|^2\,\|F\|^2_{L^2(\mathbb{R}^N)} = |h|^2\,\|f - V\,u\|_{L^2(\mathbb{R}^N)}.$$

By dividing everything by $|h|^2$ and appealing to the characterization of Sobolev spaces in terms of finite differences (see [17, chapter 8]), we get the desired regularity for u. □

2.6 Geometric Tools

In this section, we introduce a geometric tool that will be useful in the subsequent analysis.

2.6.1 Hausdorff Distance between Sets

We recall the definition of the *Hausdorff distance* between sets, which, depending on the context, will be reformulated in terms of the distance between two potential wells Σ_1 and Σ_2. This particular kind of distance provides a fundamental tool for the study of spectral stability problems in section 3.5. In what follows, we denote by

$$\operatorname{dist}_H(\Sigma_1,\Sigma_2) = \max\left\{\max_{x\in\Sigma_1}\operatorname{dist}(x,\Sigma_2),\ \max_{y\in\Sigma_2}\operatorname{dist}(y,\Sigma_1)\right\}, \tag{2.6.1}$$

the Hausdorff distance of the two compact sets $\Sigma_1, \Sigma_2 \subseteq \mathbb{R}^N$.

2.7 The Magnetic Framework

This section recalls the theoretical background necessary to address the magnetic case. We briefly review the theory of magnetic Sobolev spaces and magnetic differential operators, outlining their key properties and introducing the technical tools needed for the subsequent analysis. In particular, we discuss the *diamagnetic inequality* and a specific *Kato-type inequality*. This inequality allows the problem to be effectively reduced to the electric case.

2.7.1 Magnetic Sobolev Space

Following the existing literature, we briefly recall the functional framework of the magnetic Sobolev space $H^1_A(\mathbb{R}^N)$ (see, for example, [36, Section 7.19]).

For any vector potential A whose components satisfy

$$A_j \in L^2_{\mathrm{loc}}(\mathbb{R}^N),$$

it is natural to define the magnetic Sobolev space

$$H^1_A(\mathbb{R}^N) = \Big\{ u \in L^2(\mathbb{R}^N) \,:\, (\partial_j + iA_j)u \in L^2(\mathbb{R}^N),\, j = 1, \ldots, N \Big\},$$

equipped with the norm induced by the inner product

$$\langle u_1, u_2 \rangle_A = \langle u_1, u_2 \rangle_{L^2(\mathbb{R}^N)} + \sum_{j=1}^{N} \langle (\partial_j + iA_j)u_1, (\partial_j + iA_j)u_2 \rangle_{L^2(\mathbb{R}^N)} .$$

We note that ∇u and Au are not assumed to belong separately to $L^2(\mathbb{R}^N)$; these issues will be discussed later, whenever required. Instead, we notice that $\nabla u \in L^2_{\mathrm{loc}}(\mathbb{R}^N)$.

2.7.2 Magnetic Differential Operators: ∇_A and Δ_A

We denote the magnetic gradient and Laplacian as

$$\nabla_A = \nabla + iA = (\partial_1 + iA_1, \ldots, \partial_N + iA_N),$$

and

$$-\Delta_A = -\sum_{j=1}^{N} (\partial_j + iA_j)^2,$$

which can be expanded as

$$-(\nabla + iA)^2 = -\Delta - 2iA \cdot \nabla - i \operatorname{div} A + |A|^2. \tag{2.7.1}$$

Remark 2.7.1. *For the purposes of the present study, the following observations are useful. The magnetic gradient is not translation-invariant, unlike the standard gradient. The magnetic Laplacian Δ_A is a second-order differential operator with variable complex coefficients and its expanded form reveals that Δ_A can be interpreted as a perturbation of the standard Laplacian Δ. These features constitute some of the main challenges in extending results to the electromagnetic case, which requires suitable adaptations in the proofs.*

2.7.3 Magnetic Field and Vector Potential

From the physical standpoint, the operator $-\Delta_A$ accounts for the kinetic energy, whereas the function $V(x)$ describes the potential energy of the quantum system. The terms involving $A(x)$ encode the effect of the external magnetic field $B(x)$ on the quantum system. In dimension $N = 3$, the magnetic field is given by the curl of the vector potential, $B = \nabla \times A$, and is thus represented by a vector field in $\mathbb{R}^3$. In dimension $N = 2$, the curl reduces to a scalar quantity, so that the magnetic field can be identified with a scalar function

$B = \partial_1 A_2 - \partial_2 A_1$. For higher dimensions $N > 3$, it is more natural to employ the language of differential forms: the magnetic potential A can be regarded as a complex-valued 1-form,

$$A = \sum_{j=1}^{N} A_j \, dx_j,$$

and the corresponding magnetic field is the exterior derivative of A,

$$B = dA = \sum_{1 \le i < j \le N} (\partial_i A_j - \partial_j A_i) \, dx_i \wedge dx_j,$$

that is, a 2-form encoding the components of the magnetic field in the higher-dimensional setting (see [5, Chapter 5]).

2.7.4 Gauge Invariance

It is worth recalling that the magnetic Schrödinger operator is *gauge invariant.* More precisely, two vector potentials A and A_1 related by

$$A_1 = A + \nabla\chi,$$

where $\chi : \mathbb{R}^N \to \mathbb{R}$ is a sufficiently regular scalar function, generate the same magnetic field, since

$$B_1 = \nabla \times A_1 = \nabla \times (A + \nabla\chi) = \nabla \times A = B$$

(for $N = 3$), or equivalently $B_1 = dA_1 = dA = B$ in the differential-form formulation valid for general N. This implies that the two operators $H_{A,V}$ and $H_{A_1,V}$ have the same spectrum for any scalar potential $V(x)$ (for example, see [5, section 5.4.2]).

2.7.5 Analytics Tools

We now present a computational Lemma, which will play a role in the forthcoming proofs, giving a *Leibniz-type formula* for the magnetic gradient of the product of two functions.

Lemma 2.7.2 (Leibniz–type rule)**.** *Let $A \in L^2_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{R}^N)$ and let $u, f \in H^1_A(\mathbb{R}^N)$. Then we have*

$$\nabla_A(u\, f) = f\, \nabla_A u + u\, \nabla f = u\, \nabla_A f + f\, \nabla u$$

Proof. The proof is by direct verification, using the definition of magnetic gradient. □

At this point, we introduce two inequalities that will be useful in the following, necessary to handle certain difficulties arising in the generalization from the purely electric case to the electromagnetic case.

A fundamental tool in the study of magnetic fields is the *diamagnetic inequality.* This inequality is used to bound the kinetic energy of a quantum system in the presence of a magnetic field and shows that the introduction of a magnetic field increases the system's kinetic energy—a quantum effect related to diamagnetism. The following Proposition is proved in [36, Chapter 7, Theorem 7.21]:

Proposition 2.7.3 (Diamagnetic Inequality)**.** *Let $A(x) : \mathbb{R}^N \to \mathbb{R}^N$ be such that $A_j(x) \in L^2_{\mathrm{loc}}(\mathbb{R}^N)$, and let $u(x) \in H^1_A(\mathbb{R}^N)$. Then $|u(x)| \in H^1(\mathbb{R}^N)$, and the diamagnetic inequality*

$$|\nabla|u|(x)| \leq |\nabla_A u(x)| \tag{2.7.2}$$

holds pointwise for almost every $x \in \mathbb{R}^N$.

For further developments of diamagnetic inequality we also refer to the work of Rozenblum and Melgaard [46, Chapter 9, section 9.2]. The diamagnetic inequality is equivalent to *Kato's distributional inequality* (see [5, Chapter 5, Section 5.3]) proved by Kato in 1972 in its most general formulation for elliptic operators in [26, Lemma A]. In this work, we present a version for the Laplacian adapted to the electromagnetic framework. We state the lemma below; for the proof, the reader is referred to Kato [26, Section 5] or to Reed and Simon [43, Theorem X.27].

Lemma 2.7.4 (Kato's Distributional Inequality)**.** *Let*

$$\Delta_A = \sum_{j=1}^{N} (\partial_j - iA_j(x))^2,$$

where the components $A_j(x)$ are real-valued functions in $C^1(\mathbb{R}^N)$. If both u and $\Delta_A u$ belong to $L^1_{\mathrm{loc}}(\mathbb{R}^N)$, then the following inequality holds in the distributional sense:

$$\Delta|u(x)| \geq \Re\Big[(\operatorname{sgn} \overline{u(x)})\, \Delta_A u(x)\Big], \tag{2.7.3}$$

where

$$\operatorname{sgn} \overline{u} = \begin{cases} \frac{\overline{u}}{|u|}, & \text{if} \quad u \neq 0 \\ 0 \quad \text{if} \quad u = 0. \end{cases}$$

For further developments and generalizations of Kato's inequality, including its relativistic extensions, we refer to the works of Hess, Schrader, and Uhlenbrock [21, 22], of Rozenblum and Melgaard [46, Section 4.3] as well as to those of Hiroshima, Ichinose and Lőrinczi [23].

2.7.6 A Physically Relevant Configuration

The precise assumptions on $A(x)$ will be stated later; they are sufficiently general to encompass the case of a non-trivial uniform magnetic field, such as the physically relevant configuration determined by the vector potential

$$A(x, y, z) = \tfrac{1}{2}(-by,\, bx,\, 0), \tag{2.7.4}$$

in dimension $N = 3$, with $A \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$. This vector potential generates the uniform magnetic field

$$B = \nabla \times A = (0, 0, b),$$

with $b \geq 0$. In the physics literature, the vector potential defined in (2.7.4) corresponds to the so-called *symmetric* (or *Lorenz*) gauge.

Chapter 3

The Electric Case

3.1 Introduction to the Electric Case

In this chapter, we consider the Schrödinger operator in the purely electric case. First, we introduce the main properties of the class of potentials under consideration. Then, we study some spectral properties as well as qualitative features of the eigenfunctions, in particular their elliptic regularity, integrability estimates, and exponential decay. Moreover, we provide stability estimates for the spectrum in terms of the Hausdorff distance, concluding the section with a quantitative estimate of the distance between the eigenspaces and those of the quantum harmonic oscillator.

Special attention is devoted to the dependence of certain results on the specific geometric configuration of the potential well Σ. Most of the results are obtained via the weak variational formulation of the problem. Classical tools from functional analysis are employed, including translation techniques for Sobolev functions and their properties, standard elliptic regularity theorems in H^2, and iterative methods *a la* De Giorgi and Moser.

3.2 Definition of the Operator

We consider the elliptic operator

$$\mathcal{H}_{\omega,\Sigma}[\Psi] := -\Delta\Psi + \omega^2 V_\Sigma \Psi, \tag{3.2.1}$$

where V_Σ has the following peculiar form

$$V_\Sigma(x) = \Big(\mathrm{dist}(x,\Sigma)\Big)^2, \qquad \text{for every } x \in \mathbb{R}^N,$$

and $\Sigma \subseteq \mathbb{R}^N$ is a compact non-empty set. In particular, V_Σ is a confining potential, in the sense that

$$\lim_{|x|\to+\infty} V_\Sigma(x) = +\infty. \tag{3.2.2}$$

We point out that $\mathcal{H}_{\omega,\Sigma}$ is self-adjoint and non–negative in $L^2(\mathbb{R}^N)$, so for its spectrum we have that $\sigma(\mathcal{H}_{\omega,\Sigma}) \subseteq [0, +\infty)$.

For every such a set Σ, also called *potential well*, we will associate the following geometric parameters

$$\delta_\Sigma = \min_{x\in\Sigma} |x| \qquad \text{and} \qquad R_\Sigma = \max_{x\in\Sigma} |x|. \tag{3.2.3}$$

If no confusion is possible, we shall write $V = V_\Sigma$, $\delta = \delta_\Sigma$ and $R = R_\Sigma$ for simplicity. We observe that by definition we have that $\Sigma \subseteq \overline{B_R(0)}$. This fact implies that we have the estimates

$$V(x) \le |x - \overline{x}|^2, \qquad \text{for every } x \in \mathbb{R}^N \text{ and } \overline{x} \in \Sigma, \tag{3.2.4}$$

and

$$V(x) \ge (|x| - R)_+^2, \qquad \text{for every } x \in \mathbb{R}^N. \tag{3.2.5}$$

The case $R = 0$ is peculiar: indeed, in this case we have $\Sigma = \{0\}$. Accordingly, the confining potential is given by $V(x) = \omega^2\, |x|^2$ and our Schrödinger operator boils down to the classical *quantum harmonic oscillator* (see for example [14, Example 4.2.1] or [52, Chapter 8, Section 3] for the spectrum of this operator).

Example 3.2.1. Apart for the case $\Sigma = \{0\}$, we will be interested in considering

$$\Sigma = S_R = \left\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \,:\, x_1^2 + x_2^2 = R^2, x_3 = 0\right\}, \qquad R > 0, \tag{3.2.6}$$

for which we have $\delta_\Sigma = R_\Sigma = R$, or more generally the torus

$$\Sigma = T_{r,R} = \left\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \,:\, \left(\sqrt{x_1^2 + x_2^2} - R + r\right)^2 + x_3^2 = r^2\right\}, \qquad 0 < 2r < R. \tag{3.2.7}$$

Observe that the latter reduces to the former, when $r = 0$.

Remark 3.2.2 (Reduction to the case $\omega = 1$)**.** *For simplicity, we will always normalize the physical constant $\omega \neq 0$ to be 1. Accordingly, we will write*

$$\mathcal{H}_\Sigma[\Psi] := \mathcal{H}_{1,\Sigma}[\Psi] = -\Delta\Psi + V_\Sigma\, \Psi.$$

Actually, this is not restrictive, since one can always reduce to this case by a scale change. Indeed, observe at first that for every $t > 0$ and $\Sigma \subseteq \mathbb{R}^N$ non-empty compact set, we have

$$\mathrm{dist}(x, \Sigma) = \frac{1}{t}\,\mathrm{dist}(t\,x, t\,\Sigma), \qquad \textit{for every } x \in \mathbb{R}^N.$$

Thus, if for a smooth function Ψ we set $\Psi_t(x) = \Psi(t\,x)$, we get

$$\begin{aligned}\mathcal{H}_{\omega,\Sigma}[\Psi_t](x) &= t^2\left[-\Delta\Psi(t\,x) + \frac{\omega^2}{t^2}\Big(\mathrm{dist}(x,\Sigma)\Big)^2\,\Psi(t\,x)\right]\\ &= t^2\left[-\Delta\Psi(t\,x) + \frac{\omega^2}{t^4}\Big(\mathrm{dist}(t\,x,t\,\Sigma)\Big)^2\,\Psi(t\,x)\right]\\ &= t^2\left[-\Delta\Psi(t\,x) + \frac{\omega^2}{t^4}\,V_{t\,\Sigma}(t\,x)\,\Psi(t\,x)\right] = t^2\,\mathcal{H}_{\frac{\omega}{t^2},t\,\Sigma}[\Psi](t\,x).\end{aligned}$$

By making the choice $t = \sqrt{\omega}$ for the scale parameter, we get that

$$\Psi \text{ is an eigenstate of } \mathcal{H}_{1,\sqrt{\omega}\,\Sigma} \qquad \Longleftrightarrow \qquad \Psi_{\sqrt{\omega}} \text{ is an eigenstate of } \mathcal{H}_{\omega,\Sigma}.$$

Accordingly, we also get the following relation between the eigenvalues

$$\lambda_k(\omega^2\,V_\Sigma) = \omega\,\lambda_k(V_{\sqrt{\omega}\,\Sigma}).$$

3.3 Properties of the Scalar Potential

This section is devoted to the introduction of the basic properties of the scalar potential. We start with the following two technical lemmas of general character.

Lemma 3.3.1. *The potential V is a locally Lipschitz function such that we have*

$$|x|^2\left(\frac{\sigma-1}{\sigma}\right)^2 \le V(x), \qquad \text{for every } \sigma > 1,\ x \in \mathbb{R}^N \setminus B_{\sigma R}(0), \tag{3.3.1}$$

and

$$V(x) \le 4\,|x|^2, \qquad \text{for every } x \in \mathbb{R}^N \setminus B_\delta(0), \tag{3.3.2}$$

where δ, R are defined in (3.2.3). In particular, we have

$$\frac{1}{V} \in L^p(\mathbb{R}^N \setminus B_{\sigma R}(0)), \qquad \text{for every } \sigma > 1 \text{ and } p > \frac{N}{2}. \tag{3.3.3}$$

Proof. From (3.2.4), by arbitrariness of $\overline{x} \in \Sigma$ and recalling the definition of δ, for every $x \in \mathbb{R}^N \setminus B_\delta(0)$ we have

$$V(x) \le (|x| + \delta)^2 \le 4\,|x|^2,$$

and the upper bound (3.3.2) follows. From (3.2.5), for every $\sigma > 1$ and every $x \in \mathbb{R}^N \setminus B_{\sigma R}(0)$ we have

$$V(x) \ge (|x| - R)^2 \ge \left(|x| - \frac{|x|}{\sigma}\right)^2.$$

This gives the lower bound, as well. The claimed integrability of $1/V$ now easily follows from that of $1/|x|^2$. $\square$

Remark 3.3.2. *For later use, we also record another couple of pointwise estimates on V. By (3.2.4) and Young's inequality, we get in particular*

$$V(x) \le (1+\varepsilon)\,|x|^2 + \left(1+\frac{1}{\varepsilon}\right)\,|\overline{x}|^2, \qquad \text{for every } x \in \mathbb{R}^N \text{ and } \overline{x} \in \Sigma.$$

By minimizing with respect to $\overline{x} \in \Sigma$, this yields

$$V(x) \le (1+\varepsilon)\,|x|^2 + \left(1+\frac{1}{\varepsilon}\right)\,\delta^2, \qquad \text{for every } x \in \mathbb{R}^N. \tag{3.3.4}$$

Moreover, for every $\varepsilon > 0$

$$|x|^2 \le (1+\varepsilon)\,(|x|-R)_+^2 + \left(1+\frac{1}{\varepsilon}\right)\,R^2.$$

By using (3.2.5), we thus obtain

$$|x|^2 \le (1+\varepsilon)\,V(x) + \left(1+\frac{1}{\varepsilon}\right)\,R^2, \qquad \text{for every } x \in \mathbb{R}^N,\ \varepsilon > 0. \tag{3.3.5}$$

Lemma 3.3.3. *Let $\varphi \in L^2_{\mathrm{loc}}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N; V)$. Then*

$$\varphi \in L^q(\mathbb{R}^N), \qquad \text{for every } \frac{2\,N}{N+2} < q < 2.$$

Proof. For every $\sigma > 1$ and every $2N/(N+2) < q < 2$ we have by Hölder's inequality

$$\int_{B_{\sigma R}(0)} |\varphi|^q\,dx \le |B_{\sigma R}(0)|^{1-\frac{q}{2}} \left(\int_{B_{\sigma R}(0)} |\varphi|^2\,dx\right)^{\frac{q}{2}}.$$

On the other hand, again by Hölder's inequality we get

$$\int_{\mathbb{R}^N\setminus B_{\sigma R}(0)} |\varphi|^q\,dx \le \left(\int_{\mathbb{R}^N\setminus B_{\sigma R}(0)} V\,|\varphi|^2\,dx\right)^{\frac{q}{2}} \left(\int_{\mathbb{R}^N\setminus B_{\sigma R}(0)} \frac{1}{V^{\frac{q}{2-q}}}\,dx\right)^{\frac{2-q}{2}}.$$

We now observe that

$$q > \frac{2\,N}{N+2} \qquad \Longleftrightarrow \qquad \frac{q}{2-q} > \frac{N}{2}.$$

By Lemma 3.3.1, the last integral is finite. □

The weighted L^2 space does not depend on the particular potential V, under our assumptions. More precisely, we have the following

Lemma 3.3.4. *We have*

$$L^2_{\mathrm{loc}}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N; V) = L^2_{\mathrm{loc}}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N; |x|^2).$$

Moreover, we have:

- *for every* $\varphi \in L^2_{\rm loc}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N; V)$ *and every* $\sigma > 1$

$$\int_{\mathbb{R}^N} |x|^2 \, |\varphi|^2 \, dx \le \sigma^2 \, R^2 \int_{B_{\sigma R}(0)} |\varphi|^2 \, dx + \left(\frac{\sigma}{\sigma - 1} \right)^2 \int_{\mathbb{R}^N \setminus B_{\sigma R}(0)} V \, |\varphi|^2 \, dx; \tag{3.3.6}$$

- *for every* $\varphi \in L^2_{\rm loc}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N; |x|^2)$

$$\int_{\mathbb{R}^N} V \, |\varphi|^2 \, dx \le \left(\max_{B_\delta(0)} V \right) \int_{B_\delta(0)} |\varphi|^2 \, dx + 4 \int_{\mathbb{R}^N \setminus B_\delta(0)} |x|^2 \, |\varphi|^2 \, dx.$$

Proof. The proof is straightforward, it is sufficient to decompose the integral for every $\sigma > 1$ as follows

$$\int_{\mathbb{R}^N} |x|^2 \, |\varphi|^2 \, dx = \int_{B_{\sigma R}(0)} |x|^2 \, |\varphi|^2 \, dx + \int_{\mathbb{R}^N \setminus B_{\sigma R}(0)} |x|^2 \, |\varphi|^2 \, dx,$$

and then use (3.3.1). The second estimate can be proved similarly, by using (3.3.2): we leave the details to the reader. □

In the following, we present a simple *Poincaré-type inequality*, adapted to the case of a Schrödinger operator with a scalar potential $V(x)$, which will be useful.

Proposition 3.3.5 (Poincaré–type Inequality)**.** *There exists* $C_{N,R} > 0$ *such that for every* $u \in C^\infty_c(\mathbb{R}^N)$ *we have*

$$\frac{1}{C_{N,R}} \int_{\mathbb{R}^N} |u|^2 \, dx \le \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} V \, |u|^2 \, dx. \tag{3.3.7}$$

Moreover, the constant $C_{N,R}$ *has the following properties*

$$\lim_{R \to 0^+} C_{N,R} \ge 1 \qquad \textit{and} \qquad 0 < \lim_{R \to +\infty} \frac{C_{N,R}}{R^2} < +\infty.$$

Proof. For every $\sigma > 1$ we get from (3.3.1)

$$\begin{aligned} \int_{\mathbb{R}^N} |u|^2 \, dx &= \int_{B_{\sigma R}(0)} |u|^2 \, dx + \int_{\mathbb{R}^N \setminus B_{\sigma R}(0)} |u|^2 \, dx \\ &\le \int_{B_{\sigma R}(0)} |u|^2 \, dx + \frac{1}{(\sigma - 1)^2 \, R^2} \int_{\mathbb{R}^N} V \, |u|^2 \, dx. \end{aligned}$$

For $N \ge 3$, we can use Hölder and Sobolev inequalities (2.3.3) to estimate the L^2-norm. We obtain

$$\begin{aligned} \int_{\mathbb{R}^N} |u|^2 \, dx &\le |B_{\sigma R}(0)|^{\frac{2}{N}} \, \|u\|^2_{L^{2^*}(\mathbb{R}^N)} + \frac{1}{(\sigma - 1)^2 \, R^2} \int_{\mathbb{R}^N} V \, |u|^2 \, dx \\ &\le \frac{\omega_N^{\frac{2}{N}} \, (\sigma \, R)^2}{\mathcal{T}_N} \, \|\nabla u\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{(\sigma - 1)^2 \, R^2} \int_{\mathbb{R}^N} V \, |u|^2 \, dx. \end{aligned}$$

We choose
$$\sigma = 1 + \frac{1}{R} \qquad \text{so that} \qquad \sigma\, R = R + 1.$$
In particular, we get
$$\int_{\mathbb{R}^N} |u|^2\, dx \le \frac{\omega_N^{\frac{2}{N}}\, (R+1)^2}{\mathcal{T}_N}\, \|\nabla u\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} V\, |u|^2\, dx.$$
By defining
$$C_{N,R} := \max\left\{ \frac{\omega_N^{\frac{2}{N}}\, (R+1)^2}{\mathcal{T}_N}, 1 \right\},$$
we get the claimed estimate.

For $N = 2$, we proceed similarly, by using this time (2.3.4) with $q = 4$, i.e
$$\begin{aligned} \int_{\mathbb{R}^2} |u|^2\, dx &\le \int_{B_{\sigma R}(0)} |u|^2\, dx + \frac{1}{(\sigma-1)^2\, R^2} \int_{\mathbb{R}^2} V\, |u|^2\, dx \\ &\le |B_{\sigma R}(0)|^{\frac{1}{2}}\, \|u\|^2_{L^4(\mathbb{R}^2)} + \frac{1}{(\sigma-1)^2\, R^2} \int_{\mathbb{R}^2 \setminus B_{\sigma R}(0)} V\, |u|^2\, dx \\ &\le \frac{\sqrt{\pi\, (\sigma\, R)^2}}{\sqrt{\pi}}\, \|\nabla u\|_{L^2(\mathbb{R}^2)}\, \|u\|_{L^2(\mathbb{R}^2)} + \frac{1}{(\sigma-1)^2\, R^2} \int_{\mathbb{R}^2} V\, |u|^2\, dx. \end{aligned}$$
We can use Young's inequality on the first term, in order to absorb the L^2 norm, i.e.
$$\int_{\mathbb{R}^2} |u|^2\, dx \le \frac{\sigma^2\, R^2}{2}\, \|\nabla u\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{2}\, \|u\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{(\sigma-1)^2\, R^2} \int_{\mathbb{R}^2} V\, |u|^2\, dx,$$
which leads again to the claimed estimate, upon choosing σ as above.

The case $N = 1$ can be treated similarly, by using (2.3.5) this time. The details are left to the reader. □

Remark 3.3.6. *In light of Lemma 3.5.1 and Remark 3.5.2 below, the behavior of the constant $C_{N,R}$ with respect to R is optimal, in general.*

3.4 Spectral Analysis

In the following, we introduce the quadratic form and determine its domain of finiteness. This allows us to define the corresponding electric Schrödinger operator, whose spectrum can be shown to be discrete. This property arises from the compactness of the embedding of the weighted space $H^1(\mathbb{R}^N; V)$ into $L^2(\mathbb{R}^N)$, a result established via the Riesz–Fréchet–Kolmogorov compactness theorem for L^p spaces. Finally, we present a min–max characterization of the eigenvalues and we give an explicit lower bound for the first eigenvalue.

We introduce the following inner product for any $\varphi, \psi \in H^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N; V)$
$$\mathcal{Q}_V[\varphi, \psi] := \int_{\mathbb{R}^N} \langle \nabla\varphi, \nabla\psi \rangle\, dx + \int_{\mathbb{R}^N} V\, \varphi\, \psi\, dx.$$

Accordingly, we also set

$$\|\varphi\|_V = \sqrt{\mathcal{Q}_V[\varphi,\varphi]}, \qquad \text{for } \varphi \in H^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N;V).$$

It is not difficult to see that the latter is actually a norm.

First, we determine the domain of finiteness of the quadratic form. By referring to the weighted Sobolev spaces introduced in Section 2.3.4, we can then provide the following

Definition 3.4.1. With the notation above, we define the weighted Sobolev space

$$H^1(\mathbb{R}^N;V) := H^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N;V),$$

endowed with the norm $\|\cdot\|_V$.

The following preliminary result will be important.

Lemma 3.4.2. *The space $H^1(\mathbb{R}^N;V)$ is a Hilbert space, having $C_c^\infty(\mathbb{R}^N)$ as a dense subspace. Moreover, $H^1(\mathbb{R}^N;V)$ is contained in $L^2(\mathbb{R}^N)$, with continuous inclusion. Finally, we have*

$$H^1(\mathbb{R}^N;V) = H^1(\mathbb{R}^N;|x|^2),$$

with equivalent norms.

Proof. The density of $C_c^\infty(\mathbb{R}^N)$ follows by using standard approximation techniques. This in particular implies that the Poincaré inequality of Proposition 3.3.5 holds for functions in $H^1(\mathbb{R}^N;V)$, by density. In turn, this gives the continuity of the inclusion $H^1(\mathbb{R}^N;V) \subseteq L^2(\mathbb{R}^N)$. Thus, the norm $\|\cdot\|_V$ is equivalent to the norm $\|\cdot\|_{H^1(\mathbb{R}^N)} + \|\cdot\|_{L^2(\mathbb{R}^N;V)}$. By using that both $H^1(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N;V)$ are Hilbert spaces, we conclude that $H^1(\mathbb{R}^N;V)$ has the same property. Finally, we have

$$H^1(\mathbb{R}^N;V) = H^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N;V) = H^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N;|x|^2) = H^1(\mathbb{R}^N;|x|^2),$$

thanks to Lemma 3.3.4. As for the equivalence of the norms, this is a plain consequence of the estimates of Lemma 3.3.4, together with the Poincaré inequality of Proposition 3.3.5. $\square$

Let us now consider the self-adjoint Schrödinger operator

$$\mathcal{H}_V : \mathfrak{D}(\mathcal{H}_V) \subseteq L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$$

defined by

$$\mathcal{H}_V[\varphi] := -\Delta\varphi + V\,\varphi,$$

with domain given by[1]

$$\mathfrak{D}(\mathcal{H}_V) = \Big\{\varphi \in H^1(\mathbb{R}^N; V) \,:\, -\Delta\varphi + V\,\varphi \in L^2(\mathbb{R}^N)\Big\}.$$

Then the quadratic form

$$\mathcal{Q}_V[\varphi,\varphi] = \int_{\mathbb{R}^N} |\nabla\varphi|^2\,dx + \int_{\mathbb{R}^N} V\,|\varphi|^2\,dx, \qquad \text{for } \varphi \in H^1(\mathbb{R}^N; V),$$

is *associated with the operator* $\mathcal{H}_V$, in the sense that

$$\mathfrak{D}(\mathcal{H}_V) \subseteq H^1(\mathbb{R}^N; V),$$

and

$$\int_{\mathbb{R}^N} \mathcal{H}_V[\varphi]\,\psi\,dx = \mathcal{Q}_V[\varphi,\psi], \qquad \text{for every } \varphi \in \mathfrak{D}(\mathcal{H}_V),\ \psi \in H^1(\mathbb{R}^N; V),$$

see [6, Chapter 10].

In order to prove that our operator has a discrete spectrum, it is necessary to verify that its resolvent operator is linear, continuous, self-adjoint, positive definite, and compact. Once these properties are established, the Spectral Theorem can be directly applied to obtain the desired result. In particular, the compactness of the resolvent operator follows from the compactness of the embedding of the form domain into $L^2(\mathbb{R}^N)$, namely

$$H^1(\mathbb{R}^N; V) \hookrightarrow L^2(\mathbb{R}^N).$$

This is the content of the next result, which proves a slightly more general assertion.

Theorem 3.4.3. *Let q be an exponent such that*

$$\begin{cases} \dfrac{2\,N}{N+2} < q < 2^*, & \text{if } N \geq 3, \\[2ex] 1 < q < +\infty, & \text{if } N = 2, \\[2ex] 1 \leq q \leq +\infty, & \text{if } N = 1. \end{cases}$$

Then the embedding

$$H^1(\mathbb{R}^N; V) \hookrightarrow L^q(\mathbb{R}^N),$$

is compact.

Proof. We first prove the result for the pivotal case $q = 2$. Then, we will show how the

---

[1] The condition on the Laplacian has to be intended in distributional sense, i.e. there exists $f \in L^2(\mathbb{R}^N)$ such that

$$\mathcal{Q}_V[\varphi,\varphi] = \int_{\mathbb{R}^N} f\,\varphi\,dx, \qquad \text{for every } \varphi \in C^\infty_c(\mathbb{R}^N).$$

In Theorem 3.7.2 below, we will determine explicitly the domain $\mathfrak{D}(\mathcal{H}_V)$.

remaining cases can be deduced from this.

Case $q = 2$. We have to prove that bounded sets in $H^1(\mathbb{R}^N; V)$ are relatively compact sets in $L^2(\mathbb{R}^N)$. We just need to show that the unit ball in $H^1(\mathbb{R}^N; V)$

$$\mathcal{F} = \left\{\varphi \in H^1(\mathbb{R}^N; V)\,:\, \|\varphi\|_V \le 1\right\},$$

is relatively compact in $L^2(\mathbb{R}^N)$. We will appeal to the classical Riesz–Fréchet–Kolmogorov Theorem and proceeds as in the proof of [6, Theorem 10.6.5] (for classical results, see also [9, 12, 36]), which concerns the one-dimensional case. We thus have to verify the following three properties:

- the equi-boundeness in L^2 norm

$$\sup_{\varphi \in \mathcal{F}} \|\varphi\|_{L^2(\mathbb{R}^N)} < +\infty;$$

- the equi-continuity in L^2 norm

$$\lim_{|\mathbf{h}| \to 0} \sup_{\varphi \in \mathcal{F}} \|\tau_{\mathbf{h}}\varphi - \varphi\|_{L^2(\mathbb{R}^N)} = 0;$$

- the uniform mass concentration, i.e. that for every $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that

$$\sup_{\varphi \in \mathcal{F}} \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}(0)} |\varphi|^2\, dx < \varepsilon.$$

The first condition is a plain consequence of Proposition 3.3.5, which also holds in $H^1(\mathbb{R}^N; V)$, thanks to Lemma 3.4.2. The second condition follows because by definition and Lemma 3.4.2 we get $H^1(\mathbb{R}^N; V) \subseteq H^1(\mathbb{R}^N)$, thus we can use the classical inequality for the translates of Sobolev functions (2.3.9) and ew get

$$\|\tau_{\mathbf{h}}\varphi - \varphi\|_{L^2(\mathbb{R}^N)} \le |\mathbf{h}|\, \|\nabla\varphi\|_{L^2(\mathbb{R}^N)} \le |\mathbf{h}|\, \|\varphi\|_V \le |\mathbf{h}|.$$

Finally, for the third condition, we only have to notice that for every $\varrho \ge 2\,R$, we have from inequality (3.3.1) with $\sigma = 2$

$$\int_{\mathbb{R}^N \setminus B_\varrho(0)} |\varphi|^2 dx \le 4 \int_{\mathbb{R}^N \setminus B_\varrho(0)} \frac{V}{|x|^2}\, |\varphi|^2 dx \le \frac{4}{\varrho^2}\, \|\varphi\|_V^2 \le \frac{4}{\varrho^2}, \qquad \text{for every } \varphi \in \mathcal{F}.$$

Thus, for every $\varepsilon > 0$, if we choose

$$\varrho > \max\left\{2\,R, \frac{2}{\sqrt{\varepsilon}}\right\},$$

we also get the third condition of the Riesz–Fréchet–Kolmogorov Theorem. This concludes the proof for the case $q = 2$.

Case $q > 2$. This is quite standard, let us give the details for the case $N \geq 3$, the other cases being similar. By interpolation in Lebesgue spaces (2.3.8) and Sobolev inequality (2.3.3), we have

$$\|\varphi\|_{L^q(\mathbb{R}^N)} \leq \left(\frac{1}{\sqrt{\mathcal{T}_N}}\right)^\theta \|\nabla\varphi\|^\theta_{L^2(\mathbb{R}^N)} \|\varphi\|^{1-\theta}_{L^2(\mathbb{R}^N)}, \qquad \text{for every } \varphi \in H^1(\mathbb{R}^N).$$

Here $\theta = \theta(N, q) \in (0, 1)$ is the exponent dictated by scale invariance. This shows that any sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ converging strongly in $L^2(\mathbb{R}^N)$ and having bounded weak gradients in $L^2(\mathbb{R}^N)$, strongly converges in $L^q(\mathbb{R}^N)$, as well. In light of the first part of the proof, this is enough to conclude.

Case $q < 2$. Let $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq H^1(\mathbb{R}^N; V)$ be a bounded sequence, i.e.

$$\left(\|\varphi_n\|^2_{L^2(\mathbb{R}^N;V)} + \|\nabla\varphi_n\|^2_{L^2(\mathbb{R}^N)}\right)^{\frac{1}{2}} \leq C, \qquad \text{for every } n \in \mathbb{N}. \tag{3.4.1}$$

From the first part of the proof, we know that it strongly converges in $L^2(\mathbb{R}^N)$, up to a subsequence. Let us call $\varphi \in L^2(\mathbb{R}^N)$ this limit function. By the lower semicontinuity of the L^2–norm with respect to the weak convergence, we still have $\varphi \in H^1(\mathbb{R}^N; V)$, with the bound (3.4.1). In particular, by Lemma 3.3.3 we have $\varphi \in L^q(\mathbb{R}^N)$, with $q < 2$ as in the statement. We then proceed as in the proof of Lemma 3.3.3: we get for every $\sigma > 1$ and every $n \in \mathbb{N}$

$$\begin{aligned}\int_{\mathbb{R}^N} |\varphi_n - \varphi|^q\,dx &= \int_{B_{\sigma R}(0)} |\varphi_n - \varphi|^q\,dx + \int_{\mathbb{R}^N\setminus B_{\sigma R}(0)} |\varphi_n - \varphi|^q\,dx \\ &\leq |B_{\sigma R}(0)|^{1-\frac{q}{2}} \left(\int_{B_{\sigma R}(0)} |\varphi_n - \varphi|^2\,dx\right)^{\frac{q}{2}} \\ &+ \left(\int_{\mathbb{R}^N\setminus B_{\sigma R}(0)} V\,|\varphi_n - \varphi|^2\,dx\right)^{\frac{q}{2}} \left(\int_{\mathbb{R}^N\setminus B_{\sigma R}(0)} \frac{1}{V^{\frac{q}{2-q}}}\,dx\right)^{\frac{2-q}{2}}.\end{aligned}$$

In particular, by using (3.4.1) and the strong convergence in $L^2(\mathbb{R}^N)$, we get for every $\sigma > 1$

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |\varphi_n - \varphi|^q\,dx \leq (2\,C)^q \left(\int_{\mathbb{R}^N\setminus B_{\sigma R}(0)} \frac{1}{V^{\frac{q}{2-q}}}\,dx\right)^{\frac{2-q}{2}}.$$

By letting σ go to $+\infty$ and using Lemma 3.3.1, we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\varphi_n - \varphi|^q\,dx = 0,$$

thanks to the fact that $q/(2 - q) > N/2$. This concludes the proof. □

The following Lemma establishes the remaining properties of the resolvent operator, namely linearity, self-adjointness, continuity, and positivity.

Lemma 3.4.4. *Let*

$$R : L^2(\mathbb{R}^N) \longrightarrow L^2(\mathbb{R}^N), \qquad f \mapsto R(f) = \varphi,$$

be the resolvent operator assigning to each $f \in L^2(\mathbb{R}^N)$ *the unique weak solution* $\varphi \in H^1(\mathbb{R}^N; V) \subseteq L^2(\mathbb{R}^N)$ *of*

$$-\Delta\varphi + V\varphi = f \qquad \text{in } \mathbb{R}^N.$$

Then R *is linear, continuous, self-adjoint, positive, and compact.*

Proof. Linearity. The linearity of R follows immediately from the linearity of the Schrödinger operator $\mathcal{H}_V = -\Delta + V$.

Continuity. Let $(f_n) \subset L^2(\mathbb{R}^N)$ be a bounded sequence, i.e. $\|f_n\|_{L^2(\mathbb{R}^N)} \leq \overline{C}$. Let $\varphi_n = R(f_n)$. Using the weak formulation of the equation with the test function $\varphi = \varphi_n$, we obtain

$$\int_{\mathbb{R}^N} |\nabla\varphi_n|^2 \, dx + \int_{\mathbb{R}^N} V|\varphi_n|^2 \, dx = \int_{\mathbb{R}^N} f_n\varphi_n \, dx.$$

Applying Young's inequality to the right-hand side,

$$\int_{\mathbb{R}^N} f_n\varphi_n \leq \frac{1}{\varepsilon^2}\|f_n\|^2_{L^2(\mathbb{R}^N)} + \varepsilon^2\|\varphi_n\|^2_{L^2(\mathbb{R}^N)}$$

that holds for every $\varepsilon > 0$. Using Poincaré-type inequality (3.3.5), the term $\|\varphi_n\|^2_{L^2(\mathbb{R}^N)}$ can be estimated by the left-hand side. Choosing $\varepsilon = \frac{1}{2}$, one absorbs the last two integrals, obtaining

$$\frac{3}{4}\int_{\mathbb{R}^N} |\nabla\varphi_n|^2 \, dx + \frac{3}{4}\int_{\mathbb{R}^N} V|\varphi_n|^2 \, dx \leq C\|f_n\|_{L^2(\mathbb{R}^N)}.$$

Hence $\|\varphi_n\|_{H^1(\mathbb{R}^N;V)} \leq C\|f_n\|_{L^2(\mathbb{R}^N)}$, proving the continuity of R.

Self-adjointness. Let $f, g \in L^2(\mathbb{R}^N)$. Setting $u = R(f)$ and $v = R(g)$, the weak formulation gives

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla v \, dx + \int_{\mathbb{R}^N} V uv \, dx = \int_{\mathbb{R}^N} f v \, dx = \langle f, R(g)\rangle_{L^2(\mathbb{R}^N)}.$$

Since the left-hand side is symmetric in u and v, it follows that

$$\langle R(f), g\rangle_{L^2(\mathbb{R}^N)} = \langle f, R(g)\rangle_{L^2(\mathbb{R}^N)},$$

i.e. $R = R^*$.

Positivity. For any $f \neq 0$,

$$\langle R(f), f\rangle_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\nabla R(f)|^2 \, dx + \int_{\mathbb{R}^N} V|R(f)|^2 \, dx > 0,$$

since both integrals are nonnegative and cannot vanish simultaneously unless $f = 0$.

Compactness. As mentioned above, the embedding

$$H^1(\mathbb{R}^N; V) \hookrightarrow L^2(\mathbb{R}^N)$$

is compact by the Theorem 3.4.3. Since R maps bounded sets of $L^2(\mathbb{R}^N)$ into bounded sets of $H^1(\mathbb{R}^N; V)$, the composition with the compact embedding yields that $R : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is compact. □

By classical results in Spectral Theory (see for example [9]), we have the following[2]

Corollary 3.4.5. *Under the standing assumptions, the spectrum of the operator $\mathcal{H}_V$ is made of a countable sequence of positive eigenvalues $\{\lambda_n(V)\}_{n\in\mathbb{N}^*}$ diverging to $+\infty$, each one having an associated eigenstate $\Psi_n \in H^1(\mathbb{R}^N; V) \cap \mathcal{S}_2(\mathbb{R}^N)$. The set $\{\Psi_n\}_{n\in\mathbb{N}^*}$ forms an orthonormal basis of $L^2(\mathbb{R}^N)$. Finally, each eigenvalue has the following variational characterization*

$$\lambda_n(V) = \inf \left\{ \max_{\varphi \in W \cap \mathcal{S}_2(\mathbb{R}^N)} \int_{\mathbb{R}^N} |\nabla\varphi|^2\, dx + \int_{\mathbb{R}^N} V\, |\varphi|^2\, dx \ : \ \begin{array}{c} W \subseteq H^1(\mathbb{R}^N; V) \\ \textit{subspace with } \dim W = n \end{array} \right\},$$

and the infimum above is attained by the vector space generated by $\{\Psi_1, \dots, \Psi_n\}$.

For the variational characterization of eigenvalues, the reader is also referred to a more general version of the Spectral Theorem, valid in the context of quadratic forms (see, for example, [6, Theorem 10.1.5]).

Remark 3.4.6 (The Ground State)**.** *In particular, we recall that for $n = 1$ the above formulation reduces to*

$$\lambda_1(V) = \min_{\varphi \in H^1(\mathbb{R}^N; V) \cap \mathcal{S}_2(\mathbb{R}^N)} \left\{ \int_{\mathbb{R}^N} |\nabla\varphi|^2\, dx + \int_{\mathbb{R}^N} V\, |\varphi|^2\, dx \right\}.$$

Thus, as usual $\lambda_1(V)$ coincides with the sharp constant in the Poincaré inequality of Proposition 3.3.5 and we have

$$\lambda_1(V) \geq \frac{1}{C_{N,R}}. \tag{3.4.2}$$

We also recall a couple of classical facts: any minimizer of the previous minimization problem must have constant sign, which is going to be strict by the minimum/maximum principle, and there exists a unique positive minimizer Ψ_1. This is also called the ground state of our operator. Thus, the first eigenvalue $\lambda_1(V)$ is simple, i.e. the corresponding eigenspace is one-dimensional. We refer for example to [36, Theorem 11.8] for these facts. In particular, by uniqueness the ground state Ψ_1 must inherit all the possible symmetries of the potential well Σ.

[2] As customary, we repeat each eigenvalue according to its multiplicity.

3.5 Stability of the Spectrum

In this section, we provide some estimates of spectral stability, establishing two-sided bounds on the difference $\lambda_k(V_1) - \lambda_k(V_2)$. Using an explicit estimate, we show that this difference can be controlled in terms of the Hausdorff distance between the potential wells Σ_1 and Σ_2.

We start with the following spectral continuity estimate that will be useful. It guarantees that, at least for potential wells small enough, our operator has a spectrum quantitatively close to that of the model case $V(x) = |x|^2$.

Lemma 3.5.1. *For every $n \in \mathbb{N}^*$ we have*

$$\sqrt{\lambda_n} - R \leq \sqrt{\lambda_n(V)} \leq \sqrt{\lambda_n},$$

where λ_n is the $n-$th eigenvalue of the quantum harmonic oscillator.

Proof. We pick a point $\overline{x} \in \Sigma$, by (3.2.4) for every $\varphi \in H^1(\mathbb{R}^N; V)$ we have that

$$\int_{\mathbb{R}^N} |\nabla \varphi|^2 \, dx + \int_{\mathbb{R}^N} V \, |\varphi|^2 \, dx \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2 \, dx + \int_{\mathbb{R}^N} |x - \overline{x}|^2 \, |\varphi|^2 \, dx. \tag{3.5.1}$$

By Corollary 3.4.5, the eigenvalues have the following variational characterizations

$$\lambda_n(V) = \inf \left\{ \max_{\varphi \in W \cap \mathcal{S}_2(\mathbb{R}^N)} \int_{\mathbb{R}^N} |\nabla \varphi|^2 \, dx + \int_{\mathbb{R}^N} V \, |\varphi|^2 \, dx \ : \ \begin{array}{c} W \subseteq H^1(\mathbb{R}^N; V) \\ \text{subspace with } \dim W = n \end{array} \right\},$$

and

$$\lambda_n = \inf \left\{ \max_{\varphi \in W \cap \mathcal{S}_2(\mathbb{R}^N)} \int_{\mathbb{R}^N} |\nabla \varphi|^2 \, dx + \int_{\mathbb{R}^N} |x|^2 \, |\varphi|^2 \, dx \ : \ \begin{array}{c} W \subseteq H^1(\mathbb{R}^N; |x|^2) \\ \text{subspace with } \dim W = n \end{array} \right\},$$

where we recall the notation

$$\mathcal{S}_2(\mathbb{R}^N) = \{\varphi \in L^2(\mathbb{R}^N) \ : \ \|\varphi\|_{L^2(\mathbb{R}^N)} = 1\}.$$

Thanks to the estimate above between the quadratic forms (3.5.1), to the fact that $H^1(\mathbb{R}^N; V) = H^1(\mathbb{R}^N; |x|^2)$ (see Lemma 3.4.2) and the translation invariance of the spectrum, we then get

$$\lambda_n(V) \leq \lambda_n.$$

Thus, we verified the upper bound.

In order to get the lower bound, we proceed similarly: it is sufficient to use (3.3.5). This gives an estimate between the relevant quadratic forms: more precisely, for every $\varphi \in$

$H^1(\mathbb{R}^N;|x|^2) = H^1(\mathbb{R}^N;V)$ we get

$$\int_{\mathbb{R}^N} |\nabla\varphi|^2\,dx + \int_{\mathbb{R}^N} |x|^2\,|\varphi|^2\,dx \le (1+\varepsilon)\left[\int_{\mathbb{R}^N} |\nabla\varphi|^2\,dx + \int_{\mathbb{R}^N} V\,|\varphi|^2\,dx\right] + \left(1+\frac{1}{\varepsilon}\right) R^2 \int_{\mathbb{R}^N} |\varphi|^2\,dx.$$

As before, this entails that

$$\lambda_n \le (1+\varepsilon)\,\lambda_n(V) + \left(1+\frac{1}{\varepsilon}\right) R^2.$$

This is valid for every $\varepsilon > 0$: upon choosing $\varepsilon = R/\sqrt{\lambda_n(V)}$, we get

$$\lambda_n \le \left(\sqrt{\lambda_n(V)} + R\right)^2.$$

This concludes the proof. □

Remark 3.5.2. *The previous estimate is quite useful for R close to 0. On the other hand, for R very large, the estimate may be quite inaccurate, even in the case when the parameter δ defined in (3.2.3) is large, as well. This depends on the geometry of the potential well Σ: for example, in the case (here $R > \delta$)*

$$\Sigma = \overline{B_R(0)} \setminus B_\delta(0) \qquad \textit{for which} \qquad V(x) = \min\left\{(|x|-R)_+^2, (|x|-\delta)_+^2\right\},$$

we have

$$\lim_{(R-\delta)\nearrow+\infty} \lambda_n(V) = 0, \qquad \textit{for every } n \in \mathbb{N}^*.$$

This can be easily seen, by observing that[3] $H_0^1(B_R(0) \setminus \overline{B_\delta(0)}) \subseteq H^1(\mathbb{R}^N;V)$ *and that*

$$\mathcal{Q}_V[\varphi,\varphi] = \int_{B_R(0)} |\nabla\varphi|^2\,dx, \qquad \textit{for every } \varphi \in H_0^1(B_R(0) \setminus \overline{B_\delta(0)}),$$

since the potential V identically vanishes on $B_R(0) \setminus \overline{B_\delta(0)}$. Thus, we obtain

$$\lambda_n(V) \le \lambda_{n,\mathrm{Dir}}\left(B_R(0) \setminus \overline{B_\delta(0)}\right) \le \lambda_{n,\mathrm{Dir}}\left(B_{\frac{R-\delta}{2}}(0)\right) = \left(\frac{2}{R-\delta}\right)^2 \lambda_{n,\mathrm{Dir}}(B_1(0)), \qquad \textit{for every } n \in \mathbb{N}^*.$$

Here, by $\lambda_{n,\mathrm{Dir}}$ we intend the $n-$th eigenvalue of the Dirichlet-Laplacian of an open set and we used that the spherical shell $B_R(0) \setminus \overline{B_\delta(0)}$ contains a ball with radius $(R-\delta)/2$. Observe that the previous example does not work, in the case the potential well Σ becomes "very large" and at the same time "very thin", i.e. R goes to $+\infty$ and the difference $R-\delta$ stays bounded. Indeed, in this case it may happen that the spectrum does not trivialize: we

[3] We intend that functions in $H_0^1(B_R(0) \setminus \overline{B_\delta(0)})$ are extended by zero, outside of the open set.

give an example in Section 5.

Using the Hausdorff distance recalled in (2.6.1), we establish the following continuity estimate, expressed in terms of the distance between the potential wells.

Proposition 3.5.3. *Let $\Sigma_1, \Sigma_2 \subseteq \mathbb{R}^N$ be two non-empty compact sets. We consider the corresponding potentials*

$$V_i(x) = (\operatorname{dist}(x, \Sigma_i))^2, \qquad \text{for } x \in \mathbb{R}^N.$$

Then, for every $n \in \mathbb{N}^*$ we have

$$|\lambda_n(V_1) - \lambda_n(V_2)| \leq \operatorname{dist}_H(\Sigma_1, \Sigma_2) \max\left\{4 + \frac{\lambda_n}{2}, 2\lambda_n\right\},$$

where λ_n is the $n-$th eigenvalue of the quantum harmonic oscillator.

Proof. We first observe that if $\operatorname{dist}_H(\Sigma_1, \Sigma_2) > 1$, the estimate trivially holds true. Indeed, by Lemma 3.5.1 we have

$$|\lambda_n(V_1) - \lambda_n(V_2)| \leq (\lambda_n(V_1) + \lambda_n(V_2)) \leq 2\lambda_n \leq 2\lambda_n \operatorname{dist}_H(\Sigma_1, \Sigma_2).$$

We can thus assume that $\operatorname{dist}_H(\Sigma_1, \Sigma_2) \leq 1$. As in the previous proof, it is sufficient to estimate the relevant quadratic forms. We observe that

$$|V_1 - V_2| = \left|\sqrt{V_1} - \sqrt{V_2}\right| \left(\sqrt{V_1} + \sqrt{V_2}\right) \leq \operatorname{dist}_H(\Sigma_1, \Sigma_2) \left(\sqrt{V_1} + \sqrt{V_2}\right),$$

thanks to the triangle inequality. Thus, for any $\varphi \in H^1(\mathbb{R}^N; V_1)$ we have that

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla\varphi|^2\,dx + \int_{\mathbb{R}^N} V_1\,|\varphi|^2\,dx &\leq \left(\int_{\mathbb{R}^N} |\nabla\varphi|^2\,dx + \int_{\mathbb{R}^N} V_2\,|\varphi|^2\,dx\right)\\ &\quad + \operatorname{dist}_H(\Sigma_1, \Sigma_2)\left(\int_{\mathbb{R}^N} \sqrt{V_1}\,|\varphi|^2\,dx + \int_{\mathbb{R}^N} \sqrt{V_2}\,|\varphi|^2\,dx\right).\end{aligned} \tag{3.5.2}$$

Observe that if φ has unit L^2 norm, we have by Hölder's and Young's inequalities

$$\begin{aligned}\operatorname{dist}_H(\Sigma_1, \Sigma_2) \int_{\mathbb{R}^N} \sqrt{V_1}\,|\varphi|^2\,dx &\leq \operatorname{dist}_H(\Sigma_1, \Sigma_2) \left(\int_{\mathbb{R}^N} V_1\,|\varphi|^2\,dx\right)^{\frac{1}{2}}\\ &\leq \frac{\varepsilon}{2} \int_{\mathbb{R}^N} V_1\,|\varphi|^2\,dx + \frac{1}{2\,\varepsilon} (\operatorname{dist}_H(\Sigma_1, \Sigma_2))^2.\end{aligned}$$

Similarly, we get for $0 < \varepsilon < 1$

$$\operatorname{dist}_H(\Sigma_1, \Sigma_2) \int_{\mathbb{R}^N} \sqrt{V_2}\,|\varphi|^2\,dx \leq \frac{\varepsilon\,(1-\varepsilon)}{2} \int_{\mathbb{R}^N} V_2\,|\varphi|^2\,dx + \frac{1}{2\,\varepsilon\,(1-\varepsilon)} (\operatorname{dist}_H(\Sigma_1, \Sigma_2))^2.$$

By inserting these estimates in (3.5.2), with simple algebraic manipulations we get

$$\left(1-\frac{\varepsilon}{2}\right)\left(\int_{\mathbb{R}^N}|\nabla\varphi|^2\,dx+\int_{\mathbb{R}^N}V_1\,|\varphi|^2\,dx\right)\le\left(1+\frac{\varepsilon\,(1-\varepsilon)}{2}\right)\left(\int_{\mathbb{R}^N}|\nabla\varphi|^2\,dx+\int_{\mathbb{R}^N}V_2\,|\varphi|^2\,dx\right)$$
$$+\frac{2-\varepsilon}{2\,\varepsilon\,(1-\varepsilon)}\,(\mathrm{dist}_H(\Sigma_1,\Sigma_2))^2\,.$$

Thanks to this and the fact that $H^1(\mathbb{R}^N;V_1)=H^1(\mathbb{R}^N;V_2)$ (see again Lemma 3.4.2), we then get

$$\begin{aligned}\lambda_n(V_1)&\le\frac{2+\varepsilon-\varepsilon^2}{2-\varepsilon}\,\lambda_n(V_2)+\frac{1}{\varepsilon\,(1-\varepsilon)}\,(\mathrm{dist}_H(\Sigma_1,\Sigma_2))^2\\&=(1+\varepsilon)\,\lambda_n(V_2)+\frac{1}{\varepsilon\,(1-\varepsilon)}\,(\mathrm{dist}_H(\Sigma_1,\Sigma_2))^2\,.\end{aligned}$$

By choosing

$$\varepsilon=\frac{1}{2}\,\mathrm{dist}_H(\Sigma_1,\Sigma_2),$$

we get

$$\lambda_n(V_1)-\lambda_n(V_2)\le\frac{\lambda_n(V_2)}{2}\,\mathrm{dist}_H(\Sigma_1,\Sigma_2)+\frac{4}{2-\mathrm{dist}_H(\Sigma_1,\Sigma_2)}\,\mathrm{dist}_H(\Sigma_1,\Sigma_2).$$

The eigenvalue $\lambda_n(V_2)$ can be bounded from above thanks to Lemma 3.5.1, so to get

$$\lambda_n(V_1)-\lambda_n(V_2)\le\mathrm{dist}_H(\Sigma_1,\Sigma_2)\,\left(4+\frac{\lambda_n}{2}\right).$$

By exchanging the role for V_1 and V_2, we conclude. □

3.6 Integrability of Eigenfunctions

In this section we prove some integrability properties of the eigenstates Ψ_n. Of course, Lemma 3.3.3 applies in particular to any $\varphi\in H^1(\mathbb{R}^N;V)$. However, we will show that this integrability information can be considerably improved. We give at first a classical global L^∞ estimate for the eigenstates of our operator.

Proposition 3.6.1. *We have $\Psi_n\in L^\infty(\mathbb{R}^N)$ for every $n\in\mathbb{N}^*$. More precisely, we have the following estimate*

$$\|\Psi_n\|_{L^\infty(\mathbb{R}^N)}\le\mathrm{M}_N\,\Big(\lambda_n(V)\Big)^{\frac{N}{4}},$$

where $\mathrm{M}_N>0$ is an explicit constant depending on the dimension only.

Proof. We divide the proof in three cases, according to the value of the dimension N. For $N\ge 2$, we will use a standard iterative technique *à la* Moser.

Case $N \geq 3$. For every $\beta \geq 1$ and $M > 0$, we define the non-negative continuous function

$$f_{\beta,M}(t) = \begin{cases} \beta\, M^{\beta-1}, & \text{if } |t| > M, \\ \beta\, |t|^{\beta-1}, & \text{if } |t| \leq M, \end{cases}$$

and take the function

$$F_{\beta,M}(t) = \int_0^t f_{\beta,M}(\tau)\, d\tau.$$

Observe that

$$0 \leq t\, F_{\beta,M}(t) \leq |t|^{\beta+1}, \qquad \text{for every } t \in \mathbb{R}. \tag{3.6.1}$$

Moreover, by construction, this is a C^1 function with bounded derivative. In particular, we still have

$$F_{\beta,M}(\Psi_n) \in H^1(\mathbb{R}^N; V).$$

We can thus use this function as a feasible test function in the weak formulation of the eigenvalue equation. We get

$$\int_{\mathbb{R}^N} f_{\beta,M}(\Psi_n)\, |\nabla \Psi_n|^2\, dx + \int_{\mathbb{R}^N} V\, \Psi_n\, F_{\beta,M}(\Psi_n)\, dx = \lambda_n(V) \int_{\mathbb{R}^N} \Psi_n\, F_{\beta,M}(\Psi_n)\, dx.$$

In particular, since V is non-negative, we get

$$\int_{\mathbb{R}^N} f_{\beta,M}(\Psi_n)\, |\nabla \Psi_n|^2\, dx \leq \lambda_n(V) \int_{\mathbb{R}^N} \Psi_n\, F_{\beta,M}(\Psi_n)\, dx.$$

We now introduce the function

$$G_{\beta,M}(t) = \int_0^t \sqrt{f_{\beta,M}(\tau)}\, d\tau,$$

then the last estimate can be rewritten as

$$\int_{\mathbb{R}^N} |\nabla (G_{\beta,M} \circ \Psi_n)|^2\, dx \leq \lambda_n(V) \int_{\mathbb{R}^N} \Psi_n\, F_{\beta,M}(\Psi_n)\, dx. \tag{3.6.2}$$

Observe that by construction $G_{\beta,M}$ is again a C^1 function, with bounded derivative. Thus $(G_{\beta,M} \circ \Psi_n) \in H^1(\mathbb{R}^N)$ and we can apply the Sobolev inequality (2.3.3), so to obtain

$$\mathcal{T}_N \left(\int_{\mathbb{R}^N} |G_{\beta,M}(\Psi_n)|^{2^*}\, dx \right)^{\frac{2}{2^*}} \leq \lambda_n(V) \int_{\mathbb{R}^N} \Psi_n\, F_{\beta,M}(\Psi_n)\, dx. \tag{3.6.3}$$

In order to get that $\Psi_n \in L^q(\mathbb{R}^N)$ for every $2 \leq q < +\infty$, we define the sequence of exponents $\{\vartheta_i\}_{i \in \mathbb{N}}$ as follows

$$\vartheta_0 = 1, \qquad \vartheta_{i+1} = \frac{2^*}{2}\, \vartheta_i = \left(\frac{N}{N-2} \right)^{i+1}.$$

We then prove the following implication

$$\Psi_n \in L^{2\,\vartheta_i}(\mathbb{R}^N) \qquad \Longrightarrow \qquad \Psi_n \in L^{2\,\vartheta_{i+1}}(\mathbb{R}^N). \tag{3.6.4}$$

Indeed, let us suppose that $\Psi_n \in L^{2\,\vartheta_i}(\mathbb{R}^N)$. We use (3.6.3) with $\beta = 2\,\vartheta_i - 1$, so to obtain

$$\mathcal{T}_N \left(\int_{\mathbb{R}^N} |G_{\beta,M}(\Psi_n)|^{2^*}\,dx\right)^{\frac{2}{2^*}} \le \lambda_n(V) \int_{\mathbb{R}^N} \Psi_n\, F_{\beta,M}(\Psi_n)\,dx \le \lambda_n(V) \int_{\mathbb{R}^N} |\Psi_n|^{2\vartheta_i}\,dx.$$

We also used (3.6.1), in the second inequality. We can now pass to the limit as M goes to $+\infty$ on the left-hand side: by observing that

$$|G_{\beta,M}(t)| \le \frac{2\sqrt{\beta}}{\beta+1}\,|t|^{\frac{1+\beta}{2}} \qquad \text{and} \qquad \lim_{M\to+\infty} |G_{\beta,M}(t)| = \frac{2\sqrt{\beta}}{\beta+1}\,|t|^{\frac{1+\beta}{2}}, \quad \text{for every } t \in \mathbb{R},$$

an application of Fatou's Lemma leads to

$$\frac{2\,\vartheta_i - 1}{\vartheta_i^2}\,\mathcal{T}_N \left(\int_{\mathbb{R}^N} |\Psi_n|^{2^*\vartheta_i}\,dx\right)^{\frac{2}{2^*}} \le \lambda_n(V) \int_{\mathbb{R}^N} |\Psi_n|^{2\vartheta_i}\,dx.$$

By using that $2^*\vartheta_i = 2\,\vartheta_{i+1}$ and the fact that $\Psi_n \in L^{2\,\vartheta_i}(\mathbb{R}^N)$ we get $\Psi_n \in L^{2\,\vartheta_{i+1}}(\mathbb{R}^N)$, as desired. By observing that ϑ_i is an increasing sequence diverging to $+\infty$ and that $\Psi_n \in L^{2\,\vartheta_0}(\mathbb{R}^N)$ by assumption (observe that $2\,\vartheta_0 = 2$), we can iterate (3.6.4) as many times as we wish, so to get

$$\Psi_n \in L^q(\mathbb{R}^N), \qquad \text{for every } 2 \le q < +\infty.$$

We are now ready to prove the claimed L^∞ bound. We keep the same notation as in the previous part. We have seen that

$$\left(\int_{\mathbb{R}^N} |\Psi_n|^{2\,\vartheta_{i+1}}\,dx\right)^{\frac{2}{2^*}} \le \frac{\lambda_n(V)}{\mathcal{T}_N}\,\vartheta_i \int_{\mathbb{R}^N} |\Psi_n|^{2\,\vartheta_i}\,dx, \tag{3.6.5}$$

for every $i \in \mathbb{N}$. Observe that we used

$$\frac{2\,\vartheta_i - 1}{\vartheta_i^2} \ge \frac{1}{\vartheta_i}.$$

We take the power $1/(2\,\vartheta_i)$ on both sides of (3.6.5), so to get

$$\left(\int_{\mathbb{R}^N} |\Psi_n|^{2\,\vartheta_{i+1}}\,dx\right)^{\frac{1}{2\,\vartheta_{i+1}}} \le \left(\frac{\lambda_n(V)}{\mathcal{T}_N}\right)^{\frac{1}{2\,\vartheta_i}} \vartheta_i^{\frac{1}{2\,\vartheta_i}} \left(\int_{\mathbb{R}^N} |\Psi_n|^{2\,\vartheta_i}\,dx\right)^{\frac{1}{2\,\vartheta_i}}. \tag{3.6.6}$$

We now iterate k times (3.6.6), by starting from $i = 0$: this gives

$$\left(\int_{\mathbb{R}^N} |\Psi_n|^{2\,\vartheta_{k+1}}\,dx\right)^{\frac{1}{2\,\vartheta_{k+1}}} \le \left(\frac{\lambda_n(V)}{\mathcal{T}_N}\right)^{\sum\limits_{i=0}^{k} \frac{1}{2\vartheta_i}} \prod_{i=0}^{k} \vartheta_i^{\frac{1}{2\,\vartheta_i}} \left(\int_{\mathbb{R}^N} |\Psi_n|^2\,dx\right)^{\frac{1}{2}}. \tag{3.6.7}$$

Observe that

$$\lim_{k\to\infty}\sum_{i=0}^{k}\frac{1}{2\vartheta_i}=\frac{1}{2}\sum_{i=0}^{\infty}\left(\frac{N-2}{N}\right)^i=\frac{1}{2}\,\frac{1}{1-\dfrac{N-2}{N}}=\frac{N}{4},$$

and that

$$\lim_{k\to\infty}\prod_{i=0}^{k}\vartheta_i^{\frac{1}{2\,\vartheta_i}}=:C_N<+\infty.$$

This shows that we can take the limit as k goes to ∞ in (3.6.7). By further observing that

$$\lim_{k\to\infty}\left(\int_{\mathbb{R}^N}|\Psi_n|^{2\,\vartheta_{k+1}}\,dx\right)^{\frac{1}{2\,\vartheta_{k+1}}}=\|\Psi_n\|_{L^\infty(\mathbb{R}^N)},$$

we obtain

$$\|\Psi_n\|_{L^\infty(\mathbb{R}^N)}\le C\left(\frac{\lambda_n(V)}{\mathcal{T}_N}\right)^{\frac{N}{4}}\|\Psi_n\|_{L^2(\mathbb{R}^N)}.$$

By recalling that Ψ_n has unit L^2 norm, we conclude the proof in the case $N\ge 3$.

Case $N=2$. We already know that $\Psi_n\in L^q(\mathbb{R}^2)$, for every $2\le q<+\infty$, thanks to (2.3.4). By keeping the same notation as before, we go back to (3.6.2) and multiply both sides by

$$\int_{\mathbb{R}^2}|G_{\beta,M}(\Psi_n)|^2\,dx.$$

This yields

$$\left(\int_{\mathbb{R}^2}|\nabla(G_{\beta,M}\circ\Psi_n)|^2\,dx\right)\left(\int_{\mathbb{R}^2}|G_{\beta,M}(\Psi_n)|^2\,dx\right)$$
$$\le\lambda_n(V)\left(\int_{\mathbb{R}^2}|\Psi_n|^{\beta+1}\,dx\right)\left(\int_{\mathbb{R}^2}|G_{\beta,M}(\Psi_n)|^2\,dx\right),$$

where we also used (3.6.1). We apply (2.3.4) with $q=4$ on the left hand-side, so to get

$$\pi\int_{\mathbb{R}^2}|G_{\beta,M}(\Psi_n)|^4\,dx\le\lambda_n(V)\left(\int_{\mathbb{R}^2}|\Psi_n|^{\beta+1}\,dx\right)\left(\int_{\mathbb{R}^2}|G_{\beta,M}(\Psi_n)|^2\,dx\right).$$

We now take the limit as M goes to $+\infty$ and get

$$\pi\left(\frac{2\sqrt{\beta}}{\beta+1}\right)^2\int_{\mathbb{R}^2}|\Psi_n|^{4\frac{\beta+1}{2}}\,dx\le\lambda_n(V)\left(\int_{\mathbb{R}^2}|\Psi_n|^{\beta+1}\,dx\right)^2.$$

This time, we define the sequence of exponents $\{\vartheta_i\}_{i\in\mathbb{N}}$ as follows

$$\vartheta_0=1,\qquad \vartheta_{i+1}=2\,\vartheta_i=2^{i+1},$$

and use the above estimate with $\beta = 2\,\vartheta_i - 1$, again. This gives

$$\pi\,\frac{(2\,\vartheta_i - 1)}{\vartheta_i^2}\int_{\mathbb{R}^2} |\Psi_n|^{2\vartheta_{i+1}}\,dx \le \lambda_n(V)\left(\int_{\mathbb{R}^2} |\Psi_n|^{2\vartheta_i}\,dx\right)^2.$$

We can now proceed as in the case $N \ge 3$ and get the desired conclusion.

Case $N = 1$. This is the simplest case, the required result follows immediately from (2.3.5). □

Remark 3.6.2. *By using the estimate of Lemma 3.5.1, we can also infer the following uniform estimate*

$$\|\Psi_n\|_{L^\infty(\mathbb{R}^N)} \le \mathrm{M}_N\,\lambda_n^{\frac{N}{4}}, \qquad \textit{for every } n \in \mathbb{N}^*,$$

where λ_n is the $n-th$ eigenvalue of the quantum harmonic oscillator. Observe that the upper bound does not depend on the potential well Σ.

We now record a weighted integrability result on eigenstates. Apart for being interesting in itself, it will be useful in the sequel.

Proposition 3.6.3. *For every $n \in \mathbb{N}^*$ and $k \in \mathbb{N}$, we have*

$$\int_{\mathbb{R}^N} |\nabla\Psi_n|^2\,|x|^{2\,k}\,dx + \int_{\mathbb{R}^N} |\Psi_n|^2\,|x|^{2\,k+2}\,dx \le C_{N,n,k,R}.$$

Proof. We observe at first that the statement is true for $k = 0$, i.e. we have

$$\int_{\mathbb{R}^N} |\nabla\Psi_n|^2\,dx + \int_{\mathbb{R}^N} |\Psi_n|^2\,|x|^2\,dx \le C_{N,n,R}.$$

Indeed, we have

$$\int_{\mathbb{R}^N} |\Psi_n|^2\,|x|^2\,dx < +\infty,$$

directly from Lemma 3.3.4. More precisely, from (3.3.6) with $\sigma = 2$ we have

$$\begin{aligned}\int_{\mathbb{R}^N} |x|^2\,|\Psi_n|^2\,dx &\le 4\,R^2\int_{B_{2R}(0)} |\Psi_n|^2\,dx + 4\int_{\mathbb{R}^N\setminus B_{2R}(0)} V\,|\Psi_n|^2\,dx\\ &\le 4\,R^2 + 4\,\lambda_n(V).\end{aligned}$$

On account of Lemma 3.5.1, this gives

$$\int_{\mathbb{R}^N} |\Psi_n|^2\,|x|^2\,dx \le 4\,R^2 + 4\,\lambda_n. \tag{3.6.8}$$

In order to conclude, we will prove the following recursive gain of weighted integrability
(3.6.9)

$$\begin{aligned}&\textit{if we have} \quad \int_{\mathbb{R}^N} |\nabla\Psi_n|^2\,|x|^{2\,k}\,dx + \int_{\mathbb{R}^N} |\Psi_n|^2\,|x|^{2\,k+2}\,dx \le C_{N,n,k,R},\\ &\textit{then we have} \quad \int_{\mathbb{R}^N} |\nabla\Psi_n|^2\,|x|^{2\,k+2}\,dx + \int_{\mathbb{R}^N} |\Psi_n|^2\,|x|^{2\,k+4}\,dx \le C_{N,n,k+1,R} \quad \textit{as well.}\end{aligned}$$

This will be sufficient to get the claimed result. Let us suppose that for a $k \in \mathbb{N}$ we have

$$\int_{\mathbb{R}^N} |\nabla \Psi_n|^2 \, |x|^{2\,k} \, dx + \int_{\mathbb{R}^N} |\Psi_n|^2 \, |x|^{2\,k+2} \, dx \le C_{N,n,k,R}.$$

Let $M > 0$ and let η_M be a Lipschitz cut-off function such that

$$0 \le \eta_M \le 1, \quad \eta_M \equiv 1 \text{ on } B_M(0), \quad \eta_M \equiv 0 \text{ on } \mathbb{R}^N \setminus B_{M+1}(0),$$

and

$$|\nabla \eta_M| \le 1.$$

We then take the test function

$$\varphi = \Psi_n \, |x|^{2\,k+2} \, \eta_M^2,$$

in the weak formulation. Observe that this is feasible, thanks to the properties of both Ψ_n and η_M. We get

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla \Psi_n|^2 \, |x|^{2\,k+2} \, \eta_M^2 \, dx &+ \int_{\mathbb{R}^N} V \, |\Psi_n|^2 \, |x|^{2\,k+2} \, \eta_M^2 \, dx \\
&= \lambda_n(V) \int_{\mathbb{R}^N} |\Psi_n|^2 \, |x|^{2\,k+2} \, \eta_M^2 \, dx \\
&- (2\,k+2) \int_{\mathbb{R}^N} \langle \nabla \Psi_n, x \rangle \, \Psi_n \, |x|^{2\,k} \, \eta_M^2 \, dx \\
&- 2 \int_{\mathbb{R}^N} \langle \nabla \Psi_n, \nabla \eta_M \rangle \, \Psi_n \, |x|^{2\,k+2} \, \eta_M \, dx.
\end{aligned}$$

By using Young's inequality on the last two integrals, for every $M > 0$ and $\delta > 0$ we get

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla \Psi_n|^2 \, |x|^{2\,k+2} \, \eta_M^2 \, dx + \int_{\mathbb{R}^N} V \, |\Psi_n|^2 \, |x|^{2\,k+2} \, \eta_M^2 \, dx &\le \lambda_n(V) \int_{\mathbb{R}^N} |\Psi_n|^2 \, |x|^{2\,k+2} \, \eta_M^2 \, dx \\
&+ \delta \int_{\mathbb{R}^N} |\nabla \Psi_n|^2 \, |x|^{2\,k+2} \, \eta_M^2 \, dx \\
&+ \frac{(k+1)^2}{\delta} \int_{\mathbb{R}^N} |\Psi_n|^2 \, |x|^{2\,k} \, \eta_M^2 \, dx \\
&+ \delta \int_{\mathbb{R}^N} |\nabla \Psi_n|^2 \, |x|^{2\,k+2} \, \eta_M^2 \, dx \\
&+ \frac{1}{\delta} \int_{\mathbb{R}^N} |\nabla \eta_M|^2 \, |x|^{2\,k+2} \, |\Psi_n|^2 \, dx.
\end{aligned}$$

We can take $\delta = 1/4$ so to absorb the two integrals in the right-hand side containing $\nabla \Psi_n$.

This gives

$$\frac{1}{2}\int_{\mathbb{R}^N} |\nabla \Psi_n|^2\,|x|^{2\,k+2}\,\eta_M^2\,dx + \int_{\mathbb{R}^N} V\,|\Psi_n|^2\,|x|^{2\,k+2}\,\eta_M^2\,dx \le \lambda_n(V)\int_{\mathbb{R}^N} |\Psi_n|^2\,|x|^{2\,k+2}\,\eta_M^2\,dx$$
$$+\,4\,(k+1)^2\int_{\mathbb{R}^N} |\Psi_n|^2\,|x|^{2\,k}\,\eta_M^2\,dx$$
$$+\,4\int_{\mathbb{R}^N} |\nabla \eta_M|^2\,|x|^{2\,k+2}\,|\Psi_n|^2\,dx.$$

Thanks to the properties of η_M, this implies that

$$\begin{aligned}\frac{1}{2}\int_{B_M(0)} |\nabla \Psi_n|^2\,|x|^{2\,k+2}\,dx &+ \int_{B_M(0)} V\,|\Psi_n|^2\,|x|^{2\,k+2}\,dx\\ &\le (\lambda_n+4)\int_{\mathbb{R}^N} |\Psi_n|^2\,|x|^{2\,k+2}\,dx\\ &+ 4\,(k+1)^2\int_{\mathbb{R}^N} |\Psi_n|^2\,|x|^{2\,k}\,dx.\end{aligned} \tag{3.6.10}$$

We used again Lemma 3.5.1, to bound the eigenvalue. We observe that by Hölder's inequality and the fact that Ψ_n has unit L^2 norm, we obtain

$$\int_{\mathbb{R}^N} |\Psi_n|^2\,|x|^{2\,k}\,dx \le \left(\int_{\mathbb{R}^N} |\Psi_n|^2\,|x|^{2\,k+2}\,dx\right)^{\frac{k}{k+1}} \le (C_{N,n,k,R})^{\frac{k}{k+1}}\,.$$

By letting M go to $+\infty$ in (3.6.10) and using Lemma 3.3.4 for $\varphi = \Psi_n\,|x|^{k+1}$, we prove (3.6.9). As already explained, this concludes the proof. □

As a consequence of the previous result, we also get the following

Corollary 3.6.4. *For every $n \in \mathbb{N}^*$ and every $2 \le p < \infty$ we have*

$$V\,\Psi_n \in L^p(\mathbb{R}^N).$$

More precisely, there exists a constant $C = C(N,n,R,p) > 0$ such that

$$\int_{\mathbb{R}^N} V^p\,|\Psi_n|^p\,dx \le C, \qquad \textit{for every } R \ge 0.$$

Proof. For every $2 \le p < +\infty$, we have

$$\int_{\mathbb{R}^N} V^p\,|\Psi_n|^p\,dx \le \|\Psi_n\|_{L^\infty(\mathbb{R}^N)}^{p-2}\int_{\mathbb{R}^N} |V|^p\,|\Psi_n|^2\,dx.$$

By using (3.2.4) and Remark 3.6.2, we obtain

$$\int_{\mathbb{R}^N} V^p\,|\Psi_n|^p\,dx \le \left(\mathrm{M}_N\,\lambda_n^{\frac{N}{4}}\right)^{p-2}\int_{\mathbb{R}^N} (|x|+R)^{2\,p}\,|\Psi_n|^2\,dx.$$

The last term can be bounded from above, thanks to Proposition 3.6.3. □

3.7 Higher Regularity of Eigenfunctions

This section is devoted to the study of the higher regularity of the eigenfunctions. In particular, we establish H^2–regularity results, which allow for an explicit characterization of the domain of the electric Schrödinger operator. Building on these results, we further present classical regularity properties, obtained through standard H^2–regularity theorems, such as the Calderón–Zygmund and Schauder estimates.

The following simple result ensures that a certain test function is admissible.

Lemma 3.7.1. *For every* $\mathbf{h} \in \mathbb{R}^N$ *we have*

$$L^2_{\mathrm{loc}}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N; V) = L^2_{\mathrm{loc}}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N; \tau_{\mathbf{h}} V),$$

and

$$H^1(\mathbb{R}^N; V) = H^1(\mathbb{R}^N; \tau_{\mathbf{h}} V).$$

Proof. It is sufficient to observe that

$$\tau_{\mathbf{h}} V(x) = \Big(\mathrm{dist}(x + \mathbf{h}, \Sigma)\Big)^2 = \Big(\mathrm{dist}(x, \Sigma - \mathbf{h})\Big)^2.$$

Thus the potential $\tau_{\mathbf{h}} V$ still belongs to the same class under consideration. In light of Lemma 3.3.4, this is enough to get the first equality. The second one follows by using the same observation and Lemma 3.4.2. □

Our eigenstates belong to a higher order Sobolev space. This is the content of the following slightly more general result.

Theorem 3.7.2. *For* $f \in L^2(\mathbb{R}^N)$*, let* $\Psi \in H^1(\mathbb{R}^N; V)$ *be the weak solution of*

$$\mathcal{H}_V[\Psi] = f.$$

Then we have

$$\Psi \in H^2(\mathbb{R}^N) \cap \Big\{\psi \in L^2(\mathbb{R}^N) \,:\, V\,\psi \in L^2(\mathbb{R}^N)\Big\}.$$

In particular, the domain of $\mathcal{H}_V$ *is given by*

$$\mathfrak{D}(\mathcal{H}_V) = H^2(\mathbb{R}^N) \cap \Big\{\psi \in L^2(\mathbb{R}^N) \,:\, V\,\psi \in L^2(\mathbb{R}^N)\Big\}.$$

Proof. We know that $\Psi \in H^1(\mathbb{R}^N; V) \subseteq L^2(\mathbb{R}^N)$, thanks to Lemma 3.4.2. We first need to prove that

$$\int_{\mathbb{R}^N} V^2\, |\Psi|^2\, dx < +\infty.$$

In light of Lemma 3.3.1, it is sufficient to prove that

$$\int_{\mathbb{R}^N} |x|^4\, |\Psi|^2\, dx < +\infty. \tag{3.7.1}$$

This can be proved by repeating almost verbatim the argument of the proof of Proposition 3.6.3: we only need to treat more carefully the right-hand side. As before, let $M > 0$ and let η_M be a Lipschitz cut-off function such that

$$0 \le \eta_M \le 1, \quad \eta_M \equiv 1 \text{ on } B_M(0), \quad \eta_M \equiv 0 \text{ on } \mathbb{R}^N \setminus B_{M+1}(0),$$

and

$$|\nabla \eta_M| \le 1.$$

Upon testing the equation with $\psi = \Psi\, |x|^2\, \eta_M^2$, we get

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla\Psi|^2\, |x|^2\, \eta_M^2\, dx + \int_{\mathbb{R}^N} V\, |\Psi|^2\, |x|^2\, \eta_M^2\, dx = & \int_{\mathbb{R}^N} f\, \Psi\, |x|^2\, \eta_M^2\, dx \\ & - 2 \int_{\mathbb{R}^N} \langle \nabla\Psi, x\rangle\, \Psi\, \eta_M^2\, dx \\ & - 2 \int_{\mathbb{R}^N} \langle \nabla\Psi, \nabla\eta_M\rangle\, \Psi\, |x|^2\, \eta_M\, dx.\end{aligned}$$

By using Young's inequality on the right-hand side and (3.3.5) with $\varepsilon = 1$ on the left-hand side, for every $M > 0$ and $\delta > 0$ we get

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla\Psi|^2\, |x|^2\, \eta_M^2\, dx + \frac{1}{2} \int_{\mathbb{R}^N} |\Psi|^2\, |x|^4\, \eta_M^2\, dx \le & \, R^2 \int_{\mathbb{R}^N} |x|^2\, |\Psi|^2\, \eta_M^2\, dx + \frac{1}{2\,\delta} \int_{\mathbb{R}^N} |f|^2\, \eta_M^2\, dx \\ & + \frac{\delta}{2} \int_{\mathbb{R}^N} |\Psi|^2\, |x|^4\, \eta_M^2\, dx \\ & + \delta \int_{\mathbb{R}^N} |\nabla\Psi|^2\, |x|^2\, \eta_M^2\, dx + \frac{1}{\delta} \int_{\mathbb{R}^N} |\Psi|^2\, \eta_M^2\, dx \\ & + \delta \int_{\mathbb{R}^N} |\nabla\Psi|^2\, |x|^2\, \eta_M^2\, dx \\ & + \frac{1}{\delta} \int_{\mathbb{R}^N} |\nabla\eta_M|^2\, |x|^2\, |\Psi|^2\, dx.\end{aligned}$$

If we now take $\delta = 1/4$, we can absorb the two integrals in the right-hand side containing $\nabla\Psi$, as well as the term $|\Psi|^2\, |x|^4$. This gives

$$\begin{aligned}\frac{1}{2} \int_{\mathbb{R}^N} |\nabla\Psi|^2\, |x|^2\, \eta_M^2\, dx + \frac{3}{8} \int_{\mathbb{R}^N} |\Psi|^2\, |x|^4\, \eta_M^2\, dx \le & \, R^2 \int_{\mathbb{R}^N} |x|^2\, |\Psi|^2\, \eta_M^2\, dx + 2 \int_{\mathbb{R}^N} |f|^2\, \eta_M^2\, dx \\ & + 4 \int_{\mathbb{R}^N} |\Psi|^2\, \eta_M^2\, dx \\ & + 4 \int_{\mathbb{R}^N} |\nabla\eta_M|^2\, |x|^2\, |\Psi|^2\, dx.\end{aligned}$$

Thanks to the properties of η_M, this implies that

$$\begin{aligned}\frac{1}{2} \int_{B_M(0)} |\nabla\Psi|^2\, |x|^2\, dx + \frac{3}{8} \int_{B_M(0)} |\Psi|^2\, |x|^4\, dx \le & \, (4 + R^2) \int_{\mathbb{R}^N} |x|^2\, |\Psi|^2\, dx + 2 \int_{\mathbb{R}^N} |f|^2\, dx \\ & + 4 \int_{\mathbb{R}^N} |\Psi|^2\, dx.\end{aligned}$$

By letting M go to $+\infty$ and recalling Lemma 3.3.4, we get (3.7.1).

For the H^2-regularity of the eigenfunctions, we use the Calderón–Zygmund Theorem 2.5.1. Indeed, if we set

$$F = f - V\Psi,$$

then the function Ψ weakly solves the Poisson equation

$$-\Delta\Psi = F \in L^2(\mathbb{R}^N),$$

and it is possible to use the cited Theorem.

As for the domain of $\mathcal{H}_V$, the first part of the proof and Lemma 3.4.2 give

$$\mathfrak{D}(\mathcal{H}_V) = \Big\{\psi \in H^1(\mathbb{R}^N; V) \,:\, -\Delta\psi + V\,\psi \in L^2(\mathbb{R}^N)\Big\} \subseteq H^2(\mathbb{R}^N) \cap \Big\{\psi \in L^2(\mathbb{R}^N) \,:\, V\,\psi \in L^2(\mathbb{R}^N)\Big\}.$$

In order to prove the reverse inclusion, it is sufficient to prove that

$$\Big\{\psi \in L^2(\mathbb{R}^N) \,:\, V\,\psi \in L^2(\mathbb{R}^N)\Big\} \subseteq L^2(\mathbb{R}^N; V).$$

This easily follows from Hölder's inequality

$$\int_{\mathbb{R}^N} V\,|\psi|^2\,dx \le \left(\int_{\mathbb{R}^N} |\psi|^2\,dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} V^2\,|\psi|^2\,dx\right)^{\frac{1}{2}},$$

thus concluding the proof. □

As for classical regularity of eigenstates, we have the following result.

Proposition 3.7.3. *We have $\Psi_n \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ for every $0 < \alpha < 1$. In particular, Ψ_n solves the eigenvalue equation*

$$-\Delta\Psi_n + V\,\Psi_n = \lambda_n(V)\,\Psi_n, \qquad \textit{in } \mathbb{R}^N,$$

in classical sense.

Proof. Let us set

$$F = \lambda_n(V)\,\Psi_n - V\,\Psi_n.$$

Thanks to Proposition 3.6.1 and Corollary 3.6.4, we have $F \in L^q(\mathbb{R}^N)$ for every $2 \le q < +\infty$. Thus we have

$$-\Delta\Psi_n = F \in L^q(\mathbb{R}^N), \qquad \text{for every } 2 \le q < +\infty.$$

By the classical *Calderón-Zygmund estimates* (see for example [16, Theorem 9.9]), this implies that $\Psi_n \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N)$, for every $1 < p < +\infty$. By the *Sobolev Embedding Theorem*, this in turn implies that $\Psi_n \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$, for every $0 < \alpha < 1$. By using this regularity gain and recalling that V is locally Lipschitz, we can then infer that

$$-\Delta\Psi_n = F \in C^{0,\alpha}_{\rm loc}(\mathbb{R}^N),$$

for every $0 < \alpha < 1$. An application of *Schauder's estimates* (see for example [16, Corollary 6.3]) now gives the claimed regularity. □

Remark 3.7.4 (Maximal regularity). *In general, we can not expect the eigenstates Ψ_n to belong to C^3. Indeed, let us consider the positive ground state Ψ_1 and let us suppose that the potential V is not C^1 at the origin: this happens for example when the potential well Σ is given by the torus (3.2.7). We argue by contradiction and assume that $\Psi_1 \in C^3_{\rm loc}(\mathbb{R}^N)$. In particular, the Laplacian $\Delta\Psi_1$ would be differentiable at $x = 0$. Since Ψ_1 is a classical solution of the equation, we get the pointwise identity*

$$V = \lambda_1(V) + \frac{\Delta\Psi_1}{\Psi_1},$$

where we also used that $\Psi_1 > 0$, by the minimum principle, as already observed. The previous identity would imply that V is differentiable at the origin, while this is not the case. We thus obtain a contradiction.

3.8 Exponential Decay at Infinity

In this section, we will show exponential decay for the eigenstates, in the sup norm. This is quite a classical result (see for example [2, 24, 48]). Our proof is elementary and relies only on the weak form of the equation, in conjunction with suitable test function arguments. The result will follow by applying iterative arguments *à la* De Giorgi and Moser, thus the proof is genuinely nonlinear in nature.

As a preliminary result, we start with an exponential decay in the L^2 norm. We pay due attention to the constants involved in the estimate.

Lemma 3.8.1 (Exponential decay in L^2). *There exist an exponent $\alpha = \alpha(N) > 1$ and a constant $C_1 = C_1(N, n, R) > 0$ such that*

$$\int_{\mathbb{R}^N\setminus B_\varrho(0)} |\Psi_n|^2\,dx \le C_1\,e^{-\varrho\log\alpha}, \qquad \textit{for every } \varrho \ge 0.$$

Proof. We first observe that it is sufficient to prove that there exist $\alpha = \alpha(N) > 1$, $R_0 = R_0(N, n, R) > 0$ and $C = C(N, n, R) > 0$ such that

$$\int_{\mathbb{R}^N\setminus B_\varrho(0)} |\Psi_n|^2\,dx \le C\,e^{-\varrho\log\alpha}, \qquad \text{for every } \varrho \ge R_0. \tag{3.8.1}$$

Indeed, for $0 \leq \varrho < R_0$ we would trivially have

$$\int_{\mathbb{R}^N\setminus B_\varrho(0)} |\Psi_n|^2\,dx \leq 1 \leq \frac{e^{-\varrho\,\log\alpha}}{e^{-R_0\,\log\alpha}},$$

and thus the claimed estimate would follow by taking

$$C_1 = \max\left\{C, e^{R_0\,\log\alpha}\right\}.$$

For every $\varrho > 0$, we take the Lipschitz cuf-off function η_ϱ such that

$$\eta_\varrho \equiv 0 \text{ in } B_\varrho(0), \qquad \eta_\varrho \equiv 1 \text{ in } \mathbb{R}^N \setminus B_{\varrho+1}(0), \qquad 0 \leq \eta_\varrho \leq 1,$$

and

$$|\nabla \eta_\varrho| = 1 \qquad \text{on } B_{\varrho+1}(0) \setminus B_\varrho(0).$$

We then insert in the weak formulation the test function $\psi = \Psi_n\,\eta_\varrho^2$. We get

$$\int_{\mathbb{R}^N} |\nabla\Psi_n|^2\,\eta_\varrho^2 + \int_{\mathbb{R}^N} V\,|\Psi_n|^2\,\eta_\varrho^2\,dx = \lambda_n(V)\int_{\mathbb{R}^N} |\Psi_n|^2\,\eta_\varrho^2\,dx - 2\int_{\mathbb{R}^N} \langle\nabla\Psi_n, \nabla\eta_\varrho\rangle\,\eta_\varrho\,\Psi_n\,dx.$$

On the last integral, we apply the Cauchy-Schwarz and Young inequalities, so to get

$$-2\int_{\mathbb{R}^N} \langle\nabla\Psi_n, \nabla\eta_\varrho\rangle\,\eta_\varrho\,\Psi_n\,dx \leq \int_{\mathbb{R}^N} |\nabla\Psi_n|^2\,\eta_\varrho^2\,dx + \int_{\mathbb{R}^N} |\nabla\eta_\varrho|^2\,|\Psi_n|^2\,dx.$$

By inserting this estimate in the identity above and canceling out the common factor, we get

$$\int_{\mathbb{R}^N} V\,|\Psi_n|^2\,\eta_\varrho^2\,dx \leq \lambda_n(V)\int_{\mathbb{R}^N} |\Psi_n|^2\,\eta_\varrho^2\,dx + \int_{\mathbb{R}^N} |\nabla\eta_\varrho|^2\,|\Psi_n|^2\,dx.$$

In particular, by using the properties of η_ϱ, this entails that

$$\int_{\mathbb{R}^N\setminus B_{\varrho+1}(0)} V\,|\Psi_n|^2\,dx \leq \lambda_n(V)\int_{\mathbb{R}^N\setminus B_\varrho(0)} |\Psi_n|^2\,dx + \int_{B_{\varrho+1}(0)\setminus B_\varrho(0)} |\Psi_n|^2\,dx. \tag{3.8.2}$$

On account of the confining property (3.2.2), we have that there exists $R_0 > 0$ such that

$$\inf_{\mathbb{R}^N\setminus B_{\varrho+1}(0)} V \geq 2\,\lambda_n, \qquad \text{for every } \varrho \geq R_0.$$

More precisely, by recalling (3.2.5), we easily see that we can take

$$R_0 = R_0(N, n, R) := \sqrt{2\,\lambda_n} + R. \tag{3.8.3}$$

Thus, for every $\varrho \geq R_0$, from (3.8.2) and Lemma 3.5.1 we get

$$2\,\lambda_n\int_{\mathbb{R}^N\setminus B_{\varrho+1}(0)} |\Psi_n|^2\,dx \leq \lambda_n\int_{\mathbb{R}^N\setminus B_\varrho(0)} |\Psi_n|^2\,dx + \int_{B_{\varrho+1}(0)\setminus B_\varrho(0)} |\Psi_n|^2\,dx.$$

By decomposing

$$\int_{\mathbb{R}^N\setminus B_\varrho(0)} |\Psi_n|^2\,dx = \int_{\mathbb{R}^N\setminus B_{\varrho+1}(0)} |\Psi_n|^2\,dx + \int_{B_{\varrho+1}(0)\setminus B_\varrho(0)} |\Psi_n|^2\,dx,$$

this can be recast into

$$\int_{\mathbb{R}^N\setminus B_{\varrho+1}(0)} |\Psi_n|^2\,dx \le \left(\frac{1}{\lambda_n}+1\right) \int_{B_{\varrho+1}(0)\setminus B_\varrho(0)} |\Psi_n|^2\,dx. \tag{3.8.4}$$

We set for brevity

$$m(\varrho) = \int_{\mathbb{R}^N\setminus B_\varrho(0)} |\Psi_n|^2\,dx \qquad \text{and} \qquad \Theta = \frac{1}{\lambda_n}+1.$$

Accordingly, the estimate (3.8.4) can be rewritten as

$$m(\varrho+1) \le \Theta\Big(m(\varrho) - m(\varrho+1)\Big), \qquad \text{for every } \varrho \ge R_0.$$

In turn, we rewrite this as follows

$$m(\varrho+1) \le \frac{\Theta}{1+\Theta}\, m(\varrho), \qquad \text{for every } \varrho \ge R_0. \tag{3.8.5}$$

By iterating this estimate, we get the claimed exponential decay of $m(\varrho)$. Indeed, it is sufficient to observe that $\varrho \mapsto m(\varrho)$ is non-increasing and thus for every $\varrho \ge R_0+1$ we have

$$m(\varrho) \le m(R_0 + \lfloor \varrho - R_0 \rfloor) \le \left(\frac{\Theta}{1+\Theta}\right)^{\lfloor \varrho - R_0 \rfloor} m(R_0) \le \left(\frac{\Theta}{1+\Theta}\right)^{\varrho - R_0 - 1} m(R_0).$$

This has been obtained by applying a suitable number of times (3.8.5). By recalling that $m(R_0) \le m(0) = 1$, we then get

$$\int_{\mathbb{R}^N\setminus B_\varrho(0)} |\Psi_n|^2\,dx \le \left(\frac{\Theta}{1+\Theta}\right)^{\varrho} \left(\frac{1+\Theta}{\Theta}\right)^{R_0+1}, \qquad \text{for every } \varrho \ge R_0+1.$$

We now observe that by definition

$$\frac{1}{2} \le \frac{\Theta}{1+\Theta} \le \frac{\lambda_1+1}{2\,\lambda_1+1} =: \frac{1}{\alpha}.$$

Observe that α depends on the dimension N only, through the first eigenvalue λ_1 of the quantum harmonic oscillator. Thus, we obtain in particular

$$\int_{\mathbb{R}^N\setminus B_\varrho(0)} |\Psi_n|^2\,dx \le \alpha^{-\varrho}\, 2^{R_0+1}, \qquad \text{for every } \varrho \ge R_0+1.$$

By choosing
$$C = 2^{R_0+1},$$
we get (3.8.1), as desired. $\square$

Remark 3.8.2. *Observe that the costant C_1 is given by*
$$C_1 = \max\left\{2^{R_0+1}, e^{R_0 \log \alpha}\right\} = 2^{R_0+1},$$
with R_0 defined by (3.8.3). In particular, we see that $C_1 \nearrow +\infty$ as
$$R \nearrow +\infty \qquad \text{or} \qquad n \to \infty,$$
while C_1 stays uniformly bounded, as R goes to 0.

We can now prove the pointwise exponential decay: this is the main result of this section.

Theorem 3.8.3 (Exponential decay in L^∞)**.** *There exists a constant $C_2 = C_2(N, n, R) > 0$ such that*
$$0 \le |\Psi_n(x)| \le C_2\, e^{-\frac{|x|}{2} \log \alpha}, \qquad \text{for every } x \in \mathbb{R}^N,$$
where $\alpha = \alpha(N) > 1$ is the same exponent as in Lemma 3.8.1.

Proof. It is enough to prove a $L^\infty - L^2$ estimate, localized "at infinity", i.e. an estimate like

$$\|\Psi_n\|_{L^\infty(\mathbb{R}^N \setminus B_{\varrho+1}(0))} \le C\, \|\Psi_n\|_{L^2(\mathbb{R}^N \setminus B_\varrho(0))}, \qquad \text{for every } \varrho \ge 0, \tag{3.8.6}$$

for $C = C(N, n) > 0$. Then, by recalling that $\Psi_n \in L^\infty(\mathbb{R}^N)$, we would eventually get the conclusion by joining this estimate and Lemma 3.8.1. Indeed, if $|x| \le 1$, we would get from Remark 3.6.2
$$|\Psi_n(x)| \le \mathrm{M}_N\, \lambda_n^{\frac{N}{4}} \le \frac{\mathrm{M}_N\, \lambda_n^{\frac{N}{4}}}{e^{-\frac{\log \alpha}{2}}}\, e^{-\frac{|x|}{2} \log \alpha}.$$
On the other hand, if $|x| > 1$, then $\varrho + 1 \le |x| \le \varrho + 2$ for some $\varrho \ge 0$. Accordingly, we would get
$$\begin{aligned}
|\Psi_n(x)| \le \|\Psi_n\|_{L^\infty(\mathbb{R}^N \setminus B_{\varrho+1}(0))} &\le C\, \|\Psi_n\|_{L^2(\mathbb{R}^N \setminus B_\varrho(0))} \\
&\le C\, \sqrt{C_1}\, e^{-\varrho \frac{\log \alpha}{2}} \\
&\le C\, \sqrt{C_1}\, \frac{e^{-\varrho \frac{\log \alpha}{2}}}{e^{-(\varrho+2) \frac{\log \alpha}{2}}}\, e^{-|x| \frac{\log \alpha}{2}} = C\, \alpha\, \sqrt{C_1}\, e^{-|x| \frac{\log \alpha}{2}},
\end{aligned}$$
where C_1 is the same constant as in Lemma 3.8.1. In conclusion, we would get the claimed estimate with constant
$$C_2 = \max\left\{\sqrt{\alpha}\, \mathrm{M}_N\, \lambda_n^{\frac{N}{4}},\, C\, \alpha\, \sqrt{C_1}\right\}.$$

The estimate (3.8.6) can be obtained by appealing once again to a suitable Moser–type iteration. For every pair of radii $0 < r < R$ and every exponent $\beta \geq 1$, we insert in the weak formulation the test function

$$\psi = |\Psi_n|^{\beta-1}\, \Psi_n\, \eta_{r,R}^2,$$

where $\eta_{r,R}$ is a Lipschitz cut-off function such that

$$\eta_{r,R} \equiv 0 \text{ in } B_r(0), \qquad \eta_{r,R} \equiv 1 \text{ in } \mathbb{R}^N \setminus B_R(0), \qquad 0 \leq \eta_R \leq 1,$$

and

$$|\nabla \eta_{r,R}| = \frac{1}{R-r}, \qquad \text{in } B_R(0) \setminus B_r(0).$$

This is a feasible test function, thanks to the Chain Rule in Sobolev spaces (recall that $\Psi_n \in L^\infty(\mathbb{R}^N)$ by Proposition 3.6.1). We then get

$$\begin{aligned}\frac{4\,\beta}{(\beta+1)^2} \int_{\mathbb{R}^N} \left|\nabla |\Psi_n|^{\frac{\beta+1}{2}}\right|^2 \eta_{r,R}^2\, dx + \int_{\mathbb{R}^N} V\, |\Psi_n|^{\beta+1}\, \eta_{r,R}^2\, dx \\ = \lambda_n(V) \int_{\mathbb{R}^N} |\Psi_n|^{\beta+1}\, \eta_{r,R}^2\, dx \\ - 2 \int_{\mathbb{R}^N} \langle \nabla \Psi_n, \nabla \eta_{r,R}\rangle\, \eta_{r,R}\, |\Psi_n|^{\beta-1}\, \Psi_n\, dx.\end{aligned}$$

On the last term, we use as usual the Cauchy-Schwarz and Young inequalities, so to get

$$\begin{aligned}-2 \int_{\mathbb{R}^N} \langle \nabla \Psi_n, \nabla \eta_{r,R}\rangle\, \eta_{r,R}\, |\Psi_n|^{\beta-1}\, \Psi_n\, dx &\leq \delta \int_{\mathbb{R}^N} |\nabla \Psi_n|^2\, |\Psi_n|^{\beta-1}\, \eta_{r,R}^2\, dx \\ &+ \frac{1}{\delta} \int_{\mathbb{R}^N} |\nabla \eta_{r,R}|^2\, |\Psi_n|^{\beta+1}\, dx \\ &= \delta\, \frac{4}{(\beta+1)^2} \int_{\mathbb{R}^N} \left|\nabla |\Psi_n|^{\frac{\beta+1}{2}}\right|^2 \eta_{r,R}^2\, dx \\ &+ \frac{1}{\delta} \int_{\mathbb{R}^N} |\nabla \eta_{r,R}|^2\, |\Psi_n|^{\beta+1}\, dx,\end{aligned}$$

which holds for every $\delta > 0$. In particular, by taking $\delta = \beta/2$ we can absorb the gradient term on the right-hand side and obtain

$$\begin{aligned}\frac{2\,\beta}{(\beta+1)^2} \int_{\mathbb{R}^N} \left|\nabla |\Psi_n|^{\frac{\beta+1}{2}}\right|^2 \eta_{r,R}^2\, dx + \int_{\mathbb{R}^N} V\, |\Psi_n|^{\beta+1}\, \eta_{r,R}^2\, dx \leq \lambda_n(V) \int_{\mathbb{R}^N} |\Psi_n|^{\beta+1}\, \eta_{r,R}^2\, dx \\ + \frac{2}{\beta} \int_{\mathbb{R}^N} |\nabla \eta_{r,R}|^2\, |\Psi_n|^{\beta+1}\, dx.\end{aligned}$$

We now set for brevity $\vartheta = (\beta+1)/2$, drop the (non-negative) term containing V and use some elementary manipulations, so to get

$$\int_{\mathbb{R}^N} \left|\nabla |\Psi_n|^{\vartheta}\right|^2 \eta_{r,R}^2\, dx \leq 2\,\vartheta\, \lambda_n(V) \int_{\mathbb{R}^N} |\Psi_n|^{2\,\vartheta}\, \eta_{r,R}^2\, dx + 4 \int_{\mathbb{R}^N} |\nabla \eta_{r,R}|^2\, |\Psi_n|^{2\,\vartheta}\, dx.$$

We add on both sides the term

$$\int_{\mathbb{R}^N} |\nabla \eta_{r,R}|^2\, |\Psi_n|^{2\,\vartheta}\, dx,$$

and use that

$$\int_{\mathbb{R}^N} |\nabla \eta_{r,R}|^2\, |\Psi_n|^{2\,\vartheta}\, dx + \int_{\mathbb{R}^N} \left|\nabla |\Psi_n|^{\vartheta}\right|^2\, \eta_{r,R}^2\, dx \geq \frac{1}{2}\int_{\mathbb{R}^N} \left|\nabla\left(|\Psi_n|^{\vartheta}\,\eta_{r,R}\right)\right|^2\, dx.$$

This gives

$$\int_{\mathbb{R}^N} \left|\nabla\left(|\Psi_n|^{\vartheta}\,\eta_{r,R}\right)\right|^2\, dx \leq 4\,\vartheta\,\lambda_n(V)\int_{\mathbb{R}^N} |\Psi_n|^{2\,\vartheta}\,\eta_{r,R}^2\, dx + 10\int_{\mathbb{R}^N} |\nabla \eta_{r,R}|^2\, |\Psi_n|^{2\,\vartheta}\, dx.$$

For simplicity, we now confine ourselves to the case $N \geq 3$: the cases $N = 2$ and $N = 1$ can be treated with minor modifications, as in the proof of Proposition 3.6.1. Thus, we can bound the left-hand side from below, again thanks to the Sobolev inequality. We obtain

$$\mathcal{T}_N\left(\int_{\mathbb{R}^N}\left(|\Psi_n|^{\vartheta}\,\eta_{r,R}\right)^{2^*} dx\right)^{\frac{2}{2^*}} \leq 4\,\vartheta\,\lambda_n(V)\int_{\mathbb{R}^N} |\Psi_n|^{2\,\vartheta}\,\eta_{r,R}^2\, dx + 10\int_{\mathbb{R}^N} |\nabla \eta_{r,R}|^2\, |\Psi_n|^{2\,\vartheta}\, dx.$$

Thanks to the properties of $\eta_{r,R}$, this in turn implies that

$$\mathcal{T}_N\left(\int_{\mathbb{R}^N\setminus B_R(0)} |\Psi_n|^{2^*\vartheta}\right)^{\frac{2}{2^*}} \leq 4\,\vartheta\,\lambda_n(V)\int_{\mathbb{R}^N\setminus B_r(0)} |\Psi_n|^{2\,\vartheta}\, dx + \frac{10}{(R-r)^2}\int_{\mathbb{R}^N\setminus B_r(0)} |\Psi_n|^{2\,\vartheta}\, dx.$$

With some manipulations more, we can also obtain

(3.8.7)
$$\left(\int_{\mathbb{R}^N\setminus B_R(0)} |\Psi_n|^{2^*\vartheta}\right)^{\frac{1}{2^*\vartheta}} \leq \left(\frac{10\,\vartheta}{\mathcal{T}_N}\left(\lambda_n(V) + \frac{1}{(R-r)^2}\right)\right)^{\frac{1}{2\,\vartheta}}\left(\int_{\mathbb{R}^N\setminus B_r(0)} |\Psi_n|^{2\,\vartheta}\, dx\right)^{\frac{1}{2\,\vartheta}}.$$

We now introduce the sequence of exponents

$$\vartheta_0 = 1, \qquad \vartheta_{i+1} = \frac{2^*}{2}\,\vartheta_i = \left(\frac{N}{N-2}\right)^{i+1}, \qquad \text{for } i \in \mathbb{N}.$$

together with the sequence of radii

$$R_i = \varrho + 1 - \frac{1}{2^i}, \qquad \text{for } i \in \mathbb{N},$$

where $\varrho \geq 0$ is fixed. Observe that $R_0 = \varrho$, $R_\infty = \varrho + 1$ and $R_{i+1} - R_i = 1/2^{i+1}$. By using (3.8.7) with $\vartheta = \vartheta_i$, $r = R_i$ and $R = R_{i+1}$, we obtain

$$\left(\int_{\mathbb{R}^N\setminus B_{R_{i+1}}(0)} |\Psi_n|^{2\,\vartheta_{i+1}}\right)^{\frac{1}{2\,\vartheta_{i+1}}} \leq \left(\frac{10\,\vartheta_i}{\mathcal{T}_N}\left(\lambda_n + 4^{i+1}\right)\right)^{\frac{1}{2\,\vartheta_i}}\left(\int_{\mathbb{R}^N\setminus B_{R_i}(0)} |\Psi_n|^{2\,\vartheta_i}\, dx\right)^{\frac{1}{2\,\vartheta_i}}.$$

Observe that we also used Lemma 3.5.1, in order to estimate $\lambda_n(V)$. Starting from $i = 0$ and iterating this estimate infinitely many times, we get the desired conclusion (3.8.6). $\square$

3.9 Stability of Eigenspaces

In this section, we provide a quantitative estimate of the distance between the eigenspaces of the considered operator and those of the quantum harmonic oscillator. We analyze the limiting case $R \to 0$, that is, when the potential well Σ approaches the origin and the operator is expected to converge to the classical quantum harmonic oscillator. We rigorously show that this is indeed the case, by deriving an explicit estimate of the distance between the spectra and eigenspaces of the two operators, expressed solely in terms of the dimension N, the eigenvalue index n, and the geometric parameter R.

We already know from Lemma 3.5.1 that

$$\sqrt{\lambda_n} - R \leq \sqrt{\lambda_n(V)} \leq \sqrt{\lambda_n}, \qquad \text{for every } n \in \mathbb{N}^*.$$

In particular, we get that the spectrum of our operator collapses to the spectrum of the quantum harmonic oscillator, as R goes to 0.

In this section we want to prove a similar kind of *quantitative* result, for the relevant eigenspaces.

Theorem 3.9.1. *For $n \in \mathbb{N}^*$, we set*

$$\mathcal{W}_n = \Big\{\psi \in H^1(\mathbb{R}^N; |x|^2) \,:\, \psi \text{ is an eigenstate relative to } \lambda_n\Big\}.$$

Then there exists an explicit constant $C_3 = C_3(N, n) > 0$ such that

$$\mathrm{dist}_{L^2(\mathbb{R}^N)}(\Psi_n; \mathcal{W}_n) \leq C_3 \sqrt{R}, \qquad \text{for every } R > 0.$$

Proof. We divide the proof in various steps, for ease of readability.

Step 1: set-up. We need to fix at first some notation. We call

$$\kappa(n) = \dim \mathcal{W}_n - 1,$$

then there exists an index $j_n \in \{0, \dots, n-1\}$ such that[4]

$$\lambda_{n-j_n} = \dots = \lambda_n = \dots = \lambda_{n-j_n+\kappa(n)}. \tag{3.9.1}$$

By construction, we thus have

$$\delta_n := \lambda_{n-j_n+\kappa(n)+1} - \lambda_n > 0. \tag{3.9.2}$$

[4]For $n = 1$, we have already observed in Remark 3.4.6 that $\kappa(1) = 0$. Accordingly, in this case we have $j_1 = 0$.

We then set

$$R_1 = \frac{\sqrt{\lambda_n} - \sqrt{\lambda_{n-j_n-1}}}{2},$$

with the notation $\lambda_0 := 0$. According to Lemma 3.5.1, we have for every $0 < R \leq R_1$

$$\sqrt{\lambda_n(V)} \geq \sqrt{\lambda_n} - R \geq \frac{\sqrt{\lambda_n} + \sqrt{\lambda_{n-j_n-1}}}{2} > \sqrt{\lambda_{n-j_n-1}}. \tag{3.9.3}$$

Let us indicate by $\{\Phi_n\}_{n\in\mathbb{N}^*}$ an orthonormal basis of eigenstates for the quantum harmonic oscillator. For every $k \in \mathbb{N}^*$, we define

$$\widehat{\Psi_n}(k) = \int_{\mathbb{R}^N} \Psi_n\, \Phi_k\, dx,$$

i.e. the $k-$*th Fourier coefficient* of Ψ_n. We then have

$$1 = \|\Psi_n\|^2_{L^2(\mathbb{R}^N)} = \sum_{k=1}^{\infty} |\widehat{\Psi_n}(k)|^2. \tag{3.9.4}$$

Finally, we observe that by orthonormality

$$\begin{aligned} \Big(\mathrm{dist}_{L^2(\mathbb{R}^N)}(\Psi_n; \mathcal{W}_n)\Big)^2 &= \left\| \Psi_n - \sum_{k=n-j_n}^{n-j_n+\kappa(n)} \widehat{\Psi_n}(k)\, \Phi_k \right\|^2_{L^2(\mathbb{R}^N)} \\ &= \sum_{k=1}^{n-j_n-1} |\widehat{\Psi_n}(k)|^2 + \sum_{k=n-j_n+\kappa(n)+1}^{\infty} |\widehat{\Psi_n}(k)|^2 := \mathcal{J}_1 + \mathcal{J}_2. \end{aligned} \tag{3.9.5}$$

It is intended that the first term $\mathcal{J}_1$ is void in the case $j_n = n-1$. Thus, in order to get the claimed estimate, it is sufficient to suitably estimate the last two sums. Before going further, we observe that it is sufficient to get the claimed estimate for $R \leq R_1$: indeed, for every $R > R_1$ we would simply get

$$\Big(\mathrm{dist}_{L^2(\mathbb{R}^N)}(\Psi_n; \mathcal{W}_n)\Big)^2 = \mathcal{J}_1 + \mathcal{J}_2 \leq \sum_{k=1}^{\infty} |\widehat{\Psi_n}(k)|^2 = 1 \leq \frac{1}{R_1}\, R,$$

i.e. we have the desired estimate with $C_3 = 1/R_1$, the latter depending only on N and n, by definition.

Step 2: high frequencies estimate. We show how the estimate of $\mathcal{J}_2$ can reduced to the estimate of $\mathcal{J}_1$. By using that Ψ_n is an eigenstate corresponding to $\lambda_n(V)$, we have

$$\begin{aligned}\lambda_n(V) &= \int_{\mathbb{R}^N} |\nabla \Psi_n|^2\,dx + \int_{\mathbb{R}^N} V\,|\Psi_n|^2\,dx \\ &\ge \sum_{k=1}^{\infty} |\widehat{\Psi_n}(k)|^2\,\lambda_k + \frac{1}{1+\varepsilon}\int_{\mathbb{R}^N} |x|^2\,|\Psi_n|^2\,dx - \frac{R^2}{\varepsilon} \\ &\ge \sum_{k=1}^{\infty} |\widehat{\Psi_n}(k)|^2\,\lambda_k - \frac{R^2}{\varepsilon},\end{aligned}$$

for every $\varepsilon > 0$. In the first inequality we used (3.3.5). We estimate the series as follows

$$\begin{aligned}\sum_{k=1}^{\infty} |\widehat{\Psi_n}(k)|^2\,\lambda_k &= \sum_{k=n-j_n+\kappa(n)+1}^{\infty} |\widehat{\Psi_n}(k)|^2\,\lambda_k + \lambda_n \sum_{k=n-j_n}^{n-j_n+\kappa(n)} |\widehat{\Psi_n}(k)|^2 + \sum_{k=1}^{n-j_n-1} |\widehat{\Psi_n}(k)|^2\,\lambda_k \\ &\ge \lambda_{n-j_n+\kappa(n)+1}\,\mathcal{J}_2 + \lambda_n \sum_{k=n-j_n}^{n-j_n+\kappa(n)} |\widehat{\Psi_n}(k)|^2 + \lambda_1\,\mathcal{J}_1.\end{aligned}$$

Observe that we used the monotonicity of eigenvalues with respect to n and the multiplicity assumption (3.9.1). By further choosing $\varepsilon = R$ and subtracting λ_n, up to now we have obtained

$$\big(\lambda_n(V) - \lambda_n\big) + R \ge \lambda_{n-j_n+\kappa(n)+1}\,\mathcal{J}_2 + \lambda_n \left(\sum_{k=n-j_n}^{n-j_n+\kappa(n)} |\widehat{\Psi_n}(k)|^2 - 1\right) + \lambda_1\,\mathcal{J}_1. \tag{3.9.6}$$

Thanks to (3.9.4), we can write

$$\sum_{k=n-j_n}^{n-j_n+\kappa(n)} |\widehat{\Psi_n}(k)|^2 - 1 = -\mathcal{J}_1 - \mathcal{J}_2.$$

By injecting this identity in (3.9.6), we get

$$R + \big(\lambda_n(V) - \lambda_n\big) + (\lambda_n - \lambda_1)\,\mathcal{J}_1 \ge (\lambda_{n-j_n+\kappa(n)+1} - \lambda_n)\,\mathcal{J}_2.$$

Finally, by recalling (3.9.2) and Lemma 3.5.1, we arrive at the estimate

$$\frac{R}{\delta_n} + \frac{\lambda_n - \lambda_1}{\delta_n}\,\mathcal{J}_1 \ge \mathcal{J}_2, \tag{3.9.7}$$

which concludes this step. The previous estimate holds for every $R > 0$.

Step 3: low frequencies estimate. We now estimate the finite sum $\mathcal{J}_1$. We will derive some "almost orthogonality" relations. Indeed, by using first the equation for Φ_k and then that for Ψ_n, we get

$$\begin{aligned}\widehat{\Psi_n}(k) = \int_{\mathbb{R}^N} \Psi_n\,\Phi_k\,dx &= \frac{1}{\lambda_k}\left[\int_{\mathbb{R}^N}\langle\nabla\Phi_k,\nabla\Psi_n\rangle\,dx + \int_{\mathbb{R}^N}|x|^2\,\Phi_k\,\Psi_n\,dx\right]\\ &= \frac{1}{\lambda_k}\left[\int_{\mathbb{R}^N}\langle\nabla\Phi_k,\nabla\Psi_n\rangle\,dx + \int_{\mathbb{R}^N}V\,\Phi_k\,\Psi_n\,dx\right]\\ &+ \frac{1}{\lambda_k}\int_{\mathbb{R}^N}\left(|x|^2 - V\right)\Phi_k\,\Psi_n\,dx\\ &= \frac{\lambda_n(V)}{\lambda_k}\int_{\mathbb{R}^N}\Psi_n\,\Phi_k\,dx + \frac{1}{\lambda_k}\int_{\mathbb{R}^N}\left(|x|^2 - V\right)\Phi_k\,\Psi_n\,dx\\ &= \frac{\lambda_n(V)}{\lambda_k}\,\widehat{\Psi_n}(k) + \frac{1}{\lambda_k}\int_{\mathbb{R}^N}\left(|x|^2 - V\right)\Phi_k\,\Psi_n\,dx.\end{aligned}$$

We thus have the following estimate

$$\left|\lambda_k - \lambda_n(V)\right|\,|\widehat{\Psi_n}(k)| \le \int_{\mathbb{R}^N}\left||x|^2 - V\right|\,|\Phi_k|\,|\Psi_n|\,dx.$$

By recalling (3.9.3), for $k = 1,\dots,n-j_n-1$ and $R \le R_1$ we have

$$\left|\lambda_k - \lambda_n(V)\right| = \lambda_n(V) - \lambda_k \ge \left(\frac{\sqrt{\lambda_n} + \sqrt{\lambda_{n-j_n-1}}}{2}\right)^2 - \lambda_{n-j_n-1} =: \alpha_{N,n} > 0.$$

We thus obtain for $k = 1,\dots,n-j_n-1$ and $R \le R_1$

$$|\widehat{\Psi_n}(k)| \le \frac{1}{\alpha_{N,n}}\left(\int_{\mathbb{R}^N}\left||x|^2 - V\right|\,|\Phi_k|^2\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N}\left||x|^2 - V\right|\,|\Psi_n|^2\,dx\right)^{\frac{1}{2}}.$$

We show that the last two terms can be controlled in terms of R. Indeed, from (3.3.4) and (3.3.5), we have

$$-\frac{\varepsilon}{1+\varepsilon}\,|x|^2 - \frac{R^2}{\varepsilon} \le V(x) - |x|^2 \le \varepsilon\,|x|^2 + \left(1 + \frac{1}{\varepsilon}\right)\delta^2,$$

for every $\varepsilon > 0$. Thus, we get in particular

$$\left|V(x) - |x|^2\right| \le \max\left\{\frac{\varepsilon}{1+\varepsilon}\,|x|^2 + \frac{R^2}{\varepsilon},\varepsilon\,|x|^2 + \left(1 + \frac{1}{\varepsilon}\right)\delta^2\right\}.$$

We choose again $\varepsilon = R$, so that

$$\left|V(x) - |x|^2\right| \le R\,\max\left\{\frac{|x|^2}{R+1} + 1, |x|^2 + (R+1)\left(\frac{\delta}{R}\right)^2\right\} \le R\,(|x|^2 + R + 1). \tag{3.9.8}$$

Finally, we get for $k = 1,\dots,n-j_n-1$ and $R \le R_1$

$$|\widehat{\Psi_n}(k)| \le \frac{R}{\alpha_{N,n}}\left(\int_{\mathbb{R}^N}|x|^2\,|\Phi_k|^2\,dx + (R+1)\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N}|x|^2\,|\Psi_n|^2\,dx + (R+1)\right)^{\frac{1}{2}}.$$

In conclusion, we get

$$\mathcal{J}_1 \le \frac{R^2}{\alpha_{N,n}^2} \left(\int_{\mathbb{R}^N} |x|^2 \, |\Psi_n|^2 \, dx + (R+1)\right) \sum_{k=1}^{n-j_n-1} \left(\int_{\mathbb{R}^N} |x|^2 \, |\Phi_k|^2 \, dx + (R+1)\right), \tag{3.9.9}$$

for every $R \le R_1$.

Step 4: conclusion. By joining (3.9.5), (3.9.7) and (3.9.9) we get for every $R \le R_1$

$$\begin{aligned}
\left(\mathrm{dist}_{L^2(\mathbb{R}^N)}(\Psi_n; \mathcal{W}_n)\right)^2 &= \mathcal{J}_1 + \mathcal{J}_2 \\
&\le \frac{R}{\delta_n} + \left(1 + \frac{\lambda_n - \lambda_1}{\delta_n}\right) \mathcal{J}_1 \\
&\le \frac{R}{\delta_n} + \frac{R^2}{\alpha_{N,n}^2} \left(1 + \frac{\lambda_n - \lambda_1}{\delta_n}\right) \left(\int_{\mathbb{R}^N} |x|^2 \, |\Psi_n|^2 \, dx + (R_1+1)\right) \\
&\times \sum_{k=1}^{n-j_n-1} \left(\int_{\mathbb{R}^N} |x|^2 \, |\Phi_k|^2 \, dx + (R_1+1)\right).
\end{aligned}$$

In order to conclude, we only need to observe that for $R \le R_1$

$$\int_{\mathbb{R}^N} |x|^2 \, |\Psi_n|^2 \, dx \le 4 \, R_1^2 + 4 \, \lambda_n,$$

thanks to (4.6.1). Thus, we eventually get the desired estimate. □

With a little extra work, we can improve the metric of the previous stability estimate.

Corollary 3.9.2. *For $n \in \mathbb{N}^*$, we still set*

$$\mathcal{W}_n = \left\{\psi \in H^1(\mathbb{R}^N; |x|^2) \, : \, \psi \text{ is an eigenstate relative to } \lambda_n\right\}.$$

Then there exists an explicit constant $C_4 = C_4(N, n) > 0$ such that

$$\mathrm{dist}_{H^1(\mathbb{R}^N;|x|^2)}(\Psi_n; \mathcal{W}_n) \le C_4 \sqrt{R}, \qquad \text{for every } R > 0.$$

Proof. With the previous notation, let us set

$$\Phi = \sum_{k=n-j_n}^{n-j_n+\kappa(n)} \widehat{\Psi_n}(k) \, \Phi_k \in \mathcal{W}_n.$$

We thus have

$$\left(\mathrm{dist}_{H^1(\mathbb{R}^N;|x|^2)}(\Psi_n; \mathcal{W}_n)\right)^2 \le \int_{\mathbb{R}^N} |\nabla \Psi_n - \nabla \Phi|^2 \, dx + \int_{\mathbb{R}^N} |x|^2 \, |\Psi_n - \Phi|^2 \, dx.$$

As above, on account of the uniform estimates at our disposal, it is sufficient to bound the last two terms for $R \le R_1$. Here, the radius $R_1 = R_1(N, n)$ is the same as in the previous proof. By using the equations for both Φ and Ψ_n, we get

$$\int_{\mathbb{R}^N} \langle \nabla\Phi, \nabla(\Phi-\Psi_n)\rangle\, dx + \int_{\mathbb{R}^N} |x|^2\, \Phi\,(\Phi-\Psi_n)\, dx = \lambda_n \int_{\mathbb{R}^N} \Phi\,(\Phi-\Psi_n)\, dx,$$

and

$$\int_{\mathbb{R}^N} \langle \nabla\Psi_n, \nabla(\Phi-\Psi_n)\rangle\, dx + \int_{\mathbb{R}^N} V\, \Psi_n\,(\Phi-\Psi_n)\, dx = \lambda_n(V) \int_{\mathbb{R}^N} \Psi_n\,(\Phi-\Psi_n)\, dx.$$

By subtracting them, we get

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla\Psi_n-\nabla\Phi|^2\, dx + \int_{\mathbb{R}^N} |x|^2\, |\Psi_n-\Phi|^2\, dx &\le \int_{\mathbb{R}^N} \big|V-|x|^2\big|\, |\Psi_n|\, |\Psi_n-\Phi|\, dx \\
&\quad + \big|\lambda_n(V)-\lambda_n\big| \int_{\mathbb{R}^N} |\Psi_n|\, |\Psi_n-\Phi|\, dx \\
&\quad + \lambda_n \int_{\mathbb{R}^N} |\Psi_n-\Phi|^2\, dx \\
&\le \left(\int_{\mathbb{R}^N} \big|V-|x|^2\big|^2\, |\Psi_n|^2\, dx\right)^{\frac{1}{2}} \|\Psi_n-\Phi\|_{L^2(\mathbb{R}^N)} \\
&\quad + \big|\lambda_n(V)-\lambda_n\big|\, \|\Psi_n-\Phi\|_{L^2(\mathbb{R}^N)} \\
&\quad + \lambda_n \int_{\mathbb{R}^N} |\Psi_n-\Phi|^2\, dx.
\end{aligned}$$

The L^2 norm of $\Psi_n-\Phi$ can be estimated by Theorem 3.9.1, while for the difference of the eigenvalues we can apply

$$|\lambda_n(V)-\lambda_n| = \left(\sqrt{\lambda_n}-\sqrt{\lambda_n(V)}\right)\left(\sqrt{\lambda_n}+\sqrt{\lambda_n(V)}\right) \le 2\, R\, \sqrt{\lambda_n},$$

which follows from Lemma 3.5.1. For every $0 < R \le R_1$, these yield

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla\Psi_n-\nabla\Phi|^2\, dx + \int_{\mathbb{R}^N} |x|^2\, |\Psi_n-\Phi|^2\, dx &\le C_3\, \sqrt{R_1} \left(\int_{\mathbb{R}^N} \big|V-|x|^2\big|^2\, |\Psi_n|^2\, dx\right)^{\frac{1}{2}} \\
&\quad + 2\,\sqrt{\lambda_n}\, C_3\, \sqrt{R_1}\, R + C_3^2\, \lambda_n\, R.
\end{aligned}$$

We only need to estimate the integral containing the difference of the potentials. By (3.9.8), we get

$$\begin{aligned}
\int_{\mathbb{R}^N} \big|V-|x|^2\big|^2\, |\Psi_n|^2\, dx &\le R^2 \int_{\mathbb{R}^N} (|x|^2+R+1)^2\, |\Psi_n|^2\, dx \\
&\le 2\, R^2 \left(\int_{\mathbb{R}^N} |x|^4\, |\Psi_n|^2\, dx + (R_1+1)^2\right).
\end{aligned}$$

By using Proposition 3.6.3 with $k=1$ to bound uniformly the first term on the right-hand side, we conclude. □

Chapter 4

Extension to the Electromagnetic Case

4.1 The Electromagnetic Case

In this chapter, we extend the analysis carried out in the purely electric setting to the electromagnetic framework. The results presented here can be regarded as a natural generalization of those obtained in the previous chapter, with the inclusion of a magnetic potential.

Some proofs and arguments carry over directly from the electric case and will be briefly recalled, with precise references to the relevant sections. However, several results require substantial adaptations due to the additional structure introduced by the magnetic field. The main challenges stem from the properties of magnetic differential operators, as stated in Remark 2.7.1. These features preclude the straightforward application of translation-based techniques and many methods used in the purely electric setting. To overcome these difficulties, we employ assumptions on the magnetic vector potential, perturbative arguments on the Laplacian and a classical Kato inequality. In certain instances, the analysis can still be reduced to the electric case, while in others a distinct approach is necessary, focusing exclusively on the novel aspects and adaptations, thus avoiding unnecessary repetition of results already established for the electric operator.

Throughout this chapter, attention will also be paid to the dependence of the results on the geometric features of the scalar potential, emphasizing the similarities and differences with the electric case.

4.2 Magnetic Perturbation of the Schrödinger Operator

In what follows, we introduce the magnetic Schrödinger operator, which can be regarded as a perturbation of the purely electric case. This perturbation arises from replacing the standard Laplacian with the magnetic one, thereby incorporating the effects of the magnetic potential while preserving the overall structure and analytical framework of the electric problem.

Now, we consider the magnetic Schrödinger operator

$$\mathcal{H}_{A,V}[\Psi] := -\Delta_A \Psi + V_\Sigma \Psi, \tag{4.2.1}$$

where $\Psi(x) : \mathbb{R}^N \to \mathbb{C}$ denotes the wave function, $A = (A_1, \ldots, A_N) : \mathbb{R}^N \to \mathbb{R}^N$, with components $A_j \in L^2_{\text{loc}}(\mathbb{R}^N)$, is the magnetic vector potential, and $V(x) : \mathbb{R}^N \to \mathbb{R}$ is the electric scalar potential. The particular case $A = 0$ corresponds to the purely electric case.

4.3 Properties of Vector Potential

For the subsequent analysis, we impose the following conditions on the vector potential A. These assumptions guarantee the applicability of standard analytical tools and provide sufficient control over the growth of the magnetic terms:

A_1 Continuously differentiable: $A(x) \in C^1(\mathbb{R}^N, \mathbb{R}^N)$,

A_2 Linear growth: $\exists\, C_1 > 0 : |A(x)| \le C_1(|x| + 1), \quad \forall x \in \mathbb{R}^N$,

A_3 Lipschitz continuity: $\exists\, C_2 > 0 : |A(x) - A(y)| \le C_2|x - y|, \quad \forall x, y \in \mathbb{R}^N$.

Remark 4.3.1. *As mentioned in the introduction, we make assumptions on the vector potential $A(x)$ that are general enough to include the particular case given by*

$$A(x, y, z) = (-y, x, 0),$$

for $N = 3$, with $A \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$, which generates the magnetic field

$$B = \nabla \times A = (0, 0, 2),$$

that is, a uniform magnetic field directed along the positive z-axis. This field is solenoidal, since it has zero divergence.

Having established the Poincaré-type inequality in the electric case (Proposition 3.3.5), we can extend the result to the full electromagnetic setting. This extension relies on the diamagnetic inequality (2.7.3).

Proposition 4.3.2 (Magnetic Poincaré–type Inequality)**.** *$\exists\, C_{N,R} > 0$ such that $\forall u \in C_c^\infty(\mathbb{R}^N)$ we have*

$$\frac{1}{C_{N,R}} \int_{\mathbb{R}^N} |u|^2 \, dx \le \int_{\mathbb{R}^N} |\nabla_A u|^2 \, dx + \int_{\mathbb{R}^N} V\, |u|^2 \, dx. \tag{4.3.1}$$

Moreover, the constant $C_{N,R}$ has the following properties

$$\lim_{R\to 0^+} C_{N,R} > 1 \qquad \text{and} \qquad 0 < \lim_{R\to+\infty} \frac{C_{N,R}}{R^2} < +\infty.$$

Proof. It is sufficient to observe that

$$\int_{\mathbb{R}^N} |\nabla_A u|^2\,dx + \int_{\mathbb{R}^N} V\,|u|^2\,dx \geq \int_{\mathbb{R}^N} |\nabla|u||^2\,dx + \int_{\mathbb{R}^N} V\,|u|^2\,dx,$$

thanks to the Diamagnetic Inequality. Then the desired result follows by applying Proposition 3.3.5 to $|u|$. $\square$

4.4 Spectral Analysis

This section is devoted to the presentation of some fundamental spectral properties of the magnetic operator, as well as to the variational characterization of its eigenvalues. The domain of the operator is defined at this point and will be rigorously justified in Section 4.7.

In what follows, we define the quadratic form associated with the magnetic operator and determine its domain of finiteness, defined in a manner consistent with the framework introduced in Section 2.3.4, taking into account the new magnetic setting.

We introduce the following inner product for any $\varphi, \psi \in H^1_A(\mathbb{R}^N) \cap L^2(\mathbb{R}^N; V)$

$$\mathcal{Q}_{A,V}[\varphi,\psi] := \int_{\mathbb{R}^N} \langle \nabla_A \varphi, \overline{\nabla_A \psi}\rangle\,dx + \int_{\mathbb{R}^N} V\,\varphi\,\overline{\psi}\,dx.$$

Accordingly, we also set the following

$$\|\varphi\|_{A,V} = \sqrt{\mathcal{Q}_{A,V}[\varphi,\varphi]}, \qquad \text{for } \varphi \in H^1_A(\mathbb{R}^N) \cap L^2(\mathbb{R}^N; V).$$

It is not difficult to see that the latter is actually a norm.

Definition 4.4.1. With the notation above, we define the normed vector space

$$H^1_A(\mathbb{R}^N; V) := H^1_A(\mathbb{R}^N) \cap L^2(\mathbb{R}^N; V),$$

endowed with the norm $\|\cdot\|_{A,V}$.

The following preliminary result will be important.

Lemma 4.4.2. *The space $H^1_A(\mathbb{R}^N; V)$ is a Hilbert space, having $C^\infty_c(\mathbb{R}^N)$ as a dense subspace. Moreover, $H^1_A(\mathbb{R}^N; V)$ is contained in $L^2(\mathbb{R}^N)$, with continuous inclusion. Finally, we have*

$$H^1_A(\mathbb{R}^N; V) = H^1_A(\mathbb{R}^N; |x|^2),$$

with equivalent norms.

Proof. The proof proceeds along the same lines as that of the electric case presented in Lemma 3.4.2. $\square$

Let us now consider the self-adjoint, magnetic Schrödinger operator

$$\mathcal{H}_{A,V} : \mathfrak{D}(\mathcal{H}_{A,V}) \subseteq L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$$

defined by

$$\mathcal{H}_{A,V}[\varphi] := -\Delta_A \varphi + V(x)\,\varphi$$

with domain given by

$$\mathfrak{D}(\mathcal{H}_{A,V}) = \Big\{\varphi \in H^1_A(\mathbb{R}^N;V) \,:\, \mathcal{H}_{A,V}[\varphi] \in L^2(\mathbb{R}^N)\Big\}.$$

Then the quadratic form

$$\mathcal{Q}_{A,V}[\varphi,\varphi] = \int_{\mathbb{R}^N} |\nabla_A \varphi|^2\,dx + \int_{\mathbb{R}^N} V\,|\varphi|^2\,dx, \qquad \text{for } \varphi \in H^1_A(\mathbb{R}^N;V)$$

is *associated with the operator* $\mathcal{H}_{A,V}$, in the sense that

$$\mathfrak{D}(\mathcal{H}_{A,V}) \subseteq H^1_A(\mathbb{R}^N;V),$$

and

$$\int_{\mathbb{R}^N} \mathcal{H}_{A,V}[\varphi]\,\overline{\psi}\,dx = \mathcal{Q}_{A,V}[\varphi,\psi], \qquad \text{for every } \varphi \in \mathfrak{D}(\mathcal{H}_{A,V}),\ \psi \in H^1_A(\mathbb{R}^N;V).$$

4.4.1 Discreteness of the Spectrum

We now focus on the discreteness of the spectrum of our operator. As in the purely electric case, in the electromagnetic setting on $\mathbb{R}^N$ one may encounter a loss of compactness in the embedding of the magnetic Sobolev space $H^1_A(\mathbb{R}^N;V)$ into the Lebesgue space $L^2(\mathbb{R}^N)$. In the purely electric case, this difficulty is overcome thanks to the confining nature of the scalar potential $V(x)$, which ensures the compactness of the embedding $H^1(\mathbb{R}^N;V) \hookrightarrow L^2(\mathbb{R}^N)$. In this section, we show that, under the assumptions previously introduced on the vector potential A, the embedding

$$H^1_A(\mathbb{R}^N;V) \hookrightarrow L^2(\mathbb{R}^N). \tag{4.4.1}$$

is compact, as well. In particular, we will show that

$$H^1_A(\mathbb{R}^N;V) \hookrightarrow H^1(\mathbb{R}^N;V) \hookrightarrow L^2(\mathbb{R}^N).$$

where the first embedding is continuous and the second one is compact.

We first establish that under our standing assumptions, the magnetic Sobolev space $H^1_A(\mathbb{R}^N;V)$

is continuously embedded into $H^1(\mathbb{R}^N;V)$. This is content of the next preliminary result.

Lemma 4.4.3. *Under the assumptions on the vector potential presented in section 4.3, we have the continuous embedding*

$$H^1_A(\mathbb{R}^N;V) \hookrightarrow H^1(\mathbb{R}^N;V).$$

More precisely, there exists a constant $\overline{C}_{N,R,A} \geq 0$ such that

$$\|\varphi\|_{H^1(\mathbb{R}^N;V)} \leq \overline{C}_{N,R,A}\|\varphi\|_{H^1_A(\mathbb{R}^N;V)}, \quad \varphi \in H^1_A(\mathbb{R}^N;V). \tag{4.4.2}$$

Proof. We start by introducing an inequality that will be useful for the proof. By virtue of the triangle inequality, the standard gradient can be controlled in terms of the magnetic gradient:

$$\begin{aligned}\|\nabla\varphi\|_{L^2(\mathbb{R}^N)} &= \|((\nabla + iA)\varphi + (-iA\varphi))\|_{L^2(\mathbb{R}^N)}\\ &\leq \|\nabla_A\varphi\|_{L^2(\mathbb{R}^N)} + \|A\varphi\|_{L^2(\mathbb{R}^N)}.\end{aligned}$$

Thus, by using the elementary inequality $(a+b)^2 \leq 2a^2 + 2b^2$, we obtain

$$\|\nabla\varphi\|^2_{L^2(\mathbb{R}^N)} \leq 2\|\nabla_A\varphi\|^2_{L^2(\mathbb{R}^N)} + 2\|A\varphi\|^2_{L^2(\mathbb{R}^N)}. \tag{4.4.3}$$

We have

$$\begin{aligned}\|\varphi\|^2_{H^1(\mathbb{R}^N;V)} &= \|\varphi\|^2_{L^2(\mathbb{R}^N;V)} + \|\nabla\varphi\|_{L^2(\mathbb{R}^N)}\\ &\leq \|\varphi\|^2_{L^2(\mathbb{R}^N;V)} + 2\|\nabla_A\varphi\|^2_{L^2(\mathbb{R}^N)} + 2\|A\varphi\|^2_{L^2(\mathbb{R}^N)}\\ &\leq \|\varphi\|^2_{L^2(\mathbb{R}^N;V)} + 2\|\nabla_A\varphi\|^2_{L^2(\mathbb{R}^N)} + 2C_1^2\|(|x|+1)\varphi\|^2_{L^2(\mathbb{R}^N)},\end{aligned} \tag{4.4.4}$$

where we have used (4.4.3) and the assumption A_2 on A. Since $\nabla_A\varphi \in L^2(\mathbb{R}^N)$, its norm is finite, and we now show that the last integral is finite as well. Consider the last term:

$$\begin{aligned}4C_1^2\int_{\mathbb{R}^N}(|x|^2+1)|\varphi|^2\,dx &= 4C_1^2\int_{\mathbb{R}^N}|x|^2|\varphi|^2\,dx + 4C_1^2\int_{\mathbb{R}^N}|\varphi|^2\,dx\\ &\leq 4C_1^2\Big(\sigma^2R^2\int_{B_{\sigma R}(0)}|\varphi|^2\,dx + \Big(\frac{\sigma}{\sigma-1}\Big)^2\int_{\mathbb{R}^N}V|\varphi|^2\,dx\Big)\\ &\quad + 4C_1^2\Big(C_{N,R}\int_{\mathbb{R}^N}|\nabla_A\varphi|^2\,dx + C_{N,R}\int_{\mathbb{R}^N}V|\varphi|^2\,dx\Big),\end{aligned} \tag{4.4.5}$$

where the two integrals on the right-hand side of the equality are estimated by Lemma 3.3.4 and the Poincaré-type inequality (4.3.2) with constant $C_{N,R}$. Setting $\sigma = 2$, we obtain

$$\begin{aligned}4C_1^2\int_{\mathbb{R}^N}(|x|^2+1)|\varphi|^2\,dx &\leq 16C_1^2R^2\int_{B_{2R}(0)}|\varphi|^2\,dx + 16C_1^2\int_{\mathbb{R}^N}V|\varphi|^2\,dx\\ &\quad + 4C_1^2C_{N,R}\int_{\mathbb{R}^N}|\nabla_A\varphi|^2\,dx + 4C_1^2C_{N,R}\int_{\mathbb{R}^N}V|\varphi|^2\,dx.\end{aligned} \tag{4.4.6}$$

Estimating the integral over $B_{2R}(0)$ again by Poincaré-type inequality (4.3.2), we get

$$16C_1^2R^2 \int_{B_{2R}(0)} |\varphi|^2\,dx \leq 16C_1^2R^2C_{N,R} \int_{\mathbb{R}^N} |\nabla_A\varphi|^2\,dx + 16C_1^2R^2C_{N,R} \int_{\mathbb{R}^N} V|\varphi|^2\,dx. \tag{4.4.7}$$

Inserting these estimates into the previous inequality yields

$$\begin{aligned} \|\varphi\|^2_{H^1(\mathbb{R}^N;V)} \leq{}& (1 + 16C_1^2 + 4C_1^2C_{N,R} + 16C_1^2R^2C_{N,R})\|\varphi\|^2_{L^2(\mathbb{R}^N;V)} \\ &+ (2 + 4C_1^2C_{N,R} + 16C_1^2R^2C_{N,R})\|\nabla_A\varphi\|^2_{L^2(\mathbb{R}^N)}, \end{aligned} \tag{4.4.8}$$

and therefore

$$\|\varphi\|^2_{H^1(\mathbb{R}^N;V)} \leq \overline{C}_{N,R,A}\|\varphi\|^2_{H^1_A(\mathbb{R}^N;V)}, \tag{4.4.9}$$

with constant

$$\overline{C}_{N,R,A} = \max\left\{1 + 16C_1^2 + 4C_1^2C_{N,R} + 16C_1^2R^2C_{N,R},\ 2 + 4C_1^2C_{N,R} + 16C_1^2R^2C_{N,R}\right\}. \tag{4.4.10}$$

This concludes the proof. □

Remark 4.4.4. *We remark that the constant $\overline{C}_{N,R,A}$ depends on the vector potential A through the constant C_1 appearing in assumption A_2.*

The previous result allows us to reduce the analysis of the compactness of the embedding to the purely electric case.

Theorem 4.4.5. *Under our standing assumptions, let q be an exponent such that*

$$\begin{cases} \dfrac{2\,N}{N+2} < q < 2^*, & \text{if } N \geq 3, \\ 1 < q < \infty, & \text{if } N = 2, \\ 1 \leq q \leq \infty, & \text{if } N = 1. \end{cases}$$

Then the embedding

$$H^1_A(\mathbb{R}^N;V) \hookrightarrow L^q(\mathbb{R}^N),$$

is compact.

Proof. It is sufficient to observe that every bounded sequence in $H^1_A(\mathbb{R}^N;V)$ is bounded in $H^1(\mathbb{R}^N;V)$, as well, thanks to Lemma 4.4.3. The conclusion then follows from Theorem 3.4.3 □

As in the purely electric case, applying the Spectral Theorem requires that the resolvent operator satisfy linearity, continuity, self-adjointness, positive definiteness, and compactness. The demonstration of these properties for the resolvent of the magnetic Schroedinger operator proceeds along the same lines as in the electric case (Lemma 3.4.4), with the additional key requirement that the magnetic potential A be real-valued to ensure the self-adjointness of the magnetic Schrödinger operator. Continuity is established by employing the magnetic Poincaré inequality (4.3.2).

Building on the previous result, we can use standard findings from classical Spectral Theory to derive the following

Corollary 4.4.6. *Under the standing assumptions, the spectrum of the operator $\mathcal{H}_{A,V}$ is made of a countable sequence of positive eigenvalues $\{\lambda_{n,A}(V)\}_{n\in\mathbb{N}^*}$ diverging to $+\infty$, each one having an associated eigenstate $\Psi_n \in H^1_A(\mathbb{R}^N;V)$, $\|\Psi\|_{L^2(\mathbb{R}^N)} = 1$. The set $\{\Psi_n\}_{n\in\mathbb{N}^*}$ forms an orthonormal basis of $L^2(\mathbb{R}^N)$. Finally, each eigenvalue has the following variational characterization*

$$\lambda_{n,A}(V) = \inf\left\{ \max_{\varphi\in W\cap\mathcal{S}_2(\mathbb{R}^N)} \int_{\mathbb{R}^N} |\nabla_A\varphi|^2\,dx + \int_{\mathbb{R}^N} V\,|\varphi|^2\,dx \;:\; \begin{array}{c} W\subseteq H^1_A(\mathbb{R}^N;V) \\ \textit{subspace with} \dim W = n \end{array} \right\},$$

and the infimum above is attained by the vector space generated by $\{\varphi_1,\dots,\varphi_n\}$.

Remark 4.4.7 (The ground state)**.** *In particular, we recall that for $n = 1$ the above formulation reduces to*

$$\lambda_{1,A}(V) = \min_{\varphi\in H^1_A(\mathbb{R}^N;V)\cap\mathcal{S}_2(\mathbb{R}^N)} \left\{ \int_{\mathbb{R}^N} |\nabla_A\varphi|^2\,dx + \int_{\mathbb{R}^N} V\,|\varphi|^2\,dx \right\}.$$

Thus, as usual $\lambda_{1,A}(V)$ coincides with the sharp constant in the Poincaré inequality (4.3.2)*. Accordingly, from Proposition 4.3.2 we have*

$$\lambda_{1,A}(V) \geq \frac{1}{C_{N,R}}. \tag{4.4.11}$$

We recall that $C_{N,R} \sim R^2$, for R diverging to $+\infty$. By suitably adapting the example in Remark 3.5.2, we can show that for some physical configurations the behaviour $\lambda_{1,A}(V) \sim 1/R^2$ is optimal. For $0<\delta<R$, let us take

$$\Sigma = \overline{B_R(0)} \setminus B_\delta(0) \qquad \textit{for which} \qquad V(x) = \min\left\{(|x|-R)_+^2, (|x|-\delta)_+^2\right\},$$

and choose a vector potential A identically vanishing on the potential well Σ, i.e. the magnetic field is absent inside Σ. Then we have that[1]

$$\int_{\mathbb{R}^N} |\nabla_A\varphi|^2\,dx + \int_{\mathbb{R}^N} V\,|\varphi|^2\,dx = \int_{B_R(0)\setminus\overline{B_\delta(0)}} |\nabla\varphi|^2\,dx, \qquad \textit{for every } \varphi\in C^\infty_c(B_R(0)\setminus\overline{B_\delta(0)}),$$

[1] We intend that functions in $C^\infty_c(B_R(0)\setminus\overline{B_\delta(0)})$ are extended by zero, outside of the open set.

since both A and V identically vanishes on $B_R(0) \setminus \overline{B_\delta(0)}$. Thus, we obtain

$$\lambda_{1,A}(V) = \inf_{\varphi \in C_c^\infty(\mathbb{R}^N)} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla_A \varphi|^2 \, dx + \int_{\mathbb{R}^N} V \, |\varphi|^2 \, dx}{\displaystyle\int_{\mathbb{R}^N} |\varphi|^2 \, dx}$$
$$\le \inf_{\varphi \in C_c^\infty(B_R(0)\setminus\overline{B_\delta(0)})} \frac{\displaystyle\int_{B_R(0)\setminus\overline{B_\delta(0)}} |\nabla \varphi|^2 \, dx}{\displaystyle\int_{B_R(0)\setminus\overline{B_\delta(0)}} |\varphi|^2 \, dx} = \left(\frac{2}{R-\delta}\right)^2 \lambda_{1,\mathrm{Dir}}(B_1(0)).$$

Here, by $\lambda_{1,\mathrm{Dir}}$ we intend the first eigenvalue of the Dirichlet-Laplacian of an open set and we used that the spherical shell $B_R(0) \setminus \overline{B_\delta(0)}$ contains a ball with radius $(R-\delta)/2$. Thus, by joining this estimate and (4.4.11), we get

$$0 < \lim_{R\to+\infty} \frac{R^2}{C_{N,R}} \le \lim_{R\to+\infty} R^2 \, \lambda_{1,A}(V) \le \lim_{R\to+\infty} \left(\frac{2\,R}{R-\delta}\right)^2 \lambda_{1,\mathrm{Dir}}(B_1(0)) < +\infty,$$

in this case.

4.5 Spectral Stability

This section presents spectral continuity estimates, which ensure that, for sufficiently small potential wells Σ, the spectrum of the magnetic operator remains qualitatively close to that of the model case $V(x) = |x|^2$. Unlike in the electric case, in the result presented in Lemma 4.5.1, translation invariance of the spectrum cannot be exploited; consequently, the analysis relies on the properties of the scalar potential $V(x)$. The Lemma thus provides an estimate that generalizes the corresponding result in the electric case, also exhibiting a dependence on the geometric parameter δ. As a result, the conclusions of the subsequent Proposition 4.5.2 exhibit the same dependence.

Lemma 4.5.1. *For any $n \in \mathbb{N}^*$ we have*

$$\sqrt{\lambda_{n,A}} - R \le \sqrt{\lambda_{n,A}(V)} \le \sqrt{\lambda_{n,A}} + \delta.$$

where $\lambda_{n,A}$ is the n-th eigenvalue of the magnetic quantum harmonic oscillator.

Proof. We prove the two bounds on $\lambda_{n,A}(V)$ separately.

Lower bound. The proof is carried out in a manner analogous to that of the electric case as established in Lemma 3.5.1.

Upper bound. Differently from Lemma 3.5.1, it is sufficient to use (3.3.4), to estimate the potential $V(x)$, and observe that φ has unit $L^2(\mathbb{R}^N)$-norm, so we get

$$\int_{\mathbb{R}^N} |\nabla_A \varphi|^2 \, dx + \int_{\mathbb{R}^N} V|\varphi|^2 \, dx \le \int_{\mathbb{R}^N} |\nabla_A \varphi|^2 \, dx + (1+\varepsilon) \int_{\mathbb{R}^N} |x|^2 |\varphi|^2 \, dx + \left(1 + \frac{1}{\varepsilon}\right)^2 \delta^2.$$

In this way we obtain the following estimate

$$\lambda_{n,A}(V) \leq (1+\varepsilon)\lambda_{n,A} + \left(1+\frac{1}{\varepsilon}\right)^2 \delta^2.$$

This is valid for every $\varepsilon > 0$: now we take

$$\varepsilon = \frac{\delta}{\sqrt{\lambda_{n,A}}}$$

and we have

$$\lambda_{n,A}(V) \leq \left(1+\frac{\delta}{\sqrt{\lambda_{n,A}}}\right)\lambda_{n,A} + \left(1+\frac{\sqrt{\lambda_{n,A}}}{\delta}\right)\delta^2.$$

After the same trivial computations, we obtain

$$\lambda_{n,A}(V) \leq \left(\sqrt{\lambda_{n,A}}+\delta\right)^2$$

from which we get the upper bound

$$\sqrt{\lambda_{n,A}(V)} \leq \sqrt{\lambda_{n,A}} + \delta.$$

This conclude the proof. □

Using the Hausdorff distance recalled in (2.6.1), we establish the following continuity estimate, expressed in terms of the distance between the potential wells.

Proposition 4.5.2. *Let $\Sigma_1, \Sigma_2 \subseteq \mathbb{R}^N$ be two non-empty compact sets. We consider the corresponding potentials*

$$V_i(x) = (\mathrm{dist}(x, \Sigma_i))^2, \qquad \text{for } x \in \mathbb{R}^N.$$

Then, for every $n \in \mathbb{N}^*$ we have

$$|\lambda_{n,A}(V_1) - \lambda_{n,A}(V_2)| \leq \mathrm{dist}_H(\Sigma_1, \Sigma_2)\ \max\left\{4+\frac{\lambda_{n,A}+\delta}{2},\ 2(\lambda_{n,A}+\delta)\right\},$$

where $\lambda_{n,A}$ is the $n-th$ eigenvalue of the magnetic quantum harmonic oscillator.

Proof. As above, the proof proceeds similarly to that of the electric case in Propostion 3.5.3, observing that the dependence on the geometric parameter δ is now taken into account through the previous result. □

4.6 Integrability and Exponential Decay for the Eigenfunctions

In this section we establish some integrability properties of the eigenstates and prove their exponential decay in the sup norm.

Unlike the electric case, Moser-type iterative techniques cannot be applied directly due to the distinct algebraic structure of the magnetic gradient and Laplacian. Therefore, the main results are obtained by means of Kato's inequality (2.7.4), which allows us to reduce the analysis to the purely electric case. To this aim, we will need the following result.

Lemma 4.6.1. *For every $n \in \mathbb{N}^*$, we have that $|\Psi_n|$ is a non-negative weak subsolution of the equation*

$$-\Delta\Psi_n + V\,\Psi_n = \lambda_{n,A}\,\Psi_n.$$

Proof. We appeal to Lemma 2.7.4. We need to verify that the following conditions are satisfied:

1. $A_j \in C^1(\mathbb{R}^N, \mathbb{R}^N)$;
2. $\Psi_n \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$;
3. $\Delta_A\Psi_n \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$

The first condition is simply ensured by assumption A_1 on A. The second condition follows from the fact that $\Psi_n \in L^2(\mathbb{R}^N)$. As for the third condition, we observe that by assumption

$$-\Delta_A\Psi_n = \lambda_{n,A}\Psi_n - V\Psi_n.$$

Thus, it is sufficient to show that the right-hand side belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Of course, we have $\lambda_{n,A}\Psi_n \in L^2(\mathbb{R}^N) \subseteq L^1_{\mathrm{loc}}(\mathbb{R}^N)$. The scalar potential V is locally Lipschitz and hence bounded on any ball of finite radius R. Thus, for every $r > 0$ we have by Hölder's inequality

$$\int_{B_r(0)} V\Psi_n\,dx \leq \left(\int_{B_r(0)} V|\Psi_n|^2\,dx\right)^{\frac{1}{2}} \left(\int_{B_r(0)} V\,dx\right)^{\frac{1}{2}}.$$

Both integrals are finite, thanks to the local boundedness of V and the fact that $\Psi_n(x) \in L^2(\mathbb{R}^N; V)$. We then get $V\Psi_n \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, as well. This ensures that $\Delta_A\Psi_n \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, as desired.

We now set $F = \lambda_{n,A} - V$, so that have

$$-\Delta_A\Psi_n = F\,\Psi_n.$$

On account of Kato's inequality (2.7.4), we get

$$\Delta|\Psi_n| \geq \Re[(\operatorname{sgn}\overline{\Psi_n})\Delta_A\Psi_n] = \Re\left[\frac{\overline{\Psi_n}}{|\Psi_n|}(-F\Psi_n)\right].$$

By using that F is real-valued, we then obtain

$$-\Delta|\Psi_n| \leq F\left(\Re\frac{\Psi_n\overline{\Psi_n}}{|\Psi_n|}\right) = F|\Psi_n|.$$

Hence, we get that the modulus $|\Psi_n|$ is a subsolution of the corresponding purely electric problem, as claimed. □

4.6.1 Integrability of Eigenfunctions

An explicit $L^\infty-$bound on eigenstates is now an easy consequence of the previous result. Namely, we have the following

Proposition 4.6.2. *We have $\Psi_n \in L^\infty(\mathbb{R}^N)$ for every $n \in \mathbb{N}^*$. More precisely, we have the following estimate*

$$\|\Psi_n\|_{L^\infty(\mathbb{R}^N)} \leq \mathrm{M}_N \left(\lambda_{n,A}(V)\right)^{\frac{N}{4}},$$

where $\mathrm{M}_N > 0$ is an explicit constant depending only on the dimension.

Proof. We obviously have

$$\|\Psi_n\|_{L^\infty(\mathbb{R}^N)} = \||\Psi_n|\|_{L^\infty(\mathbb{R}^N)}.$$

Thanks to Lemma 4.6.1, we know that $|\Psi_n|$ is a non-negative subsolution of the eigenvalue equation, for the purely electric case. Then the claimed result follows by repeating verbatim the proof of Proposition 3.6.1, based on the *Moser iteration technique*, noting that the proof there applies to subsolutions, as well. □

Remark 4.6.3. *By using the estimate of Lemma 3.5.1, we can also infer the following uniform estimate*

$$\|\Psi_n\|_{L^\infty(\mathbb{R}^N)} \leq \mathrm{M}_N \left(\lambda_{n,A} + \delta\right)^{\frac{N}{4}}, \qquad \textit{for every } n \in \mathbb{N}^*,$$

where $\lambda_{n,A}$ is the $n-$th eigenvalue of the magnetic quantum harmonic oscillator.

At this point, we recall the corresponding weighted integrability result established in the purely electric case, suitably adapted to the electromagnetic setting. Note that the constant now also depends on the parameter δ and on the vector potential $A(x)$.

Proposition 4.6.4. *For every $n \in \mathbb{N}^*$ and $k \in \mathbb{N}$, we have*

$$\int_{\mathbb{R}^N} |\nabla_A \Psi_n|^2 \, |x|^{2\,k} \, dx + \int_{\mathbb{R}^N} |\Psi_n|^2 \, |x|^{2\,k+2} \, dx \leq C_{N,n,k,R,\delta,A}.$$

Proof. We observe at first that the statement is true for $k = 0$, i.e. we have

$$\int_{\mathbb{R}^N} |\nabla_A \Psi_n|^2 \, dx + \int_{\mathbb{R}^N} |\Psi_n|^2 \, |x|^2 \, dx \leq C_{N,n,R,\delta,A}.$$

Indeed, we have

$$\int_{\mathbb{R}^N} |\Psi_n|^2 \, |x|^2 \, dx < +\infty,$$

directly from Lemma (3.3.4). More precisely, from inequality (3.3.6) with $\sigma = 2$ we have

$$\int_{\mathbb{R}^N} |\Psi_n|^2 \, |x|^2 \, dx \le 4\, R^2 \int_{B_{2R}(0)} |\Psi_n|^2 \, dx + 4 \int_{\mathbb{R}^N \setminus B_{2R}(0)} V\, |\Psi_n|^2 \, dx$$
$$\le 4\, R^2 + 4\, \lambda_{n,A}(V).$$

On account of Lemma (3.5.1), this gives

$$\int_{\mathbb{R}^N} |\Psi_n|^2 \, |x|^2 \, dx \le 4\, R^2 + 4\, (\lambda_{n,A} + \delta). \tag{4.6.1}$$

In order to conclude, we will prove the following recursive gain of weighted integrability
(4.6.2)

$$\textit{if we have} \quad \int_{\mathbb{R}^N} |\nabla_A \Psi_n|^2 \, |x|^{2\,k} \, dx + \int_{\mathbb{R}^N} |\Psi_n|^2 \, |x|^{2\,k+2} \, dx \le C_{N,n,k,R,\delta,A},$$
$$\textit{then we have} \quad \int_{\mathbb{R}^N} |\nabla_A \Psi_n|^2 \, |x|^{2\,k+2} \, dx + \int_{\mathbb{R}^N} |\Psi_n|^2 \, |x|^{2\,k+4} \, dx \le C_{N,n,k+1,R,\delta,A} \quad \textit{as well.}$$

This will be sufficient to get the claimed result. Let us suppose that for a $k \in \mathbb{N}$ we have

$$\int_{\mathbb{R}^N} |\nabla_A \Psi_n|^2 \, |x|^{2\,k} \, dx + \int_{\mathbb{R}^N} |\Psi_n|^2 \, |x|^{2\,k+2} \, dx \le C_{N,n,k,R,\delta,A}.$$

Let $M > 0$ and let η_M be a Lipschitz cut-off function such that

$$0 \le \eta_M \le 1, \quad \eta_M \equiv 1 \text{ on } B_M(0), \quad \eta_M \equiv 0 \text{ on } \mathbb{R}^N \setminus B_{M+1}(0),$$

and

$$|\nabla \eta_M| \le 1.$$

We then take the test function

$$\psi = \Psi_n \, |x|^{2\,k+2} \, \eta_M^2,$$

in the weak formulation. Observe that this is feasible, thanks to the properties of both Ψ_n and η_M. We get

$$\int_{\mathbb{R}^N} \nabla_A \Psi_n \overline{\nabla_A \psi} \, dx + \int_{\mathbb{R}^N} V\, \Psi_n \psi \, dx = \lambda_{n,A}(V) \int_{\mathbb{R}^N} \Psi_n \psi \, dx,$$

$$\int_{\mathbb{R}^N} \nabla_A \Psi_n \overline{\nabla_A (\Psi_n \, |x|^{2\,k+2} \, \eta_M^2)} \, dx + \int_{\mathbb{R}^N} V |\Psi_n|^2 \, |x|^{2\,k+2} \, \eta_M^2 \, dx = \lambda_{n,A}(V) \int_{\mathbb{R}^N} |\Psi_n|^2 \, |x|^{2\,k+2} \, \eta_M^2 \, dx.$$

We consider the first integral. Thanks to Lemma 2.7.2 and some straightforward calculations, we obtain

$$\begin{aligned}\nabla_A\Psi_n\overline{\nabla_A(\Psi_n\,|x|^{2\,k+2}\,\eta_M^2)} &= \nabla_A\Psi_n((\overline{\nabla_A\Psi_n})|x|^{2k+2}\eta_M^2 + (2k+2)\Psi_n|x|^{2k+1}\eta_M^2\\ &\quad + 2\eta_M\Psi_n|x|^{2k+2}\nabla\eta_M)\\ &= |\nabla_A\Psi_n|^2|x|^{2k+2}\eta_M^2 + (2k+2)\left\langle\nabla_A\Psi_n, x\right\rangle\Psi_n|x|^{2k}\eta_M^2\\ &\quad + 2\left\langle\nabla_A\Psi_n,\nabla\eta_M\right\rangle\Psi_n|x|^{2k+2}\eta_M.\end{aligned}$$

Now we get

$$\begin{aligned}\int_{\mathbb{R}^N}|\nabla_A\Psi_n|^2\,|x|^{2\,k+2}\,\eta_M^2\,dx + \int_{\mathbb{R}^N}V\,|\Psi_n|^2\,|x|^{2\,k+2}\,\eta_M^2\,dx&\\ = \lambda_{n,A}(V)\int_{\mathbb{R}^N}|\Psi_n|^2\,|x|^{2\,k+2}\,\eta_M^2\,dx&\\ -(2\,k+2)\int_{\mathbb{R}^N}\left\langle\nabla_A\Psi_n, x\right\rangle\Psi_n\,|x|^{2\,k}\,\eta_M^2\,dx&\\ -2\int_{\mathbb{R}^N}\left\langle\nabla_A\Psi_n,\nabla\eta_M\right\rangle\Psi_n\,|x|^{2\,k+2}\,\eta_M\,dx.&\end{aligned}$$

By using Young's inequality on the last two integrals, for every $M>0$ and $\mu>0$ we get

$$\begin{aligned}\int_{\mathbb{R}^N}|\nabla_A\Psi_n|^2\,|x|^{2\,k+2}\,\eta_M^2\,dx + \int_{\mathbb{R}^N}V\,|\Psi_n|^2\,|x|^{2\,k+2}\,\eta_M^2\,dx &\le \lambda_{n,A}(V)\int_{\mathbb{R}^N}|\Psi_n|^2\,|x|^{2\,k+2}\,\eta_M^2\,dx\\ &\quad+\mu\int_{\mathbb{R}^N}|\nabla_A\Psi_n|^2\,|x|^{2\,k+2}\,\eta_M^2\,dx\\ &\quad+\frac{(k+1)^2}{\mu}\int_{\mathbb{R}^N}|\Psi_n|^2\,|x|^{2\,k}\,\eta_M^2\,dx\\ &\quad+\mu\int_{\mathbb{R}^N}|\nabla_A\Psi_n|^2\,|x|^{2\,k+2}\,\eta_M^2\,dx\\ &\quad+\frac{1}{\mu}\int_{\mathbb{R}^N}|\nabla\eta_M|^2\,|x|^{2\,k+2}\,|\Psi_n|^2\,dx.\end{aligned}$$

We can take $\mu = 1/4 \Rightarrow \frac{1}{\mu} = 4$ so to absorb the two integrals in the right-hand side containing $\nabla_A\Psi_n$. This gives

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^N}|\nabla_A\Psi_n|^2\,|x|^{2\,k+2}\,\eta_M^2\,dx + \int_{\mathbb{R}^N}V\,|\Psi_n|^2\,|x|^{2\,k+2}\,\eta_M^2\,dx &\le \lambda_{n,A}(V)\int_{\mathbb{R}^N}|\Psi_n|^2\,|x|^{2\,k+2}\,\eta_M^2\,dx\\ &\quad+4\,(k+1)^2\int_{\mathbb{R}^N}|\Psi_n|^2\,|x|^{2\,k}\,\eta_M^2\,dx\\ &\quad+4\int_{\mathbb{R}^N}|\nabla\eta_M|^2\,|x|^{2\,k+2}\,|\Psi_n|^2\,dx.\end{aligned}$$

Thanks to the properties of η_M, this implies that

$$\begin{aligned}\frac{1}{2}\int_{B_M(0)} |\nabla_A \Psi_n|^2\,|x|^{2\,k+2}\,dx + \int_{B_M(0)} V\,|\Psi_n|^2\,|x|^{2\,k+2}\,dx\\ \le (\lambda_{n,A}+\delta+4)\int_{\mathbb{R}^N} |\Psi_n|^2\,|x|^{2\,k+2}\,dx\\ + 4\,(k+1)^2\int_{\mathbb{R}^N} |\Psi_n|^2\,|x|^{2\,k}\,dx.\end{aligned} \tag{4.6.3}$$

We used again Lemma 3.5.1, to bound the eigenvalue. We observe that by Hölder's inequality and the fact that Ψ_n has unit L^2-norm, we obtain

$$\int_{\mathbb{R}^N} |\Psi_n|^2\,|x|^{2\,k}\,dx \le \left(\int_{\mathbb{R}^N} |\Psi_n|^2\,|x|^{2\,k+2}\,dx\right)^{\frac{k}{k+1}} \le (C_{N,n,k,R,\delta,A})^{\frac{k}{k+1}}\,.$$

By letting M go to $+\infty$ in (4.6.3) and using Lemma 3.3.4, we prove (3.6.9). As already explained, this concludes the proof. □

As a consequence of the previous result, we also get the following

Corollary 4.6.5. *For every $n \in \mathbb{N}^*$ and every $2 \le p < \infty$ we have*

$$V\,\Psi_n \in L^p(\mathbb{R}^N).$$

More precisely, there exists a constant $C = C(N,n,R,p,\delta,A) > 0$ such that

$$\int_{\mathbb{R}^N} |V|^p\,|\Psi_n|^p\,dx \le C, \qquad \textit{for every } R \ge 0.$$

Proof. For every $2 \le p < \infty$, we have

$$\int_{\mathbb{R}^N} |V|^p\,|\Psi_n|^p\,dx \le \|\Psi_n\|^{p-2}_{L^\infty(\mathbb{R}^N)}\int_{\mathbb{R}^N} |V|^p\,|\Psi_n|^2\,dx.$$

By using (3.2.4) and Remark 3.6.2, we obtain

$$\int_{\mathbb{R}^N} |V|^p\,|\Psi_n|^p\,dx \le \left(\mathrm{M}_N\left(\lambda_{n,A}+\delta\right)^{\frac{N}{4}}\right)^{p-2}\int_{\mathbb{R}^N} (|x|+R)^{2\,p}\,|\Psi_n|^2\,dx.$$

The last term can be bounded from above, thanks to Proposition 3.6.3. □

4.6.2 Exponential Decay at Infinity.

In an analogous way of Proposition 4.6.2, we can establish the pointwise exponential decay of the eigenstates. We can rely once again on Kato's inequality, which allows us to reduce the problem to the purely electric case.

Theorem 4.6.6 (Exponential decay in L^∞). *There exists an exponent $\alpha = \alpha(N) > 1$ and a constant $C_1 = C_1(N, n, R) > 0$ such that*

$$0 \leq |\Psi_n(x)|^2 \leq C_1 \, e^{-\frac{|x|}{2} \log \alpha}, \qquad \textit{for every } x \in \mathbb{R}^N.$$

Proof. As in the previous result, it is sufficient to use Lemma 4.6.1 to get that $|\Psi_n|$ is a subsolution. Then the result follows by reproducing verbatim the proof of Theorem 3.8.3, which equally applies to subsolutions. □

4.7 Higher Regularity of Eigenfunctions

This section establishes regularity properties of the eigenstates of the magnetic Schrödinger operator and, in particular, defines its domain.

The main difficulty compared to the electric case lies in establishing the H^2 regularity of the eigenfunctions, which involves translations within the framework of the Calderón–Zygmund Theorem 2.5.1. For this reason, the Theorem cannot be applied directly; instead, a perturbative approach based on the standard Laplacian is employed, allowing the application of Calderón–Zygmund Theorem.

Our eigenstates belong to a higher order Sobolev space. This is the content of the following slightly more general result.

Theorem 4.7.1. *For $f \in L^2(\mathbb{R}^N)$, let $\Psi \in H^1_A(\mathbb{R}^N; V)$ be the weak solution of*

$$\mathcal{H}_{A,V}[\Psi] = f.$$

Then we have

$$\Psi \in H^2(\mathbb{R}^N) \cap \Big\{ \psi \in L^2(\mathbb{R}^N) \, : \, V\,\psi \in L^2(\mathbb{R}^N) \Big\}.$$

In particular, the domain of $\mathcal{H}_{A,V}$ is given by

$$\mathfrak{D}(\mathcal{H}_{A,V}) = H^2(\mathbb{R}^N) \cap \Big\{ \psi \in L^2(\mathbb{R}^N) \, : \, V\,\psi \in L^2(\mathbb{R}^N) \Big\}.$$

Proof. First inclusion:

$$\mathfrak{D}\left(\mathcal{H}_{A,V}\right) \subseteq H^2\left(\mathbb{R}^N\right) \cap \left\{ \Psi \in L^2\left(\mathbb{R}^N\right) \, : \, V\Psi \in L^2\left(\mathbb{R}^N\right) \right\}.$$

That is, it suffices to verify that $V\Psi \in L^2(\mathbb{R}^N)$ and $-\Delta\Psi \in L^2(\mathbb{R}^N)$.

First, we want to prove that $V\Psi \in L^2(\mathbb{R}^N)$. It is already known that $\Psi \in H^1_A\left(\mathbb{R}^N, V\right) \subseteq L^2(\mathbb{R}^{\mathbb{N}})$, due to Lemma 3.4.2. We first need to prove this

$$\int_{\mathbb{R}^N} V^2 |\Psi|^2 dx < +\infty. \tag{4.7.1}$$

Thanks to Lemma 3.3.1 it is sufficient to prove that

$$\int_{\mathbb{R}^N} |x|^4|\Psi|^2 dx < +\infty. \tag{4.7.2}$$

Let $M > 0$ and let η_M be a Lipschitz cut-off function such that

$$0 \le \eta_M \le 1, \quad \eta_M \equiv 1 \text{ on } B_M(0), \quad \eta_M \equiv 0 \text{ on } \mathbb{R}^N \setminus B_{M+1}(0),$$

and

$$|\nabla \eta_M| \leqslant 1.$$

We recall the setting in which we are working

$$\mathcal{H}_{A,V}[\Psi] = f \quad \text{i.e.} \quad -(\nabla + iA)^2\Psi + V\Psi = f$$

which leads to the following weak formulation

$$\int_{\mathbb{R}^n} \left\langle \nabla_A \Psi, \overline{\nabla_A \psi} \right\rangle dx + \int_{\mathbb{R}^n} V\Psi\overline{\psi}\, dx = \int_{\mathbb{R}^n} f\overline{\psi}\, dx.$$

The following function is now employed as a test function, $\psi = \Psi|x|^2\eta_M^2$

$$\int_{\mathbb{R}^N} \left\langle \nabla_A \Psi, \overline{\nabla_A \left(\Psi|x|^2\eta_M^2\right)} \right\rangle dx + \int_{\mathbb{R}^N} V\Psi \left(\Psi|x|^2\eta_M^2\right) dx = \int_{\mathbb{R}^N} f \left(\Psi|x|^2\eta_M^2\right) dx$$

Attention is now focused on the following term $\nabla_A \Psi \overline{\nabla_A \left(\Psi|x|^2\eta_M^2\right)}$ and thanks to Lemma 2.7.2 we get

$$\begin{aligned}
\nabla_A \Psi \left(\overline{\nabla_A \left(\Psi|x|^2\eta_M^2\right)}\right) &= \nabla_A \Psi \left[\left(\overline{\nabla_A \Psi}\right)|x|^2\eta_M^2 + 2x\Psi\eta_M^2 + 2\Psi|x|^2\eta_M\nabla\eta_M\right] \\
&= |\nabla_A \Psi|^2 |x|^2\eta_M^2 + 2\left\langle \nabla_A \Psi, x\right\rangle \Psi\eta_M^2 \\
&+ 2\left\langle \nabla_A \Psi, \nabla\eta_M\right\rangle \Psi|x|^2\eta_M
\end{aligned}$$

and one has

$$\begin{aligned}
\int |\nabla_A \Psi|^2 |x|^2\eta_M^2\, dx + \int V|\Psi|^2|x|^2\eta_M^2\, dx = &\int f\Psi|x|^2\eta_M^2 dx \\
&- 2\int \left\langle \nabla_A \Psi, x\right\rangle \Psi\eta_M^2\, dx \\
&- 2\int \left\langle \nabla_A \Psi, \nabla\eta_M\right\rangle \Psi|x|^2\eta_M\, dx.
\end{aligned}$$

By using (3.3.5) with $\varepsilon = 1$ on the left-hand side

$$|x|^2 \le 2V(x) + 2R^2 \quad \Rightarrow \quad \frac{|x|^2}{2} - R^2 \le V(x)$$

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla_A \Psi|^2 |x|^2 \eta_M^2 \, dx + \int_{\mathbb{R}^n} \left(\frac{|x|^2}{2} - R^2 \right) |\Psi|^2 |x|^2 \eta_M^2 \, dx = \\
&= \int_{\mathbb{R}^n} |\nabla_A \Psi|^2 |x|^2 \eta_M^2 \, dx + \int_{\mathbb{R}^n} \frac{|x|^2}{2} |\Psi|^2 |x|^2 \eta_M^2 \, dx + \int_{\mathbb{R}^n} -R^2 |\Psi|^2 |x|^2 \eta_M^2 \, dx \\
&= \int_{\mathbb{R}^n} |\nabla_A \Psi|^2 |x|^2 \eta_M^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^n} |x|^2 |\Psi|^2 |x|^2 \eta_M^2 \, dx - R^2 \int_{\mathbb{R}^n} |\Psi|^2 |x|^2 \eta_M^2 \, dx \\
&= \int_{\mathbb{R}^n} |\nabla_A \Psi|^2 |x|^2 \eta_M^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^n} |\Psi|^2 |x|^4 \eta_M^2 \, dx - R^2 \int_{\mathbb{R}^n} |\Psi|^2 |x|^2 \eta_M^2 \, dx
\end{aligned}
$$

and Young's inequality on the right-hand side for every $M > 0$ and $\mu > 0$, one gets

$$
\begin{aligned}
\int_{\mathbb{R}^N} |\nabla_A \Psi|^2 |x|^2 \eta_M^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} |\Psi|^2 |x|^4 \eta_M^2 \, dx \leq R^2 & \int_{\mathbb{R}^N} |\Psi|^2 |x|^2 \eta_M^2 \, dx \\
& + \frac{1}{2\mu} \int_{\mathbb{R}^N} |f|^2 \eta_M^2 \, dx + \frac{\mu}{2} \int_{\mathbb{R}^N} |\Psi|^2 |x|^4 \eta_M^2 \, dx \\
& + \mu \int_{\mathbb{R}^n} |\nabla_A \Psi|^2 |x|^2 \eta_M^2 \, dx \\
& + \frac{1}{\mu} \int_{\mathbb{R}^N} |\Psi|^2 \eta_M^2 \, dx \\
& + \mu \int_{\mathbb{R}^N} |\nabla_A \Psi|^2 |x|^2 \eta_M^2 \, dx \\
& + \frac{1}{\mu} \int_{\mathbb{R}^N} |\nabla \eta_M|^2 \Psi|^2 |x|^2 \, dx.
\end{aligned}
$$

If we now take $\mu = 1/4 \Rightarrow \frac{1}{\mu} = 4$, we can absorbe the two integrals in the right-hand side containing $\nabla_A \Psi$, as well as the term $|\Psi|^2 |x|^4$. This gives

$$
\begin{aligned}
\int |\nabla_A \Psi|^2 |x|^2 \eta^2 \, dx + \frac{1}{2} \int |\Psi|^2 |x|^4 \eta_M^2 \, dx \leqslant R^2 & \int |\Psi|^2 |x|^2 \eta_M^2 \, dx \\
& + 2 \int |f|^2 \eta_M^2 \, dx + \frac{1}{8} \int |\Psi|^2 |x|^4 \eta_M^2 \, dx \\
& + \frac{1}{4} \int |\nabla_A \Psi|^2 |x|^2 \eta_M^2 \, dx \\
& + 4 \int |\Psi|^2 \eta_M^2 \, dx \\
& + \frac{1}{4} \int |\nabla_A \Psi|^2 |x|^2 \eta_M^2 \, dx + 4 \int |\nabla \eta_M|^2 |\Psi|^2 ||x|^2 \, dx
\end{aligned}
$$

and one has

$$
\begin{aligned}
\frac{1}{2} \int_{\mathbb{R}^N} |\nabla_A \Psi|^2 |x|^2 \eta^2 \, dx + \frac{3}{8} \int_{\mathbb{R}^N} |\Psi|^2 |x|^4 \eta_M^2 \, dx \leqslant R^2 & \int_{\mathbb{R}^N} |\Psi|^2 |x|^2 \eta_M^2 \, dx \\
& + 2 \int_{\mathbb{R}^N} |f|^2 \eta_M^2 \, dx \\
& + 4 \int_{\mathbb{R}^N} |\Psi|^2 \eta_M^2 \, dx \\
& 4 \int_{\mathbb{R}^N} |\nabla \eta_M|^2 |\Psi|^2 |x|^2 \, dx.
\end{aligned}
$$

Thanks to the properties of η_M, this implies that

$$\begin{aligned}\frac{1}{2}\int_{B_M(0)} |\nabla_A \Psi|^2\,|x|^2\,dx + \frac{3}{8}\int_{B_M(0)} |\Psi|^2|x|^4\,dx \leqslant (4+R^2)\int_{B_M(0)} |\Psi|^2|x|^2\,dx \\ + 2\int_{B_M(0)} |f|^2\,dx \\ + 4\int_{B_M(0)} |\Psi|^2\,dx.\end{aligned} \tag{4.7.3}$$

By letting $M \to \infty$ and recalling Lemma 3.3.4 we get the claimed estimate. The previous result guarantees that both integrals on the left-hand side of the inequality are finite, a remark that will be useful in what follows.

In order to establish the H^2-regularity of the eigenstates, it suffices to verify that $-\Delta\Psi \in L^2(\mathbb{R}^N)$. By the Calderón–Zygmund Theorem, this condition ensures that $\Psi \in H^2(\mathbb{R}^N)$. To this end, we employ a perturbative argument: the magnetic Laplacian can be rewritten as a perturbation of the standard Laplacian, and under the assumptions imposed on the vector potential, the additional terms can be shown to belong to $L^2(\mathbb{R}^N)$, which allows the application of the theorem.

We start from the weak formulation

$$\int_{\mathbb{R}^N} -\Delta_A \Psi\,\overline{\psi}\,dx + \int_{\mathbb{R}^N} V\Psi\,\overline{\psi}\,dx = \int_{\mathbb{R}^N} F\,\overline{\psi}\,dx, \quad \psi \in H^1_A(\mathbb{R}^N; V). \tag{4.7.4}$$

We first focus on the magnetic Laplacian term on the left-hand side. Expanding Δ_A yields

$$\begin{aligned}\int_{\mathbb{R}^N} -\Delta_A \Psi\,\overline{\psi}\,dx = \int_{\mathbb{R}^N} \langle \nabla\Psi, \nabla\overline{\psi}\rangle\,dx + \int_{\mathbb{R}^N} \langle \nabla\Psi, (-iA\overline{\psi})\rangle\,dx \\ + \int_{\mathbb{R}^N} \langle (iA\Psi), \nabla\overline{\psi}\rangle\,dx + \int_{\mathbb{R}^N} |A|^2\Psi\,\overline{\psi}\,dx.\end{aligned} \tag{4.7.5}$$

Substituting this expansion into (4.7.4), we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} \langle \nabla\Psi, \nabla\overline{\psi}\rangle\,dx - i\int_{\mathbb{R}^N} \langle \nabla\Psi, A\rangle\overline{\psi}\,dx + i\int_{\mathbb{R}^N} \langle A\Psi, \nabla\overline{\psi}\rangle\,dx \\ + \int_{\mathbb{R}^N} |A|^2\Psi\,\overline{\psi}\,dx + \int_{\mathbb{R}^N} V\Psi\,\overline{\psi}\,dx = \int_{\mathbb{R}^N} F\,\overline{\psi}\,dx.\end{aligned} \tag{4.7.6}$$

Integrating by parts the term involving $\nabla\overline{\psi}$ and passing to the strong formulation, we obtain

$$-\Delta\Psi - 2i\langle \nabla\Psi, A\rangle - i(\mathrm{div}A)\Psi + |A|^2\Psi + V\Psi = F \tag{4.7.7}$$

from which follows

$$-\Delta\Psi = F - V\Psi + 2i\langle \nabla\Psi, A\rangle + i(\mathrm{div}A)\Psi - |A|^2\Psi \tag{4.7.8}$$

To conclude that $\Psi \in H^2(\mathbb{R}^N)$, it suffices to prove that each term on the right-hand side of the equation belongs to $L^2(\mathbb{R}^N)$. We treat them separately.

We first show that $\langle \nabla\Psi, A\rangle \in L^2(\mathbb{R}^N)$. Using the the vector potential's linear growth A_2, we have

$$\begin{aligned}\int_{\mathbb{R}^N} |A|^2|\nabla\Psi|^2\,dx &\le 2C_1^2\int_{\mathbb{R}^N}(|x|^2+1)|\nabla\Psi|^2\,dx\\ &\le 2C_1^2\int_{\mathbb{R}^N}|x|^2|\nabla\Psi|^2\,dx + 2C_1^2\int_{\mathbb{R}^N}|\nabla\Psi|^2\,dx.\end{aligned}\tag{4.7.9}$$

The second term on the right-hand side is finite since we have proved $H^1_A(\mathbb{R}^N;V) \hookrightarrow H^1(\mathbb{R}^N;V)$. To control the first integral, we use inequality (4.4.3), obtaining

$$\le 2C_1^2\int_{\mathbb{R}^N}|x|^2|\nabla_A\Psi|^2\,dx + 2C_1^2\int_{\mathbb{R}^N}|x|^2|A|^2|\Psi|^2\,dx + 2C_1^2\|\nabla\Psi\|^2_{L^2(\mathbb{R}^N)}.\tag{4.7.10}$$

By the results obtained in 4.7.3, the first integral is finite. The second term is estimated using again the linear growth condition A_2 on $A(x)$, yielding

$$\begin{aligned}&\le 2C_1^2\|x\nabla_A\Psi\|^2_{L^2(\mathbb{R}^N)} + 2C_1^2\|\nabla\Psi\|^2_{L^2(\mathbb{R}^N)}\\ &+ 4C_1^4\int_{\mathbb{R}^N}|x|^4|\Psi|^2\,dx + 4C_1^4\int_{\mathbb{R}^N}|x|^2|\Psi|^2\,dx.\end{aligned}$$

The integral involving the factor $|x|^4$ is finite thanks to the previous estimate (4.7.3), while the remaining integral can be bounded using assumption (3.3.6). Indeed, we have

$$\begin{aligned}&\le 2C_1^2\|x\nabla_A\Psi\|^2_{L^2(\mathbb{R}^N)} + 2C_1^2\|\nabla\Psi\|^2_{L^2(\mathbb{R}^N)} + 4C_1^4\|x^2\Psi\|^2_{L^2(\mathbb{R}^N)}\\ &+ 4C_1^4\sigma^2R^2\int_{B_{\sigma R}(0)}|\Psi|^2\,dx + 4C_1^4\Big(\frac{\sigma}{\sigma-1}\Big)^2\int_{\mathbb{R}^N\setminus B_{\sigma R}(0)}V|\Psi|^2\,dx.\end{aligned}\tag{4.7.11}$$

Taking $\sigma = 2$ and applying Poincaré's inequality (3.3.5), we obtain

$$\begin{aligned}&\le 2C_1^2\|x\nabla_A\Psi\|^2_{L^2(\mathbb{R}^N)} + 2C_1^2\|\nabla\Psi\|^2_{L^2(\mathbb{R}^N)}\\ &+ 4C_1^4\|x^2\Psi\|^2_{L^2(\mathbb{R}^N)} + 16C_1^4R^2C_{N,R}\|\nabla_A\Psi\|^2_{L^2(\mathbb{R}^N)} + (R^2+1)16C_1^4\|\Psi\|^2_{L^2(\mathbb{R}^N;V)}.\end{aligned}\tag{4.7.12}$$

Hence, $\langle \nabla\Psi, A\rangle \in L^2(\mathbb{R}^N)$.

Now we show that $\mathrm{div}A\Psi \in L^2(\mathbb{R}^N)$. Since A is Lipschitz continuous by assumption A_3 and $\Psi \in L^2(\mathbb{R}^N)$, we have $\mathrm{div}A \in L^\infty(\mathbb{R}^N)$ and, consequently, $(\mathrm{div}A)\Psi \in L^2(\mathbb{R}^N)$.

We now prove that $|A|^2\Psi \in L^2(\mathbb{R}^N)$. Using again the linear growth property A_2 we have

$$\int_{\mathbb{R}^N}|A|^4|\Psi|^2\,dx \le 4C_1^4\int_{\mathbb{R}^N}|x|^4|\Psi|^2\,dx + 4C_1^4\int_{\mathbb{R}^N}|\Psi|^2\,dx,\tag{4.7.13}$$

and the first integral on the right-hand side is finite thanks to (4.7.3).

Finally, $V\Psi \in L^2(\mathbb{R}^N)$ once again thanks to (4.7.3).

Combining all the above, we conclude that the standard Laplacian $-\Delta\Psi$ equals the sum of $L^2(\mathbb{R}^N)$ terms, implying that $-\Delta\Psi \in L^2(\mathbb{R}^N)$. Hence, $\Psi \in H^2(\mathbb{R}^N)$, which completes the proof.

Reverse inclusion. In order to prove the reverse inclusion, it is sufficient to prove that

$$\left\{\Psi \in L^2\left(\mathbb{R}^N\right) \,:\, V\Psi \in L^2\left(\mathbb{R}^N\right)\right\} \subseteq L^2\left(\mathbb{R}^N; V\right).$$

It follows from Holder's inequality

$$\begin{aligned}\int_{\mathbb{R}^N} V|\Psi|^2\,dx &= \int_{\mathbb{R}^N} V\Psi\overline{\Psi}\,dx \\ &\leq \left(\int_{\mathbb{R}^N} |\Psi|^2 dx\right)^{1/2} \left(\int_{\mathbb{R}^N} V^2|\Psi|^2 dx\right)^{1/2}.\end{aligned}$$

This completes the proof. □

4.8 Stability of Eigenspaces

We already know from Lemma 3.5.1 that

$$\sqrt{\lambda_{n,A}} - R \leq \sqrt{\lambda_{n,A}(V)} \leq \sqrt{\lambda_{n,A}} + \delta, \qquad \text{for every } n \in \mathbb{N}^*.$$

In particular, we get that the spectrum of our operator collapses to the spectrum of the magnetic quantum harmonic oscillator, as R (and thus δ) goes to 0.

In this section, we analyze the behavior of the relevant eigenspaces and provide similar *quantitative* estimates to measure their distance from those associated with the model case. This distance is understood in an integral sense; in particular, results are given in terms of the L^2-norm and the H^1_A-norm. The following result extends Theorem 3.9.1 and Corollary 3.9.2. The proof techniques employed in the absence of a magnetic field remain applicable when the vector potential A is present. Only minor adjustments are required, due to the fact that the estimates in Lemma 4.5.1 now also depend on the geometric parameter δ; for this reason, the proofs are omitted.

Theorem 4.8.1. *For $n \in \mathbb{N}^*$, we set*

$$\mathcal{W}_n = \Big\{\psi \in H^1_A(\mathbb{R}^N; |x|^2) \,:\, \psi \textit{ is an eigenstate relative to } \lambda_{n,A}\Big\}.$$

Then there exists an explicit constant $C_3 = C_3(N, n, A) > 0$ such that

$$\operatorname{dist}_{L^2(\mathbb{R}^N)}(\Psi_n; \mathcal{W}_n) \leq C_3\sqrt{R}, \qquad \textit{for every } R > 0.$$

With a little extra work, we can improve the metric of the previous stability estimate.

Corollary 4.8.2. *For $n \in \mathbb{N}^*$, we set*

$$\mathcal{W}_n = \Big\{ \psi \in H^1_A(\mathbb{R}^N; |x|^2) \,:\, \psi \textit{ is an eigenstate relative to } \lambda_{n,A} \Big\}.$$

Then there exists an explicit constant $C_4 = C_4(N, n, \delta, A) > 0$ such that

$$\mathrm{dist}_{H^1_A(\mathbb{R}^N;|x|^2)}(\Psi_n; \mathcal{W}_n) \leq C_4 \sqrt{R}, \qquad \textit{for every } R > 0.$$

Chapter 5

An Illustrative Example: The Case of $\mathbb{S}^1$ in $\mathbb{R}^3$

Now, we want to briefly comment on the case of the $\mathbb{S}^1-$type potential well (3.2.6), in the case R becoming larger and larger. Observe that the explicit universal lower bound (4.4.11) trivializes in the limit as R goes to $+\infty$. We will show that this behavior is not optimal, in this peculiar case: the ground state energy must stay uniformly bounded from below, even when R goes to $+\infty$.

To this aim, we are going to exploit the ideas of [39, Chapter 16].

Proposition 5.0.1. *Let $\Sigma \subseteq \mathbb{R}^3$ be given by* (3.2.6)*. There exists a universal constant $\beta > 0$ such that*

$$\lambda_1(V) \geq \beta, \qquad \textit{for every } R > 0.$$

Proof. In view of the lower bound from Proposition 3.3.5, it is not restrictive to assume that $R \geq 2$. It is sufficient to prove the following Poincaré inequality

$$\beta \int_{\mathbb{R}^3} |\phi|^2 \, dx \leq \int_{\mathbb{R}^3} |\nabla \phi|^2 \, dx + \int_{\mathbb{R}^3} V \, |\phi|^2 \, dx, \qquad \text{for every } \phi \in C_c^\infty(\mathbb{R}^3),$$

with a constant $\beta > 0$ not depending on R. We proceed as in the proof of [39, Theorem 16.2.1]. We first tile the whole space $\mathbb{R}^3$ with the family of cubes

$$Q_{\mathbf{n}} := \mathbf{n} + \left(-\frac{1}{2}, \frac{1}{2}\right)^3, \qquad \text{with } \mathbf{n} \in \mathbb{Z}^3.$$

Observe that each $Q_{\mathbf{n}}$ has edges of length $d = 1$, parallel to the coordinate axes. Accordingly, for every $\phi \in C_c^\infty(\mathbb{R}^3)$ we can write

$$\int_{\mathbb{R}^3} |\phi|^2 \, dx = \sum_{\mathbf{n} \in \mathbb{Z}^3} \int_{Q_{\mathbf{n}}} |\phi|^2 \, dx.$$

On each cube $Q_{\mathbf{n}}$, we apply the following weighted Poincaré inequality of [39, Lemma

16.1.1]

$$\int_{Q_{\mathbf{n}}} |\phi|^2\,dx \le \frac{C}{\lambda}\int_{Q_{\mathbf{n}}} |\nabla\phi|^2\,dx + \frac{C}{\displaystyle\inf_{e\in\mathscr{N}_\lambda(Q_{\mathbf{n}})}\int_{Q_{\mathbf{n}}\setminus e} V\,dx}\int_{Q_{\mathbf{n}}} V\,|\phi|^2\,dx, \tag{5.0.1}$$

where:

- $C > 0$ is a universal constant;
- $\lambda > 0$ is an arbitrary positive number;
- $\mathscr{N}_\lambda(Q_{\mathbf{n}})$ is the collection of all compact sets $e \subseteq \overline{Q_{\mathbf{n}}}$ such that

  $$\mathrm{cap}(e; 2\,Q_{\mathbf{n}}) \le \lambda,$$

  where $2\,Q_{\mathbf{n}}$ denotes the cube having the same center as $Q_{\mathbf{n}}$, dilated by a factor 2, and

  $$\mathrm{cap}(e;\Omega) = \inf_{\phi\in C_c^\infty(\Omega)}\left\{\int_\Omega |\nabla\phi|^2\,dx \;:\; \phi \ge 1 \text{ on } e\right\},$$

  is the *capacity of a compact set e relative to the open set* Ω containing it.

In light of this inequality, it is sufficient to assure that we can choose $\lambda > 0$ small enough such that

$$\inf_{e\in\mathscr{N}_\lambda(Q_{\mathbf{n}})}\int_{Q_{\mathbf{n}}\setminus e} V\,dx \ge c_0, \tag{5.0.2}$$

for a constant $c_0 > 0$, independent of R. By summing up (5.0.1) and using the tiling property of the cubes, we will eventually get the conclusion.

In order to choose λ such that (5.0.2) holds, we first observe that if $B_{\mathbf{n}}$ is the ball having the same center as $Q_{\mathbf{n}}$ and radius $\sqrt{3}$, then $2\,Q_{\mathbf{n}} \subseteq B_{\mathbf{n}}$. Thus, for every $e \in \mathscr{N}_\lambda(Q_{\mathbf{n}})$ we have

$$\mathrm{cap}(e; B_{\mathbf{n}}) \le \mathrm{cap}(e; 2\,Q_{\mathbf{n}}) \le \lambda.$$

By [39, equation (2.2.10)] we have

$$\mathrm{cap}(e; B_{\mathbf{n}}) \ge (4\,\pi)^{\frac{2}{3}}\ \sqrt[3]{3}\,\frac{|B_{\mathbf{n}}|^{\frac{1}{3}}\,|e|^{\frac{1}{3}}}{|B_{\mathbf{n}}|^{\frac{1}{3}} - |e|^{\frac{1}{3}}}.$$

By observing that

$$0 < |B_{\mathbf{n}}|^{\frac{1}{3}} - |e|^{\frac{1}{3}} \le |B_{\mathbf{n}}|^{\frac{1}{3}},$$

we get in particular

$$\lambda \ge (4\,\pi)^{\frac{2}{3}}\ \sqrt[3]{3}\,|e|^{\frac{1}{3}} =: \gamma\,|e|^{\frac{1}{3}}.$$

Observe that both λ and γ are universal constant and do not depend on anything. We choose λ as follows

$$\frac{\lambda}{\gamma} = \left(\frac{1}{4}\right)^{\frac{1}{3}} \qquad \text{that is} \qquad \lambda = \gamma \left(\frac{1}{4}\right)^{\frac{1}{3}}.$$

This discussion guarantees that for such a choice of λ, we have

$$|e| \le \frac{1}{4} = \frac{|Q_{\mathbf{n}}|}{4}, \qquad \text{for every } e \in \mathscr{N}_\lambda(Q_{\mathbf{n}}). \tag{5.0.3}$$

By using this property and the fact that $V(x) = (\mathrm{dist}(x, \Sigma))^2$, with Σ given by (3.2.6), we can now easily get (5.0.2). Indeed, for every $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{Z}^3$ with $|n_3| \ge 1$, we have

$$\mathrm{dist}(x, \Sigma) \ge \frac{1}{2}, \qquad \text{for every } x \in Q_{\mathbf{n}},$$

which implies that

$$V(x) \ge \frac{1}{4}, \qquad \text{for every } x \in Q_{\mathbf{n}}.$$

Accordingly, for every such $\mathbf{n} \in \mathbb{Z}^3$ we get

$$\inf_{e \in \mathscr{N}_\lambda(Q_{\mathbf{n}})} \int_{Q_{\mathbf{n}} \setminus e} V\,dx \ge \frac{1}{4} \inf_{e \in \mathscr{N}_\lambda(Q_{\mathbf{n}})} |Q_{\mathbf{n}} \setminus e| \ge \frac{3}{16},$$

thanks to (5.0.3). We now take $\mathbf{n} \in \mathbb{Z}^3$ such that $n_3 = 0$. Accordingly, it may happen that the cube $Q_{\mathbf{n}}$ intersects the potential well Σ, where V vanishes. However, for every such a cube, we observe that

$$V(x) \ge \frac{9}{256}, \qquad \text{for every } x \in Q_{\mathbf{n}} \text{ such that } |x_3| \ge \frac{3}{16},$$

thanks to the fact that Σ lies in the hyperplane $\{x \in \mathbb{R}^3 \,:\, x_3 = 0\}$. In particular, we get in this case

$$\begin{aligned}
\inf_{e \in \mathscr{N}_\lambda(Q_{\mathbf{n}})} \int_{Q_{\mathbf{n}} \setminus e} V\,dx &\ge \inf_{e \in \mathscr{N}_\lambda(Q_{\mathbf{n}})} \int_{\{x \in Q_{\mathbf{n}}\,:\,|x_3| \ge 3/16\} \setminus e} V\,dx \\
&\ge \frac{9}{256} \inf_{e \in \mathscr{N}_\lambda(Q_{\mathbf{n}})} \left|\{x \in Q_{\mathbf{n}} \,:\, |x_3| \ge 3/16\} \setminus e\right| \\
&\ge \frac{9}{256} \left(2\left(\frac{1}{2} - \frac{3}{16}\right) - \frac{1}{4}\right) = \frac{9}{256} \cdot \frac{3}{8},
\end{aligned}$$

where we used again (5.0.3). This finally gives the claimed fact (5.0.2), with $c_0 = 27/2048$.

$\square$

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