



Chaotic dynamics of a continuous and discrete generalized Ziegler pendulum

Stefano Disca · Vincenzo Coscia

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Abstract We present analytical and numerical results on integrability and transition to chaotic motion for a generalized Ziegler pendulum, a double pendulum subject to an angular elastic potential and a follower force. Several variants of the original dynamical system, including the presence of gravity and friction, are considered, in order to analyze whether the integrable cases are preserved or not in presence of further external forces, both potential and non-potential. Particular attention is devoted to the presence of dissipative forces, that are analyzed in two different formulations. Furthermore, a study of the discrete version is performed. The analysis of periodic points, that is presented up to period 3, suggests that the discrete map associated to the dynamical system has not dense sets of periodic points, so that the map would not be chaotic in the sense of Devaney for a choice of the parameters that corresponds to a general case of chaotic motion for the original system.

Keywords Double pendulum · Non-integrability · Transition to chaos · Friction · Devaney-chaos

1 Introduction

The Ziegler pendulum is a classical dynamical system introduced by Ziegler [1], in order to study the applicability of the stability criteria to non-conservative systems [2]. In his treatment, Ziegler studied the motion for small angles of a planar physical double pendulum composed by two rigid rods and subject to gravity, angular elastic potentials on the pins and a follower force along the lower rod. Such as the classical double pendulum [3–5], the system shows itself to be chaotic. The simpler version considered in [6], that is a mathematical double pendulum not subject to gravity, exhibits chaotic motion too, but existence of regular solutions is found in presence of particular symmetries, regardless the initial conditions on the canonical variables of the system. Different treatments of integrability and chaotic dynamics of this system can be found in [7, 8]. The Ziegler pendulum is well-known for the paradoxical destabilization of the system when subject to damping [9–11], while in general a non-conservative system gains stability in presence of a dissipative force. The role of dissipation in the dynamics of the Ziegler pendulum has been studied, among others, in [12], where it is numerically found the occurrence of stable limit-cycles in presence of damping. For an optimization

S. Disca
Department of Physics and Earth Science, University
of Ferrara, Via Saragat 1, 44122 Ferrara, Italy

V. Coscia (✉)
Department of Mathematics and Computer Science,
University of Ferrara, Via Machiavelli 30, 44121 Ferrara,
Italy
e-mail: cos@unife.it

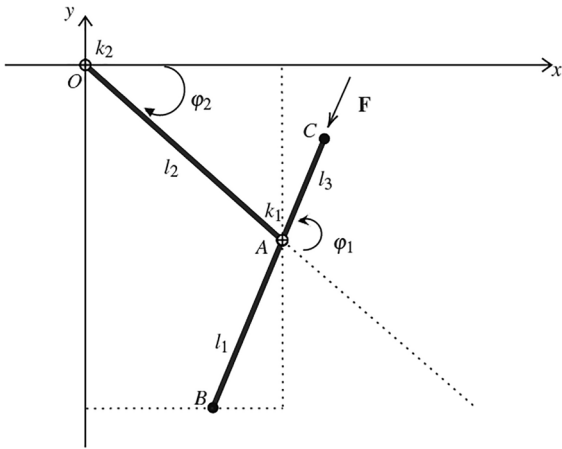


Fig. 1 A generalized Ziegler pendulum: three material points are joined by two massless rigid rods, with two cylindrical springs located on the pins and a follower force acting on the lower rod

point of view of damped and undamped system see also [13].

In this work we define the Ziegler pendulum as a planar mathematical double pendulum consisting of three material points A, B, C having mass m_1, m_2, m_3 respectively. The point A is held at constant distance from the point O by means of a massless rod of length l_2 , while the points B and C are positioned at the ends of a second massless rod hinged in A in such a way that $\overline{BA} = l_1$ and $\overline{AC} = l_3$ (see Fig. 1). On the system acts a force of size F always direct as the vector $B - C$ and two elastic forces produced by two cylindrical springs placed on the hinges O and A and having elastic constants k_1 and k_2 . Furthermore, we write $M := m_A + m_B + m_C$ for the total mass and put $\Delta := m_B l_1 - m_C l_3$, that quantifies the distribution of mass on the lower rod.

For the rest of the paper we refer to the above definition of Ziegler pendulum, originally proposed in [6]. Notice that, by taking $l_1 = 0$ or $l_3 = 0$, this generalized version returns the Ziegler pendulum defined in [1], except for the presence of gravity.

By choosing φ_1, φ_2 as generalized variables, we have the following expressions for the velocities:

$$v_A^2 = l_2^2 \dot{\varphi}_2^2 \tag{1a}$$

$$v_B^2 = l_2^2 \dot{\varphi}_2^2 + l_1^2 (\dot{\varphi}_1 + \dot{\varphi}_2)^2 - 2l_1 l_2 \dot{\varphi}_2 (\dot{\varphi}_1 + \dot{\varphi}_2) \cos \varphi_1 \tag{1b}$$

$$v_C^2 = l_2^2 \dot{\varphi}_2^2 + l_3^2 (\dot{\varphi}_1 + \dot{\varphi}_2)^2 + 2l_2 l_3 \dot{\varphi}_2 (\dot{\varphi}_1 + \dot{\varphi}_2) \cos \varphi_1. \tag{1c}$$

The elastic potential energy is given by

$$\Pi = \frac{1}{2} k_1 \varphi_1^2 + \frac{1}{2} k_2 \varphi_2^2. \tag{2}$$

Furthermore the system is subject to a follower force with constant magnitude

$$\mathbf{F} = F \frac{\mathbf{r}_A - \mathbf{r}_C}{|\mathbf{r}_A - \mathbf{r}_C|} = \begin{pmatrix} -F \cos(\varphi_1 + \varphi_2) \\ F \sin(\varphi_1 + \varphi_2) \end{pmatrix}, \tag{3}$$

to which the generalized forces

$$Q_1 = \mathbf{F} \cdot \frac{\partial \mathbf{r}_C}{\partial \varphi_1} = 0 \tag{4a}$$

$$Q_2 = \mathbf{F} \cdot \frac{\partial \mathbf{r}_C}{\partial \varphi_2} = -Fl_2 \sin \varphi_1 \tag{4b}$$

are associated.

The kinetic energy has the form

$$T = \frac{1}{2} \left(A_{11} \dot{\varphi}_1^2 + (A_{12} + A_{21}) \dot{\varphi}_1 \dot{\varphi}_2 + A_{22} \dot{\varphi}_2^2 \right), \tag{5}$$

where

$$A_{11} = m_B l_1^2 + m_C l_3^2 \tag{6a}$$

$$A_{12} = A_{21} = A_{11} - \Delta l_2 \cos \varphi_1 \tag{6b}$$

$$A_{22} = A_{11} + M l_2^2 - 2\Delta l_2 \cos \varphi_1 \tag{6c}$$

for the standard Ziegler pendulum.

The Euler–Lagrange equations can be written as follows:

$$A_{11} \ddot{\varphi}_1 + A_{12} \ddot{\varphi}_2 = r_1 \tag{7a}$$

$$A_{21} \ddot{\varphi}_1 + A_{22} \ddot{\varphi}_2 = r_2, \tag{7b}$$

where

$$r_1 = -k_1 \varphi_1 + \Delta l_2 \dot{\varphi}_2^2 \sin \varphi_1 \tag{8a}$$

$$r_2 = -k_2 \varphi_2 - \Delta l_2 \dot{\varphi}_1 (\dot{\varphi}_1 + 2\dot{\varphi}_2) \sin \varphi_1 - Fl_2 \sin \varphi_1. \tag{8b}$$

An alternative form for the system (7), useful also for the numerical calculations, is the following one:

$$\dot{\varphi}_1 = v_1 \tag{9a}$$

$$\dot{v}_1 = \frac{A_{22}r_1 - A_{12}r_2}{A_{11}A_{22} - (A_{12})^2} \tag{9b}$$

$$\dot{\varphi}_2 = v_2 \tag{9c}$$

$$\dot{v}_2 = \frac{A_{11}r_2 - A_{12}r_1}{A_{11}A_{22} - (A_{12})^2}. \tag{9d}$$

It has been proved in [6] that for $k_2 = 0$ and in presence of two particular symmetries the system is integrable: in the sense of Liouville [14] for the Hamiltonian case $F = 0$, in the sense of Jacobi [15] for the non-Hamiltonian case $\Delta = 0$. Furthermore, in these two cases there exists a family of periodic solutions that intersect the plane $\varphi_1 = 0$, and the integrability survives for small breaking of these symmetries (i.e. F or Δ sufficiently small). The integrability of the system results from the cyclicity of the variable φ_2 in the case $k_2 = 0$; in particular for the Hamiltonian case ($F = 0$) this cyclicity gives rise to a further first integral for the system, that is the conjugate momentum $K = \frac{\partial L}{\partial \dot{\varphi}_2}$.

In view of the results indicated above, in the sequel we analyze the role in the dynamics of further external forces, both conservative and not, in order to verify whether or not the integrability of the system survives in presence of them and for $k_2 = 0$. Moreover, we study a discrete map associated to the system (9) in order to check if it verifies the definition of chaoticity in the sense of Devaney [16, 17] for an arbitrary choice on k_2 . All the numerical simulations presented have been produced by a Runge–Kutta method of fourth order, with a size step $dt = 0.005$ and a number of iteration $n = 50000-500000$, implemented in Python; we also report the computation of the Lyapunov exponents for some cases, by using a proper MATLAB code¹ that exploits an algorithm proposed in [18] for the calculation of Lyapunov exponents of systems of ordinary differential equations.

The paper is organized as follows. In Sect. 2 we analyze the role of gravity on the dynamics. The effect of linear elastic potentials is treated in Sect. 3 in Sect. 5, while a brief analysis of three simple geometric variants. Then, the role of different kinds of friction on the system is discussed in Sect. 4. Moreover, in Sect. 6 we study a discrete version of the system, where we propose a conjecture on properties of the sets of periodic points. Finally in Sect. 7 we point out the concluding remarks and research perspectives.

2 Gravity

In this section we consider a vertical Ziegler pendulum subject to gravity, that is we include in the equations of motion a further potential energy

$$V_G = g \sum_{j=A,B,C} m_j y_j = -Mgl_2 \sin \varphi_2 + \Delta g \sin(\varphi_1 + \varphi_2). \tag{10}$$

The terms A_{ij} defined in the (6) do not change, while for r_j we have

$$r_1 = -k_1 \varphi_1 + \Delta l_2 \dot{\varphi}_2^2 \sin \varphi_1 + \Delta g \cos(\varphi_1 + \varphi_2) \tag{11a}$$

$$r_2 = -\Delta l_2 \dot{\varphi}_1 (\dot{\varphi}_1 + 2\dot{\varphi}_2) \sin \varphi_1 - Fl_2 \sin \varphi_1 - Mgl_2 \cos \varphi_2 + \Delta g \cos(\varphi_1 + \varphi_2). \tag{11b}$$

The integrability of the system is lost for the Hamiltonian case $F = 0$, since the variable φ_2 is not cyclic anymore. This is not unexpected, since in absence of the external force the system is actually a double pendulum subject to an elastic potential, that does not introduce terms in order to compensate the chaotic behavior (Fig. 2).

Anyway, for the non-Hamiltonian case $\Delta = 0$ we have

$$A_{11} = A_{12} = A_{21} = m_B l_1^2 + m_C l_3^2 \tag{12a}$$

$$A_{22} = A_{11} + Ml_2^2 \tag{12b}$$

$$r_1 = -k_1 \varphi_1 \tag{12c}$$

$$r_2 = -Fl_2 \sin \varphi_1 - Mgl_2 \cos \varphi_2. \tag{12d}$$

If we think of an angle-dependent form

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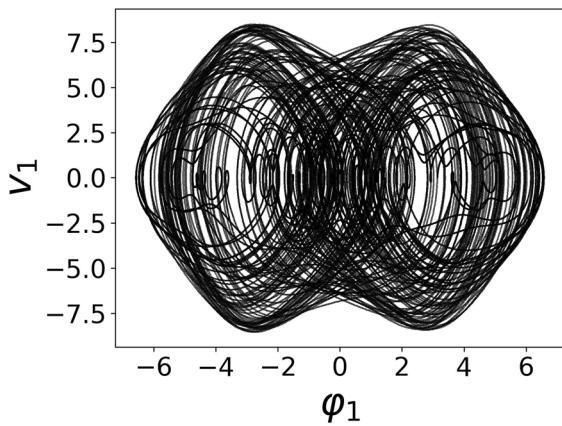


Fig. 2 Chaotic motion in presence of gravity. Parameters: $m_A = 1, m_B = 1.5, m_C = 1, l_1 = 1, l_2 = 1, l_3 = 1, k_1 = 2, F = 0$. Initial conditions: $\varphi_1(0) = \pi, \varphi_2(0) = 0.1, v_1(0) = v_2(0) = 0$

$$F(\varphi_1, \varphi_2) = -Mg \frac{\cos \varphi_2}{\sin \varphi_1} \tag{13}$$

for the force, the term r_2 vanishes and the system (7) reduces to

$$A_{11}(\ddot{\varphi}_1 + \ddot{\varphi}_2) = -k_1 \varphi_1 \tag{14a}$$

$$A_{11}\ddot{\varphi}_1 + A_{22}\ddot{\varphi}_2 = 0, \tag{14b}$$

that is (in this case all coefficients are constant)

$$\ddot{\varphi}_1 + \ddot{\varphi}_2 = \frac{-k_1}{A_{11}} \varphi_1 \tag{15a}$$

$$A_{11}\ddot{\varphi}_1 + A_{22}\ddot{\varphi}_2 = K. \tag{15b}$$

By substituting (15b) in (15a) we obtain the following equation independent of φ_2

$$\ddot{\varphi}_1 \left(1 - \frac{A_{11}}{A_{22}} \right) + \frac{k_1}{A_{11}} \varphi_1 = 0, \tag{16}$$

that is a one-dimensional harmonic oscillator with frequency

$$\omega^2 := \frac{k_1}{A_{11}} \frac{A_{22}}{A_{22} - A_{11}}. \tag{17}$$

Given that, the motion for φ_2 is solved by quadratures

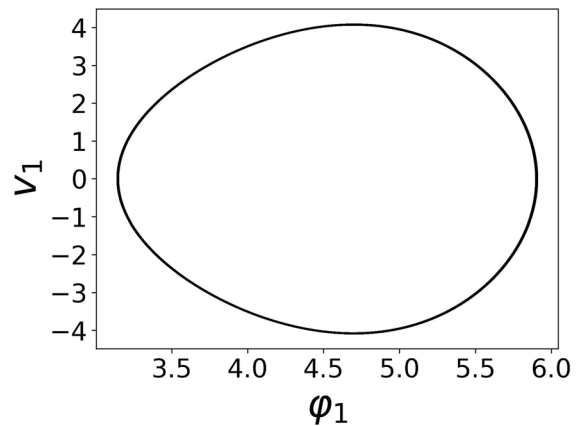


Fig. 3 Periodic motion in presence of gravity (compare with Fig. 4). Parameters: $m_A = 1, m_B = 1, m_C = 1, l_1 = 1, l_2 = 1, l_3 = 1.46, k_1 = 1, F = 0$. Initial conditions: $\varphi_1(0) = \pi, \varphi_2(0) = 0, v_1(0) = v_2(0) = 0$

$$\varphi_2(t) = \frac{K}{A_{11}} t - \int_{t_0}^t A_{11} \dot{\varphi}_1(t') dt'. \tag{18}$$

This is a very simple result, but it is worth to observe that, in order to obtain it, we must impose a non constant magnitude for the force, that is an *ad hoc* form for the external force that allows r_2 to vanish. Furthermore, the expression (13) is singular if $\varphi_1 = k\pi$ and $\varphi_2 \neq k\pi + \frac{\pi}{2}$ with $k \in \mathbb{N}$, that is for an infinite set of values and initial conditions.

However, even if the gravity destroys the integrable cases known for the Ziegler pendulum, the system shows isolated cases of regular motion for a proper choice of parameters and initial conditions. An interesting feature, that can be found in other chaotic systems [19], is the fast transition shown from a chaotic regime to a regular one. As an example, the motion for two close values of l_3 is shown in Fig. 3 and 4 (initial conditions and all the other parameters are kept the same). The transition from a regular motion to a chaotic one is confirmed by the computation of the Lyapunov exponents: null Lyapunov exponents are associated to the motion shown in Fig. 3, while the motion shown in Fig. 4 exhibits a positive Lyapunov exponent $\lambda_{\varphi_1} \sim 0.19$.

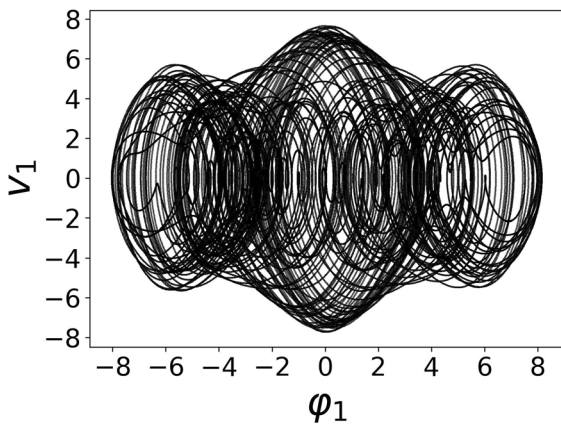


Fig. 4 Chaotic motion in presence of gravity (compare with Fig. 3). Parameters: $m_A = 1, m_B = 1, m_C = 1, l_1 = 1, l_2 = 1, l_3 = 1.45, k_1 = 1, F = 0$. Initial conditions: $\varphi_1(0) = \pi, \varphi_2(0) = 0, v_1(0) = v_2(0) = 0$

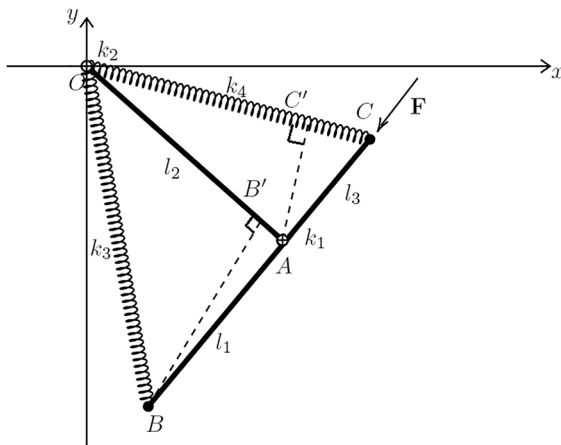


Fig. 5 The Ziegler pendulum with two linear springs (angles defined in Fig. 1)

3 Linear springs

In this section we consider Ziegler pendulum with two further linear springs that join the fixed point with the two ends of the lower rods (Fig. 5).

The system is now subject to a further elastic potential

$$V_{EL} = \frac{1}{2}k_{OB}|\overline{OB}|^2 + \frac{1}{2}k_{OC}|\overline{OC}|^2. \tag{19}$$

Since the elongation of these linear springs depends only on the rotation of the lower rod, as one can figure

out from a physical point of view, the elastic potential must be a function of φ_1 ; therefore the variable φ_2 remains cyclic and all the previous results still hold for this model. The quadratic elastic potential is easily evaluated by computing the lengths of the springs. If we denote B' the orthogonal projection of B on the rod \overline{OA} , we have

$$|\overline{OB}|^2 = l_1^2 + l_2^2 - 2l_1l_2 \cos \varphi_1 \tag{20}$$

and in analogous way, if we call A' the orthogonal projection of A on the spring \overline{OC} , we have

$$|\overline{OC}|^2 = l_2^2 + l_3^2 + 2l_2l_3 \cos \varphi_1, \tag{21}$$

so that the quadratic elastic potential can be written as

$$V_{EL} = l_2(k_{OC}l_3 - k_{OB}l_1) \cos \varphi_1 \tag{22}$$

up to a constant. The equations of motion change as follows: the terms A_{ij} and r_2 still follow the (6) and (8), while r_1 earns a term due to $\frac{\partial V_{EL}}{\partial \varphi_1}$

$$r_1 = -k_1\varphi_1 + \Delta l_2\dot{\varphi}_2^2 \sin \varphi_1 - l_2(k_{OC}l_3 - k_{OB}l_1) \sin \varphi_1. \tag{23}$$

Due to the cyclicity of φ_2 , the cases $F = 0$ and $\Delta = 0$ are still integrable in the sense previously discussed. Furthermore, a new symmetry arises, associated to the case $k_{OB}l_1 = k_{OC}l_3$. This situation physically corresponds to have two external elastic forces that compensate each other, so the motion is the same that we would have for a standard Ziegler pendulum (Figs. 6, 7, 8).

However, the system is integrable for all the possible values of k_{OB} and k_{OC} , in the sense of Liouville if $F = 0$ or in the sense of Jacobi if $\Delta = 0$. In particular all the results proved in [6] still hold, such as for example the existence of a family of periodic solutions that intersect the plane $\varphi_1 = 0$. The presence of the linear springs of course modifies the periodic solutions of the system; in particular, they show to have intersections, absent in the standard case, as depicted in Fig. 9.

4 Three geometric variants

Let us consider a Ziegler pendulum with two physical homogeneous rods, as introduced in [1] (see Fig. 10). We denote by $C_U = (x_U, y_U)$ and $C_D = (x_D, y_D)$ the

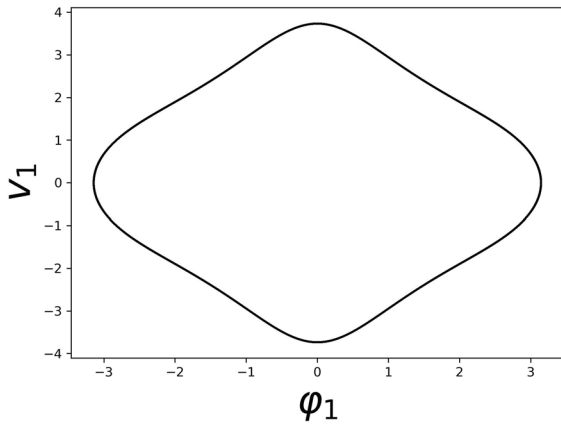


Fig. 6 Closed trajectory in presence of two linear springs, $F = 0$ and $k_{OB}l_1 = k_{OC}l_3$. Parameters: $m_A = 1, m_B = 1, m_C = 2, l_1 = 1, l_2 = 1, l_3 = 2, k_1 = 2, k_{OB} = 1, k_{OC} = 0.5$. Initial conditions: $\varphi_1(0) = \pi, \varphi_2(0) = 0.2, v_1(0) = 0.1, v_2(0) = 0$

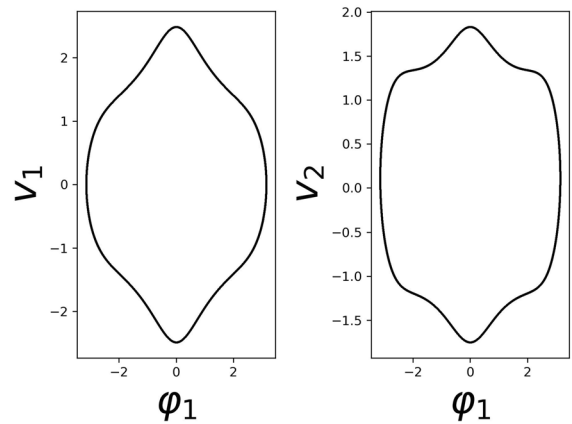


Fig. 8 Closed trajectories in presence of two linear springs, $F = 0$ and $k_{OB}l_1 \neq k_{OC}l_3$. Parameters: $m_A = 2, m_B = 1, m_C = 3, l_1 = 1, l_2 = 1, l_3 = 3, k_1 = 2.5, k_{OB} = 3, k_{OC} = 0$. Initial conditions: $\varphi_1(0) = \pi, \varphi_2(0) = 0.2, v_1(0) = 0.1, v_2(0) = 0$

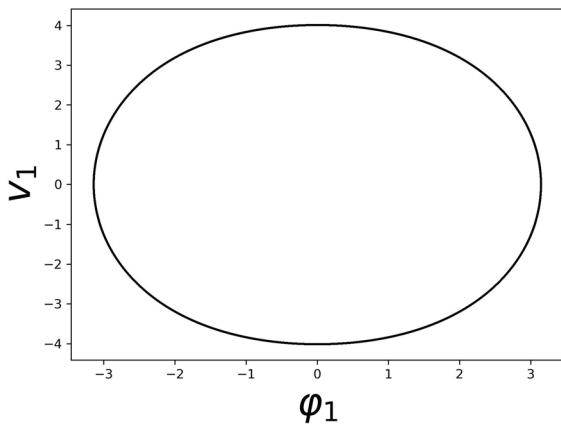


Fig. 7 Closed trajectory in presence of two linear springs, $\Delta = 0$ and $k_{OB}l_1 = k_{OC}l_3$. Parameters: $m_A = 1, m_B = 1, m_C = 2, l_1 = 1, l_2 = 1, l_3 = 0.5, k_1 = 2, F = 2, k_{OB} = 1, k_{OC} = 2$. Initial conditions: $\varphi_1(0) = \pi, \varphi_2(0) = 0.2, v_1(0) = 0.1, v_2(0) = 0$

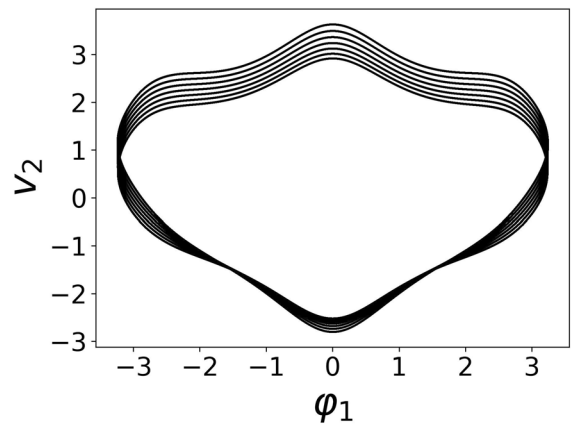


Fig. 9 A family of periodic solutions in presence of two linear springs and $F = 0$. Parameters: $m_A = 2, m_B = 1, m_C = 3, l_1 = 1, l_2 = 1, l_3 = 3, k_1 = 2.5, k_{OB} = 3, k_{OC} = 1$. Initial conditions: $\varphi_1(0) = \pi, \varphi_2(0) = \frac{\pi}{2}, v_1(0) = 0.5, v_2(0) = 0.1j$ with $j = 0, \dots, 6$

centers of mass of the upper and lower pendulum, with masses m_U and m_D respectively; we still use $\varphi_{1,2}$ as generalized coordinates of the system and refer to the lengths previously introduced.

Denoting by $\theta = \varphi_1 + \varphi_2$ the angle between the lower rod and the horizontal axis, it is easy to find

$$\begin{cases} x_U = \frac{l_2}{2} \cos \varphi_2 \\ y_U = -\frac{l_2}{2} \sin \varphi_2 \end{cases} \quad (24)$$

and

$$\begin{cases} x_D = l_2 \cos \varphi_2 - \left(\frac{l_1 - l_3}{2}\right) \cos(\varphi_1 + \varphi_2) \\ y_D = -l_2 \sin \varphi_2 + \left(\frac{l_1 - l_3}{2}\right) \sin(\varphi_1 + \varphi_2). \end{cases} \quad (25)$$

The kinetic energy of the upper pendulum, that rotates around the fixed point O , is

$$T_U = \frac{1}{6} m_U l_2^2 \dot{\varphi}_2^2, \quad (26)$$

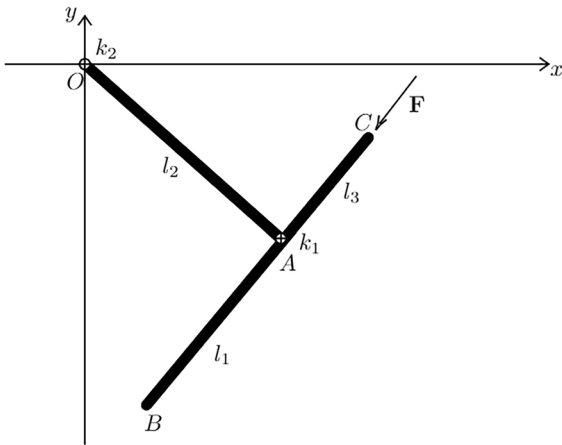


Fig. 10 A Ziegler pendulum with two physical rods (angles defined in Fig. 1)

while the kinetic energy of the lower pendulum is found to be

$$T_D = \frac{1}{2}m_D \left(l_2^2 \dot{\varphi}_2^2 + \left(\frac{l_1 - l_3}{2} \right)^2 (\dot{\varphi}_1 + \dot{\varphi}_2)^2 - l_2(l_1 - l_3) \dot{\varphi}_2 (\dot{\varphi}_1 + \dot{\varphi}_2) \cos \varphi_1 \right) + \frac{1}{24}m_D(l_1 + l_3)^2 \dot{\varphi}_1^2. \tag{27}$$

The parameters in the equations of motion (7) are modified in the following way:

$$A_{11} = \frac{m_D}{3}(l_1^2 + l_3^2 - l_1 l_3) \tag{28a}$$

$$A_{12} = A_{21} = \frac{m_D}{4}(l_1 - l_3)^2 - \frac{m_D}{2}l_2(l_1 - l_3) \cos \varphi_1 \tag{28b}$$

$$A_{22} = \frac{m_U}{3}l_2^2 + \frac{m_D}{4}(4l_2^2 + (l_1 - l_3)^2) - m_D l_2(l_1 - l_3) \cos \varphi_1 \tag{28c}$$

$$r_1 = -k_1 \varphi_1 + \frac{m_D}{2}l_2(l_1 - l_3) \dot{\varphi}_2^2 \sin \varphi_1 \tag{28d}$$

$$r_2 = -k_2 \varphi_2 - \frac{m_D}{2}l_2(l_1 - l_3) \dot{\varphi}_1 (\dot{\varphi}_1 + 2\dot{\varphi}_2) \sin \varphi_1 - Fl_2 \sin \varphi_1. \tag{28e}$$

We formally recover the same equations of motion valid for the standard Ziegler pendulum. The symmetry $m_1 l_1 - m_3 l_3 = 0$ is replaced by the symmetry $l_1 - l_3 = 0$, which again corresponds to have the

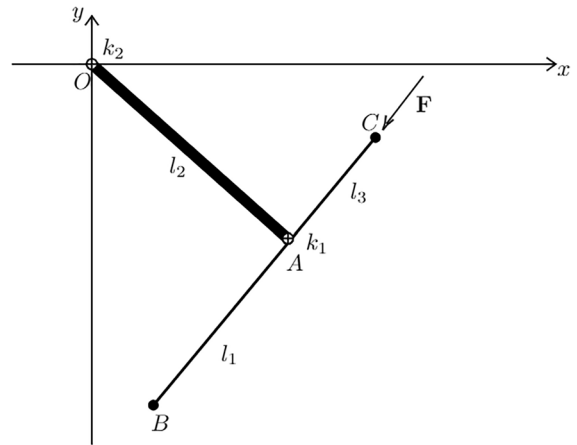


Fig. 11 A Ziegler pendulum with a physical upper rod and a mathematical lower rod (angles defined in Fig. 1)

lower center of mass located on the pin between the two rods. The usual two integrable cases hold and the

motion is formally the same as for the standard Ziegler pendulum.

For a Ziegler pendulum with a physical upper rod and a mathematical lower rod (see Fig. 11), we recover the standard Ziegler pendulum with just a constant correction on the coefficient A_{22} , that is ($m_U = m_A$)

$$A_{22} = A_{11} + \left(\frac{1}{3}m_A + m_B + m_C \right) l_2^2 - 2\Delta l_2 \cos \varphi_2. \tag{29}$$

Conversely, for a Ziegler pendulum with a mathematical upper rod and a physical lower rod (see Fig. 12), we recover the physical Ziegler pendulum with just a constant correction on the coefficient A_{22} , that is ($m_U = m_A$)

$$A_{22} = m_U l_2^2 + \frac{m_D}{4}(4l_2^2 + (l_1 - l_3)^2) - m_D l_2(l_1 - l_3) \cos \varphi_1. \tag{30}$$

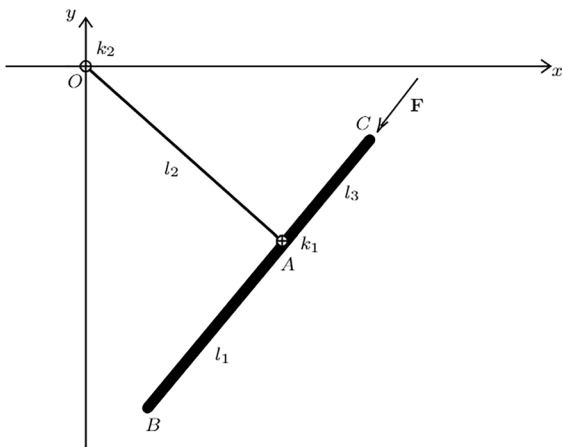


Fig. 12 A Ziegler pendulum with a mathematical upper rod and a physical lower rod (angles defined in Fig. 1)

For both mixed versions the correction does not change the formal shape of the equations of motion and there are no differences in the qualitative motion of the system.

Finally, all these three variants of the Ziegler pendulum (a physical double pendulum and two mixed double pendulums) exhibit the same features of the standard Ziegler pendulum. This is not surprising, since these variants simply change the positions of the centers of mass of the two rods without adding relevant terms in the equations of motion.

5 Friction

In this section we analyze the Ziegler pendulum subject to friction in two possible senses: a viscous friction that acts on the three material points and a friction force that acts on the pins of the pendulum being opposite to the rotation.

5.1 Stokes friction

Let's consider a friction force that follows

$$\mathbf{A} = -\mu\mathbf{v}, \quad \mu \geq 0, \tag{31}$$

that is, with a proper dimensional choice for μ , the Stokes law for the friction due to a viscous fluid (e.g. the air). From a physical point of view, the presence of three friction coefficients corresponds

to having three material points made of three different materials, or even to have the pendulum located in a box with three different separate fluids. We do that in order to consider the most general perturbation to the dynamical system; a realistic model can be obtained for example by equating all the three friction coefficients.

In order to calculate the generalized force associated to the friction (31), we consider the following Rayleigh dissipation function

$$R := \frac{1}{2}\mu v^2 \implies \mathbf{A} = -\nabla_v R. \tag{32}$$

Since

$$\frac{\partial \mathbf{v}}{\partial \dot{\varphi}_j} = \frac{\partial \mathbf{r}}{\partial \varphi_j}, \tag{33}$$

the components of the generalized force associated to \mathbf{A} are given by

$$Q_j = -\frac{\partial R}{\partial \dot{\varphi}_j} \tag{34}$$

and the Euler–Lagrange equations become

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\varphi}_j} \right) - \frac{\partial L}{\partial \varphi_j} + \frac{\partial R}{\partial \dot{\varphi}_j} = 0. \tag{35}$$

By recalling (1a), (1b) and (1c), the following Rayleigh function and generalized forces are associated to the friction acting on A

$$R_A = \frac{1}{2}\mu_A l_2^2 \dot{\varphi}_2^2, \tag{36}$$

$$Q_1^A = 0 \tag{37a}$$

$$Q_2^A = -\mu_A l_2^2 \dot{\varphi}_2, \tag{37b}$$

to the friction acting on B

$$R_B = \frac{1}{2}\mu_B \left(l_2^2 \dot{\varphi}_2^2 + l_1^2 (\dot{\varphi}_1 + \dot{\varphi}_2)^2 - 2l_1 l_2 \dot{\varphi}_2 (\dot{\varphi}_1 + \dot{\varphi}_2) \cos \varphi_1 \right), \tag{38}$$

$$Q_1^B = -\mu_B \left(l_1^2 (\dot{\varphi}_1 + \dot{\varphi}_2) - l_1 l_2 \dot{\varphi}_2 \cos \varphi_1 \right) \tag{39a}$$

$$Q_2^B = -\mu_B \left(l_1^2(\dot{\varphi}_1 + \dot{\varphi}_2) + l_2^2\dot{\varphi}_2 - l_1l_2(\dot{\varphi}_1 + 2\dot{\varphi}_2) \cos \varphi_1 \right), \tag{39b}$$

and to the friction acting on C

$$R_C = \frac{1}{2}\mu_C \left(l_2^2\dot{\varphi}_2^2 + l_3^2(\dot{\varphi}_1 + \dot{\varphi}_2)^2 + 2l_2l_3\dot{\varphi}_2(\dot{\varphi}_1 + \dot{\varphi}_2) \cos \varphi_1 \right), \tag{40}$$

$$Q_1^C = -\mu_C \left(l_3^2(\dot{\varphi}_1 + \dot{\varphi}_2) + l_2l_3\dot{\varphi}_2 \cos \varphi_1 \right) \tag{41a}$$

$$Q_2^C = -\mu_C \left(l_3^2(\dot{\varphi}_1 + \dot{\varphi}_2) + l_2^2\dot{\varphi}_2 + l_3l_2(\dot{\varphi}_1 + 2\dot{\varphi}_2) \cos \varphi_1 \right). \tag{41b}$$

The terms A_{ij} in the equations of motion still follow the (6), while $r_{1,2}$ earn some terms due to the presence of frictions

$$r_1 = -k_1\varphi_1 + \Delta l_2\dot{\varphi}_2^2 \sin \varphi_1 - (\mu_B l_1^2 + \mu_C l_3^2)(\dot{\varphi}_1 + \dot{\varphi}_2) + (\mu_B l_1 - \mu_C l_3)l_2\dot{\varphi}_2 \cos \varphi_1, \tag{42a}$$

$$r_2 = -\Delta l_2\dot{\varphi}_1(\dot{\varphi}_1 + 2\dot{\varphi}_2) \sin \varphi_1 - Fl_2 \sin \varphi_1 - (\mu_B l_1^2 + \mu_C l_3^2)(\dot{\varphi}_1 + \dot{\varphi}_2) + (\mu_B l_1 - \mu_C l_3)l_2(\dot{\varphi}_1 + 2\dot{\varphi}_2) \cos \varphi_1 - (\mu_A + \mu_B + \mu_C)l_2^2\dot{\varphi}_2. \tag{42b}$$

We point out that the presence of a general friction does not re-introduce the variable φ_2 in the equations of motion, so that φ_2 is still cyclic, but this further non-conservative force destroys the possible Hamiltonian case we had for $F = 0$.

The symmetry of the problem is clearly visible, since the friction coefficients introduce terms analogous to A_{11} , Δ and M , with the mass m_j replaced by μ_j . In particular we observe that a term $\mu_B l_1 - \mu_C l_3$ arises in the parameters $r_{1,2}$, formally equal to Δ . We therefore define

$$\begin{aligned} A_{11\mu} &:= \mu_B l_1^2 + \mu_C l_3^2 \\ M_\mu &:= \mu_A + \mu_B + \mu_C \\ A_{22\mu} &:= A_{11\mu} + M_\mu l_2^2 \\ \Delta_\mu &:= \mu_B l_1 - \mu_C l_3 \end{aligned} \tag{43}$$

and require that $\Delta = \Delta_\mu = 0$, so that

$$A_{11} = A_{12} = A_{21} = m_B l_1^2 + m_C l_3^2 \tag{44a}$$

$$A_{22} = A_{11} + M l_2^2 \tag{44b}$$

$$r_1 = -k_1\varphi_1 - A_{11\mu}(\dot{\varphi}_1 + \dot{\varphi}_2) \tag{44c}$$

$$r_2 = -Fl_2 \sin \varphi_1 - A_{11\mu}(\dot{\varphi}_1 + \dot{\varphi}_2) - M_\mu l_2^2\dot{\varphi}_2. \tag{44d}$$

The equations of motion become

$$A_{11}(\ddot{\varphi}_1 + \ddot{\varphi}_2) = -k_1\varphi_1 - A_{11\mu}(\dot{\varphi}_1 + \dot{\varphi}_2) \tag{45a}$$

$$\begin{aligned} A_{11}\ddot{\varphi}_1 + (A_{11} + M l_2^2)\ddot{\varphi}_2 \\ = -Fl_2 \sin \varphi_1 - A_{11\mu}(\dot{\varphi}_1 + \dot{\varphi}_2) - M_\mu l_2^2\dot{\varphi}_2. \end{aligned} \tag{45b}$$

If we subtract the first equation from the second one we get

$$M l_2^2\ddot{\varphi}_2 = k_1\varphi_1 - Fl_2 \sin \varphi_1 - M_\mu l_2^2\dot{\varphi}_2. \tag{46}$$

If the following condition on the coefficients

$$\frac{A_{11}}{M} = \frac{A_{11\mu}}{M_\mu} \tag{47}$$

holds, a φ_2 -independent equation can be extracted from the system, by substituting the (46) in the first equation of (45)

$$\ddot{\varphi}_1 = -\frac{k_1}{M l_2^2} \left(1 + \frac{M l_2^2}{A_{11}} \right) \varphi_1 - \frac{F}{M l_2} \sin \varphi_1 - \frac{A_{11\mu}}{A_{11}} \dot{\varphi}_1, \tag{48}$$

that is the equation of a one-dimensional damped harmonic oscillator. We then substitute the previous one in the second equation of (45) to get

$$\ddot{\varphi}_2 = \frac{k_1}{M l_2^2} \varphi_1 - \frac{F}{M l_2} \sin \varphi_1 - \frac{A_{22\mu}}{A_{22}} \dot{\varphi}_2 \tag{49}$$

and the system given by Eqs. (48) and (49) is formally integrable.

In Fig. 13 is shown the expected spiral motion, that is the system has an attractive point. In the case of breaking of the three symmetries considered above, i.e. $\Delta = \Delta_\mu = 0$ and (47), also limit cycles can arise, as shown in Fig. 14 and 15; furthermore, the approaching to the attractive point can be unusual, as shown in Fig. 16.

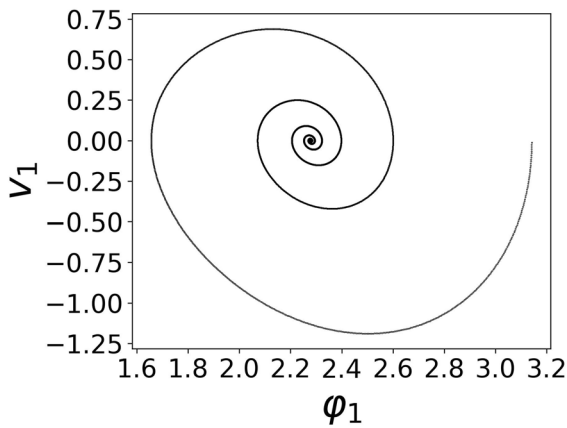


Fig. 13 Attractive point in presence of Stokes friction, $\Delta = 0$, $\Delta_\mu = 0$ and $\frac{A_{11}}{M} = \frac{A_{11\mu}}{M_\mu}$. Parameters: $m_A = 1, m_B = 2, m_C = 1, l_1 = 1, l_2 = 1, l_3 = 2, k_1 = 2, F = 10, \mu_A = 0.5, \mu_B = 1, \mu_C = 0.5$. Initial conditions: $\varphi_1(0) = \pi, \varphi_2(0) = 0, v_1(0) = v_2(0) = 0$

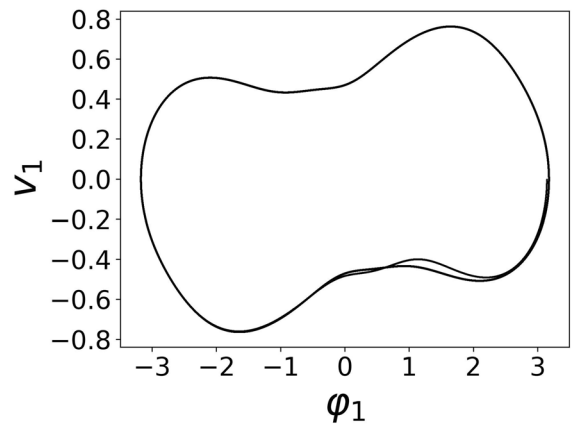


Fig. 15 Limit cycle in presence of Stokes friction, $\Delta \neq 0, \Delta_\mu \neq 0$ and $\frac{A_{11}}{M} \neq \frac{A_{11\mu}}{M_\mu}$. Parameters: $m_A = 1, m_B = 2.5, m_C = 5, l_1 = 1, l_2 = 2, l_3 = 2, k_1 = 2, F = 10, \mu_A = 0.4, \mu_B = 2, \mu_C = 0.5$. Initial conditions: $\varphi_1(0) = \pi, \varphi_2(0) = 0, v_1(0) = v_2(0) = 0$

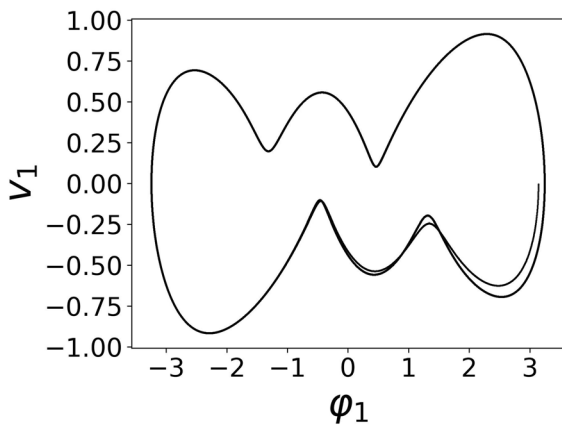


Fig. 14 Limit cycle in presence of Stokes friction, $\Delta \neq 0, \Delta_\mu \neq 0$ and $\frac{A_{11}}{M} \neq \frac{A_{11\mu}}{M_\mu}$. Parameters: $m_A = 1, m_B = 2.5, m_C = 5, l_1 = 1, l_2 = 1, l_3 = 2, k_1 = 2, F = 10, \mu_A = 0.4, \mu_B = 2, \mu_C = 0$. Initial conditions: $\varphi_1(0) = \pi, \varphi_2(0) = 0, v_1(0) = v_2(0) = 0$

that result in adding a torque opposite to the rotation of the system. The friction on the pin O acts on the rod OA , so it acts on the point A ; the friction on the pin A acts on the rod BC , so it acts on the center of mass of the material points B and C . In this case it is more useful to calculate the generalized forces by using the definition.

We calculate the generalized force associated to the friction acting on the pin O

$$\mathbf{A}_O = -\mu_O \hat{\varphi}_2 \hat{\varphi}_2 = -\mu_O \hat{\varphi}_2 \begin{pmatrix} -\sin \varphi_2 \\ \cos \varphi_2 \end{pmatrix}, \tag{51}$$

$$Q_1^O = \mathbf{A}_O \cdot \frac{\partial \mathbf{r}_A}{\partial \varphi_1} = 0 \tag{52a}$$

$$Q_2^O = \mathbf{A}_O \cdot \frac{\partial \mathbf{r}_A}{\partial \varphi_2} = \mu_O l_2 \varphi_2 \cos(2\varphi_2) \tag{52b}$$

and the generalized force associated to the friction acting on the pin A

$$\mathbf{A}_A = -\mu_A \hat{\varphi}_1 \hat{\varphi}_1 = -\mu_A \hat{\varphi}_1 \begin{pmatrix} -\sin(\varphi_1 + \varphi_2) \\ \cos(\varphi_1 + \varphi_2) \end{pmatrix}, \tag{53}$$

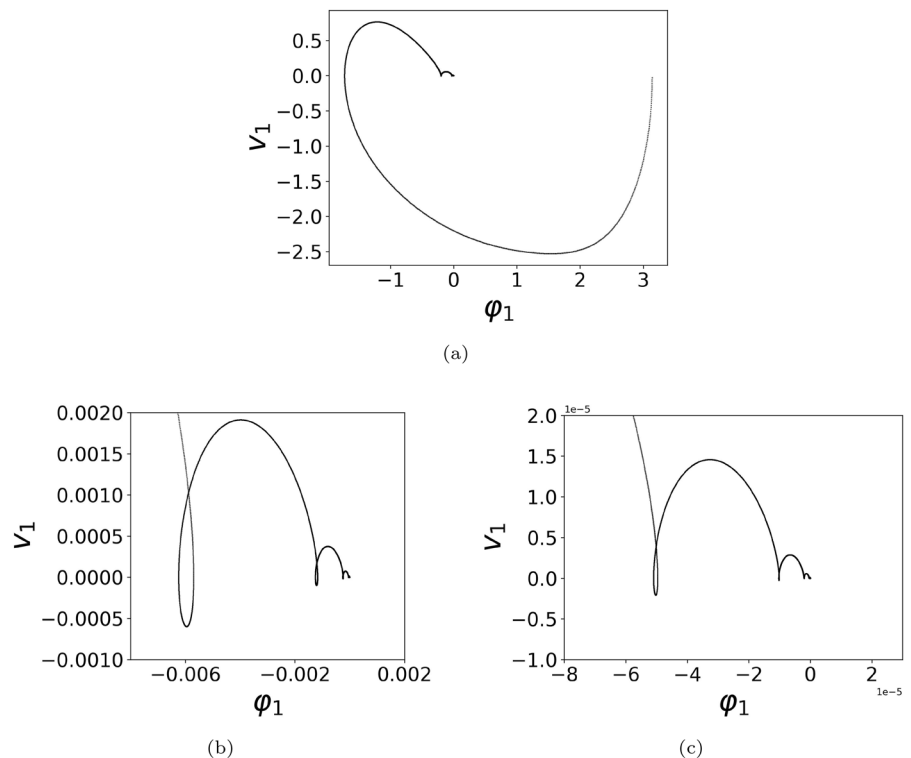
$$\mathbf{r}_{BC} = \frac{m_B \mathbf{r}_B + m_C \mathbf{r}_C}{m_B + m_C} = \begin{pmatrix} l_2 \cos \varphi_2 - \frac{\Delta}{m_B + m_C} \cos(\varphi_1 + \varphi_2) \\ -l_2 \sin \varphi_2 + \frac{\Delta}{m_B + m_C} \sin(\varphi_1 + \varphi_2) \end{pmatrix}, \tag{54}$$

5.2 Friction on the pins

Let us now consider two friction forces generated on the pins O and A represented by

$$\mathbf{A} = -\mu \hat{\varphi} \hat{\varphi}, \quad \mu \geq 0 \tag{50}$$

Fig. 16 "Jumping" motion approaching to an attractive point in presence of Stokes friction; two zooms up to a scale of 10^{-5} are shown. Parameters: $m_A = 1$, $m_B = 1.5$, $m_C = 1$, $l_1 = 1.1$, $l_2 = 3$, $l_3 = 1$, $k_1 = 4$, $F = 10$, $\mu_A = 0.5$, $\mu_B = 0.5$, $\mu_C = 0.5$. Initial conditions: $\varphi_1(0) = \pi$, $\varphi_2(0) = 0$, $v_1(0) = v_2(0) = 0$



$$Q_1^A = \mathbf{A}_A \cdot \frac{\partial \mathbf{r}_{BC}}{\partial \varphi_1} = -\frac{\mu_A \Delta}{m_B + m_C} \dot{\varphi}_1 \cos(2\varphi_1 + 2\varphi_2) \tag{55a}$$

$$Q_2^A = \mathbf{A}_A \cdot \frac{\partial \mathbf{r}_{BC}}{\partial \varphi_2} = -\mu_A \dot{\varphi}_1 \left(-l_2 \cos(\varphi_1 + 2\varphi_2) + \frac{\Delta}{m_B + m_C} \cos(2\varphi_1 + 2\varphi_2) \right). \tag{55b}$$

The equations of motion change as follows: the terms A_{ij} still follow the (6), while $r_{1,2}$ earn some terms due to the presence of frictions

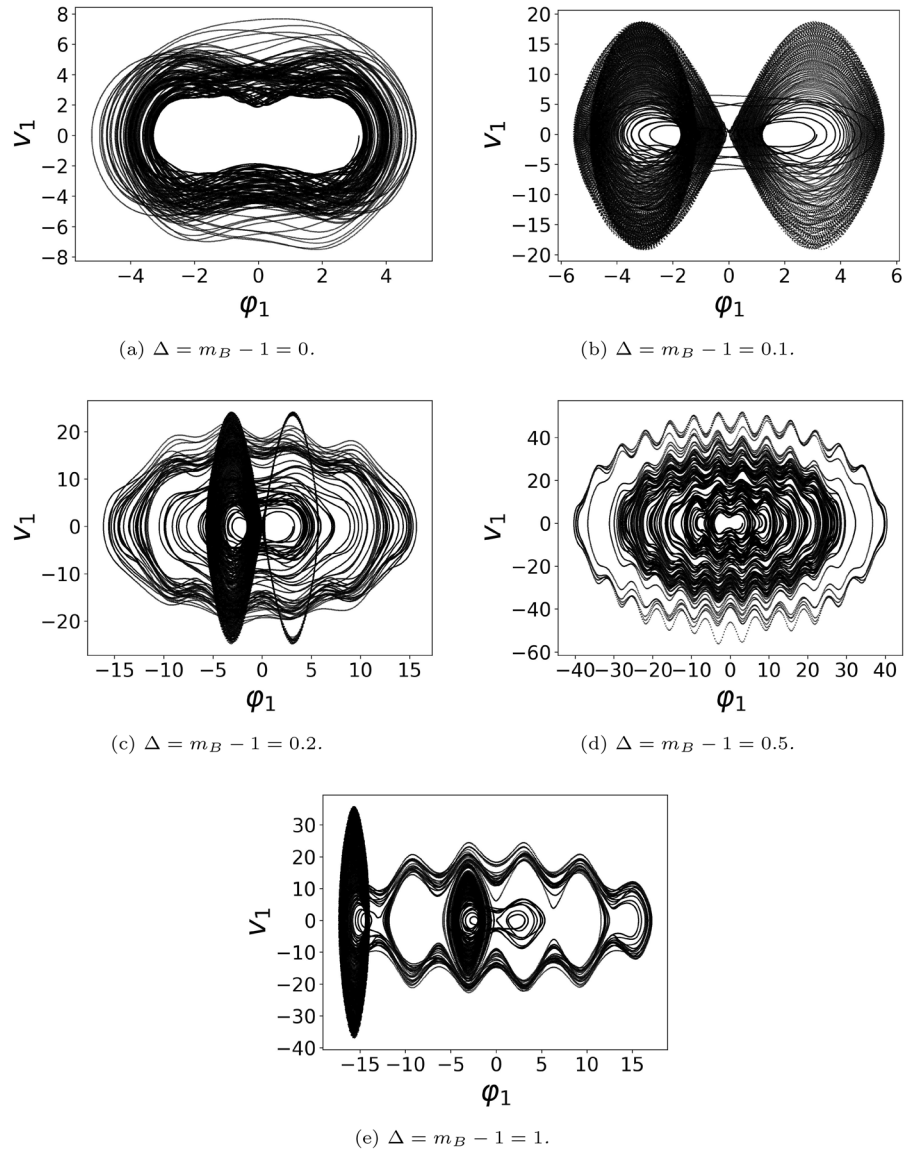
$$r_1 = -k_1 \varphi_1 + \Delta l_2 \dot{\varphi}_2^2 \sin \varphi_1 - \frac{\mu_A \Delta}{m_B + m_C} \dot{\varphi}_1 \cos(2\varphi_1 + 2\varphi_2) \tag{56a}$$

$$r_2 = -\Delta l_2 \dot{\varphi}_1 (\dot{\varphi}_1 + 2\dot{\varphi}_2) \sin \varphi_1 - Fl_2 \sin \varphi_1 + \mu_O l_2 \dot{\varphi}_2 \cos(2\varphi_2) + \mu_A l_2 \dot{\varphi}_1 \cos(\varphi_1 + 2\varphi_2) - \frac{\mu_A \Delta}{m_B + m_C} \dot{\varphi}_1 \cos(2\varphi_1 + 2\varphi_2). \tag{56b}$$

In this case there's no way to obtain the previous integrable cases, so we may expect that the system shows a chaotic behavior, with the holding or the breaking of the known symmetries.

Anyway it could be interesting to analyze the behavior of the system under a progressive breaking of the symmetry $\Delta = 0$; we will do that by holding fixed the initial conditions and all the parameters except for m_B , that will be progressively increased. As shown in Fig. 17a, with the holding of the symmetry the motion starts from the initial point and seems trying to build a limit cycle that expands more and more, up to generate irregular motion. A first small breaking of the symmetry, shown in Fig. 17b, gives rise to a very different behavior; in particular the chaotic motion accumulates on pseudo-spherical shapes, symmetric with respect to the axis $\varphi_1 = 0$. This behavior is mixed with the previous one under a further increasing of Δ , as shown in Fig. 17c. Notice that this pseudo-spherical shape has been already found in [6], for a general non-integrable case of the system. By increasing Δ , as shown in Fig. 17d, the system seems again trying to build limit cycles that expand as the time

Fig. 17 Progressive breaking of the symmetry $\Delta = 0$. Parameters: $m_A = 1, m_C = 1, l_1 = 1, l_2 = 1, l_3 = 1, k_1 = 2, F = 10, \mu_O = 1, \mu_A = 2$. Initial conditions: $\varphi_1(0) = \pi, \varphi_2(0) = 0, v_1(0) = v_2(0) = 0$



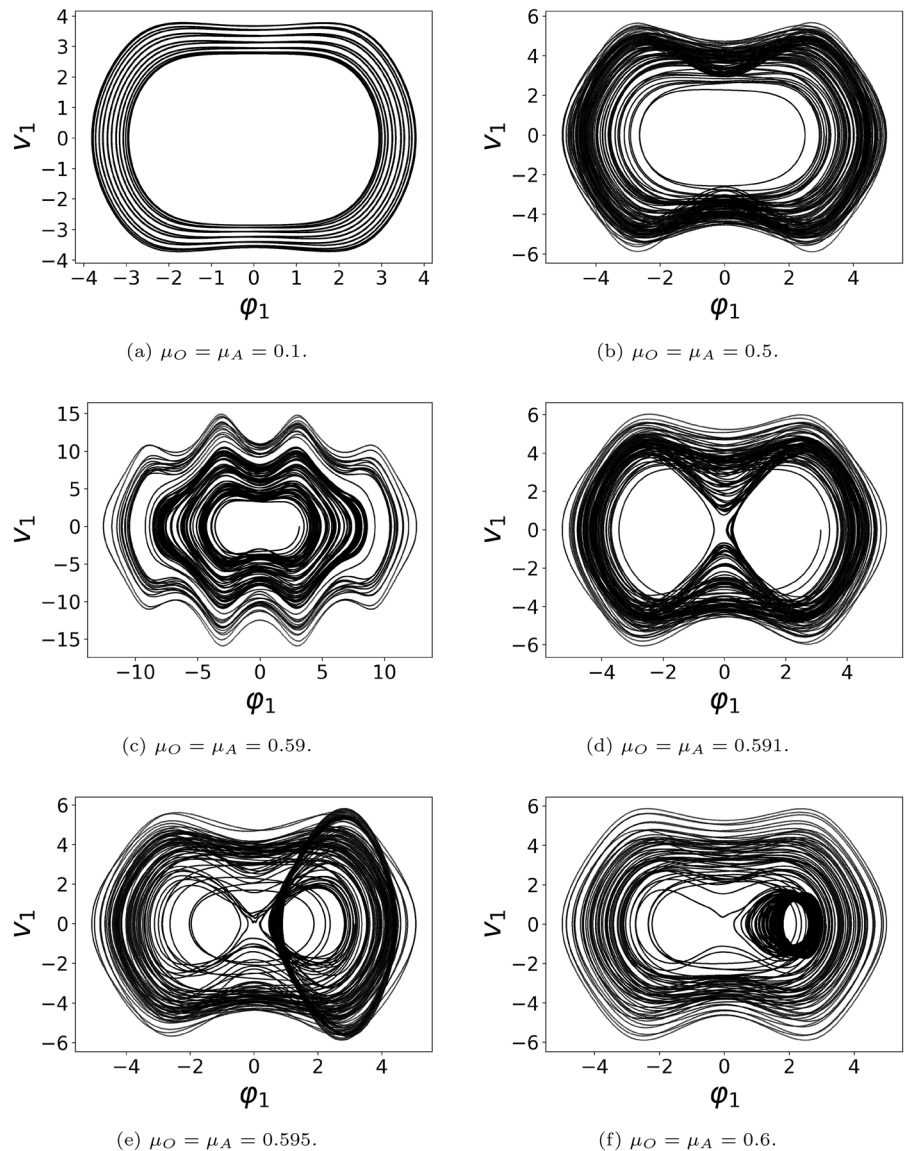
increases, but in a more complex way, while the oscillations of the motion are now more visible. A further increase of Δ gives rise again to a mixed behavior, shown in Fig. 17e; we have now a pseudo-cyclic limit interrupted by the presence of two new chaotic orbits, similar to the previous ones.

The simulations shown in Fig. 17 suggest that, for relatively small breaking of the symmetry $\Delta = 0$, the system mixes two different behaviors: a phase in which it tries to build limit cycles that expand as the time increase, and a phase in which the completely chaotic motion is dominant and it accumulates on a well defined pseudo-spherical shape. This behavior

is confirmed even if we increase Δ (i.e. m_B for our considerations) up to relatively large values; the system keeps to mix the two previous behaviors, with the chaotic orbit that moves along the φ_1 -axis and the possible rising of a second smaller chaotic orbit, even if this progressively leads to a deformation of the pseudo-spherical shape.

A further suggested interpretation is the following: from Fig. 17b, c, we see that the motion accumulates on a symmetrical shape, by going from one side to the other (for a sufficiently large number of iterations we would see that in Fig. 17c the right side will be filled). This back and forth motion reminds

Fig. 18 Progressive rising of friction on the pins. Parameters: $m_A = 1$, $m_B = 2$, $m_C = 1$, $l_1 = 1$, $l_2 = 1$, $l_3 = 1$, $k_1 = 2$, $F = 0$. Initial conditions: $\varphi_1(0) = \pi$, $\varphi_2(0) = 0$, $v_1(0) = v_2(0) = 0$



the Lorenz strange attractor [20], and the fact that this finite portion of the phase space is recurrent for several choices of the parameters could suggest the presence of a strange attractor for the Ziegler pendulum; the appearance of this attractor is actually different from the usual one, since the motion is not totally accumulated on it, as is (for instance) for the Lorenz attractor.

Let us now briefly look for an estimate of the threshold values of the friction coefficients for the transition to chaos. If we set $F = 0$ and $\mu_O = \mu_A = 0$, we of course obtain the integrable Hamiltonian case studied in [6] and the related periodic orbits. By

keeping $F = 0$, as shown in Fig. 18a, a small value for μ_O and μ_A is sufficient to transform the periodic orbit in a quasi-periodic motion, then the system passes through several phases, more or less chaotic; in particular, as μ_O and μ_A increase, the initial closed orbit is split in several and more complex semi-orbits, up to the fast rising of an effective chaotic motion. The value $\mu_O = \mu_A = 0.59$ can be seen as a threshold value for the system, in the sense that it distinguishes between an approximately regular motion (Fig. 18a, b) and an effective chaotic motion (Fig. 18d–f). Furthermore, a comparison between the Lyapunov exponents associated to the different choices of parameters

suggests to interpret this value as related to a maximal chaotic orbit; indeed a Lyapunov exponent $\lambda_{\varphi_1} \sim 0.25$ is associated to the case $\mu_O = \mu_A = 0.59$, while a smaller value is related to lesser and greater friction coefficients.

6 A discrete version

In this section we consider a discrete version of the equations of motion (9). For simplicity, we rename the canonical variables as $(\varphi_1, p_1, \varphi_2, p_2) \mapsto (x, y, z, \omega)$ and the parameters as

$$A_{11} \mapsto a \tag{57a}$$

$$A_{12} \mapsto a - \Delta \cos x \tag{57b}$$

$$A_{22} \mapsto a + b - 2\Delta \cos x \tag{57c}$$

$$r_1 \mapsto -k_1 x + \Delta \omega^2 \sin x \tag{57d}$$

$$r_1 \mapsto -k_2 y - \Delta y(y + 2\omega) \sin x - c \sin x, \tag{57e}$$

with $a > 0, b > 0$; in this way we preserve the physical meaning of the equations. A discrete version of the system (9) can be obtained by simply replacing the derivatives with respect to time with the $(n + 1)$ -th term in the sequences, that is we define a map

$$f : \mathbb{R}^4 \rightarrow \mathbb{R}^4 \tag{58}$$

$$x_{n+1} = f(x_n), \quad n \in \mathbb{N}$$

such that

$$x_{n+1} = y_n \tag{59a}$$

$$y_{n+1} = -(a + b)k_1 x_n + ak_2 z_n + [(a + b)\Delta \omega_n^2 + a\Delta y_n(y_n + 2\omega_n) + ac] \sin x_n + (2k_1 x_n - k_2 z_n)\Delta \cos x_n - [2\Delta \omega_n^2 + \Delta y_n(y_n + 2\omega_n) + c]\Delta \sin x_n \cos x_n \tag{59b}$$

$$z_{n+1} = \omega_n \tag{59c}$$

$$\omega_{n+1} = ak_1 x_n - ak_2 z_n - [a\Delta \omega_n^2 + a\Delta y_n(y_n + 2\omega_n) + ac] \sin x_n - k_1 \Delta x_n \cos x_n + \Delta^2 \omega_n^2 \sin x_n \cos x_n, \tag{59d}$$

where all the variables of the system are defined on the entire real set. The constraint $\Delta = 0$, that represents a non-Hamiltonian integrable case for the standard Ziegler pendulum, is again a symmetry that simplifies the system; in this case the equations (59) become

$$x_{n+1} = y_n \tag{60a}$$

$$y_{n+1} = \frac{1}{ab}[-(a + b)k_1 x_n + ak_2 z_n + ac \sin x_n] \tag{60b}$$

$$z_{n+1} = \omega_n \tag{60c}$$

$$\omega_{n+1} = \frac{1}{ab}[ak_1 x_n - ak_2 z_n - ac \sin x_n] \tag{60d}$$

and they are well defined $\forall (x, y, z, \omega) \in \mathbb{R}^4$, since we imposed $a > 0, b > 0$ from the beginning. We now look for fixed and periodic points of the map (60) in the attempt to prove that the discrete map associated to the Ziegler pendulum is chaotic in the sense of Devaney, i.e. it is topologically transitive and has a dense set of periodic points ([16, 17]).

Before going through the calculations, we compute the Jacobian associated to the system (60)

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{a+b}{ab}k_1 + \frac{c}{b} \cos x_n & 0 & \frac{k_2}{b} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{b} - \frac{c}{b} \cos x_n & 0 & -\frac{k_2}{b} & 0 \end{pmatrix} \tag{61}$$

that has determinant

$$|J| = \frac{k_1 k_2}{ab}, \tag{62}$$

so it is contractive if $k_1 k_2 < ab$, conservative if $k_1 k_2 = ab$ and expansive if $k_1 k_2 > ab$. The eigenvalues of J satisfy

$$\lambda^4 + \lambda^2 \left(\frac{k_2}{b} + \frac{a+b}{ab}k_1 - \frac{c}{b} \cos x_n \right) = \frac{k_2}{b} \left(-\frac{a+b}{ab}k_1 + \frac{k_1}{b} \right). \tag{63}$$

In the sequel the variable written without index is automatically taken as the n -th term of the sequence, that is $x_n =: x, y_n =: y$, etc.

6.1 Fixed points

We look for the fixed points of the map (60), that is $x_{n+1} = x_n$, etc. A trivial one is $(0, 0, 0, 0)$, while for a generic fixed point we have

$$y = x \tag{64a}$$

$$-(a + b)k_1x + ak_2z + ac \sin x = aby \tag{64b}$$

$$\omega = z \tag{64c}$$

$$ak_1x - ak_2z - ac \sin x = ab\omega. \tag{64d}$$

Proposition 1 *The set of fixed points of the map (60) is not dense in \mathbb{R}^4 .*

Proof The fixed points of the map (60) solve the (64), that is

$$-(a + b)k_1x + ak_2z + ac \sin x = abx \tag{65a}$$

$$ak_1x - ak_2z - ac \sin x = abz. \tag{65b}$$

By summing the (65a) and the (65b) we obtain

$$z = -\frac{k_1 + a}{a}x, \tag{66}$$

that replaced in the original system returns

$$\sin x = \frac{[b(k_1 + a) + ak_1 + k_2(k_1 + a)]}{ac}x =: \frac{\alpha}{c}x, \tag{67}$$

where $\alpha > 0$; so a generic fixed points of the map (60) is given by

$$(x_0, y_0, z_0, \omega_0) = \tilde{x}(\alpha)(1, 1, \beta, \beta), \tag{68}$$

where $\tilde{x}(\alpha)$ is one solution of the (67) for a given value of α and $\beta := -\frac{k_1+a}{a} < 0$.

If we restrict ourselves on the interval $x \in [-\pi, \pi]$, the (67) has the only solution $x = 0$ if $\alpha \geq |c|$, while it admits the solution $x = 0$ and two non null symmetric solutions if $\alpha < |c|$, that is

$$b(k_1 + a) + ak_1 + k_2(k_1 + a) < |c|. \tag{69}$$

By extending the domain to $x \in [-n\pi, n\pi]$ with $n \in \mathbb{N}$, we have that for a sufficiently small value of α the (67) admits an odd number of solutions, more and more large as α is close to zero; indeed $x = 0$ is always a solution of (67), while if $x \neq 0$ is a solution, then $-x$ is a solution too, thanks to the odd symmetry of the sine function. Therefore we can write the set of fixed points of the map (60) for a certain choice on α as follows

$$U(\alpha) = \left\{ \tilde{x}(\alpha)(1, 1, \beta, \beta) \right\}_{\sin(\tilde{x}(\alpha)) = \alpha\tilde{x}(\alpha)}. \tag{70}$$

Since $\alpha > 0$, (70) is always a finite set of points. For a certain value of α , the closure of U is equal to the closed interval that joins the smallest and largest solutions of (67); therefore $\bar{U} \neq \mathbb{R}^4$ and U is not dense for any $\alpha > 0$. □

Disregarding for a moment the original domains we imposed for the parameters, we have the following.

Corollary 1 *The set of fixed points of the map (67) is not dense in \mathbb{R}^4 for any real value of the parameters.*

Proof Proposition 1 is symmetrically generalizable for $\alpha < 0$. For $\alpha = 0$ and remembering (70), the set of fixed points of the map (60) is given by

$$U(0) = \left\{ 2k\pi(1, 1, \beta, \beta) \right\}_{k \in \mathbb{Z}}. \tag{71}$$

The points of U are uniquely defined by the real number \tilde{x} , so it is sufficient to find an open interval of \mathbb{R} that contains \tilde{x} but does not contain $\beta\tilde{x}$; furthermore, it is sufficient to find this interval for just one point of U . A trivial choice for $k = 0$ is $I = (0, 2\tilde{x})$ if $\tilde{x} > 0$, $I = (-2\tilde{x}, 0)$ if $\tilde{x} < 0$ or $I = (-\epsilon, \epsilon)$ with $\epsilon < \pi$ if $\tilde{x} = 0$; in any case there exists an open set that does not intersect U , therefore U is not dense. □

As an example we briefly discuss the stability of the fixed points through the following numerical example. If we put

$$\begin{aligned} a &= b = 1 \\ k_1 &= k_2 = 0.5 \\ c &= 3 \end{aligned} \tag{72}$$

the (69) holds and the system (60) admits the following fixed points

$$(0, 0, 0, 0); \quad \pm 0.71623935 \left(1, 1, -\frac{3}{2}, -\frac{3}{2} \right). \quad (73)$$

We can easily establish the stability of these fixed points is verified by substituting in the (63) the previous values for the parameters, to get

$$4\lambda^4 - 12\lambda^2 \left(\cos x_n - \frac{1}{2} \right) + 1 = 0. \quad (74)$$

The fixed point associated to $x = 0$ has four real eigenvalues and two of them have magnitude greater than one, so it is a hyperbolic point; the two fixed points associated to $x = \pm 0.71623935$ have instead four complex eigenvalues, so they are not hyperbolic. Notice that the parity of the cosine in the (63) ensures us that the two symmetric fixed points have always the same stability.

We consider the density of the set of the fixed points since, as we will see, it is strongly suggested that the sets of periodic points of the map can be generally reduced to the set (70), so that from Proposition 1 immediately follows that the map (60) is not chaotic in the sense of Devaney for any choice of the parameters.

6.2 Periodic points of period 2

We look for the 2-periodic points of the map (60), that is $x_{n+2} = x_n$, etc. We have

$$x_{n+2} = y_{n+1} = \frac{1}{ab} \left\{ -(a+b)k_1x_n + ak_2z_n + ac \sin x_n \right\} = x_n \quad (75a)$$

$$y_{n+2} = \frac{1}{ab} \left\{ -(a+b)k_1x_{n+1} + ak_2z_{n+1} + ac \sin x_{n+1} \right\} = y_n \quad (75b)$$

$$z_{n+2} = \omega_{n+1} = \frac{1}{ab} \left\{ ak_1x_n - ak_2z_n - ac \sin x_n \right\} = z_n \quad (75c)$$

$$\omega_{n+2} = \frac{1}{ab} \left\{ ak_1x_{n+1} - ak_2z_{n+1} - ac \sin x_{n+1} \right\} = \omega_n. \quad (75d)$$

Proposition 2 *The set of 2-periodic points of the map (60) is not dense in \mathbb{R}^4 .*

Proof We substitute the terms $x_{n+1} = y_n, z_{n+1} = \omega_n$ in the (75), to get

$$-(a+b)k_1x + ak_2z + ac \sin x = abx \quad (76a)$$

$$-(a+b)k_1y + ak_2\omega + ac \sin y = aby \quad (76b)$$

$$ak_1x - ak_2z - ac \sin x = abz \quad (76c)$$

$$ak_1y - ak_2\omega - ac \sin y = ab\omega. \quad (76d)$$

By summing the (76a) and the (76c) and summing the (76b) and the (76d) we obtain

$$z = -\frac{k_1 + a}{a}x \quad (77a)$$

$$\omega = -\frac{k_1 + a}{a}y, \quad (77b)$$

that is we get exactly the set (70), that has been shown to be not dense; so the map (60) has not a dense set of 2-periodic points. \square

6.3 Periodic points of period 3 and conjecture

We now look for the 3-periodic points of the map (60), that is $x_{n+3} = x_n$, etc. We have

$$x_{n+3} = y_{n+2} = \frac{1}{ab} \left\{ -(a+b)k_1x_{n+1} + ak_2z_{n+1} + ac \sin x_{n+1} \right\} = x_n \quad (78a)$$

$$y_{n+3} = \frac{1}{ab} \left\{ -(a+b)k_1x_{n+2} + ak_2z_{n+2} + ac \sin x_{n+2} \right\} = y_n \quad (78b)$$

$$z_{n+3} = \omega_{n+2} = \frac{1}{ab} \left\{ ak_1x_{n+1} - ak_2z_{n+1} - ac \sin x_{n+1} \right\} = z_n \quad (78c)$$

$$\omega_{n+3} = \frac{1}{ab} \left\{ ak_1x_{n+2} - ak_2z_{n+2} - ac \sin x_{n+2} \right\} = \omega_n. \quad (78d)$$

Proposition 3 *The set of 3-periodic points of the map (60) is not dense in \mathbb{R}^4 .*

Proof We substitute the terms $x_{n+1} = y_n, z_{n+1} = \omega_n$ and $x_{n+2} = y_{n+1}, z_{n+2} = \omega_{n+1}$ in the (78), to get

$$-(a + b)k_1y_n + ak_2\omega_n + ac \sin y_n = abx_n \tag{79a}$$

$$-(a + b)k_1y_{n+1} + ak_2\omega_{n+1} + ac \sin y_{n+1} = aby_n \tag{79b}$$

$$ak_1y_n - ak_2\omega_n - ac \sin y_n = abz_n \tag{79c}$$

$$ak_1y_{n+1} - ak_2\omega_{n+1} - ac \sin y_{n+1} = ab\omega_n. \tag{79d}$$

By summing and subtracting the (79a) and the (79c) we obtain

$$-k_1y = a(x + z) \tag{80a}$$

$$k_2\omega = -(a + 2b)z - ax - c \sin \left[-\frac{a}{k_1}(x + z) \right]. \tag{80b}$$

We can finally explicit y_{n+1} and ω_{n+1} in (79b) and (79d), substitute the previous constraints and get

$$P(x, z) + A \sin x + B \sin \left[P(x, z) + A \sin x \right] = 0 \tag{81a}$$

$$Q(x, z) + C \sin x + D \sin \left[Q(x, z) + C \sin x \right] = E \sin \left[F(x + z) \right], \tag{81b}$$

where A, \dots, F are constants (that depend on the usual parameters) and $P(x, z), Q(x, z)$ are polynomial functions.

Regardless the specific form of the system, we have the following set of 3-periodic points:

$$U = \left\{ \left(\tilde{x}, \tilde{y}(\tilde{x}, \tilde{z}), \tilde{z}, \tilde{\omega}(\tilde{x}, \tilde{z}) \right) \right\}, \tag{82}$$

where (\tilde{x}, \tilde{z}) are solutions of the (81) and $\tilde{y}, \tilde{\omega}$ are obtained from the (80), for a certain choice of the parameters. Once we take a proper redefinition of the sine's arguments, we get for the 3-periodic points an equation analogous to (67); therefore we can generalize the Proposition 1 to this case and conclude that the set of 3-periodic points is not dense in \mathbb{R}^4 .

□

Given the previous results, we are in position to propose the following.

Proposition 4 (conjectured) *The map (60) has not a dense set of periodic points.*

The validity of the previous conjecture is suggested by the fact that, by increasing the period of the periodic points, we get a series of concatenate sine functions, so that the previous considerations can be extended to an arbitrary period. If the Proposition 4 is true, the map (60) is not chaotic in the sense of Devaney for a choice of parameters ($k_2 \neq 0$) that has been numerically proven as a general case of chaotic motion for the Ziegler pendulum. Anyway, there is no reason for the regular or chaotic behavior to be preserved by passing from the continuous version to a discrete version of a specific dynamical system, or viceversa. As an example, the logistic map [21] shows the opposite behavior, since the original discrete system shows a very chaotic behavior for a certain choice of the parameter, as one can see from the associated bifurcation diagram, while its continuous version is a simple ordinary equation with simple and regular solutions.

We may ask ourselves whether the discrete map associated to the Ziegler pendulum is chaotic in the sense of Devaney if we add a dissipative force in the equations, for instance the terms associated to the friction on the pins. Assuming as before $\Delta = 0$ and with a proper redefinition of the parameters, the equations for the discrete system become

$$x_{n+1} = y_n \tag{83a}$$

$$y_{n+1} = \frac{1}{ab} \left[-(a + b)k_1x_n + ak_2z_n + ac \sin x_n - a\mu_O\omega \cos(2z_n) - a\mu_Ay \cos(x_n + 2z_n) \right] \tag{83b}$$

$$z_{n+1} = \omega_n \tag{83c}$$

$$\omega_{n+1} = \frac{1}{ab} \left[ak_1x_n - ak_2z_n - ac \sin x_n + a\mu_O\omega \cos(2z_n) + a\mu_Ay \cos(x_n + 2z_n) \right], \tag{83d}$$

that is we modify the original equations (60) by adding the same term in the sequence defining y_n (with a negative sign) and in the sequence defining ω_n (with a positive sign). It is not difficult to see that this symmetry in the equations allows us to generalize all the previous considerations. Given that, regardless an explicit study of this case, we can conjecture that also

the map (83) is not chaotic in the sense of Devaney, since it has the same identical set of fixed points and similar sets of periodic points when compared with the map (60).

7 Conclusion

In this paper we studied the dynamics of a generalized Ziegler pendulum, when subject to further external forces, both potential and non-potential.

We found that, in general, the presence of gravity destroys the integrability of the system that is present in the original dynamical system if two specific symmetries on the parameters hold, even if isolated periodic orbits can be found for certain choices of parameters and initial conditions. By adding instead a further elastic potential energy that preserves the cyclicity of one of the two generalized coordinates, the integrable cases of the original system survive, so that closed trajectories and families of periodic solutions arise. The physical version of the system does not suggest any relevant considerations, that is the regular or chaotic behavior of the system is essentially the same as if we consider a mathematical, physical or mixed double pendulum.

An interesting behavior is observed if the system is subject to a dissipative force. In the case of fluid-like friction we found a specific symmetry on the friction coefficients under which the system admits attractive points, while in general it approaches to limit cycles. Another formulation of dissipative force has been studied, by considering torsional friction forces on the pendulum pins. By numerically studying a progressive breaking of the non-Hamiltonian symmetry $\Delta = 0$, we observe the emergence of a possible strange attractor, that seems to arise in a different way if compared with the strange attractors such as the Lorenz attractor. Furthermore, a brief study of the transition to chaos has been performed, with the aim of finding a threshold value for the friction coefficients able to distinguish between an approximately regular motion and chaotic orbits.

Finally, we found that the discrete map associated to the Ziegler pendulum, if the non-Hamiltonian symmetry $\Delta = 0$ holds, does not have dense sets of periodic points up to period 3. A qualitative analysis of the periodic points suggests that the map, with this

constraint, is not chaotic in the sense of Devaney, for a choice of parameters that corresponds generally to chaotic orbits for the continuous Ziegler pendulum.

The original Ziegler pendulum introduced in [1] has found several applications in the last decades. The variants analyzed in this work, in particular the one that takes into account the presence of friction on the pins, may be considered as more realistic models in order to improve the mechanical and engineering applications of this dynamical system; furthermore, the occurrence of a limit cycle in presence of fluid-like friction may be taken into account for further studies on optimization and control of the system. Finally, the results here presented relating the discrete map associated to the dynamical system may be developed to further analyze the definition of chaos in the sense of Devaney.

In view of possible future developments several questions can now be asked.

- Given the nature of the sets of periodic points, it should be possible, even if not necessary easy, to prove the Proposition 4 by induction on the period; the main difficulty is related to the quite complex form associated to the equations solved by the k -periodic points for periods $k > 2$.
- For all the variants considered, it is evident that the non-Hamiltonian symmetry $\Delta = 0$ plays a fundamental role in the distinction between regular and chaotic motion, but the reason is not trivial.
- We may wonder why we have a periodic orbit in presence of gravity for the choices of parameters shown in Fig. 3; for these considerations a deep study of the applications of KAM theory for this specific dynamical system could be considered.
- The presence of dissipative forces on the pins seems to give rise to a strange attractor, at least for a specific choice of parameters; in order to state this in a rigorous way, an extensive and detailed numerical study for several values of parameters and initial conditions must be performed.
- Since the Hamilton equations of the standard Ziegler pendulum present a periodic generalized force, the Poincaré–Mel’nikov method [22, 23] can be applied to the system, in order to verify in different fashion the chaotic behavior of the system with the breaking of the mentioned symmetries.

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