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EDITORS
Ceneida Fernández / Salvador Llinares
Ángel Gutiérrez / Núria Planas

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Research Reports (A – H)

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RESEARCH REPORTS (A-H)

UNIVERSITY STUDENTS' DISCOURSE ABOUT IRREDUCIBLE POLYNOMIALS

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The purpose of this paper is to analyse, following Sfard's theoretical framework, the students' discourse about irreducible polynomials in order to understand their difficulties. Through the students' explanations, given during the interviews conducted after carrying out the questionnaire, it is possible to study which processes were triggered by them to solve the task. In particular, we focused on the concept of the irreducible polynomials and the link between the roots of the polynomial and its reducibility in university students.

INTRODUCTION

The focus of this research is to analyse university students' discourse about irreducible polynomials: the link with the roots of the polynomial, the field being worked on, the idea of algebraic or graphic representations of a polynomial. The data were analysed through the theoretical framework of Anna Sfard's commognitive approach (Sfard, 2008).

Nowadays there are not many studies at university level that use this approach apart from a few works concerning Calculus, Group Theory, and the shift to mathematical proof (Nardi et al., 2014). More generally, the literature about the learning of irreducible polynomials appears scarce (if any) and not study about this topic has been carried from the perspective of the commognitive approach.

Adopting a commognitive perspective, the process of teaching/learning can be described as initiation to a discourse. In the case of irreducible polynomial (in the Italian case) such initiation starts in high school and then narratives about such polynomials are often used in scientific faculties (like Physics, Chemistry, Information Technologies, etc.) without discussing again the rules of the discourse. The discourse of lecturers (to which we refer as academic discourse) may be quite different than the discourse developed by students in high school. For instance, Güçler (2013) shows that when a lecturer shifts the discourse on limits (from limit as a number to limit as a process) without making such shifts explicit to the students, they do not even notice it. On the contrary, lecturers' understanding of the difficulties met by students in taking part to the university mathematical discourse may help them in making explicit – at least – their use of words. This appears as a necessary, even if not sufficient, condition to realize an effective communication between students shifting from high school to university and their lecturers (Stadler, 2011; Nardi et al., 2014).

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THEORETICAL FRAMEWORK

Sfard's theoretical framework is based on the notion that thinking is an interpersonal form of communication and she coins the new word 'commognition' to denote the combination of communication and cognition (Sfard, 2008). According to Sfard, mathematics is a discourse and, as such, it is characterized by four features. First, a discourse is determined by the *word use,* meaning the keywords characterizing a discourse. Word use refers both to the use of mathematical terms and of more colloquial words having a specific meaning within mathematics (such as 'field' or 'roots').

Discourses in general, and the mathematical discourse in particular, have specific *visual mediators*: these are visible objects intervening during the communication process (like gestures, inscriptions, drawings, and so on). In this feature we consider mediators of mathematical meaning (such as symbol and algebraic notation) and material objects useful during teaching of the mathematics.
In this context, it is useful to refer to Zazkis and Liljedahl's studies, who studied the representation of prime and irrational numbers, analysing how students perceive and understand these concepts (Zazkis & Liljedahl, 2004, Zazkis, 2005). They distinguish between transparent representations, which completely shows the meaning of the represented structures, and opaque representations, that highlights some aspects of the structures while hiding others. Recent studies of Zazkis and Lilhedal (2004) show that the difficulty of finding a transparent representation prime numbers or irrationals leads to difficulties in learning the concepts themselves. We can conjecture that the same could apply to polynomials.

A discourse is not characterized only by the objects of the discourse, but also by the rules of production of narratives. In Sfard's framework, *narratives* are texts, written or oral, such as descriptions of the objects and links between them. Narratives are submitted to endorsement or rejection, using the processes and rules accepted by the community (such as axioms, deduction rules, accepted definitions, etc.). Furthermore, the discourse is produced following established *routines*. These are repetitive schemes that characterize a discourse. Sfard distinguishes three types of routines:

- Deeds: a routine is called in this way if there is a physical change in the objects. Deeds may be defined as a set of rules that produce or modify the objects;
- Explorations: a routine is called this way if it helps produce endorsed narratives. You can divide it into constructive explorations, which lead to approvable narratives, justificatory explorations, which help to decide to approve a narrative, and recall narrative, which are processes applied to evoke an endorsed narrative;

- Rituals: when there is a sequence of discursive actions, which are intended more to create and maintain a relationship with people (for instance meeting expectations) rather than exploring within the discourse.

Sfard highlights two types of learning:

- Object-level learning: occurs when there is an expansion of the discourse, expanding vocabulary, building new routines, producing new endorsed narratives;
- Meta-level-learning: causes changes in the discourse metarules (these rules define the models for producing and validating object-level narratives). This change means that some familiar tasks will be performed differently and some familiar words will change their use.

Sfard does not believe that students start a meta change on their own (Sfard, 2008). It is possible, in fact, that this change originates from the direct meeting between student and new discourse. This meeting brings a *commognitive conflict*, that is a situation in which individuals apply different metarules. When students move from high school (when irreducible polynomials are usually introduced in the Italian school context) to university, their discourse about irreducible polynomial meet the discourse of scholar experts, which may be incommensurable, meaning that "they do not share criteria for deciding whether a given narrative should be endorsed" (Sfard, 2008, p. 257). Students and experts can use words differently without being aware of such differences. Characterizing university students' discourse when they enter university appear important to then understand how to help them and their lecturers towards a "gradual mutual adjusting of their discursive ways" (Sfard, 2008, p.145).

METHODOLOGY

Inspired by the research of Zazkis and Liljedahl (2004), we created a questionnaire to characterize students' discourse on the irreducibility of polynomials. This research was conducted at the level of university students in science faculties. They had different backgrounds due to different courses they followed and their previous studies. In particular, all of them studied the concept of irreducibility of polynomials at the high school level and not in the university, therefore their answers to the questionnaire were based on previous studies.

In a similar fashion to what Park (2013) did for the concept of derivative, we investigate students' discourse about irreducible polynomials in terms of object-level learning.

The subjects of this study are 14 students from Chemistry, Physics, and Computer Science courses. The questionnaire consists of five open-ended questions preceded by a definition of irreducible polynomials and an example of reducible polynomial. The subjects responded individually to the questionnaire in 30 minutes, and they had to take note of each personal consideration and to explain their answers. Afterwards, we conducted an unstructured interview individually to better understand their explanations. During the interview, we asked them to explicit the processes implemented.

In this paper we focus on two questions from the questionnaire:

- Is the polynomial $x^3 + 6x^2 + 7$ reducible or irreducible?
- Is the polynomial $(x^2 + 1)(x^2 + 2)$ reducible or irreducible?

The example given before the questions is the following:

The polynomial $x^2 – 1 = (x + 1) (x – 1)$ is reducible.

ANALYSIS OF THE COLLECTED DATA AND DISCUSSION

Use of the words

A first feature characterizing the discourse is the use of the words. Analysing the students' productions and their interviews, we noticed an ambiguity in the use of the words roots and solutions. For instance, a student of the degree course in Physics, answering to first question, tried to decompose the polynomial by collecting x^2, but he did not know whether this decomposition was valid. Furthermore, he did not remember how to use Ruffini's rule. He stated:

> "I always forget Ruffini's rule, I have not memorized the process".

From this statement, we conducted the interview to find out what the student thought about the relationship between *reducibility* of polynomial and *roots* of polynomial. The student stated:

> "I do not remember if there is a relationship between the roots of the polynomial and its reducibility. If the polynomial is reduced, this leaves it easier to see the solutions, but if I do not see the solution, that does not mean the polynomial does not have them".

In this second statement, we can see the use of the terms roots and solutions as synonyms. This inappropriate use may be a consequence of the use of the words *equation* and *polynomial* as synonyms.

It is possible to observe that an incorrect use of the word guided the actions of the students. Indeed, another student, answering to second question, stated:

> "It does not say: "is the reduced polynomial reducible?" but only "the polynomial". That is why I did not think that the question was about the initial polynomial."

From this statement we can observe that the use of the term *polynomial* without the word 'reduced' confused the student. This underlines that student did not recognise the product written as a polynomial. This difficulty also crops up in others, in fact, there are some students who solved the product and then adopted known ***routines*** to decompose the obtained polynomial. Many students interpreted the second question in this way: "is the polynomial further decomposable?", this can be seen from their answers, in fact, they answered "not further decomposable" or "irreducible, because it is already reduced".

Visual mediators

We can than explain this behaviour as a misunderstanding in the task (because of the word use), but we could also interpret it as a lack of transparency (for the students) of the provided visual mediator. This representation is not transparent to them, as they interpreted the product as something to be calculated and not as a polynomial.

Referring to visual mediators, we also noticed that no one referred to graphical representation to address the questions. Some students stated that they had difficulty in understanding the presence of roots from given graph.

One student of the degree course in Chemistry answered to the first question:

"I have no tools to say that"

During the interview, we reflected with him on techniques for determining the reducibility of third-degree polynomials. We explored his knowledge about graphical representation of the polynomial.

Interviewer: […] It may be useful to represent it graphically. Have you thought about this possibility?

Student: No, we calculate at most the derivative… but to take it to the graph and understand whether the polynomial is reducible or not, I am not able to.

Interviewer: Are you able to say if there are roots by having the graph?

Student: No… I do not think.

Therefore, some students do not have established routines for using the graphical representation of these mathematical concepts, or they do not consider the production of a graphical representation as an endorsed narrative. A consequence of this is that students are not even able to interpret provided graphs.

Endorsed narrative

We can see the role of endorsed narrative in the discourse also when students try to remember the link between the presence of roots and the reducibility of the polynomial (or the reducibility of the polynomial in the different fields) for example:

"[…] If the polynomial is reduced, this leaves it easier to see the solutions, but if I do not see the solution, that does not mean the polynomial does not have them".

Another student of the degree course in Physics stated:

"Irreducible in real field, it can be reducible in complex field".

We investigated the motivation that led him to this statement. He explained that he tried to use Ruffini's rule, but he did not find any suitable constant to use this technique, therefore he concluded that the polynomial was irreducible in real field. He thought it was possible to extend Ruffini's rule to searche roots in the complex field too.

Routines

The use of rote-learned theorems or deduction rules are the most common routines to perform the tasks. The students applied different procedures to solve the questions: Ruffini’s rule, decomposition with factoring trinomial, total or partial collecting and finding the solutions of the equation associated to the polynomial. As noted above, for the second question, students needed to solve the product and then applied the chosen routine. One student of the degree course in Chemistry answered, for example:

> “This is the result of the reducible polynomial $x^4 + 3x^2 + 2$, through decomposition with factoring trinomial”.

Thanks to this argument, we can deduce that student was able to recognise the reducibility of this polynomial after calculating the product and decomposing it with factoring trinomial. Moreover, he said that he was not able to decompose more this polynomial with the knowledge at his disposal.

This underlines the difficulty of the students in identifying the polynomial written as a product. Moreover, we can observe that some of them had problems with the equality operator. In fact, one student of the same course, after solving the product, stated:

> Student: Reducible because this is given by the decomposition of $x^4 + 3x^2 + 2$?
>
> Interviewer: What are your doubts about this solving process?
>
> Student: I do not know if this decomposition is right. If this decomposition is right, the polynomial is reducible.

This highlights the difficulty of the students in interpreting the equal symbol as an equivalence, but they see it as a one-way procedural operator.

Many of the students’ routines can be classified as “deeds” because they appear to be acting on the algebraic symbolism more than on the involved mathematical objects.

Some students stated that they did not remember the Ruffini’s rule:

> “I always forget Ruffini’s rule, I have not memorized the process”.

From this statement it can be observed that students sometimes apply repetitive procedures, in a ritualistic way, apparently without understanding the meaning behind them. In fact, some of them tried to use Ruffini’s rule to find the roots of the polynomial in the real field and in some cases in the complex field.

> “Irreducible in real field, it can be reducible in complex field”.

These activities underline how students interpret the Ruffini’s rule like a decomposition technique and not a procedure to find the roots of the polynomial. These routines can be classified as “rituals” because they applied the repetitive sequences to solve the task without questioning when this procedure might be applied. Most likely, this procedure was introduced by their high school teachers (considered as the "ultimate substantiators" of their narratives, Sfard, 2008, p. 234) and applied in a ritualistic way during high school.

CONCLUSIONS

This study focused on university students' discourse about the concept of polynomial reducibility according to Sfard's theoretical framework.

Some of students' arguments are inconsistent with the academic discourse about reducible polynomials, this highlights the importance of their motivations to understand their discourse and then their difficulties in approaching the academic discourse about this concept.

To the first question almost all participants stated that $x^3 + 6x^2 + 7$ is irreducible, but they used different motivations to justify their answer.

This question was used to investigate the students' ability to apply techniques different than Ruffini's rule. Analysing the arguments, it can be seen that many students used Ruffini's rule in a ritualistic way and some students were confused about the relationship between roots of the polynomial and the reducibility of the polynomial. We explained this difficulty as an ambiguity in their use of words as 'polynomial' and 'equation'.

Taking into account the second task, most of the students answered that $(x^2 + 1)(x^2 + 2)$ is reducible, but there were also few of them saying that the polynomial is not reducible or that it depends on the field. This question was designed to investigate the students' ability to work with fourth-degree polynomials and their ability to recognise properties highlighted by (what we considered as) a transparent visual mediator. This is the why we gave them a decomposed polynomial. However, many students had problem with the meaning of the question, in fact, they interpreted the question in this way: "is the polynomial further decomposable?". This may depend on the lack of transparency (for them) of the visual mediator, or on the different metarules applied to the discourse. Apparently, for some of them, factorization routines must be applied to a polynomial that is not already expressed as a product of polynomials.

Thanks to interviews conducted, we were able to characterize the discourse they use to justify their solutions. The difference between their discourse and the academic discourse explain why they are not able to reason in unfamiliar context or have difficulties with questions posed in different way than what they saw in high school. As pointed out by Sfard (2001) "one has no chance to modify one's discursive habits on her own. In order to change them, one has to be led outside her own discourse by others. Only then can the conflict necessary to create the learning-engendering experience of incomprehension eventually arise" (p. 47). The conducted analysis suggests that, for narrowing the distance between students' discourse and the academic one, several lines of intervention could be adopted by university lecturers. It would be important to expand their discourse both by discussing word use, visual mediators, routines, and endorsed narratives. Results from this study provide significant hints for designing teaching experiments with such goal.

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